BROADCASTING IN HARARY GRAPHS

Shreelekha Tanna

A thesis

in

The Department

of

Computer Science And Software Engineering

Presented in Partial Fulfillment of the Requirements

For the Degree of Master of Computer Science

Concordia University

Montréal, Québec, Canada

January 2015

Concordia University

School of Graduate Studies

This is to certify that the thesis prepared

By: **Shreelekha Tanna**

Entitled: **Broadcasting in Harary Graphs**

and submitted in partial fulfillment of the requirements for the degree of

Master of Computer Science

complies with the regulations of this University and meets the accepted standards with respect to originality and quality.

Signed by the final examining commitee:

______________________________ Chair
Dr. O. Ormandjieva

______________________________ Examiner
Dr. B. Jaumard

______________________________ Examiner
Dr. D. Goswami

______________________________ Supervisor
Dr. H. A. Harutyunyan

Approved ______________________________

Chair of Department or Graduate Program Director

__________ 20 ____ ______________________________

Amir Asaf, Ph.D., Dean

Faculty of Engineering and Computer Science

Abstract

Broadcasting in Harary Graphs

Shreelekha Tanna

With the increasing popularity of interconnection networks, efficient information dissemination has become a popular research area. Broadcasting is one of the information dissemination primitives. Broadcasting in a graph is the process of transmitting a message from one vertex, the originator, to all other vertices of the graph. We follow the classical model for broadcasting.

This thesis studies the Harary graph in depth. First, we find the diameter of Harary graph. We present an additive approximation algorithm for the broadcast problem in Harary graph. We also provide some properties for the graph like vertex transitivity, circulant graph and regularity.

In the next part we introduce modified harary graph. We calculate the diameter and broadcast time for the graph. We will also provide 1-additive approximation algorithm to find the broadcast time in the modified harary graph.

Acknowledgments

I would like to express my sincere gratitude to my supervisor Prof. Hovhannes Harutyunyan for the continuous support of masters study and research, for his patience, motivation, enthusiasm, and immense knowledge. His guidance helped me in all the time of research and writing of this thesis. I could not have imagined having a better advisor and mentor.

I would like to dedicate this thesis to my husband for his constant influence and encouragement. I would also like to thank my parents and my sister for their endless love and support.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

The need for high performance computing is constantly increasing. Examples of applications which require high performance computing are medical imaging, financial trading, data warehousing, bioscience, data security and many more. Parallelism where more than one processors work on solving the problem, has been employed for many years in high performance computing. One of the most common model for parallel computing is MIMD(Multiple Instruction and Multiple Data) which is referred as multicomputers or multiprocessors. The different processors working in parallel will require to exchange data with each other. This can be either done by shared memory or through an interconnection network. Shared memory multicomputers have a limitation on the number of processors that can be connected together. Thus, it is more practical if processors, which have their own memory, communicate with each other through an interconnection network. The performance of multicomputers depends not only on the processing power but also on the performance of the interconnection network in terms of information dissemination. To improve the performance of the interconnection network, structural properties like ease of routing, fault tolerance, cost, etc. must be taken into account.

Our main interest in harary graph lies in the role they play in design of networks. In this context, connectivity is of interest as a measure of reliability, while diameter is the measure of transmission delay. Hence given certain parameters for graph, we will be interested in maximizing the connectivity and minimizing the diameter of the graph. As well as from the expense point of view, it is more practical to keep the links of the network to a minimum.

Models of Broadcasting

The distribution of information can be classified into four main classes:

- Routing, distribute information from one to one;
- Broadcasting, distribute information from one to all;
- Multi-casting, distribute information from one to multiple, but not all;
- Gossiping, distribute information from all to all.

Broadcasting is the problem of information dissemination in which one message needs to be transmitted to the group of nodes connected by an interconnection network [36]. This problem finds applications in areas such as fixed and mobile networks, internet messaging, supercomputing, multimedia, epidemic algorithms, replicated databases, etc. Broadcasting is assumer to take place in distinct time units. Each time unit is called a round. During each round, a series of calls, i.e. message-exchanging takes place between adjacent vertices in parallel. The total number of rounds after which all nodes of the network are informed is used to measure the broadcast time.

To simplify the analysis of broadcast process, we assume that broadcasting is done according to the following constraints. This broadcast model is known as the classical or original model.

- Each call requires one unit of time, i.e. one round.
- A vertex can participate in only one call per time unit.
- Each call involves only one informed vertex and one of its uninformed neighbors.
- In one unit of time, many calls can be performed in parallel.

Initially, only one vertex u, called the *originator*, is informed. During each round, all informed vertices will send information to one of their uninformed adjacent vertices. A *call* is the process of transferring information from informed node to an uninformed neighbor.

A *broadcast scheme* of an originator u is a sequence of parallel calls that completes broadcasting in the network originated at u. An optimal broadcast scheme informs all the vertices in the minimum number of time units.

In general, any interconnection network can be represented as a connected graph $G(V, E)$. Here the vertices V represent the nodes or the processors inside interconnection network, and the edges E represents the communication links between the processors. Next paragraph presents some of graph theoretic definitions that are used in the rest of this thesis.

Given a graph $G = (V, E)$ and originator $u \in V(G)$, The broadcast time of u which is represented by $b(u)$ or $b(u, G)$, is the minimum required time to finish broadcasting from u.The broadcast time of graph G, is denoted as $b(G)$ is defined as $b(G) = max\{b(u) \mid u \in V\}$. The broadcast center of the given graph $G = (V, E)$, denoted $BC(G)$, is defined as $BC(G) = \{v \mid v \in V, b_{(v,G)} = b_{min}(G)\}$. Thus, $BC(G)$ is the set of vertices in G, with the smallest broadcast time.

A vertex $v \in V$ is said to be a *neighbor* of $u \in V$, if there exists an edge $e \in E$ between u and v such that $e = (v, u)$. A path between two vertices is the sequence of edges that connect the two vertices together. The distance between two vertices u and v, $d(u, v)$ is the length of the shortest path between the two vertices. The diameter of a graph G, denoted by $D(G)$, is the longest distance between any pair of vertices in the graph. A graph G is called a *connected graph*, if there exists at least one path between any pair of vertices of graph G. The degree of vertex v, $\delta(v)$, in an undirected graph $G(V, E)$, is the number of neighbors that v have. The degree of $G(V, E)$, $\Delta(G)$, is the maximum degree of its vertices, $\Delta(G) = max_{v \in V}\{\delta(v)\}$. Let $V(G)$ be the vertex set of a simple graph G and $E(G)$ its edge set. Then a graph *isomorphism* from a simple graph G to a simple graph H is a bijection $f : V(G) \to V(H)$ such that $u, v \in E(G)$ if and only if $f(u)f(v) \in E(H)$. An *automorphism* of a graph is a graph isomorphism with itself, i.e., a mapping from the vertices of the given graph G back to vertices of G such that the resulting graph is isomorphic with G. A graph $G(V, E)$ is called *vertex transitive* if for any two vertices $u, v \in V$ there is an automorphism of G mapping u to v. For all other definitions of graph theoretic terms refer to [61].

Finding the broadcast time of a given vertex in random graph is proved to be *NP-complete* in [59]. The proof used a reduction of the *three-dimensional matching (3DM)* problem to the broadcast time problem. In [54] it is shown that the broadcast time problem remains *NP-complete* even for regular planar graphs of degree 3. The minimum broadcast time problem is addressed by many different approximation algorithms and heuristics with reasonably good proximities and running times.

Only for very few graph families, an algorithm for the broadcast time problem with a polynomial time complexity is known. An algorithm with linear time complexity for the exact solution of the broadcast time problem in trees is presented by Slater, Cocakyne and Hedetniemi in [59]. Their algorithm finds in linear time the broadcast center of the given tree, and using it, determines the broadcast times of all vertices of the tree. In [60] Proskurowski suggested another linear time algorithm for the same problem, which without finding the broadcast center, determines the broadcast time of a vertex in a tree. A linear time algorithm is also known for the unicyclic graphs (connected graphs with only one cycle) [33, 34]. Algorithms for the exact solution of the broadcast time problem for a few other tree-like graph families are presented in [52].

In the literature, there are two main broadcasting problem. The first one, called the *minimum broadcast time* problem, can be described as follows: given an originator u, find a broadcast scheme such that the time required to transmit information inside graph G is minimized. It is easy to conclude that for any vertex u in connected graph G with n vertices, $\lceil \log n \rceil \leq b(u) \leq n - 1$, since during each time unit, number of informed vertices can at most doubled.

The second one called, *minimum broadcast graph* problem, can be defined as follows: find a network architecture on n vertices with broadcast time $\lceil \log n \rceil$ and a minimum number of edges. Clearly the answer to above problems is highly dependent on the model of communication used.

At the communication link level, there are two models discussed in literature [36]:

- **One way mode -** also called *telegraph mode* or *half duplex*. In this mode, one link many be used in only one direction during a communication process involving two adjacent nodes. This process can be modeled using directed graphs as the underlying topology.

- **Two way mode -** also called *telephone mode* or *full-duplex*. In this mode, one link may be used in both direction during a communication process involving two adjacent nodes. This process can be modeled using undirected graphs as the underlying topology.

Following [15], communication in interconnection networks can be classified based on the ability of the vertices to communicate simultaneously with their neighbors:

- **Processor bound:** also called *1-port* or *whispering*, in which a vertex can communicate only with one neighbor at a time.

- **Link-bound:** also called *n-port* or *shouting*, in which a vertex can call all its neighbors simultaneously.

Here we have to mention *k-broadcasting* which is a model of broadcasting that fits between the two above-mentioned models. Some authors also refer to it as *c-broadcasting* [19, 46, 45, 65]. In this model, a vertex can call simultaneously up to k of its neighbors. A considerable amount of literature [42, 26, 25, 27, 28, 64] is dedicated to this model which is useful to the study of DMA-bound system [44] or on computing functions in network [3, 7].

Another issue in characterizing communication in networks in the necessary time for a message to be prepared, to travel along an edge, and to be received. There are two widely used models addressing this issue:

- The **constant model**, in which the time needed to transmit and receive a message is constant, $T = const$.

- The **linear model**, in which the time needed to communicate is modeled as $T = \beta + L\tau$ where β is the cost of preparing the message, L the length of the message, and τ the propagation time of a data unit length.

Using different assumption regarding the communication model employed, other broadcasting models have been developed and analyzed:

- **Broadcasting with universal lists**

  In *broadcasting with universal lists*, also known as *orderly broadcasting*, a (universal) broadcasting scheme is a function of assigning a single ordered list of its neighbors, called the universal list, to every node. The list is determined regardless of source and the node will transmit the received message in the order of the list.

  This model has two versions: *adaptive* and *nonadaptive*. In the adaptive version, each informed vertex tracks the vertices from which it receives the message. During sending the message, it skips them from its universal list. In the nonadaptive version an informed vertex does not know from which neighbor the message comes from. It sends the message to all of its neighbors.

  The adaptive model of broadcasting with universal lists was introduced in [62] where the broadcasting in trees under this model was studied. In [11, 12], authors introduced the non-adaptive model of broadcasting with universal lists. They studied both adaptive and non-adaptive models in trees, rings, grids, toruses and complete graphs. An upper bound on nonadaptive broadcast time for two dimensional tori is presented in [35], where this broadcast model was studied under the name orderly broadcasting. In [40] the nonadaptive version of this model was studied in paths, complete k-ary trees, grids, complete graphs, and hypercubes. The nonadaptive broadcasting g in trees was recently studied in [32].

- **Messy broadcasting**

  The messy broadcasting model was introduced in [2]. The main difference of this model, from the classical model, is that here the protocols of nodes are not coordinated. In this model, a node knows nothing about the topology of the network or the broadcast originator. The behavior of a node only depends on its all or some neighbors. Depending on the amount of information available for each node about its neighbors, 3 sub models were considered. Below, these sub models are listed in the decreasing order of information available in each node.

  - **Model** M_1 **:** Each node knows the state of its all neighbors, i.e which are informed and which are not. Using this information, in each time unit, an informed node sends the message to one of its uninformed neighbors.
  - **Model** M_2 **:** Each node keeps a list of all neighbors from which it received a message or sent a message. In each time unit, it sends the message to a neighbor not present in the list.
  - **Model** M_3 **:** Each node keeps a list of all neighbors to which it sent a message. In each time unit, it sends the message to a neighbor not present in the list.

  We note that in M_2 and M_3 models, a node may send a message to a node which is already informed. This is a price paid for the simplicity of the broadcast protocols.

The exact values for the worst-case broadcast time of various graphs such as complete graphs, paths, cycles, and complete d-ary trees for all three sub models are presented in [31]. More recent results are presented in [30] . The exact values of the worst-case broadcast time in M_1 and M_2 models and bounds in M_3 model are given for the hypercube. In [10, 24] multidimensional directed tori and complete bipartite graphs are studied. The average-case broadcast time of stars (claws), paths, cycles, complete d-ary trees and hypercubes are studied in [47].

The same model under the name *broadcasting in unknown networks* is studied in [18]. Very similar model with a limited knowledge of the network topology is studied in [53], where a vertex knows the topology of the network only within knowledge radius r from it.

- **Radio broadcasting**

  In the radio broadcasting model, an informed vertex in each time unit can send the message to all its neighbors simultaneously. Note that a vertex cannot send the message to a strict subset of its neighbors. A vertex is considered as informed if it receives the messages from precisely one neighbor in a certain time unit. The intuition behind this constraint is that a message received from more than one neighbor in the same time unit gets corrupted. There is a considerable amount of literature regarding this model.

- **Fault-tolerant broadcasting**

  The fault-tolerant communication is a huge area of research. It is assumed that the nodes and/or links of the network are not reliable. A faulty link may stop transmit messages and a faulty node may stop to send or receive messages. Numbers of results are presented specially for broadcasting and gossiping in faulty networks.

  The *fault-tolerant broadcasting* model was introduced in [48], where the k-tolerant broadcast function $B_k(n)$ is defined. $B_k(n)$ is the minimum number of links in a network supporting k-tolerant broadcasting from any originator in theoretical smallest possible time. This research was continued in [9].

  In [1], the *minimal k-fault tolerant broadcast graphs* were studied. Authors generalize the minimum broadcast graph problem. They study the networks where up to k links may fail, but any originator still must be able to complete the broadcasting in optimal time.

The survey article [58] provides detailed overview of the fault-tolerant communication problem in the context of broadcasting and gossiping. Large source of information about broadcasting and related problems can be found in [37].

Chapter 2

Literature Review

First section of this chapter briefly reviews the commonly used interconnection topologies. A topology is a schematic or geometric description of the arrangement of an interconnected network (graph), including its processors (vertices) and connecting links (edges). Section 2.2 describes broadcast graph problem and lists currently known values of broadcast functions.

2.1 Commonly used Topologies

This section reviews the commonly used topologies on basis of three important communication parameters: (i) the degree, (ii) the diameter, and (iii) the broadcast time. Table 2.1 summarizes properties of all the topologies. It also includes number of vertices and edges in respective topologies.

The path P_n is a tree with two end nodes of vertex degree 1, and the remaining $n-2$ nodes of vertex degree 2, thus the maximum degree of P_n is 2. The $D(P_n) = b(P_n) = n-1$. A path is therefore a graph that can be drawn so that all of its vertices and edges lie on a single straight line. Figure 1 shows a path with seven vertices, where $D(P_7) = b(P_7) = 6$.

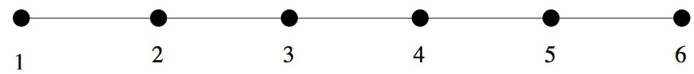

Figure 1: The Path graph for $n = 6$

The Cycle C_n

Cycle C_n, $n \geq 3$, is a simple graph with vertices $v_1, \ldots, v_n$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$. In other words cycle C_n is a path such that the start vertex and end vertex are also connected by an edge. C_n has n vertices and the maximum degree is 2. The $\mathrm{D}(C_n) = \lfloor \frac{n}{2} \rfloor$ and the $\mathrm{b}(C_n) = \lceil \frac{n}{2} \rceil$. Figure 2 demonstrates C_4 and C_6, where the diameter and broadcast time of C_4 is 2 and that of C_6 is 3.

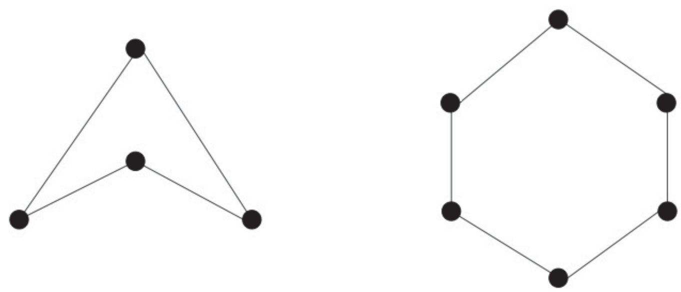

Figure 2: The Cycle graph for $n = 4$ and $n = 6$.

The Complete graph K_n

A complete graph K_n is a simple graph with exactly one edge between any pair of distinct vertices. K_n has n vertices and degree $n - 1$. The diameter of K_n is 1. K_n is a broadcast graph because during each time unit the number of informed vertices is doubled, thus $b(K_n) = \lceil \log_2 n \rceil$. Figure 3 demonstrates K_4 and K_6, where the broadcast time of K_4 is 2 and that of K_6 is 3.

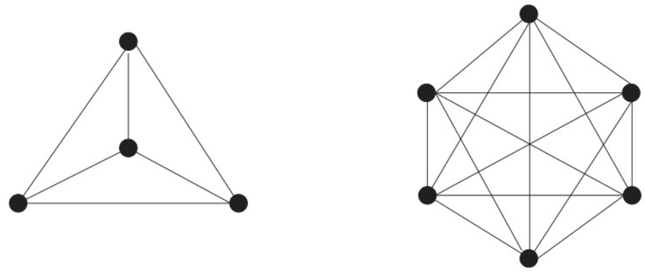

Figure 3: The Complete graph for $n = 4$ and $n = 6$.

The Hypercube H_n

The hypercube of dimension n, denoted by H_n, is a simple graph with vertices representing 2^n bit strings of length $n, n \geq 1$ such that adjacent vertices have bit strings differing in exactly one bit position. H_n has 2^n vertices and $n \cdot 2^n - 1$ edges. The diameter of H_n is n and each vertex has exactly degree n. As illustrated in Figure 4 $(n+1)$-dimensional hypercube can be constructed from two $n-$dimensional hypercubes by connecting each pair of the corresponding vertices. H_n is the minimum broadcast graph. The b$(H_n) = \lceil \log_2 2^n \rceil = n$.

The Cube-Connected Cycles CCC_n

CCC_n is a modification of the hypercube H_n by replacing each vertex of the hypercube with a cycle of n vertices. The i^{th} dimensional edge incident to a node of the hyper-node is then connected to the i^{th} node of corresponding cycle of the CCC_n. Thus, CCC_n has $n \cdot 2^n$ nodes and the maximum degree is 3. The D$(CCC_n) = 2n + \lceil \frac{n}{2} \rceil - 2$. The b$(CCC_n) = \lceil \frac{5n}{2} \rceil - 1$, first every informed vertex sends the message to the hypercube neighbor, then to the right neighbor on the ring, and finally to the left one. Figure 5 shows a 3-dimensional cube connected cycle.

The Shuffle-Exchange SE_n

SE_n is the graph whose vertices can be represented by binary strings of length n. Each edge of SE_n connects vertex βa, where β is a binary string of length $n - 1$ and a is in $\{0, 1\}$, with vertex

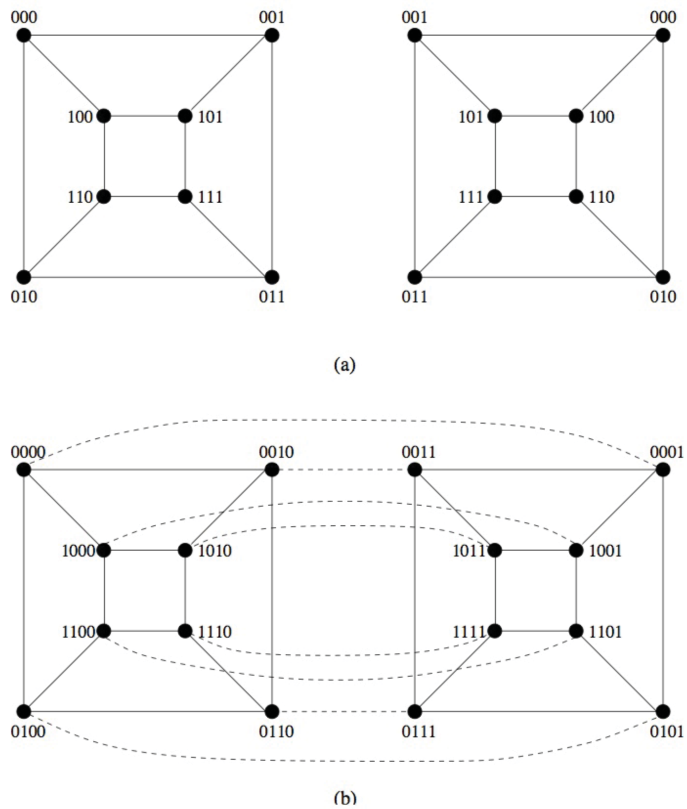

Figure 4: The Hypercube graphs

βc and vertex βa, where c is the binary complement of a. SE_n has 2n vertices and the maximum degree is 3. The D(SE_n)=$2n-1$ and it is provided that $b(SE_n) \leq 2n-1$. Figure 6 represents a Shuffle-Exchange graph SE_3.

The DeBruijn DB_n

DB_n is the graph, whose nodes can be represented by binary strings of length n and whose edges connect each string βa, where β is a binary string of length $n-1$ and a is in $\{0,1\}$, with the strings βb, where b is a symbol in $\{0,1\}$. DB_n has 2^n vertices with the maximum degree 4 and the diameter is n. [57] provides the lower bound $b(DB_n) \geq 1.3171n$, and [5] proves the upper bound, $b(DB_n) \leq 1.5n + 1.5$. Figure 7 illustrates a DeBruijn graph of dimension 3.

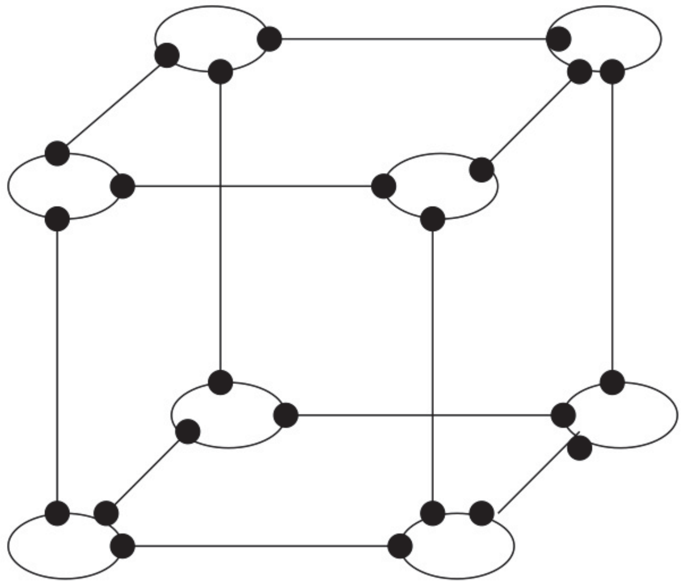

Figure 5: The CCC_3

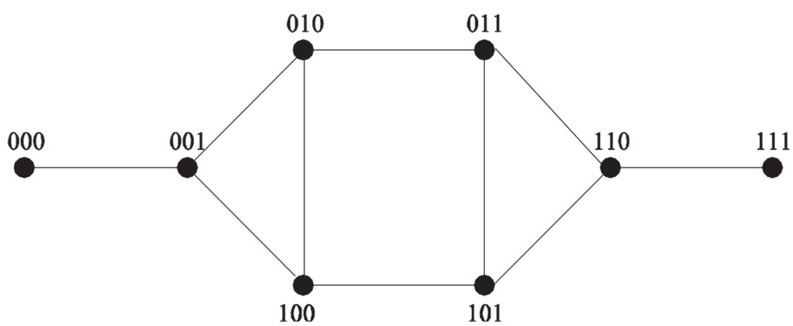

Figure 6: Shuffle Exchange Graph SE_3

The d-Grid $G[a_1 \times a_2 \times \cdots \times a_d]$

The d-dimensional grid (or mesh) is the graph whose nodes are all d-tuples of positive integers $(z_1, z_2, \ldots, z_d)$, where $0 \leq z_i < a_i$ for all $i(1 \leq i \leq d)$, and whose edges connect d -tuples, which differ in exactly by coordinate one. For example, in $G[3,3]$, vertex $(1,1)$ is connected to vertices $(0,1), (2,1), (1,0)$ and $(1,2)$. $G[a_1 \times a_2 \times \cdots \times a_d]$ has $a_1 \times a_2 \times \cdots \times a_d$ vertices with the maximum degree $2d$, if each a_i is at least 3. The diameter of d-Grid G$[a_1 \times a_2 \times \cdots \times a_d]$ is $(a_1 - 1) + (a2 - 1) + \cdots + (a_d - 1)$ and [reference] provides the $b(G[a_1 \times a_2])$=$a_1 + a_2 - 2$. Figure 8 shows a 2-Grid graph $G[4 \times 5]$.

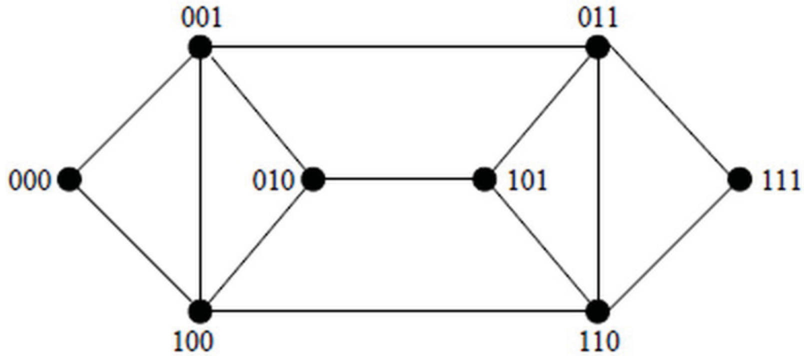

Figure 7: DeBruijn Graph DB_3

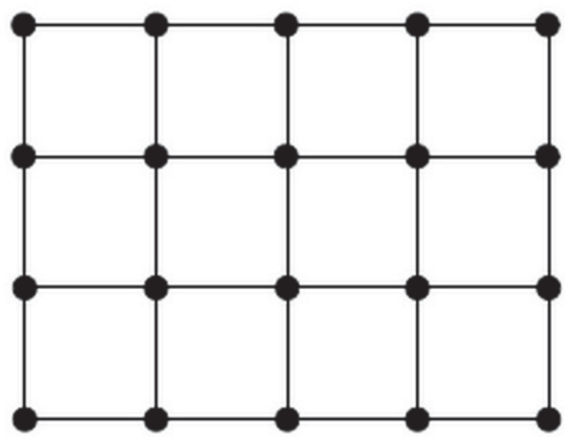

Figure 8: 2- Grid graph $G[4 \times 5]$

The d-Torus T

A d-Torus graph is a d-grid graph with both ends of rows and columns connected. $T[a_1 \times a_2 \times \cdots \times a_d]$ denotes the d-Torus graph. The diameter of $k \times k$ X-Torus is given in [reference], that is $\lfloor \frac{k}{2} \rfloor + 1$ if k is odd, and $\lfloor \frac{k}{2} \rfloor$ if k is even. It is proven in [reference] that the optimal broadcast time of 2-Torus graph is $\lceil \frac{a_1}{2} \rceil + \lceil \frac{a_2}{2} \rceil$, when a_1 or a_2 is even; and it is $\lceil \frac{a_1}{2} \rceil + \lceil \frac{a_2}{2} \rceil - 1$ when both a_1 and a_2 are odd. The bounds on the broadcast time of Torus are $D \leq b(T[a_1 \times a_2 \times \cdots \times a_d]) \leq D \; + \; max(0, m-1)$, where $D = \sum_{i=1}^{d} a_i - d$, and m is the number of odd a_i. Figure 9 shows a 2-Torus graph $T[4 \times 3]$.

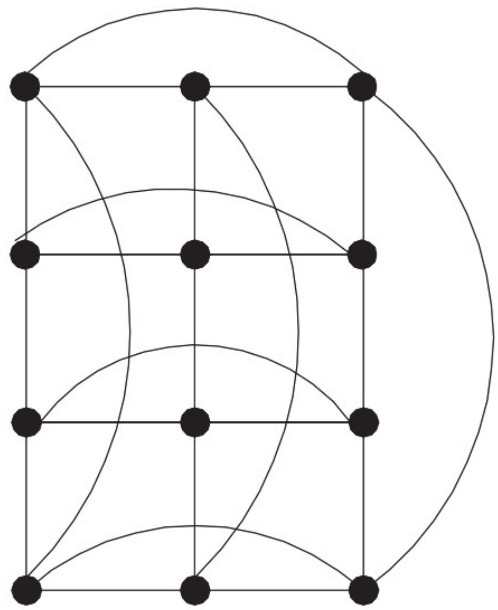

Figure 9: 2-Torus graph $T[4 \times 3]$

Recursive Circulant graph $RC(n, d)$

The recursive circulant graph $RC(N, d)$ is introduced by Park and Chwa [56]. We define the recursive circulant graph $RC(N, d) = (V, E)$ with $d \geq 2$, to be a graph where, $V = 0, 1, \ldots, n-1$, and the edge set $E = \{uv | \exists i, 0 \leq i \leq \lceil \log(n) \rceil - 1$, such that $u + d^i \equiv v (\text{mod } n)\}$. $RC(N, d)$ has recursive structure when $N = cd^m$, $1 \leq c < d$. The [reference] provides the diameter as follows: if d is odd, $D(RC(cd^m, d)) = \lfloor \frac{d}{2} \rfloor m + \lfloor \frac{c}{2} \rfloor$. When d is even and c is odd, the diameter is $\lceil \frac{d-1}{2} \rceil m + \lfloor \frac{c}{2} \rfloor$ Finally, when both d and c are even, the diameter is $\lfloor \frac{d-1}{2} \rfloor m + \lfloor \frac{c}{2} \rfloor$. $RC(2^m, 4)$, whose degree is m, compares favorably to the hypercube H_m. $RC(2^m, 4)$ has the maximum possible connectivity, and its diameter is $\lceil \frac{3m-1}{4} \rceil$. The broadcast time of $RC(2^m, 4)$ is m. Figure 10 shows the two recursive circulant graphs, $RC(8, 4)$ and $RC(16, 4)$.

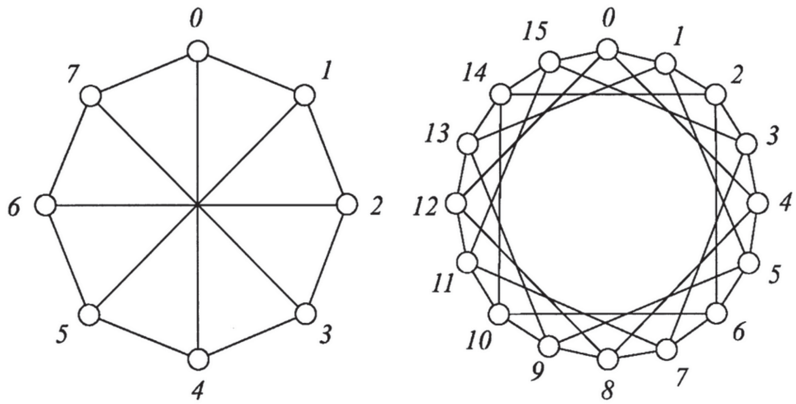

Figure 10: Recursive Circulant graphs $G(8,4)$ and $G(16,4)$

Graph	Vertices	Edges	Diameter	Degree	$b(G)$
$P_n (n \geq 3)$	n	$n-1$	n	2	$n-1$
$C_n (n \geq 3)$	n	n	$\lfloor \frac{n}{2} \rfloor$	2	$\lceil \frac{n}{2} \rceil$
K_n	n	$\frac{n(n-1)}{2}$	1	$n-1$	$\lceil \log_2 n \rceil$
H_d	2^d	$d \cdot 2^{d-1}$	d	d	d
$T(n_1, \ldots, n_d)$	$\prod_{i=1}^d n_i$	$d \prod_{i=1}^d n_i$	$\sum_{i=1}^d \lfloor \frac{n_i}{2} \rfloor$	$2d$	$\sum_{i=1}^d \lfloor \frac{n_i}{2} \rfloor \leq D \leq \sum_{i=1}^d \lfloor \frac{n_i}{2} \rfloor + max\{0, \sum_{i=1}^d n_i \bmod 2 \text{ -1 } \}$. [16]
$G(n_1, \ldots, n_d)$	$\prod_{i=1}^d n_i$	$\prod_{i=1}^d (n_i - 1)$	$\sum_{i=1}^d n_i - d$	$2d$	$\sum_{i=1}^d n_i - d$. [14]
CCC_n	$n \cdot 2^n$	$3n \cdot 2^{n-1}$	$\lfloor \frac{5n}{2} \rfloor - 2$	3	$\lfloor \frac{5n}{2} \rfloor - 2$. [49]
DB_n	2^n	2^{n+1}	n	4	$1.3171n \leq D(DB_n) \leq \frac{3}{2}(n+1)$
SE_n	2^n	$3 \cdot 2^{n-1}$	$2n-1$	3	$2n-1$
$RC(2^m, 4)$	2^m	$m \cdot 2^{m-1}$	$\lceil \frac{3m-1}{4} \rceil$	m	m

Table 1: Properties of the commonly used topologies

2.2 Broadcast graph problem

A *broadcast graph* is a connected graph on n vertices such that $b(G) = \lceil \log n \rceil$. A *minimum broadcast graph* on n vertices is a broadcast graph with the minimum number of edges over all broadcast graphs on n vertices. This minimum number of edges is denoted $B(n)$. There has been significant research on the problem of finding minimum broadcast graphs. Finding $B(n)$ is not easy even for small values of n. The exct value of $B(n)$ is known mostly for small values of n. Only three non isomorphic infinite graph families are known as minimum broadcast graphs. These families are the hypercube of dimension k [13], the recursive circulant graph $G(2^k, 4)$ [56] and Knödel graph $W_{k,2^k}$ [41].

To determine the value of $B(n)$, we need to find a solution of the corresponding MBG problem, which means that determining $B(n)$ is a difficult problem.

The value of the broadcast function, obtained by the above mentioned three graph families is:

$$B(2^k) = k \cdot 2^{k-1},$$
$$B(2^k - 2) = (k-1) \cdot (2^{k-1} - 1)$$

For some small values of n, specially constructed minimum broadcast graphs are also known. Figure 11 illustrates minimum broadcast graphs for $2 \leq n \leq 15$ from [14]. All the currently known values of $B(n)$ and the corresponding references are presented in Table 1.

Since it is extremely difficult to find the exact values of the broadcast function, a considerable effort has been made on finding tight upper and lower bounds on $B(n)$. Usually to get an upper bound on $B(n)$ we should construct a broadcast graph on n vertices and pick the number of its edges. For a lower bound we should use the graph-theoretic properties of the minimum broadcast graphs, e.g. the minimum possible vertex degree and derive from them a lower bound on its number of edges.

n	$B(n)$	**Ref.**
1	0	[14]
2	1	[14]
3	2	[14]
4	4	[14]
5	5	[14]
6	6	[14]
7	8	[14]
9	10	[14]
10	12	[14]
11	13	[14]
12	15	[14]
13	18	[14]
14	21	[14]
15	24	[14]
17	22	[55]
18	23	[38, 68]
19	25	[38, 68]
20	26	[51]
21	28	[51]
22	31	[51]
23	33 or 34	[8, 51]
24	35 or 36	[17, 38]
25	38, 39 or 40	[17, 38]
26	42	[63, 69]
27	44	[63]
28	48	[63]
29	52	[63]
30	60	[38]
31	65	[38]
58	121	[63]
59	124	[63]
60	130	[63]
61	136	[63]
63	162	[43]
127	389	[29]
1023	4650	[64]
4095	22680	[64]
2^p	$p2^{p1}$	[14, 41, 56]
2^p2	$(p1)(2^{p1}1)$	[50, 39]

Table 2: Known values of broadcast function $B(n)$.

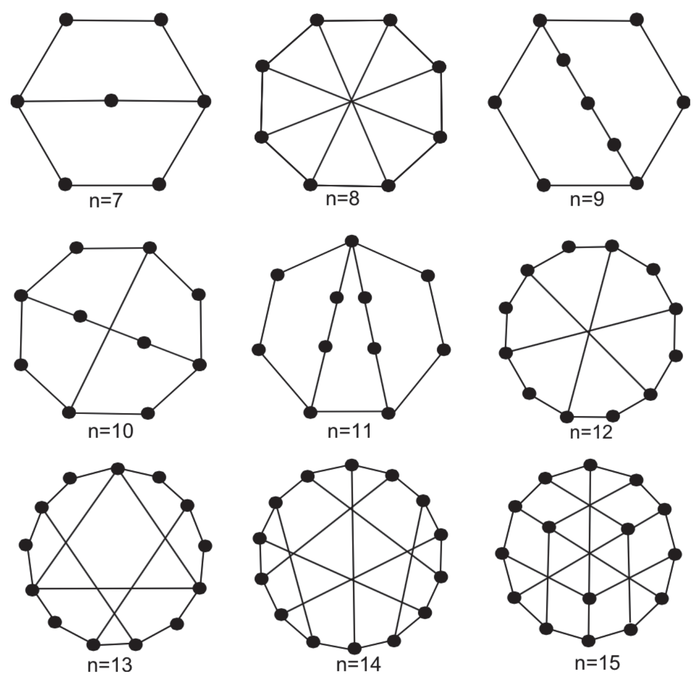

Figure 11: Minimum broadcast graphs for $7 \leq n \leq 15$.

Chapter 3

New Results on Harary Graph

The first section of this chapter defines Harary graphs and describes some properties of the graphs. After that each section describes different case of Harary graphs. Each section is divided in two parts. In the first part we calculate the diameter of respective graph and second part describes broadcast scheme, which will help to find the upper bound on the broadcast time of respective graph.

3.1 Harary Graph

We are always interested in maximizing connectivity to make network more reliable. Along with that we are also interested to minimize the number of edges to reduce the cost of network. Therefore we always try to balance both factors cost and reliability.

In general, as the connectivity of graph increases, number of edges increases. F. Harary introduced Harary graph $H_{k,n}$ in 1962 [22]. It is a k-connected graph on n vertices has degree at least k and $\lceil \frac{kn}{2} \rceil$ edges.

The structure of $H_{k,n}$ depends on the parities of k and n.

Classification of Harary Graph

Harary graph $H_{k,n}$, given $1 < k < n$, can be constructed by placing n vertices around the circle, equally spaced, and joining them based on the parities of k and n.

There are three different cases as mentioned below:

Case 1: k even

Form $H_{k,n}$ by joining each vertex to its adjacent $\frac{k}{2}$ neighbors in each direction around the circle [66]. Graph $H_{4,8}$ is shown in Figure 12 .

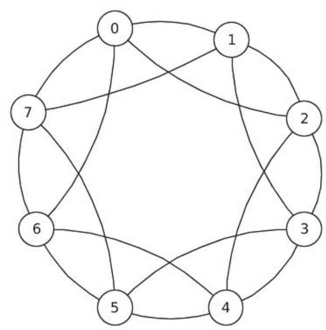

Figure 12: Graph $H_{4,8}$

As we can see in Figure 12, each vertex is connected to its $\frac{k}{2}$ neighbors in both directions. Therefore, vertices $\{i, i+1, \ldots, i+\frac{k}{2}\}$and $\{i, i-1, \ldots, 1-\frac{k}{2}\}$ form clique in clockwise and counter-clockwise directions respectively. There are many cliques in Harary graph. In fact, from every vertex there is a clique in each direction.

Case 2: k odd, n even

Let $k = 2r+1$. Then $H_{2r+1,n}$ is constructed by first drawing $H_{2r,n}$ and then adding edges joining vertex i to $i+\frac{n}{2}$ for $1 \leq i \leq \frac{n}{2}$ [66]. Figure 13 shows $H_{5,8}$. In graph $H_{5,8}$, every vertex is connected to its $\frac{k-1}{2}$ neighbors. Therefore vertices between i to $i+\frac{k-1}{2}$ form a clique.

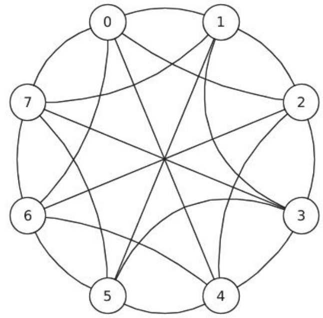

Figure 13: Graph $H_{5,8}$

Case 3: k odd, n odd

Let $k = 2r + 1$. then $H_{2r+1,n}$ is constructed by first drawing $H_{2r,n}$ and then adding edges joining vertex 0 to vertices $\frac{n-1}{2}$ and $\frac{n+1}{2}$ and vertex i to vertex $i + \frac{n+1}{2}$ for $1 \leq i < \frac{n-1}{2}$ [66]. $H_{5,9}$ is shown in Figure 14.

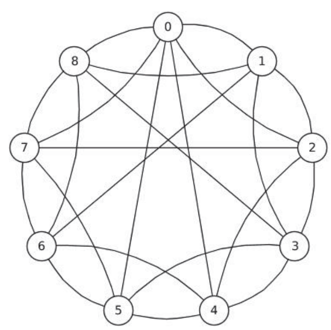

Figure 14: Graph $H_{5,9}$

In figure 14 each vertex is connected to its $\frac{k-1}{2}$ neighbors which forming a clique with vertices i to $i + \frac{k-1}{2}$. Vertex 0 is connected to $\frac{k-1}{2}$ neighbors in both directions and to the vertices $\frac{n-1}{2}$ and $\frac{n+1}{2}$. Therefore, the degree of vertex 0 is $k + 1$, while degree of any other vertex is k.

Properties of Harary graph

- Harary graph $H_{k,n}$ is vertex transitive except for both n and k are odd.
- Harary graph $H_{k,n}$ with k or n even is circulant graph, which implies that it is regular.

- Harary graph $H_{k,n}$ has $\lceil \frac{kn}{2} \rceil$ edges.
- In third case when both n and k are odd, all the vertices have degree k except one vertex which has degree $k+1$.
- Harary graph $H_{n-1,n}$ gives complete graph K_n, and $H_{2,n}$ gives cycle C_n.

Theorem 3.1.1 *Harary graph $H_{k,n}$ with n or k even is vertex transitive .*

Proof A graph is vertex transitive, if for any two vertices u and v there exists some automorphism mapping u to v. Define the family of functions $f(u) = u + \alpha$ where α can be any integer other than 0. Let i and j be any two vertices. The automorphism satisfying $f(i) = j$ is $f(u) = u + (j - i)$. Applying this function on any two vertices x and y, we get $f(x) = x + j - i$ and $f(y) = y + j - i$. $(f(x), f(y))$ will be an edge only if $f(y) = f(x) \pm m$ where $l = 0, ..., n-1$. Here when k is even, $m \in \{(l - \frac{k}{2}) \bmod n, (l + \frac{k}{2}) \bmod n\}$ and when k is odd and n is even, $m \in \{(l - \frac{k-1}{2}) \bmod n, (l + \frac{k-1}{2}) \bmod n\} \cup \{l + \frac{n}{2}\}$. Substituting the values of the functions, we get $y + b - a = x + b - 1 \pm m$ which is equivalent to $y = x \pm m$. Which means that transformed vertices will be adjacent only if the original ones were too. Thus, the suggested family of function preserves the structure of the graph. Hence there is an automorphism. ∎

Theorem 3.1.2 proves that Harary graph achieves the smallest possible number of edges. This is proved in [67] for the case when k is even. For completeness we will present the proofs for all cases.

Theorem 3.1.2 $\kappa(H_{k,n}) = k$*, and hence the minimum number of edges in k-connected graph on n vertices is* $\lceil \frac{kn}{2} \rceil$.

Proof Case 1: k is even $k = 2r$, Let $G = H_{k,n}$. Since $\delta(G) = k$, it suffices to prove $\kappa(G) \geq k$. For $S \subseteq V(G)$ with $\mid S \mid < k$, we prove that $G - S$ is connected. Consider $u, v \in V(G) - S$. The original circular arrangement has a clockwise u, v - path and a counterclockwise u, v - path along the circle; let A and B be the sets of internal vertices on these two paths.

Since $\mid S \mid < k$, the pigeonhole principle implies that in one of $\{A, B\}$, S has fewer than $\frac{k}{2}$ vertices. Since in G each vertex has edges to the next $\frac{k}{2}$ vertices in each direction, deleting fewer than $\frac{k}{2}$ consecutive vertices cannot block travel in that direction. Thus we can find a u, v-path in $G - S$ via the set A or B in which S has fewer than $\frac{k}{2}$ vertices.

Case 2: k **is odd** For $n > k = 2r + 1$ and $r \geq 1$, we show that the Harary graph $H_{k,n}$ is k-connected. The graph consists of n vertices $v_0, ..., v_{n-1}$ spaced equally around a circle, with each vertex adjacent to the r nearest vertices in each direction, plus the "special" edges $(v_i, v_(i + \lceil \frac{n}{2} \rceil))$. When n is odd, $v_{\lceil \frac{n}{2} \rceil}$ has two incident special edges.

To prove that $\kappa(G) = k$, consider a separating set S. Since $G - S$ is disconnected, there are nonadjacent vertices x and y such that every x, y-path passes through S. Let $C(u, v)$ denote the vertices encountered when moving from u to v clockwise along the circle (except u and v). The cut S must contain r consecutive vertices from each of $C(x, y)$ and $C(y, x)$ in order to break every x, y-path(otherwise one could start at x and always take a step in the direction of y). Hence $\mid S \mid \geq k$ unless S contains exactly r consecutive vertices in each of $C(x, y)$ and $C(y, x)$.

In this case, we claim that there remains an x, y-path using a special edge involving x or y. Let x' and y' be the neighbors of x and y along the special edges, using v_0 as the neighbor when one of these is $v_{\lceil \frac{n}{2} \rceil}$. Label x and y so that $C(x, y)$ is smaller than Cy, x (diametrically opposite vertices require n even and are adjacent). Note that $\mid C(x', y') \mid \geq \mid C(x, y) \mid -1$ (the two sets has different sizes when n is odd if $x = v_i$ and $y = v_j$) with $0 \leq j < \lceil \frac{n}{2} \rceil \leq i \leq n - 1$. Because $\mid C(x, y) \mid \geq r$, we have $\mid C(x', y') \mid \geq r - 1$. Therefore, when we delete r consecutive vertices from $C(y, x)$, all of $C(y, x') \cup \{x'\}$ or $\{y'\} \cup C(y\prime, x)$ remains. Therefore at least one of the two x, y-paths with these sets as internal vertices remains in $G - S$. ∎

Theorem 3.1.3 *The graph $H_{k,n}$ with n or k even is circulant graph.*

Proof There are two conditions which needs to be satisfied in order to prove if the graph is Circulant. First, a circulant graph is a graph of n vertices in which each vertex u is adjacent to the vertices $(u + v)$ and $(u - v)$ for each v in a list l. Second, a graph G is a circulant iff the automorphism group of G contains at least one permutation consisting of a minimal cycle of length $\mid V(G) \mid$. To show that $MH_{k,n}$ is circulant, we will prove that it satisfies these two conditions.

As described in section 3.1 , each vertex u in $H_{k,n}$ is connected to vertices $u+\frac{k}{2}$ and $u-\frac{k}{2}$ where $u = \{0, \dots, n-1\}$ when k is even. When k is odd and n is even, each vertex is connected to vertices $u+\frac{k}{2}$, $u-\frac{k}{2}$ and $u+\frac{n}{2}$.Which satisfies the first condition.

Now as we described before, each vertex u in $H_{k,n}$ is adjacent to vertices $u+r$ and $u-r$ where $u = \{0, \dots, n-1\}$ and $r = \{\ 1, \dots, \frac{k}{2}\ \}$ when k is even or $r = \{\ 1, \dots, \frac{k-1}{2}\ \} \cup \{u+\frac{n}{2}\}$ when k is odd. Here if we take $r = 1$, then vertex u is connected to vertices $u+1$ and $u-1$. Let's take automorphism group where vertex u is connected to node $u+1$. As proved in theorem 4.1.1 that $H_{k,n}$ is vertex transitive, we will start from node 0 and traverse through vertices in clockwise direction.

$$0 \to 1 \to \dots \to u \to (u+1) \to \dots \to (n-1) \to 0$$

As vertex 0 repeats in above path, there is a cycle consisting of vertices $0, \dots n-1$. Thus, the automorphism group has cycle of length $\mid V(G) \mid$. This satisfies the second condition. Hence $H_{k,n}$ is circulant graph. ∎

Third case of Harary graph is not a vertex transitive graph. Thus, it cannot be a circulant graph.

3.2 $H_{k,n}$ with k even

$H_{6,20}$ is the example of the first case which is shown in Figure 15. Since $H_{k,n}$ is circulant graph, it is vertex transitive. Therefore, it is easy to consider vertex 0 as originator. As we already know, from every vertex v starts a clique in each direction. It is graphically convenient to consider cliques as shown in Figure 16 for simplicity.

Since for every vertex i, where $i \in \{0, \dots, (n-1) \bmod n\}$, there are two cliques in $H_{k,n}$ formed by the set of vertices V_1 and V_2, then from i we can visit vertices $(i+1) \bmod n, \dots, (i+\frac{k}{2}) \bmod n$ and $(i-1) \bmod n, \dots, (i-\frac{k}{2})$ in just one time unit. From node i if we visit node $(i+\frac{k}{2}) \bmod n$ or $(i-\frac{k}{2}) \bmod n$, then it is called a city-tour, otherwise it is called a village-tour.

Lemma 3.2.1 *To find the shortest distance between any pair of vertices in the Harary graph $H_{k,n}$, it is always optional to first take the city-tours as much as possible.*

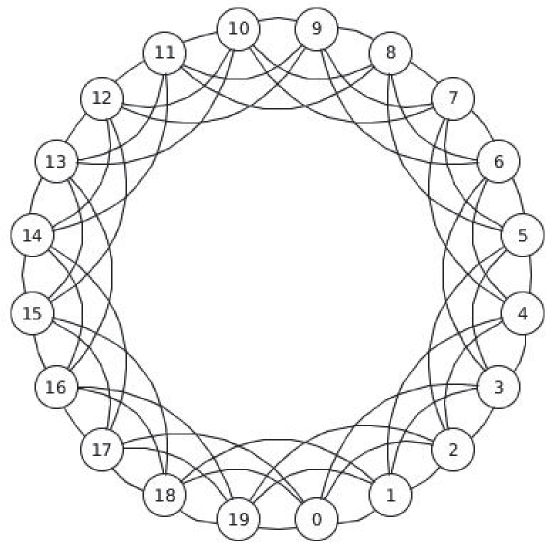

Figure 15: Example Graph $H_{6,20}$ of first case

Proof: Let us consider the pair of vertices be 0 and X. Let us assume a scheme S_u where we have taken maximum possible city-tours from 0 in order to reach vertex X. In one city-tour we can cover $\lfloor \frac{k}{2} \rfloor$ vertices. Thus we can have at most $\lfloor \frac{2X}{k} \rfloor$ city-tours. From there at most one village-tour will take us to vertex X. Hence, $dist_{S_u}(0, X) = \lceil \frac{2X}{k} \rceil$.

Let us consider another scheme S_v, where after m city-tours from vertex 0, where $m < \lfloor \frac{2X}{k} \rfloor$, there is a village-tour which covers c vertices and $c < \lfloor \frac{k}{2} \rfloor$. Then after $m + 1$ passes, the vertices still need to be covered are $X - \frac{mk}{2} - c$. If the remaining are all city-tours then $dist_{S_v}(0, X) = m + 1 + \lceil \frac{X - \frac{mk}{2} - c}{\frac{k}{2}} \rceil = \lceil \frac{2X+k-2c}{k} \geq \rceil \frac{2X}{k}$ (as $c < \lfloor \frac{k}{2} \rfloor$)$= dist_{S_v}(0, X)$. Now we apply the method of induction.

The base case, when there is only one village-tour before we have completed the maximum possible city-tours has been shown to be true for t such village-tours and let in total c_t vertices are being covered in t village tours. Similarly, we can claim $distS_v(0, X) = m+t+\lceil \frac{X-mk/2-c_t}{k/2} \rceil \geq dist_{S_u}(0, X)$ is true. We are now going to consider the case for $t + 1$ village-tours. Let us assume in the $(t + 1)^{th}$ village-tour c_{t+1} vertices will be covered and $c_{t+1} < \frac{k}{2}$. Then, $dist_{S_{v(t+1)}}(0, X) = m + t + 1 + \lceil \frac{X-mk/2-c_t-c_{t+1}}{k/2} \rceil = m + t + \lceil \frac{X-mk/2-c_t}{k/2} \rceil + (1 - \frac{c_{t+1}}{k/2}) = dist_{S_{v(t)}}(0, X) + (1 - \frac{c_{t+1}}{k/2}) \geq dist_{S_u}(0, X) + (1 - \frac{c_{t+1}}{k/2})$. Now, $1 - \frac{c_{t+1}}{k/2} > 0$ as $c_{t+1} < \frac{k}{2}$. Thus, $dist_{s_{v(t+1)}}(0, X) = dist_{S_u}(0, X)$.

$\blacksquare$

3.2.1 Diameter

A shortest $u-v$ (where $u, v \in G$) path is often called a geodesic. The diameter $D(G)$ of a connected graph G is the length of any longest geodesic [23]. A clique starts from every vertex in each direction. So vertex $i \in H_{k,n}$ can reach vertices $\{(i+1) \text{mod } n \ldots (i+\frac{k}{2}) \text{mod } n\}$ and $\{(i-1) \text{mod } n \ldots (1-\frac{k}{2})$ mod $n\}$ in just one time unit through a village-tour. Therefore diametral vertex from i will be inside the farthest clique from i.

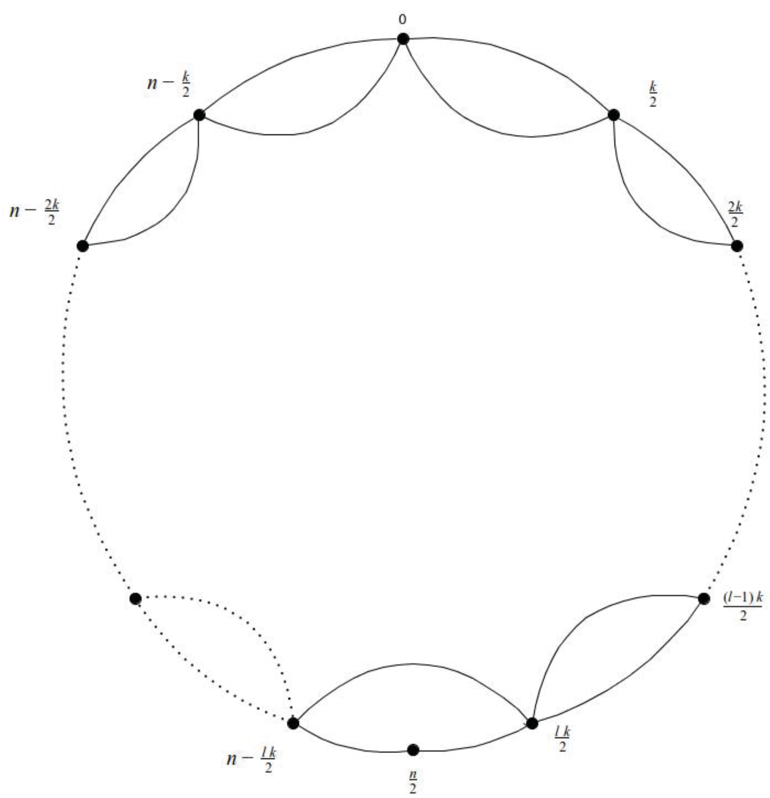

Figure 16: Diameter path for the graph

Figure 16 shows the diametral path. We will take a city-tour from originator 0 and visit vertex $\frac{k}{2}$. Then from vertex $\frac{k}{2}$ to vertex$\frac{2k}{2}$ and so on until we arrive at vertex $\frac{lk}{2}$. The path for diameter will be

$$0 \to \frac{k}{2} \to \frac{2k}{2} \to \ldots \to \frac{lk}{2}$$

or

$$0 \to n - \frac{k}{2} \to n - \frac{2k}{2} \to \ldots \to n - \frac{lk}{2}$$

Where l is an integer.

Here $\frac{n}{2} = \frac{lk}{2}$ when $n \bmod k = 0$, we reached at farthest vertex from 0 using above path in l time units. If $\lfloor \frac{n}{2} \rfloor = \frac{lk}{2} + m$, where $m < \frac{k}{2}$, then vertex $\lfloor \frac{n}{2} \rfloor$ is inside the clique which starts at $\frac{lk}{2}$. So the diameter for that case will be $l+1$. l is the distance to reach vertex $\lfloor \frac{n}{2} \rfloor$ or its neighbor $\frac{lk}{2}$ based on the value of n and k. Vertices $\frac{lk}{2}$ and $n - \frac{lk}{2}$ both can be reached after traveling l edges.

$$\frac{lk}{2} = n - \frac{lk}{2}$$

$$l = \lfloor \frac{n}{k} \rfloor$$

We used floor function because in l time units we can reach halfway through the $H_{k,n}$ or to the clique which contains $\frac{n}{2}$ vertex. Here $l = \lfloor \frac{n}{k} \rfloor$ is obvious, as graph $H_{k,n}$ has k-connectivity on n vertices and we can reach farthest vertex in $\lfloor \frac{n}{k} \rfloor$ time units. (We know cycle graph has 2-connectivity, and farthest vertex is at $\lfloor \frac{n}{2} \rfloor$).

Diameter of $H_{k,n}$ can be either l or $l+1$, which depends on parity of k, n and value of $\frac{n}{k}$.

Case 1: n even, $\frac{n}{k} = P$ (P is integer)

As we saw before, $l = \frac{n}{k}$, which means vertex $\frac{n}{2}$ can be reached in $\frac{n}{k}$ time units by making city-tours, which is the diameter of the Harary Graph where n and k are even and n is divisible by k.

Case 2: n even, $\frac{n}{k} \neq P$ (P is integer)

As shown in Figure 16, $\lfloor \frac{n}{2} \rfloor$ is inside the clique. Therefore, in $l+1$ time units, we can reach vertex $\frac{n}{2}$.

$$\frac{n}{2} = l \cdot \frac{k}{2} + m \ , \ m < \frac{k}{2}$$

From above, we can clearly say that vertex $l \cdot \frac{k}{2}$ is nearly halfway in the graph. As we have seen previously, Harary Graph is a circulant graph with k connectivity, but n is not divisible by k. Hence, the diameter would be $\lfloor \frac{n}{k} \rfloor + 1 = \lceil \frac{n}{k} \rceil \quad (\text{Since } l = \lfloor \frac{n}{k} \rfloor)$

Case 3: n odd, and $n \ mod \ k = 1$ (where $n = i \cdot k + 1$, i is an integer.)

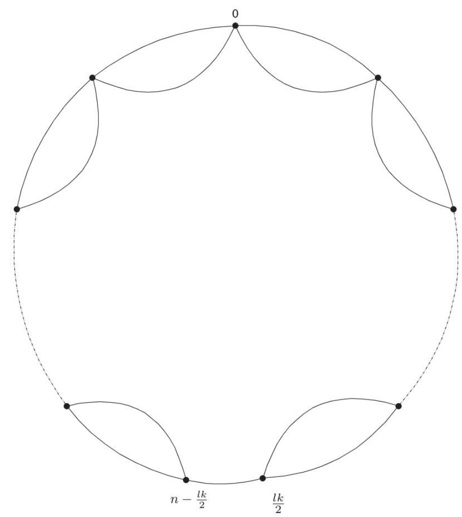

Figure 17: $H_{k,n}$ when n mod k =1

Here, since n is odd, there are two vertices from node 0 at diametric distance. As shown in Figure 17 farthest vertices are at distance $l \cdot \frac{k}{2}$, so the diameter of the graph in this case would be $\lfloor \frac{n}{k} \rfloor$.

Case 4: n odd, $n \ mod \ k > 1$.

This case is similar in nature with the case 2 because the farthest vertices are inside the clique. As we have seen before, these vertices are at distance $l + 1$ from node 0. Hence, as case 2, the diameter would be following:$\lfloor \frac{n}{k} \rfloor + 1 = \lceil \frac{n}{k} \rceil \quad (\text{Since } l = \lfloor \frac{n}{k} \rfloor)$

Summary:

$$d(H_{k,n}) = \lfloor \tfrac{n}{k} \rfloor, \text{ if } n \text{ mod } k = 1$$
$$= \lceil \tfrac{n}{k} \rceil, \ otherwise$$

Lemma 3.2.2 *Let $H_{k,n}$ be a Harary graph on n and k is even. Then the diameter of $H_{k,n}$ is $\lceil \frac{n}{k} \rceil$.*

Proof This result is a consequence of Lemma 3.2.1. From vertex 0, at most $\lfloor \frac{n}{2} \rfloor$ vertices can be covered from either clockwise or anti-clockwise direction. At first we will take maximum possible city-tours from vertex 0, and in one city-tour we can cover $\frac{k}{2}$ vertices. Initially we will traverse through $\lceil \frac{n}{k} \rceil$ city-tours. The number of vertices being covered through this traversal will be $\lfloor \frac{n}{k} \rfloor \frac{k}{2} = \frac{n-c_1}{2}$, where $c_1 = n \bmod k$. If $\frac{n}{k} = p$, where p is any positive integer, then a vertex can be at maximum $\frac{n}{k}$ distance apart from 0. We need one more village tour to reach such a vertex. This makes the diameter of the graph to be $\lceil \frac{n}{k} \rceil$. ∎

Lemma 3.2.3 *Let $H_{k,n}$ be a Harary graph on n vertices and k is even. The broadcast time of $H_{k,n}$ from any originator is*

(i) $b(H_{k,n}) \geq \lceil \frac{n}{k} \rceil$ *when* $\frac{2n}{k} = p$

(ii) $b(H_{k,n}) \geq \lceil \frac{n}{k} \rceil + 1$ *when* $\frac{2n}{k} \neq p$

Proof This lemma gives lower bound on the first case of Harary graph.

(i) This is a consequence of the result in [36], claiming that the diameter of the graph is a trivial lower bound on the broadcast time . This is true when value of n very much bigger than th value of k.

Suppose there is a graph $H_{k,n}$ which can finish broadcasting in less than $\lceil \frac{n}{k} \rceil$ time units. From Lemma 3.2.2, the longest distance between any two vertices in graph is $\lceil \frac{n}{k} \rceil$ which contradicts that every vertex in $H_{k,n}$ can be informed in less than $\lceil \frac{n}{k} \rceil$.

(ii) Since $\frac{2n}{k} \neq p$, there are at least two vertices in $H_{k,n}$ at the diametral distance from originator. As described in [15], if there exists at least two vertices at a diametral distance D from vertex u in graph G, then $b(G) \geq D + 1$. Hence, $b(G) \geq \lceil \frac{n}{k} \rceil + 1$. ∎

3.2.2 Broadcasting

In this section we will give a broadcast scheme that completes broadcasting in $\lceil \frac{n}{k} \rceil + \lceil \log(\frac{k}{2}+1) \rceil$ time units. Figure 18 shows the broadcast scheme for the first case of Harary graph. Since the graph is vertex transitive we will describe the broadcast scheme only for the originator 0.

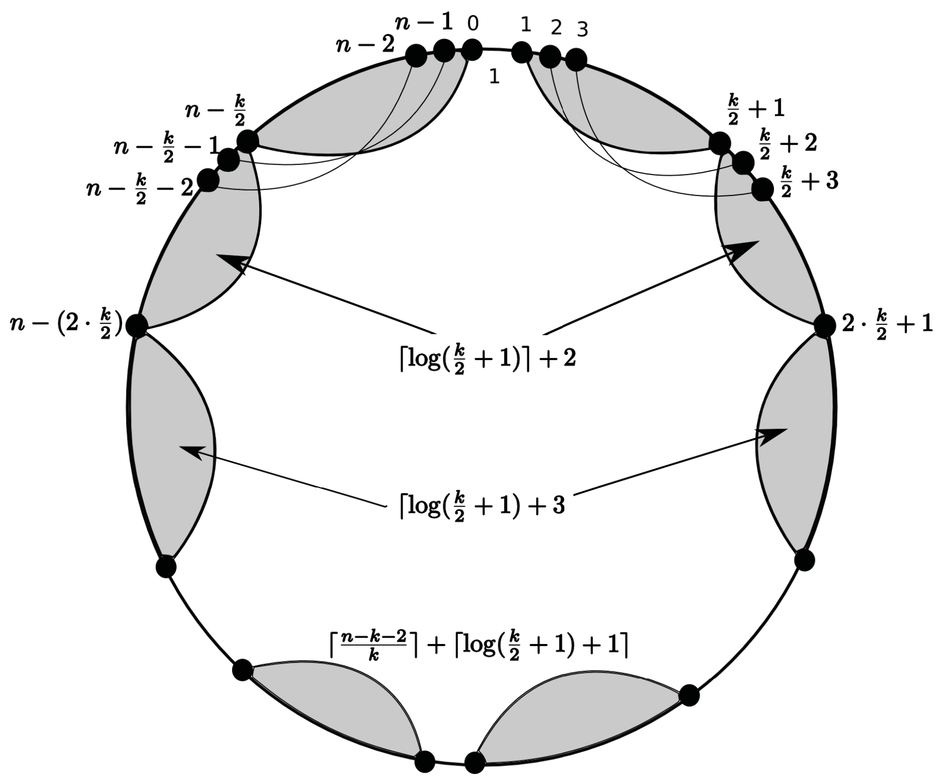

Figure 18: Broadcast scheme of the first case of Harary graph

Broadcast Scheme for Originator 0

- At time unit 1, vertex 0 sends the message to its adjacent vertex 1.
- Starting time unit 2, vertex 0 and vertex 1 start broadcasting inside cliques with vertices $\{0,\ldots,n-\frac{k}{2}\}$ and $\{1,\ldots,\frac{k}{2}+1\}$ respectively.
- Cliques $\{0,\ldots,n-\frac{k}{2}\}$ and $\{1,\ldots,\frac{k}{2}+1\}$, each have $\frac{k}{2}+1$ vertices. Thus broadcasting within the cliques will take $\lceil \log(\frac{k}{2}+1) \rceil$ time units.

- After time $1 + \lceil \log(\frac{k}{2} + 1) \rceil$, both cliques will be informed. In other words, the graph will have in total $2 + 2 \cdot \frac{k}{2} = k + 2$ informed vertices.

- Starting at time $\lceil \log(\frac{k}{2} + 1) \rceil + 2$, every informed vertex v in these 2 cliques except vertices 0 and 1, will inform a new vertex $v + \frac{k}{2}$ in clockwise direction and $v - \frac{k}{2}$ in counter-clockwise direction.

- At time unit $\lceil \log(\frac{k}{2} + 1) \rceil + 2$, k more vertices will be informed.

- Each informed vertex u continues to inform an uninformed vertex $u + \frac{k}{2}$ or $u - \frac{k}{2}$,depending on its position in graph, until all the vertices in the graph are informed.

- Starting at time unit $\lceil \log(\frac{k}{2} + 1) \rceil + 2$, k new vertices are informed at each time unit until all the vertices are informed.

- Thus, the total broadcast time of the graph will be $\lceil \frac{n-2}{k} \rceil + \lceil \log(\frac{k}{2} + 1) \rceil$.

- During the last time unit of broadcasting, some uninformed vertices may receive the message from clockwise and counter-clockwise directions.

Algorithm 1 gives the broadcast scheme for $H_{k,n}$ when k is even. Let's take set $U = \{u_1, u_2, \ldots, u_p\}$ as set of informed vertices in clockwise direction, and set $W = \{w_1, w_2, \ldots, w_q\}$ as set of informed vertices in counter-clockwise direction.

Algorithm 1 *Broadcast Algorithm S_1*

```
procedure BROADCAST-SCHEME-S1(H_{k,n}, 0, U, W)
    Vertex 0 is the originator , k is even
    Initially U = {} and W = {}
    Vertex 0 informs vertex 1
```

Starting at time unit 2, vertex 0 and vertex 1 start to inform vertices inside cliques $\{0, (n-1) \bmod n, \ldots, (n-\frac{k}{2}) \bmod n\}$ and $\{1, 2, \ldots, \frac{k}{2}+1\}$ respectively.

After time $\lceil \log(\frac{k}{2}+1) \rceil + 1$ do

$U = U \cup \{1, 2, \ldots, \frac{k}{2}+1\}$ and $W = W \cup \{0, (n-1) \bmod n, \ldots, (n-\frac{k}{2}) \bmod n\}$

for $i = 1, \ldots, \lceil \frac{n-k-2}{k} \rceil$ in clockwise direction **do**

  Represent all the informed vertices in $U = \{u_1, u_2, \ldots, u_p\}$ as u_m where $m \in \{1, 2, \ldots, p\}$

  if $u_m + \frac{k}{2} \notin U$ **then**

    u_m informs $u_m + \frac{k}{2}$ at time unit i

    $U = U \cup \{u_m + \frac{k}{2}\}$

  end if

end for

for $j = 1, \ldots, \lceil \frac{n-k-2}{k} \rceil$ in counter-clockwise direction **do**

  Represent all the informed vertices in $W = \{w_1, w_2, \ldots, w_q\}$ as w_n where $n \in \{1, 2, \ldots, q\}$

  if $w_n - \frac{k}{2} \notin W$ **then**

    w_n informs $w_n - \frac{k}{2}$ at time unit j

    $W = W \cup \{w_m - \frac{k}{2}\}$

  end if

end for

During the last time unit $\lceil \frac{n-k-2}{k} \rceil$, some uninformed vertices may receive information from both clockwise and anti-clockwise directions

end procedure

end

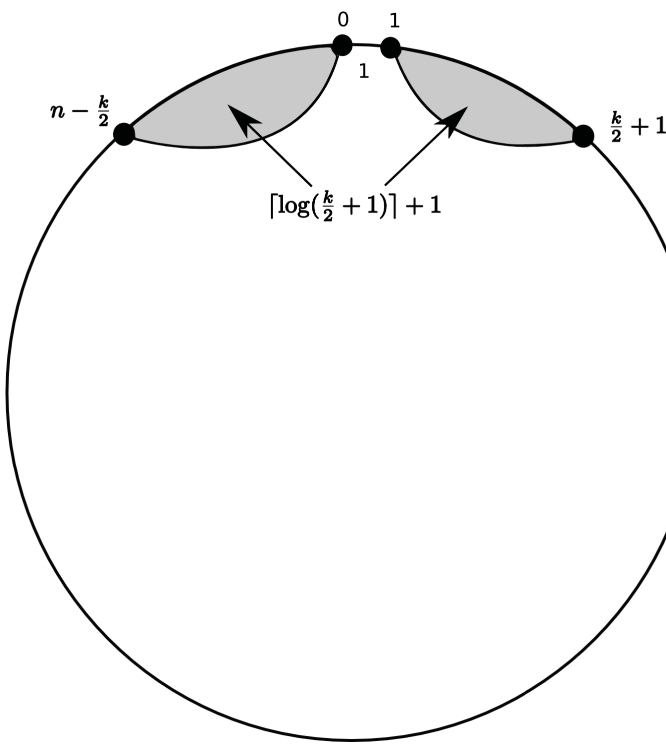

Figure 19: Broadcasting inside $H_{k,n}$ after $\lceil\log(\frac{k}{2}+1)\rceil+1$ time units

Algorithm 1 gives the broadcast scheme for the first case of Harary graph. At time unit 1 vertex 0 sends message to vertex 1. After that they both start to broadcast inside their respective cliques.

Now as we know that both cliques have $\frac{k}{2}+1$ vertices, so it will take $\lceil\log(\frac{k}{2}+1)\rceil$ time units to broadcast inside those cliques. Figure 19 describes step 1-7 of algorithm 1.

After $\lceil\log(\frac{k}{2}+1)\rceil+1$ time units, in total $k+2$ vertices are informed. In other words, there are $n-k-2$ uninformed vertices. These uninformed vertices receives message by following step 8-20 of algorithm 1. Starting from $\lceil\log(\frac{k}{2}+1)\rceil+2$ time unit, k new vertices receives message at each time unit. Figure 20 shows, the graph after $\lceil\log(\frac{k}{2}+1)\rceil+2$ time units. Therefore, it takes $\lceil\frac{n-k-2}{k}\rceil$ time units to send message to $n-k-2$ vertices.

Thus, broadcast scheme takes in total $\lceil\frac{n-2}{k}\rceil+\lceil\log(\frac{k}{2}+1)$ time units.

Below we will show that our algorithm 1 will always generate $\lceil\frac{n-2}{k}\rceil+\lceil\log(\frac{k}{2}+1)$ time units for broadcasting in $H_{k,n}$.

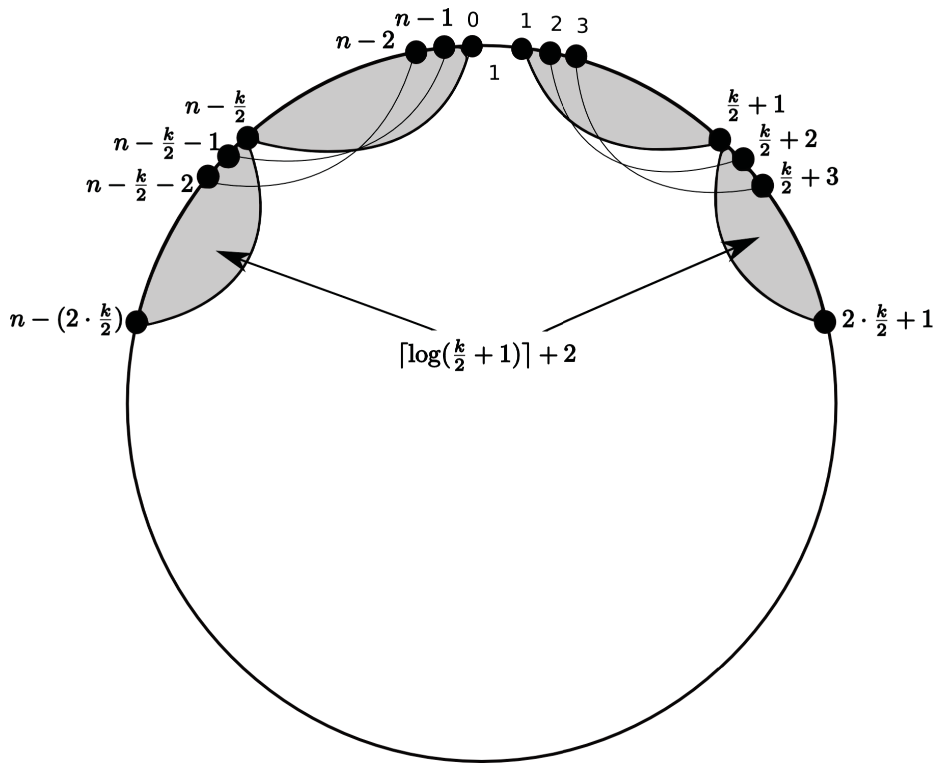

Figure 20: Broadcasting inside $H_{k,n}$ after $\lceil\log(\frac{k}{2}+1)\rceil+2$ time units

Theorem 3.2.4 *Algorithm 1 gives* ($\lceil\log\frac{k+2}{2}\rceil$)*-additive approximation for* $b(H_{k,n})$*.*

Proof Under the algorithm 1, vertex 0 first informs vertex 1. Starting at time unit 2, vertices 0 and 1 start sending the message to vertices inside clique $\{0,\ldots,n-\frac{k}{2}\}$ and $\{1,\ldots,\frac{k}{2}+1\}$ respectively. There are $\frac{k}{2}+1$ vertices inside both cliques. So, it will take $\lceil\log\frac{k+2}{2}\rceil$ time units to broadcast in each of those cliques. By $\lceil\log(\frac{k+2}{2})\rceil+1$ time units, in total $2+2\cdot\frac{k}{2}=k+2$ vertices will be informed.

Now vertices 0 and 1 do not have any uninformed neighbors. At time unit $\lceil\log(\frac{k+2}{2})\rceil+2$ vertices $v\in\{2,\ldots,\frac{k}{2}+1\}$ and $u\in\{n-1,\ldots,n-\frac{k}{2}-1\}$ will inform their neighbors $v+\frac{k}{2}$ and $u-\frac{k}{2}$ respectively as shown in Figure 20. At time unit $\lceil\log(\frac{k+2}{2})\rceil+2$, k new vertices are informed. In next time unit, these new informed k vertices will send the message to next k uninformed vertices at distance $\frac{k}{2}$ by following steps 8-21 of algorithm 1. At each time unit k vertices are informed, therefore it takes $\lceil\frac{n-k-2}{k}\rceil$ additional time units to inform all the remaining $n-k-2$ vertices.

Thus, $b_{S_1}(H_{k,n}) = 1+\lceil\log(\frac{k+2}{2})\rceil+\lceil\frac{n-k-2}{k}\rceil = \lceil\frac{n-2}{k}\rceil+\lceil\log(\frac{k+2}{2})\rceil \le \lceil\frac{n}{k}\rceil+\lceil\log(\frac{k+2}{2})\rceil \le b(H_{k,n})+\lceil\log(\frac{k+2}{2})\rceil$ using Lemma 3.2.3(i). ∎

Lemma 3.2.5 *Let $H_{k,n}$ be a Harary graph on n vertices where k is even. The broadcast time of $H_{k,n}$ from any originator is $b(H_{k,n}) \leq \lceil \frac{n-2}{k} \rceil + \lceil \log(\frac{k+2}{2}) \rceil$.*

Proof This is direct consequence of the proof of theorem 3.2.4. ∎

3.3 $H_{k,n}$ with n even k odd

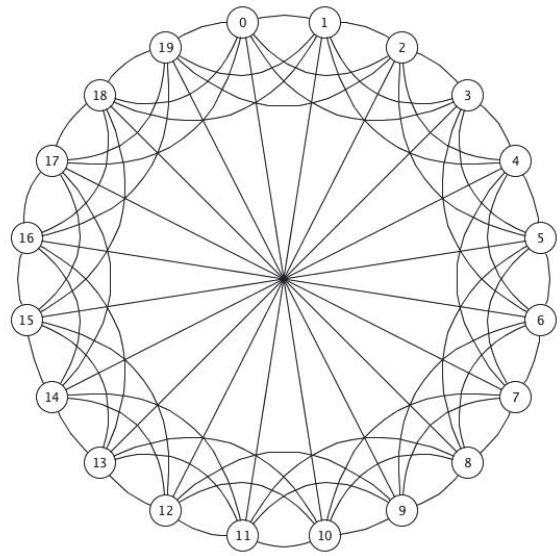

Figure 21: Example Graph $H_{5,20}$ of second case

Figure 21 shows the example graph $H_{5,20}$ of second case. Here n is even and k is odd. Every vertex in $H_{k,n}$ is connected to $\frac{k-1}{2}$ consecutive neighbors in both direction and to the diametrically opposite vertex. Each vertex has k degrees.

As we discusses in section 3.1 for every vertex i, where $i \in \{0, \ldots, (n-1) \bmod n\}$, there are two cliques in $H_{k,n}$ formed by the set of vertices V_1 and V_2, then from i we can visit vertices $(i+1)$ mod n,...,$(i + \frac{k-1}{2})$ mod n and $(i-1)$ mod n,...,$(i - (\frac{k-1}{2})$ mod $n)$ in just one time unit. From node i if we visit node $(i + \frac{k-1}{2})$ mod n or $(i - \frac{k-1}{2})$ mod n, then it is called a city-tour, otherwise it is called village-tour.

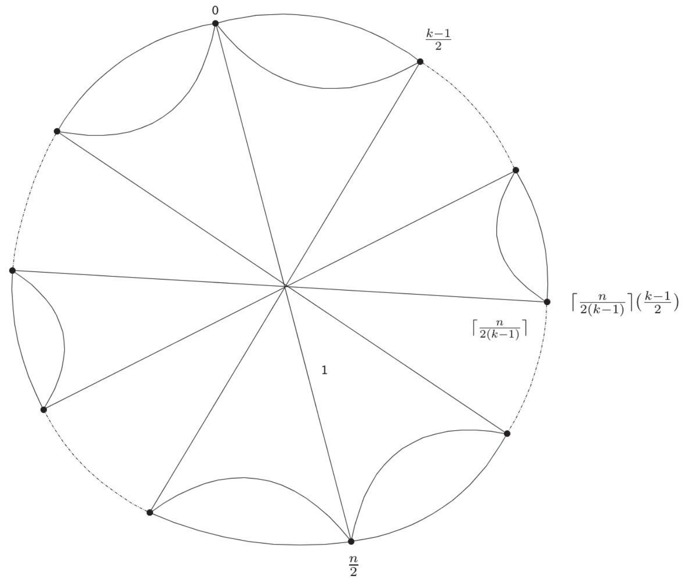

Figure 22: Diametral path of second Case

3.3.1 Diameter

Figure 22 shows the diametral path of the second case on Harary graph. As $H_{k,n}$ is vertex transitive, we can start from any vertex v in the graph. For simplicity, we will take vertex 0 as originator.

.Vertex 0 is directly connected vertex $\frac{n}{2}$, which divides graph $H_{k,n}$ into half. So the diametral vertex will be inside the farthest clique from vertex 0 and $\frac{n}{2}$. First it is necessary to reach the clique that contains the diametral vertex. After that in just one village-tour one can reach the vertex at diametral distance from node 0.

- Vertex 0 visits node $\frac{n}{2}$ at time unit 1.
- Starting at time unit 2 node $\frac{n}{2}$ and 0 start to make city tours to reach the farthest clique.
- As we described before, every city-tour connects vertex $v \in V$ in $H_{k,n}$ to vertex $v + \frac{k-1}{2}$.
- So the farthest clique will be at distance $\lceil\frac{n/4}{(k-1)/2}\rceil = \lceil\frac{n}{2(k-1)}\rceil$ from either vertex 0 or vertex $\frac{n}{2}$.
- Vertex $\lceil\frac{n}{2k-2}\rceil(\frac{k-1}{2})$ can be reached at time unit $\lceil\frac{n}{2k-2}\rceil$.

- After that one can visit diametral vertex in just one time unit.

Lemma 3.3.1 *Let $H_{k,n}$ be a Harary graph on n vertices where k is odd and n is even then diameter of $H_{k,n}$ is denoted as $D(H_{k,n})$ is $\lceil \frac{n}{2k-2} \rceil + 1$.*

Proof Vertex 0 is directly connected to vertex $\frac{n}{2}$, here n is even. We are going to consider several cases based on value of $\frac{n}{k-1}$.

(i)When $\frac{n}{k-1}$ is even:

Let $\frac{n}{k-1} = 2p$, where p is positive integer. In other words in either direction, starting from vertex 0 we can make 2p city-tours, since at most $\frac{n}{2}$ vertices can be covered. Let us label the city-tours as $1, 2, \ldots, 2p$ from vertex 0 along any particular direction. From vertex 0 we can make city-tours along $1, 2, \ldots, p$ in p moves. During the same p moves we can first visit vertex $\frac{n}{2}$ and then traverse through city-tours labeled as $2p, 2p-1, \ldots, p+1$. The vertices covered by $(p+1)$th city-tour can be reached through one more move, therefore making the total moves to be $p+1$. Hence, diameter of $H_{k,n} = p + 1 = \frac{n}{2(k-1)} + 1$.

(ii) When $\frac{n}{k-1}$ is odd:

Let $\frac{n}{k-1} = 2p - 1$. Similar to the proof in part (i), from vertex 0 we will consider city-tours along $1, 2, \ldots, p$ in p moves. During the same p moves we can first visit vertex $\frac{n}{2}$ and then traverse through city-tours labeled as $2p-1, \ldots, p+1$, thus making the total moves to be p. Hence, diameter of $H_{k,n} = p = \frac{n}{2(k-1)} + \frac{1}{2} = \lceil \frac{n}{2k-2} \rceil + 1$ (Since $\frac{n}{k-1}$ is odd).

(iii) When $\lceil \frac{n}{k-1} \rceil$ is even:

Let $\lceil \frac{n}{k-1} \rceil = 2p$. Similar to proof in part (i), from vertex 0 we will consider city-tours along $1, 2, \ldots, p$ in p moves. During the same p moves we can first visit vertex $\frac{n}{2}$ and the traverse through city tours labeled as $2p, \ldots, p+2$. Now, $(p+1)$th city tour covers less then $\frac{k-1}{2}$ vertices. we need one more city tour to reach a vertex at diametrall distance from 0, therefore, making total moves to be $p+1$. Thus, diameter of $H_{k,n} = \lceil \frac{n}{2k-2} \rceil + 1$.

(iv) When $\lceil \frac{n}{k-1} \rceil$ is odd:

Let $\lceil \frac{n}{k-1} \rceil = 2p - 1$. Similar to the proof in part (ii), from vertex 0 we will consider city-tours along $1, 2, \ldots, p$ in p moves. During the same $p-1$ moves we can first visit vertex $\frac{n}{2}$ and then traverse through city-tours labeled as $2p-1, \ldots, p+2$. Now $(p+1)$th city tour covers less than $\frac{k-1}{2}$ vertices. We need one more city-tour to reach at vertex at a diametral distance from 0, therefore making total moves to be p. Similarly, diameter of $H_{k,n} = \lceil \frac{n}{2k-2} \rceil + 1$. ∎

Lemma 3.3.2 *Let $H_{k,n}$ be a Harary graph on n vertices where degree of each vertex is at least k, here k is odd and n is even. The broadcast time of $H_{k,n}$ from any originator is*

(i) $b(H_{k,n}) \geq \lceil \frac{n}{2k-2} \rceil + 1$ *when* $\frac{2n}{k-1} = p$

(ii) $b(H_{k,n}) \geq \lceil \frac{n}{2k-2} \rceil + 2$ *when* $\frac{2n}{k-1} \neq p$

Proof This lemma gives the proof for lower bound on the second case of Harary graph $H_{k,n}$.

(i) This is direct consequence of the result in [36], that the diameter of the graph is a trivial lower bound on the broadcast time. Therefore from Lemma 3.3.1, $b(G) \geq \lceil \frac{n}{2k-2} \rceil + 1$.

(ii) Since $\frac{2n}{k-1} \neq p$, there are at least two vertices in $H_{k,n}$ at the diametral distance from originator. As described in [15], if there exists at least two vertices at a diametral distance D from vertex u in graph G, then $b(G) \geq D + 1$. Hence, $b(G) \geq \lceil \frac{n}{2k-2} \rceil + 2$. ∎

3.3.2 Broadcast Scheme

This section illustrates the broadcast scheme inside $H_{k,n}$ where n is even and k is odd.

Broadcast Scheme for Originator 0

- At time unit 1 originator 0 sends message to vertex $\frac{n}{2}$. Let us take $v = \frac{n}{2}$, here n is even.
- At time unit 2 vertex 0 sends message to vertex 1 and vertex v sends message to vertex $v + 1$.

- Starting at time unit 3, vertex 0 informs vertices inside clique $\{0, n-1, \ldots, n-\frac{k-1}{2}\}$,vertex 1 informs vertices inside clique $\{1, 2, \ldots, \frac{k+1}{2}\}$, vertex v informs vertices in clique $\{ v, v-1, \ldots, v-\frac{k-1}{2} \}$ and vertex $v+1$ informs vertices inside clique $\{v+1, v+2, \ldots, v+\frac{k+1}{2}\}$.

- Each of these four clique has $\frac{k-1}{2}+1=\frac{k+1}{2}$ vertices.

- By the time $\lceil \log \frac{k+1}{2} \rceil + 2$, in total $4(\frac{k+1}{2}) = 2k+2$ vertices will have information including vertices 0,1, v and $v+1$.

- Let us assume that $U = \{2, \ldots, \frac{k+1}{2}, v+1, \ldots, v+\frac{k-1}{2}\}$ and $W = \{n-1, \ldots, n-\frac{k-1}{2}, v-1, \ldots, v-\frac{k+1}{2}\}$. Starting at time unit $\lceil \log(\frac{k-1}{2}+1) \rceil + 3$, all the vertices in vertices $u_m \in U$ informs vertices $u_m + \frac{k-1}{2}$ and all the vertices inside $w_n \in W$ informs vertices $w_n - \frac{k-1}{2}$. Vertex 0,1, v and $v+1$ do not have any uninformed neighbor from this step.

- At time unit $\lceil \log(\frac{k-1}{2}+1) \rceil + 3$, new $2k-2$ vertices will receive information.

- After that at each time unit $2k-2$ more vertices will receive message until all the vertices in $H_{k,n}$ are informed.

- Therefore, it will take $\lceil \frac{n-2k-2}{2k-2} \rceil$ time units to inform $n-2k-2$ vertices.

- Thus, the total broadcast time of the graph will be $\lceil \frac{n-2k-2}{2k-2} \rceil + \lceil \log \frac{k+1}{2} \rceil + 2 = \lceil \frac{n-4-2k+2}{2k-2} \rceil + \lceil \log \frac{k+2}{2} \rceil + 2 = \lceil \frac{n-4}{2k-2} \rceil - \lceil \frac{2k-2}{2k-2} \rceil + \lceil \log \frac{k+1}{2} \rceil + 2 = \lceil \frac{n-4}{2k-2} \rceil + \lceil \log(\frac{k+1}{2}) \rceil + 1$.

Figure 23 shows the broadcast scheme inside $H_{k,n}$ when n is even and k is odd. In the second case of Harary graph node 0 is connected to node $\frac{n}{2}$. By sending the message to vertex $\frac{n}{2}$ at time unit 1, we divide the graph into half. Thus we can observe that broadcast time of the first case of Harary graph is almost double than the second case of Harary graph.

Algorithm 2 describes the broadcast scheme of the second case of Harary graph. Let's take set $U = \{u_1, u_2, \ldots, u_p\}$ as the set of informed vertices in clockwise direction and set $W = \{w_1, w_2, \ldots, u_q\}$ as the set of unformed vertices in counter-clockwise direction.

Algorithm 2 *Broadcast Algorithm S_2*

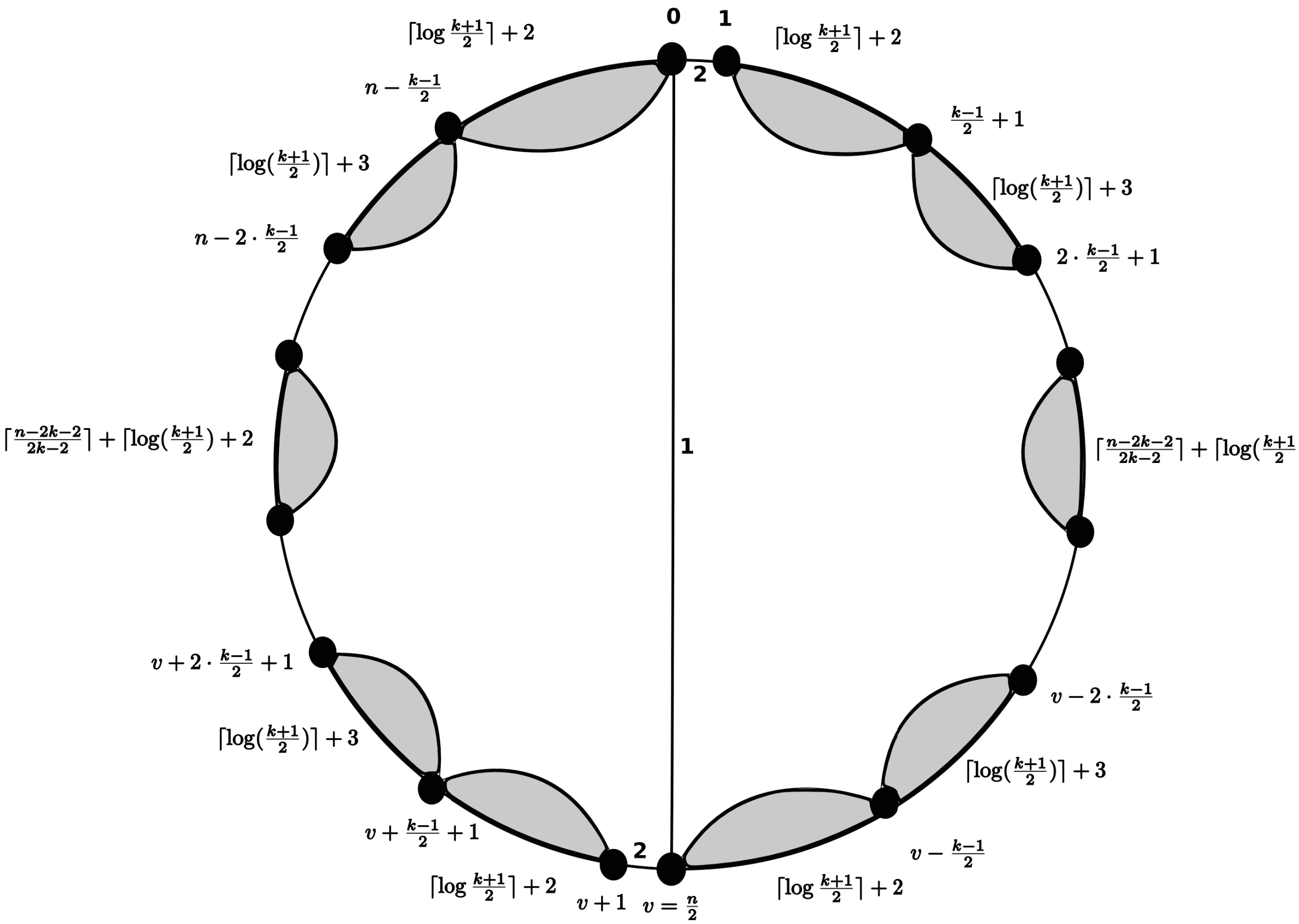

Figure 23: Broadcast Scheme inside $H_{k,n}$ when n is even and k is odd

procedure BROADCAST-SCHEAM S_2 $(H_{k,n}, 0, U, W)$

Vertex 0 is the originator, n is even and k is odd

Vertex 0 informs vertex $\frac{n}{2}$ at time 1. Let $v = \frac{n}{2}$

At time unit 2, vertex 0 informs vertex 1 and vertex v informs vertex $v + 1$

Starting at time unit 3, vertex 0 informs vertices inside clique. $\{0, n-1, \ldots, n - \frac{k-1}{2}\}$,vertex 1 informs vertices inside clique $\{1, 2, \ldots, \frac{k+1}{2}\}$, vertex v informs vertices in clique $\{\ v, v-1, \ldots, v - \frac{k-1}{2}\ \}$ and vertex $v+1$ informs vertices inside clique $\{v+1, v+2, \ldots, v + \frac{k+1}{2}\}$.

After time $\lceil \log(\frac{k+1}{2}) + 2 \rceil + 1$ do

$U = U \cup \{1, 2, \ldots, \frac{k+1}{2}, v+1, v+2, \ldots, v+\frac{k+1}{2}\}$ and $W = W \cup \{0, (n-1), \ldots, (n-\frac{k-1}{2}), v-1, v-2, \ldots, v-\frac{k-1}{2}\}$

for $i = 1, \ldots, \lceil \frac{n-2k-2}{2k-2} \rceil$ in clockwise direction **do**

  Represent all the informed vertices in $U = \{u_1, u_2, \ldots, u_p\}$ as u_m where $m \in \{1, 2, \ldots, p\}$

  if $u_m + \frac{k-1}{2} \notin U$ **then**

    u_m informs $u_m + \frac{k-1}{2}$ at time unit i

    $U = U \cup \{u_m + \frac{k-1}{2}\}$

  end if

end for

for $j = 1, \ldots, \lceil \frac{n-2k-2}{2k-2} \rceil$ in counter-clockwise direction **do**

  Represent all the informed vertices in $W = \{w_1, w_2, \ldots, w_q\}$ as w_n where $n \in \{1, 2, \ldots, q\}$

  if $w_n - \frac{k-1}{2} \notin W$ **then**

    w_n informs $w_n - \frac{k-1}{2}$ at time unit j

    $W = W \cup \{w_m - \frac{k-1}{2}\}$

  end if

end for

end procedure

end

Algorithm 2 gives the broadcast scheme for the second case of Harary graph. At time unit 1 vertex 0 sends information to vertex $v = \frac{n}{2}$. At time unit 2 vertices 0 and v informs vertices 1 and $v+1$ respectively. Starting at time unit 3, vertex 0 informs vertices inside clique $\{0, n-1, \ldots, n-\frac{k-1}{2}\}$,vertex 1 informs vertices inside clique $\{1, 2, \ldots, \frac{k+1}{2}\}$, vertex v informs vertices in clique $\{v, v-1, \ldots, v-\frac{k-1}{2}\}$ and vertex $v+1$ informs vertices inside clique $\{v+1, v+2, \ldots, v+\frac{k+1}{2}\}$.

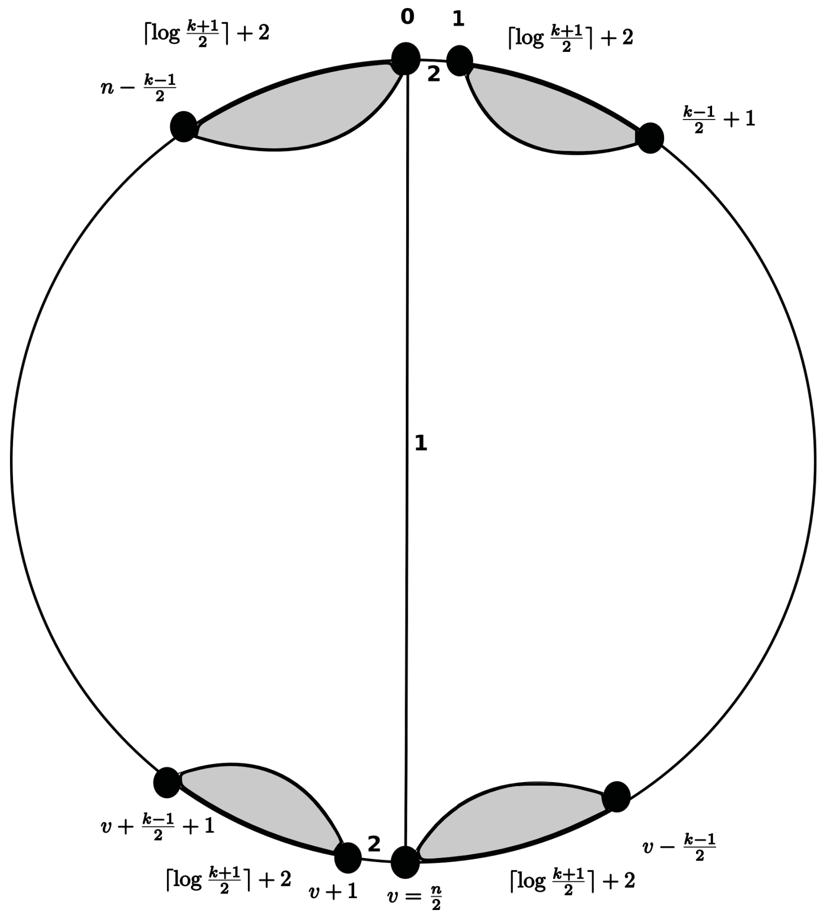

Figure 24: Second case of graph $H_{k,n}$ after $\lceil \log(\frac{k+1}{2}) \rceil + 2$ time units

Figure 24 shows $H_{k,n}$ after $\lceil \log(\frac{k+1}{2}) \rceil + 2$ time units.

After $\lceil \log(\frac{k+1}{2}) \rceil + 2$ time units, in total $2k + 2$ vertices will have information. In other words, there are $n - 2k - 2$ uninformed vertices. Figure 25 shows $H_{k,n}$ after $\lceil \log(\frac{k+1}{2}) \rceil + 3$ time units. Starting at time unit $\lceil \log(\frac{k+1}{2}) \rceil + 3$, $2k - 2$ vertices are informed at each time unit new .So It takes $\lceil \frac{n-2k-2}{2k-2} \rceil$ time units to inform rest of the $n - 2k - 2$ vertices. Figure 23 shows the complete broadcast scheme inside the second case of Harary graph.

Theorem 3.3.3 *Algorithm 2 gives* ($\lceil \log \frac{k+1}{2} \rceil$)*-additive approximation.*

Proof Under the algorithm 2, vertex 0 first informs vertex $v = \frac{n}{2}$. Vertices 1 and $v + 1$ receive the message at time unit 2 form vertex 0 and 1 respectively. Starting at time unit 3, vertex 0 informs vertices inside clique $\{0, n-1, \ldots, n - \frac{k-1}{2}\}$,vertex 1 informs vertices in clique $\{1, 2, \ldots, \frac{k+1}{2}\}$,

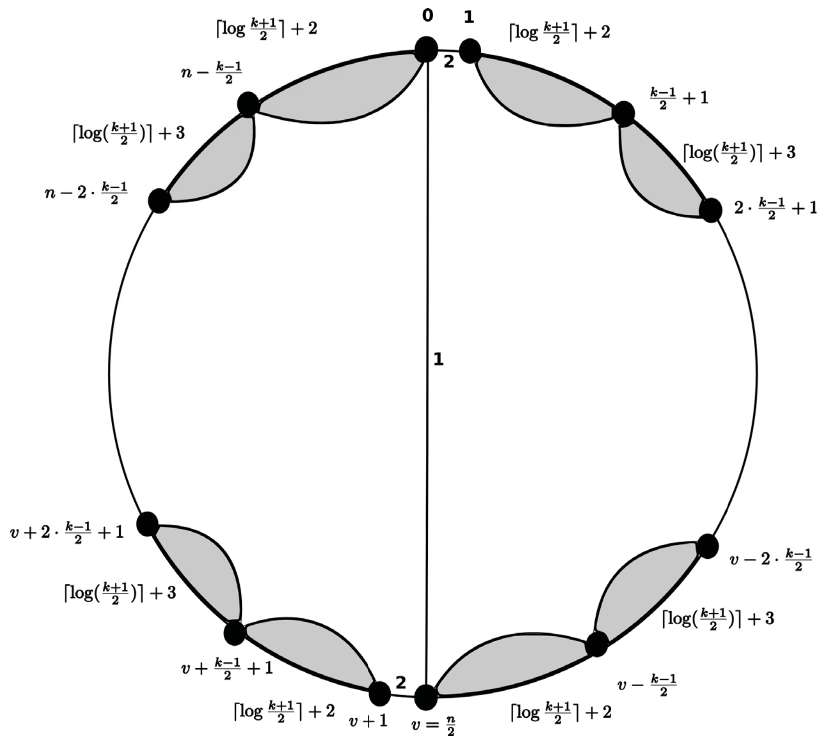

Figure 25: Second case of graph $H_{k,n}$ after $\lceil \log(\frac{k+1}{2}) \rceil + 3$ time units

vertex v informs vertices in clique $\{v, v-1, \ldots, v - \frac{k-1}{2}\}$ and vertex $v+1$ informs vertices in clique $\{v+1, v+2, \ldots, v + \frac{k+1}{2}\}$. There are $\frac{k-1}{2} + 1$ vertices inside all four cliques. So it will take $\lceil \log \frac{k+1}{2} \rceil$ time units to broadcast in each of those cliques. By $\lceil \log(\frac{k+1}{2}) + 2 \rceil$ time units, total $4(\frac{k-1}{2} + 1) = 2k - 2 + 4 = 2k + 2$ vertices will be informed.

Now vertices 0,1, v and $v+1$ do not have any uninformed neighbor. At time unit $\lceil \log(\frac{k+1}{2}) \rceil + 3$ vertices $v \in \{2, \ldots, \frac{k-1}{2} + 1, v+2, \ldots, v + \frac{k-1}{2} + 1\}$ and $u \in \{n-1, \ldots, n - \frac{k-1}{2}, v-1, \ldots v - \frac{k-1}{2}\}$ will inform their neighbor $v + \frac{k}{2}$ and $u - \frac{k}{2}$ respectively as shown in Figure 25. At time unit $\lceil \log(\frac{k+1}{2}) \rceil + 3$, $2k-2$ new vertices are informed. In next time unit, these new informed vertices will send message to next $2k-2$ uninformed vertices by following steps 8-21 of algorithm 2. At each time unit $2k-2$ vertices are informed, therefore it takes $\lceil \frac{n-2k-2}{2k-2} \rceil$ time units to inform all $n-2k-2$ vertices.

Thus, $b_{S_2}(H_{k,n}) = \lceil \frac{n-2k-2}{2k-2} \rceil + \lceil \log \frac{k+1}{2} \rceil + 2 = \lceil \frac{n-4-2k+2}{2k-2} \rceil + \lceil \log \frac{k+2}{2} \rceil + 2 = \lceil \frac{n-4}{2k-2} \rceil - \lceil \frac{2k-2}{2k-2} \rceil + \lceil \log \frac{k+1}{2} \rceil + 2 = \lceil \frac{n-4}{2k-2} \rceil + \lceil \log(\frac{k+1}{2}) \rceil + 1 \leq \lceil \frac{n}{2k-2} \rceil + \lceil \log(\frac{k+1}{2}) \rceil + 1 \leq b(H_{k,n}) + \lceil \log(\frac{k+1}{2}) \rceil$ using

Lemma 3.3.2(i). Note that when $\lceil \frac{n-4}{2k-2} \rceil < \lceil \frac{n}{2k-2} \rceil$, the approximation will be $\lceil \log(\frac{k+1}{2}) \rceil - 1$. ∎

Lemma 3.3.4 *Let $H_{k,n}$ be a Harary graph on n vertices where k is odd and n is even. The broadcast time of $H_{k,n}$ from any originator is $b(H_{k,n}) \leq \lceil \frac{n-4}{2k-2} \rceil + \lceil \log(\frac{k+1}{2}) \rceil + 1$.*

Proof This is direct consequence of the proof of theorem 3.3.3. ∎

3.4 $H_{k,n}$ with n,k odd

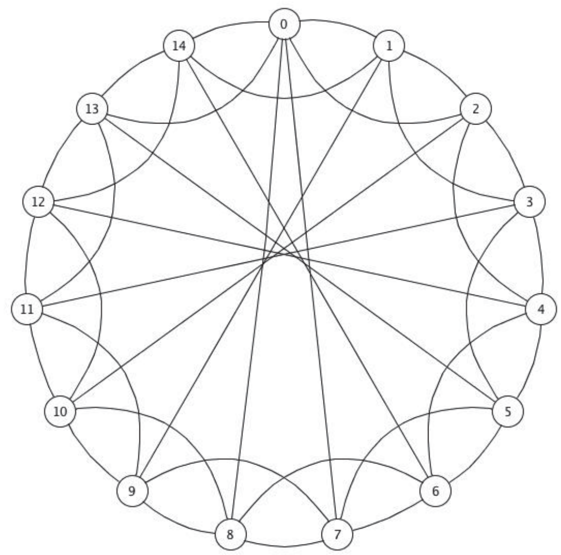

Figure 26: Example Graph $H_{5,15}$ of third case

Figure 26 represents graph $H_{5,15}$, which is example of the third case of Harary graph. This case is nearly similar to the second case of Harary graph. Every vertex v in $H_{k,n}$ is connected to $v + \frac{k-1}{2}$ to $v - \frac{k-1}{2}$ vertices, and then adding edges joining vertex 0 to vertices $\frac{n-1}{2}$ and $\frac{n+1}{2}$ and vertex i to vertex $i + \frac{n+1}{2}$ for $1 \leq i < \frac{n-1}{2}$.

As represented in Lemma 3.2.1, there are two cliques in $H_{k,n}$ formed by the set of vertices V_1 and V_2. From i we can visit vertices $(i+1) \bmod n, \ldots, (i + \frac{k-1}{2}) \bmod n$ and $(i-1) \bmod n, \ldots, (i - \lfloor \frac{k-1}{2} \rfloor)$ in just one time unit. From node i if we visit node $(i + \frac{k-1}{2}) \bmod n$ or $(i - \frac{k-1}{2}) \bmod n$, then it is called a city-tour, otherwise it is called village-tour.

3.4.1 Diameter

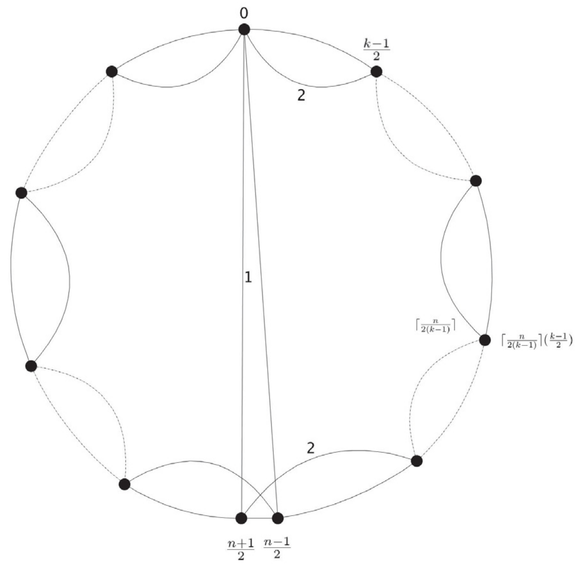

Figure 27: Diameter path of third case

Above Figure 27 shows the diametral path of the third case on Harary graph. Here $H_{k,n}$ is not vertex transitive. Every vertex in this graph has k degree except node 0, which has degree $k+1$. The graph is similar to the second case except vertex 0 is connected to one more vertex $\frac{n-1}{2}$.

In this graph vertex 0 is directly connected vertex $\frac{n+1}{2}$, which divides graph $H_{k,n}$ into almost half. So the diametral vertex will be inside the farthest clique from vertex 0 and $\frac{n+1}{2}$. First of all, it is necessary to reach the clique that contains the diametral vertex. After that in just one village-tour one can reach the vertex at diametral distance from node 0.

- Vertex 0 visits node $\frac{n+1}{2}$ at time unit 1.
- At time unit 2, node $\frac{n+1}{2}$ and 0 starts to make city tours to reach farthest clique.
- Every city-tour connects vertex each vertex $v \in V$ in $H_{k,n}$ to vertex $v + \frac{k-1}{2}$.
- Therefore the farthest clique will be at distance $\lceil \frac{n/4}{(k-1)/2} \rceil = \lceil \frac{n}{2(k-1)} \rceil$ from either vertex 0 or vertex $\frac{n+1}{2}$.
- Vertex $\lceil \frac{n}{2k-2} \rceil (\frac{k-1}{2})$ can be reached at time unit $\lceil \frac{n}{2k-2} \rceil$.

- After that, one can visit diametral vertex in just one time unit by making a village-tour.

Lemma 3.4.1 describes the proof for the diameter of the third case of Harary graph. Diameter of the graph changes depending on the value of $\frac{n}{k-1}$.

Lemma 3.4.1 *Let $H_{k,n}$ be a Harary graph on n vertices where n and k are odd, then $D(H_{k,n}) = \left\lceil \frac{n}{2k-2} \right\rceil + 1$.*

Proof When n is odd, 0 is directly connected with $\frac{n+1}{2}$. We are going to consider several cases:

(i) when $\frac{n}{k-1} = 2p$, for some positive integer p:

Let $\frac{n}{k-1} = 2p$, for some positive integer p. In other words in either direction, starting from vertex 0, we can make $2p$ city-tours, since at most $\frac{n+1}{2}$ vertices can be covered. Let us label the city-tours as $1, 2, ..., 2p$ from vertex 0 along any particular direction. From vertex 0 one can make p city-tours. During the same p moves, one can first visit vertex $\frac{n+1}{2}$ and then traverse through $p-1$ city-tours. We need at most one more move to reach a diametral vertex. Hence, the diameter of $H_{k,n} = p + 1$ $= \frac{n}{2(k-1)} + 1$.

(ii) when $\frac{n}{k-1} = 2p - 1$:

Similar to the proof in part (i), in total, p moves are enough to reach a diametral vertex from vertex 0. Hence, the diameter of $H_{k,n} = p = \frac{n}{2(k-1)} + 1/2 = \left\lceil \frac{n}{2k-2} \right\rceil + 1$ (since $\frac{n}{k-1}$ is odd).

(iii) When $\left\lceil \frac{n}{k-1} \right\rceil = 2p$:

Similar to the proof in part(i), from vertex 0 we need $p+1$ moves to reach a vertex at a diametral distance. Thus, diameter of $H_{k,n} = \left\lceil \frac{n}{2k-2} \right\rceil + 1$.

(iv) When $\left\lceil \frac{n}{k-1} \right\rceil = 2p - 1$:

Similar to the proof in part(ii), from vertex 0 we will consider city-tours along $1, 2, ..., p$ in p moves. During the same $p - 1$ moves one can first visit vertex $\frac{n+1}{2}$ and then traverse through city-tours labeled as $2p-1, ..., p+2$. Now, the $(p+1)$th city-tour covers less than $\frac{k-1}{2}$ vertices. We need one more village-tour to reach a vertex at a diametral distance from 0, thus making total moves to be p. Similarly, diameter of $H_{k,n} = \left\lceil \frac{n}{2k-2} \right\rceil + 1$. ∎

Lemma 3.4.2 *Let $H_{k,n}$ be a Harary graph on n vertices where degree of each vertex is at least k, here k and n are odd. The broadcast time of $H_{k,n}$ from any originator is*

(i) $b(H_{k,n}) \geq \lceil \frac{n}{2k-2} \rceil + 1$ *when* $\frac{2n}{k-1} = p$

(ii) $b(H_{k,n}) \geq \lceil \frac{n}{2k-2} \rceil + 2$ *when* $\frac{2n}{k-1} \neq p$

Proof This lemma gives the proof for lower bound on the third case of Harary graph $H_{k,n}$.

(i) This is direct consequence of the result in [36], that the diameter of the graph is a trivial lower bound on the broadcast time. Therefore , $b(G) \geq \lceil \frac{n}{2k-2} \rceil + 1$.

(ii) Since $\frac{2n}{k-1} \neq p$, there are at least two vertices in $H_{k,n}$ at the diametral distance from originator. As described in [15], if there exists at least two vertices at a diametral distance D from vertex u in graph G, then $b(G) \geq D + 1$. Hence, $b(G) \geq \lceil \frac{n}{2k-2} \rceil + 2$. ∎

3.4.2 Broadcast Scheme

This section illustrates the broadcast scheme inside $H_{k,n}$ where n and k are odd. Here, the broadcast scheme is almost similar to the second case of Harary graph.

Broadcast Scheme for Originator 0

- At time unit 1 originator 0 sends message to vertex $\frac{n-1}{2}$. Let us take $v = \frac{n-1}{2}$, here n and k are odd.
- At time unit 2 vertex 0 sends message to vertex 1 and vertex v sends message to vertex $v + 1$.
- Starting at time unit 3, vertex 0 informs vertices inside clique $\{0, n-1, \ldots, n - \frac{k-1}{2}\}$,vertex 1 informs vertices inside clique $\{1, 2, \ldots, \frac{k+1}{2}\}$, vertex v informs vertices in clique $\{ v, v - 1, \ldots, v - \frac{k-1}{2} \}$ and vertex $v+1$ informs vertices inside clique $\{v+1, v+2, \ldots, v + \frac{k+1}{2}\}$.
- Each of four clique has $\frac{k-1}{2} + 1 = \frac{k+1}{2}$ vertices.

- By the time $\lceil \log(\frac{k+1}{2} + 2)$, total $4(\frac{k+1}{2}) = 2k+2$ vertices will have information including vertices 0,1, v and $v+1$.

- Let us assume that $U = \{2, \ldots, \frac{k+1}{2}, v+1, \ldots, v+\frac{k-1}{2}\}$ and $W = \{n-1, \ldots, n-\frac{k-1}{2}, v-1, \ldots, v-\frac{k+1}{2}\}$. Starting at time unit $\lceil \log(\frac{k-1}{2}+1) \rceil + 3$, all the vertices in vertices $u_m \in U$ informs vertices $u_m + \frac{k-1}{2}$ and all the vertices inside $w_n \in W$ informs vertices $w_n - \frac{k-1}{2}$. Vertices 0,1,v and $v+1$ will not participate in the broadcasting process from this step because they do not have any uninformed neighbor.

- At time unit $\lceil \log(\frac{k-1}{2}+1) \rceil + 3$, new $4(\frac{k-1}{2}) = 2k-2$ vertices will receive information.

- After that at each time unit $2k-2$ more vertices will receive message until all the vertices in $H_{k,n}$ are informed.

- Therefore, it will take $\lceil \frac{n-2k-2}{2k-2} \rceil$ time units to inform $n-2k-2$ vertices.

- Thus, the total broadcast time of the graph will be $\lceil \frac{n-2k-2}{2k-2} \rceil + \lceil \log \frac{k+1}{2} \rceil + 2 = \lceil \frac{n-4-2k+2}{2k-2} \rceil + \lceil \log \frac{k+2}{2} \rceil + 2 = \lceil \frac{n-4}{2k-2} \rceil - \lceil \frac{2k-2}{2k-2} \rceil + \lceil \log \frac{k+1}{2} \rceil + 2 = \lceil \frac{n-4}{2k-2} \rceil + \lceil \log(\frac{k+1}{2}) \rceil + 1$.

Figure 23 shows the broadcast scheme inside $H_{k,n}$ when n and k are odd. In the Third case of Harary graph node 0 is connected to node $\frac{n-1}{2}$.

Algorithm 3 describes the broadcast scheme of the third case of Harary graph. Let's take set $U = \{u_1, u_2, \ldots, u_p\}$ as the set of informed vertices in clockwise direction and set $W = \{w_1, w_2, \ldots, u_q\}$ as the set of informed vertices in counter-clockwise direction.

Algorithm 3 *Broadcast Algorithm* S_3

procedure BROADCAST-SCHEAM S_3 $(H_{k,n}, 0, U, W)$
  vertex 0 is the originator, n is odd and k is odd
  vertex 0 informs vertex $\frac{n-1}{2}$ at time 1. Let $v = \frac{n-1}{2}$
  At time unit 2, vertex 0 informs vertex 1 and vertex v informs vertex $v+1$

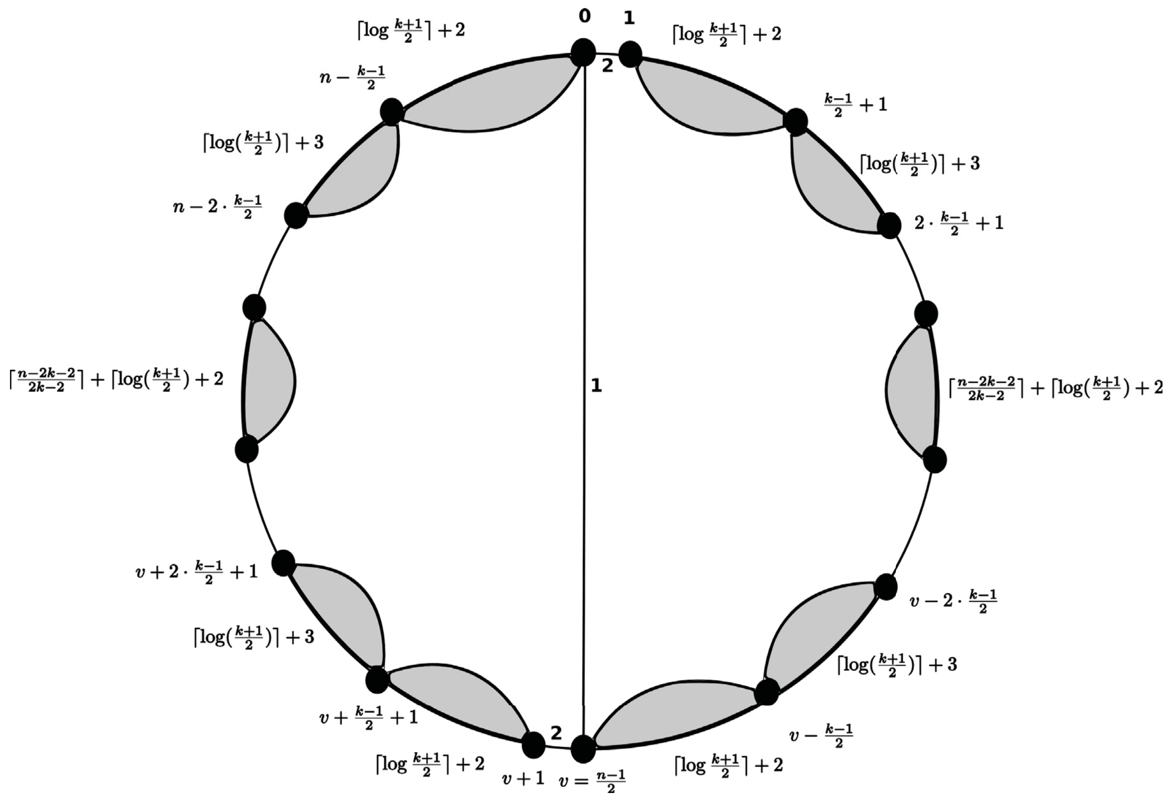

Figure 28: Broadcast Scheme inside $H_{k,n}$ when n and k are odd

Starting at time unit 3, vertex 0 informs vertices inside clique. $\{0, n-1, \ldots, n - \frac{k-1}{2}\}$,vertex 1 informs vertices inside clique $\{1, 2, \ldots, \frac{k+1}{2}\}$, vertex v informs vertices in clique $\{\ v, v-1, \ldots, v - \frac{k-1}{2}\ \}$ and vertex $v+1$ informs vertices inside clique $\{v+1, v+2, \ldots, v + \frac{k+1}{2}\}$.

After time $\lceil \log(\frac{k+1}{2}) + 2 \rceil + 1$ do

$U = U \cup \{1, 2, \ldots, \frac{k+1}{2}, v+1, v+2, \ldots, v + \frac{k+1}{2}\}$ and $W = W \cup \{0, (n-1), \ldots, (n - \frac{k-1}{2}), v-1, v-2, \ldots, v - \frac{k-1}{2}\ \}$

for $i = 1, \ldots, \lceil \frac{n-2k-2}{2k-2} \rceil$ in clockwise direction **do**

Represent all the informed vertices in $U = \{u_1, u_2, \ldots, u_p\}$ as u_m where $m \in \{1, 2, \ldots, p\}$

if $u_m + \frac{k-1}{2} \notin U$ **then**

u_m informs $u_m + \frac{k-1}{2}$ at time unit i

$U = U \cup \{u_m + \frac{k-1}{2}\}$

end if

end for

for $j = 1, \ldots, \lceil \frac{n-2k-2}{2k-2} \rceil$ in counter-clockwise direction **do**

Represent all the informed vertices in $W = \{w_1, w_2, \ldots, w_q\}$ as w_n where $n \in \{1, 2, \ldots, q\}$

if $w_n - \frac{k-1}{2} \notin W$ **then**

w_n informs $w_n - \frac{k-1}{2}$ at time unit j

$W = W \cup \{w_m - \frac{k-1}{2}\}$

end if

end for

end procedure

end

Algorithm 3 gives the broadcast scheme for the third case of Harary graph. At time unit 1 vertex 0 sends information to vertex $v = \frac{n-1}{2}$. At time unit 2 vertices 0 and v informs vertices 1 and $v+1$ respectively. Starting at time unit 3, vertex 0 informs vertices inside clique $\{0, n-1, \ldots, n - \frac{k-1}{2}\}$,vertex 1 informs vertices inside clique $\{1, 2, \ldots, \frac{k+1}{2}\}$, vertex v informs vertices in clique $\{ v, v-1, \ldots, v - \frac{k-1}{2} \}$ and vertex $v+1$ informs vertices inside clique $\{v+1, v+2, \ldots, v + \frac{k+1}{2}\}$. Figure 29 shows $H_{k,n}$ after $\lceil \log(\frac{k+1}{2}) + 2 \rceil$ time units.

After $\lceil \log(\frac{k+1}{2}) \rceil + 2$ time units, in total $2k+2$ vertices will have information. In other words, there are $n - 2k - 2$ uninformed vertices. Figure 30 shows $H_{k,n}$ after $\lceil \log(\frac{k+1}{2}) \rceil + 3$ time units. Starting at time unit $\lceil \log(\frac{k+1}{2}) \rceil + 3$, at each time unit new $2k-2$ vertices are informed. So It takes $\lceil \frac{n-2k-2}{2k-2} \rceil$ time units to inform rest of the $n - 2k - 2$ vertices. Figure 28 shows the complete broadcast scheme inside the third case of Harary graph.

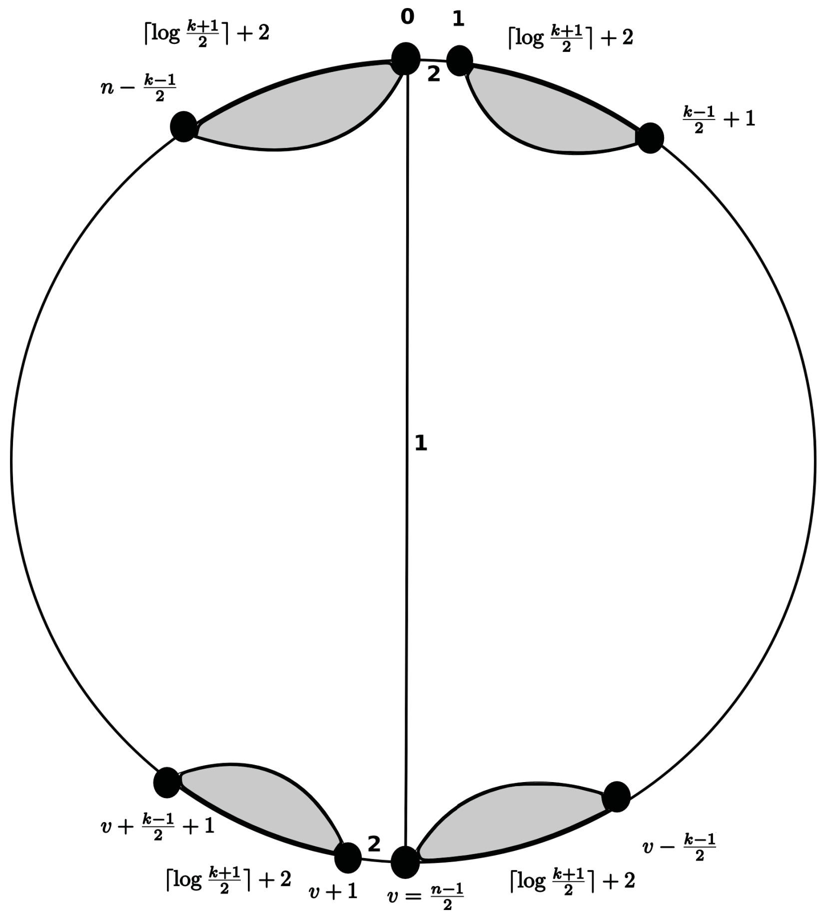

Figure 29: Third case of graph $H_{k,n}$ after $\lceil\log(\frac{k+1}{2})\rceil+2$ time units

Theorem 3.4.3 *Algorithm 3 gives* ($\lceil\log\frac{k+1}{2}\rceil$)*-additive approximation.*

Proof Under the algorithm 2, vertex 0 first informs vertex $v=\frac{n-1}{2}$. Vertices 1 and $v+1$ receives information at time unit 2 form vertex 0 and 1 respectively. Starting at time unit 3, vertex 0 informs vertices inside clique $\{0, n-1, \dots, n-\frac{k-1}{2}\}$,vertex 1 informs vertices inside clique $\{1, 2, \dots, \frac{k+1}{2}\}$, vertex v informs vertices in clique $\{\ v, v-1, \dots, v-\frac{k-1}{2}\ \}$ and vertex $v+1$ informs vertices inside clique $\{v+1, v+2, \dots, v+\frac{k+1}{2}\}$. There are $\frac{k-1}{2}+1$ vertices inside all four cliques. So it will take $\lceil\log\frac{k+1}{2}\rceil$ time units to broadcast in each of those cliques. By $\lceil\log(\frac{k+1}{2})\rceil+2$ time units, total $4(\frac{k-1}{2}+1)=2k+2$ vertices will be informed.

Now vertices 0,1, v and $v+1$ do not have any uninformed neighbor. At time unit $\lceil\log(\frac{k+2}{2})\rceil+3$ vertices $v\in\{2,\dots,\frac{k-1}{2}+1, v+2, \dots, v+\frac{k-1}{2}+1\ \}$ and $u\in\{n-1,\dots,n-\frac{k-1}{2}, v-1, \dots v-\frac{k-1}{2}$

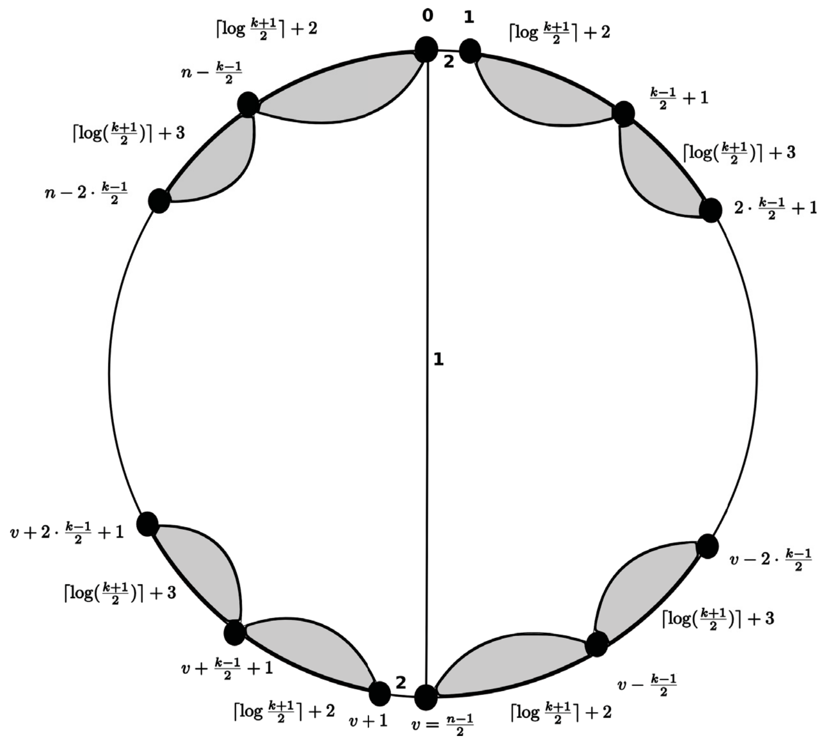

Figure 30: Third case of graph $H_{k,n}$ after $\lceil \log(\frac{k+1}{2}) \rceil + 3$ time units

$\}$ will inform their neighbor $v + \frac{k}{2}$ and $u - \frac{k}{2}$ respectively as shown in Figure 30. At time unit $\lceil \log(\frac{k+1}{2}) \rceil + 3$, $2k - 2$ new vertices are informed. In next time unit, these new informed vertices will send message to next $2k - 2$ uninformed vertices by following steps 8-21 of algorithm 3. At each time unit $2k - 2$ vertices are informed, therefore it takes $\lceil \frac{n-2k-2}{2k-2} \rceil$ time units to inform all $n - 2k - 2$ vertices.

Thus, $b_{S_3}(H_{k,n}) = \lceil \frac{n-2k-2}{2k-2} \rceil + \lceil \log \frac{k+1}{2} \rceil + 2 = \lceil \frac{n-4-2k+2}{2k-2} \rceil + \lceil \log \frac{k+2}{2} \rceil + 2 = \lceil \frac{n-4}{2k-2} \rceil - \lceil \frac{2k-2}{2k-2} \rceil + \lceil \log \frac{k+1}{2} \rceil + 2 = \lceil \frac{n-4}{2k-2} \rceil + \lceil \log(\frac{k+1}{2}) \rceil + 1 \leq \lceil \frac{n}{2k-2} \rceil + \lceil \log(\frac{k+1}{2}) \rceil + 1 \leq b(H_{k,n}) + \lceil \log(\frac{k+1}{2}) \rceil$ using Lemma 3.4.2(i). Note that when $\lceil \frac{n-4}{2k-2} \rceil < \lceil \frac{n}{2k-2} \rceil$, the approximation will be $\lceil \log(\frac{k+1}{2}) \rceil - 1$. ∎

Lemma 3.4.4 *Let $H_{k,n}$ be a Harary graph on n vertices where k is odd and n is odd. The broadcast time of $H_{k,n}$ from any originator is $b(H_{k,n}) \leq \lceil \frac{n-4}{2k-2} \rceil + \lceil \log(\frac{k+1}{2}) \rceil + 1$.*

Proof This is direct consequence of the proof of theorem 3.4.3. ∎

3.5 Analytical Observations

We saw in section 3.1 that $H_{n-1,n}$ is a complete graph and $H_{2,n}$ is a cycle graph. Thus,

$$b(H_{n-1,n}) = \lceil \log n \rceil$$

$$b(H_{2,n}) = \lceil \tfrac{n}{2} \rceil$$

Now let us consider different value of k based on n such that broadcast time of the graph is minimum.

In general theorem 3.2.4 gives $\lceil \log \frac{k+1}{2} \rceil$ approximation in the worst case. However, our analysis below will show that for many values of k our algorithm generates much better approximation. Moreover, for many values of k our algorithm generates the exact broadcast time of Harary graph. If $k = \lceil \frac{n}{2} - 1 \rceil$, where k is even, then

$$\begin{aligned}
b(H_{k,n}) &= \lceil \frac{n-2}{k} \rceil + \lceil \log \frac{k+2}{2} \rceil \\
b(H_{\lceil \frac{n}{2}-1 \rceil, n} &= \lceil \frac{n-2}{\lceil \frac{n}{2} - 1 \rceil} \rceil + \lceil \log(\frac{\lceil \frac{n}{2} - 1 \rceil + 2}{2}) \rceil \\
&= \lceil \frac{2(n-2)}{n-2} \rceil + \lceil \log(\frac{n+2}{4}) \rceil \\
&= 2 + \lceil \log(n+2) \rceil - 2 \\
&= \lceil \log(n+2) \rceil \\
&= \lceil \log n \rceil \quad \text{for all } n \text{ except } n = 2^l, 2^l - 1, 2^l - 2 \text{ where } l \text{ is integer}
\end{aligned}$$

Since $b(G) \geq \lceil \log n \rceil$ for any graph G on n vertices and then algorithm 1 generates the exact broadcast time for $H_{k,n}$ when $k = \lceil \frac{n}{2} - 1 \rceil$. It is also clear that when $\lceil \frac{n}{2} - 1 \rceil \leq k \leq n - 1$, $b(H_{k,n}) \leq b(H_{\lceil \frac{n}{2}-1 \rceil, n}) = \lceil \log n \rceil$. As when k increases, the size of every clique in the graph gets bigger and more vertices are part of the same cliques.

Now let's take $k = \lceil \frac{n}{4} \rceil$.

$$
\begin{aligned}
b(H_{k,n}) &= \lceil \frac{n-2}{k} \rceil + \lceil \log \frac{k+2}{2} \rceil \\
b(H_{\lceil \frac{n}{4} \rceil,n} &= \lceil \frac{n-2}{\frac{n}{4}} \rceil + \lceil \log(\frac{\frac{n}{4}+2}{2}) \rceil \\
&= \lceil \frac{4(n-2)}{n} \rceil + \lceil \log(\frac{n+8}{8}) \rceil \\
&= 4 + \lceil \log(n+8) \rceil - 3 \\
&= \lceil \log(n+8) \rceil + 1 \\
&= \lceil \log n \rceil + 1 \quad \text{for all n except } 2^l - 8 \leq n \leq 2^l \text{ where } l \text{ is integer}
\end{aligned}
$$

Based on above calculations, we can say that when $k = \lceil \frac{n}{4} \rceil$, broadcasting in $H_{k,n}$ takes just one more time unit than in $H_{k,n}$ where $k = \lceil \frac{n}{2} - 1 \rceil$. It is clear that $b(H_{k,n}) \leq b(H_{\lceil \frac{n}{4} \rceil,n} = \lceil \log n \rceil + 1$ when $\lceil \frac{n}{2} - 1 \rceil > k \geq \lceil \frac{n}{4} \rceil$.

Theorem 3.3.3 gives $\lceil \log \frac{k+1}{2} \rceil + 1$ approximation. However the analysis below will show that algorithm 2 gives exact broadcast time. Let's take $k = \lceil \frac{n}{4} \rceil$ where k is odd. Then

$$
\begin{aligned}
b(H_{k,n}) &= \lceil \frac{n-4}{2k-2} \rceil + \lceil \log \frac{k+1}{2} \rceil + 1 \\
b(H_{\frac{n}{4},n}) &= \lceil \frac{n-4}{2(\frac{n}{4})-2} \rceil + \lceil \log \frac{\frac{n}{4}+1}{2} \rceil + 1 \\
&= \lceil \frac{2(n-4)}{n-4} \rceil + \lceil \log \frac{n+4}{8} \rceil + 1 \\
&= 2 + \lceil \log \frac{n+4}{8} \rceil + 1 \\
&= 3 + \lceil \log \frac{n+4}{8} \rceil \\
&= \lceil \log 8 \cdot \frac{n+4}{8} \rceil \\
&= \lceil \log(n+4) \rceil \\
&= \lceil \log n \rceil \quad \text{except } 2^l - 4 \leq n \leq 2^l \text{where } l \text{ is an integer}
\end{aligned}
$$

As k increases, the size of each clique gets bigger. So there will be more vertices in each clique, which helps to broadcast faster in the graph. Thus, it is clear that $b(H_{k,n}) \leq b(H_{\frac{n}{4},n}) = \lceil \log n \rceil$

when $\lceil \frac{n}{4} \rceil \leq k \leq n-1$. Now let's consider $k = \lceil \frac{n}{8} + 1 \rceil$ where k is odd.

$$
\begin{aligned}
b(H_{k,n}) &= \lceil \frac{n-4}{2k-2} \rceil + \lceil \log \frac{k+1}{2} \rceil + 1 \\
b(H_{\frac{n}{8}+1,n}) &= \lceil \frac{n-4}{2(\frac{n}{8}+1)-2} \rceil + \lceil \log \frac{\frac{n}{8}+1+1}{2} \rceil + 1 \\
&= \lceil \frac{4(n-4)}{n} \rceil + \lceil \log \frac{n+16}{16} \rceil + 1 \\
&= 4 + \lceil \log \frac{n+16}{16} \rceil + 1 \\
&= 4 + \lceil \log(n+16) \rceil - 4 + 1 \\
&= \lceil \log(n+16) \rceil + 1 \\
&\geq \lceil \log n \rceil + 1 \quad \text{except } 2^l - 16 \leq n \leq 2^l \text{ where } l \text{ is an integer}
\end{aligned}
$$

Broadcasting in $H_{k,n}$ takes just 1 more time unit when $k = \frac{n}{8} + 1$ than in $H_{k,n}$ with $k = \lceil \frac{n}{4} \rceil$. So it is clear that if $\lceil \frac{n}{4} \rceil > k \geq \lceil \frac{n}{8} + 1 \rceil$, then $b(H_{k,n}) \leq b(H_{\frac{n}{8}+1,n}) = \lceil \log n \rceil + 1$.

Chapter 4

Results on Modified Harary Graph

This chapter introduces new Modified Harary Graph $MH_{k,n}$. We will first see the definition and construction of $MH_{k,n}$. Section 4.2 illustrates the diameter of modified harary graph. Section 4.3 describes the broadcast scheme and gives upper bound on broadcast time.

4.1 New Modified Harary Graph

Definition 1 *Modified Harary graph, $MH_{k,n}$ on even number of vertices with even degree is constructed as follows: Let the vertices be labeled as $0, 1, ..., n-1$. The two vertices i and j are connected by an edge if vertices $j \in \{(i - 2^r + 1) mod\ n, (i + 2^r - 1) mod\ n\}$ for $i = 0, ..., n-1$ and $r = 1, ..., \frac{k}{2}$ (see Figure 31). Here n, k are even.*

Modified Harary graph is more practical in termes of cost as it has less numbers of edges than Harary graph. In this graph not all consecutive vertices are connected. As the value of k increases, Every vertex gets connected to the farthest possible vertex. Which can speed up the broadcasting process in the graph.

Properties of Modified Harary Graph:

- $MH_{k,n}$ is a circulant graph.

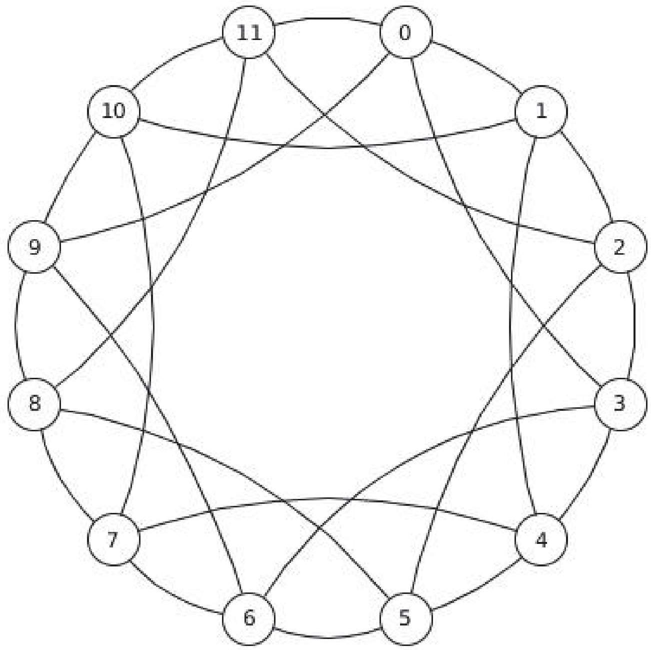

Figure 31: Graph $MH_{4,12}$

- It is vertex transitive.
- Each vertex is connected to $\frac{k}{2}$ vertices in the clockwise direction and another $\frac{k}{2}$ vertices in the anti-clockwise direction.
- Every vertex has degree k.

Theorem 4.1.1 *Graph $MH_{k,n}$ is vertex transitive .*

Proof A Graph is vertex transitive if for any two vertices u and v there exists some automorphism mapping u to v. Define the family of functions $f(u) = u + \alpha$ where α can be any integer other than 0. let i and j be any two vertices. The automorphism satisfying $f(i) = j$ is $f(u) = u + (j - i)$. Applying this function on any two vertices x and y we get $f(x) = x + j - i$ and $f(y) = y + j - i$. $(f(x), f(y))$ will be an edge only if $f(y) = f(x) \pm m$ where $m \in \{(l - 2^r + 1) \bmod n, (l + 2^r - 1) \bmod n\}$ where $l = 0, ..., n - 1$ and $r = 1, ..., \frac{k}{2}$. Substituting the values of the functions we get $y + b - a = x + b - 1 \pm m$ which is equivalent to $y = x \pm m$. Which means that transformed vertices

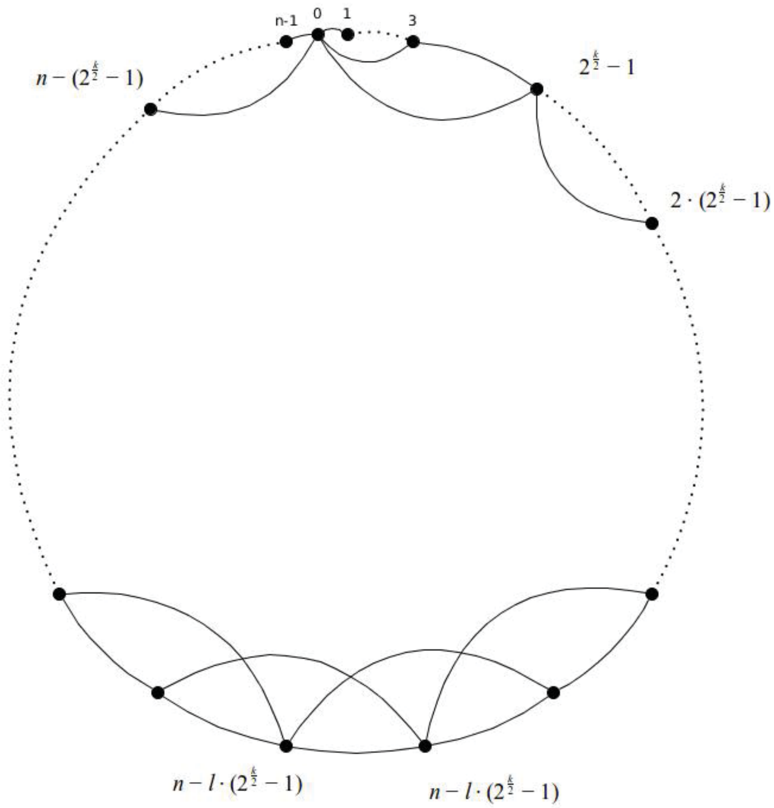

Figure 32: Modified Harary graph $M_{k,n}$

will be adjacent only if the original ones were too. Thus, the suggested family of function preserves the structure of the graph hence there is an automorphism. ∎

Theorem 4.1.2 *The graph $MH_{k,n}$ is circulant graph.*

Proof There are two conditions which needs to be satisfied in order to prove if the graph Circulant. First, a circulant graph is a graph of n vertices in which each vertex u is adjacent to the vertices $(u+v)$ and $(u-v)$ for each v in a list l.Second, a graph G is a circulant iff the automorphism group of G contains at least one permutation consisting of a minimal cycle of length $\mid V(G) \mid$. To show that $MH_{k,n}$ is circulant, we will prove that it satisfies these two conditions.

As described in definition 1, each vertex u in $MH_{k,n}$ is connected to vertices $u+(2^r-1)$ and $u-(2^r-1)$ where $u=\{0,\ldots,n-1\}$ and $r=\{\ 1,\ldots,\frac{k}{2}\ \}$. Which satisfies the first condition.

Now as we described before, each vertex u in $MH_{k,n}$ is adjacent to vertices $u+(2^r-1)$ and

$u - (2^r - 1)$ where $u = \{0, \dots, n-1\}$ and $r = \{\ 1, \dots, \frac{k}{2}\ \}$. Here if we take $r = 1$, then vertex u is connected to vertices $u + 1$ and $u - 1$. Let's take automorphism group where vertex u is connected to node $u + 1$ .As proved in theorem 4.1.1 that $MH_{k,n}$ is vertex transitive, we will start from node 0 and traverse through vertices in clockwise direction.

$$0 \to 1 \to \dots \to u \to (u+1) \to \dots \to (n-1) \to 0$$

As vertex 0 repeats in above path, there is a cycle consisting of vertices $0, \dots n-1$. Thus, the automorphism group has cycle of length $\mid V(G) \mid$. Which satisfies the second condition. Hence $MH_{k,n}$ is circulant graph. $\blacksquare$

Figure 32 shows that i is connected to $\frac{k}{2}$ vertices in clockwise direction and $\frac{k}{2}$ in counter-clockwise direction. Which concludes that every vertex in $MH_{k,n}$ have k degree. We can say that $MH_{k,n}$ is circulant graph $C_n(2^1 - 1, 2^2 - 1, \dots, 2^{\frac{k}{2}} - 1)$ with k connectivity. Figure 31 shows the example of modified harary graph.

As the graph $MH_{k,n}$ is vertex transitive, we will consider vertex 0 as the originator. It is necessary to modify the definitions of city-tours and village-tours for $MH_{k,n}$. From any node i in $MH_{k,n}$, if we visit either node $(i + 2^{\frac{k}{2}} - 1) \bmod n$ or $(i - 2^{\frac{k}{2}} + 1) \bmod n$, then it is called a city-tour; otherwise it is termed as a village-tour. For any vertex i, we will refer the set of vertices $\{i, i+1, ..., i + 2^{\frac{k}{2}} - 2\}$ as the region of i. Figure 32 shows that with each vertex there starts a region with $2^{\frac{k}{2}}$ vertices in both direction, where $\frac{k}{2} = r$, which is shown in Figure 33.

4.2 Diameter of Modified Harary Graph

As described in chapter 3, diameter is the longest shortest distance between any two vertices in graph. $MH_{k,n}$ is vertex transitive, therefore we can find diameter from any vertex. It is easier to start from vertex 0. Figure 34 shows the diametral path from vertex 0.

Here we will take $\frac{k}{2} = r$ for the simplicity. A city-tour in clockwise and anticlockwise direction from vertex 0 will cover at least $2 \cdot (2^r - 1)$ vertices. So the farthest vertex from 0 will be in the

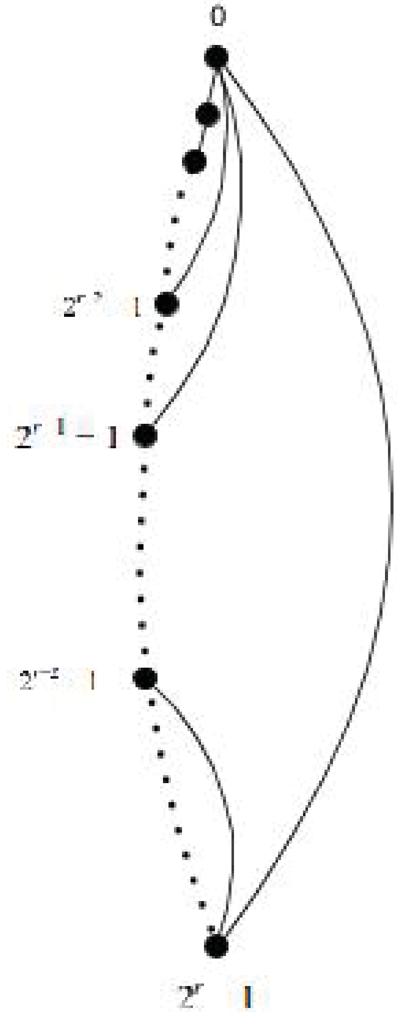

Figure 33: region inside $MH_{k,n}$

region which starts from vertex $\lceil \frac{n}{2\cdot(2^r-1)} \rceil \cdot (2^r-1)$. To find the diameter of the $MH_{k,n}$, it is necessary to first reach vertex $\lceil \frac{n}{2\cdot(2^r-1)} \rceil \cdot (2^r-1)$. As shown in Figure 34, it will take $\frac{n}{2\cdot(2^r-1)}$ city-tours from originator to visit vertex $\lceil \frac{n}{2\cdot(2^r-1)} \rceil \cdot (2^r-1)$. There are 2^r-1 vertices in between nodes $\lceil \frac{n}{2(2^r-1)} \rceil \cdot (2^r-1)$ and $(\lceil \frac{n}{2(2^r-1)} \rceil - 1)(2^r-1)$. So the diameter of the $MH_{k,n}$ will be :

$D(MH_{k,n}) = \lceil \frac{n}{2\cdot(2^r-1)} \rceil$ + number of village-tours to cover (2^r-1) vertices

In order to cover (2^r-1) vertices, we will take recursive approach. With each village tour the number of vertices to be covered will be reduced in half, because number of vertices are in the power

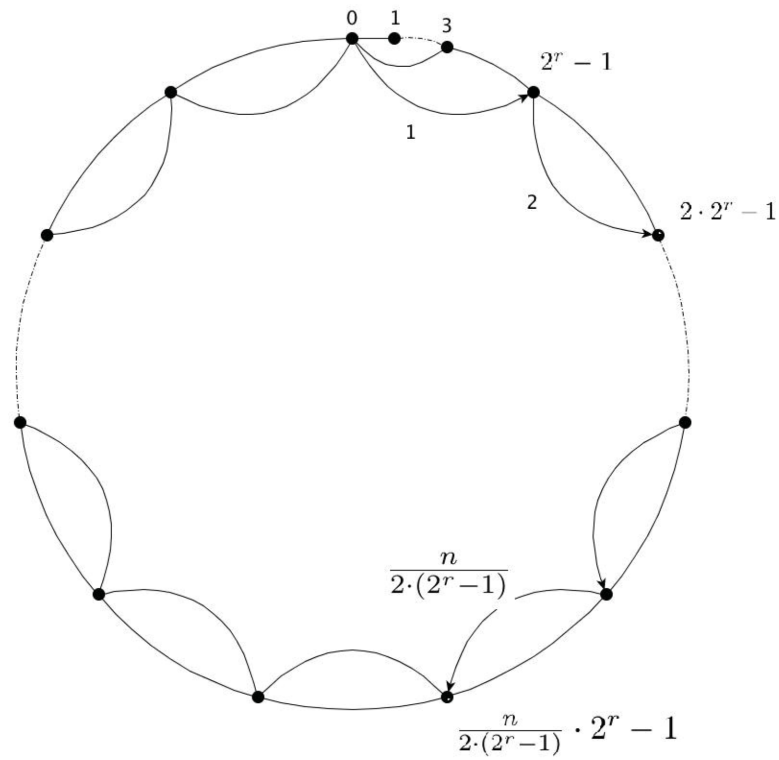

Figure 34: Diametral path of $MH_{k,n}$

of 2. So it will take $r-1$ village tours to cover (2^r-1) vertices. This result is proven in Lemma 4.3.1. Therefore diameter of the graph is :

$D(MH_{k,n}) = \lceil \frac{n}{2\cdot(2^r-1)} \rceil + r - 1.$

Lemma 4.2.1 *Let $MH_{k,n}$ be a modified Harary graph on n vertices. Then $D(MH_{k,n}) = \left\lceil \frac{n}{2(2^r-1)} \right\rceil + r - 1$, where $r = \frac{k}{2}$.*

Proof The result in Lemma 3.2.1 is also applicable in the modified Harary graph $MH_{k,n}$. This is because of its similar properties to Harary graph. Like $H_{k,n}$, $MH_{k,n}$ is also a minimal k-connected graph on n vertices and each vertex in $MH_{k,n}$is connected to $\frac{k}{2}$ vertices in both clockwise and anti-clockwise directions. From vertex 0, at most $\frac{n}{2}$ vertices can be covered from either clockwise or anti-clockwise direction. To start, we will take the maximum possible city-tours from vertex 0. In one city-tour we can cover $2^r - 1$ vertices. Thus, initially we will traverse through $\left\lceil \frac{n}{2(2^r-1)} \right\rceil$ city-tours. There are $2^r - 1$ vertices in between nodes $\left\lceil \frac{n}{2(2^r-1)} \right\rceil (2^r - 1)$ and $(\left\lceil \frac{n}{2(2^r-1)} \right\rceil - 1)(2^r - 1)$.

From either of these 2 nodes, we can make village-tours which will cover 2^{r-1} vertices from either node. So in a recursive way, with each village-tour, the number of vertices to be covered reduces to half. Thus, at most we need $r-1$ village-tours in order to cover 2^{r-1} vertices. This makes the diameter of the graph to be $\left\lceil \frac{n}{2(2^r-1)} \right\rceil + r - 1$. ∎

Lemma 4.2.2 *Let $MH_{k,n}$ be a modified Harary graph on n vertices, where the degree of each vertex is k. The broadcast time of $MH_{k,n}$ from any originator,*

(i) $b(MH_{k,n}) \geq \left\lceil \frac{n}{2(2^r-1)} \right\rceil + r - 1$, if $\frac{n}{(2^r-1)} = p$

(ii) $b(MH_{k,n}) \geq \left\lceil \frac{n}{2(2^r-1)} \right\rceil + r$, if $\frac{n}{(2^r-1)} \neq p$

where $r = \frac{k}{2}$ and p is any positive integer.

Proof (i) Since $\frac{n}{(2^r-1)} = p$, there is exactly one vertex in $MH_{k,n}$ which is at a diametral distance from the original vertex. Thus, the proof is a direct consequence of the result in [36], where it has been shown that $b(G) \geq D(G)$ for any connected graph.

(ii) Since $\frac{n}{(2^r-1)} \neq p$, there are at least two vertices in $MH_{k,n}$ which are at a diametral distance from the original vertex. It has been shown in [15], if there exists at least two vertices at a diametral distance D from vertex u in graph G, then $b(G) \geq D + 1$. Hence, $b(MH_{k,n}) \geq \left\lceil \frac{n}{2(2^r-1)} \right\rceil + r$. ∎

4.3 Broadcasting in Modified Harary Graph

This section illustrates broadcast scheme of $MH_{k,n}$ and gives upper bound and lower bound on broadcast time.

As described before, in $MH_{k,n}$ with every vertex there starts a region with 2^k vertices in both clockwise and counter-clockwise direction. For simplicity we will refer it as a region.

In order to find the upper bound on broadcast time, It is necessary to consider two broadcast schemes. First we will describe broadcast scheme inside the region of $MH_{k,n}$ and after that we will do broadcasting in the whole graph.

Broadcast scheme inside region

Figure 35 shows the broadcast scheme inside the region from the originator 0. Here $\frac{k}{2} = r$.

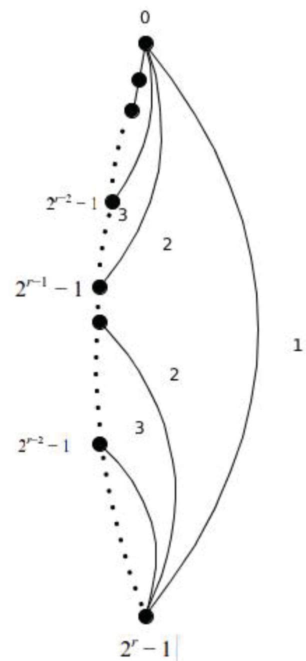

Figure 35: Broadcast scheme in region

- Vertex 0 sends the message to its uninformed neighbor $2^r - 1$ at time unit 1.
- Then node 0 and $2^r - 1$ will send message to its next farthermost uninformed neighbor as shown in Figure 35.
- In short, every informed vertex will send message to its farthermost uninformed neighbor until all the vertices in the region are informed.

  Here Farthermost vertex from the originator is referred in terms of its assigned integer value.

We can see in Figure 35, when node 0 sends message to $2^r - 1$ vertex, it divides internal region with $2^{r-1} - 1$ vertices. Every time we send message to uninformed neighbor using above broadcast scheme, we divide internal region into 2 half regions with same number of vertices. By this, we can say that broadcasting in this region can be done by $\log 2^r = r$ time units.

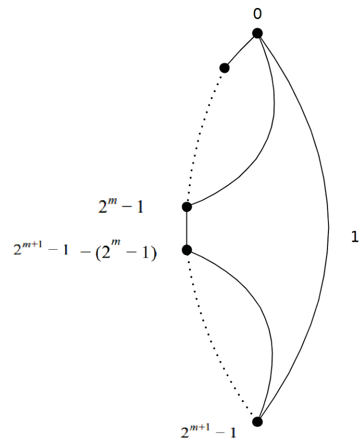

Figure 36: Broadcasting in region with $2^{m+1} - 1$ vertices can be done by m+1 time units

Lemma 4.3.1 *Broadcasting in region can be done by $\frac{k}{2}$ time units.*

Proof We will prove by induction that broadcasting in region can be done by $\frac{k}{2}$ time units.

Base Case: When $\frac{k}{2} = 1$, region will have only 2 vertices 0 and 1. Therefore broadcasting can be done in just one time unit.

Induction Hypothesis: Assume when $\frac{k}{2} = m$, broadcasting in region can be done by m time units.

Induction step: Let $\frac{k}{2} = m + 1$. Figure 36 shows the region with 2^{m+1} vertices. By following the broadcast scheme, node 0 will send message to vertex $2^{m+1} - 1$.That divides inner region into two regions with 2^m vertices. By the induction hypothesis we can say that broadcasting in two regions with 2^m vertices can be done by m time units. As node 0 sends message to vertex $2^{m+1} - 1$ at first time unit, broadcasting in inner two regions can be done simultaneously. The broadcast time is $m + 1$ when $\frac{k}{2} = m + 1$. Hence by induction we proved broadcasting in region can be done by $\frac{k}{2}$ time units. ∎

Broadcast scheme $MH_{k,n}$

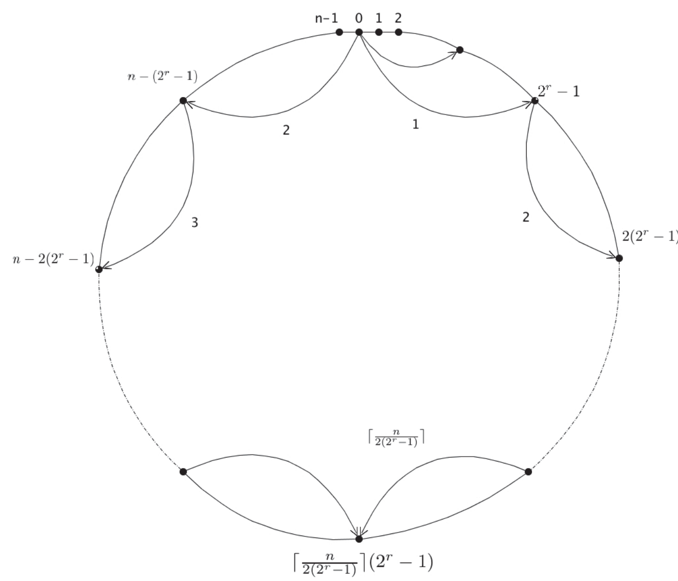

Figure 37: Broadcasting in Modified Harary graph

Figure 37 shows the broadcast scheme inside $MH_{k,n}$ with vertex 0 as originator.

- Let $\frac{k}{2} = r$. At time unit 1, vertex 0 will send information to node $2^r - 1$.

- At time unit 2, the originator sends message to vertex $n - (2^r - 1)$, and vertex $2^r - 1$ sends message to $2(2^r - 1)$.

- As shown in Figure 37, in each direction vertex $i \cdot (2^r - 1)$ or $n - i \cdot (2^r - 1)$ will respectively send message to vertex $(i + 1) \cdot (2^r - 1)$ or $n - (i + 1)(2^r - 1)$, where $0 \leq i \leq \lceil \frac{n}{2(2^r-1)} \rceil - 1$, at i+1 time unit until every region has at least one informed vertex.

- An informed vertex v will always first send message to its farthest uninformed vertex and then it will start broadcasting inside its respective region as described in Lemma 4.3.1.

- A region which starts with vertex $\lceil \frac{n}{2(2^r-1)} \rceil \cdot (2^r - 1)$, will be the last one to have a informed vertex. After that it will start broadcasting inside its region at $\lceil \frac{n}{2(2^r-1)} \rceil + 1$ time unit.

Above broadcast scheme suggests that it is optimal to first inform at least one vertex inside each region, so that they can start broadcasting inside their region. Above broadcast scheme finishes broadcasting inside MHk, n in $\lceil \frac{n}{2(2^r-1)} \rceil + r$ time units.

Algorithm 4 is developed using both broadcast schemes inside region and in Modified harary graph [6].

Approximation Algorithm to broadcast in Modified Harary graph

Algorithm 4 *Broadcast Algorithm S_m*

procedure BROADCAST-SCHEME-$S_m(MH_{k,n}, 0)$

  vertex 0 is the originator and let $r = \frac{k}{2}$

  for $i = 1, ..., \left\lceil \frac{n}{2(2^r-1)} \right\rceil$ in clockwise direction **do**

    vertex $(i-1)(2^r-1)$ informs vertex $i(2^r-1)$ at time i.

  end for

  for $j = 2, ..., \left\lceil \frac{n}{2(2^r-1)} \right\rceil$ **do**

    vertex $(j-1)(2^r-1)$ informs the uninformed vertices in its region starting at time $j+1$. REGION-BROADCAST-RB$((j-1)(2^r-1), r, j)$

  end for

  Starting at time $\left\lceil \frac{n}{2(2^r-1)} \right\rceil + 1$, vertex $\left\lceil \frac{n}{2(2^r-1)} \right\rceil (2^r-1)$ informs its region. REGION-BROADCAST-RB$(\left\lceil \frac{n}{2(2^r-1)} \right\rceil (2^r-1), r, \left\lceil \frac{n}{2(2^r-1)} \right\rceil)$

  for $i = 2, ..., \left\lceil \frac{n}{2(2^r-1)} \right\rceil$ in anti-clockwise direction **do**

    vertex $(n-(i-2)(2^r-1))\text{mod } n$ informs vertex $(n-(i-1)(2^r-1))$ mod n at time i.

  end for

  for $j = 2, ..., \left\lceil \frac{n}{2(2^r-1)} \right\rceil$ **do**

    vertex $(n-(j-2)(2^r-1))\text{mod } n$ informs the uninformed vertices in its region starting at time $j+1$. REGION-BROADCAST-RB$((j-1)(2^r-1), r, j)$.

  end for

Starting at time $\left\lceil \frac{n}{2(2^r-1)} \right\rceil + 1$, vertex $(n - (\left\lceil \frac{n}{2(2^r-1)} \right\rceil - 1)(2^r - 1)) \bmod n$ informs its region. REGION-BROADCAST-RB($(n - (\left\lceil \frac{n}{2(2^r-1)} \right\rceil - 1)(2^r - 1)) \bmod n, r,$ $\left\lceil \frac{n}{2(2^r-1)} \right\rceil$)

end procedure

end

Algorithm 5 *Region Broadcast Algorithm*

procedure REGION-BROADCAST-RB(u, r, τ)

vertex 0 is the originator and let $r = \frac{k}{2}$

if $r = 1$ **then** $MH_{k,n}$ is a cycle and there will be no uninformed vetex in the region of u.

end if

if $r = 2$ **then** $u \xrightarrow{\tau+1} (u + 2^2 - 1)$ and $u \xrightarrow{\tau+2} (u + 2^1 - 1)$, $(u + 2^2 - 1) \xrightarrow{\tau+2} (u + 2^2 - 2)$

end if

if $r \geq 3$ **then** Let $c = 1$

while $r \geq 4$ **do**

vertex u informs vertex $u + 2^{r-1} - 1$ at time unit $\tau + c$.

SUB-REGION-BROADCAST-SRB $(u + 2^{r-1} - 1, \tau + c)$

$r = r - 1$ and $c = c + 1$

end while

if $r = 3$ **then** the set of uninformed vertices within the region $\{u, ..., u + 2^3 - 1\}$ are $u + 1, ..., u + 6$

$u \xrightarrow{\tau+c+1} (u + 3)$

$u \xrightarrow{\tau+c+2} (u + 1)$, $(u + 3) \xrightarrow{\tau+c+2} (u + 6)$

$(u + 3) \xrightarrow{\tau+c+3} (u + 4)$, $(u + 1) \xrightarrow{\tau+c+3} (u + 2)$ and $(u + 6) \xrightarrow{\tau+c+3} (u + 5)$

end if

end if

end procedure

end

Algorithm 6 *Sub-Region Broadcast Algorithm*

procedure SUB-REGION-BROADCAST-SRB $(u + 2^{r-1} - 1, \tau + c)$

We will consider the vertices within the region R=$\{u$+$2^{r-1}-1$,...,u+$2(2^{r-1}-1)$ =u+$2^r-2\}$. Let the set of informed vertices in R be I. Initially $I = \{u$+$2^{r-1}-1\}$ and R is the region for $u + 2^{r-1} - 1$

for every vertex, $v_1 \in I$ **do**

v_1 informs the farthest uninformed vertex v_2 within its own region.

The set of vertices within $\{v_1, ..., v_2\}$ becomes the region for both v_1 and v_2.

Update $I = I + v_2$.

end for

end procedure

end

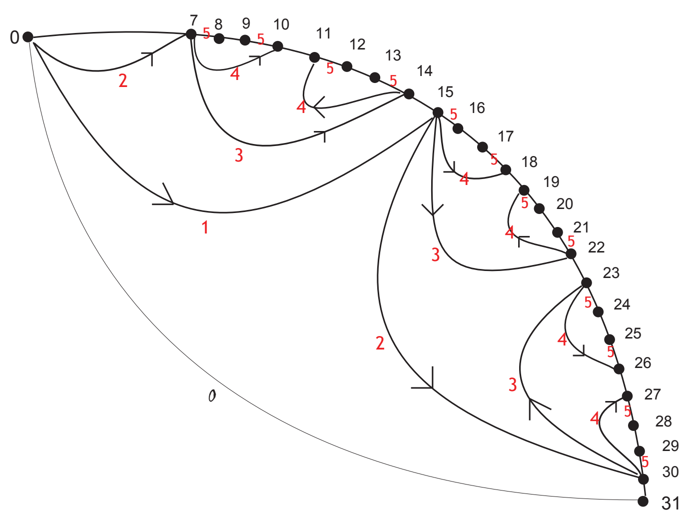

Figure 38: Graph $MH_{4,12}$

The REGION-BROADCAST scheme for the region containing vertices $\{0, 1, ..., 2^5 - 1\}$ except when $r = 3$. The figures in black are the labels of the vertices and the figures in red show the broadcast times. We assume in the figure that vertex 0 informs vertex 31 at time 0. The broadcast times for the regions $\{15, ..., 30\}$ and $\{7, ..., 14\}$ are based on the SUB-REGION-BROADCAST schema. Initially $I = \{15\}$ for the region containing the vertices $\{15, ..., 30\}$ and region for vertex 15 is $\{15, ..., 30\}$. 15 informs the farthest uninformed vertex within its own region (in this case 30) at time 2. The region for vertex 30 is $\{15, ..., 30\}$ and $I = \{15, 30\}$. In the next time unit, both 15 and 30 respectively inform the farthest uninformed vertices in their regions. Thus, 15 informs vertex 22 and 30 informs vertex 23. The region for both vertices 15 and 22 now becomes $\{15, ..., 22\}$ and that for vertices 30 and 23 is $\{23, ..., 30\}$ and $I = \{15, 30, 22, 23\}$. Similarly in the next time unit, all the vertices in set I inform the farthest uninformed vertex in their own regions.

Under the Sub-Region-Broadcast scheme, when node, $u+2^{r-1}-1$ sends a message to its farthest uninformed vertex $u+2^r-2$, it divides the region with $u+2^r-2-(u+2^{r-1}-1)+1 = 2^{r-1}$ vertices. In the next time unit when both vertices $u+2^{r-1}-1$ and $u+2^r-2$ inform their respective farthest uninformed vertices within their region, the new regions formed will each have 2^{r-2} vertices. Thus, every time we send a message to the farthest uninformed vertex, we divide the region into two new regions with same number of vertices.

Complexity: In all the broadcast schemes S_m, Region-Broadcast and Sub-Region-Broadcast, at a given time, a set of informed vertices are informing another set of uninformed vertices and will be part of informed vertices in the next round. This makes the total complexity of the algorithms to be $O(|V|)$.

Lemma 4.3.2 *Let $MH_{k,n}$ be a modified Harary graph on n vertices and let $r = \frac{k}{2}$. The Sub-Region-Broadcast scheme takes r time units to broadcast in a region of $MH_{k,n}$ with 2^r vertices.*

Proof This proof is similar to the proof of Lemma 4.3.1.

Lemma 4.3.3 *Let $MH_{k,n}$ be a modified Harary graph on n vertices and let $r = \frac{k}{2}$. The Region-Broadcast scheme takes r time units to broadcast in a region of $MH_{k,n}$ with 2^r vertices.*

Proof When $r = 2$, it is clear from step 5 of Region-Broadcast that it will take 2 time units to complete broadcasting. When $r = 3$, steps 13-16 show 3 time units are enough to broadcast. When

$r \geq 4$, in step 9, u informs vertices $u + 2^{r-1} - 1$, $u + 2^{r-2} - 1$,...,$u + 2^{r-i} - 1$ at times 1,2,...,i respectively (we assume $\tau = 0$ here). In other words, after time i, we have a region containing 2^{r-i} vertices which will be operated upon by the Sub-Region-Broadcast scheme. From Lemma 4.3.2, we know the Sub-Region-Broadcast takes $r - i$ time units to finish broadcasting in this region. In total, $i + r - i = r$ time units are necessary. ∎

Theorem 4.3.4 *Algorithm S_m gives 1-additive approximation when $\frac{n}{(2^r-1)} = p$ for some positive integer p.*

Proof We assume $\frac{k}{2} = r$.

Case 1: when $\frac{n}{2^r-1} = 2q$:

In other words in either direction, starting from vertex 0, we can make q city-tours. Let us label the city-tours as $1, 2, ..., 2q$ from vertex 0 in a clockwise direction. Under algorithm S_m, starting at time 1 in a clockwise direction, vertex 0 makes $\frac{n}{2(2^r-1)} = q$ city-tours to inform vertex $\frac{n}{2(2^r-1)}(2^r - 1) = \frac{n}{2}$ at time $\frac{n}{2(2^r-1)}$. Similarly, starting at time 2 in an anti-clockwise direction, vertex 0 makes $q-1$ city-tours to inform vertex $(n - (\frac{n}{2(2^r-1)} - 1)(2^r - 1))\text{mod}n$=$\frac{n}{2} + 2^r - 1$ at time $\frac{n}{2(2^r-1)}$. All the informed vertices will start informing the uninformed vertices in their respective regions no later than $\frac{n}{2(2^r-1)} + 1$ time units. Similarly, vertex $\frac{n}{2}$ will inform the vertices covered by the $(q+1)$th city-tour. Since there are 2^r vertices in that region, we know from Lemma 4.3.3 that Region-Broadcast scheme will take r time units to finish broadcasting. Thus, $b_{S_m(MH_{k,n})} \leq \frac{n}{2(2^r-1)} + r \leq b(MH_{k,n}) + 1$ from Lemma 4.2.2(i).

Case 2: $\frac{n}{2^r-1} = 2q - 1$ is not possible.

Let us assume by contradiction that $\frac{n}{2^r-1} = 2q-1$ is possible. Since $2^r - 1$ is odd, n is also odd. This contradicts as in $MH_{k,n}$, n is even. ∎

Theorem 4.3.5 *Algorithm S_m is optimal when $\frac{n}{(2^r-1)} \neq p$ for some positive integer p.*

Proof We assume $\frac{k}{2} = r$.

This is similar to the result we proved in Case 2 of Theorem 4.3.4. Similarly, depending on whether $\left\lceil \frac{n}{2^r-1} \right\rceil$ is odd or even in scheme S_m, the uninformed vertices in the regions of $\frac{n}{2(2^r-1)}(2^r-1)$

and $(n-(\frac{n}{2(2^r-1)}-1)(2^r-1))$ either share c_1 common vertices or do not share any common vertex. Instead, there are exactly c_2 vertices between the regions, where $1 \leq c_1, c_2 < 2^r$. Thus, these vertices will take less than r time units to inform the uninformed vertices in their regions using the Region-Broadcast scheme. However starting at time $\left\lceil \frac{n}{2(2^r-1)} \right\rceil + 1$, vertex $(n-(\frac{n}{2(2^r-1)}-2)(2^r-1))$ takes exactly r time units to inform the 2^r uninformed vertices using Region-Broadcast (from Lemma 4.3.3). Thus, $b_{S_m(MH_{k,n})} \leq \left\lceil \frac{n}{2(2^r-1)} \right\rceil + r \leq b(MH_{k,n})$ from Lemma 4.2.2(ii). ∎

Theorem 4.3.6 *$MH_{2\lceil \log n \rceil - 2, n}$ is a broadcast graph.*

Proof Definition of broadcast graph is described in section 1.2 of chapter 1.

From Lemma 4.2.2(i), we know that $b(MH_{k,n}) \geq \left\lceil \frac{n}{2^{\frac{k}{2}+1}-2} \right\rceil + \frac{k}{2} - 1$.

When $k = 2\lceil \log n \rceil - 2$, $2^{\frac{k}{2}+1} = 2^{\lceil \log n \rceil} = n + c$ for some positive integer c.

Thus, $b(MH_{2\lceil \log n \rceil - 2, n}) \geq \left\lceil \frac{n}{n-2} \right\rceil + \lceil \log n \rceil - 1 - 1 = 2 + \lceil \log n \rceil$ - 2 $= \lceil \log n \rceil$. Hence, $MH_{2\lceil \log n \rceil - 2, n}$ is a broadcast graph. ∎

Chapter 5

Conclusion and Future Work

In this thesis we studied the problem of broadcasting in Harary graph $H_{k,n}$ and new modified harary graph $MH_{k,n}$. We proved that $H_{k,n}$ is vertex transitive and circulant graph when at least n and k are even or at least one of them is even. It has smallest possible number of edges $\lceil \frac{kn}{2} \rceil$ for $\kappa(n) = k$. Then we studied each case of Harary graph in depth.

Chapter 3 illustrates diameter and the broadcast scheme of each case. We calculated the diameter D of each case of Harary graph. When k is even, the diameter of the graph was calculated to be $\lceil \frac{n}{k} \rceil$. We developed an algorithm to compute the broadcast time on n vertices, which completes broadcasting inside $H_{k,n}$ by $\lceil \frac{n}{k} \rceil + \lceil \log(\frac{k}{2} + 1) \rceil +$ time units. Then we proved that this algorithm gives $\lceil \log(\frac{k}{2} + 1) \rceil$ additive approximation.

The diameter of $H_{k,n}$ is $\lceil \frac{n}{2(k-1)} \rceil + 1$ when n is even and k is odd. Since in the second case of Harary graph every vertex is connected to its diametral opposite vertex, The diameter is almost half as compared to diameter in first case of Harary graph. We also developed an algorithm for broadcasting in the second case of Harary graph. Using this algorithm we can finish broadcasting in $H_{k,n}$ by $\lceil \frac{n}{2k-2} \rceil + 1 + \lceil \log \frac{k+1}{2} \rceil$ time units. It gives $\lceil \log \frac{k+1}{2} \rceil$-additive approximation for broadcasting.

Third case of $H_{k,n}$ when n and k are odd, gives the same diameter and broadcast time as the second case of Harary graph. Since here both n and k are odd and every vertex in the graph does not have same degree, the graph is not vertex transitive and circulant. Lower bound of Harary graph is same as the diameter of the graph when value of n is much bigger than k. In future one can find out proportionality between n and k such that broadcast time is same as the diameter of the graph.

In section 3.5, we analyzed the $H_{k,n}$ based on different value of k. In the first case of Harary graph, $b(H_{k,n}) \geq \lceil \log n \rceil$, when $n-1 \geq k \geq \lceil \frac{n}{2} - 1 \rceil$. When $\lceil \frac{n}{2} - 1 \rceil > k \geq \rceil \frac{n}{4} \rceil$, the broadcast time of Harary graph is $b(H_{k,n}) \geq \lceil \log n \rceil + 1$. In the second case of Harary graph when $n-1 \geq k \geq \lceil \frac{n}{4} \rceil$, broadcast time of Harary graph is $b(H_{k,n}) \geq \lceil \log n \rceil$. When $\lceil \frac{n}{4} \rceil > k \geq \lceil \frac{n}{8} + 1 \rceil$, broadcast time of Harary graph is $b_(H_{k,n}) \geq \lceil \log n \rceil + 1$.

Here by observing all three cases of Harary graph, we can say that second case of Harary graph gives the best topology in terms of diameter and broadcast time comparing to the first case of the graph. It is vertex transitive, circulant and symmetric which is advantageous over the third case of graph, as it does not have such properties.

In chapter 4, we defined a new modified harary graph $MH_{k,n}$ when n is even. We proved that $MH_{k,n}$ is vertex transitive and circulant graph. We calculated the diameter of $MH_{k,n}$ to be $\lceil \frac{n}{2(2^{\frac{k}{2}}-1)} \rceil + \frac{k}{2} - 1$ which is also lower bound on the graph. To calculate the broadcast time of the graph, it is necessary to find broadcast time of the region. By induction, we proved that broadcasting inside region can be finished in $\frac{k}{2}$ time units. We developed algorithms to broadcast inside the region and inside whole $MH_{k,n}$. Using these algorithms, broadcasting inside $MH_{k,n}$ can be finished in $\lceil \frac{n}{2(2^{\frac{k}{2}}-1)} \rceil + \frac{k}{2}$ time units, just 1 more time unit than the diameter of the graph. This proves that this algorithm gives 1-additive approximation. Here approximation is very close to optimal. We also showed that graph $MH_{2\lceil \log n \rceil - 2, n}$ is a broadcast graph.

The future work here will be to determine the exact broadcast time of Harary graph for any value of k. One can provide polynomial algorithm for the broadcast time problem in that subclass of modified harary graph. Many different graphs can be created by modifying Harary graphs.

Bibliography

[1] R. Ahlswede, L. Gargano, H.S. Haroutunian, and L.H. Khachatrian. Fault-tolerant minimum broadcast networks. *Networks*, 27(4):293–307, 1996.

[2] R. Ahlswede, H.S. Haroutunian, and L.H. Khachatrian. Messy broadcasting in networks. In *Communications and Cryptography*, volume 276, pages 13–24. Springer, 1994.

[3] A. Bar-Noy, J. Bruck, C. T. Ho, S. Kipnis, and B. Schieber. Computing global combine operations in the multi-port postal model. *IEEE Trans. Par. Distr. Syst.*, 6:896–900, 1995.

[4] Hervé Baumann, Pierre Fraigniaud, Hovhannes A Harutyunyan, and Rémi de Verclos. The worst case behavior of randomized gossip protocols. *Theoretical Computer Science*, 560:108–120, 2014.

[5] J.C. Bermond and C. Peyrat. Broadcasting in debruijn networks. In *19th Southeastern Conference on Combinatorics, Graph Theory and Computing, Congressus Numerantium*, 1988.

[6] P. Bhabak, H. A. Harutyunyan, and S. Tanna. Broadcasting in harary-like graphs. In *13th International Symposium on Pervasive Systems, Algorithms, and Networks (I-SPAN 2014)*, Chengdu, 2014.

[7] J. Bruck and C. T. Ho. Efficient global combine operations in the multi-port message passing system. *Parallel Processing Letters*, 3:335–346, 1993.

[8] L. Changhong and Z. Kemin. The broadcast function value b(23) is 33 or 34. *Acta Mathematicae Applicatae Sinica (English Series)*, 16(3):329–331, 2000.

[9] S.C. Chau and A.L. Liestman. Constructing fault-tolerant minimal broadcast networks. *Journal of Combinatorics, Information and System Sciences*, 11:1–18, 1986.

[10] F. Comellas, H.A. Harutyunyan, and A.L. Liestman. Messy broadcasting in multidimensional directed tori. *Journal of Interconnection Networks*, 4(1):37–51, 2003.

[11] K. Diks and A. Pelc. Broadcasting with universal lists. In *Proceedings of the 28th Hawaii International Conference on System Sciences (HICSS 1995)*, volume 2, pages 564–573. IEEE, 1995.

[12] K. Diks and A. Pelc. Broadcasting with universal lists. volume 27(3), pages 183–196, 1996.

[13] A. M. Farley. Minimal broadcast networks. *Networks*, 9(4):313–332, 1979.

[14] A.M. Farley and S.T. Hedetniemi. Broadcasting in grid graphs. In *In Proceedings of the 9th Southeastern Conference on Combinatorics, Graph Theory, and Computing*, pages 275–288, 1978.

[15] P. Fraigniaud and E. Lazard. methods and problems of communication in usual networks. *Discrete Appl. Math*, 53:79–133, 1994.

[16] P. Fraigniaud and E. Lazard. Methods and problems of communication in usual networks. *Discrete Applied Mathematics*, 53(1-3):79–133, 1994.

[17] H. Grigoryan G. Barsky and H.A. Harutyunyan. Lower bounds on broadcast function for n = 24 and 25. *Discrete Applied Mathematics*, 175:109–114, 2014.

[18] L. Gargano, A. Pelc, S. Pérennes, and U. Vaccaro. Efficient communication in unknown networks. *Networks*, 38(1):39–49, 2001.

[19] M. Grigni and D. Peleg. Tight bounds on minimum broadcast networks. *SIAM j. Discr. Math.*, 4:207–222, 1991.

[20] Hayk Grigoryan and Hovhannes A. Harutyunyan. Diametral broadcast graphs. *Discrete Applied Mathematics*, 171(0):53 – 59, 2014.

[21] Hayk Grigoryan, Hovhannes A. Harutyunyan, and Georgy Barsky. Tight lower bounds on broadcast function for n=24 and 25. *Discrete Applied Mathematics*, 175(0):109–114, 2014.

[22] F. Harary. The maximum connectivity of a graph. In *National Academy of Sciences of the United States of America*, volume 48, pages 1142–1146, 1962.

[23] F. Harary. *Graph Theory*. MA: Addison-Wesley, 1994.

[24] T.E. Hart and H.A. Harutyunyan. Improved messy broadcasting in hypercubes and complete bipartite graphs. In *Proceedings of the 33rd Southeastern Conference on Combinatorics, Graph Theory, and Computing*, pages 181–196, Boca Raton, 2002.

[25] H. A. Harutyunyan and A. L. Liestman. k-broadcasting in trees. *Networks*, 38(3):163–168, 2001.

[26] H. A. Harutyunyan and A. L. Liestman. Improved upper and lower bounds for k-broadcasting. *Networks*, 37:94–101, 2001.

[27] H. A. Harutyunyan and B. Shao. A heuristic for k-broadcasting in arbitrary networks. In *IEEE Conference on Applications of Graph Theory IV03*, pages 287–292, London, England, 2003.

[28] H. A. Harutyunyan and B. Shao. Optimal k-broadcast in trees. *Congressus Numerantium*, 64, 2003.

[29] H.A. Harutyunyan. An efficient vertex addition method for broadcast networks. *Internet Mathematics*, 5(3):211–225, 2008.

[30] H.A. Harutyunyan, P. Hell, and A.L. Liestman. Messy broadcasting - decentralized broadcast schemes with limited knowledge. *Discrete Applied Mathematics*, 159(5):322–327, 2011.

[31] H.A. Harutyunyan and A.L. Liestman. Messy broadcasting. *Parallel Processing Letters*, 8(2):149–159, 1998.

[32] H.A. Harutyunyan, A.L. Liestman, K. Makino, and T. C. Shermer. Nonadaptive broadcasting in trees. *Networks*, 57(2):157–168, 2011.

[33] H.A. Harutyunyan and E. Maraachlian. Linear algorithm for broadcasting in unicyclic graphs. In *Computing and Combinatorics*, volume 4598, pages 372–382. Springer, 2007.

[34] H.A. Harutyunyan and E. Maraachlian. On broadcasting in unicyclic graphs. *Journal of Combinatorial Optimization*, 16(3):307–322, 2008.

[35] H.A. Harutyunyan and P. Taslakian. Orderly broadcasting in a 2d torus. In *Proceedings of the 8th International Conference on Information Visualization (IV 2004)*, pages 370–375, London, UK, 2004.

[36] J. Hromkovic, R. Klasing, B. Monien, and R. Peine. Dissemination of information in interconnection networks. *Combinatorial Network Theory*, pages 125–212, 1996.

[37] J. Hromkovic, R. Klasing, A. Pelc, P. Ruzicka, and W. Unger. *Dissemination of information in communication networks: broadcasting, gossiping, leader election, and fault-tolerance.* Springer, 2005.

[38] A.L. Liestman J.C. Bermond, P. Hell and J.G. Peters. Sparse broadcast graphs. *Discrete Applied Mathematics*, 36(2):97–130, 1992.

[39] L.H. Khachatrian and O.S. Harutounian. Construction of new classes of minimal broadcast networks. *Conference on Coding Theory*, pages 69–77, 1990.

[40] J.H. Kim and K.Y. Chwa. Optimal broadcasting with universal lists based on competitive analysis. *Networks*, 45:224–231, 2005.

[41] W. Knödel. New gossips and telephones. *Discrete Mathematics*, 13(1):95, 1975.

[42] J.-C. König and E.Lazard. Minimum k-broadcast graphs. *Discr. Appl. Math.*, 53:199–209, 1994.

[43] R. Labahn. A minimum broadcast graph on 63 vertices. *Discrete Applied Mathematics*, 53(1-3):247–250, 1994.

[44] E. Lazard. Broadcasting in dma-bound bounded degree graphs. *Discr. Appl. Math.*, 37/38, 1992.

[45] S. Lee. *Information dissemination theory in communication networks: Design of c Broadcast Networks*. PhD thesis, Pennsylvania State University, 1999.

[46] S. Lee and J.A. Ventura. An algorithm for constructing minimal c-broadcast networks. *Networks*, 38(1):6–21, 2001.

[47] C. Li, T.E. Hart, K.J. Henry, and I.A. Neufeld. Average-case messy broadcasting. *Journal of Interconnection Networks*, 9(4):487–505, 2008.

[48] A.L. Liestman. Fault-tolerant broadcast graphs. *Networks*, 15(2):159–171, 1985.

[49] A.L. Liestman and J.G. Peters. Broadcast networks of bounded degree. *SIAM Journal on Discrete Mathematics*, 1(4):531–540, 1988.

[50] M. Fellows M. Dinneen and V. Faber. Algebraic constructions of efficient broadcast networks. *Applied Algebra, Algebraic Algorithms and Error- Correcting Codes*, 539:152–158, 1991.

[51] M. Maheo and J.F. Sacle. Some minimum broadcast graphs. *Discrete Applied Mathematics*, 53(1-3):285, 1994.

[52] E. Maraachlian. *Optimal broadcasting in treelike graphs*. PhD thesis, Concordia University, Montreal, P.Q., Canada, 2010.

[53] G. Marco and A. Pelc. Deterministic broadcasting time with partial knowledge of the network. In *Algorithms and Computation*, volume 1969, pages 374–385. Springer, 2000.

[54] M. Middendorf. Minimum broadcast time is np-complete for 3-regular planar graphs and deadline 2. *Information Processing Letters*, 46(6):281–287, 1993.

[55] S. Mitchell and S. Hedetniemi. A census of minimum broadcast graphs. *Journal of Combinatorics, Information and System Sciences*, 5:141–151, 1980.

[56] J.H. Park and K.Y. Chwa. Recursive circulant: a new topology for multicomputer networks (extended abstract). In *International Symposium on Parallel Architectures, Algorithms and Networks (ISPAN 1994)*, pages 73–80, Kanazawa, 1994.

[57] R. Peine, E.A. Stohr, R. Klasing, and B. Monien. Broadcasting in butterfly and debruijn networks. *Discrete Applied Mathematics*, 53, 1994.

[58] A. Pelc. Fault-tolerant broadcasting and gossiping in communication networks. *Networks*, 28(3):143–156, 1996.

[59] Ernest J. Cockayne Peter J. Slater and S. T. Hedetniemi. Information dissemination in trees. *SIAM Journal on Computing*, 10(4):692–701, 1981.

[60] A. Proskurowski. Minimum broadcast trees. *IEEE Transactions on Computers*, 100(30):363–366, 1981.

[61] Diestel. R. *Graph Theory*. Springer-Verlag, 3^{rd} edition, 2005.

[62] A. Rosenthal and P. Scheuermann. Universal rankings for broadcasting in tree networks. In *Proceedings of the 25th Allerton Conference on Communication, Control and Computing*, pages 641–649, USA, 1987.

[63] J.F. Sacle. Lower bounds for the size in four families of minimum broadcast graphs. *Discrete Mathematics*, 150(1):359–369, 1996.

[64] B. Shao. *On k-broadcasting in graphs.* PhD thesis, Concordia University, Montreal, P.Q., Canada, 2006.

[65] A. Shastri and S. Gaur. Multi-broadcasting in communication networks. In *Proc. Int. Symp. on Communications (ISCOM97)*, pages 167–170, 1997.

[66] S. Skiena. *Implementing Discrete Mathematics: Combinatorics and Graph Theory with Mathematica*. Addison-Wesley, 1990.

[67] D. B. West. *Introduction to Graph Theory.* NJ: Prentice-Hall, 2^{nd} edition, 2000.

[68] J. Xiao and X. Wang. A research on minimum broadcast graphs. *Chinese Journal of Computers*, 11:99–105, 1988.

[69] J. Zhou and K. Zhang. A minimum broadcast graph on 26 vertices. *Applied Mathematics Letters*, 14(8):1023–1026, 2001.