Orthogonal polynomials, equilibrium measures and quadrature domains associated with random matrix models

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Abstract

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Motivated by asymptotic questions related to the spectral theory of complex random matrices, this work focuses on the asymptotic analysis of orthogonal polynomials with respect to quasi-harmonic potentials in the complex plane. The ultimate goal is to develop new techniques to obtain strong asymptotics (asymptotic expansions valid uniformly on compact subsets) for planar orthogonal polynomials and use these results to understand the limiting behavior of spectral statistics of matrix models as their size goes to infinity. For orthogonal polynomials on the real line the powerful Riemann–Hilbert approach is the main analytic tool to derive asymptotics for the eigenvalue correlations in Hermitian matrix models. As yet, no such method is available to obtain asymptotic information about planar orthogonal polynomials, but some steps in this direction have been taken.

The results of this thesis concern the connection between the asymptotic behavior of orthogonal polynomials and the corresponding equilibrium measure. It is conjectured that this connection is established via a quadrature identity: under certain conditions the weak-star limit of the normalized zero counting measure of the orthogonal polynomials is a quadrature measure for the support of the equilibrium measure of the corresponding two-dimensional electrostatic variational problem of the underlying potential.

Several results are presented on equilibrium measures, quadrature domains, orthogonal polynomials and their relation to matrix models. In particular, complete strong asymptotics are obtained for the simplest nontrivial quasi-harmonic potential by a contour integral reduction method and the Riemann-Hilbert approach, which confirms the above conjecture for this special case.

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Special symbols and notations

$\mathbb{C}$	complex numbers
$\mathbb{R}$	real numbers
$\mathbb{N}$	natural numbers $\{1, 2, \dots\}$
$\mathbb{C}[z]$	ring of polynomials over $\mathbb{C}$
$\mathbb{C}_n[z]$	the subspace of polynomials of degree at most n
$\mathbb{E}(X)$	expectation value of a random variable X
$\mathcal{H}_n$	space of $n \times n$ Hermitian matrices
$\mathcal{N}_n$	space of $n \times n$ normal matrices
$\mathbb{C}P^1$	Riemann sphere $\mathbb{C} \cup \{\infty\}$
dA	area measure on $\mathbb{C}$
$A(S)$	area of the set S
$\operatorname{cap}(T)$	logarithmic capacity of a set T
$\operatorname{Tr}(M)$	trace of a matrix M
$\operatorname{supp}(\mu)$	support of a measure μ
$\overline{z}$	complex conjugate of z
$\operatorname{Re}(z)$	real part of z
$\operatorname{Im}(z)$	imaginary part of z
S^c	the complement of the set S
χ_S	the characteristic function of the set S
∂S	the boundary of the set S
$\operatorname{int}(S)$	the interior of the set S
$\operatorname{cl}(S)$	the closure of the set S
$\partial_{\overline{z}}$	dbar operator $\partial_{\overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ for $z = x + iy$
$\Phi_{\pm}$	nontangential boundary value of Φ along the $+/-$ side of a contour

Special symbols and notations (continued)

f^+	positive part $f^+ = \max\{f, 0\}$ of the real function f
$\operatorname*{res}_{\zeta=z} \omega(\zeta)$	the residue of the one-form $\omega(\zeta)$ at $\zeta = z$
$D(a, r)$	open disk of radius r centered at a
$\mathrm{Co}(S)$	convex hull of a set $S \subset \mathbb{C}$
$\mathrm{Pc}(S)$	polynomial convex hull of a set $S \subset \mathbb{C}$
$\mathcal{M}(\Sigma)$	set of probability measures on the set $\Sigma \subset \mathbb{C}$
σ_3	Pauli matrix $\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Chapter 1

Introduction

1.1 Random matrix models

A *random matrix* is a real or complex matrix whose entries are random variables corresponding to a probability distribution described in one of the following ways:

(a) The joint probability distribution of the individual entries is given.

(b) A probability density is given in terms of matrix invariants (typically, for $n \times n$ matrices, powers of traces or coefficients of the characteristic polynomial) with respect to a reference measure on a subset of $\mathbb{C}^{n\times n}$.

Random matrices were studied independently by Wishart [102] in the 1920's and by Wigner [99, 100, 101] in the 1950's from completely different points of view. Motivated by questions in multivariate statistics, Wishart considered empirical $p \times p$ covariance matrices of the form

$$S = X^t X \tag{1.1}$$

where X is an $n \times p$ sample matrix whose rows are independent p-dimensional vector-valued Gaussian random variables and n is the sample size. Wigner investigated the eigenvalues of *random sign symmetric matrices* whose diagonal elements are 0 and the non-diagonal elements have the same absolute value $|v| > 0$ and random signs (respecting the symmetry).

This and subsequent works of Wigner were motivated by the statistical analysis of energy levels of large atomic nucleii, where precise quantum many-body calculations were unfeasible and a statistical analysis of spectral properties was more realistic phenomenologically. His idea was to model the unknown quantum Hamiltonian by a large $n \times n$ Hermitian matrix whose probability distribution is invariant under a *change of coordinates*, i.e., conjugations by $n \times n$ unitary matrices. The simplest such model is the *Gaussian Unitary Ensemble* for which the underlying probability measure can be characterized in two equivalent ways:

(a) The entries of the random $n \times n$ Hermitian matrix

$$M = \begin{bmatrix} x_{11} & x_{12} + iy_{12} & \cdots & x_{1n} + iy_{1n} \\ x_{12} - iy_{12} & x_{22} & \cdots & x_{2n} + iy_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} - iy_{1n} & x_{2n} - iy_{2n} & \cdots & x_{nn} \end{bmatrix}, \tag{1.2}$$

are given in terms of the real coordinates

$$\{x_{kl}\}_{k \leq l}\ ,\ \{y_{kl}\}_{k<l} \tag{1.3}$$

that form an n^2-dimensional random Gaussian vector variable

$$[x_{11}, \ldots, x_{nn}, x_{12}, x_{13}, \ldots, x_{n-1n}, y_{12}, y_{13}, \ldots, y_{n-1n}] \tag{1.4}$$

with zero mean and covariance matrix Σ:

$$\Sigma = \begin{bmatrix} 1 & 0 & & & \\ & \ddots & & 0 & \\ 0 & & 1 & & \\ & & & \frac{1}{2} & 0 \\ & 0 & & & \ddots & \\ & & & 0 & \frac{1}{2} \end{bmatrix} . \tag{1.5}$$

In other words, the entries are independent Gaussian variables with zero mean and variance specified by Σ, i.e. the diagonal elements x_{kk} have variance 1 and the off-diagonal elements x_{kl}, y_{kl} $(k < l)$ have variance $\frac{1}{2}$.

(b) The probability measure given on the linear space $\mathcal{H}_n$ of $n \times n$ Hermitian matrices by

$$d\mu(M) = \frac{1}{Z_n} \exp\left(-\frac{1}{2} Tr(M^2)\right) dM \tag{1.6}$$

where dM is the Lebesgue measure on $\mathcal{H}_n$

$$dM := \prod_{k=1}^{n} dx_{kk} \prod_{k<l} dx_{kl} dy_{kl}, \tag{1.7}$$

and Z_n is the normalization constant (or *partition function*)

$$Z_n = \int_{\mathcal{H}_n} \exp\left(-\frac{1}{2} Tr(M^2)\right) dM \ . \tag{1.8}$$

The Gaussian Unitary Ensemble and the closely related Gaussian Orthogonal and Symplectic Ensembles were studied extensively by researchers from both the mathematical and the theoretical physics communities: the pioneering works of Dyson [36, 33, 34, 35], Gaudin and Mehta [71, 70] were of essential importance in the development of the subject.

As a straightforward generalization of the Gaussian Unitary Ensemble they also considered unitary invariant probability densities on the space of Hermitian matrices of the special form

$$\frac{1}{Z_n} \exp\left(-Tr(V(M))\right) dM \ , \tag{1.9}$$

in terms of a *potential function* V that has suitable regularity and growth properties (see 2.1). Since the eigenvalues of a Hermitian matrix are real, the corresponding eigenvalue distributions are supported on the real line. The invariance under unitary transformations implies that that the essential information is encoded into the eigenvalues: written in terms of a suitable radial-angular coordinate system on $\mathcal{H}_n$ corresponding to the group action of $U(n)$, the probability density is independent of the angular $U(n)$ part which is therefore integrated out. The Jacobian of this coordinate change is a expressible in terms of the Vandermonde matrix evaluated on the eigenvalues and therefore the eigenvalue statistics are expressible in terms of orthogonal polynomials with respect to the measure $e^{-V(x)}dx$ (see 3.6). This allows the asymptotic questions on random matrix observables to be rephrased (and solved) in terms of orthogonal polynomials.

In recent years there has been an explosion of new results in this area; in particular, the introduction of methods originating in the theory of integrable systems has proven to be extremely fruitful and gave an insight into several phenomena characteristic to such random matrix models. Research activity on large n asymptotics of unitarily invariant random matrix models has largely been concentrated around the following main topics:

- global asymptotic behavior of the eigenvalues related to a variational problem and an associated "spectral curve" [58, 26],
- "universality" in the local limiting eigenvalue correlations in the bulk and the edge of the spectrum [28],
- fluctuations in linear statistics of the eigenvalues [58],

- gap probabilities and fluctuations of the largest eigenvalue [94],
- scaling limits at critical points [19].

In the meantime, the theory of random matrices has found a vast array of applications and several connections have been made linking it to diverse fields of mathematics and physics such as:

- combinatorial enumeration of maps into Riemann surfaces [106],
- intersection theory on moduli spaces [62],
- L-functions and related topics in number theory [60],
- random tilings, random words, random permutations [59, 8]
- solid state physics [44],
- two-dimensional topological gravity theories [32]

and numerous other fields.

It is also possible to consider *pairs* or *chains* of Hermitian random matrices of the above form coupled together, forming stochastically dependent matrix systems. This approach gives rise to certain types of two-matrix (Itzykson-Zuber coupling [57] or Cauchy coupling[17]) and, more generally, multi-matrix models [40, 41].

Ginibre [43] considered an analogue of the GUE model for complex non-selfadjoint matrices with a Gaussian density. A natural generalization of this matrix model is provided by *random normal matrix models* [21] (see Section 2.2). A completely new feature of these models is that the eigenvalues are not constrained to the real axis. These matrix ensembles are relevant in modeling two-dimensional physical phenomena including two-dimensional Coulomb plasmas, noninteracting fermions and electrons in a magnetic field [104].

Similarly to the Hermitian case, the spectral statistics of normal matrix distributions can be expressed in terms of the corresponding orthogonal polynomials. However, these polynomials are orthogonal with respect to a measure that is absolutely continuous with respect to the area measure of the plane and there is no similarly general and effective method to extract asymptotic information regarding such polynomials. Therefore, apart from exactly solvable special cases, it is considerably harder to obtain detailed spectral asymptotics for normal matrix ensembles.

1.2 Asymptotic analysis of orthogonal polynomials

For any positive measure μ in the complex plane for which the monomials z^n are square integrable with respect to μ and form a linearly independent system (i.e., μ is not a finite linear combination of point masses) the Gram-Schmidt orthogonalization algorithm produces *orthonormal polynomials* with respect to μ. These polynomials are unique up to a phase factor. The *monic orthogonal polynomials* obtained from normalizing the orthonormal polynomials are characterized by a minimimization problem for the $L^2(\mu)$-norm.

The well-known *classical orthogonal polynomials* (Hermite, Laguerre, Jacobi, Legendre, Chebyshev and a few others) corresponding to special orthogonality weights on the real line appear in several applications in mathematics and physics. These polynomials are regarded as special functions since each of them satisfies a second order linear differential equation, can be calculated in terms of a generating function and has an integral representation in the complex plane. Therefore, standard complex analysis techniques may be applied to the available representations and the asymptotic behavior of these polynomials can be obtained as the degree tends to infinity [2, 91].

For general orthogonality measures on the real line, such representations are not available. However, the symmetry of the scalar product on $\mathbb{R}$ has strong algebraic consequences such as the *three-term recurrence relation*, the *Christoffel-Darboux telescopic identities* and the *interlacing properties of the zeroes* of consecutive orthogonal polynomials [91]. Since there is no general integral representation for these polynomials, a standard steepest descent method [1] is not applicable to the study of their detailed asymptotics.

Fortunately, there is a non-local analogue of the contour integral formalism based on Riemann-Hilbert factorization problems for matrix-valued analytic functions; the orthogonal polynomials are characterized by a 2×2 matrix Riemann-Hilbert problem discovered by Fokas, Its and Kitaev in the 90's [55]. This factorization problem may be analysed using the *nonlinear steepest descent method* developed by Deift and Zhou [29] of which the standard linear steepest descent approach is just a special case, as shown in [15]. This gives asymptotic information about the polynomials in *every* region of the complex plane with error bounds that are uniform on compact subsets. For applications to random matrix models the resulting asymptotic expansions are sufficient to obtain the limiting behavior of the eigenvalue correlation functions and prove *universality* results [30, 28]. The Deift-Zhou nonlinear steepest descent method has proven to be a very powerful and versatile tool for solving several other problems that have a representation in terms of a Riemann-Hilbert factorization, many of which originate in the area of integrable systems [30].

Apart from orthogonal polynomials on the unit circle [86, 87] where the relation $\overline{z} = 1/z$ implies a somewhat similar algebraic framework, the asymptotic properties of orthogonal polynomials with respect to *general* measures in $\mathbb{C}$ are much less understood (and studied). There are results on upper and lower bounds and on limit points of the normalized counting measures of the zeroes for a fairly general class of orthogonal polynomials, mostly based on potential theory and on the L^2-optimality of the orthogonal polynomials [88, 81]. For some special types of planar orthogonal polynomials (such as Szegő polynomials [91], orthogonal with respect to a weight along a curve, or Bergman polynomials [90], orthogonal with respect to the area measure on a planar domain) there are fairly general asymptotic results based on various specific methods.

In a recent work Its and Takhtajan [56] formulated a 2×2 matrix-valued d-bar problem that, similarly to the Fokas-Its-Kitaev Riemann-Hilbert problem, characterizes the orthogonal polynomials for a fairly general class of weights with respect to area measure in the complex plane. However, the analogue of the nonlinear steepest descent method is not developed as yet for this inherently non-local problem. The spectral asymptotics of normal matrix models are one of the most important motivating problems in the development of new techniques; new asymptotic results, at least as strong as the ones provided by the Riemann-Hilbert approach for weights on the real line, are needed to get a better understanding of the local behavior of eigenvalues in normal matrix ensembles.

Parallel to the L^2-theory of orthogonality, there are many problems that lead to *non-Hermitian orthogonal polynomials* for weights along contours in the complex plane. For that setting, the orthogonality is not with respect to a Hermitian inner product, i.e., there is no complex conjugation in the orthogonality relations. This has several disadvantages: for instance, the existence of a monic polynomial of degree n is not guaranteed automatically, and depends on the non-singularity of a moment matrix determinant. On the other hand, the clear advantage is that some of the algebraic features of the orthogonal polynomials are preserved, including the Riemann-Hilbert characterization. One of the main results of the present thesis relies on the fact that the linear system of L^2-orthogonality relations for the orthogonal polynomials can be reduced to a system of non-Hermitian orthogonality relations along certain contours of the complex plane in the special case considered and therefore the Riemann-Hilbert approach may be applied.

1.3 Electrostatic variational problems on the real line and in the complex plane

It is a well-known fact [30] that the asymptotic distribution of the eigenvalues of large Hermitian matrices from the Gaussian unitary ensemble is given by the *Wigner semicircle distribution* [99, 101]. It is also known that this limiting measure is the unique solution of an electrostatic variational problem for logarithmic potentials in the presence of an external field generated by a quadratic potential [81]. For Hermitian matrix models it is true under very general assumptions that the global asymptotic behavior of the eigenvalues is governed by the solution of the same energy problem with respect to the external field that is given by the potential function $V(x)$. This is a consequence of the fact that the joint probability density of the eigenvalues of such random matrices may be viewed as the Boltzmann weight of a particle system with pairwise logarithmic (planar Coulomb) interaction confined to the real line in the presence of an external field generated by the same potential function. The equilibrium measure of the corresponding electrostatic variational problem [81] is essential in understanding the asymptotics of the eigenvalue distributions as the matrix size goes to infinity in a certain scaling limit: for instance, the expectation value of the (random) normalized counting measure of the eigenvalues converges to the equilibrium measure in the weak-star sense. Also, in the large n asymptotic expansion of the partition function Z_n the leading term is given by the minimum energy attained by the equilibrium measure [58, 30].

Since the n-th monic orthogonal polynomial for the weight $e^{-V(x)}$ gives the averaged characteristic polynomial of the corresponding $n \times n$ matrix model (see 3.39), it is not surprising that the expected value of the normalized counting measure of the zeroes of the orthogonal polynomials converges to the same equilibrium measure [58, 30]. As a consequence of the Riemann-Hilbert analysis it is easy to show that the nth root of the monic orthogonal polynomial behaves essentially like the exponentiated (complex) logarithmic potential of the equilibrium measure.

The natural setting for the logarithmic variational problem with external field is actually the complex plane, since the logarithmic kernel is associated to the Laplace operator in two dimensions. The existence and uniqueness of the equilibrium measure is guaranteed under very mild assumptions of the underlying potential function [81] and one may regard equilibrium measures on the real line as charge distributions confined to the 'wire' $\mathbb{R}$.

1.4 Moving boundary problems and normal matrix models

In recent works [72, 3, 64, 92], Wiegmann, Zabrodin et al. considered random normal matrix models in connection with a certain two-dimensional moving boundary problem called *Laplacian growth.* The term 'Laplacian growth' or 'growth by harmonic measure' refers to an idealized mathematical setup to study the dynamics of a moving boundary curve in the plane satisfying the condition that the normal velocity of the boundary is proportional to the gradient of the Green's function of the unbounded exterior domain. There are several physical problems that are associated to the Laplacian growth model, such as

- viscous flows in a Hele-Shaw cell in a zero surface tension limit [48],
- diffusion limited aggregation (DLA) in the plane [103],

- semiclassical electronic droplets associated to the Quantum Hall Effect [3] .

(See [48, 52] for a more complete account on the variety of applications.)

A heuristic description of the growth can be given as follows: *if $D(t)$ denotes the growing domain at time t, the width of the infinitesimal layer $D(t+\Delta t) \setminus D(t)$ that is added in the course of the evolution is proportional to the harmonic measure of the domain $D(t)$*. (The *harmonic measure* is the measure supported on the boundary $\partial D(t)$ whose density with respect to the arclength measure is given by the normal derivative of the Green's function on $\partial D(t)$.)

This law of motion has certain consequences:

- Even for non-singular analytic initial data, the solution may develop *cusp-type singularities* in finite time preventing the continuation of the solution [48] (see Fig. 1.1).

- There is an infinite number of conserved quantities (the exterior harmonic moments) [79, 97]. This brings in the tools of *integrable systems*: the evolution can be associated to the dispersionless limit of the two-dimensional Toda hierarchy of integrable equations realized on the infinite dimensional 'manifold' of conformal mappings [98].

- The classes of polynomial, rational and logarithmic exterior conformal mappings are all invariant under the flow on the space of conformal mappings associated to Laplacian growth [49].

Laplacian growth appears naturally in the context of random normal matrix models, in the following way. Consider the unitary invariant probability measure on the set of $n \times n$ complex normal matrices corresponding to a quasi-harmonic potential $V(z)$. The equilibrium measure of a quasi-harmonic potential is given by the area measure restricted to a compact set D. Therefore the one-parameter family of potentials

$$t \mapsto \frac{1}{t} V(z) \qquad t > 0 \tag{1.10}$$

gives rise to a one-parameter family of equilibrium supports D_t, where the area grows linearly in t. It can be shown [104] that, under certain regularity assumptions, the exterior harmonic moments are conserved and the moving boundary ∂D_t undergoes Laplacian growth. Since the average density of the eigenvalues converges to the equilibrium measure in a certain semiclassical limit [50] as the matrix size n goes to infinity, the finite n (deterministic) averaged eigenvalue density provides a smooth approximation to the domain, or more precisely, to the characteristic function of D_t. This means that the evolution of the one-parameter family of eigenvalue densities, referred to as *quantum Hele-Shaw flow* [50], provides a smooth approximation to the increasing family of evolving domains D_t.

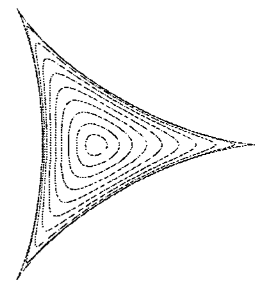


Figure 1.1: Cusp formation for Laplacian growth in a special case

Since the averaged eigenvalue density is expressible in terms of a Christoffel-Darboux-type kernel associated to the orthogonal polynomials with respect to the potential V [37], the study of asymptotic properties of such approximating flows relies on the analysis of the limiting behavior of the underlying orthogonal polynomials. This has motivated many results on the asymptotics of the Christoffel-Darboux reproducing kernels and orthogonal polynomials [50, 6, 7, 38, 37, 39, 56].

1.5 Schwarz functions and quadrature domains

The *Schwarz reflection principle* in complex analysis says that if a function f is holomorphic in a domain D in the upper half-plane whose boundary contains a segment L of the real line and f has a continuous real boundary value there then f admits an analytic continuation to the domain $D \cup L \cup \overline{D}$ where $\overline{D}$ is the reflected domain in the lower half-plane [80]. This theorem and its generalization involving circular arcs and inversions in circles is very useful in constructing special conformal mappings, referred to as *Schwarz-Christoffel mappings*, uniformizing polygonal domains and domains bounded by circular arcs [75]. This principle relies on the fact that reflections and inversions both provide a globally defined (except the isolated pole for the inversion) anti-conformal involution that leaves the line or circle of reflection fixed and establishes an involutive map that exchanges the two domains separated by the curve.

This construction can be generalized to arbitrary non-singular analytic arcs and Jordan curves, but only in a local sense: it can be shown that for such a curve L there exists an anti-conformal reflection φ in a neighborhood of L that leaves the curve fixed and exchanges the two sides of the curve in sufficiently small neighborhoods around the points of L [25, 85]. The conjugate function of this anti-conformal reflection $S = \overline{\varphi}$ is called the *Schwarz function*: it is analytic in a neighborhood of L and satisfies $S(z) = \overline{z}$ [25, 85].

The Schwarz function plays an important role in *inverse balayage problems* for planar domains in potential theory. This involves finding harmonic continuations of the logarithmic potential of a planar domain to the interior of the domain [46]. Motivated by inverse problems in electrostatics and geophsyics [51, 105], the main question is to what extent is it possible to continue the logarithmic potential (also referred to as electrostatic or gravitational potential [85, 97]) through the boundary inwards without hitting singularities that prevent proceeding any further. The complex gradient field of the potential can be expressed in terms of the Schwarz function of the boundary [85] and therefore the singularity structure of the Schwarz function indicates the nature and location of the singularities that the logarithmic potential may have inside the domain [84].

The inverse problem described above is closely related to *quadrature domains* [25, 85, 83, 47]. A domain is said to satisfy a *quadrature identity for the area integral* if there exist a finite number of points in the domain such that the area integral, as a linear functional on the space of integrable analytic functions, can be written equivalently as a linear combination of point evaluations (possibly including evaluations of derivatives of arbitrary orders also). The simplest such domain is a disk in the complex plane: by Cauchy's theorem, the integral of any holomorphic function with respect to the area measure on the disk divided by the area is equal to the value of the integrand at the center of the disk. A classical quadrature domain is characterized by the property that the Schwarz function of its boundary admits a meromorphic continuation to the interior of the domain [25]. Allowing general measures in the quadrature identity representing the area integral leads to *generalized quadrature domains* [83]. For instance, any ellipse is a generalized quadrature domain with a Wigner semicircle-type measure supported along the focal segment of the ellipse [25].

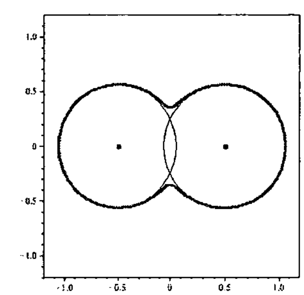


Figure 1.2: A symmetric two-point quadrature domain

There is an associated forward problem associated to the inverse balayage problem: given a compactly supported positive measure in the plane, an equivalent planar domain is to be found that generates the logarithmic potential, with respect to the standard area measure, as the prescribed measure. In other words, in the forward problem quadrature domains are to be constructed corresponding to a given fixed measure. The potential theoretic background of the questions on existence and uniqueness in the forward balayage problem makes it necessary to extend the notion of quadrature domains involving larger classes of test functions, leading to the notion of *harmonic* and *subharmonic quadrature domains* [83].

The relevance of the Schwarz function to normal matrix models is that, given the quasi-harmonic potential associated to the matrix model, the determination of the support of the equilibrium measure uses Schwarz function techniques to find the conformal mapping that describes the boundary of the support [63].

Also, for matrix models corresponding to special cases of quasi-harmonic potentials it was observed [37, 12] that the zeroes of the corresponding orthogonal polynomials are accumulating along certain curve segments and the asymptotic distribution satisfies a generalized quadrature property with respect to the equilibrium measure. In other words, the normalized counting measure of the zeroes provides an *approximate quadrature measure* supported on a finite number of points. It is conjectured [37, 12] that this is a general phenomenon for a large class of quasi-harmonic potentials. One of the main results of this thesis is the confirmation of this conjecture for quasi-harmonic potentials of a special type (see Fig. 1.5 and Chap. 8 for a detailed description of the conjecture and Chap. 12 for the proof).

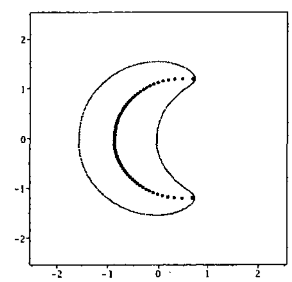


Figure 1.3: Numerical plot of the zeroes of orthogonal polynomials and the support of the equilibrium measure for the case studied in Chap. 12

1.6 Summary of the thesis

The thesis is based on the following publications, preprints and manuscripts:

1. F. Balogh, J. Harnad, *Superharmonic perturbations of a Gaussian measure, equilibrium measures and orthogonal polynomials*, Complex Analysis and Operator Theory, **3** (2): 333–360, 2009.

2. F. Balogh, M. Bertola, *Regularity of a vector potential problem and its spectral curve*, Journal of Approximation Theory, **161** (1):353–370, 2009.

3. F. Balogh, M. Bertola, *On the norms and roots of orthogonal polynomials in the plane and L^p optimal polynomials with respect to varying weights*, arXiv:0910.4223, 2009.

4. F. Balogh, *External potentials for two-point quadrature domains*, in preparation.

5. F. Balogh, M. Bertola, K. T-R. McLaughlin, S. Y. Lee, *Riemann-Hilbert analysis of the Bratwurst orthogonal polynomials*, in preparation.

It consists of two main parts: the first part provides a general overview of the mathematical background necessary to understand the results; a detailed description of the results is presented in the second part.

Each chapter of the first part is a brief summary of basic definitions and standard theorems on the following topics: random matrix models, orthogonal polynomials, equlibrium measures for logarithmic energy problems, Schwarz functions, quadrature domains and Riemann–Hilbert problems. Most of this material is standard and therefore most proofs of cited results are omitted in this part. However, every nontrivial statement is accompanied with references to standard monographs or research papers where a detailed proof can be found.

The second part devotes an individual chapter to each work mentioned above. Each chapter contains a short summary and explanation of the results, the re-print of the paper itself and an appendix containing extra calculations, if necessary.

The first paper (see Chap. 8) introduces the notion of superharmonic perturbations of the quadratic potential and the structure of the supports of equilibrium measures corresponding to such potentials is studied in detail. For orthogonal polynomials with respect to superharmonic perturbations the validity of a matrix d-bar problem, introduced by Its and Takhtajan [56] for a special case of cut-off potentials, is extended. Based on numerical calculations, a general conjecture is stated concerning the connection between the asymptotic distribution of the zeroes of orthogonal polynomials and the support of the equilibrium measure via a quadrature identity.

The second set of results (see Chap. 9) provides a generalized setting for a vector potential problem and the existence and uniqueness of the corresponding vector equilibrium measure is established. Motivated by matrix models, a special case of the vector potential problem is considered and the regularity properties of the components of the corresponding vector equilibrium measure is discussed. It is shown that the resolvents corresponding to the equilibrium measure satisfy a pseudo-algebraic equation. As an illustration, a pseudo-algebraic curve associated to a special case is calculated explicitly.

The third topic considered (see Chap. 10) concerns optimal weighted monic polynomials with respect to the L^p-norm corresponding to a a measure on the complex plane. The results address the asymptotic location of the zeroes of optimal L^p-polynomials (and hence include orthogonal polynomials corresponding to the case $p = 2$) and the nth root asymptotics of the L^p-norms of the optimal weighted polynomials.

In Chap. 11 a special case of the following inverse problem is addressed: given a compact set in the plane, find a quasi-harmonic potential such that the equilibrium measure corresponding to the potential is equal to the normalized area measure restricted to the prescribed set. Three different types of two-point quadrature domains are considered: bicircular quartics (two distinct points with equal weights), limacons (two confluent quadrature nodes resulting in a node of second order) and a pair of two disjoint congruent disks. It is shown that the quasi-harmonic potential can be written as a superharmonic perturbation of the quadratic potential in all three cases.

The fifth and final topic considered (see Chap. 12) is the complete Riemann-Hilbert analysis of the *Bratwurst* orthogonal polynomials corresponding to a quasi-harmonic potential with a single logarithmic singularity. It is shown that the system of two-dimensional orthogonality relations can be reduced to an equivalent system of non-Hermitian orthogonality relations on a contour by constructing a piecewise solution of an associated scalar d-bar problem. Therefore the orthogonal polynomials are characterized by the Fokas-Its-Kitaev Riemann–Hilbert problem which allows the Deift–Zhou nonlinear steepest descent method to be applied. The strong asymptotics of the orthogonal polynomials is calculated explicitly for every region in the complex plane for all values of the parameters. As an application, the quadrature conjecture on the zeroes of the orthogonal polynomials is confirmed for this special case.

Part I

Background

Chapter 2

Random matrix models

The standard references on random matrix models are [70, 30]; the survey paper [104] gives an introduction to normal matrix models.

2.1 Hermitian matrix models

2.1.1 Probability measures on matrices

For fixed $n \in \mathbb{N}$ let $\mathcal{H}_n$ denote the space of $n \times n$ Hermitian matrices with complex entries. $\mathcal{H}_n$ is a linear space over $\mathbb{R}$ of dimension n^2 and therefore admits the n^2-dimensional Lebesgue measure. In terms of the real coordinate system

$$\{M_{kk}\}_{1\le k\le n}\ ,\{\mathrm{Re}(M_{kl})\}_{1\le k<l\le n}\ ,\{\mathrm{Im}(M_{kl})\}_{1\le k<l\le n} \tag{2.1}$$

this flat measure can be written as

$$dM := \prod_{k=1}^{n} dM_{kk} \prod_{1\le k<l\le n} d\mathrm{Re}(M_{kl}) d\mathrm{Im}(M_{kl})\ . \tag{2.2}$$

This measure is not normalizable to a probability measure but can be used as a reference measure that is multiplied by suitable probability densities in order to define a probability measure.

In the theory of random Hermitian matrices the following standard type of probability densities with respect to dM are considered:

$$\rho_V(M) := \frac{1}{\tilde{Z}_n^V} \exp(-\mathrm{Tr}(V(M))) \tag{2.3}$$

where $V: \mathbb{R} \longrightarrow \mathbb{R}$ is called the *potential function.* The term $V(M)$ in the density has the following meaning: if $V(x)$ has a Taylor series expansion around $x = 0$

$$V(x) = \sum_{k=0}^{\infty} a_k x^k \tag{2.4}$$

that converges everywhere in $\mathbb{R}$ then the matrix series

$$V(M) = \sum_{k=0}^{\infty} a_k M^k \tag{2.5}$$

is convergent for every $M \in \mathcal{H}_n$. If $V(x)$ is real analytic $V(M)$ can be defined by the formula [23]

$$V(M) = \frac{1}{2\pi i} \int_{\Gamma} V(z)(zI - M)^{-1} dz \tag{2.6}$$

where Γ is a simple positively oriented contour in the domain of analyticity of V enclosing the eigenvalues of M.

The potential is assumed to grow sufficiently fast as $|x| \longrightarrow \infty$ such that the normalization constant

$$\tilde{Z}_n^V := \int_{\mathcal{H}_n} \exp(-\mathrm{Tr}(V(M)))dM \tag{2.7}$$

(also referred to as the *partition function*) is finite.

For example, choosing $V(x) = \frac{1}{2}x^2$ gives the *Gaussian Unitary Ensemble* (GUE). Similarly one may take $V(x)$ to be a real polynomial of even degree.

2.1.2 Symmetry reduction to the eigenvalues

Since

$$(zI - UMU^{\star})^{-1} = U(zI - M)^{-1}U^{\star} \tag{2.8}$$

using (2.6) gives

$$V(UMU^\star) = UV(M)U^\star \,. \tag{2.9}$$

The trace is invariant under unitary conjugations and therefore these measures are invariant under the action of the unitary group $U(n)$. This suggests that the coordinates be separated into a *radial* and an *angular* part (using the terminology reminiscent of the $SO(N)$ spherical coordinate system in $\mathbb{R}^N$) such that the measure can be written as a product measure and the density does not depend on the angular coordinates. The angular part can then be integrated out to give a constant multiple of the volume of the compact group $U(n)$ with respect to the Haar measure. Since Hermitian matrices are diagonalizable, we can use the coordinate system given by the matrix entries of U and D such that

$$M = UDU^\star \tag{2.10}$$

where U is unitary and D is diagonal. Since the eigenvalues of a generic $M \in \mathcal{H}_n$ are simple and the columns of U are unique up to phases, the diagonal elements $\lambda_1, \dots, \lambda_n$ of D up to permutations and the entries of U modulo n phase factors constitute the so-called *spectral coordinates* [30]. These are provide a well-defined coordinate system on an open dense subset of $\mathcal{H}_n$. The Jacobian of this change of coordinates has a simple form:

$$\prod_{k=1}^{n} dM_{kk} \prod_{1\le k<l\le n} d\mathrm{Re}(M_{kl})d\mathrm{Im}(M_{kl}) = \frac{1}{(2\pi)^n n!} \prod_{1\le i<j\le n} (\lambda_i - \lambda_j)^2 \prod_{k=1}^{n} d\lambda_k dU \,. \tag{2.11}$$

Therefore, if $f\colon \mathcal{H}_n \to \mathbb{C}$ is a function that is invariant under unitary conjugation,

$$\begin{aligned}
&\int_{\mathcal{H}_n} f(M)\exp(-\mathrm{Tr}(V(M)))dM \\
&= \frac{1}{(2\pi)^n n!}\int_{\mathbb{R}^n}\left(\int_{U(n)} f(UDU^\star)\exp(-\mathrm{Tr}(V(UDU^\star)))dU\right)\prod_{1\le i<j\le n}(\lambda_i-\lambda_j)^2\prod_{k=1}^{n} d\lambda_k \\
&= \frac{1}{(2\pi)^n n!}\mathrm{Vol}(U(n))\int_{\mathbb{R}^n} f(\mathrm{diag}\{\lambda_1,\dots,\lambda_n\})e^{-\sum_{k=1}^n V(\lambda_k)}\prod_{1\le i<j\le n}(\lambda_i-\lambda_j)^2\prod_{k=1}^{n} d\lambda_k\,.
\end{aligned} \tag{2.12}$$

The volume of the unitary group is [106]

$$\mathrm{Vol}(U(n)) = \frac{(2\pi)^{\frac{n(n+1)}{2}}}{\prod_{k=1}^{n-1} k!}\,. \tag{2.13}$$

This means that taking expected values of such conjugation invariant functions with respect to the original matrix measure is essentially the same as taking expected values of its evaluation on the space of diagonal matrices with respect to the reduced joint probability density on the unordered eigenvalues:

$$\mathcal{P}_n^V(\lambda_1,\dots,\lambda_n) = \frac{1}{Z_n^V}\prod_{1\le i<j\le n}(\lambda_i-\lambda_j)^2 e^{-\sum_{k=1}^n V(\lambda_k)}\,, \tag{2.14}$$

where the partition function is written as

$$Z_n^V = \frac{(2\pi)^n n!\tilde{Z}_n^V}{\mathrm{Vol}(U(n))}\,. \tag{2.15}$$

2.1.3 Correlation functions

The *m-point correlation function* [70] is defined to be

$$R_{n,m}^V(x_1,\dots,x_m) := \frac{n!}{(n-m)!}\underbrace{\int\!\!\int\cdots\int}_{n-m}\mathcal{P}_n^V(x_1,\dots,x_m,x_{m+1},\dots,x_n)dx_{m+1}\cdots dx_n\,. \tag{2.16}$$

For example, the one-point function is the density of the expectation value of the counting measure of the eigenvalues: for any $B \subset \mathbb{R}$ Borel set,

$$\begin{aligned}
\mathbb{E}(\#\text{eigenvalues in } B) &= \mathbb{E}\left(\sum_{k=1}^{n} \chi_B(\lambda_k)\right) \\
&= \int\!\!\int \cdots \int \sum_{k=1}^{n} \chi_B(\lambda_k) \mathcal{P}_n^V(\lambda_1, \ldots, \lambda_n) d\lambda_1 \cdots d\lambda_n \\
&= n \int\!\!\int \cdots \int \chi_B(\lambda_1) \mathcal{P}_n^V(\lambda_1, \ldots, \lambda_n) d\lambda_1 \cdots d\lambda_n \\
&= \int_B R_{n,1}^V(\lambda_1) d\lambda_1 \; .
\end{aligned} \tag{2.17}$$

2.1.4 Heuristic asymptotics

The joint probability density of the eigenvalues may be rewritten in the form

$$\mathcal{P}_n^V(\lambda_1, \ldots, \lambda_n) = \frac{1}{Z_n^V} \exp\left(- \sum_{k \neq l} \log \frac{1}{|\lambda_k - \lambda_l|} - \sum_{k=1}^{n} V(\lambda_k) \right) , \tag{2.18}$$

whose total integral is the partition function Z_n^V. In the terminology of statistical physics the density (2.18) is a static Boltzmann-type weight for a particle system with logarithmic interaction in the presence of an external potential $V(z)$. Therefore one expects the eigenvalues of a random matrix M drawn from the above distribution to behave like charged particles of positive unit charge in two dimensions in the presence of a confining external electrostatic potential. This fact is referred to as *eigenvalue repulsion* which is consequence of the unitary symmetry and the reduction procedure described above. In probabilistic terms, the probability of finding two eigenvalues close to each other vanishes quadratically as a function of their mutual distance.

Moreover, these eigenvalues are confined to the real line since M is Hermitian; this particle system lives on a one-dimensional 'wire' (the real line $\mathbb{R}$) embedded into a two-dimensional planar geometry.

If one looks for the eigenvalue configurations that maximize the probability, we have to minimize the logarithmic energy expression

$$\sum_{k\neq l} \log \frac{1}{|\lambda_k - \lambda_l|} + \sum_{k=1}^{n} V(\lambda_k) \ . \tag{2.19}$$

This suggests that as $n \to \infty$ the dominant contribution to the partition function comes from configurations that are close to the solution of a continuum variational problem given by the logarithmic energy functional

$$I_V(\mu) := \iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y) + \int V(x) d\mu(x) \tag{2.20}$$

where μ is a probability measure on $\mathbb{R}$. Notice that the discrete energy functional has to be rescaled properly to expect that this gives the correct continuum limit:

$$\sum_{k\neq l} \log \frac{1}{|\lambda_k - \lambda_l|} + \sum_{k=1}^{n} V(\lambda_k) = n^2 \left(\sum_{k\neq l} \frac{1}{n^2} \log \frac{1}{|\lambda_k - \lambda_l|} + \frac{1}{n} \sum_{k=1}^{n} \frac{1}{n} V(\lambda_k) \right) \ . \tag{2.21}$$

In the present form the external potential term (as a Riemann sum) is suppressed by the factor $\frac{1}{n}$. This shows that the potential V has to be rescaled with n if we are interested in the global asymptotic properties of the eigenvalue distribution. This suggests including the additional scaling factor $N > 0$ attached to the potential V:

$$\sum_{k\neq l} \log \frac{1}{|\lambda_k - \lambda_l|} + N \sum_{k=1}^{n} V(\lambda_k) = n^2 \left(\sum_{k\neq l} \frac{1}{n^2} \log \frac{1}{|\lambda_k - \lambda_l|} + \frac{N}{n} \sum_{k=1}^{n} \frac{1}{n} V(\lambda_k) \right) \ . \tag{2.22}$$

In the limit

$$n \to \infty \ , \quad N \to \infty \ , \quad \frac{n}{N} \to t \ , \tag{2.23}$$

where t is a positive constant, we may expect that the partition function gets the dominant contribution from configurations that are close to the equilibrium configuration of the potential $\frac{1}{t} V(z)$.

The above mentioned heuristics have their mathematically rigorous counterparts; a formal statement of each of these will be given in the subsequent sections once all the necessary ingredients are properly introduced.

2.2 Normal matrix models

The fact that every Hermitian matrix is diagonalizable by conjugation by a unitary matrix makes it possible to reduce the unitary invariant measures considered above by a symmetry reduction to the eigenvalue space: the relevant expectation values are expressible in terms of expectation values with respect to $\mathcal{P}_n$ in terms of the eigenvalues. This is of considerable advantage because such random matrix ensembles are made more accessible using available asymptotic methods available as a result of this simplification. A natural generalization is therefore to consider probability measures on the algebraic variety of complex normal matrices

$$\mathcal{N}_n := \{N \in \mathbb{C}^{n\times n} : NN^\star = N^\star N\} \subset \mathbb{C}^{n\times n} . \tag{2.24}$$

To be able to consider measures of the form (2.3) we need a reference measure dN on $\mathcal{N}$ and a spectral coordinate system that makes a symmetry reduction possible. The details of this procedure can be found in [104] and [37]; the resulting joint probability density on the space of eigenvalues is shown to be

$$\mathcal{P}_n^V(z_1, \dots, z_n) = \frac{1}{Z_n^V} \prod_{1\le i<j\le n} |z_i - z_j|^2 e^{-\sum_{k=1}^n V(z_k)} . \tag{2.25}$$

From a purely probabilistic point of view one may simply start with joint probability densities of complex random variables of the above form without any reference to the underlying matrix models and study their asymptotic behavior as $n \to \infty$.

Chapter 3

Orthogonal polynomials

The standard reference on orthogonal polynomials is the monograph of Szegő [91]. We also refer to the books [30, 88] and the survey papers [95, 93].

3.1 Standard L^2-orthogonality

Let μ be a Borel measure on the complex z-plane and assume that

(i) μ is a finite positive measure,

(ii) the support of μ consists of an infinite number of points,

(iii) the absolute moments are finite:

$$\int |z|^k d\mu(z) < \infty \qquad k = 0, 1, \ldots \tag{3.1}$$

Then the monomials $1, z, z^2, \ldots$ all belong to $L^2(\mathbb{C}, d\mu)$ and they are linearly independent since the support of μ is not a finite set. The *Gram–Schmidt orthogonalization procedure* guarantees the existence of the *orthonormal polynomials*

$$p_n(d\mu; z) = \gamma_n z^n + \mathcal{O}\left(z^{n-1}\right) \ , \quad \gamma_n(d\mu) > 0 \qquad (n = 0, 1, \ldots) \tag{3.2}$$

satisfying

$$\int_{\mathbb{C}} p_n(d\mu;z)\overline{p_m(d\mu;z)}d\mu(z) = \delta_{nm} \qquad n,m = 0,1,\dots \tag{3.3}$$

These are unique up to a phase factor; choosing the leading coefficient $\gamma_n(d\mu)$ to be real and positive eliminates this ambiguity.

The *monic orthogonal polynomial* of degree n

$$P_n(d\mu;z) := \frac{1}{\gamma_n(d\mu)}p_n(d\mu;z) \tag{3.4}$$

satisfies the orthogonality conditions

$$\int_{\mathbb{C}} P_n(d\mu;z)\bar{z}^k d\mu(z) = \delta_{nm}h_n(d\mu) \qquad k = 0,1,\dots,n\ , \tag{3.5}$$

with

$$P_n(d\mu;z) = z^n + \mathcal{O}\left(z^{n-1}\right) \qquad n = 0,1,\dots\ , \tag{3.6}$$

where

$$h_n(d\mu) = \frac{1}{\gamma_n^2(d\mu)} \tag{3.7}$$

is the n-th *normalization constant* of μ.

3.1.1 Moment matrices

The *(complex) moments* of the measure μ are

$$m_{kl}(d\mu) := \int_{\mathbb{C}} z^l\bar{z}^k d\mu(z) \qquad k,l = 0,1,\dots\ . \tag{3.8}$$

These form the semi-infinite *moment matrix*

$$M(d\mu) := \left(m_{kl}(d\mu)\right)_{k,l\in\mathbb{N}_0} \tag{3.9}$$

The truncated moment matrices (or Gram matrices [95])

$$M^{(n)}(d\mu) := \left(m_{kl}(d\mu)\right)_{0\le k,l\le n} \tag{3.10}$$

are Hermitian and positive definite [95].

If the support of the orthogonality measure is a subset of an algebraic curve $E(z,\overline{z}) = 0$ for some $E(x,y) \in \mathbb{C}[x,y]$ the entries of the moment matrix satisfy some algebraic relations. For example, if $\mathrm{supp}(\mu) \subset \mathbb{R}$ (i.e., the polynomial is $E(z,\overline{z}) = z - \overline{z}$) then

$$m_{kl}(d\mu) = \int_{\mathbb{R}} z^k \overline{z}^l d\mu(z) = \int_{\mathbb{R}} x^{k+l} d\mu(x) \tag{3.11}$$

and therefore $M(d\mu)$ is a semi-infinite *Hänkel matrix.* If μ is supported on the unit circle

$$\mathrm{supp}(\mu) \subset \{z \in \mathbb{C} \, : \, |z| = 1\} \tag{3.12}$$

the moments form a semi-infinite *Toeplitz matrix*:

$$m_{kl}(d\mu) = \int_{|z|=1} z^k \overline{z}^l d\mu(z) = \int_{|z|=1} z^{k-l} d\mu(z) \; . \tag{3.13}$$

In both cases there is a *recurrence relation* for consecutive orthogonal polynomials as a consequence of these algebraic structures.

3.1.2 Determinant and multiple integral representations

(i) The n-th monic orthogonal polynomial is given explicitly by the determinantal formula

$$P_n(d\mu; z)$$

$$= \frac{1}{\det(M^{(n-1)}(d\mu))} \begin{vmatrix} m_{00}(d\mu) & m_{01}(d\mu) & m_{02}(d\mu) & \cdots & m_{0n}(d\mu) \\ m_{10}(d\mu) & m_{11}(d\mu) & m_{12}(d\mu) & \cdots & m_{1n}(d\mu) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{n-1\,0}(d\mu) & m_{n-1\,1}(d\mu) & m_{n-1\,2}(d\mu) & \cdots & m_{n-1\,n}(d\mu) \\ 1 & z & z^2 & \cdots & z^n \end{vmatrix} \tag{3.14}$$

and hence

$$h_n(d\mu) = \frac{\det(M^{(n)}(d\mu))}{\det(M^{(n-1)}(d\mu))} \quad n = 1, 2, \ldots \tag{3.15}$$

This formula is easily obtained easily by solving the linear system expressing the orthogonality conditions for the coefficients of $P_n(d\mu; z)$ [95].

(ii) The moment determinants are expressible as multiple integrals [30]:

$$\det(M^{(n)}) = \frac{1}{(n+1)!} \underbrace{\iint \cdots \int}_{n+1} \prod_{0 \le k < l \le n} |z_k - z_l|^2 \prod_{k=0}^{n} d\mu(z_k) \tag{3.16}$$

This identity may be proved using the well-known expression for the determinant of Vandermonde matrices:

$$\prod_{0 \le k < l \le n} |z_k - z_l|^2 = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_0 & z_2 & \cdots & z_n \\ \vdots & \vdots & \ddots & \vdots \\ z_0^n & z_1^n & \cdots & z_n^n \end{vmatrix} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \overline{z_0} & \overline{z_2} & \cdots & \overline{z_n} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{z_0}^n & \overline{z_1}^n & \cdots & \overline{z_n}^n \end{vmatrix} . \tag{3.17}$$

For the idea of the proof see [30].

(iii) The *Heine formula* expresses the monic orthogonal polynomials in terms of multiple integrals [30, 91]:

$$P_n(d\mu; z) = \frac{\underbrace{\iint \cdots \int}_{n} \prod_{k=1}^{n}(z - z_k) \prod_{1 \le k < l \le n} |z_k - z_l|^2 \prod_{k=1}^{n} d\mu(z_k)}{\underbrace{\iint \cdots \int}_{n} \prod_{1 \le k < l \le n} |z_k - z_l|^2 \prod_{k=1}^{n} d\mu(z_k)} . \tag{3.18}$$

3.2 Christoffel–Darboux kernels

The n-th *Christoffel–Darboux kernel* is defined by

$$K_n(d\mu; z, w) := \sum_{k=0}^{n-1} p_k(d\mu; z)\overline{p_k(d\mu; w)} \tag{3.19}$$

This is a *reproducing kernel* for the subspace $\mathbb{C}_{n-1}[z] \subset L^2(\mathbb{C}, d\mu)$ of polynomials of degree at most $n-1$. Every $Q(z) \in \mathbb{C}_{n-1}[z]$ can be written as a linear combination

$$Q(z) = \sum_{k=0}^{n-1} a_k p_k(d\mu; z) \ , \tag{3.20}$$

and therefore

$$\begin{aligned} \langle Q, K_n(d\mu; \cdot, w)\rangle_{L^2(\mathbb{C},d\mu)} &= \int_{\mathbb{C}} Q(z)\overline{K_n(d\mu; z, w)}d\mu(z) \\ &= \sum_{k,l=0}^{n-1} a_k \left[\int_{\mathbb{C}} p_k(d\mu; z)\overline{p_l(d\mu; z)}d\mu(z)\right] p_l(d\mu; w) \\ &= \sum_{k=0}^{n-1} a_k p_k(d\mu; w) \\ &= Q(w) \ . \end{aligned} \tag{3.21}$$

Orthogonal polynomials on the real line satisfy a three term recurrence relation [95] which, in turn, implies the *Christoffel-Darboux identity*:

$$K_n(d\mu; x, y) := \frac{\gamma_{n-1}(d\mu)}{\gamma_n(d\mu)} \frac{p_n(d\mu; x)p_{n-1}(d\mu; y) - p_{n-1}(d\mu; x)p_n(d\mu; y)}{x-y} \tag{3.22}$$

(see [95]). This identity is not valid for general orthogonal polynomials if the support is off the real line.

3.3 L^2-minimality of orthogonal polynomials

The monic orthogonal polynomial $P_n(d\mu; z)$ is the unique solution of the following minimization problem [95]:

$$\begin{cases} \int |Q_n(z)|^2 d\mu(z) \to \min. \\ Q_n(z) \in \mathbb{C}_n[z] \ , \ Q_n(z) = z^n + \mathcal{O}\left(z^{n-1}\right) \ . \end{cases} \tag{3.23}$$

Similarly, for any fixed $z_0 \in \mathbb{C}$, the evaluated Christoffel-Darboux kernel

$$\frac{K_n(d\mu; z, z_0)}{K_n(d\mu; z_0, z_0)} \tag{3.24}$$

is the unique solution of the minimization problem

$$\begin{cases} \displaystyle\int |Q_n(z)|^2 d\mu(z) \to \min. \\ Q_n \in \mathbb{C}_n[z]\ ,\ Q_n(z_0) = 1\ . \end{cases} \tag{3.25}$$

3.4 Non-Hermitian orthogonality

Let Γ be a system of oriented contours in the complex plane and $w(z)$ a complex weight function analytic in a neighbourhood of Γ. Assume that

(i) The non-Hermitian moments

$$\nu_k(w,\Gamma) := \int_\Gamma z^k w(z)dz \tag{3.26}$$

exists for all $k = 0, 1, \ldots$.

(ii) The Hänkel matrices generated by the moments $\{\nu_k(w,\Gamma)\}_{k=0}^\infty$

$$M^{(n)}(w,\Gamma) := \begin{vmatrix} \nu_0(w,\Gamma) & \nu_1(w,\Gamma) & \cdots & \nu_n(w,\Gamma) \\ \nu_1(w,\Gamma) & \nu_2(w,\Gamma) & \cdots & \nu_{n+1}(w,\Gamma) \\ \vdots & \vdots & \ddots & \vdots \\ \nu_n(w,\Gamma) & \nu_{n+1}(w,\Gamma) & \cdots & \nu_{2n}(w,\Gamma) \end{vmatrix} \tag{3.27}$$

are nonsingular:

$$\det(M^{(n)}(w,\Gamma)) \neq 0 \qquad n = 0, 1, \ldots \tag{3.28}$$

The *monic orthogonal polynomial* of degree n with respect to the weight w on the contour Γ is characterized by the orthogonality relations

$$\int_{\mathbb{C}} P_n(w;z) z^k w(z)dz = \delta_{nk} h_n(w,\Gamma) \qquad k = 0, 1, \ldots, n\ , \tag{3.29}$$

and the condition on its leading coefficient:

$$P_n(w;z) = z^n + \mathcal{O}\left(z^{n-1}\right)\ . \tag{3.30}$$

Since the matrices $M^{(n)}(w, \Gamma)$ are not necessarily positive definite, the existence of a polynomial of degree *at most n* that satisfies the orthogonality relations is equivalent to the nonsingularity condition $\det(M^{(n)}(w, \Gamma)) \neq 0$. The solution has exact degree n if and only if $\det(M^{(n-1)}(w, \Gamma)) \neq 0$. As above, the n-th monic orthogonal polynomial is given by the determinantal formula

$$P_n(w; z) = \frac{1}{\det(M^{(n-1)}(w))} \begin{vmatrix} \nu_0(w,\Gamma) & \nu_1(w,\Gamma) & \nu_2(w,\Gamma) & \cdots & \nu_n(w,\Gamma) \\ \nu_1(w,\Gamma) & \nu_2(w,\Gamma) & \nu_3(w,\Gamma) & \cdots & \nu_{n+1}(w,\Gamma) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \nu_{n-1}(w,\Gamma) & \nu_n(w,\Gamma) & \nu_{n-2}(w,\Gamma) & \cdots & \nu_{2n-1}(w,\Gamma) \\ 1 & z & z^2 & \cdots & z^n \end{vmatrix} \tag{3.31}$$

Also,

$$h_n(w, \Gamma) = \frac{\det(M^{(n)}(w, \Gamma))}{\det(M^{(n-1)}(w, \Gamma))} \quad n = 1, 2, \ldots \tag{3.32}$$

3.5 Hermitian vs. non-Hermitian orthogonality

In many special cases it happens that a sequence of monic polynomials orthogonal with respect to a positve measure in the L^2-sense (or Hermitian sense) also satisfies non-Hermitian orthogonality relations with respect to a weight function w on a contour Γ.

A special example, that illustrates an important idea later on, is given by the Gaussian orthogonality measure

$$d\mu(z) := e^{-|z|^2 + \frac{T}{2}z^2 + \frac{\overline{T}^2}{2}\overline{z}^2} dA(z) \tag{3.33}$$

supported on the whole plane. It is well-known [31] that the orthogonal polynomials with respect to $w(z)$ are expressible in terms of classical Hermite polynomials:

$$P_n(z) = \frac{\overline{T}^n}{2^{\frac{n}{2}}(1-|T|^4)^{\frac{n}{2}}} H_n\left(\sqrt{\frac{1-|T|^4}{2|T|^4}} Tz\right) . \tag{3.34}$$

The proof in [31] uses the generating function of the Hermite polynomials. There is a different approach [69] which is of relevance since the same idea will be applied in the subsequent chapters. The basic idea is to find a solution to the scalar d-bar problem

$$\partial_{\overline{z}} f_k(z,\overline{z}) = -\overline{z}^k e^{-z\overline{z}+\frac{\overline{T}^2}{2}\overline{z}^2+\frac{T^2}{2}z^2} \tag{3.35}$$

in the whole complex plane. As described in App. B, a piecewise solution of this d-bar problem can be constructed in terms of contour integrals on a set of domains that cover the plane. Then, by using Stokes' Theorem, the two-dimensional integrals are expressible as linear combinations of contour integrals. This method gives the following result:

Theorem 3.5.1 ([69]) *The system of two-dimensional orthogonality relations*

$$\int_{\mathbb{C}} P_n(z)\overline{z}^k e^{-|z|^2+\frac{T^2z^2}{2}+\frac{\overline{T}^2\overline{z}^2}{2}} dA(z) = 0 \qquad k = 0,1,\ldots,n-1 \tag{3.36}$$

is equivalent to the system of non-Hermitian orthogonality relations

$$\int_{Im(Tz)=0} P_n(z) z^k e^{\frac{z^2}{2}\left(T^2-\frac{1}{T^2}\right)} dz = 0 \qquad k = 0,1,\ldots,n-1 \tag{3.37}$$

along the straight line $Re(Tz) = 0$.

3.6 Orthogonal polynomials in matrix models

Consider a Hermitian matrix model with a potential function $V(x)$. The measure

$$e^{-NV(x)}dx \tag{3.38}$$

gives a positive measure; we assume that $V(x)$ grows sufficiently fast so that the absolute moments exist and the corresponding orthogonal polynomials are well-defined. The Heine formula states that

$$P_n(e^{-NV(x)}dx;x) = \frac{\int\int\cdots\int\prod_{k=1}^{n}(x-x_k)\prod_{0\leq k<l\leq n}(x_k-x_l)^2 e^{-N\sum_{k=0}^{n}V(x_k)}dx_1\cdots dx_n}{\int\int\cdots\int\prod_{1\leq k<l\leq n}(x_k-x_l)^2 e^{-N\sum_{k=0}^{n}V(x_k)}dx_1\cdots dx_n}$$
$$= \mathbb{E}(\det(xI_n - M)) \ . \tag{3.39}$$

This means that the averaged characteristic polynomial of a Hermitian random matrix ensemble corresponding to the potential $V(x)$ is given by the n-th monic orthogonal polynomial with respect to the measure $e^{-NV(x)}dx$.

The algebraic structure of the joint probability density $\mathcal{P}_n^V$ implies that the m point correlation function $R_{n,m}^V(x_1,\ldots,x_n)$ is expressible as an $m\times m$ determinant for every $1\leq m\leq n$ [70]:

$$R_{n,m}^{NV}(x_1,\ldots,x_n) = \begin{vmatrix} K_n(e^{-NV}dx;x_1,x_1) & \cdots & K_n(e^{-NV}dx;x_1,x_n) \\ \vdots & \ddots & \vdots \\ K_n(e^{-NV}dx;x_n,x_1) & \cdots & K_n(e^{-NV}dx;x_n,x_n) \end{vmatrix} \tag{3.40}$$

In particular, the one point function is given by

$$R_{n,1}^{NV}(x) = K_n(e^{-NV}dx;x,x) \ . \tag{3.41}$$

These formulae hold for normal matrix models as well, simply by taking the absolute value of the Vandermonde term and changing dx_i to $dA(z_i)$ where dA stands for the area measure in the plane [104, 37].

Chapter 4

Equilibrium measures for logarithmic energy with external fields

In this section we briefly describe both the classical and weighted logarithmic energy problems of potential theory. The standard references are [78] and [81]. We use the conventions of [81] for the sign of the logarithmic kernel.

4.1 Potential theory preliminaries

Let D be an open subset of $\mathbb{C}$. A function $f\colon D \to [-\infty, \infty)$ is called *upper semicontinuous* if

$$\{z \in D : \ f(z) < \alpha\} \tag{4.1}$$

is an *open* subset of D for every $\alpha \in \mathbb{R}$. This condition is equivalent to

$$\limsup_{w \to z} f(w) \le f(z) \tag{4.2}$$

for all $z \in D$. Similarly, $g\colon D \longrightarrow (-\infty, \infty]$ is *lower semicontinuous* if $-g$ is upper semicontinuous.

A function $f\colon D \longrightarrow \mathbb{R}$ is continuous if and only if it is both upper and lower semicontinuous.

An example of an upper semicontinuous function is

$$f\colon \mathbb{C} \longrightarrow (-\infty, \infty] \qquad f(z) = \log|z| \,. \tag{4.3}$$

An upper semicontinuous function restricted to a compact set is bounded from above and attains its maximum (but it may be unbounded from below) [78].

A function $u\colon D \longrightarrow [-\infty, \infty)$ is said to be *subharmonic* if it is upper semicontinuous and satisfies the *local submean inequality*: for all $a \in D$ there exists a $\rho > 0$ such that

$$u(a) \leq \frac{1}{2\pi}\int_0^{2\pi} u\left(a + re^{i\theta}\right) d\theta \qquad 0 \leq r < \rho \,. \tag{4.4}$$

A function g is *superharmonic* if $-g$ is subharmonic. Again, $\log|z|$ is subharmonic.

4.2 Logarithmic potentials and Cauchy transforms of measures

4.2.1 Logarithmic potential

Let μ be a compactly supported finite positive Borel measure in the complex plane. The *logarithmic potential* of μ is

$$U^{\mu}(z) := \int_{\mathbb{C}} \log\frac{1}{|z-w|} d\mu(w) \qquad z \in \mathbb{C} \,. \tag{4.5}$$

The logarithmic potential of a positive measure is superharmonic on $\mathbb{C}$ and harmonic outside the support of μ [81]; moreover

$$U^{\mu}(z) = \mu(\mathbb{C})\log\frac{1}{|z|} + \mathcal{O}\left(\frac{1}{z}\right), \qquad |z| \to \infty \tag{4.6}$$

where $\mu(\mathbb{C})$ is the total mass of μ [78].

If $U^\mu(z)$ is smooth enough the density of the measure μ can be recovered from this potential by taking the Laplacian of $U^\mu(z)$ [81]: if in a region $R \subseteq \mathbb{C}$ the logarithmic potential $U^\mu(z)$ of the measure μ has continuous second partial derivatives, then μ is absolutely continuous with respect to the planar Lebesgue measure in R and the density is given by

$$d\mu = -\frac{1}{2\pi}\Delta U^\mu dA \,, \tag{4.7}$$

where dA denotes the area measure in the plane.

4.2.2 Cauchy transform

The *Cauchy transform* of a finite Borel measure μ is given by

$$C_\mu(z) := \int \frac{d\mu(t)}{t-z} \,. \tag{4.8}$$

The Cauchy transform is holomorphic outside the support of μ. If μ is compactly supported then

$$C_\mu(z) \sim \sum_{k=0}^{\infty} \frac{m_{0k}}{z^{k+1}} \qquad |z| \to \infty \,. \tag{4.9}$$

Outside the support of μ the logarithmic potential $U^\mu(z)$ is harmonic and therefore justified to consider $\partial_z U^\mu(z)$ there. The following equality holds:

$$\partial_z U^\mu(z) = \frac{1}{2} C_\mu(z) \qquad z \in \mathbb{C} \setminus \mathrm{supp}(\mu) \,. \tag{4.10}$$

This means that the *complex electric field* generated by the charge distribution μ is given by

$$\vec{E}(z) = -\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) U^\mu(z) = -2\partial_{\bar{z}} U^\mu(z) = -\overline{C_\mu(z)} \tag{4.11}$$

outside the support of μ.

If $K \subset \mathbb{C}$ is compact and $A(K) > 0$ then the Cauchy transform (using a simplified notation) is

$$C_K(z) = \int_K \frac{dA(\zeta)}{\zeta - z} \,. \tag{4.12}$$

This function is continuous in $\mathbb{C}$ and satisfies [24]

$$|C_K(z)| \leq \sqrt{\pi A(K)} \,. \tag{4.13}$$

4.3 Classical energy problem and capacity

Let K be a compact subset of $\mathbb{C}$ and let $\mathcal{M}(K)$ denote the set of all Borel probability measures supported on K. In classical potential theory, the *logarithmic energy* of a measure $\mu \in \mathcal{M}(K)$ is defined to be

$$I_K(\mu) := \int_K U^\mu(z) d\mu(z) = \int_K \int_K \log \frac{1}{|z-t|} d\mu(t) d\mu(z). \tag{4.14}$$

The quantity

$$E_K := \inf_{\mu \in \mathcal{M}(K)} I_K(\mu) \tag{4.15}$$

is either finite or $+\infty$. The *logarithmic capacity* of K is defined as

$$\mathrm{cap}(K) := e^{-E_K}. \tag{4.16}$$

The capacity of an arbitrary Borel set $B \subset \mathbb{C}$ is

$$\mathrm{cap}(B) := \sup\{\mathrm{cap}(K) \mid K \subseteq B, K \text{ compact}\} \,. \tag{4.17}$$

A property is said to hold *quasi-everywhere* if the set of exceptional points (i.e. those where it does not hold) is of capacity zero.

If $E_K < \infty$ Frostman's theorem (see e.g. [78]) implies that there exists a unique measure ω_K in $\mathcal{M}(K)$ minimizing the energy functional $I_K(\cdot)$ and this measure is called the *equilibrium measure* of K. The equilibrium measure is supported on the boundary ∂K and

$$\begin{aligned} U^{\omega_K}(z) &= E_K \qquad \text{quasi-every } z \in \partial K \ , \\ U^{\omega_K}(z) &\leq E_K \qquad z \in \mathbb{C} \ . \end{aligned} \tag{4.18}$$

[78]. The minimal energy value $E_K = I_K(\omega_K)$ is referred to as *Robin constant* [81].

Example. The equilibrium measure of the interval $K = [-1, 1]$ is

$$d\omega_K(x) = \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}} dx \qquad -1 < x < 1 \ . \tag{4.19}$$

4.3.1 Green's function

For a domain $D \subset \mathbb{C}P^1$ a function $g_D \colon D \times D \to (-\infty, \infty]$ is a *Green's function* if

(i) $g_D(z, a)$ is harmonic in $D \setminus \{a\}$,

(ii) $g_D(a, a) = \infty$ and

$$g_D(z,a) = \begin{cases} -\log|z-a| + \mathcal{O}(1) & a \neq \infty \\ \log|z| + \mathcal{O}(1) & a = \infty \end{cases} \qquad \text{as } z \to a \ , \tag{4.20}$$

(iii) $\lim\limits_{z \to w, \ z \in D} g_D(z, a) = 0$ for quasi-every $w \in \partial D$.

It is well-known [78] that if ∂D is of nonzero capacity then there is a unique Green's function associated to D. If $\infty \in D$ then the Green's function at infinity can be expressed in terms of the logarithmic potential of the equilibrium measure:

$$g_D(z, \infty) = -U^{\omega_{\partial D}}(z) - \log \operatorname{cap}(\partial D) \ . \tag{4.21}$$

4.4 Weighted energy problem

In the weighted setting we have a closed set $\Sigma \subseteq \mathbb{C}$ and a function $w \colon \Sigma \longrightarrow [0, \infty)$ on Σ called the *weight function*, usually given in the form

$$w(z) = \exp(-Q(z)) \tag{4.22}$$

where $Q\colon \Sigma \to (-\infty, \infty]$ is the called the *external potential.*

The weight function w is said to be *admissible* [81] if

(i) w is upper semi-continuous,

(ii) the set $\{z \in \Sigma \mid w(z) > 0\}$ is of positive capacity,

(iii) $\lim\limits_{|z|\to\infty} |z|w(z) = 0$.

In terms of the potential, $w(z) = \exp(-Q(z))$ is admissible if and only if Q is lower semi-continuous, the set $\{z \in \Sigma \mid Q(z) < \infty\}$ has nonzero capacity and $\lim\limits_{|z|\to\infty} (Q(z) - \log|z|) = \infty$.

Let $\mathcal{M}(\Sigma)$ denote the set of all Borel probability measures supported on $\Sigma \subseteq \mathbb{C}$. The *weighted energy functional* I_w is defined for all $\mu \in \mathcal{M}(\Sigma)$ by

$$\begin{aligned} I_w(\mu) &:= \int_\Sigma\int_\Sigma \log\left[|z-t|w(z)w(t)\right]^{-1} d\mu(z)d\mu(t) \\ &= \int_\Sigma\int_\Sigma \log\frac{1}{|z-t|} d\mu(z)d\mu(t) + 2\int_\Sigma Q(z)d\mu(z)\ . \end{aligned} \tag{4.23}$$

It is important to note that the factor 2 in the potential term appears in the convention of [81] but is not present in the random matrix literature (see [30]). In what follows we use the energy functional (4.23).

The solution of the electrostatic variational problem is a probability measure that minimizes this functional on $\mathcal{M}(\Sigma)$.

Theorem 4.4.1 ([81]) *For an admissible weight w*

$$E_w := \inf_{\mu\in\mathcal{M}(\Sigma)} I_w(\mu) \tag{4.24}$$

is finite and there exists a unique measure, denoted by μ_w, that has finite logarithmic energy and minimizes I_w. Moreover, the support of μ_w, denoted by S_w, is compact and has positive capacity.

The measure μ_w is called the *equilibrium measure* of the weight function w. The logarithmic potential of μ_w satisfies the equilibrium conditions

$$\begin{cases} U^{\mu_w}(z) + Q(z) \geq F_w \text{ quasi-every } z \in \Sigma \ , \\ U^{\mu_w}(z) + Q(z) \leq F_w \text{ for } z \in S_w \ , \end{cases} \tag{4.25}$$

where F_w is the *modified Robin constant*:

$$F_w = E_w - \int Q d\mu_w. \tag{4.26}$$

Therefore, the *effective potential* $U^{\mu} + Q$ is constant quasi-everywhere on the support of μ_w. The physical interpretation of (4.25) is that the charge distribution μ is in equilibrium and hence the effective electric field has to vanish on the support.

Moreover, the equilibrium condition (4.25) characterizes the equilibrium measure:

Theorem 4.4.2 ([81]) *If for a compactly supported measure μ with finite logarithmic energy there exists a constant F such that*

$$\begin{cases} U^{\mu}(z) + Q(z) \geq F & \textit{quasi-every } z \in \Sigma \\ U^{\mu}(z) + Q(z) \leq F & \textit{for all } z \in \mathrm{supp}(\mu) \ , \end{cases} \tag{4.27}$$

then $\mu = \mu_w$ and $F = F_w$.

The classical equilibrium problem corresponds to the special choice of weight function of the form

$$w(z) = \chi_K(z) \tag{4.28}$$

where $K \subset \mathbb{C}$ is a compact set of positive capacity.

4.4.1 A one-parameter family of equilibrium measures

Assume that $\{w^t(z)\}$ a one-parameter family of admissible weight functions defined for $t > 0$ and consider the corresponding one-parameter family of equilibrium measures

$$t \mapsto t\mu_{w^t} \qquad t > 0 \; . \tag{4.29}$$

The parameter t is often referred to as *time* or *total charge*: it is easy to see that the electrostatic variational problem with a modified total charge constraint

$$\begin{cases} \displaystyle\int_\Sigma\int_\Sigma \log \frac{1}{|z-t|} d\sigma(z) d\sigma(t) + 2\int_\Sigma Q(z) d\sigma(z) \longrightarrow \min. \\ \sigma(\mathbb{C}) = t \end{cases} \tag{4.30}$$

is equivalent to the standard minimization problem

$$\begin{cases} \displaystyle\int_\Sigma\int_\Sigma \log \frac{1}{|z-t|} d\mu(z) d\mu(t) + \frac{2}{t}\int_\Sigma Q(z) d\mu(z) \longrightarrow \min. \\ \mu(\mathbb{C}) = 1 \; , \end{cases} \tag{4.31}$$

where the solutions are related by $\sigma = t\mu$. In other words, $t\mu_{w^t}$ gives the time evolution of the equilibrium configuration with linearly growing total charge t in the presence of the *fixed* potential Q corresponding to the weight w.

4.5 Fekete points

Along with the continuous energy problem for measures of finite logarithmic energy, one can consider a discrete energy problem for an n-tuple of positive unit charges. Of course, the self-energy of such a measure is infinite, but, by removing the diagonal terms in the energy expression, one may consider the variational problem of minimizing the functional

$$\mathcal{I}_{n,Q}(z_1, \ldots, z_n) := \sum_{k \neq l} \log \frac{1}{|z_k - z_l|} + \sum_{k=1}^{n} Q(z_k) \; . \tag{4.32}$$

Definition 4.5.1 *An n-*Fekete point configuration *is a set of points for which the minimum of $\mathcal{I}_{n,Q}$ is attained.*

The admissibility conditions on Q guarantee the existence of such minimizing configurations; a Fekete point configuration may not be unique (for instance, if the potential has certain symmetries). The distribution of the Fekete point configurations is close to the equilibrium measure as n gets large:

Theorem 4.5.1 ([81]) *The normalized counting measure of any sequence n-Fekete point configurations converges to the corresponding equilibrium measure in the weak-star sense.*

4.6 Regularity of the equilibrium measure

It is important to understand how the regularity properties of the external potential influence the regularity of the corresponding equilibrium measure. In many cases the *explicit* recovery of the equilibrium measure uses tools that need some *a priori* regularity assumptions on the potential.

Theorem 4.6.1 ([26]) *If the potential*

$$V\colon [-1,1] \to \mathbb{R} \tag{4.33}$$

of the weight function $w(x) = \exp(-V(x))$ with $\Sigma = [-1,1]$ is in $C^2[-1,1]$ then the corresponding equilibrium measure μ_w is absolutely continuous with respect to the Lebesgue measure,

$$d\mu_w = \psi(x)dx \qquad \psi(x) \geq 0 \tag{4.34}$$

and the density $\psi(x)$ has the following properties:

- *The inequality*

$$\psi(x) \leq \frac{c}{\sqrt{1-x^2}} \tag{4.35}$$

  holds for some $c \in (0,\infty)$.

- *The density ψ is continuous on $(-1,1)$.*
- *If $V \in C^k[-1,1]$ for some $k \geq 3$ then ψ is Hölder continuous of order $\frac{1}{2}$.*
- *If V is real analytic in a neighborhood of $[-1,1]$ then ψ is supported on a* finite *number of closed intervals.*

This result holds in any bounded closed interval and since the equilibrium measure is always compactly supported for an admissible potential, the regularity results of 4.6.1 are valid for potentials defined on $\mathbb{R}$. This allows to use algebraic techniques in finding the support and the density *explicitly* for a large class of potentials.

4.7 Vector potential problems

There is an extended version of the weighted energy problem relevant to certain questions in approximation theory (multiple orthogonal polynomials [96], Nikishin systems [76]) and multi-matrix models. A *vector potential system* involves a finite collection of closed sets (*conductors*)

$$\Sigma_k \subseteq \mathbb{C} \qquad k = 1, \dots, n \;, \tag{4.36}$$

a family of functions (*background potentials*)

$$V_k \colon \Sigma_k \longrightarrow (-\infty, \infty] \qquad k = 1, \dots, n \;, \tag{4.37}$$

and a real symmetric positive definite matrix (*matrix of interactions*)

$$A := [a_{ij}]_{1 \leq i,j \leq n} \;. \tag{4.38}$$

Consider the functions

$$h_k(z) := \log \frac{1}{d(z, \Sigma_k)} \;, \qquad z \in \mathbb{C} \tag{4.39}$$

where $d(\cdot, K)$ denotes the *distance function* from the closed subset K of the complex plane:

$$d(z, K) := \inf_{t \in K} |z - t|. \tag{4.40}$$

A vector potential system

$$(\{\Sigma_k\}_{k=1}^n, \{V_k\}_{k=1}^n, A) \tag{4.41}$$

is said to be *admissible* [9] if

(a.1) The intersection $\Sigma_k \cap \Sigma_l$ has zero logarithmic capacity whenever $a_{kl} < 0$.

(a.2) The potentials V_k are lower semi-continuous on Σ_k for $k = 1, \ldots, n$.

(a.3) The sets $\{z \in \Sigma_k \ : \ V_k(z) < \infty\}$ are of positive logarithmic capacity for all k.

(a.4) The functions

$$H_{kl}(z,t) := \frac{V_k(z) + V_l(t)}{n} + a_{kl} \log \frac{1}{|z-t|} \tag{4.42}$$

are uniformly bounded from below, i.e. there exists an $L \in \mathbb{R}$ such that

$$H_{kl}(z,t) \geq L \tag{4.43}$$

on $\{(z,t) \in \Sigma_k \times \Sigma_l \ : \ z \neq t\}$ for all $k, l = 1, \ldots, n$.

(a.5) There are constants $0 \leq c < 1$ and $C > 0$ such that

$$H_{kl}(z,t) \geq \frac{1-c}{R}(V_k(z) + V_l(t)) - \frac{C}{n^2} \ . \tag{4.44}$$

(a.6) The functions

$$Q_k(z) := \sum_{l:\ a_{kl}<0} \left(\frac{1}{n} V_l(z) + a_{kl} h_l(z) \right) \tag{4.45}$$

are bounded from below on Σ_k.

Note that, without loss of generality, one may assume that all the potentials are nonnegative and that $L = 0$ by adding a common constant to all the potentials.

The weighted energy with interaction matrix A is defined for a vector of measures

$$\vec{\mu} = [\mu_1, \ldots, \mu_n] \qquad \mu_k \in \mathcal{M}(\Sigma_k) \tag{4.46}$$

as

$$I_{A,\vec{V}}(\vec{\mu}) := \sum_{k,l}^{n} a_{kl} \int_{\Sigma_k}\int_{\Sigma_l} \log \frac{1}{|z-t|} d\mu_k(z) d\mu_l(t) + 2\sum_{k=1}^{n} \int_{\Sigma} V_k(z) d\mu_k(z) \; . \tag{4.47}$$

The following theorem is a joint result of the author with M. Bertola:

Theorem 4.7.1 ([9]) *For an admissible system*

$$(\{\Sigma_k\}_{k=1}^n, \{V_k\}_{k=1}^n, A) \tag{4.48}$$

the following statements hold:

- *The extremal value*

$$\mathcal{I}_{A,\vec{V}} := \inf_{\vec{\mu}} I_{A,\vec{V}}(\vec{\mu}) \tag{4.49}$$

of the functional $I_{A,\vec{V}}(\cdot)$ is finite and there exists a unique vector measure $\vec{\mu}^\star$ such that

$$\mathcal{I}_{A,\vec{V}} := I_{A,\vec{V}}(\vec{\mu}^\star) \; . \tag{4.50}$$

- *The components of $\vec{\mu}^\star$ have finite logarithmic energy and compact support. Moreover, the potential V_k and the logarithmic potential $U^{\mu_k^\star}$ is bounded on the support of $\mu_k^\star$ for all $k = 1, \ldots, n$.*

- *For $j = 1, \ldots, n$ the effective potential*

$$U_k^{\mathrm{eff}}(z) := U^{\mu_k^\star}(z) + V_k(z) \geq F_k \tag{4.51}$$

for some real constant F_k, and equality holds quasi-everywhere on the support of μ_k.

It is important to note that this vector potential problem is a slight generalization of the potential problem considered in [81]. There the following stronger assumptions are made:

- The condensers Σ_k and Σ_l are of positive distance for all $k, l = 0, \ldots, n, k \neq l$.
- The interaction matrix is of the form

$$A = \vec{m}^T \otimes \vec{m} \qquad \vec{m} = [\varepsilon_1 m_1, \varepsilon_2 m_2, \cdots, \varepsilon_n m_n] \ , \tag{4.52}$$

  where $m_k > 0$ and $\varepsilon_k = \pm 1$ are the mass and the sign of μ_k respectively.

Under a weaker form of the admissibility conditions the existence and uniqueness is guaranteed by Thm. VIII.1.4. [81] which is generalized by Thm. 4.7.1.

4.8 Equilibrium measures in matrix models

4.8.1 Asymptotics of the correlation functions

In what follows we assume that the weight function

$$w(x) = e^{-\frac{1}{2}V(x)} \tag{4.53}$$

satisfies the following conditions:

- the potential function $V(z)$ is admissible (see Sec. 4.4) and

$$\frac{V(x)}{\log(x^2+1)} \geq 1 + \delta \tag{4.54}$$

  for some δ for sufficiently large x.

-

$$\int_{\mathbb{R}} |x|^n e^{-\frac{N}{2}V(x)} dx < \infty \qquad n = 0, 1, 2, \ldots \tag{4.55}$$

  for all $N > 0$.

Theorem 4.8.1 **([58, 27])** *The one-point density of the matrix model corresponding to $w(x)$ converges to the equilibrium measure in the weak-star sense*

$$\frac{1}{n} R_{n,1}^{NV}(x)dx \xrightarrow{w*} d\mu_{w^t} \tag{4.56}$$

in the scaling limit

$$n \to \infty , \qquad N \to \infty , \qquad \frac{n}{N} \to t . \tag{4.57}$$

The convergence is valid in the pointwise sense also, i.e., the one-point density function converges pointwise to the density of the equilibrium measure.

Also, the normalized m-point correlation function of the matrix model corresponding to $w(x)$ converges to the m-fold product of the equilibrium measure in the weak-star sense:

$$\frac{n!}{(n-m)!} R_{m,1}^{NV}(x_1, \ldots, x_m) dx_1 \cdots dx_m \xrightarrow{w*} \underbrace{d\mu_{w^t} \times \cdots \times d\mu_{w^t}}_{m} . \tag{4.58}$$

4.8.2 Asymptotics of the averaged the characteristic polynomial

It follows from (3.39) that the averaged characteristic polynomial in a matrix model is given by the corresponding monic orthogonal polynomial. In this way the following result on the zeroes of orthogonal polynomials is of considerable importance for matrix models:

Theorem 4.8.2 **([58, 27])** *The normalized counting measure*

$$\nu_{n,N} := \frac{1}{n} \sum_{k=1}^{n} \delta_{z_{k,n,N}} \qquad P_{n,N}(z) = \prod_{k=1}^{n} (z - z_{k,n,N}) \tag{4.59}$$

of the monic orthogonal polynomial $P_{n,N}(z)$ converges to the corresponding equilibrium measure in the weak-star sense:

$$\nu_{n,N} \xrightarrow{w*} \mu_{w^t} \tag{4.60}$$

in the scaling limit (2.23).

4.8.3 Algebraic curves associated to Hermitian matrix models

Assume for simplicity that $V(x)$ is a non-constant polynomial of even degree $2n$ with positive leading coefficient. The equilibrium measure μ_w is supported on a finite number of intervals with a density function $\psi(x)$. The equilibrium condition for μ_w reads as

$$\int \log \frac{1}{|x-t|}\psi(t)dt + V(x) \equiv F \qquad x \in \text{supp}(\psi) \ . \tag{4.61}$$

The regularity properties of ψ allow the differentiation of the logarithmic potential [30]

$$\frac{d}{dx}\int \log \frac{1}{|x-t|}\psi(t)dt = \text{p.v.}\int \frac{\psi(t)dt}{t-x} \qquad x \in \text{supp}(\psi) \ , \tag{4.62}$$

and therefore

$$\text{p.v.}\int \frac{\psi(t)dt}{t-x} + V'(x) \equiv 0 \qquad x \in \text{supp}(\psi) \ . \tag{4.63}$$

In the context of matrix models, the Cauchy transform of μ_w

$$W(z) := C_{\mu_w}(z) = \int \frac{\psi(s)ds}{s-z} \qquad z \in \mathbb{C} \setminus \text{supp}(\psi) \tag{4.64}$$

is called the *resolvent function* since, by Thm. 4.8.1, this gives the limiting form the normalized expectation of the trace of the resolvent:

$$\frac{1}{n}\mathbb{E}(\text{Tr}((zI-M)^{-1})) \ . \tag{4.65}$$

By the Sokhotski-Plemelj formulae (5.29) we get

$$\begin{aligned} W_+(x) - W_-(x) &= 2\pi i \psi(x) \\ W_+(x) + W_-(x) &= 2\text{p.v.}\int \frac{\psi(t)dt}{t-x} \ . \end{aligned} \tag{4.66}$$

The function $W^2(z)$ is analytic in $\mathbb{C} \setminus \text{supp}(\psi)$ and has the jump

$$W_+^2(x) - W_-^2(x) = -4\pi i \psi(x) V'(x) \ . \tag{4.67}$$

Therefore,

$$W^2(z) = -2\int \frac{\psi(s)V'(s)ds}{s-z} \tag{4.68}$$

(The difference

$$W^2(z) + 2\int \frac{\psi(s)V'(s)ds}{s-z} \tag{4.69}$$

that has no jump and therefore it an entire function converging to zero as $z \longrightarrow 0$ and hence it is identically zero by Liouville's theorem. Since

$$\int \frac{\psi(s)V'(s)ds}{s-z} = \int \frac{\psi(s)(V'(s)-V'(z))ds}{s-z} + W(z)V'(z) \tag{4.70}$$

and

$$P(z) := \frac{1}{2}\int \frac{\psi(s)(V'(s)-V'(z))ds}{s-z} \tag{4.71}$$

is a polynomial of degree $2n-2$ we obtain an *algebraic equation* for the resolvent:

$$W^2(z) - 2V'(z)W(z) - P(z) = 0\ . \tag{4.72}$$

The equation (4.72) is valid for non-polynomial potentials $V(x)$ as long as $\psi(x)$ is regular enough. In that case (4.72) is a *pseudo-algebraic curve.*

4.8.4 Asymptotic results for normal matrix models

$$w^N(z)dA(z) = \exp(-NQ(z))dA(z)\ . \tag{4.73}$$

Theorem 4.8.3 ([50]) *Assume that the potential function $Q\colon \mathbb{C} \to \mathbb{R}$ is C^2-smooth and*

$$Q(z) \geq A\log|z| + \mathcal{O}(1) \qquad z \longrightarrow \infty \tag{4.74}$$

for every $A > 0$ and the orthogonal polynomials exist:

$$\int_{\mathbb{C}} |z|^n e^{-NQ(z)} dA(z) < \infty \qquad n, N > 0\ . \tag{4.75}$$

The one-point function $R_{n,1}^{NQ}(z)$ with respect to the measure $e^{-NQ(z)}dA(z)$ converges to the equilibrium measure of $\frac{1}{t}Q(z)$ in the weak-star sense: Then

$$\frac{1}{n}R_{n,1}^{NQ}(z)dA(z) \xrightarrow{w*} d\mu_{w^{1/t}} \tag{4.76}$$

in the scaling limit (4.57).

Chapter 5

Schwarz functions

5.1 Uniformizing maps

The standard reference on conformal mappings is [75].

5.1.1 Riemann mapping theorem

According to the *Riemann mapping theorem* any simply connected domain $D \subset \mathbb{C}P^1$ whose complement contains at least two points is conformally equivalent to the unit disk; i.e., there exists a univalent (one-to-one) holomorphic function $f\colon \mathbb{D} \to D$ mapping onto the domain D (see e.g. [80]). This uniformizing map is unique up to compositions with biholomorphic maps of the unit disk onto itself. The freedom of choosing a nonsingular linear fractional transformation $\mathbb{D} \to \mathbb{D}$ can be eliminated by fixing the standard normalization conditions

$$f(0) = a \quad \text{and} \quad f'(0) \in \mathbb{R}^+ \tag{5.1}$$

which give a unique conformal map from $\mathbb{D}$ to D.

The Green's function of D can be written in terms of the conformal map f nor-

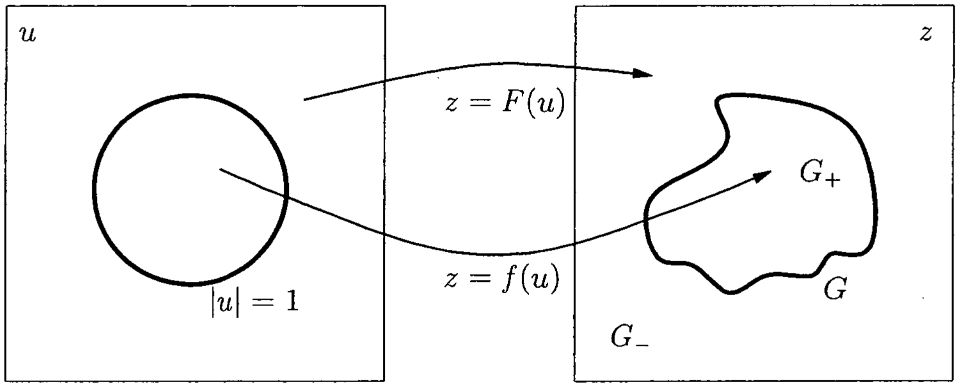


Figure 5.1: Interior and exterior conformal maps of a Jordan curve

malized as above [81]:

$$g_D(z, a) = \log |f^{-1}(z)| \, . \tag{5.2}$$

Let $G \subset \mathbb{C}$ be a Jordan curve. The interior and exterior domains are denoted by G_+ and G_- respectively. An *interior uniformizing map* is a conformal map $f\colon \mathbb{D} \longrightarrow G_+$ while an *exterior uniformizing map* is a conformal map $F\colon \operatorname{int}(\mathbb{D}^c) \longrightarrow G_-$.

The exterior uniformizing map is usually normalized at $u = \infty$ such that $F(\infty) = \infty$ and

$$F(u) = ru + \sum_{k=0}^{\infty} a_k \frac{1}{u^k} \qquad u \longrightarrow \infty \tag{5.3}$$

where r is assumed to be real and positive. This uniquely defined r is referred to as the *conformal radius* of the domain G_+.

Since

$$g_{G_-}(z, \infty) = \log |F^{-1}(z)| \tag{5.4}$$

and

$$g_{G_-}(z, \infty) = \begin{cases} -U^{\omega_G}(z) - \log \operatorname{cap}(G) = \log |z| - \log \operatorname{cap}(G) + \mathcal{O}\left(\dfrac{1}{|z|}\right) \\ \log |F^{-1}(z)| = \log |z| - \log r + \mathcal{O}\left(\dfrac{1}{|z|}\right) \end{cases} \quad z \longrightarrow \infty \tag{5.5}$$

we get

$$r = \mathrm{cap}(G) \ . \tag{5.6}$$

5.1.2 Boundary behavior

Carathéodory's theorem states that the conformal map $f\colon \mathbb{D} \to G_+$ can be extended to a homeomorphism between the closures $\mathrm{cl}(\mathbb{D})$ and $\mathrm{cl}(G_+)$ ([80], Theorem 14.19). Moreover if G is an analytic curve then there exists an open set U such that $\mathbb{D} \subset U$ and f has an analytic continuation to U [5].

5.1.3 Area of the interior region

Assume that G_- is uniformized by the exterior conformal map

$$F(u) = ru + \sum_{k=0}^{\infty} a_k \frac{1}{u^k} \qquad u \to \infty \ . \tag{5.7}$$

Then [24]

$$A(G_+) = \pi \left(r^2 - \sum_{k=1}^{\infty} k|a_k|^2 \right) \ . \tag{5.8}$$

5.2 Schwarz function of an analytic curve

For the details of the following, we refer to the monographs [25, 85, 97].

5.2.1 Definition of the Schwarz function

Assume that Γ is a nonsingular analytic arc in the complex plane. This means that Γ is the image of the interval $[0,1]$ under a function $F\colon [0,1] \to \mathbb{C}$ that is one-to-one and analytic in a neighborhood of $[0,1]$ and whose derivative $F'(t)$ does not vanish on $[0,1]$.

Then, for every point $z_0 \in \Gamma$, the implicit function theorem implies [85] that there exists a neighborhood U of z_0 and an analytic function $S(z)$ such that

$$s_{z_0}(z) = \overline{z} \qquad z \in \Gamma \cap U \ . \tag{5.9}$$

Since this can be done in a neighborhood of every point on the arc, by analytic continuation these locally defined analytic functions give a function $S(z)$ holomorphic in a neighborhood of Γ and satisfying the equality

$$S(z) = \overline{z} \qquad z \in \Gamma \ . \tag{5.10}$$

This is function is called the *Schwarz function* of the nonsingular analytic arc Γ.

If Γ is a closed curve then a single-valued Schwarz function can be constructed in an annular neighborhood A of Γ. The function obtained by analytic continuation cannot be multi-valued along Γ: if the analytic continuation starts from a function element $S_1(z)$ in a neighborhood $U$of $z_0 \in \Gamma$ then the analytic continuation $S_2(z)$ satisfies

$$S_1(z) = \overline{z} = S_2(z) \qquad z \in \Gamma \cap \tilde{U} \tag{5.11}$$

for some neighborhood $\tilde{U}$ of z_0 but then $S_1(z) = S_2(z)$ for $z \in \tilde{U}$.

The *Schwarz reflection*

$$T(z) := \overline{S(z)} \tag{5.12}$$

is an anti-conformal involution defined for z sufficiently close to Γ satisfying

$$T(z) = z \qquad z \in \Gamma \ . \tag{5.13}$$

5.2.2 Examples

The straight line L that passes through the points $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is given in terms of the complex notation $z = x + iy$ by

$$(y_1 - y_2)x + (x_2 - x_1)y + y_2 x_1 - x_2 y_1 = 0 \ , \tag{5.14}$$

which is equivalent to the complex form

$$\overline{z} = \frac{\overline{z}_2 - \overline{z}_1}{z_2 - z_1} z + \frac{z_2\overline{z_1} - \overline{z_2} z_1}{z_2 - z_1} . \tag{5.15}$$

Therefore the Schwarz function is the linear function

$$S(z) = \frac{\overline{z}_2 - \overline{z}_1}{z_2 - z_1} z + \frac{z_2\overline{z_1} - \overline{z_2} z_1}{z_2 - z_1} . \tag{5.16}$$

The circle C of radius r centered at the point $a \in \mathbb{C}$ is

$$|z - a|^2 = r^2 , \tag{5.17}$$

which implies that

$$S(z) = \frac{r^2}{z - a} + \overline{a} . \tag{5.18}$$

The standard *inversion* in the circle C is therefore given by

$$T(z) = \frac{r^2}{\overline{z} - \overline{a}} + a . \tag{5.19}$$

5.3 Schwarz function in terms of a uniformizing map

For a holomorphic function $F \colon D \subset \mathbb{C}$ defined on a domain D consider the *conjugate function*[1]

$$\overline{F} \colon \overline{D} \to \mathbb{C} , \qquad \overline{F}(z) := \overline{F(\overline{z})} \tag{5.20}$$

where $\overline{D}$ stands for the reflected domain

$$\overline{D} := \{\overline{z} \in \mathbb{C} : \ z \in D\} . \tag{5.21}$$

Assume now that Γ is a nonsingular analytic curve and therefore its interior conformal map extends analytically to a domain $U \supset \mathbb{D}$. Along Γ the parametrization

$$z = f(u) \qquad |u| = 1 \tag{5.22}$$

[1]The term *conjugate function* is taken from [25]; this definition should not be confused with the notion of a harmonic conjugate function [1].

implies that

$$\overline{z} = \overline{f(u)} = \overline{f}(\overline{u}) = \overline{f}\left(\frac{|u|^2}{u}\right) = \overline{f}\left(\frac{1}{u}\right) . \tag{5.23}$$

The inverted conformal map $\overline{f}\left(\frac{1}{u}\right)$ is analytic in

$$\overline{U}^{-1} := \left\{\frac{1}{\overline{u}} : u \in U\right\} . \tag{5.24}$$

Therefore in the set

$$A := f\left(U \cap \overline{U}^{-1}\right) \tag{5.25}$$

we can define

$$\tilde{S}(z) = \overline{f}\left(\frac{1}{f^{-1}(z)}\right) \tag{5.26}$$

It is easy to see that $\tilde{S}$ is analytic on A and that $\tilde{S}(z) = \overline{z}$ for $z \in \Gamma$. Therefore $S = \tilde{S}$ on A .

5.4 The decomposition of $S(z)$

5.4.1 Sokhotski-Plemelj formulae

Let γ be a smooth positively oriented Jordan curve in $\mathbb{C}$. Assume that $\varphi(t)$ is a complex valued function that satisfies the *Hölder condition* with *Hölder constant* $0 < \alpha \leq 1$; namely,

$$|\varphi(t_1) - \varphi(t_2)| < C|t_1 - t_2|^\alpha \qquad t_1, t_2 \in \gamma . \tag{5.27}$$

Then the contour integral

$$\Phi(z) := \frac{1}{2\pi i}\int_\gamma \frac{\varphi(t)dt}{t - z} \tag{5.28}$$

is holomorphic in $\mathbb{C} \setminus \gamma$ and has nontangential boundary values $\Phi_+(t)$ and $\Phi_-(t)$ for $t \in \gamma$; The *Sokhotskii–Plemelj formulae* [74, 1] for the boundary values give

$$\begin{cases} \Phi_+(t) = \frac{1}{2}\varphi(t) + \frac{1}{2\pi i}\text{p.v.} \int_\gamma \frac{\varphi(t)dt}{t-z} \\ \Phi_-(t) = -\frac{1}{2}\varphi(t) + \frac{1}{2\pi i}\text{p.v.} \int_\gamma \frac{\varphi(t)dt}{t-z} \, . \end{cases} \tag{5.29}$$

This means that

$$\Phi_+(t) - \Phi_-(t) = \phi(t) \qquad t \in \gamma \, . \tag{5.30}$$

For a nonsingular analytic curve Γ let Ω_+ and Ω_- denote the interior and the exterior domains of Γ respectively. The function $\varphi(t) = \bar{t}$ obviously satisfies the Hölder condition and therefore the function

$$\Phi(z) := \frac{1}{2\pi i} \int_\Gamma \frac{\overline{w}dw}{w-z} \tag{5.31}$$

has nontangential boundary values along Γ satisfying

$$\Phi_+(z) - \Phi_-(z) = \overline{z} = S(z) \qquad z \in \Gamma \, . \tag{5.32}$$

Let

$$S^\pm := \Phi|_{\Omega_\pm} \, . \tag{5.33}$$

The functions $S^+(z)$ and $S^-(z)$ are holomorphic in Ω_+ and Ω_- respectively.

Therefore the function $S(z) + S^-(z)$ is holomorphic in $A \cap \Omega_-$ and its boundary value is the same as the boundary value of $S^+(z)$ along γ. This means that $S^+(z)$ has the analytic continuation $S(z) + S^-(z)$ to the set $\Omega_+ \cup A$. Similarly, $S^-(z)$ has the analytic continuation $S^+(z) - S(z)$ to the set $\Omega_- \cup A$.

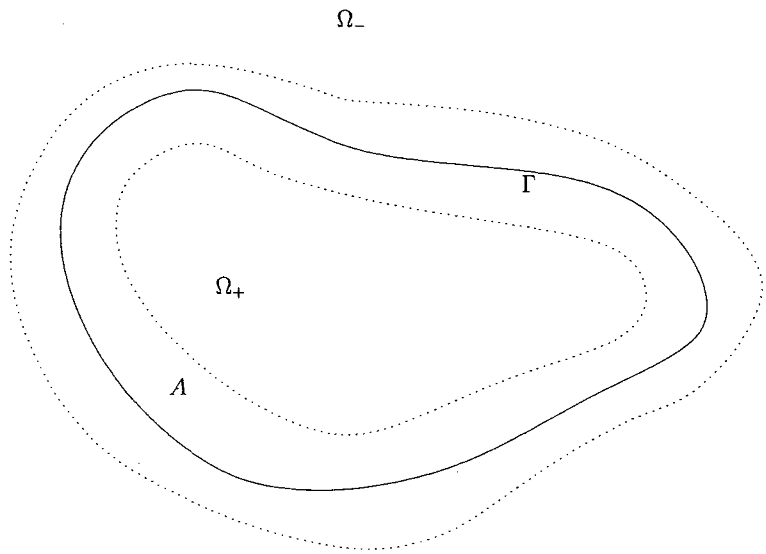


Figure 5.2: The domains Ω_+, Ω_- and A

5.5 An algebraic curve associated to a Schwarz function

Assume that for a nonsingular analytic curve Γ the exterior domain Ω_- is an *algebraic domain*; [97], that is, assume that its Riemann mapping is given by a rational function

$$f(u) = \frac{P(u)}{Q(u)} \tag{5.34}$$

for some polynomials P and Q with $\deg(P) = n$ and $\deg(Q) = n-1$ for some $n \in \mathbb{N}$. Then the

$$\overline{f}\left(\frac{1}{u}\right) = \frac{\tilde{P}(u)}{\tilde{Q}(u)} \tag{5.35}$$

for some polynomials $\tilde{P}$ and $\tilde{Q}$ with $\deg(\tilde{P}) = n$ and $\deg(\tilde{Q}) = n-1$. The rational map

$$u \mapsto \left(f(u), \overline{f}\left(\frac{1}{u}\right)\right) \tag{5.36}$$

embeds the Riemann sphere $\mathbb{C}P^1$ as a curve $\mathcal{L} \subset \mathbb{C}P^2$. A point (z, w) is on the curve if there exists a value of the parameter u such that

$$\begin{cases} P(u) - zQ(u) = 0 \\ \tilde{P}(u) - w\tilde{Q}(u) = 0 \; . \end{cases} \tag{5.37}$$

Therefore the resultant of the two polynomials has to vanish at u.

$$E(z, w) := \text{resultant}(P(u) - zQ(u), \tilde{P}(u) - w\tilde{Q}(u); u) \tag{5.38}$$

This means that along the curve Γ, $E(z, \overline{z}) = 0$. Therefore

$$E(z, S(z)) = 0 \qquad z \in A \; . \tag{5.39}$$

This curve is of genus zero which is obvious from its construction. The same curve can be obtained also from the *Schottky double* contruction [104, 97].

5.6 The Cauchy transform and the Schwarz function

Let K be a compact set of area $A(K)$ whose boundary ∂K is a nonsingular analytic curve. The Cauchy Transform of the measure $\chi_K dA$ (the area measure restricted to K) may be written as (using a simplified notation for the Cauchy transform as)

$$C_K(z) := \int_K \frac{dA(\zeta)}{\zeta - z} = \frac{1}{2i} \iint_K \frac{d\overline{\zeta} \wedge d\zeta}{\zeta - z} \; . \tag{5.40}$$

If $z \in K^c$ then the one-form

$$\frac{\overline{\zeta} d\zeta}{\zeta - z} \tag{5.41}$$

has no singularities inside ∂K and its exterior derivative is

$$d\left(\frac{\overline{\zeta} d\zeta}{\zeta - z}\right) = \frac{d\overline{\zeta} \wedge d\zeta}{\zeta - z} \; . \tag{5.42}$$

Stokes' Theorem gives that

$$C_K(z) = \frac{1}{2i} \iint_K \frac{d\overline{\zeta} \wedge d\zeta}{\zeta - z} = \frac{1}{2i} \int_{\partial K} \frac{\overline{\zeta} d\zeta}{\zeta - z} \; . \tag{5.43}$$

If $z \in \text{int}(K)$ then the above one-form has a singularity which contributes an extra term. The extended Cauchy formula ([42], Lemma II.2.2) applied to the function $u(z) = \bar{z}$ implies that

$$2\pi i\bar{z} = \int_{\partial K} \frac{\bar{\zeta} d\zeta}{\zeta - z} + \iint_K \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z} , \tag{5.44}$$

and hence

$$C_K(z) = -\pi\bar{z} + \pi\frac{1}{2\pi i}\int_{\partial K} \frac{\bar{\zeta} d\zeta}{\zeta - z} \tag{5.45}$$

for $z \in \text{int}K$. Therefore the Cauchy transform of K may be expressed in terms of the components of the Schwarz function of ∂K:

$$\frac{1}{\pi}C_K(z) = \begin{cases} -\bar{z} + S^+(z) & z \in \text{int}(K) \\ S^-(z) & z \in K^c . \end{cases} \tag{5.46}$$

5.7 Singularity Correspondence

Let Γ be a nonsingular analytic Jordan curve. Let f and F denote the interior and exterior conformal maps respectively. Observe that [97]

$$S(f(u)) = \bar{f}\left(\frac{1}{u}\right) \tag{5.47}$$

along $|u| = 1$ and therefore

$$\bar{f}\left(\frac{1}{u}\right) = S(f(u)) = S^+(f(u)) - S^-(f(u)) \qquad |u| = 1 . \tag{5.48}$$

Since $S^+(f(u))$ is holomorphic inside the unit disk of the u-plane, the function

$$\psi(u) := \bar{f}\left(\frac{1}{u}\right) + S^-(f(u)) \tag{5.49}$$

defined on the boundary $|u| = 1$ admits analytic continuation to the exterior of the unit disk. This implies that there is a *singularity correspondence* between the analytic continuation of the exterior component $S^-(z)$ into Γ_+ and the inverted interior conformal mapping [79, 45, 97]. For example, if $S^-(z)$ has a pole of order k at $z_0 \in \Gamma_+$ then

$\overline{f}\left(\frac{1}{u}\right)$ also has a pole of order k at the pre-image $u_0 = f^{-1}(z_0)$ and vice versa. Since the isolated singularities and the branch points of the function $f(u)$ and the inverted function $\overline{f}\left(\frac{1}{u}\right)$ are in bijective correspondence with each other via the mapping

$$u \longrightarrow \frac{1}{\overline{u}}\,, \tag{5.50}$$

the poles and branch points of the analytic continuations of f and of $S^-(z)$ are in bijective correspondence with each other.

For example, the uniformizing map f is rational if and only if $S^-(z)$ admits a *rational* analytic continuation into Γ_+.

Similarly,

$$S(F(u)) = \overline{F}\left(\frac{1}{u}\right) \tag{5.51}$$

along $|u| = 1$ and therefore

$$\overline{F}\left(\frac{1}{u}\right) = S(F(u)) = S^+(F(u)) - S^-(F(u)) \qquad |u| = 1\,. \tag{5.52}$$

Now $S^-(F(u))$ is holomorphic outside the unit disk of the u-plane and therefore the function

$$\tilde{\psi}(u) := \overline{F}\left(\frac{1}{u}\right) + S^+(F(u))\,, \tag{5.53}$$

defined on the boundary $|u| = 1$, admits an analytic continuation to the interior of the unit disk. Consequently a similar *singularity correspondence* holds between the analytic continuation of the interior component $S^+(z)$ into Γ_- and the inverted exterior conformal mapping.

For example, the interior component $S^+(z)$ is polynomial of degree n if and only if the only singularity of the exterior conformal mapping is a pole of order n at $u = 0$ apart from the simple pole at $u = \infty$. This class of mappings is investigated in detail in [37].

5.8 Harmonic moments

There are two sequences of *harmonic moments* associated to a nonsingular analytic curve Γ.

5.8.1 Interior harmonic moments

The *interior harmonic moments* of Γ are given by

$$C_{-k} := \int_{\Gamma_+} \zeta^k dA(\zeta) = m_{0k}(\chi_{\Gamma_+} dA) \qquad (k = 0, 1, \dots). \tag{5.54}$$

It is easy to see that for large z

$$\left.\begin{matrix} S^-(z) \\ \frac{1}{\pi} C_{\Gamma_+}(z) \end{matrix}\right\} \sim -\frac{1}{\pi} \sum_{k=0}^{\infty} C_{-k} \frac{1}{z^{k+1}} \qquad z \to \infty \,. \tag{5.55}$$

In terms of the electrostatic interpretation these moments can be thought of as the coefficients of a complex multipole expansion of the charge distribution $\chi_{\Gamma_+} dA$ at infinity.

There is a well-known *inverse problem* associated to these parameters: given the interior harmonic moments of a bounded simply connected domain D is it possible to reconstruct the domain D uniquely? In other words, the data given is the Cauchy transform (or the electric field generated by the uniformly charged plate D) in a neighborhood of infinity.

The answer to this question is negative: there are known constructions of pairs of different simply connected domains whose Cauchy transforms match around infinity (see [97], Section 2.3 and [82]). However, a classical result of Novikov [77] says that if two bounded domains are star-shaped with respect to a common center and their moments are the same then they are equal.

5.8.2 Exterior harmonic moments

There is a certain ambiguity in the definition of the *exterior harmonic moments*. For $a \in \Gamma_+$ fixed, the Taylor expansion of the holomorphic function $S^+(z)$ is given by

$$S^+(z) = \frac{1}{\pi} \sum_{k=0}^{\infty} C_{k+1}^a (z-a)^k. \tag{5.56}$$

where C_k^a are the corresponding coefficients of the series expansion. Assuming that $0 \in \Gamma_+$ (which is not a generic assumption) one can make the standard choice $a = 0$ and define the exterior harmonic moments as

$$C_k := C_k^0 \, . \tag{5.57}$$

In terms of the Schwarz function,

$$C_k = \frac{1}{2i} \int_\Gamma z^{-k} S(z) dz \qquad (k \in \mathbb{Z}). \tag{5.58}$$

Chapter 6

Quadrature domains

For details regarding this subsection, see [83, 25, 85, 46].

6.1 Two simple examples

Let $D(a,r)$ denote the open disk of radius $r > 0$ centered at $a \in \mathbb{C}$. For any function $f(z)$ holomorphic in a neighborhood of $D(a,r)$, we have that

$$\begin{aligned}\int_{D(a,r)} f(z)dA(z) &= \frac{1}{2i}\int_{D(a,r)} f(z)d\bar{z}\wedge dz \\ &= \frac{1}{2i}\int_{\partial D(a,r)} f(z)\bar{z}dz \quad \text{(by Stokes' Theorem)} \\ &= \frac{1}{2i}\int_{\partial D(a,r)} f(z)\left(\frac{r^2}{z-a}+\bar{a}\right)dz \\ &= \frac{r^2\pi}{2\pi i}\int_{\partial D(a,r)} \frac{f(z)}{z-a}dz \\ &= r^2\pi f(a)\ .\end{aligned} \tag{6.1}$$

Since the area integral functional is expressed as a point evaluation functional (which is a quadrature formula), the identity

$$\int_{D(a,r)} f dA = r^2\pi f(a) \tag{6.2}$$

is called a *quadrature identity*. The disk $D(a,r)$ is the simplest example of a *classical quadrature domain.*

Let $E(a,b)$ denote the ellipse

$$\left\{ z = x + iy \;:\; \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\} \tag{6.3}$$

with $a, b, \in \mathbb{R},\; a > b$. In terms of the complex parameters z and $\overline{z}$, the boundary of $E(a,b)$ is given by the equation

$$\left(\frac{z+\overline{z}}{2a} \right)^2 + \left(\frac{z-\overline{z}}{2bi} \right)^2 = 1 \;, \tag{6.4}$$

and therefore

$$\overline{z} = \frac{(a^2+b^2)z - 2ab\sqrt{z^2-c^2}}{c^2} \;, \tag{6.5}$$

where $c^2 = a^2 - b^2$ (the foci of the ellipse are $-c$ and c) and $\sqrt{z^2-c^2}$ has its branch cut along the segment $[-c, c]$ and the sheet is chosen by the condition

$$\sqrt{z^2-c^2} \sim z \qquad z \longrightarrow \infty \;. \tag{6.6}$$

For any function $f(z)$ holomorphic in a neighborhood of $E(a,b)$, we have that

$$\begin{aligned} \int_{E(a,b)} f(z) dA(z) &= \frac{1}{2i} \int_{E(a,b)} f(z) d\overline{z} \wedge dz \\ &= \frac{1}{2i} \int_{\partial D(a,r)} f(z) \overline{z} dz \quad \text{(by Stokes' Theorem)} \\ &= -\frac{ab}{c^2 i} \int_{\partial D(a,r)} f(z) \sqrt{z^2-c^2} dz \\ &= \frac{2ab}{c^2} \int_{-c}^{c} f(x) \sqrt{c^2-x^2} dx \;. \end{aligned} \tag{6.7}$$

In the last step the positively oriented contour is deformed to the segment $[-c, c]$ along which

$$\sqrt{z^2-c^2} dz = i\sqrt{c^2-x^2} dx \;, \tag{6.8}$$

where $\sqrt{c^2-x^2}$ is the positive root.

The ellipse $E(a,b)$ is a *generalized quadrature domain* for the measure $d\nu = \frac{2ab}{c^2}\sqrt{c^2-x^2}dx$ along the focal segment $[-c,c]$.

In both examples the boundary is a nonsingular analytic curve whose Schwarz function is particularly simple. This allows us to change the $\overline{z}$ term in the boundary integral to $S(z)$ and then deform the contour to obtain the quadrature identity.

6.2 Test function classes

The formal definition of a quadrature domain is given by Sakai in [83]. Let ν be a positive Borel measure on the complex plane; for a nonempty domain, i.e., an open, connected set Ω in $\mathbb{C}$ we consider the following *test function* spaces (as subspaces of the Banach space $L^1(\Omega, dA)$):

$$\begin{aligned} AL^1(\Omega) &= \{\text{Re} f \in L^1(\Omega, dA) \mid f \text{ is holomorphic in } \Omega\} \\ HL^1(\Omega) &= \{h \in L^1(\Omega, dA) \mid h \text{ is harmonic in } \Omega\} \\ SL^1(\Omega) &= \{s \in L^1(\Omega, dA) \mid s \text{ is subharmonic in } \Omega\} \,. \end{aligned} \tag{6.9}$$

$AL^1(\Omega)$ is just the real part of the *Bergman space* $L^1_a(\Omega)$. In general,

$$AL^1(\Omega) \subset HL^1(\Omega) \subset SL^1(\Omega) \,. \tag{6.10}$$

The first two classes are equal if Ω is simply connected: every harmonic function on a simply connected domain can be represented as the real part of an analytic function [78]. The function $h(z) = \log|z|$ on $\Omega = \{z \,:\, 1 < |z| < 2\}$ is harmonic, but it is not a real part of a single-valued analytic function on Ω.

The inclusion $HL^1(\Omega) \subset SL^1(\Omega)$ is always proper. (Consider the subharmonic function $s_a(z) := \log|z-a|$ for some $a \in \Omega$, which satisfies $s_a \in SL^1(\Omega) \setminus HL^1(\Omega)$.)

6.3 Quadrature domains

The domain Ω is called a *quadrature domain for a test function class* $F(\Omega)$ of the measure ν if

(i) ν is concentrated on Ω, i.e. $\nu(\Omega^c) = 0$,

(ii) for every $f \in F(\Omega)$

$$\int_\Omega f^+ d\nu < \infty \quad \text{and} \quad \int_\Omega f d\nu \leq \int_\Omega f dA \,, \tag{6.11}$$

where $f^+ := \max\{f, 0\}$.

Note that if $F(\Omega)$ is a function class such that $-f \in F(\Omega)$ whenever $f \in F(\Omega)$ then the second condition is equivalent to

$$\int_\Omega |f| d\nu < \infty \quad \text{and} \quad \int_\Omega f d\nu = \int_\Omega f dA \tag{6.12}$$

for every $f \in F(\Omega)$. Let $Q(\nu, F)$ denotes the set of quadrature domains of ν for the function class F. For a measure ν, the quadrature domains corresponding to the classes $AL^1(\Omega)$, $HL^1(\Omega)$ and $SL^1(\Omega)$ are called holomorphic, harmonic and subharmonic quadrature domains respectively. We have the obvious inclusion relations

$$Q(\nu, SL^1) \subseteq Q(\nu, HL^1) \subseteq Q(\nu, AL^1) \,. \tag{6.13}$$

The study of questions related to the existence and uniqueness of holomorphic and harmonic quadrature domains shows the importance of the notion of subharmonic quadrature domains [83]. It turns out that subharmonic quadrature domains are unique up to sets of zero Lebesgue measure [83].

To illustrate the possible non-uniqueness of quadrature domains even for simple measures, consider the measure ν supported on $\{z : |z| = 1\}$ given by

$$d\nu = \frac{t}{2\pi} d\theta \qquad 0 \leq \theta < 2\pi \tag{6.14}$$

in terms of the parameter $z = e^{i\theta}$ for some $t > 0$ (the total mass). Introduce the auxiliary notation for the annuli

$$R_\alpha := \left\{ z \in \mathbb{C} : \sqrt{\frac{\alpha}{\pi}} < |z| < \sqrt{\frac{\alpha + t}{\pi}} \right\} . \tag{6.15}$$

Depending on the values of the parameter t, these are the sets of quadrature domains for the three distinguished test function classes:

$$Q(\nu, AL^1) = \begin{cases} \{R_\alpha : \pi - t < \alpha < \pi\} & 0 < t \leq \pi \\ \{R_\alpha : 0 \leq \alpha < \pi\} \cup \{D(0, \sqrt{t/\pi})\} & t > \pi \end{cases}$$

$$Q(\nu, HL^1) = \begin{cases} \{R_{\alpha(t)}\} & 0 < t \leq \pi \\ \{R_{\alpha(t)} \ , \ D(0, \sqrt{t/\pi})\} & \pi < t \leq e\pi \\ \{D(0, \sqrt{t/\pi})\} & t > e\pi \end{cases} \tag{6.16}$$

$$Q(\nu, SL^1) = \begin{cases} \{R_{\alpha(t)}\} & 0 < t \leq e\pi \\ \{\{z\colon 0 < |z| < \sqrt{e}\} \ , \ D(0, \sqrt{e})\} & t = e\pi \\ \{D(0, \sqrt{t/\pi})\} & t > e\pi \ , \end{cases}$$

where $\alpha(t)$ stands for the unique solution of the equation

$$\int_{\sqrt{\alpha/\pi}}^{\sqrt{(\alpha+t)/\pi}} r \log r dr = 0 \tag{6.17}$$

in the interval $0 < \alpha < 1$ for $0 < t \leq e\pi$ (see [83], Ex. 1.2).

6.4 Quadrature identities in terms of the external field

For every point z in the complement of a domain Ω the test function

$$a_z(w) = \frac{1}{w - z} \tag{6.18}$$

belongs to $L^1_a(\Omega, dA)$. If $\Omega \in Q(\nu, AL^1)$ for some measure then

$$C_\Omega(z) = C_\nu(z) \qquad z \in \Omega^c . \tag{6.19}$$

As a consequence of an approximation theorem of Bers [14], the linear combinations of the test functions $\{a_z\}_{z\in\Omega^c}$ are dense in $L^1_a(\Omega, dA)$. Therefore the equality of the Cauchy transforms outside is equivalent to the holomorphic quadrature property for ν.

Similarly, using the harmonic test functions $h_z(w) = \log\frac{1}{|z-w|}$ for $z \in \Omega^c$ and the fact that the linear combinations of the h_z's and their derivatives are dense in $HL^1(\Omega)$ [83], the harmonic quadrature property is equivalent to

$$U^\nu(z) = U^\Omega(z) \qquad z \in \Omega^c . \tag{6.20}$$

The quadrature inequality holds for the class of subharmonic functions if and only if

$$\begin{aligned} U^\nu(z) &= U^\Omega(z) \qquad & z \in \Omega^c \\ U^\nu(z) &\geq U^\Omega(z) \qquad & z \in \mathbb{C} . \end{aligned} \tag{6.21}$$

6.5 Classical quadrature domains

A domain Ω is a *classical quadrature domain* if it is a holomorphic quadrature domain for a positive linear combination of point masses:

$$\nu = \sum_{k=1}^{n} \beta_k \delta_{a_k} . \tag{6.22}$$

Confluence of points in the above formula may be considered: this gives rise to the more general form of classical quadrature identities:

$$\int_\Omega f dA = \sum_{k=1}^{n} \sum_{l=1}^{m_k} \beta_{k,l} f^{(l)}(a_k) \qquad f \in L^1_a(\Omega) . \tag{6.23}$$

It was shown in [4] that if a domain Ω satisfies a classical quadrature identity of the form (6.23) then there exists an irreducible polynomial $Q(x, y) \in \mathbb{C}[x, y]$ such that

$$\partial\Omega \subset \{z \in \mathbb{C} : \ Q(z, \bar{z}) = 0\} . \tag{6.24}$$

In particular, if Ω is simply connected then (6.23) is equivalent to the condition that the interior uniformizing map is rational (see also [25]). Obviously, the rationality of the interior conformal map implies the existence of the polynomial Q as we saw above.

If $\partial\Omega$ is a nonsingular analytic curve then the corresponding Schwarz function satisfies the equation

$$Q(z, S(z)) = 0 \tag{6.25}$$

by analytic continuation.

6.6 Potentials associated to analytic curves

Definition 6.6.1 ([104]) *An admissible potential is said to be* quasi-harmonic *if it is of the form*

$$Q\colon \Sigma \to (-\infty, \infty] \qquad Q(z) = \alpha|z|^2 - 2Re(H(z)) \tag{6.26}$$

where Σ has non-empty interior, $\alpha > 0$ and $H(z)$ is holomorphic in int(Σ).

Definition 6.6.2 ([68]) *The potential*

$$Q(z) = \alpha|z|^2 - 2Re(H(z)) \tag{6.27}$$

is called semiclassical *if $H'(z)$ is a rational function on the Riemann sphere.*

The relevance of these potentials is justified by the following result:

Theorem 6.6.1 ([63]) *For a nonsingular analytic curve G the negative of the logarithmic potential of the area measure of the interior region*

$$-U^{G_+}(z) = -\int_{G_+} \log\frac{1}{|z-w|} dA(w) \tag{6.28}$$

is quasi-harmonic in G_+.

Proof. Let $z_0 \in G$ fixed. Since $\partial_z U^{G_+}(z) = \frac{1}{2} C_{G_+}(z)$ and

$$C_{G_+}(z) = -\pi\overline{z} + \pi S^+(z) \tag{6.29}$$

for $z \in G_+$, the logarithmic potential inside G is given by

$$-U^{G_+}(z) = \frac{\pi}{2}|z|^2 - \pi \mathrm{Re}\left(\int_{z_0}^{z} S^+(\zeta)d\zeta\right) + \mathcal{C} \ , \tag{6.30}$$

($\mathcal{C}$ is a ration) which is obviously quasi-harmonic.

■

It is easy to see that we get semiclassical potentials in (6.28) if and only if $S^+(z)$ is rational. By the singularity correspondence, this is equivalent to the requirement that the *exterior* uniformizing map of the domain G_- be rational.

The normalized area measure on G_+

$$d\mu := \frac{1}{A(G_+)}\chi_{G_+} dA \tag{6.31}$$

has the logarithmic potential

$$U^\mu(z) = -\frac{\pi}{2A(G_+)}|z|^2 + \frac{\pi}{A(G_+)}\mathrm{Re}\left(\int_{z_0}^{z} S^+(\zeta)d\zeta\right) + C \quad z \in G_+ \ . \tag{6.32}$$

Assume that $H(z)$ is an analytic continuation of the function

$$\frac{\pi}{2A(G_+)}\int_{z_0}^{z} S^+(\zeta)d\zeta \tag{6.33}$$

to $\mathrm{int}(\Sigma)$ for some closed set Σ containing $\mathrm{cl}(G_+)$. Then the external potential

$$V(z) := \alpha|z|^2 - 2\mathrm{Re}(H(z)) \tag{6.34}$$

with

$$\alpha := \frac{\pi}{2A(G_+)} \tag{6.35}$$

satisfies the first half of the equilibrium condition (4.25)

$$U^\mu(z) + V(z) = \mathrm{const.} \qquad z \in \mathrm{cl}(G_+) \ . \tag{6.36}$$

Theorem 6.6.2 ([63]) *Consider the quasi-harmonic potential*

$$Q(z) = \frac{\pi}{2}|z|^2 - 2Re(H(z)) \qquad z \in \Sigma \tag{6.37}$$

and assume that on the interval (t_0, t_1) *(*$0 \le t_0 < t_1 \le \infty$*) there is a monotonically increasing one-parameter family of compact sets*

$$K_t \subset \Sigma \qquad t \in (t_0, t_1) \quad t_0 > 0 \tag{6.38}$$

with nonsingular analytic boundaries such that

- $0 \in K_t$ *for all* $t \in (t_0, t_1)$.
- *The area grows linearly in* t: $A(K_t) = t$ *for all* $t \in (t_0, t_1)$.
- *The measure*
$$d\mu_t := \frac{1}{A(K_t)} \chi_{K_t} dA \tag{6.39}$$
is the equilibrium measure for the potential $\frac{1}{t}Q(z)$ *for* $t \in (t_0, t_1)$.

Then the exterior harmonic moments are preserved in the course of the evolution:

$$C_k(t) \equiv C_k \qquad t \in (t_0, t_1) \ . \tag{6.40}$$

Proof. The exterior harmonic moments of K_t are given by the Taylor expansion of the Schwarz function $S_t^+(z)$ of ∂K_t at $z = 0$:

$$S_t^+(z) = \sum_{k=1}^{\infty} C_{k+1}(t) z^k \ . \tag{6.41}$$

Since μ_t is the equilibrium measure of $\frac{1}{t}Q(z)$ we have that

$$\frac{1}{t}Q(z) + U^{\mu_t}(z) \equiv F_t \qquad z \in K_t \tag{6.42}$$

for some time-dependent constant F_t. This equality is equivalent to

$$\frac{\pi}{t}\mathrm{Re}\left(\int_0^z S_t^+(\zeta) d\zeta\right) - \frac{2}{t}\mathrm{Re}(H(z)) \equiv G_t \qquad z \in K_t \tag{6.43}$$

for another constant G_t in z. Therefore applying ∂_z gives

$$\frac{\pi}{2t} S_t^+(z) - \frac{1}{t} H'(z) \equiv 0 \qquad z \in K_t \tag{6.44}$$

which means that

$$\sum_{k=1}^{\infty} C_{k+1}(t) z^k = \frac{2}{\pi} H'(z) \ . \tag{6.45}$$

The expression on the right hand side is independent of t which is enough to conclude that the exterior harmonic moments are preserved.

$\blacksquare$

6.6.1 Polynomial curves

Consider analytic Jordan curves given by rational exterior conformal maps $F(u)$ with pole at $u = \infty$ only:

$$F(u) = ru + \sum_{k=0}^{n} \frac{a_k}{u^k} \qquad r > 0 \ , a_n \neq 0 \ , n \in \mathbb{N} \ . \tag{6.46}$$

In this context these curves appear in the works by Wiegmann and Zabrodin. Elbau refers to them as *polynomial curves* in [37]. The above mappings are univalent and produce a nonsingular boundary G only for a certain subset of the parameter space $(r, a_0, \cdots, a_n)$.

The following observation is due to Makarov [67]: The interior component $S^+(z)$ of the Schwarz function of the image curve G is given by

$$S^+(z) = \frac{1}{2\pi i} \int_{|u|=1} \frac{\overline{F}\left(\frac{1}{u}\right) F'(u) du}{F(u) - z} \ . \tag{6.47}$$

It is known that the generating function

$$\frac{F'(u)}{F(u) - z} = \sum_{k=0}^{\infty} \frac{P_k(z)}{u^{k+1}} \tag{6.48}$$

for large u gives the *Faber polynomials* $\{P_k(z)\}_{k=0}^{\infty}$ corresponding to the exterior conformal mapping F [90]. Also

$$\overline{F}\left(\frac{1}{u}\right) = \frac{r}{u} + \sum_{k=0}^{n} \overline{a}_k u^k \,. \tag{6.49}$$

Comparing the coefficients in the contour integral representation we get

$$S^+(z) = \sum_{k=0}^{n} \overline{a}_k P_k(z) \,. \tag{6.50}$$

Therefore the corresponding Schwarz function is a polynomial of degree at most n.

The corresponding potentials are not admissible in the Saff–Totik sense unless $n \leq 2$.

6.6.2 Polynomial curves of degree two

Assume that

$$H(z) = az + \frac{T^2}{2} z^2 \tag{6.51}$$

where $|T| < 1$. In [38] it is proven that the equilibrium measure μ_w is given by

$$d\mu_w = \frac{1}{A(G+)} \chi_{G_+} dA \,, \tag{6.52}$$

where G is a polynomial curve. Since

$$S^+(z) = H'(z) = T^2 z \tag{6.53}$$

has a simple pole at $z = \infty$ the singularity correspondence implies that F has to be of the form

$$F(u) = ru + a_0 + \frac{a_1}{u} \,, \tag{6.54}$$

where $r > 0$. To find the coefficients r, a_0 and a_1 we use (6.49):

$$a + T^2 z = \overline{a}_0 + \frac{\overline{a}_1 (z - a_0)}{r} \,. \tag{6.55}$$

Therefore

$$a_1 = r\overline{T}^2 \ , \quad a_0 = \frac{\overline{a} + T^2 a}{1 - |T|^4} \ . \tag{6.56}$$

The conformal radius r is fixed by the area equation (5.8):

$$\frac{\pi}{2\alpha} = \pi \left(r^2 - |a_1|^2\right) \implies r = \sqrt{\frac{1}{2\alpha(1 - |T|^4)}} \ . \tag{6.57}$$

The image G of the unit circle under F is the boundary of an ellipse in the z-plane. The ramification points of F where $F'(u) = 0$ are $u = \overline{T}$ and $u = \overline{T}$. The condition $|T| < 1$ ensures that $F(u)$ is univalent in the exterior of the unit disk. The focal points of the ellipse are the images of these points under F:

$$z_{\pm} = a_0 \pm 2r\overline{T} \ . \tag{6.58}$$

Note that if $T = 0$ then G is a circle of radius

$$r = \sqrt{\frac{1}{2\alpha}} \tag{6.59}$$

centered at $z = \overline{a}$.

Chapter 7

Riemann–Hilbert approach

For a more detailed description of the Riemann-Hilbert approach, see [30, 53, 54, 20, 65, 15].

7.1 Riemann–Hilbert problems

A *Riemann–Hilbert (factorization) problem* is given by a pair (Γ, G) where Γ is a system of oriented contours in the complex plane and $M\colon \Gamma \to GL(n, \mathbb{C})$ is an $n \times n$ matrix-valued function defined on Γ. We seek a function $Y(z)$ satisfying

(i) $Y(z)$ is holomorphic in $\mathbb{C} \setminus \Gamma$ and has continuous non-tangential boundary values $Y_+(z)$ and $Y_-(z)$ along Γ,

(ii) $Y_+(z) = Y_-(z)M(z)$ for $z \in \Gamma$,

(iii) $Y(z) \sim I$ as $z \to \infty$.

The solution to a Riemann–Hilbert problem may not exist or it may not be unique.

7.2 The Fokas–Its–Kitaev RH problem for orthogonal polynomials

Let Γ be a system of oriented contours and $w(z)$ be a weight function analytic in a neighborhood of Γ such that the complex moments $\nu_k(w;\Gamma)$ exist for all $k \geq 0$. Consider the following Riemann–Hilbert problem for a 2×2 matrix-valued function $Y(z)$ [55]:

(i) $Y(z)$ is holomorphic in $\mathbb{C} \setminus \Gamma$,

(ii) $Y(z)$ has continuous boundary values $Y_+(z)$ and $Y_-(z)$ along Γ and

$$Y_+(z) = Y_-(z) \begin{bmatrix} 1 & w(z) \\ 0 & 1 \end{bmatrix}, \tag{7.1}$$

(iii)

$$Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) \begin{bmatrix} z^n & 0 \\ 0 & z^{-n} \end{bmatrix} \qquad z \to \infty . \tag{7.2}$$

The following results are known to be valid in this case [55]:

(i) If a solution $Y(z)$ exists then it is unique.

(ii) The solution exists if and only if $\det(M^{(n)}(w,\Gamma)) \neq 0$ and the solution is given by the 2×2 matrix-valued function

$$Y(z) = \begin{bmatrix} p_n(z) & \frac{1}{2\pi i}\int_\Gamma \frac{p_n(t)(t)dt}{t-z} \\ q_{n-1}(z) & \frac{1}{2\pi i}\int_\Gamma \frac{q_{n-1}(t)w(t)dt}{t-z} \end{bmatrix} \tag{7.3}$$

where the first column is expressed in terms of non-Hermitian monic orthogonal polynomials of degree n and $n-1$ with respect to the weight w as

$$p_n(z) = P_n(w;z) \qquad q_{n-1}(z) = -\frac{2\pi i}{h_{n-1}(w,\Gamma)} P_{n-1}(w;z) \tag{7.4}$$

(see Sec. 3.4).

7.3 Deift–Zhou nonlinear steepest descent method for the Fokas-Its-Kitaev Riemann-Hilbert problem

We provide a very brief introduction to the main steps of the Deift-Zhou nonlinear steepest descent method for orthogonal polynomials on the real line, following the presentation of [53].

We study the asymptotic behavior of the matrix-valued function

$$Y(z) := Y_{n,N}(z) = \begin{bmatrix} P_{n,N}(z) & \dfrac{1}{2\pi i}\displaystyle\int_{\mathbb{R}} \frac{P_{n,N}(t)e^{-NV(t)}dt}{t-z} \\ -\dfrac{2\pi i}{h_{n-1,N}} P_{n-1,N}(z) & -\dfrac{1}{h_{n-1,N}}\displaystyle\int_{\mathbb{R}} \frac{P_{n-1,N}(t)e^{-NV(t)}dt}{t-z} \end{bmatrix} \tag{7.5}$$

in the scaling limit

$$n \to \infty\ , \quad N \to \infty\ , \quad \frac{n}{N} \to t > 0 \tag{7.6}$$

for some $t > 0$. For simplicity, we assume that

$$N = \frac{n}{t}\ . \tag{7.7}$$

7.3.1 The g-function and the 'undressing'

The first difficulty arising in the analysis of the Fokas-Its-Kitaev Riemann-Hilbert problem is the non-standard asymptotic condition

$$Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)\begin{bmatrix} z^n & 0 \\ 0 & z^{-n} \end{bmatrix} \qquad z \to \infty\ . \tag{7.8}$$

One might try to eliminate this problem by considering the *undressed* matrix

$$U(z) := Y(z)z^{-n\sigma_3} \qquad z \in \mathbb{C}\setminus\mathbb{R}\ . \tag{7.9}$$

where the abbreviated notation $z^{-n\sigma_3}$ uses the *Pauli matrix*

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\ . \tag{7.10}$$

The matrix $U(z)$ satisfies a Riemann-Hilbert problem with the desired asymptotics

$$U(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \qquad z \to \infty \ . \tag{7.11}$$

However, a serious drawback of this approach is that by removing the singular behavior at $z = \infty$ we introduce a new singularity at $z = 0$ and therefore this new Riemann-Hilbert problem is no simpler than the previous one. On the other hand, to achieve the normalization for large z one can use *any* matrix $G(z) = G_n(z)$ depending on n that is holomorphic and invertible in $\mathbb{C} \setminus \mathbb{R}$ and

$$z^{n\sigma_3} G_n(z)^{-1} \sim I \qquad n \to \infty \ . \tag{7.12}$$

The jump of the undressed matrix

$$U(z) := Y(z) G^{-1}(z) \qquad z \in \mathbb{C} \setminus \mathbb{R} \tag{7.13}$$

is given by

$$\begin{aligned} U_+(x) &= Y_+(x) G_+^{-1}(x) \\ &= Y_-(x) M(x) G_+^{-1}(x) \\ &= Y_-(x) G_-(x)^{-1} G_-(x) M(x) G_+^{-1}(x) \\ &= U_-(x) \left[G_-(x) M(x) G_+^{-1}(x) \right] \ . \end{aligned} \tag{7.14}$$

We seek $G(z)$ of the special form

$$G(z) = G_n(z) = z^{ng(z)\sigma_3} \qquad z \in \mathbb{C} \setminus \mathbb{R} \ , \tag{7.15}$$

where $g(z)$ is a holomorphic function in $\mathbb{C} \setminus \mathbb{R}$ satisfying

$$g(z) = \log z + \mathcal{O}\left(\frac{1}{z}\right) \qquad z \to \infty \ . \tag{7.16}$$

In terms of $g(z)$, the jump matrix is of the form

$$e^{ng_-(x)} M(x) e^{-ng_+(x)} = \begin{bmatrix} e^{n(g_-(x) - g_+(x))} & e^{n\left(-\frac{1}{t}V(x) + g_+(x) + g_-(x)\right)} \\ 0 & e^{n(g_+(x) - g_-(x))} \end{bmatrix} \tag{7.17}$$

Note that for points $x \in \mathbb{R}$ where $e^{ng(z)}$ has no jump, the jump of the undressed matrix has the form

$$\begin{bmatrix} 1 & e^{n\left(-\frac{1}{t}V(x)+2g(x)\right)} \\ 0 & 1 \end{bmatrix} \tag{7.18}$$

The goal is to find a function $g(z)$ such that this jump matrix is sufficiently close to the identity matrix for large n; it is reasonable to expect that then the solution is close to the identity as well, i.e.,

$$Y(z) \sim G_n(z) \qquad n \longrightarrow \infty\ . \tag{7.19}$$

Note that if $g(z)$ has continuous boundary functions $g_+(x)$ and $g_-(x)$ there must be a non-empty set $I \subset \mathbb{R}$ where

$$h(x) := g_+(x) - g_-(x) \qquad x \in I \tag{7.20}$$

is non-zero because otherwise the asymptotic condition (7.16) cannot be satisfied.

Following the steps of the non-linear steepest descent method [30], we seek a function $g(z)$ (the *g-function* corresponding to this Riemann–Hilbert problem) such that the following conditions are satisfied:

(g.1) The function $g(z)$ is holomorphic in $\mathbb{C}\backslash\mathbb{R}$ and has continuous boundary functions $g_+(x)$ and $g_-(x)$ on $\mathbb{R}$.

(g.2)

$$g(z) = \log z + \mathcal{O}\left(\frac{1}{z}\right) \qquad z \longrightarrow \infty\ . \tag{7.21}$$

(g.3) There exists a set $I \subset \mathbb{R}$ such that

$$g_+(x) + g_-(x) - \frac{1}{t}V(x) \equiv \ell \qquad x \in I \tag{7.22}$$

for some constant $\ell \in \mathbb{R}$ and

$$\mathrm{Re}(g_+(x) - g_-(x)) \equiv 0 \qquad x \in I\ . \tag{7.23}$$

(g.4) The inequality

$$\mathrm{Re}\left(g_+(x) + g_-(x) - \frac{1}{t}V(z) - \ell\right) < 0 \qquad x \in \mathbb{R} \setminus I \tag{7.24}$$

holds and

$$e^{n(g_+(x)-g_-(x))} = 1 \qquad x \in \mathbb{R} \setminus I \ . \tag{7.25}$$

(g.5) The function

$$h(z) = g_+(z) - g_-(z) \tag{7.26}$$

has an analytic continuation in a thin *lens-shaped region* D around I such that

$$\begin{cases} \mathrm{Re}(h(z)) > 0 & z \in D \cap \{\mathrm{Re}(z) > 0\} \\ \mathrm{Re}(h(z)) < 0 & z \in D \cap \{\mathrm{Re}(z) < 0\} \ . \end{cases} \tag{7.27}$$

The relevance of the assumptions (g.3), (g.4) and (g.5) is not clear at this stage; the meaning of these will be clarified by the steps of the analysis below. Also, the existence such a g-function is not at all obvious from the above set of conditions. It can be shown that the conditions (g.1), (g.2) and (g.3) lead to a scalar Riemann-Hilbert problem for $g'(z)$ and, for real analytic potentials I consists of a finite number of compact intervals [26]. The remaining conditions of the g-function are then used to find the endpoints of the intervals where $g(z)$ has jumps. This solution is intimately connected to the equilibrium measure of the potential

$$V_t(x) = \frac{1}{t}V(x) \ . \tag{7.28}$$

Heuristically, the theorem on the asymptotics of the zeroes of the orthogonal polynomials 4.8.2 suggests that

$$\frac{1}{n}\log\frac{1}{|P_{n,N}(z)|} \sim \int \log\frac{1}{|z-s|}d\mu_{V/t}(s) \qquad n \to \infty \ , \tag{7.29}$$

where $\mu_{V/t}$ is the equilibrium measure corresponding to $\frac{1}{t}V$, since the normalized counting measure of the zeroes converges to the equilibrium measure. Therefore it is expected that

$$P_{n,N}(z) \sim n \int \log(z-s) d\mu_{V/t}(s) \qquad n \to \infty\,. \tag{7.30}$$

Comparing this to (7.19) it is not surprising that the g-function is given by the complex logarithmic potential of the equilibrium measure [30]:

$$g(z) = \int \log(z-s) d\mu_{V/t}(s) \qquad z \in \mathbb{C} \setminus \mathbb{R}\,. \tag{7.31}$$

As we indicated above, the g-function can be constructed *without considering the minimization problem* just by solving a scalar Riemann-Hilbert problem implied by the g-function conditions. It is important to note that for certain types of non-Hermitian orthogonal polynomials the g-function can be constructed based on a similar set of conditions. This construction is not always associated to the solution of a variational problem (see [16] for a general scheme and Chap. 12 for a special case of such a situation).

7.3.2 Lens opening

From now on, for simplicity, assume that the jump set of g-function consists of a single bounded closed interval

$$I = [a,b] \qquad -\infty < a < b < \infty\,. \tag{7.32}$$

(The analysis can be carried through for the case of several intevals in a similar way as described below.)

On the interval $I \subset \mathbb{R}$ where (7.22) is prescribed, the jump matrix of the undressed matrix is

$$\begin{bmatrix} e^{n(g_-(x)-g_+(x))} & e^{n\ell} \\ 0 & e^{n(g_+(x)-g_-(x))} \end{bmatrix}, \tag{7.33}$$

where ℓ is the related to the modified Robin constant of the variational problem. It is easy to see that the slightly modified undressed matrix

$$U(z) := e^{\frac{n\ell}{2}} Y(z) e^{-ng(z)\sigma_3} e^{-\frac{n\ell}{2}\sigma_3} \qquad z \in \mathbb{C} \setminus \mathbb{R} \tag{7.34}$$

gives the jump

$$\begin{bmatrix} e^{n(g_-(x)-g_+(x))} & 1 \\ 0 & e^{n(g_+(x)-g_-(x))} \end{bmatrix} \qquad x \in (a,b)\ . \tag{7.35}$$

Since $g_+(x) - g_-(x)$ is purely imaginary on I, the diagonal terms exhibit rapid oscillations as $n \to \infty$. Note that if x is slightly moved off the segment I into $\mathbb{C} \setminus \mathbb{R}$, by our assumption on the sign of the real part of the auxiliary function $h(z)$, one of the diagonal terms decays while the other grows exponentially (the rate of convergence depends on the location of x) as $n \to \infty$. Therefore a simple deformation of the contour is not effective in making *both diagonal terms* exponentially small. Instead, by using the matrix identity

$$\begin{bmatrix} A & 1 \\ 0 & A^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ A^{-1} & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ A & 1 \end{bmatrix} \qquad A \neq 0\ , \tag{7.36}$$

we can *split* the jump matrix and use *both sides* of I at the same time. More precisely, we can fix two contours Γ_+ and Γ_- lying entirely in D enclosing the domains Ω_+ and Ω_- for which we define the following modified matrix-valued function:

$$T(z) := \begin{cases} U(z) \begin{bmatrix} 1 & 0 \\ -e^{-nh(z)} & 1 \end{bmatrix} & z \in \Omega_+ \\[2ex] U(z) \begin{bmatrix} 1 & 0 \\ e^{nh(z)} & 1 \end{bmatrix} & z \in \Omega_- \\[2ex] U(z) & z \in \mathbb{C} \setminus (\Omega_+ \cup \Omega_- \cup \mathbb{R} \cup \Gamma_+ \cup \Gamma_-)\ . \end{cases} \tag{7.37}$$

Straightforward calculation gives that $T_+(z) = T_-(z)H(z)$, where

$$H(z) = \begin{cases} \begin{bmatrix} 1 & 0 \\ e^{-nh(z)} & 1 \end{bmatrix} & z \in \Gamma_+ \ , \\ \\ \begin{bmatrix} 1 & 0 \\ e^{nh(z)} & 1 \end{bmatrix} & z \in \Gamma_- \ , \\ \\ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & z \in I \ . \end{cases} \tag{7.38}$$

This shows that the resulting matrix satisfies a Riemann-Hilbert problem whose jump matrix is exponentially close to the identity at every point of $(\mathbb{R} \cup \Gamma_1 \cup \Gamma_2) \setminus I$. The exponential rate is uniform outside small neighborhoods of the endpoints a and b.

7.3.3 Model problem

The above seen asymptotic properties of the jump matrix that determines $T(z) = T_n(z)$ suggest that it approximates the solution of the following *model problem* on the interval $[a, b]$:

(1) $\Psi(z)$ is holomorphic in $\mathbb{C} \setminus [a, b]$ and has continuous boundary values on (a, b) ,

(2)

$$\Psi_+(x) = \Psi_-(x) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad x \in (a, b) \tag{7.39}$$

(3) $\Psi(z) = I + \mathcal{O}\left(\frac{1}{z}\right)$ as $z \to \infty$.

Note that the solution of this Riemann-Hilbert problem is *not unique*: without prescribing the asymptotic behavior at the endpoints $x = a$ and $x = b$, the solution is defined up to adding meromorphic matrix-valued functions $E(z)$ holomorphic on $\mathbb{C} \setminus (\{a\} \cup \{b\})$ such that

$$E(z) = \mathcal{O}\left(\frac{1}{z}\right) \qquad z \to \infty . \tag{7.40}$$

The relevant solution to this constant jump problem can be constructed explicitly [30]:

$$\Psi(z) = \begin{bmatrix} \dfrac{B(z) + B(z)^{-1}}{2} & \dfrac{B(z) - B(z)^{-1}}{2i} \\ \dfrac{-B(z) + B(z)^{-1}}{2i} & \dfrac{B(z) + B(z)^{-1}}{2} \end{bmatrix} , \tag{7.41}$$

where

$$B(z) = \left(\frac{z-b}{z-a}\right)^{\frac{1}{4}} \qquad z \in \mathbb{C} \setminus [a, b] , \tag{7.42}$$

where the sheet of $B(z)$ is fixed by the asymptotic condition

$$B(z) = 1 + \mathcal{O}\left(\frac{1}{z}\right) \qquad z \to \infty . \tag{7.43}$$

Note that the exact same construction can be used if we replace $[a, b]$ with a simple smooth contour Σ joining two arbitrary points $z = a$ and $z = b$, except that the branch cut of $B(z)$ has to be placed on Σ.

7.3.4 Small norm Riemann–Hilbert problems

Since jump matrix of $T(z)$ is exponentially close to the jump matrix of $\Psi(z)$ pointwise except at the endpoints of I, we expect that the *error matrix*

$$S(z) = S_n(z) := T_n(z)\Psi^{-1}(z) \tag{7.44}$$

is close to the identity matrix as $n \to \infty$. The following result is of central importance in the conclusion of the asymptotic analysis [53, 22]:

Theorem 7.3.1 ([53, 22]) *Consider a Riemann-Hilbert factorization problem given by the data (Γ, G_n) and assume that*

$$\|G_n(z) - I\|_{L^2(\Gamma)\cap L^\infty(\Gamma)} \leq \frac{C}{n^\delta} \qquad n \geq n_0 \tag{7.45}$$

for some $n_0 \in \mathbb{N}$ *and* $\delta > 0$. *Then, for sufficiently large* n, *the Riemann–Hilbert problem associated to* (Γ, G_n) *is uniquely solvable and the solution* $Y_n(z)$ *satisfies the following uniform estimate*

$$\|Y_n(z) - I\| \leq \frac{C}{(1+|z|^{\frac{1}{2}})n^\delta} \qquad z \in K \tag{7.46}$$

for $n \geq n_1$ *for some* $n_1 \geq n_0$, *on any closed set* $K \subset \mathbb{C}$ *that satisfies*

$$\inf_{z\in K} \frac{d(z,\Gamma)}{1+|z|} > 0 \ . \tag{7.47}$$

The behavior of $S(z)$ at the endpoints imply that the $L^\infty(\Gamma)$-norm condition of Thm. 7.3.1 *does not hold* for $S(z)$. This problem is addressed by the construction of *local parametrices* to find a modified error matrix that does satisfy a small-norm Riemann–Hilbert problem.

7.3.5 Local parametrices

The basic idea to treat the error matrix $S(z)$ at the endpoint $x = a$ (and similarly at the endpoint $x = b$) is to solve the jump conditions for $T(z)$ *explicitly* in a sufficiently small disk $D(a, \varepsilon)$ around $z = a$, i.e., to find a matrix-valued function $P^a(z) = P^a_n(z)$ in $D(a, \varepsilon)$ such that $P^a(z)$ has exactly the same jump conditions as $T(z)$ and

$$P^a(z) = \left(I + \mathcal{O}\left(\frac{1}{n^\delta}\right)\right)\Psi(z) \tag{7.48}$$

uniformly on $\partial D(a, \varepsilon)$ as $n \longrightarrow \infty$. Using a conformal change of coordinates such solution $P^a(z)$ can be constructed using *Airy functions* [2]; The steps of this procedure are not detailed here; it can be found in [30, 53]. In Chap. 12 the construction of two special local parametrices can be found.

7.3.6 Conclusion of the asymptotic analysis

Let Σ_T denote the union of the contours

$$\Sigma_T := \mathbb{R} \cup \Gamma_1 \cup \Gamma_2 \,. \tag{7.49}$$

If we consider the modified error matrix

$$\tilde{S}(z) := \begin{cases} T(z)\Psi^{-1}(z) & z \in \mathbb{C} \setminus (\Sigma_T \cap D(a,\varepsilon) \cap D(b,\varepsilon) \\ \\ T(z)(P^a(z))^{-1} & z \in D(a,\varepsilon) \\ \\ T(z)(P^b(z))^{-1} & z \in D(b,\varepsilon) \end{cases} \tag{7.50}$$

Then it is shown [30, 53] that $\tilde{S}$ satisfies a small Riemann-Hilbert problem in both the L^2 and L^∞-sense and therefore

$$S(z) = I + \mathcal{O}\left(\frac{1}{n^\delta}\right) \,. \tag{7.51}$$

This means that, by tracing back the steps $Y \mapsto U \mapsto T \mapsto \Psi$ and using the error term guaranteed by the error matrix, we obtain strong asymptotics for the original matrix function Y as $n \to \infty$ uniformly on compact subsets depending on the region in question [30, 53]. For a special case of such asymptotics, see Chap. 12.

7.4 Quadratic differentials

We restrict ourselves to give the definition quadratic differentials an some of their basic properties. For an extensive treatment of the theory of quadratic differentials see [89]. See also [73] for a brief and elementary introduction with examples.

Definition 7.4.1 *Let $\mathcal{R}$ be a Riemann surface with a fixed atlas $\{(U_k, z_k)\}$ consisting of open sets $U_k \subset \mathcal{R}$ whose union covers $\mathcal{R}$ and local coordinates $z_k\colon U_k \to \mathbb{C}P^1$*

with holomorphic transition functions on the regions of overlap. A meromorphic quadratic differential φ *on* $\mathcal{R}$ *is a collection of meromorphic functional elements* φ_k *that transform by the rule*

$$\varphi_k(z_k)dz_k^2 = \varphi_l(z_l)dz_l^2 \tag{7.52}$$

with respect to a coordinate change on overlapping coordinate neighborhoods U_k *and* U_l*, i.e.,*

$$\varphi_k(z_k) = \varphi_l(z)\left(\frac{dz_l}{dz_k}\right)^2 . \tag{7.53}$$

This definition is analogous to the definition of one-forms on an arbitrary Riemann surface [42]. A more abstract definition can also be given in the framework of algebraic geometry (see [73]).

Example. On the Riemann sphere $\mathcal{R} = \mathbb{C}P^1$ we consider the two overlapping coordinate neighborhoods

$$U_1 := \mathbb{C}P^1 \setminus \{\infty\} , \qquad U_2 := \mathbb{C}P^1 \setminus \{0\} \tag{7.54}$$

with local coordinates z and $\tilde{z}$ respectively, such that the transition function between the two local coordinates is given by

$$\tilde{z}(P) = \frac{1}{z(P)} \qquad P \in U_1 \cap U_2 . \tag{7.55}$$

Any rational function $R(z)$ gives a quadratic differential denoted by $R(z)dz^2$ on $\mathbb{C}P^1$ given by the functional elements

$$\varphi_1(z) = R(z) , \qquad \varphi_2(\tilde{z}) = R\left(\frac{1}{\tilde{z}}\right)\frac{1}{\tilde{z}^4} . \tag{7.56}$$

A meromorphic quadratic differential φ on a Riemann surface $\mathcal{R}$ does not associate functional values to the points of $\mathcal{R}$. However, it makes sense to consider *poles* and *zeroes* of φ since these are independent of the choice of the local coordinate. The order of a zero or a pole is also invariant under change of coordinates; the leading term of φ is invariant only for poles of order two [89] (analogously to the case of one-forms for which only the residues are invariant).

Definition 7.4.2 *The poles and zeroes of a meromorphic quadratic differential φ are referred to as the* critical points *of φ.*

Given a point $P_0 \in \mathcal{R}$, a distinguished local parameter w can be defined corresponding to $\varphi(z)dz^2$ by considering the one-form

$$\omega := \sqrt{\varphi(z)}dz \tag{7.57}$$

that is integrated to

$$w(P) := \int_{P_0}^{P} \sqrt{\varphi(z)}dz \; . \tag{7.58}$$

Evidently, the one-form ω and therefore the coordinate w is defined up to sign in a neighborhood of a non-critical point because of the sign ambiguity of the square root.

The *metric* associated to a quadratic differential $\varphi(z)dz^2$ on $\mathcal{R}$ is given by the length element

$$ds^2 = |\varphi(z)||dz|^2 \; . \tag{7.59}$$

The *Strebel length of a curve* γ with respect to φ is given by

$$l_\varphi(\gamma) = \int_\gamma |\varphi(z)|^{\frac{1}{2}}|dz| \; . \tag{7.60}$$

For the properties of this metric, see [89]. The following notions associated to quadratic differentials are needed in Chap. 12.

Definition 7.4.3 *Let*

$$\gamma\colon (a,b) \rightarrow \mathcal{R} \tag{7.61}$$

be a parametric curve on $\mathcal{R}$. γ is called a horizontal arc *of φ if*

$$\varphi(\gamma(t)) \left(\frac{d\gamma(t)}{dt}\right)^2 > 0 \qquad t \in (a,b) \; . \tag{7.62}$$

If

$$\varphi(\gamma(t))\left(\frac{d\gamma(t)}{dt}\right)^2 < 0 \qquad t \in (a,b) , \tag{7.63}$$

the curve is called a vertical arc *of* φ.

The horizontal and vertical arcs emanating from a point are distinguished geodesics of the metric $|\varphi(z)||dz|^2$. A maximal horizontal or vertical geodesic is called a *horizontal trajectory* or a *vertical trajectory* respectively. A *critical horizontal/vertical trajectory* is a horizontal/vertical trajectory that emanates from a critical point.

For every non-critical point P of $\varphi(z)dz^2$ there exists a unique horizontal and a unique vertical geodesic arc passing through P. In the vicinity of critical points, however, there is a completely different behavior depending on the nature and order of the singularity of $\varphi(z)dz^2$:

- In the vicinity of a simple zero P of $\varphi(z)$ there are three horizontal trajectories emanating from P with asymptotic angles $\frac{2\pi}{3}$.
- In the vicinity of a double zero P of $\varphi(z)$ there are four horizontal trajectories emanating from P with asymptotic angles $\frac{\pi}{2}$.
- In a neighborhood around a double pole P of $\varphi(z)$ with negative leading coefficient (in any local parameter), the horizontal trajectories are concentric circles in terms of the distinguished parameter w defined above.

For a complete list of the local behavior of the trajectories near critical points, see [89].

Part II

Results

Chapter 8

Superharmonic perturbations of a Gaussian measure, equilibrium measures and orthogonal polynomials

8.1 Summary

This chapter presents the results of the published paper [12]. Four main topics are addressed in this work:

1. A special class of quasi-harmonic potentials (superharmonic perturbations of the quadratic potential) is introduced (Sec. 2).

2. The structure of the supports of equilibrium measures corresponding to superharmonic perturbations is studied (Sec. 3).

3. For orthogonal polynomials with respect to superharmonic perturbations of a

Gaussian weight the validity of the Its-Takhtajan matrix d-bar problem (to be defined below) is extended (Sec. 4 and 5).

4. A general conjecture is stated concerning the connection between the asymptotic distribution of the zeroes of orthogonal polynomials and the support of the equilibrium measure via a quadrature identity (Sec. 6).

8.1.1 Superharmonic perturbations

In several works on random normal matrix models [92, 38, 37, 56], polynomial quasi-harmonic potentials of the form

$$Q(z) = \alpha|z|^2 - 2\mathrm{Re}(P(z)) \tag{8.1}$$

are considered, where $P(z)$ is an arbitrary polynomial. However, unless the degree of this polynomial is at most two, the corresponding matrix integrals do not converge since the quadratic term is no longer dominant, and there will be asymptotic sectors in the complex plane where the reduced matrix integrals are divergent. For the same reason, these potentials are not admissible for the weighted energy problem so there is no corresponding equilibrium measure. A standard way of regularizing these potentials is to introduce a *cut-off domain* $\Sigma \subset \mathbb{C}$ outside of which the potential is assumed to be infinite, i.e., the weight function is set to be zero (similar to the notion of an infinite well potential in quantum mechanics). This approach has the drawback that Σ can be chosen quite arbitrarily (as long as it is bounded) and the dependence on the choice of Σ of the matrix integrals, the orthogonal polynomials and the equilibrium measure is not completely understood.

As an alternative, a special class of quasi-harmonic potentials is introduced:

Definition 8.1.1 *A* superharmonic perturbation *of the Gaussian potential is a quasi-*

harmonic potential of the form

$$Q(z) := \alpha|z|^2 + U^\nu(z) \ , \tag{8.2}$$

where U^ν is the logarithmic potential of a compactly supported finite positive Borel measure ν in the complex plane.

These potentials are shown to be admissible (Prop. 2.4) and the corresponding orthogonal polynomials exist (Prop. 4.1).

8.1.2 Equilibrium measures for superharmonic perturbation potentials

The paper [12] contains three results on the structure of the supports of equilibrium measures for superharmonically perturbed Gaussian weights, which are presented below in Thms. 8.1.1, 8.1.2 and 8.1.3.

First, as an illustration of the complicated structure of the equilibrium domains, the following lemma presents a detailed calculation of the support of the equilibrium measure for the simplest non-trivial case

$$Q(z) = \alpha|z|^2 + \beta \log \frac{1}{|z-a|} \ , \tag{8.3}$$

where the perturbing measure ν is a Dirac point measure of mass $\beta > 0$ concentrated at $a \in \mathbb{C}$. It is important to note that although the result was known [92, 67] and is elementary to derive, there seems to have been no complete proof available in the existing literature.

Lemma 8.1.1 (Prop 3.3, [12]) *Define two radii R and r as*

$$R := \sqrt{\frac{1+\beta}{2\alpha}} \qquad r := \sqrt{\frac{\beta}{2\alpha}} \ . \tag{8.4}$$

The equilibrium measure μ_Q is absolutely continuous with respect to the Lebesgue measure with constant density $\frac{2\alpha}{\pi}$. The shape of $\mathrm{supp}(\mu_Q)$ depends on the geometric arrangement of the disks $D(0,R)$ and $D(a,r)$ in the following way:

(i) If $D(a,r) \subset D(0,R)$ then

$$S_Q = \overline{D}(0,R) \setminus D(a,r) \ . \tag{8.5}$$

(ii) If $D(a,r) \not\subset D(0,R)$ then S_Q is simply connected and uniformized by a rational exterior conformal mapping of the Joukowski-type [1]:

$$F(u) = ru + a_0 + \frac{v}{u-A} \ , \tag{8.6}$$

where the coefficients $r \in \mathbb{R}^+, 0 < |A| < 1$ and $a_0, v \in \mathbb{C}$ are uniquely determined by the parameters α, β and a.

The different possible configurations are illustrated in Fig. 8.1. The equilibrium measure of the special potential considered above has a simple structure if $D(a,r) \subset D(0,R)$: the equilibrium support is the difference of two disks. There is a simple explanation of this from a potential theoretic point of view, as follows. The potential $Q(z)$ is the sum of a quadratic term

$$\tilde{Q}(z) := \alpha|z|^2 \tag{8.7}$$

with Laplacian

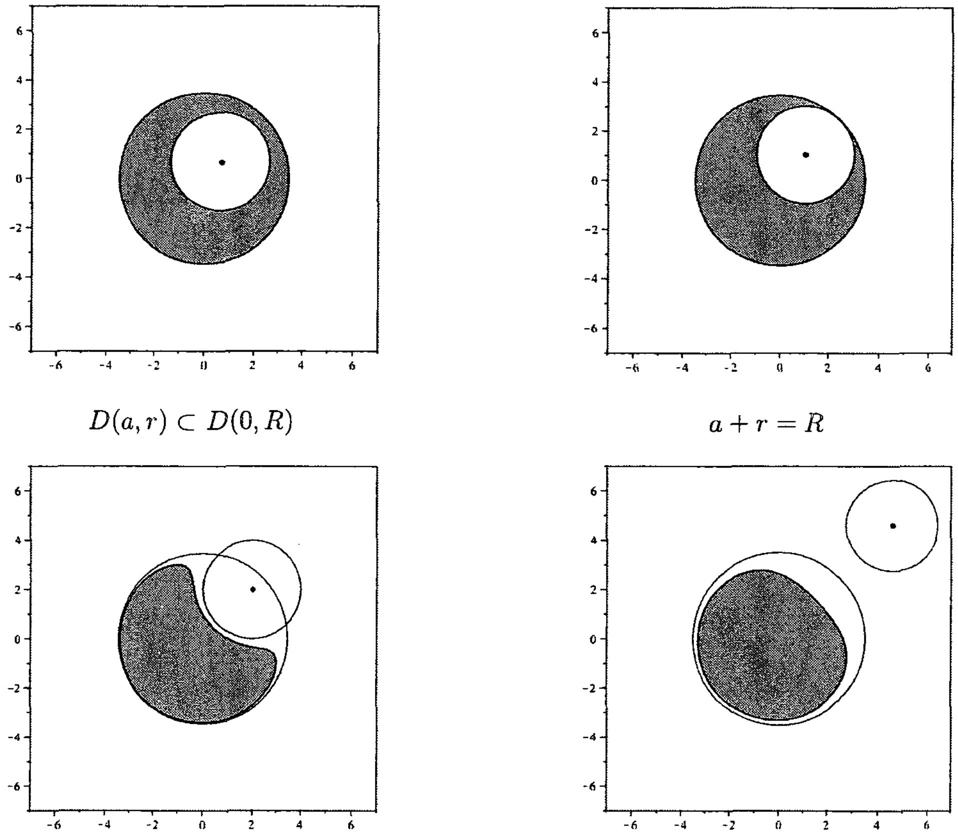
$$\Delta\tilde{Q}(z) = 4\alpha \ , \tag{8.8}$$

and a pure logarithmic potential term $U^\nu(z)$ corresponding to the measure $\nu = \beta\delta_a$. Note that the modified variational problem

$$\begin{cases} \displaystyle\iint \log\frac{1}{|z-t|} d\tilde{\mu}(z)d\tilde{\mu}(t) + 2\int \alpha|z|^2 d\tilde{\mu}(z) \rightarrow \text{min.} \\ \tilde{\mu}(\mathbb{C}) = 1+\beta \end{cases} \tag{8.9}$$

with *increased total charge* $1+\beta$ is solved by the measure

$$d\mu_{\tilde{Q}} = \frac{1}{R^2\pi}\chi_{D(0,R)}dA \ , \tag{8.10}$$



$D(a,r) \subset D(0,R)$ $\quad$ $a + r = R$

$D(a,r) \setminus D(0,R) \neq \emptyset$ $\quad$ $D(a,r) \cap D(0,R) = \emptyset$

Figure 8.1: The shape of the support S_Q for the different configurations

where the radius R is the same as (8.4). The excised disk $D(a,r)$ with density $\frac{2\alpha}{\pi}$ is equivalent to the point measure $\nu = \beta\delta_a$ outside $D(a,r)$, in the following sense:

$$\begin{aligned} &\frac{2\alpha}{\pi}U^{D(a,R)}(z) = U^{\nu}(z) \qquad z \in \mathbb{C} \setminus D(a,r) \\ &\frac{2\alpha}{\pi}U^{D(a,R)}(z) \geq U^{\nu}(z) \qquad z \in D(a,r)\ . \end{aligned} \tag{8.11}$$

This pair of relations is equivalent to the subharmonic quadrature property of the disk $D(a,r)$ with respect to the measure $r^2\delta_a$.

Therefore, using the variational inequalities (4.25) for $\tilde{Q}$, the following inequalities hold:

$$
\begin{aligned}
Q(z)+U^{\mu_Q}(z) &= \tilde{Q}(z)+U^{\nu}(z)+\frac{2\alpha}{\pi}U^{\overline{D}(0,R)}(z)-\frac{2\alpha}{\pi}U^{D(a,r)}(z) \\
&= \tilde{Q}(z)+U^{\mu_{\tilde{Q}}}(z)+U^{\nu}(z)-\frac{2\alpha}{\pi}U^{D(a,r)}(z) \\
&= \begin{cases}
\tilde{Q}(z)+U^{\mu_{\tilde{Q}}}(z) \quad \geq \quad F_{\tilde{Q}} & z\in\mathbb{C}\setminus\overline{D}(0,R) \\
F_{\tilde{Q}} & z\in\overline{D}(0,R)\setminus D(a,r) \\
F_Q+U^{\nu}(z)-\frac{2\alpha}{\pi}U^{D(a,r)}(z) \quad \geq \quad F_{\tilde{Q}} & z\in D(a,r)\ .
\end{cases}
\end{aligned}
\tag{8.12}
$$

In the electrostatic configuration described by the external potential the perturbing charge ν can be regarded as a 'fixed deposit' of charge β in the presence of the 'pure' quadratic potential $\tilde{Q}$. The electrostatic effect of the fixed charge ν is the same as that of the disk $D(a,r)$ with uniform charge $\frac{2\alpha}{\pi}$ outside $D(a,r)$. Hence as long as $D(a,r)\subset D(0,R)$, the movable charge μ in the variational problem can be arranged in such a way that the electrostatic effect of the sum $\mu+\nu$ is the same as that of the equilibrium measure $\mu_{\tilde{Q}}$ and therefore $\mu+\nu$ is in equilibrium in the presence of the potential $\tilde{Q}$.

The generalization of this idea motivated the following

Theorem 8.1.1 (Thm. 2.4, [12]) *Assume that the perturbation measure ν can be decomposed into a sum*

$$\nu=\sum_{k=1}^{m}\nu_k \tag{8.13}$$

where the measures ν_k are all finite positive Borel measures satisfying the following conditions:

(i) The supports of the measures ν_k are pairwise disjoint and each ν_k has positive total mass.

(ii) Each rescaled measure $\frac{\pi}{2\alpha}\nu_k$ has a unique maximal subharmonic quadrature domain D_k for $k = 1, \ldots, m$.

(iii) The domains D_k are pairwise disjoint and $D_k \subset D(0, R)$ for $k = 1, \ldots, m$ where

$$R := \sqrt{\frac{1 + \nu(\mathbb{C})}{2\alpha}} . \tag{8.14}$$

Then the equilibrium measure μ_Q is absolutely continuous with respect to the Lebesgue measure with constant density $\frac{2\alpha}{\pi}$, supported on the compact set

$$K := \overline{D}(0, R) \setminus \left(\bigcup_{k=1}^{m} D_k \right) . \tag{8.15}$$

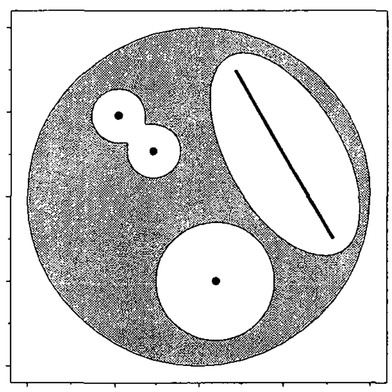


Figure 8.2: A configuration involving subharmonic quadrature domains: a disk, a two-point quadrature domain and an ellipse

The situation described in Thm. 8.1.1 is illustrated in Fig. 8.2 for the configuration involving the enclosed quadrature domains, corresponding to a point charge, a pair of point charges with overlapping disks and a line of charge corresponding to an elliptic quadrature domain. The conclusion of Theorem 8.1.1 does not hold if some of the domains D_k overlap or intersect the exterior of $D(0, R)$. The complete description of all possible configurations becomes quite complicated geometrically if we relax any of the conditions above. However, it is plausible to expect that the support of the equilibrium measure is *always* contained in the disk $\overline{D}(0, R)$.

The following result confirms this assertion if ν is a positive *rational* linear combination of point masses.

Theorem 8.1.2 (Thm. 3.5, [12]) *Let ν is a measure of the form*

$$\nu = \sum_{k=1}^{m} r_k \delta_{a_k} \tag{8.16}$$

where $r_1, r_2, \ldots, r_m$ are positive rational numbers. *Then the support S_w of the corresponding equilibrium measure is entirely contained in the closed disk $\overline{D}(0, R)$ where*

$$R = \sqrt{\frac{1 + \nu(\mathbb{C})}{2\alpha}} \ . \tag{8.17}$$

8.1.3 The Its-Takhtajan d-bar problem for orthogonal polynomials

The nonlinear steepest descent method of Deift and Zhou applied to the Riemann-Hilbert problem of Fokas, Its and Kitaev (Sec. 7.2) provided the strong asymptotics for orthogonal polynomials on the real line. No comparable method is as yet developed that is similarly applicable to the asymptotics of orthogonal polynomials in the plane. However, some partial results in this direction are known, in which, for certain classes of potentials, instead of an associated Riemann–Hilbert problem, the orthogonal polynomials are uniquely determined by a suitable d-bar problem. Consider quasi-harmonic potentials in the plane

$$Q(z) = \alpha |z|^2 - 2\mathrm{Re}(H(z)) \qquad z \in \Sigma \tag{8.18}$$

that are admissible (see Sec. 4.4) and such that the corresponding orthogonal polynomials $P_{n,N}(z)$ satisfying

$$\int_{\Sigma} P_{n.N}(z) \overline{P_{m,N}(z)} e^{-NQ(z)} dA(z) = h_{n,N} \delta_{nm} \tag{8.19}$$

exist. In a recent attempt to find an analog of the Riemann–Hilbert approach valid for for polynomial quasi-harmonic potentials with a compact cut-off set Σ, Its and Takhtajan [56] considered the 2×2 matrix-valued function

$$Y_{n,N}(z) := \begin{bmatrix} P_{n,N}(z) & \frac{1}{\pi}\int_{\mathbb{C}} \frac{\overline{P_{n,N}(w)}}{w-z} e^{-NQ(w)} dA(w) \\ \\ -\frac{\pi}{h_{n-1,N}} P_{n-1,N}(z) & -\frac{1}{h_{n-1,N}} \int_{\mathbb{C}} \frac{\overline{P_{n-1,N}(w)}}{w-z} e^{-NQ(w)} dA(w) \end{bmatrix}. \tag{8.20}$$

that satisfies the following d-bar problem:

$$\partial_{\bar{z}} Y_{n,N}(z) = \overline{Y_{n,N}(z)} \begin{bmatrix} 0 & -e^{-NQ(z)} \\ \\ 0 & 0 \end{bmatrix} \qquad z \in \mathbb{C} . \tag{8.21}$$

They showed in [56] that the d-bar problem

(1) $M(z)$ is continuously differentiable on $\mathbb{C}$,

(2) M satisfies the d-bar equation

$$\partial_{\bar{z}} M(z) = \overline{M(z)} \begin{bmatrix} 0 & -\chi_\Sigma(z) e^{-NQ(z)} \\ \\ 0 & 0 \end{bmatrix} \qquad z \in \mathbb{C} \tag{8.22}$$

(3)

$$M(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) z^{-n\sigma_3} \qquad |z| \to \infty . \tag{8.23}$$

characterizes the matrix function $Y_{n,N}(z)$ uniquely and therefore determines the monic orthogonal polynomial $P_{n,N}(z)$. Note that Σ is compact and therefore $Y_{n,N}(z)$ is *holomorphic outside* Σ.

It is shown in [12] that the validity of this d-bar problem can be extended to superharmonic perturbations of the Gaussian weight with no cut-off:

Theorem 8.1.3 (Prop. [12]) *Assume that*

$$Q(z) = \alpha|z|^2 + U^\nu(z) \tag{8.24}$$

is a superharmonic perturbation of the quadratic potential. Then the 2×2 *matrix-valued function* $Y_{n,N}(z)$

$$Y_{n,N}(z) = \begin{bmatrix} P_{n,N}(z) & \dfrac{1}{\pi}\displaystyle\int_{\mathbb{C}} \frac{\overline{P_{n,N}(w)}}{w-z} e^{-NV(w)} dA(w) \\ & \\ -\dfrac{\pi}{h_{n-1,N}} P_{n-1,N}(z) & -\dfrac{1}{h_{n-1,N}} \displaystyle\int_{\mathbb{C}} \frac{\overline{P_{n-1,N}(w)}}{w-z} e^{-NV(w)} dA(w) \end{bmatrix} . \tag{8.25}$$

is the unique solution of the following matrix d-bar problem:

(1) $M(z)$ *is continuously differentiable on* $\mathbb{C}$,

(2) M *satisfies the d-bar equation*

$$\partial_{\bar{z}} M(z) = \overline{M(z)} \begin{bmatrix} 0 & -e^{-NV(z)} \\ & \\ 0 & 0 \end{bmatrix} \qquad z \in \mathbb{C} \tag{8.26}$$

(3)

$$M(z) = \left(I + \mathcal{O}\left(\frac{1}{|z|}\right)\right) z^{-n\sigma_3} \qquad |z| \to \infty . \tag{8.27}$$

The details of the proof are given in the paper [12], comprising the next subsection. Similarly to the Fokas-Its-Kitaev Riemann-Hilbert problem, the Its-Takhtajan d-bar problem uniquely characterizes the orthogonal polynomials. The first steps of a method to analyse this d-bar problem are laid down in [56]; however, at present, this approach is not sufficiently developed to provide asymptotic expansions of such orthogonal polynomials.

8.1.4 Zeroes of orthogonal polynomials and quadrature domains

Recall (Sec. 4.8) that, under suitable regularity assumptions on the potential, $V(x)$ on the real line, the following three sequence of measures converge to the equilibrium measure of V in the appropriate scaling limit:

- the density of the one-point function of the matrix model (Thm. 4.8.1),
- the normalized counting measure of a fixed Fekete point configuration (Def. 4.5.1 and Thm. 4.5.1),
- the normalized counting measure of the zeroes of the corresponding orthogonal polynomials (Thm. 4.8.2).

Recall also that the analogues of the first and the second statements are valid for normal matrix models as well (Thms. 4.8.3 and 4.5.1).

However, the following two examples indicate that the analogous statement to Theorem 4.8.2 does not hold for planar orthogonal polynomials. These examples however suggest a suitable generalization of Thm. 4.8.2 that forms the basis of the above mentioned conjecture.

Example 1. Circular symmetric planar Gaussian weights. For the potential

$$Q(z) = \alpha|z|^2 \tag{8.28}$$

the equilibrium measure is given by (recall Sec. 6.6.2)

$$d\mu_Q = \frac{1}{R^2\pi}\chi_{\overline{D}(0,R)}dA \tag{8.29}$$

where

$$R = \sqrt{\frac{1}{2\alpha}}\ . \tag{8.30}$$

By the circular symmetry of the potential,

$$\begin{aligned}\int_{\mathbb{C}} z^n \bar{z}^k e^{-\alpha|z|^2} dA(z) &= \int_0^{2\pi}\int_0^{\infty} r^{n+k+1} e^{i(n-k)\theta} e^{-\alpha r^2} dr d\theta \\ &= \frac{\pi}{\alpha^{n+1}}\delta_{nk}\int_0^{\infty} s^n e^{-s} ds = \frac{\pi n!}{\alpha^{n+1}}\delta_{nk} \ ,\end{aligned} \tag{8.31}$$

where $z = re^{i\theta}$. Therefore the corresponding monic orthogonal polynomials are just the monomials:

$$P_n(e^{-|z|^2}dA; z) = z^n \ . \tag{8.32}$$

(Note that this is true for every circular symmetric potential.) The normalized counting measure of the zeroes is δ_0 for every n, so the asymptotic distribution is also δ_0. This means that

$$\nu_{n,N} \xrightarrow{w*} \delta_0 \neq \mu_Q \ . \tag{8.33}$$

Example 2. General planar Gaussian weights. The quadratic potential function

$$Q(z) = |z|^2 - \frac{1}{2}\left(T^2 z^2 + \overline{T}^2 \bar{z}^2\right) \ , \qquad 0 < |T| < 1 \ , \tag{8.34}$$

corresponds to the Gaussian weight

$$w(z) =$$

$$\exp\left(-\left[(1+\mathrm{Im}(T)^2 - \mathrm{Re}(T)^2)x^2 + (1+\mathrm{Re}(T)^2 - \mathrm{Im}(T)^2)y^2 + 4\mathrm{Re}(T)\mathrm{Im}(T)xy\right]\right) \ , \tag{8.35}$$

where $z = x + iy$. The covariance matrix of this Gaussian density is positive definite if and only if $|T| < 1$.

The equilibrium measure is given by (Sec. 6.6.2)

$$d\mu_Q = \frac{1}{A(E)}\chi_E dA \ , \tag{8.36}$$

where E is the ellipse given by the exterior uniformizing map

$$F(u) = ru + \frac{r\overline{T}^2}{u} \ , \tag{8.37}$$

with conformal radius

$$r = \sqrt{\frac{1}{2(1-|T|^4)}} . \tag{8.38}$$

The foci are given by

$$\pm c = \pm \frac{a}{T} \tag{8.39}$$

where we use the abbreviated notation

$$a := \sqrt{\frac{2|T|^4}{1-|T|^4}} . \tag{8.40}$$

The monic polynomials $P_{n,N}(z)$ with respect to the scaled weight function

$$w(z) = e^{-N\left(|z|^2 - \frac{1}{2}(T^2 z^2 + \overline{T}^2 \overline{z}^2)\right)} \tag{8.41}$$

are expressible in terms of the Hermite polynomials [31] (see Thm. 3.5.1)

$$P_{n,N}(z) = \frac{a^n}{2^n N^{\frac{n}{2}} T^n} H_n\left(\sqrt{N}\frac{T}{a}z\right) . \tag{8.42}$$

Since

$$\frac{T^n}{a^n} P_{n,N}\left(\frac{az}{T}\right) = \frac{1}{2^n N^{\frac{n}{2}}} H_n\left(\sqrt{N}z\right) \tag{8.43}$$

is the monic orthogonal polynomial *on the real line* with respect to the weight

$$\tilde{w}^N(x) = \exp\left(-Nx^2\right) , \tag{8.44}$$

Thm. 4.8.1 implies that the normalized counting measure of the zeroes of the rescaled Hermite polynomials converges to the *Wigner semicircle distribution*

$$d\mu_{\tilde{w}}(x) = \frac{2}{\pi}\chi_{[-1,1]}(x)\sqrt{1-x^2}dx , \tag{8.45}$$

as $n, N \to \infty$, $\frac{n}{N} \to \frac{1}{2}$. Therefore the normalized counting measure of the zeroes of $P_{n,N}(z)$ converges to the probability measure

$$d\sigma(z) := \frac{2a}{T\pi}\sqrt{1-\left(\frac{az}{T}\right)^2}dz \tag{8.46}$$

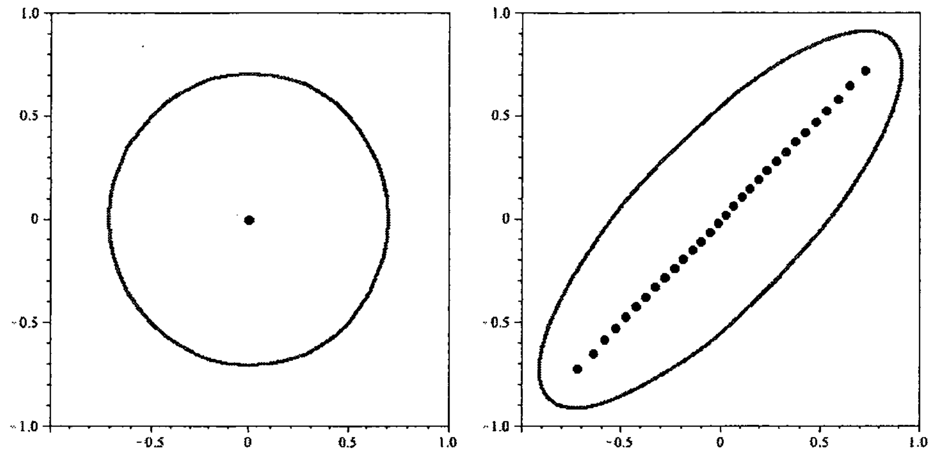


Figure 8.3: The zeroes for the orthogonal polynomials for the special cases above

along the focal segment $[-c, c]$ of E. Note that this density gives the *normalized quadrature measure* for E; for any integrable holomorphic function f on E,

$$\frac{1}{A(E)} \int_E f\, dA = \int_{-c}^{c} f\, d\sigma \ . \tag{8.47}$$

The configuration of the preceding two examples is illustrated in Fig. 8.3. Based on these examples one can formulate the following

Conjecture 8.1.1 (Sec. 6, [12]) *Assume that $Q(z)$ is an admissible quasi-harmonic potential of the form*

$$Q(z) = |z|^2 - 2Re(H(z)) \qquad z \in \Sigma \tag{8.48}$$

with the corresponding monic orthogonal polyomials $P_{n,N}(z)$ characterized by the orthogonality relations

$$\int_{\mathbb{C}} P_{n,N}(z) \bar{z}^k e^{-N|z|^2 + 2N Re(H(z))} dA(z) = h_{n,N} \delta_{nk} \qquad k = 0, \ldots, n-1 \ . \tag{8.49}$$

(1) *For every $t > 0$ there exists a probability measure σ_t supported in the polynomial convex hull of $S_{Q/t}$*

$$\mathrm{supp}(\sigma_t) \subseteq \mathrm{Pc}(S_{Q/t}) \tag{8.50}$$

such that the full sequence *of normalized counting measures of the zeroes of $P_{n,N}(z)$ converges to σ_t in the weak-star sense:*

$$\nu_{n,N} \xrightarrow{w*} \sigma_t \tag{8.51}$$

in the scaling limit

$$n \to \infty, \quad N \to \infty, \quad \frac{n}{N} \to t\ . \tag{8.52}$$

(2) *The Cauchy transforms of the equilibrium measure and the asymptotic zero distribution are the same outside the polynomial convex hull of the support $S_{Q/t}$:*

$$C_{\mu_{Q/t}}(z) = C_{\sigma_t}(z) \qquad z \in \mathbb{C} \setminus \mathrm{Pc}(S_{Q/t})\ . \tag{8.53}$$

(3) *The measure σ_t is supported along the union of a collection of curve segments $\cup_j \mathcal{B}_j^t$ depending the value of t for which $C_{\mu_{Q/t}}(z)$ admits an analytic continuation $F^t(z)$ to $\mathbb{C} \setminus \cup_j \mathcal{B}_j^t$ such that the one-form*

$$Re(F_+^t(z) - F_-^t(z))dz \tag{8.54}$$

vanishes along each curve $\cup_j \mathcal{B}_j^t$. Some of the curves of $\cup_j \mathcal{B}_j$ may degenerate to points.

For the case of polynomial quasi-harmonic potentials with a compact cut-off set Σ, essentially the same conjecture appears in [37].

Note that this conjecture consists of three distinct statements:

(1) It is conjectured that there is a unique accumulation point of the sequence of measures $\{\nu_{n,N}\}_n$. There are many examples of orthogonality measures in [88] for

which the sequence of normalized counting measures of the zeroes of orthogonal polynomials have more than one accumulation points.

(2) The *weak* part of the conjecture is that the asymptotic distribution of the zeroes and the equilibrium measure have the same Cauchy transforms outside the polynomial convex hull of $S_{Q/t}$. In [37] this part is proved for cut-off polynomial quasi-harmonic potentials using the multiple integral representation (3.18) of Heine for $P_{n,N}$ and the large deviation method of Johansson [58]. This fact is also known for certain classes of general orthogonal polynomials [88].

(3) The *strong* part is the localization of the asymptotic distribution along curves. Conceptually this is the same problem as finding a *minimal quadrature measure* for a given bounded domain in the plane [46, 84]. The one-form condition (8.54) is just a tautology because it is implied by the positivity of σ_t via the Sokhotski-Plemelj formulae (5.29).

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Complex Analysis and Operator Theory

Superharmonic Perturbations of a Gaussian Measure, Equilibrium Measures and Orthogonal Polynomials

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Abstract. This work concerns superharmonic perturbations of a Gaussian measure given by a special class of positive weights in the complex plane of the form $w(z) = \exp(-|z|^2 + U^{\mu}(z))$, where $U^{\mu}(z)$ is the logarithmic potential of a compactly supported positive measure μ. The equilibrium measure of the corresponding weighted energy problem is shown to be supported on subharmonic generalized quadrature domains for a large class of perturbing potentials $U^{\mu}(z)$. It is also shown that the 2×2 matrix d-bar problem for orthogonal polynomials with respect to such weights is well-defined and has a unique solution given explicitly by Cauchy transforms. Numerical evidence is presented supporting a conjectured relation between the asymptotic distribution of the zeroes of the orthogonal polynomials in a semi-classical scaling limit and the Schwarz function of the curve bounding the support of the equilibrium measure, extending the previously studied case of harmonic polynomial perturbations with weights $w(z)$ supported on a compact domain.

1. Introduction

This work mainly concerns equilibrium problems in potential theory, but its motivation derives largely from two related domains: random matrix theory and interface dynamics of incompressible fluids. Recent work of Wiegmann, Zabrodin and their collaborators [18, 20] connected the spectral distributions of random normal matrices to the *Laplacian Growth* model for the interface dynamics of a pair of two-dimensional incompressible fluids. The unitarily invariant probability measure on the set of $n \times n$ complex normal matrices in [18] is determined by a potential

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function

$$V(z,\overline{z}) = z\overline{z} + A(z) + \overline{A(z)} \tag{1.1}$$

and the corresponding density is of the form

$$f(M,M^*) = e^{-\frac{1}{\hbar}\mathrm{Tr}V(M,M^*)} \tag{1.2}$$

with respect to a unitarily invariant reference measure on the set of normal matrices. The function $A(z)$ is assumed to have a single-valued derivative $A'(z)$ meromorphic in some domain $D \subset \mathbb{C}$. Under suitable assumptions on $A(z)$, the large n limit of the averaged normalized eigenvalue distribution in the scaling limit

$$n \to \infty\,, \quad \hbar \to 0\,, \quad n\hbar = t \tag{1.3}$$

tends to a probability measure $\mu_{V,t}$ where t is some fixed positive number (*quantum area*). It turns out that $\mu_{V,t}$ is the unique solution of a two-dimensional electrostatic equilibrium problem in the presence of the external potential V in the complex plane. In most cases, $\mu_{V,t}$ is absolutely continuous with respect to the planar Lebesgue measure with constant density. It can be shown that the support $\mathrm{supp}(\mu_{V,t})$ undergoes Laplacian growth in terms of the scaling constant t and the area of the support is linear in t. However, this evolution problem is ill-defined; even initial domains with analytic boundaries may develop cusp-like singularities in finite time and the solution cannot be continued in the strong sense (see [17,18]). The averaged eigenvalue density of the corresponding normal matrix model for finite matrix size n can be viewed as describing a sort of discretized version whose contiuum limit may be interpreted as a semiclassical limit (1.3) that tends to $\mu_{V,t}$ as shown in [1,9].

In studying these questions, it is important to understand first the possible shapes of compact sets which are supports of equilibrium measures for potentials of the form (1.1). In this work we show the support of the equilibrium measure for a class of perturbed Gaussian potentials of the form

$$V_{\alpha,\nu}(z) := \alpha|z|^2 + U^{\nu}(z)\,, \tag{1.4}$$

where $\alpha > 0$ and ν is a compactly supported finite positive Borel measure, to be so-called *generalized quadrature domains* [8,14].

To understand the asymptotic behaviour of the averaged eigenvalue density of normal matrix models in different scaling regimes, one has to study the asymptotics of the corresponding orthogonal polynomials for the weight $e^{-V(z)}$. The well-known Riemann–Hilbert approach is not applicable directly in this case because the orthogonality weight is not constrained to the real axis. However, there is a sort of matrix $\overline{\partial}$-bar problem, introduced by Its and Takhtajan [10], which is a candidate to replacing the matrix Riemann–Hilbert problem in the study of strong asymptotics of the orthogonal polynomials. In the present work it is shown that the $\overline{\partial}$-bar problem is also well-defined and characterizes the orthogonal polynomials for the class of perturbed Gaussian potentials considered above.

To fix notations, let m denote the two-dimensional Lebesgue measure in the complex plane $\mathbb{C}$. We denote respectively by $\overline{H}$ and H^c the closure and the complement of a set $H \subset \mathbb{C}$ and by I_H the indicator function. The open disk of radius r centered at $c \in \mathbb{C}$ is denoted by $B(c,r)$, and the Riemann sphere by $\hat{\mathbb{C}}$.

2. Weighted energy problem and logarithmic potentials

In this section we briefly describe both the *classical and weighted energy problems* of logarithmic potential theory (see [12] and [13]) and specify the class of background potentials we are concerned with in this paper.

Definition 2.1. Let μ be a compactly supported finite, positive Borel measure in the complex plane. The *logarithmic potential* produced by μ is defined as

$$U^\mu(z) := \int_{\mathbb{C}} \log \frac{1}{|z-w|} d\mu(w) \quad (z \in \mathbb{C}). \tag{2.1}$$

In particular, for a bounded subset S of the plane with $m(S) > 0$, we consider the measure η_S given by

$$d\eta_S = I_S dm. \tag{2.2}$$

Thus η_S is the Lebesgue measure restricted to S. In the following, we use the simplified notation $U^S(z)$ for the logarithmic potential $U^{\eta_S}(z)$ of the measure η_S.

The logarithmic potential of a positive measure μ is harmonic outside the support of μ and superharmonic on $\mathrm{supp}(\mu)$ (see [13], Theorem 0.5.6). Moreover, it has the asymptotic behaviour

$$U^\mu(z) = \mu(\mathbb{C}) \log \frac{1}{|z|} + \mathcal{O}\left(\frac{1}{z}\right), \quad (|z| \to \infty) \tag{2.3}$$

where $\mu(\mathbb{C})$ is the total mass of μ. If $U^\mu(z)$ is smooth enough the density of the measure μ can be recovered from this potential by taking the Laplacian of $U^\mu(z)$:

Theorem 2.2 ([13], II.1.3). *If in a region $R \subseteq \mathbb{C}$ the logarithmic potential $U^\mu(z)$ of the measure μ has continuous second partial derivatives, then μ is absolutely continuous with respect to the planar Lebesgue measure m in R and we have the formula*

$$d\mu = -\frac{1}{2\pi} \Delta U^\mu dm. \tag{2.4}$$

Now, let K be a compact subset of $\mathbb{C}$ and let $\mathcal{M}(K)$ denote the set of all Borel probability measures supported on K. In classical potential theory, the *logarithmic energy* of a measure $\mu \in \mathcal{M}(K)$ is defined to be

$$I(\mu) := \int_K U^\mu(z) d\mu(z) = \int_K \int_K \log \frac{1}{|z-t|} d\mu(t) d\mu(z). \tag{2.5}$$

The quantity

$$E_K := \inf_{\mu \in \mathcal{M}(K)} I(\mu) \tag{2.6}$$

is either finite or $+\infty$. The *logarithmic capacity* of K is

$$\operatorname{cap}(K) := e^{-E_K}. \tag{2.7}$$

If $E_K < \infty$ then, by a well-known theorem of Frostman (see e.g. [12]), there exists a unique measure μ_K in $\mathcal{M}(K)$ minimizing the energy functional $I(\cdot)$ and this measure is called the *equilibrium measure* of K. The capacity of an arbitrary Borel set $B \subset \mathbb{C}$ is defined as

$$\operatorname{cap}(B) := \sup\left\{\operatorname{cap}(K) \mid K \subseteq B, K \text{ compact}\right\}.$$

A property is said to hold *quasi-everywhere* if the set of exceptional points (i.e. those where it does not hold) is of capacity zero.

In the more general setting we have a closed set $\Sigma \subseteq \mathbb{C}$ and a function $w\colon \Sigma \to [0, \infty)$ on Σ called the *weight function*. Usually the weight function is given in the form

$$w(z) = \exp\big(-Q(z)\big), \tag{2.8}$$

where $Q\colon \Sigma \to (-\infty, \infty]$. In the electrostatical interpretation, the set Σ is called the *conductor* and Q is called the *background potential*.

Definition 2.3 ([13]). The weight function w is said to be *admissible* if

- w is upper semi-continuous,
- $\{z \in \Sigma \mid w(z) > 0\}$ has nonzero capacity,
- $\lim_{|z|\to\infty} |z|w(z) = 0$.

The admissibility conditions can be rephrased in terms of the potential Q; $w(z) = \exp(-Q(z))$ is admissible if and only if Q is lower semi-continuous, the set $\{z \in \Sigma \mid Q(z) < \infty\}$ has nonzero capacity and $\lim_{|z|\to\infty}(Q(z) - \log|z|) = \infty$.

Let $\mathcal{M}(\Sigma)$ denote the set of all Borel probability measures supported on $\Sigma \subseteq \mathbb{C}$. The *weighted energy functional* I_Q is defined for all $\mu \in \mathcal{M}(\Sigma)$ by

$$I_Q(\mu) := \int_\Sigma\int_\Sigma \log\left[|z-t|w(z)w(t)\right]^{-1} d\mu(z)d\mu(t) \tag{2.9}$$

$$= \int_\Sigma\int_\Sigma \log\frac{1}{|z-t|} d\mu(z)d\mu(t) + 2\int_\Sigma Q(z)d\mu(z). \tag{2.10}$$

The goal is then to find a probability measure that minimizes this functional on $\mathcal{M}(\Sigma)$. If Q is admissible it can be shown (see [13], Theorem I.1.3) that

$$E_Q := \inf_{\mu\in\mathcal{M}(\Sigma)} I_Q(\mu) \tag{2.11}$$

is finite and there exists a unique measure, denoted by μ_Q, that has finite logarithmic energy and minimizes I_Q. Moreover, the support of μ_Q, denoted by S_Q, is compact and has positive capacity. The measure μ_Q is called the *equilibrium measure* of the background potential Q. The logarithmic potential satisfies the equilibrium conditions

$$\begin{aligned} U^{\mu_Q}(z) + Q(z) &\geq F_Q \quad \text{quasi-everywhere on } \Sigma, \\ U^{\mu_Q}(z) + Q(z) &\leq F_Q \quad \text{for all} \quad z \in S_Q, \end{aligned} \tag{2.12}$$

where F_Q is the *modified Robin constant*:

$$F_Q = E_Q - \int Q d\mu_Q \,. \tag{2.13}$$

Motivated by random normal matrix models (see [17,18]), we are interested in background potentials of the following type:

$$V_{\alpha,\nu}(z) := \alpha|z|^2 + U^\nu(z)\,, \tag{2.14}$$

where α is a positive real number and ν is a compactly supported finite positive Borel measure. These potentials have a planar Gaussian leading term controlled by the positive parameter α and this is perturbed by a fixed positive charge distribution given by the measure ν.

Proposition 2.4. *The potential $V_{\alpha,\nu}(z)$ is admissible for all possible choices of α and ν.*

Proof. $V_{\alpha,\nu}(z)$ is lower semi-continuous because $U^\nu(z)$ is superharmonic in the whole complex plane. The set where $V_{\alpha,\nu}(z)$ is finite contains at least $\mathbb{C}\setminus\operatorname{supp}(\nu)$, which is of positive capacity since $\operatorname{supp}(\nu)$ is compact. Finally, the required boundary condition is also fulfilled:

$$V_{\alpha,\nu}(z) - \log|z| = \alpha|z|^2 + U^\nu(z) - \log|z| \tag{2.15}$$

$$= \alpha|z|^2 - \big(\nu(\mathbb{C}) + 1\big)\log|z| + \mathcal{O}\left(\frac{1}{z}\right), \tag{2.16}$$

so the difference $V_{\alpha,\nu}(z) - \log|z|$ goes to $+\infty$ as $|z| \to \infty$. □

We are especially interested in cases for which the perturbing measure ν is singular with respect to the planar Lebesgue measure m. In particular, ν can be chosen to be a positive linear combination of point masses, i.e.

$$\nu := \sum_{k=1}^{m} \beta_k \delta_{a_k}\,, \quad \beta_k \in \mathbb{R}^+ \tag{2.17}$$

where $a_1, a_2, \dots, a_m \in \mathbb{C}$ are the locations and $\beta_1, \beta_2, \dots, \beta_m$ are the charges of the fixed point masses.

3. Supports of equilibrium measures and quadrature domains

The determination of the support of the equilibrium measure for a background potential $V(z) = V_{\alpha,\nu}(z)$ of the form (2.14) above is closely related to finding *generalized quadrature domains* of some measures in the complex plane. Let us recall the definition of quadrature domains given by Sakai [14].

Definition 3.1. Let ν be a positive Borel measure on the complex plane. For a nonempty open, connected domain Ω in $\mathbb{C}$ let $F(\Omega)$ be a subset of the space

$$\operatorname{Re}L^1(\Omega) := \{\operatorname{Re} f \mid f \in L^1(\Omega, m)\}\,. \tag{3.1}$$

of real-valued integrable functions on Ω.

The domain Ω is called a *(generalized) quadrature domain* of the measure ν for the function class $F(\Omega)$ if

(a) ν is concentrated on Ω, i.e. $\nu(\Omega^c) = 0$,

(b)
$$\int_\Omega f^+ d\nu < \infty \quad \text{and} \quad \int_\Omega f d\nu \le \int_\Omega f dm \tag{3.2}$$
for every $f \in F(\Omega)$ where $f^+ := \max\{f, 0\}$.

Note that if $F(\Omega)$ is a function class such that $-f \in F(\Omega)$ whenever $f \in F(\Omega)$ then the second condition is equivalent to

$$\int_\Omega |f| d\nu < \infty \quad \text{and} \quad \int_\Omega f d\nu = \int_\Omega f dm \tag{3.3}$$

for every $f \in F(\Omega)$. We are interested in the following subclasses:

$$\mathrm{Re}AL^1(\Omega) = \{\mathrm{Re} f \in L^1(\Omega, m) \mid f \text{ is holomorphic in } \Omega\} \tag{3.4}$$
$$HL^1(\Omega) = \{h \in L^1(\Omega, m) \mid h \text{ is harmonic in } \Omega\} \tag{3.5}$$
$$SL^1(\Omega) = \{s \in L^1(\Omega, m) \mid s \text{ is subharmonic in } \Omega\}. \tag{3.6}$$

For a measure ν the quadrature domains corresponding to these classes are called generalized classical (holomorphic), harmonic and subharmonic quadrature domains, respectively. We have the obvious inclusions

$$Q(\nu, SL^1) \subseteq Q(\nu, HL^1) \subseteq Q(\nu, \mathrm{Re}AL^1), \tag{3.7}$$

where $Q(\nu, F)$ denotes the set of quadrature domains of ν for the function class F. It is important to note that if the domain Ω belongs to $Q(\nu, F)$ then its *saturated set* or *areal maximal set*

$$[\Omega] := \left\{z \in \mathbb{C} \mid m(B(z,r) \cap \Omega^c) = 0 \text{ for some } r > 0\right\} \tag{3.8}$$

also belongs to $Q(\nu, F)$.

For example, it can be shown that the disk $B(c, R)$ is the only classical generalized quadrature domain for the point measure $\nu = R^2\pi\delta_c$ (see [14], Example 1.1). The simplest examples are the *classical quadrature domains* whose quadrature measure is a positive linear combination of point masses:

$$\nu = \sum_{k=1}^{m} \beta_k \delta_{a_k}, \quad \beta_k \in \mathbb{R}^+. \tag{3.9}$$

This means that for every holomorphic function f that is integrable on Ω we have the identity

$$\int_\Omega f dm = \sum_{k=1}^{m} \beta_k f(a_k). \tag{3.10}$$

An immediate generalization is obtained by allowing points a_k of higher multiplicities $j_k \geq 1$ in the above sum, this means allowing derivatives of finite order to appear in the sum representing the area integral functional:

$$\int_\Omega f dm = \sum_{k=1}^{m} \sum_{l=0}^{j_k} \beta_{k,l} f^{(l)}(a_k)\,. \tag{3.11}$$

However, in this work we do not consider such quadrature domains.

It is easy to see that if Ω is a subharmonic quadrature domain of the measure ν then, using subharmonic test functions of the form

$$s_z(w) = -\log \frac{1}{|z-w|} \quad (z \in \mathbb{C})\,, \tag{3.12}$$

we have

$$\begin{aligned} U^\Omega(z) &\leq U^\nu(z) \quad \text{if} \quad z \in \mathbb{C}\,, \\ U^\Omega(z) &= U^\nu(z) \quad \text{if} \quad z \in \mathbb{C} \setminus \Omega\,. \end{aligned} \tag{3.13}$$

To illustrate the structure of the equilibrium measure of a potential of the form (2.14) above, we consider a simple but nontrivial example:

$$V(z) = \alpha |z|^2 + \beta \log \frac{1}{|z-a|}\,, \tag{3.14}$$

where $\alpha \in \mathbb{R}^+, \beta \in \mathbb{R}^+$ and $a \in \mathbb{C}$. The calculation of the equilibrium measure for this potential is quite standard (see, for example, [17, 18]) but the details will be of importance in suggesting generalizations. For the sake of completeness, the statement of the result and a short sketch of its proof are therefore included here.

To find the equilibrium measure for this potential one can use the following characterization theorem:

Theorem 3.2 ([13], I.3.3). *Let $Q\colon \Sigma \to (-\infty, \infty]$ be an admissible background potential. If a measure $\sigma \in \mathcal{M}(\Sigma)$ has compact support and finite logarithmic energy, and there is a constant $F \in \mathbb{R}$ such that*

$$U^\sigma(z) + Q(z) = F \qquad \textit{quasi-everywhere on } supp(\sigma) \tag{3.15}$$

and

$$U^\sigma(z) + Q(z) \geq F \qquad \textit{quasi-everywhere on } \Sigma\,, \tag{3.16}$$

then σ coincides with the equilibrium measure μ_Q.

The logarithmic potential of the uniform measure $\eta_{B(c,R)}$ on a disk $B(c,R)$ is easily calculated to be

$$U^{B(c,R)}(z) = \begin{cases} \frac{1}{2} R^2 \pi \left(\log \frac{1}{R^2} + 1 - \frac{|z-c|^2}{R^2} \right) & |z-c| \leq R \\ R^2 \pi \log \frac{1}{|z-c|} & |z-c| > R\,. \end{cases} \tag{3.17}$$

Proposition 3.3. *Define two radii R and r as*

$$R := \sqrt{\frac{1+\beta}{2\alpha}} \quad \text{and} \quad r := \sqrt{\frac{\beta}{2\alpha}}. \tag{3.18}$$

The equilibrium measure μ_V is absolutely continuous with respect to the Lebesgue measure and its density is the constant $\frac{2\alpha}{\pi}$. The support S_V of μ_V depends on the geometric arrangement of the disks $B(a,r)$ and $B(0,R)$ in the following way:

(a) *If $B(a,r) \subset B(0,R)$ then*

$$S_V = \overline{B(0,R)} \setminus B(a,r) \tag{3.19}$$

(see (a.1) and (a.2) in Figure 3.1).

(b) *If $B(a,r) \not\subset B(0,R)$ then $\hat{\mathbb{C}} \setminus S_V$ is given by a rational exterior conformal mapping of the form*

$$f : \hat{\mathbb{C}} \setminus \{\zeta \,:\, |\zeta| \leq 1\} \to \hat{\mathbb{C}} \setminus S_V\,, \quad f(\zeta) = \rho\zeta + u + \frac{v}{\zeta - A}\,, \tag{3.20}$$

where the coefficients $\rho \in \mathbb{R}^+, 0 < |A| < 1$ and $u, v \in \mathbb{C}$ of the mapping $f(\zeta)$ are uniquely determined by the parameters α, β and a of the potential $V(z)$ (see (b.1) and (b.2) in Figure 3.1).

Proof. Suppose first that $B(a,r) \subset B(0,R)$.

Let σ be the measure given by

$$d\sigma := \frac{1}{m(K)} I_K dm \tag{3.21}$$

where $K = \overline{B(0,R)} \setminus B(a,r)$. The area of K is

$$m(K) = (R^2 - r^2)\pi = \frac{\pi}{2\alpha}\,. \tag{3.22}$$

Therefore the logarithmic potential of σ is

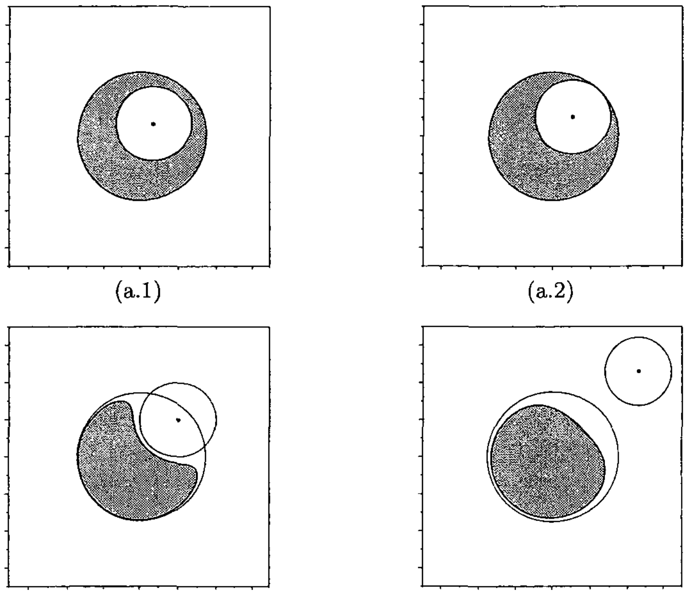
$$U^\sigma(z) = \frac{2\alpha}{\pi}\left(U^{B(0,R)}(z) - U^{B(a,r)}(z)\right). \tag{3.23}$$

Now, for $z \in K$ the effective potential at z is

$$\begin{aligned} U^\sigma(z) + V(z) &= \alpha R^2\left(\log\frac{1}{R^2} + 1 - \frac{|z|^2}{R^2}\right) - 2\alpha r^2 \log\frac{1}{|z-a|} \\ &\quad + \alpha|z|^2 + \beta\log\frac{1}{|z-a|} \\ &= \alpha R^2\left(\log\frac{1}{R^2} + 1\right). \end{aligned} \tag{3.24}$$

Define

$$F := \alpha R^2\left(\log\frac{1}{R^2} + 1\right). \tag{3.25}$$



(a.1) (a.2) (b.1) (b.2)

FIGURE 3.1. The shape of the support S_V (shaded area) is illustrated for the different disk configurations (a) and (b) in Proposition 3.3.

If $z \notin K$ a short calculation gives

$$U^\sigma(z) + V(z) = \begin{cases} F + 2\alpha r^2 f\left(\frac{|z-a|}{r}\right) & z \in B(a,r) \\ F + 2\alpha R^2 f\left(\frac{|z|}{R}\right) & |z| > R\,. \end{cases} \tag{3.26}$$

where

$$f(x) := \frac{x^2-1}{2} - \log x\,. \tag{3.27}$$

Since f is nonnegative the effective potential satisfies

$$U^\sigma(z) + V(z) \geq F\,. \tag{3.28}$$

By Theorem 3.2, we conclude that the equilibrium measure for the background potential V is σ.

Now suppose that $B(a,r) \not\subset B(0,R)$. For this case, we only sketch the calculation giving the system of equations that relate the parameters of the potential $V(z)$ and the conformal map $f(\zeta)$. The potential is smooth in the domain $\mathbb{C} \setminus \{a\}$

and its Laplacian there is constant which suggests, by Theorem 2.2, that the density of the equilibrium measure is equal to

$$\frac{1}{2\pi}\Delta V(z) = \frac{2\alpha}{\pi}. \tag{3.29}$$

We expect the equilibrium measure therefore to be the normalized Lebesgue measure restricted to some compact set $S \subset \mathbb{C} \setminus \{a\}$; that is,

$$d\mu_V = \frac{1}{m(S)} dm, \tag{3.30}$$

where $m(S)$ is given by (3.22). The first equilibrium condition (3.15) is

$$U^{\mu_V}(z) + V(z) = F \quad (z \in S) \tag{3.31}$$

for some constant F. Assuming the necessary smoothness of $U^{\mu_V}(z)$ and applying the differential operator ∂_z, this gives the necessary condition

$$-\frac{\alpha}{\pi}\int_S \frac{dm(w)}{z-w} + \left(\alpha\bar{z} - \frac{\beta}{2}\frac{1}{z-a}\right) = 0 \quad (z \in S), \tag{3.32}$$

which is the same as

$$\frac{\alpha}{2\pi i}\int_S \frac{d\bar{w}\wedge dw}{z-w} - \alpha\bar{z} + \frac{\beta}{2}\frac{1}{z-a} = 0 \quad (z \in S). \tag{3.33}$$

By Green's Theorem we get

$$\frac{1}{2\pi i}\int_{\partial S} \frac{\bar{w}dw}{z-w} = -\frac{\beta}{2\alpha}\frac{1}{z-a} \quad (z \in S). \tag{3.34}$$

Following [18], we seek to express $\mathbb{C} \setminus S$ as the image of the exterior of the unit disk under a conformal mapping of the form

$$f\colon \hat{\mathbb{C}} \setminus \{\zeta : |\zeta| \le 1\} \to \hat{\mathbb{C}} \setminus S, \quad f(\zeta) = \rho\zeta + u + \frac{v}{\zeta - A} \tag{3.35}$$

where $\rho > 0$ and $0 < |A| < 1$. By rewriting the equilibrium condition eq. (3.32), we obtain

$$\frac{1}{2\pi i}\int_{|\zeta|=1} \frac{\overline{f(\zeta)}f'(\zeta)d\zeta}{z - f(\zeta)} = -\frac{\beta}{2\alpha}\frac{1}{z-a} \quad (z \in S). \tag{3.36}$$

Along the positively oriented simple circular contour $|\zeta| = 1$ we have

$$\overline{f(\zeta)} = \rho\frac{1}{\zeta} + \bar{u} + \frac{\bar{v}\zeta}{1-\bar{A}\zeta}. \tag{3.37}$$

Therefore the rational function

$$T(\zeta; z) := \frac{\left(\rho\frac{1}{\zeta} + \bar{u} + \frac{\bar{v}\zeta}{1-\bar{A}\zeta}\right)\left(\rho - \frac{v}{(\zeta-A)^2}\right)}{z - \rho\zeta - u - \frac{v}{\zeta-A}} \tag{3.38}$$

must satisfy the equation

$$\frac{1}{2\pi i}\int_{|\zeta|=1} T(\zeta; z)d\zeta = -\frac{\beta}{2\alpha}\frac{1}{z-a} \quad (z \in S) \tag{3.39}$$

and the parameters $\rho; u; v; A$ must be chosen to satisfy the area normalization condition (3.22).

The differential $T(\zeta; z)d\zeta$ has four fixed poles in ζ at $\zeta = 0, \infty, A, \frac{1}{A}$ and two other poles depending on z via the equation $z = f(\zeta)$. (These extra poles may coincide with some of the fixed poles above.) This equation can be rewritten as a quadratic equation in ζ:

$$\rho\zeta^2 + (u - z - A\rho)\zeta + A(z - u) + v = 0\,. \tag{3.40}$$

If $z \in S$ both solutions of this equation are *inside* the unit disk $\{\zeta \mid |\zeta| < 1\}$. To calculate the integral of $T(\zeta; z)d\zeta$ in terms of residues, we write the contour integral in the standard local coordinate $\xi = \frac{1}{\zeta}$ around $\zeta = \infty$:

$$\frac{1}{2\pi i}\int_{|\zeta|=1} T(\zeta; z)d\zeta = \frac{1}{2\pi i}\int_{|\xi|=1} T\left(\frac{1}{\xi}; z\right)\frac{d\xi}{\xi^2}\,. \tag{3.41}$$

The two simple poles inside the disk $\{\xi \mid |\xi| < 1\}$ are $\xi = 0$ and $\xi = \overline{A}$ and hence

$$\frac{1}{2\pi i}\int_{|\xi|=1} T\left(\frac{1}{\xi}; z\right)\frac{d\xi}{\xi^2} = \mathrm{Res}_{\xi=0}T\left(\frac{1}{\xi}; z\right)\frac{1}{\xi^2} + \mathrm{Res}_{\xi=\overline{A}}T\left(\frac{1}{\xi}; z\right)\frac{1}{\xi^2}\,. \tag{3.42}$$

Since

$$T\left(\frac{1}{\xi}; z\right)\frac{1}{\xi^2} = \frac{1}{\xi}\frac{(\rho\xi + \overline{u} + \frac{\overline{v}}{\xi - \overline{A}})(\rho - \frac{v\xi^2}{(1-A\xi)^2})(1 - A\xi)}{(1 - A\xi)(z\xi - u\xi - \rho) - v\xi^2}\,, \tag{3.43}$$

the residues are

$$\mathrm{Res}_{\xi=0}T\left(\frac{1}{\xi}; z\right)\frac{1}{\xi^2} = \frac{\overline{v}}{\overline{A}} - \overline{u}\,, \tag{3.44}$$

$$\mathrm{Res}_{\xi=\overline{A}}T\left(\frac{1}{\xi}; z\right)\frac{1}{\xi^2} = \frac{\overline{v}}{\overline{A}}\frac{(\rho - \frac{v\overline{A}^2}{(1-|A|^2)^2})(1 - |A|^2)}{(1 - |A|^2)(z\overline{A} - u\overline{A} - \rho) - v\overline{A}^2}\,. \tag{3.45}$$

A short calculation gives the area of S in terms of the mapping parameters:

$$m(S) = \pi\left(\rho^2 - \frac{|v|^2}{(1 - |A|^2)^2}\right). \tag{3.46}$$

Finally we obtain the following system of equations:

$$\begin{aligned} \rho^2 - \frac{|v|^2}{(1 - |A|^2)^2} &= \frac{1}{2\alpha} \\ \frac{\overline{v}}{\overline{A}} - \overline{u} &= 0 \\ u + \frac{\rho}{\overline{A}} + \frac{v\overline{A}}{1 - |A|^2} &= a \\ \frac{\overline{v}}{\overline{A}^2}\left(\rho - \frac{v\overline{A}^2}{(1 - |A|^2)^2}\right) &= -\frac{\beta}{2\alpha}\,. \end{aligned} \tag{3.47}$$

We must prove that if we assume $|a| + r > R$ there exists a unique solution ρ, u, v, A to this system in terms of the parameters α, β, a. Eliminating u from the third equation gives

$$a = \frac{\rho}{A} + \frac{v}{A(1 - |A|^2)} . \tag{3.48}$$

The last equation of (3.47) shows that $\frac{v}{A^2}$ is real. The phases of v and A are therefore fixed by (3.48). Writing a in polar form

$$a = te^{i\phi} \tag{3.49}$$

(with $t > 0$ because $B(a, r) \not\subset B(0, R)$), we obtain

$$v = se^{2i\phi} , \quad A = Ke^{i\phi} , \tag{3.50}$$

where s and K are positive real numbers. We can express ρ and s in terms of K:

$$\rho = \frac{1}{2Kt}\left(K^2t^2 + \frac{1}{2\alpha}\right) \tag{3.51}$$

$$s = \frac{1-K^2}{2Kt}\left(K^2t^2 - \frac{1}{2\alpha}\right) . \tag{3.52}$$

Setting $x = K^2$, this must be a solution of the the cubic equation

$$2t^4x^3 - \left(t^4 + \frac{1+2\beta}{\alpha}t^2\right)x^2 + \frac{1}{4\alpha^2} = 0 . \tag{3.53}$$

The condition $|a| + r > R$ means that

$$t^4 - \frac{1+2\beta}{\alpha}t^2 + \frac{1}{4\alpha^2} < 0 . \tag{3.54}$$

Defining the function

$$g(x) := 2t^4x^3 - \left(t^4 + \frac{1+2\beta}{\alpha}t^2\right)x^2 + \frac{1}{4\alpha^2} , \tag{3.55}$$

we have

$$g(0) = \frac{1}{4\alpha^2} > 0 \quad \text{and} \quad g(1) = t^4 - \frac{1+2\beta}{\alpha}t^2 + \frac{1}{4\alpha^2} < 0 \tag{3.56}$$

by (3.54). Since

$$\begin{aligned} g'(x) &= 6t^4x^2 - 2\left(t^4 + \frac{1+2\beta}{\alpha}t^2\right)x \\ &= 6t^4x\left(x - \frac{1}{3t^4}\left(t^4 + \frac{1+2\beta}{\alpha}t^2\right)\right) \end{aligned} \tag{3.57}$$

is negative in the interval $[0, 1]$, $g(x)$ has a unique root in $(0, 1)$, and therefore K is uniquely determined by (3.53). This means that there is a unique solution for ρ, u, v and A of (3.47) in terms of α, β, a.

To conclude the proof one should show that the logarithmic potential $\frac{2\alpha}{\pi}U^S(z)$ satisfies the inequality (3.16) of Theorem 3.2. This part of the proof is omitted. □

The electrostatic interpretation of case (a) in Proposition 3.3 is simple. If we replace the point charge $\beta\delta_a$ by a uniform charge distribution of density $\frac{2\alpha}{\pi}$ on $B(a,r)$, the resulting configuration is in equilibrium in the presence of the pure Gaussian potential $\alpha|z|^2$. The disk $B(a,r)$ is a quadrature domain for the measure $r^2\pi\delta_a$, so

$$\frac{2\alpha}{\pi}U^{B(a,r)}(z) = \beta U^{\delta_a}(z) \quad \text{in} \quad z \in B(a,r)^c, \tag{3.58}$$

which means that the electric fields of $\frac{2\alpha}{\pi}\eta_{B(a,r)}$ and $\beta\delta_a$ are indistinguishable in the exterior of $B(a,r)$. A quadrature domain shaped cavity emerges in the support of the equilibrium measure of the unperturbed Gaussian potential since the fixed perturbing measure substitutes the portion of uniform charge placed in the cavity of the original equilibrium configuration, as illustrated in (a.1) and (a.2) of Figure 3.1. A useful generalization of this idea turns out to be valid in a more general setting:

Theorem 3.4. *Let*

$$V(z) := \alpha|z|^2 + U^\nu(z) \tag{3.59}$$

be a background potential. Assume that the measure ν can be decomposed into a sum

$$\nu = \sum_{k=1}^{m} \nu_k \tag{3.60}$$

where the measures ν_k are all finite positive Borel measures satisfying the following conditions:

(a) *The supports of the measures ν_k are pairwise disjoint and each ν_k has positive total mass.*
(b) *The measure $\frac{\pi}{2\alpha}\nu_k$ has an essentially unique (i.e. unique up to sets of measure zero) subharmonic quadrature domain, and D_k denotes the saturated element of $Q(\nu_k, SL^1)$ for all $k = 1, 2, \dots, m$ respectively.*
(c) *The domains D_k are pairwise disjoint and $D_k \subset B(0,R)$ for all $k = 1, 2, \dots, m$ where*

$$R := \sqrt{\frac{1+\nu(\mathbb{C})}{2\alpha}}. \tag{3.61}$$

Then the equilibrium measure μ_V is absolutely continuous with respect to the Lebesgue measure with constant density $\frac{2\alpha}{\pi}$ and is supported on the set

$$K := \overline{B(0,R)} \setminus \left(\bigcup_{k=1}^{m} D_k\right). \tag{3.62}$$

(The situation is illustrated for a simple configuration of point and line charges in Figure 3.2.)

Proof. Let σ be the measure given by

$$d\sigma := \frac{1}{m(K)} I_K dm. \tag{3.63}$$

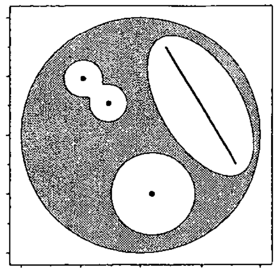


FIGURE 3.2. A typical configuration involving subharmonic quadrature domains: a disk, a so-called *bicircular quartic* (see [16]) and an ellipse.

To calculate the area of K, we note that the area of D_k is given by

$$m(D_k) = \int_{D_k} dm = \frac{\pi}{2\alpha}\int d\nu_k = \frac{\pi}{2\alpha}\nu_k(\mathbb{C})\,. \tag{3.64}$$

Therefore

$$m(K) = m\big(B(0,R)\big) - \sum_{k=1}^{m} m(D_k) \tag{3.65}$$

$$= \frac{\pi}{2\alpha}\left(1 + \nu(\mathbb{C}) - \sum_{k=1}^{m}\nu_k(\mathbb{C})\right) \tag{3.66}$$

$$= \frac{\pi}{2\alpha}\,. \tag{3.67}$$

For each measure ν_k the corresponding logarithmic potential $U^{\nu_k}(z)$ satisfies

$$\frac{2\alpha}{\pi}U^{D_k}(z) \le U^{\nu_k}(z) \quad \text{if} \quad z \in \mathbb{C}\,, \tag{3.68}$$

$$\frac{2\alpha}{\pi}U^{D_k}(z) = U^{\nu_k}(z) \quad \text{if} \quad z \in \mathbb{C}\setminus D_k\,. \tag{3.69}$$

The logarithmic potential of σ is

$$U^{\sigma}(z) = \frac{2\alpha}{\pi}\left(U^{B(0,R)}(z) - \sum_{k=1}^{m} U^{D_k}(z)\right). \tag{3.70}$$

Now, if $z \in K$ then the effective potential at z is

$$\begin{aligned} U^\sigma(z) + V(z) &= \alpha R^2 \left(\log \frac{1}{R^2} + 1 - \frac{|z|^2}{R^2}\right) - \frac{2\alpha}{\pi} \sum_{k=1}^{m} U^{D_k}(z) \\ &\quad + \alpha|z|^2 + U^\nu(z) \\ &= \alpha R^2 \left(\log \frac{1}{R^2} + 1\right) - \sum_{k=1}^{m} U^{\nu_k}(z) + U^\nu(z) \\ &= \alpha R^2 \left(\log \frac{1}{R^2} + 1\right). \end{aligned} \tag{3.71}$$

Define $F := \alpha R^2 \left(\log \frac{1}{R^2} + 1\right)$. If $z \notin K$ then either $z \in D_k$ for some k or $|z| > R$. If $z \in D_k$ then we have the inequality

$$\begin{aligned} U^\sigma(z) + V(z) &= \alpha R^2 \left(\log \frac{1}{R^2} + 1 - \frac{|z|^2}{R^2}\right) - \frac{2\alpha}{\pi} \sum_{k=1}^{m} U^{D_k}(z) \\ &\quad + \alpha|z|^2 + U^\nu(z) \\ &= F + U^{\nu_k}(z) - \frac{2\alpha}{\pi} U^{D_k}(z) \\ &\geq F. \end{aligned} \tag{3.72}$$

On the other hand, if $|z| > R$ then

$$\begin{aligned} U^\sigma(z) + V(z) &= 2\alpha R^2 \log \frac{1}{|z|} - \frac{2\alpha}{\pi} \sum_{k=1}^{m} U^{D_k}(z) \\ &\quad + \alpha|z|^2 + U^\nu(z) \\ &= \alpha|z|^2 + \alpha R^2 \log \frac{1}{|z|^2} \\ &= F + 2\alpha R^2 f\left(\frac{|z|}{R}\right) \\ &\geq F, \end{aligned} \tag{3.73}$$

where $f(x) = \frac{x^2-1}{2} - \log x$. Since f is nonnegative the effective potential satisfies

$$U^\sigma(z) + V(z) \geq F. \tag{3.74}$$

By Theorem 3.2, we conclude that the equilibrium measure for the background potential V is σ. □

The conclusion of Theorem 3.4 does not hold if some of the domains D_k overlap or intersect the exterior of $B(0, R)$. In the first case we have to find a new decomposition of the perturbing measure and the corresponding domains; in the second the outer boundary no longer coincides with the boundary of $B(0, R)$ as we saw in Proposition 3.3. It is hard to give a complete description of the support of the equilibrium measure in the general case. If ν is a finite linear combination of point masses then the methods of Crowdy and Marshall in [3] used in the fluid dynamical context of rotating vortex patches are applicable to recovering the corresponding supports.

In the cases considered above, the support S_V of the equilibrium measure is contained in the closed disk $\overline{B(0,R)}$. It seems plausible that the same is true for all perturbing measures ν. The following theorem states that $S_V \subset \overline{B(0,R)}$ is valid if ν is a positive rational linear combination of point masses.

Theorem 3.5. *Let $V(z)$ be a potential of the form*

$$V(z) = \alpha|z|^2 + U^\nu(z) \tag{3.75}$$

where ν is a measure of the form

$$\nu = \sum_{k=1}^{m} r_k \delta_{a_k} \tag{3.76}$$

where $r_1, r_2, \ldots, r_m$ are positive rational numbers. *Then the support S_V of the corresponding equilibrium measure is contained in the closed disk $\overline{B(0,R)}$ where*

$$R = \sqrt{\frac{1+\nu(\mathbb{C})}{2\alpha}}\,. \tag{3.77}$$

Proof. For a continuous weight $w(z) = \exp(-V(z))$ in the complex plane, $z \in \mathbb{C}$ belongs to the support S_V if and only if for every neighborhood B of z there exists a *weighted polynomial* $w^n P_n$ of degree $\deg P_n \leq n$, such that $w^n P_n$ attains its maximum modulus only in B (see [13], Corollary IV.1.4). Since our $w(z)$ is continuous this characterization is applicable to this setting.

Let $z \in S_V$ and suppose B is a neighborhood of z. Then there exists a polynomial of degree at most n for some $n \in \mathbb{N}$ such that $P_n(z)w^n(z)$ attains its maximum modulus only in B. Let q be the least common denominator of the rational numbers $r_1, r_2, \ldots, r_n$ such that

$$r_k = \frac{p_k}{q} \tag{3.78}$$

where $p_k \in \mathbb{N}$ for all $k = 1, 2, \ldots, m$. Then

$$\begin{aligned}\left(|P_n(z)|w^n(z)\right)^q &= |P_n(z)^q| \left|\prod_{k=1}^{m}(z-a_k)^{np_k}\right| e^{-nq\alpha|z|^2}\,.\\ &= \left|P_n(z)^q \prod_{k=1}^{m}(z-a_k)^{np_k}\right| \exp\left(-n(q+L)\frac{q\alpha}{q+L}|z|^2\right),\end{aligned} \tag{3.79}$$

where

$$L = \sum_{k=1}^{m} p_k\,. \tag{3.80}$$

Since all the p_k's are assumed to be positive integers,

$$Q_{n(q+L)}(z) := P_n(z)^q \prod_{k=1}^{m} (z - a_k)^{np_k} \tag{3.81}$$

is a polynomial of degree at most $n(q + L)$. If we consider the modified weight $v(z) = \exp\left(- \frac{q\alpha}{q+L} |z|^2 \right)$ the corresponding weighted polynomial

$$Q_{n(q+L)}(z) v^{n(q+L)}(z) \tag{3.82}$$

attains its maximum modulus only in B. Therefore z belongs to the support of the equilibrium measure of the weight $v(z)$ which is exactly $\overline{B(0,R)}$, where

$$R = \sqrt{\frac{q+L}{2q\alpha}} = \sqrt{\frac{1+\nu(\mathbb{C})}{2\alpha}}\,. \tag{3.83}$$

This proves that $S_V \subseteq \overline{B(0,R)}$. □

4. Orthogonal polynomials

In random normal matrix models, the correlation functions are expressed in terms of planar orthogonal polynomials with respect to *scaled weight functions* of the form $\exp(-NQ(z))$ associated to a potential $Q(z)$ where $N > 0$ is a scaling parameter (N has the same role as $\frac{1}{\hbar}$). For our special potentials of the form $V(z) = V_{\alpha,\nu}(z)$ defined in (2.14) above we have the weights

$$e^{-NV(z)} = \exp\left(-N \left[\alpha|z|^2 + \int \log \frac{1}{|z-w|} d\nu(w) \right] \right), \tag{4.1}$$

where $N > 0$ is the scaling parameter.

Proposition 4.1. *We have*

$$e^{-NV(z)} \in L^1(\mathbb{C}, dm) \cap L^\infty(\mathbb{C}, dm) \tag{4.2}$$

for all choices of the parameters N, α, ν. Moreover, the absolute moments

$$\int_{\mathbb{C}} |z|^k e^{-NV(z)} dm(z) \tag{4.3}$$

are all finite for $k = 0, 1, \ldots$

Proof. The exponent in the weight can be decomposed as

$$N \left[\alpha|z|^2 + \int \log \frac{1}{|z-w|} d\nu(w) \right] = \frac{N\alpha}{2}|z|^2 + N \left[\frac{\alpha}{2}|z|^2 + \int \log \frac{1}{|z-w|} d\nu(w) \right], \tag{4.4}$$

in which the second term is lower semicontinuous and satisfies

$$N \left[\frac{\alpha}{2}|z|^2 + \int \log \frac{1}{|z-w|} d\nu(w) \right] = N \left[\frac{\alpha}{2}|z|^2 + \nu(\mathbb{C}) \log \frac{1}{|z|} \right] + \mathcal{O}\left(\frac{1}{z} \right), \tag{4.5}$$

as $|z| \to \infty$. This means that the expression is bounded from below in $\mathbb{C}$:

$$N\left[\frac{\alpha}{2}|z|^2 + \int \log\frac{1}{|z-w|}d\nu(w)\right] \geq L \tag{4.6}$$

for some constant $L \in \mathbb{R}$ depending on the parameters N, α and on the measure ν. So

$$0 \leq e^{-NV(z)} \leq e^{-\frac{N\alpha}{2}|z|^2}e^{-L}\,, \tag{4.7}$$

which implies that

$$e^{-NV(z)} \in L^1(\mathbb{C}, dm) \cap L^\infty(\mathbb{C}, dm)\,. \tag{4.8}$$

Moreover,

$$\int_{\mathbb{C}} |z|^k e^{-NV(z)}dm(z) \leq e^{-L}\int_{\mathbb{C}} |z|^k e^{-\frac{N\alpha}{2}|z|^2}dm(z) < \infty \tag{4.9}$$

for all $k = 0, 1, \ldots$ □

It follows from this and the positivity of the weight that the *monic orthogonal polynomials*

$$P_{n,N}(z) = z^n + \mathcal{O}\left(z^{n-1}\right) \quad (n = 0, 1, \ldots) \tag{4.10}$$

satisfying

$$\int_{\mathbb{C}} P_{k,N}(z)\overline{P_{l,N}(z)}e^{-NV(z)}dm(z) = h_{k,N}\delta_{kl}\,, \quad k, l = 0, 1, \ldots \tag{4.11}$$

exist and are unique where $h_{n,N}$ denotes the square of the L^2-norm of $P_{n,N}(z)$.

5. Matrix $\overline{\partial}$-problem for orthogonal polynomials

In this section we show that the 2×2 matrix $\overline{\partial}$-problem for orthogonal polynomials introduced by Its and Takhtajan [10] in the case of measures supported within a finite radius (cut-off exponentials of polynomial potentials) is also well-defined for the class of potentials considered above and determines the polynomials uniquely. In [10] the same family of potentials is considered as in [5].

To be able to formulate the $\overline{\partial}$-problem, we need some estimates of Cauchy transforms of measures with unbounded supports. For a given potential $V(z) = V_{\alpha.\nu}(z)$ of the form (2.14) considered above let λ be the measure, absolutely continuous with respect to the Lebesgue measure in $\mathbb{C}$, having the form

$$d\lambda = e^{-NV(z)}dm\,, \tag{5.1}$$

where and $N > 0$. Note that $\lambda(\mathbb{C})$ is finite because $e^{-NV(z)} \in L^1(\mathbb{C}, dm)$.

The Cauchy transform of λ is defined to be

$$[C\lambda](z) := \int_{\mathbb{C}} \frac{d\lambda(w)}{z-w}\,. \tag{5.2}$$

We need to control the asymptotic behaviour of $[C\lambda](z)$ for such measures at infinity allowing the possibility that $[C\lambda](z)$ is not holomorphic in any neighborhood of $z = \infty$.

First of all, it follows from Proposition 4.1 that the density is bounded from above: $e^{-NV(z)} < K$ for some $K \in \mathbb{R}$. For a fixed positive radius R, we have

$$\begin{aligned}\int_{\mathbb{C}} \frac{d\lambda(w)}{|z-w|} &\leq \int_{|z-w|>R} \frac{d\lambda(w)}{R} + \int_{|z-w|\leq R} \frac{K\,dm(w)}{|z-w|} \\ &\leq \frac{1}{R}\int_{\mathbb{C}} d\lambda(w) + \int_0^{2\pi}\int_0^R \frac{K}{r} r\,dr\,d\theta \\ &= \frac{1}{R}\lambda(\mathbb{C}) + 2\pi RK \end{aligned} \tag{5.3}$$

for all $z \in \mathbb{C}$. Thus, there exists an upper bound H_λ of $[C\lambda](z)$ depending only on N, α, ν and independent of z (One can get rid of R in the last expression e.g. by minimizing the bound in R):

$$\int_{\mathbb{C}} \frac{d\lambda(w)}{|z-w|} < H_\lambda \quad (z \in \mathbb{C}). \tag{5.4}$$

Now

$$\begin{aligned}[C\lambda](z) - \frac{\lambda(\mathbb{C})}{z} &= \int_{\mathbb{C}} \left(\frac{1}{z-w} - \frac{1}{z}\right) d\lambda(w) \\ &= \frac{1}{z^2}\int_{\mathbb{C}} \left(\frac{wz}{z-w}\right) d\lambda(w) \\ &= \frac{1}{z^2}\int_{\mathbb{C}} \left(w + \frac{w^2}{z-w}\right) d\lambda(w). \end{aligned} \tag{5.5}$$

Hence the absolute value of the difference satisfies

$$\begin{aligned}\left| z^2 \left[[C\lambda](z) - \frac{\lambda(\mathbb{C})}{z} \right] \right| &\leq \int_{\mathbb{C}} |w| d\lambda(w) + \int_{\mathbb{C}} \frac{1}{|z-w|} |w|^2 d\lambda(w) \\ &= \int_{\mathbb{C}} d\hat{\lambda}(w) + \int_{\mathbb{C}} \frac{d\tilde{\lambda}(w)}{|z-w|} \leq \hat{\lambda}(\mathbb{C}) + H_{\tilde{\lambda}}, \end{aligned} \tag{5.6}$$

where $\hat{\lambda}$ and $\tilde{\lambda}$ correspond to the measures

$$\hat{\nu} := \nu + \frac{1}{N}\delta_0 \quad \tilde{\nu} := \nu + \frac{2}{N}\delta_0, \tag{5.7}$$

respectively. This means that

$$[C\lambda](z) = \frac{\lambda(\mathbb{C})}{z} + \mathcal{O}\left(\frac{1}{z^2}\right). \tag{5.8}$$

If $P_{n,N}(z)$ denotes the nth monic orthogonal polynomial with respect to the measure λ, the modified measure

$$d\lambda_n(z) := |P_{n,N}(z)|^2 d\lambda(z) \tag{5.9}$$

corresponds to the perturbing measure

$$\nu_n := \nu + \frac{2}{N} \sum_{k=1}^{n} \delta_{a_k^{(n,N)}} \tag{5.10}$$

where $\{a_1^{(n,N)}, a_2^{(n,N)}, \dots, a_n^{(n,N)}\}$ are the zeroes of $P_{n,N}(z)$ and $h_{n,N} = \lambda_n(\mathbb{C})$. An easy calculation gives

$$\begin{aligned} \frac{1}{z-w} &= \frac{1}{P_{n,N}(z)} \frac{P_{n,N}(z) - P_{n,N}(w)}{z-w} + \frac{1}{P_{n,N}(z)} \frac{P_{n,N}(w)}{z-w} \\ &= \frac{1}{P_{n,N}(z)} Q_{n-1}(z,w) + \frac{1}{P_{n,N}(z)} \frac{P_{n,N}(w)}{z-w} \end{aligned} \tag{5.11}$$

where $Q_n(z,w)$ is a symmetric polynomial in z and w of degree $n-1$ with leading order z^{n-1} in the variable z. Therefore, by orthogonality, we get

$$\begin{aligned} \int_{\mathbb{C}} \frac{\overline{P_{n,N}(w)}}{z-w} d\lambda(w) &= \frac{1}{P_{n,N}(z)} \int_{\mathbb{C}} \frac{|P_{n,N}(w)|^2}{z-w} d\lambda(w) \\ &= \frac{1}{P_{n,N}(z)} \int_{\mathbb{C}} \frac{d\lambda_n(w)}{z-w} \\ &= \frac{1}{z^n} \frac{z^n}{P_{n,N}(z)} \left[\frac{h_{n,N}}{z} + \mathcal{O}\left(\frac{1}{z^2}\right) \right] \\ &= \frac{h_{n,N}}{z^{n+1}} + \mathcal{O}\left(\frac{1}{z^{n+2}}\right). \end{aligned} \tag{5.12}$$

Following the approach of Its and Takhtajan in [10], we consider the following 2×2 matrix-valued function in the complex plane:

$$Y_{k,N}(z) := \begin{bmatrix} P_{k,N}(z) & \frac{1}{\pi} \int_{\mathbb{C}} \frac{\overline{P_{k,N}(w)}}{w-z} e^{-NV(w)} dm(w) \\ -\frac{\pi}{h_{k-1,N}} P_{k-1,N}(z) & -\frac{1}{h_{k-1,N}} \int_{\mathbb{C}} \frac{\overline{P_{k-1,N}(w)}}{w-z} e^{-NV(w)} dm(w) \end{bmatrix}. \tag{5.13}$$

The $\overline{\partial}$-derivative is

$$\frac{\partial}{\partial \overline{z}} Y_{k,N}(z) = \begin{bmatrix} 0 & -\overline{P_{k,N}(z)} e^{-NV(z)} \\ 0 & \frac{\pi}{h_{k-1,N}} \overline{P_{k-1,N}(z)} e^{-NV(z)} \end{bmatrix} = \overline{Y_{k,N}(z)} \begin{bmatrix} 0 & -e^{-NV(z)} \\ 0 & 0 \end{bmatrix}. \tag{5.14}$$

Using the asymptotic behaviour of the Cauchy transforms as $|z| \to \infty$ proven above, we have that

$$\begin{aligned}
&\begin{bmatrix} P_{k,N}(z) & \frac{1}{\pi}\int_{\mathbb{C}} \frac{\overline{P_{k,N}(w)}}{w-z} e^{-NV(w)} dm(w) \\ -\frac{\pi}{h_{k-1,N}} P_{k-1,N}(z) & -\frac{1}{h_{k-1,N}} \int_{\mathbb{C}} \frac{\overline{P_{k-1,N}(w)}}{w-z} e^{-NV(w)} dm(w) \end{bmatrix} \begin{bmatrix} z^{-k} & 0 \\ 0 & z^k \end{bmatrix} \\
&= \begin{bmatrix} \frac{P_{k,N}(z)}{z^k} & \frac{z^k}{\pi}\int_{\mathbb{C}} \frac{\overline{P_{k,N}(w)}}{w-z} e^{-NV(w)} dm(w) \\ -\frac{\pi}{h_{k-1,N}} \frac{P_{k-1,N}(z)}{z^k} & -\frac{z^k}{h_{k-1,N}} \int_{\mathbb{C}} \frac{\overline{P_{k-1,N}(w)}}{w-z} e^{-NV(w)} dm(w) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \mathcal{O}\left(\frac{1}{z}\right).
\end{aligned} \tag{5.15}$$

So $Y_{k,N}(z)$ is a solution of the following 2×2 matrix $\overline{\partial}$-problem:

$$\begin{cases} \dfrac{\partial}{\partial \overline{z}} M(z) = \overline{M(z)} \begin{bmatrix} 0 & -e^{-NV(z)} \\ 0 & 0 \end{bmatrix} & (z \in \mathbb{C}) \\ M(z) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \mathcal{O}\left(\frac{1}{z}\right) \right) \begin{bmatrix} z^k & 0 \\ 0 & z^{-k} \end{bmatrix} & (|z| \to \infty). \end{cases} \tag{5.16}$$

The important point made in [10] is that the $\overline{\partial}$-problem in that setting has a unique solution and therefore it characterizes the matrix $Y_{k,N}(z)$ and the corresponding orthogonal polynomials. Although we cannot assume that the relevant Cauchy transform entries are holomorphic around $z = \infty$, we nevertheless can prove that the solution is unique in this case as well.

Proposition 5.1. *The matrix $Y_{k,N}(z)$ is the unique solution of the $\overline{\partial}$-problem* (5.16).

Proof. We have seen that $Y_{k,N}$ solves the $\overline{\partial}$-problem (5.16). Conversely, assume that the matrix $M(z)$ has continuous entries with continuous partial derivatives and $M(z)$ solves (5.16) with the prescribed asymptotic conditions. Then $M_{11}(z)$ and $M_{21}(z)$ are entire functions with asymptotic forms for large z

$$M_{11}(z) = z^k + \mathcal{O}\left(z^{k-1}\right), \tag{5.17}$$

$$M_{21}(z) = \mathcal{O}\left(z^{k-1}\right) \quad |z| \to \infty. \tag{5.18}$$

Hence $M_{11}(z)$ is a monic polynomial of degree k and $M_{21}(z)$ is a polynomial of degree at most $k-1$. The $\overline{\partial}$-equation in (5.16) can be written in terms of the entries of $M(z)$ as

$$\frac{\partial}{\partial \bar{z}} M_{12}(z) = -\overline{M_{11}(z)} e^{-NV(z)}\,, \tag{5.19}$$

$$\frac{\partial}{\partial \bar{z}} M_{22}(z) = -\overline{M_{21}(z)} e^{-NV(z)}\,. \tag{5.20}$$

Taking into account the fact that $M_{12}(z) \to 0$ and $M_{22}(z) \to 0$ as $|z| \to \infty$, this implies

$$M_{12}(z) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{\overline{M_{11}(w)}}{w - z} e^{-NV(w)} dm(w)\,, \tag{5.21}$$

$$M_{22}(z) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{\overline{M_{21}(w)}}{w - z} e^{-NV(w)} dm(w) \tag{5.22}$$

(see [2]). Using the expansion

$$\begin{aligned} \frac{1}{z-w} &= \frac{1}{z^k} \frac{z^k - w^k}{z - w} + \frac{1}{z^k} \frac{w^k}{z - w} \\ &= \sum_{l=0}^{k-1} \frac{1}{z^{l+1}} w^l + \frac{1}{z^k} \frac{w^k}{z - w}\,, \end{aligned} \tag{5.23}$$

we get

$$\begin{aligned} M_{12}(z) &= \frac{1}{\pi} \int_{\mathbb{C}} \frac{\overline{M_{11}(w)}}{w - z} e^{-NV(w)} dm(w) \\ &= \sum_{l=0}^{k-1} \frac{1}{z^{l+1}} \frac{1}{\pi} \int_{\mathbb{C}} w^l \overline{M_{11}(w)} e^{-NV(w)} dm(w) \\ &\quad + \frac{1}{z^k} \frac{1}{\pi} \int_{\mathbb{C}} \frac{w^k \overline{M_{11}(w)}}{w - z} e^{-NV(w)} dm(w)\,. \end{aligned} \tag{5.24}$$

The prescribed asymptotic behaviour

$$M_{12}(z) = \mathcal{O}\left(\frac{1}{z^{k+1}}\right) \quad \text{as} \quad |z| \to \infty \tag{5.25}$$

implies the following equations:

$$\int_{\mathbb{C}} w^l \overline{M_{11}(w)} e^{-NV(w)} dm(w) = 0\,, \quad l = 0, 1, \ldots, k-1\,.$$

Hence $M_{11}(z) = P_{k,N}(z)$, because $M_{11}(z)$ is a monic polynomial of degree k. Similarly for $M_{21}(z)$ we have

$$\begin{aligned} M_{22}(z) &= \frac{1}{\pi}\int_{\mathbb{C}} \frac{\overline{M_{21}(w)}}{w-z} e^{-NV(w)} dm(w) \\ &= \sum_{l=0}^{k-1} \frac{1}{z^{l+1}} \frac{1}{\pi} \int_{\mathbb{C}} w^l \overline{M_{21}(w)} e^{-NV(w)} dm(w) \\ &+ \frac{1}{z^k}\frac{1}{\pi}\int_{\mathbb{C}} \frac{w^k \overline{M_{21}(w)}}{w-z} e^{-NV(w)} dm(w)\,, \end{aligned} \tag{5.26}$$

and

$$M_{22}(z) = \frac{1}{z^k} + \mathcal{O}\left(\frac{1}{z^{k+1}}\right) \quad \text{as} \quad |z| \to \infty \tag{5.27}$$

implies

$$\int_{\mathbb{C}} w^l \overline{M_{21}(w)} e^{-NV(w)} dm(w) = 0 \quad l = 0, 1, \ldots, k-2\,, \tag{5.28}$$

and

$$\int_{\mathbb{C}} w^{k-1} \overline{M_{21}(w)} e^{-NV(w)} dm(w) = 1\,. \tag{5.29}$$

Now, if $M_{21}(z) = az^{k-1} + \mathcal{O}\left(z^{k-2}\right)$, where $a \in \mathbb{C}$, then

$$\int_{\mathbb{C}} |M_{21}(w)|^2 e^{-NV(w)} dm(w) = a \int_{\mathbb{C}} w^{k-1} \overline{M_{21}(w)} e^{-NV(w)} dm(w) = a\,. \tag{5.30}$$

Clearly $a \neq 0$ (because otherwise $M_{21}(z)$ vanishes and hence also $M_{22}(z)$ would be zero which is impossible). So $M_{21}(z)$ is a polynomial of degree $k-1$, and from (5.28) we have

$$M_{21}(z) = aP_{k-1,N}(z)\,. \tag{5.31}$$

By the asymptotic relation (5.12),

$$\begin{aligned} M_{22}(z) &= \frac{1}{\pi}\int_{\mathbb{C}} \frac{\overline{M_{21}(w)}}{w-z} e^{-NV(w)} dm(w) \\ &= -\frac{a}{\pi}\int_{\mathbb{C}} \frac{\overline{P_{k-1,N}(w)}}{z-w} e^{-NV(w)} dm(w) \\ &= -\frac{ah_{k-1,N}}{\pi}\frac{1}{z^k} + \mathcal{O}\left(\frac{1}{z^{k+1}}\right)\,, \end{aligned} \tag{5.32}$$

which forces the constant a to equal $-\frac{\pi}{h_{k-1,N}}$, and hence

$$M_{21}(z) = -\frac{\pi}{h_{k-1,N}} P_{k-1,N}(z)\,, \tag{5.33}$$

which completes the proof. □

6. Zeros of orthogonal polynomials and quadrature domains

In this final section we briefly discuss some known and conjectured relations between the asymptotics of orthogonal polynomials in the plane, equilibrium measures of the type studied in Sections 1 – 3 and generalized quadrature domains. These concern relations between the asymptotics of the zeros of orthogonal polynomials and the associated equilibrium measures that have previously been studied by Elbau [5, 6] for a certain class measures with bounded support. Their validity for the class of measures considered here is supported by numerical calculations.

To relate orthogonal polynomials with the support of the equilibrium measure we have, for finite values of n, three measures which, in the case of random Hermitian matrices are known to approach the same equilibrium measure on the real line in the scaling limit (cf. (1.3))

$$n \to \infty\,, \quad N \to \infty\,, \quad \frac{N}{n} \to \gamma = \frac{1}{t}\,, \quad \hbar := \frac{1}{N} \in \mathbb{R}^+\,. \tag{6.1}$$

All the limiting relations below are understood in this scaled sense. We introduce the notation

$$Q(z) := \frac{\gamma}{2} V(z) \tag{6.2}$$

for the rescaled potential corresponding to the scaling parameter γ. Then for a large class of real potentials and $z \in \mathbf{R}$, all three of the following measures converge weakly to the equilibrium measure $d\mu_Q(z)$ of $Q(z)$:

1) The normalized counting measure of the zeros

$$\mathcal{Z}_{n,N} := \{z_1^{(n,N)}, z_2^{(n,N)}, \dots, z_n^{(n,N)}\} \tag{6.3}$$

of the orthogonal polynomials $P_{n,N}(z)$ with respect to the weight $\exp(-NV(z))$

$$\nu_{n,N} := \frac{1}{n} \sum_{z \in \mathcal{Z}_{n,N}} \delta_z\,, \tag{6.4}$$

$$\nu_{n,N} \xrightarrow{w*} \mu_Q \quad (n \to \infty)\,. \tag{6.5}$$

2) The expected density of eigenvalues (or *one-point function*) of random Hermitian matrices

$$\rho_{n,N}(z) = \frac{1}{n} \sum_{k=0}^{n-1} |p_{k,N}(z)|^2 e^{-NV(z)}\,, \tag{6.6}$$

derived from the probability density

$$\frac{1}{Z_{n,N}} \exp\Big(-N\mathrm{Tr}\big(V(H)\big)\Big) dH\,, \tag{6.7}$$

$$Z_{n,N} := \int \exp\Big(-N\mathrm{Tr}\big(V(H)\big)\Big) dH \tag{6.8}$$

$$\rho_{n,N}(z)dz \xrightarrow{w*} d\mu_Q(z) \quad (n \to \infty)\,. \tag{6.9}$$

3) The normalized counting measure of equilibrium point configurations

$$\mathcal{F}_n^Q := \{z_1^Q, \dots, z_n^Q\} \tag{6.10}$$

of the two-dimensional Coulomb energy

$$\mathcal{E}_n(z_1, \dots z_N) = \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \log \frac{1}{|z_i - z_j|} + \sum_{i=1}^{N} Q(z_i) \tag{6.11}$$

(which in the plane become the so-called *weighted Fekete points*)

$$\eta_n = \frac{1}{n} \sum_{z \in \mathcal{F}_n^Q} \delta_z \tag{6.12}$$

$$\eta_n \xrightarrow{w*} \mu_Q \quad (n \to \infty)\,. \tag{6.13}$$

For random normal matrices the eigenvalues are not confined to the real axis. In this case it is known [1, 6, 9] that in the scaling limit (6.1), the analogs of $\rho_{n,N}(z)dz$ and η_n also approach the equilibrium measure μ_Q.

It is also known [6], for cut-off measures of the form

$$e^{-NV(z)} I_{\mathcal{D}}(z)\,, \quad V(z) = -\alpha|z|^2 + P_{\mathrm{harm}}(z)\,, \tag{6.14}$$

where $I_{\mathcal{D}}$ is the indicator function of a compact domain $\mathcal{D}$ containing the origin whose boundary curve $\partial\mathcal{D}$ is twice continuously differentiable and P_{harm} is a harmonic polynomial, that

$$\lim_{N \to \infty} \frac{1}{n} \log \frac{1}{|P_{n,N}(z)|} = \int \log \frac{1}{|z - \zeta|} d\mu_Q(\zeta)\,, \quad z \in \mathbb{C} \setminus \mathrm{supp}(\mu_Q)\,. \tag{6.15}$$

That is, the limit of the zeros acts effectively as an equivalent source of the external Coulomb potential.

For such potentials, the boundary $\partial(\mathrm{supp}(\mu_Q))$ of the support of the equilibrium measure is determined through the Riemann mapping theorem as the image of the unit circle under a rational conformal map, whose inverse therefore has a finite number of branch points. The Schwarz function $S(z)$, defined along the boundary, determines the curve via the equation

$$\bar{z} = S(z)\,. \tag{6.16}$$

It has a unique analytic continuation to the interior on the complement of any tree $\mathcal{C}^{\mathrm{tree}}$ whose vertices include the branch points, with edges formed from curve segments. It is shown in [6] that, assuming there is a condensation limit $\mathcal{C}^Q$ for the orthogonal polynomial zeros supported on a tree-like graph $\mathcal{C}^{\mathrm{tree}}$ whose edges are curve segments, $\mathcal{C}^{\mathrm{tree}}$ may be chosen so that $\mathcal{C}^Q \subset \mathcal{C}^{\mathrm{tree}}$. Moreover, denoting by $\delta S(z)$ the jump discontinuity of $S(z)$ away from the nodes, $\mathcal{C}^{\mathrm{tree}}$ may be chosen as an integral curve of the direction field annihilated by the real part $\mathrm{Re}[\delta S(z)dz]$ of the differential $(\delta S(z))dz$; i.e. such that the tangents X to the curve segments forming the edges satisfy

$$\mathrm{Re}\big[\delta S(z)dz\big](X) = 0\,. \tag{6.17}$$

We refer to such integral curves as *critical trajectories.*

Based on computational evidence, and general results known for other cases [15], there is good reason to believe that the same result holds for the class of superharmonic perturbed Gaussian measures studied in Sections 1–3, without the

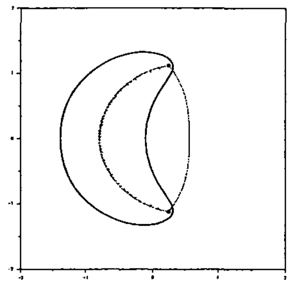


FIGURE 6.1. Zeroes of $P_{n,N}(z)$ for $n = 50$ and the critical trajectories.

need for introducing the cutoff factor $I_{\mathcal{D}}$. This statement, for some suitable restrictions on the permissible superharmonic perturbations, forms the first part of the conjectured relation between the zeros of the orthogonal polynomials considered in Sections 4–5 and the equilibrium measure μ_Q.

The second part gives a more detailed relation; namely, the effective density $\kappa_Q(z)$ along C_Q of the condensed orthogonal polynomial zeros is given, within a suitable scaling constant by

$$d\kappa_Q(z) \sim \frac{1}{2\pi i}(\delta S(z))dz = \frac{1}{2\pi}\mathrm{Im}\Big[(\delta S(z))dz\Big]. \tag{6.18}$$

Explicitly, this means that the external potential due to a uniform, normalized charge supported in $\mathrm{supp}(\mu_Q)$ is

$$\int_{C_Q} \log\frac{1}{|z-\zeta|}d\kappa_Q(\zeta) = \int \log\frac{1}{|z-\zeta|}d\mu_Q(\zeta). \tag{6.19}$$

To support the validity of these conjectures, we take the case of the potential

$$V(z) = \alpha|z|^2 + \beta\log\frac{1}{|z-a|}, \tag{6.20}$$

in the simply connected case considered in Section 3 above and compare the locus of the zeros of the corresponding orthogonal polynomial $P_{n,N}(z)$ with the two different integral curves of the direction field defined by (6.17) joining the branch points (Figure 6.1). (The two other critical trajectories emanating from the branch points are omitted from the graph.)

We also compare, in Figure 6:2, the value of the logarithmic potential $-\frac{1}{n}\log|P_{n,N}(z)|$ created by the normalized counting measure $\nu_{n,N}$ of the zeroes of $P_{n,N}(z)$ with $n = 30, N = 2n = 60$ and the external potential as given by (6.19) in the external region $z \in \mathbb{C}\setminus\mathrm{supp}(\mu_V)$.

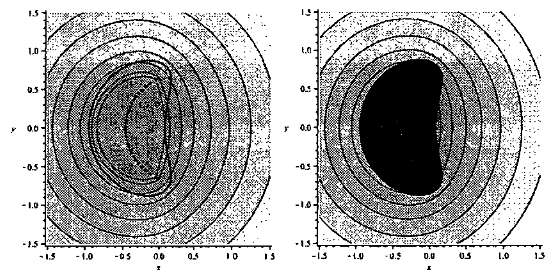


FIGURE 6.2. The figure on the left is a contour plot of the potential $-\frac{1}{n}\log P_{n,N}(z)|$ arising from equal charges at the roots of the orthogonal polynomial $P_{n,N}(z)$; the one on the right is the potential $\int \log \frac{1}{|z-\zeta|} d\mu_Q(\zeta)$ on the exterior region due to a normalized uniform charge on $\mathrm{supp}(\mu_Q)$.

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Chapter 9

Regularity of a vector potential problem and its spectral curve

9.1 Summary

This chapter is based on the published article [9]. The paper is organized as follows:

1. A generalized setting of the vector potential problem (see 4.7) is introduced (Sec. 2).

2. The existence and uniqueness of a vector equilibrium measure is established (Sec. 3).

3. A special case of the vector potential problem is considered and the regularity properties of the components of the corresponding vector equilibrium measure is discussed (Sec 4.)

4. It is shown that the resolvents (Cauchy transforms) of the components of the vector equilibrium measure satisfy a pseudo-algebraic equation, i.e., there is a *spectral curve* associated to the problem (Sec. 5).

5. The pseudo-algebraic curve is calculated explicitly for a special case involving two symmetric semiclassical potentials (Sec. 6).

9.1.1 The generalized vector potential problem

We recall the setting of a vector potential problem (see Sec. 4.7)

$$(\{\Sigma_k\}_{k=1}^n, \{V_k\}_{k=1}^n, A) \tag{9.1}$$

consisting of a finite collection of conductors Σ_k, the corresponding potentials

$$V_k \colon \Sigma_k \to (-\infty, \infty] \ , \tag{9.2}$$

and the interaction matrix A that describes the strength and the sign (meaning either attraction or repulsion) of the coupling between the measures on the different supports.

Consider the following (electrostatic) variational problem corresponding to this vector potential setting:

$$\begin{cases} I_{A,\vec{V}}(\vec{\mu}) := \displaystyle\sum_{k,l}^{n} a_{kl} \int_{\Sigma_k}\int_{\Sigma_l} \log \frac{1}{|z-t|} d\mu_k(z) d\mu_l(t) + 2\sum_{k=1}^{n} \int_{\Sigma} V_k(z) d\mu_k(z) \to \min. \\ \mu_k(\Sigma_k) \in \mathcal{M}(\Sigma_k) \qquad k = 1, \ldots, n \ . \end{cases} \tag{9.3}$$

(Recall that $\mathcal{M}(\Sigma)$ means the set of probability measures on Σ).

It is necessary to find sufficient conditions on the ingredients of the vector potential problem to ensure the existence and uniqueness of a vector equilibrium measure corresponding to this coupled system.

9.1.2 Existence and uniqueness of the vector equilibrium measure

As it was indicated in Sec. 4.7, the admissibility conditions discussed in [81] are sufficient but they require that the conductors be pairwise disjoint; this constraint turns out to be too restrictive, keeping in mind the mathematical applications targeted in the paper (to be described below).

Assuming the admissibility conditions introduced in Sec. 4.7 we recall the

Theorem 9.1.1 (4.7.1, Thm. 3.2, [9]) *For an admissible system*

$$(\{\Sigma_k\}_{k=1}^n, \{V_k\}_{k=1}^n, A) \tag{9.4}$$

the following statements hold:

- *The extremal value*

$$\mathcal{I}_{A,\vec{V}} := \inf_{\vec{\mu}} I_{A,\vec{V}}(\vec{\mu}) \tag{9.5}$$

  of the functional $I_{A,\vec{V}}(\cdot)$ is finite and there exists a unique vector measure $\vec{\mu}^\star$ such that

$$\mathcal{I}_{A,\vec{V}} := I_{A,\vec{V}}(\vec{\mu}^\star) \ . \tag{9.6}$$

- *The components of $\vec{\mu}^\star$ have finite logarithmic energy and compact support. Moreover, the potential V_k and the logarithmic potential $U^{\mu_k^\star}$ is bounded on the support of $\mu_k^\star$ for all $k = 1, \ldots, n$.*

- *For $k = 1, \ldots, n$ the effective potential satisfies the variational inequalities*

$$U_k^{\text{eff}}(z) := U^{\mu_k^\star}(z) + V_k(z) \geq F_k \tag{9.7}$$

  for some real constant F_k, and equality holds quasi-everywhere on the support of μ_k.

9.1.3 Nearest-neighbor interactions

The aim of the paper [9] that follows is to lay down the necessary background to the nonlinear steepest descent analysis of orthogonal polynomials related to a certain

Hermitian multi-matrix model of the form

$$\mathrm{d}\mu(M_1,\ldots,M_n) = \frac{1}{\mathcal{Z}_N}\frac{\prod_{j=1}^{n} e^{-V_k(M_k)}\mathrm{d}M_k}{\prod_{j=1}^{n-1}\det(M_k+M_{K+1})^N} \tag{9.8}$$

where all of the matrices M_k are assumed to be positive definite. It is not in the scope of the thesis to describe these matrix models in detail; we refer to the works [18, 17] instead for a complete description of the motivation and the setup of the matrix chain problem. Therefore we consider the special *nearest-neighbor* interaction matrix

$$A := \underbrace{\begin{bmatrix} 2 & -1 & 0 & \ldots & 0 \\ -1 & 2 & -1 & \ldots & 0 \\ 0 & -1 & 2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 2 \end{bmatrix}}_{n}, \tag{9.9}$$

also known as a Nikishin-type interaction matrix [96]. The specific matrix model in question (after a suitable procedure involving the reflection $x \mapsto -x$ on the eigenvalues on each matrix with odd index) requires the measures to be supported on the sets

$$\Sigma_k := \begin{cases} [0,\infty) & k \text{ odd} \\ (-\infty,0] & k \text{ even} \end{cases} \tag{9.10}$$

for $k = 1,\ldots,n$.

Apart from the admissibility conditions on the potentials $V_k(x)$, we assume that the derivative $V_k'(x)$ of each potential on Σ_k is the restriction of a real analytic function defined in a neighborhood of the real axis possessing at most polar singularities on $\mathbb{R}\setminus\Sigma_k$.

For the vector equilibrium measure $\vec{\mu}^\star$ corresponding to the above problem we note that the individual variational problems

$$I_{\widehat{V}_k}(\sigma) := \int_{\Sigma_k}\int_{\Sigma_k}\log\frac{1}{|z-t|}d\sigma(z)d\sigma(t) + 2\int_{\Sigma_k}\widehat{V}_k(z)\mathrm{d}\sigma(z) \tag{9.11}$$

corresponding to fixing all components of $\mu^\star$ but $\mu_k^\star$, are minimized by $\sigma = \mu_k^\star$, where the effective potentials $\widehat{V}_k(z)$ are given by

$$
\begin{aligned}
\widehat{V}_1(z) &:= \frac{1}{2}V_1(z) - \frac{1}{2}\int_{\Sigma_2} \log\frac{1}{|z-t|}d\mu_2^\star(t) \\
\widehat{V}_k(z) &:= \frac{1}{2}V_k(z) - \frac{1}{2}\int_{\Sigma_{k+1}} \log\frac{1}{|z-t|}d\mu_{k+1}^\star(t) - \frac{1}{2}\int_{\Sigma_{k-1}} \log\frac{1}{|z-t|}d\mu_{k-1}^\star(t) \\
& \qquad k = 2, \dots, n-1 \\
\widehat{V}_n(z) &:= \frac{1}{2}V_n(z) - \frac{1}{2}\int_{\Sigma_{n-1}} \log\frac{1}{|z-t|}d\mu_{n-1}^\star(t) \,.
\end{aligned}
\tag{9.12}
$$

Therefore, by applying Thm. 4.6.1 to this standard weighted problem for a single measure, we have that the components $\mu_k^\star$ are absolutely continuous with respect to the Lebesgue measure with densities ψ_k at least Hölder–$\frac{1}{2}$ continuous and the supports of these equilibrium measures have a positive distance from the origin. Since $x = 0$ is the only common point of the nearest neighbor conductors because of the odd-even alternation of the supports Σ_k and the tridiagonal structure of A, we have that

$$
\operatorname{supp}(\psi_k) \cap \operatorname{supp}(\psi_{k+1}) = \emptyset \qquad k = 1, \dots, n-1 \,. \tag{9.13}
$$

As a consequence of the variational equations for $\mu_k^\star$, the Sokhotski-Plemelj identities imply that the following variational equations

$$
\left.\begin{aligned}
(W_k)_+(x) + (W_k)_+(x) &= V_k'(x) + W_{k+1}(x) + W_{k-1}(x) \\
(W_k)_+(x) - (W_k)_-(x) &= -2i\pi\psi_k(x)
\end{aligned}\right. \qquad x \in \operatorname{supp}(\psi_k) \tag{9.14}
$$

are satisfied for $k = 1, \dots, n$ for the resolvent functions (see (4.64))

$$
W_k(z) := \int_{\Sigma_k} \frac{\psi_k(x)dx}{z-x} \qquad z \in \mathbb{C} \setminus \operatorname{supp}(\psi_k) \,. \tag{9.15}
$$

and the convention $W_0 \equiv W_{n+1} \equiv 0$ in the notation is used.

Since the support of ψ_k is disjoint from the supports of $\psi_{k\pm1}$, the resolvents $W_{k\pm1}$ have no jumps on $\mathrm{supp}(\psi_k)$.

9.1.4 The construction of the spectral curve

Consider the vector functions

$$\vec{W}(z) := \begin{bmatrix} W_1(z) \\ W_2(z) \\ \vdots \\ W_n(z) \end{bmatrix}, \qquad \vec{V}'(z) := \begin{bmatrix} V_1'(z) \\ V_2'(z) \\ \vdots \\ V_n'(z) \end{bmatrix} \tag{9.16}$$

and construct the vector function

$$\vec{Z}(z) := M(A^{-1}\vec{V}'(z) + \vec{W}(z)) \tag{9.17}$$

where M is the $(n+1) \times n$ matrix

$$M := \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & -1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{9.18}$$

The key result is the following:

Theorem 9.1.2 (Prop. 5.1, [9]) *The power-sum symmetric polynomials*

$$\sum_{k=1}^{n} Z_k^m(z) \qquad m = 0, 1, \ldots \tag{9.19}$$

of the components $Z_k(z)$ of the vector function $\vec{Z}(z)$ are real analytic in a common domain of analyticity of the potentials, namely, they have no discontinuities on the supports of the densities ψ_k. Therefore the elementary symmetric polynomials of the Z_k's have no jumps on the supports of the densities ψ_k.

This immediately implies that

Theorem 9.1.3 (Thm. 5.1, [9]) *The functions $Z_k(z)$ are all solutions of a pseudo-algebraic equation of the form*

$$E(w, z) = w^{n+1} + C_2(z)w^{n-1} + \cdots + C_{n+1}(z) = 0 \ , \tag{9.20}$$

i.e., the coefficients $C_2(x), \ldots, C_{n+1}(x)$ are real analytic functions on the common domain of analyticity of the potentials.

The set of endpoints of the jumps of $Z_k(z)$ is contained in the zero set of the discriminant of $E(w, z)$ as a function of z. Since this discriminant cannot have an infinite number of zeroes on a compact set, the discontinuities of the functions occur along a finite union of compact intervals. Since the sets of points of discontinuity for $\vec{Y}(z)$ and $\vec{Z}(z)$ are the same, we conclude that

Theorem 9.1.4 (Cor, 5.1, [9]) *The density $\psi_k(x)$ is supported on a finite union of compact intervals in Σ_k for all $k = 1, \ldots, n$.*

The practical importance of this theorem is that it allows us to find the supports and the densities *explicitly* for admissible potential problems, as illustrated by the example given in Sec. 6 of [9], motivated by a simple choice of potentials corresponding to a matrix model of the type (9.8) for $n = 2$.

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Regularity of a vector potential problem and its spectral curve

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Abstract

In this note we study a minimization problem for a vector of measures subject to a prescribed interaction matrix in the presence of external potentials. The conductors are allowed to have zero distance from each other but the external potentials satisfy a growth condition near the common points.

We then specialize the setting to a specific problem on the real line which arises in the study of certain biorthogonal polynomials (studied elsewhere) and we prove that the equilibrium measures solve a pseudo-algebraic curve under the assumption that the potentials are real analytic. In particular, the supports of the equilibrium measures are shown to consist of a finite union of compact intervals.

1. Introduction

In this short paper we consider a vector potential problem of relevance in the study of the asymptotic behavior of multiple orthogonal polynomials for the so-called Nikishin systems [15]. The original problem was introduced in [8] (without external fields) and further questions have been addressed in [9,16,10,1,11]. The main motivation of our interest for this problem arises in

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a recently introduced set of biorthogonal polynomials [2]. These polynomials are related on one side to the spectral theory of the "cubic string" and the Degasperis–Procesi peakon solutions of the homonymous nonlinear differential equation [5]; on the other end they are related to a two-matrix model [3] with a measure of the form

$$d\mu(M_1, M_2) = \frac{1}{\mathcal{Z}_N} dM_1 dM_2 \frac{\alpha(M_1)\beta(M_2)}{\det(M_1 + M_2)^N} \tag{1.1}$$

where the M_j's are positive-definite Hermitian matrices of size $N \times N$, α, β are some positive densities on $\mathbb{R}_+$ and the expressions $\alpha(M_1)$, $\beta(M_2)$ stand for the product of those densities on the spectra of M_j.

The relation between the relevant biorthogonal polynomials and the above-mentioned matrix model is on the identical logical footing as the relation between ordinary orthogonal polynomials and the Hermitian random matrix model [14].

In [2] a Riemann–Hilbert formulation (similar to the formulation of multiple-orthogonal polynomials as explained in [21] but adapted to the peculiarities of the model) was derived and in [3] the correlation functions of the spectra of the two matrices were completely characterized in terms of the matrix solution of that Riemann–Hilbert problem.

In [4] the analysis of the strong asymptotics with respect to varying weight (following [7]) will be carried out. A pre-requisite of that analysis is the existence and regularity of the solution of a suitable potential problem, namely the one which we explain in the second part of the paper.

In fact, the present paper is addressing a wider class of potential problems that will be necessary for the study of the spectral statistics in the limit of large sizes of the *multi-matrix model*

$$d\mu(M_1, \dots, M_R) = \frac{1}{\mathcal{Z}_N} \frac{\prod_{j=1}^{R} \alpha_j(M_j) dM_j}{\prod_{j=1}^{R-1} \det(M_j + M_{j+1})^N} \tag{1.2}$$

corresponding to a *chain* of positive-definite Hermitian matrices M_j with densities α_j as above.

In Section 2 we consider the problem as a vector potential problem in the complex plane with a prescribed interaction matrix. Under a suitable growth condition for the external potentials $V_j(z)$ near the overlap region of the conductors (in particular the common points on the boundaries) it is shown that the minimizing vector of equilibrium measures has supports for the components separated by positive distances.

In Section 4 we specialize the setting to the situation in which the conductors $\Sigma_j = (-1)^{j-1}[0, \infty)$ (so that they have the origin in common), with an interaction matrix of Nikishin type as in [21]. We prove that the minimizing measure is regular and supported in the interior of the condensers (under our assumption of growth of the potentials).

This result allows to proceed in Section 5 with a manipulation of algebraic nature involving the Euler–Lagrange equations for the *resolvents* (Cauchy transforms) $W_j(x)$ of the equilibrium measures. It is shown that certain auxiliary quantities Z_j that depend linearly on the resolvents and the potentials (see (5.4) for the precise formula) enter a pseudo-algebraic equation of the form

$$z^R + C_2(x)z^{R-1} + \cdots + C_{R+1}(x) = 0 \tag{1.3}$$

where the functions $C_j(x)$ are analytic functions with the same singularities as the derivative of the potentials $V_k'(x)$ in the common neighborhood of the real axis where all the potentials are real analytic. In particular the coefficients $C_j(x)$ do not have jumps on the real axis and the various branches of Eq. (1.3) are precisely the $Z_j(x)$ defined above. For example, if the derivative potentials are rational functions, then so are the coefficients of (1.3). This immediately implies that the branchpoints of (1.3) on the real axis (i.e. the zeroes of the discriminant) are nowhere dense and hence *a priori* the supports of the measures must consist of a finite union of intervals (since they must be compact as shown in Section 2 in the general setting).

The role of the pseudo-algebraic curve (1.3) is exactly the same as the well-known pseudo-hyperelliptic curve that appears in the one-matrix model [6,18]: in the context of the study of asymptotic properties of multiple-orthogonal polynomials it has been pointed out since the fundamental work [17] that the Cauchy transforms of the extremal measures solve an algebraic equation.

We also mention the recent work [13], in which the limiting behavior of Hermitian random matrices with external source is investigated and the presented asymptotic analysis relies on a set of conditions which are shown to be equivalent to the existence of a particular algebraic curve. The methods used in that paper to prove the existence for some special cases are very similar to our approach.

As it was pointed out by one of the referees, examples of of algebraic curves for special external fields were also obtained in the recent papers [11,12].

1.1. Connection to a Riemann–Hilbert problem

The principal motivation to the present paper is the application to the study of biorthogonal (multiply orthogonal) polynomials that arise in the study of the model hinted at by Eq. (1.1). In [2,3] we introduced the biorthogonal polynomials

$$\int_{\mathbb{R}_+^2} p_n(x)q_m(y)\frac{\mathrm{e}^{-N(V_1(x)+V_2(y))}}{x+y}\mathrm{d}x\mathrm{d}y = c_n^2\delta_{mn}\,,$$
$$p_n(x) = x^n + \cdots,\ q_n(y) = y^n + \cdots. \tag{1.4}$$

In [3] it was shown how a natural vector potential problem (for two measures) arises in that context and leads to a three-sheeted spectral curve of the form (1.3). Such problem enters in a natural way in the normalization of the 3×3 Riemann–Hilbert problem considered in [2] characterizing those polynomials (and some accessory ones) in the limit $N \to \infty$, $n \to \infty$, $\frac{N}{n} \to T > 0$. The notation V_1, V_2 is meant here to reflect the notation that will be used in Sections 5 and 6 (up to a reflection $V_2(y) \mapsto V_2(-y)$, as explained in [2,3]).

In perspective the more general situation with several measures considered in Sections 4 and 5 will be associated to the polynomials appearing in the study of the random-matrix chain (1.2) and biorthogonal polynomials for pairings of the form

$$\int_{\mathbb{R}_+^K} p_n(x_1)q_m(x_K)\frac{\mathrm{e}^{-N\sum\limits_{j=1}^{K} V_j(x_j)}}{\prod\limits_{j=1}^{K-1}(x_j+x_{j+1})}\prod_{j=1}^{K}\mathrm{d}x_j = c_n^2\delta_{mn} \tag{1.5}$$

$$p_n(x) = x^n + \cdots,\ q_n(y) = y^n + \cdots. \tag{1.6}$$

The details are to appear in forthcoming publications [4].

2. The vector potential problem

In this section we consider the vector potential problem which is a slightly generalized form of the weighted energy problem of signed measures ([20], Chapter VIII; [16], Chapter 5).

Let $A = (a_{ij})_{i,j=1}^{R}$ be an $R \times R$ real symmetric *positive-definite* matrix (in particular it has positive diagonal entries), referred to as the **interaction matrix**, containing the information on the total charges of the measures and their pair interaction coefficients. Suppose $\Sigma_1, \Sigma_2, \ldots, \Sigma_R$ is a collection of non-empty, *not necessarily disjoint* closed subsets of $\mathbb{C}$ such that $\Sigma_k \cap \Sigma_l$ has zero logarithmic capacity whenever $a_{kl} < 0$. Define the functions $h_k: \mathbb{C} \to (-\infty, \infty]$ for each Σ_k to be

$$h_k(z) := \ln \frac{1}{d(z, \Sigma_k)}, \quad (z \in \mathbb{C}) \tag{2.1}$$

where $d(\cdot, K)$ is the *distance function* from the closed subset K of the complex plane:

$$d(z, K) := \inf_{t \in K} |z - t|.$$

The function $d(z, K)$ is non-negative, uniformly continuous on $\mathbb{C}$ so $h_k(z)$ is upper semi-continuous and $h_k(z) = \infty$ on Σ_k.

Definition 2.1. A collection of background potentials

$$V_k: \Sigma_k \to (-\infty, \infty], \quad k = 1, 2, \ldots, R \tag{2.2}$$

is said to be *admissible with respect to the (positive definite) interaction matrix A* if the following conditions hold:

[A1] the potentials V_k are lower semi-continuous on Σ_k for all k,

[A2] the sets $\{z \in \Sigma_k : V_k(z) < \infty\}$ are of positive logarithmic capacity for all k,

[A3] the functions

$$H_{jk}(z, t) := \frac{V_j(z) + V_k(t)}{R} + a_{jk} \ln \frac{1}{|z - t|} \tag{2.3}$$

are uniformly bounded from below, i.e. there exists an $L \in \mathbb{R}$ such that

$$H_{jk}(z, t) \geq L \tag{2.4}$$

on $\{(z, t) \in \Sigma_j \times \Sigma_k : z \neq t\}$ for all $j, k = 1, \ldots, R$. Without loss of generality we can assume $L = 0$ by adding a common constant to all the potentials so that

$$H_{jk}(z, t) \geq 0. \tag{2.5}$$

We will also assume (again, without loss of generality) that **all the potentials are non-negative.**

[A4] There exist constants $0 \leq c < 1$ and C such that (recall that $a_{kk} > 0$)

$$H_{jk}(z, t) \geq \frac{(1 - c)}{R}(V_j(z) + V_k(t)) - \frac{C}{R^2}. \tag{2.6}$$

The constant C can be chosen to be *positive*.

[A5] The potentials are given such that the functions

$$Q_k(z) := \sum_{l:\, a_{kl} < 0} \left(\frac{1}{R} V_l(z) + a_{kl} h_l(z)\right) = \frac{s_k}{R} V_k(z) + \sum_{l:\, a_{kl} < 0} a_{kl} h_l(z) \tag{2.7}$$

are **bounded from below on** Σ_k (here $s_k \leq R - 1$ is the number of negative a_{kl}'s).

Note that in the above sum $k \neq l$ because of the assumption that $a_{kk} > 0$.

Definition 2.2. We define the **weighted energy with interaction matrix** A of a measure $\vec{\mu} = [\mu_1, \ldots, \mu_R]$ with $\mu_j \in \mathcal{M}_1(\Sigma_j)$ by

$$I_{A,\vec{V}}(\vec{\mu}) := \sum_{j,k}^{R} a_{jk} \iint \ln \frac{1}{|z-t|} \mathrm{d}\mu_j(z)\mathrm{d}\mu_k(t) + 2\sum_{k=1}^{R} \int V_k(z)\mathrm{d}\mu_k(z)$$
$$= \sum_{j,k} \iint H_{jk}(z,t)\mathrm{d}\mu_j(z)\mathrm{d}\mu_k(t), \tag{2.8}$$

where $\mathcal{M}_1(K)$ stands for the set of all Borel probability measures supported on some set $K \subset \mathbb{C}$.

Remark 2.1. The assumption [A3] is a *sufficient* requirement to ensure that the definition of the functional $I_{A,\vec{V}}(\cdot)$ is well-posed and it is a rather mild assumption on the growth of the potentials near the overlap regions and infinity. Indeed (with $L = 0$)

$$I_{A,\vec{V}}(\vec{\mu}) = \sum_{j,k} \iint H_{jk}(z,t)\mathrm{d}\mu_j(z)\mathrm{d}\mu_k(t) \geq 0. \tag{2.9}$$

Note also that if a conductor Σ_j is unbounded the condition (2.6) implies that

$$\frac{c}{R} V_j(z) \geq a_{jj} \ln|z - t_0| - \frac{c}{R} V_j(t_0) - \frac{C}{R^2} \tag{2.10}$$

and hence V_j grows at least like a logarithm. In [20] the usual requirement is the stronger one that $V_j(z)/\ln|z| \to \infty$ as $z \to \infty$.

Remark 2.2 (*A4*). is a stronger requirement which will be used for proving tightness (and therefore relative compactness) of a certain subfamily of measures over which $I_{A,\vec{V}}(\cdot)$ is guaranteed to attain its minimum value.

Remark 2.3 (*A5*). is yet stronger and assumes that all potentials have a suitable logarithmic growth near the common boundaries with those condensers carrying an opposite charge. This condition could be relaxed in some settings.

3. Existence and uniqueness of the equilibrium measure

In this section we prove the existence and uniqueness of the equilibrium measure for the vector potential problem described above. Before stating our main theorem, we recall that a family of measures $\mathcal{F}$ on a metric space X is said to be *tight* if for all $\varepsilon > 0$ there exists a compact set $K \subset X$ such that $\mu(X \setminus K) < \varepsilon$ for all measures $\mu \in \mathcal{F}$. The following theorem is a standard result in probability theory:

Theorem 3.1 (*Prokhorov's Theorem [19]*). *Let (X, d) be a separable metric space and $\mathcal{M}_1(X)$ the set of all Borel probability measures on X.*

- *If a subset $\mathcal{F} \subset \mathcal{M}_1(X)$ is a **tight family** of measures, then $\mathcal{F}$ is relatively compact in $\mathcal{M}_1(X)$ in the topology of weak convergence.*
- *Conversely, if there exists an equivalent complete metric d_0 on X then every relatively compact subset $\mathcal{F}$ of $\mathcal{M}_1(X)$ is also a tight family.*

We will use the following little lemma:

Lemma 3.1. *Let $F: X \to [0, \infty]$ be a non-negative lower semi-continuous function on the locally compact metric space X satisfying*

$$\lim_{x\to\infty} F(x) = \infty, \tag{3.1}$$

i.e. for all $H > 0$ there exists a compact set $K \subset X$ such that $F(x) > H$ for all $x \in X \setminus K$. Then for all $H > \inf F$ the family

$$\mathcal{F}_H := \left\{\mu \in \mathcal{M}_1(X) : \int_X F \mathrm{d}\mu < H\right\} \tag{3.2}$$

is a non-empty tight subset of $\mathcal{M}_1(X)$.

Proof. F attains its minimum at some point $x_0 \in X$ since F is lower semi-continuous and $\lim_{x\to\infty} F(x) = \infty$ and therefore the Dirac measure δ_{x_0} belongs to $\mathcal{F}_H$. To prove the tightness of $\mathcal{F}_H$, let $\varepsilon > 0$ be given. Since F goes to infinity "at the boundary" of X there exists a compact set $K \subset X$ such that $F(x) > \frac{2H}{\varepsilon}$ for all $x \in X \setminus K$. If $\mu \in \mathcal{F}_H$ we have

$$\mu(X \setminus K) = \int_{X\setminus K} \mathrm{d}\mu \le \frac{\varepsilon}{2H}\int_{X\setminus K} F\mathrm{d}\mu \le \frac{\varepsilon}{2H}\int_X F\mathrm{d}\mu \le \frac{\varepsilon}{2H}H = \frac{\varepsilon}{2} < \varepsilon. \quad \square \tag{3.3}$$

Define

$$U_k^{\vec{\mu}}(z) := \sum_{k=1}^{R} a_{kl} \int \ln\frac{1}{|z-t|}\,\mathrm{d}\mu_l(t), \tag{3.4}$$

which is the logarithmic potential (external terms and self-potential together) experienced by the kth charge component in the presence of $\vec{\mu}$ only.

Theorem 3.2 *(See [20], Thm. VIII.1.4). With the admissibility assumptions* [A1]–[A5] *above the following statements hold:*

1. *The extremal value*

$$\mathcal{V}_{A,\vec{V}} := \inf_{\vec{\mu}} I_{A,\vec{V}}(\vec{\mu}) \tag{3.5}$$

of the functional $I_{A,\vec{V}}(\cdot)$ is finite and there exists a unique (vector) measure $\vec{\mu}^\star$ such that $I_{A,\vec{V}}(\vec{\mu}) = \mathcal{V}_{A,\vec{V}}$.

2. *The components of $\vec{\mu}^\star$ have finite logarithmic energy and compact support. Moreover, the V_j's and the logarithmic potentials $U_k^{\vec{\mu}^\star}$ are bounded on the support of μ_k for all $k = 1, \dots, R$.*
3. *For $j = 1, \dots, R$ the* effective potential

$$\varphi_j(z) := U_j^{\vec{\mu}^\star}(z) + V_j(z) \tag{3.6}$$

is bounded from below by a constant F_j (Robin's constant) on Σ_j, with the equality holding a.e. on the support of μ_j.

Remark 3.1. The content of Theorem 3.2 is probably neither completely new nor very surprising and the proof is a rather straightforward generalization: the main improvement over the most common literature is the fact that we allow the condensers to overlap even if the corresponding term in the interaction matrix is negative. The assumption on the potentials that they provide a screening effect so that the equilibrium measures will not have support on the overlap region. The theorem will be instrumental in the proof of Theorem 5.1, which is the main result of the paper.

Proof of Theorem 3.2. First of all, we have to prove that

$$\mathcal{V}_{A,\vec{V}} = \inf_{\vec{\mu}} I_{A,\vec{V}}(\vec{\mu}) < \infty \tag{3.7}$$

by showing that there exists a vector measure with finite weighted energy. To this end, let $\vec{\eta}$ be the R-tuple of measures whose kth component η_k is the equilibrium measure of the standard weighted energy problem (in the sense of [20]) with potential $V_k(z)/a_{kk}$ on the conductor Σ_k for all k. (The potential $V_k(z)/a_{kk}$ is admissible in the standard sense on Σ_k since

$$\frac{1}{a_{kk}} V_k(z) - \ln|z| \geq \frac{R}{c} \ln|z - t_0| - \frac{1}{a_{kk}} V_k(t_0) - \frac{C}{c a_{kk} R} - \ln|z| \to \infty \tag{3.8}$$

as $|z| \to \infty$ for $z \in \Sigma_k$ if Σ_k is unbounded.) We know that η_k is supported on a compact set of the form

$$\left\{ z \in \Sigma_k : \frac{V_k(z)}{a_{kk}} \leq K_k \right\} \tag{3.9}$$

for some $K_k \in \mathbb{R}$. These sets are mutually disjoint by the growth condition (2.7) imposed on the potentials. The sum of the "diagonal" terms and the potential terms in the energy functional are finite for $\vec{\eta}$ since this is just a linear combination of the individual weighted energies of the equilibrium measures η_k. The "off-diagonal" terms with positive interaction coefficient a_{kl} are bounded from above because the supports of η_k and η_l are separated by a positive distance; the terms with negative interaction coefficient are also bounded from above since η_k and η_l are compactly supported. Therefore

$$\mathcal{V}_{A,\vec{V}} \leq I_{A,\vec{V}}(\vec{\eta}) < \infty. \tag{3.10}$$

Integrating the inequalities (2.6) it follows that

$$I_{A,\vec{V}}(\vec{\mu}) = \sum_{j,k=1}^{R} \iint H_{jk}(z,t) \mathrm{d}\mu_j(z) \mathrm{d}\mu_k(t) \geq (1-c) \sum_{k=1}^{R} \int V_k(z) \mathrm{d}\mu_k(z) - C. \tag{3.11}$$

We then study the minimization problem over the following set of probability measures:

$$\begin{aligned} \mathcal{F} &:= \left\{ \vec{\mu} : \sum_{k=1}^{R} \int V_k(z) \mathrm{d}\mu_k(z) \leq \frac{1}{(1-c)} (\mathcal{V}_{A,\vec{V}} + C + 1) \right\} \\ &\subset \mathcal{M}_1(\Sigma_1) \times \cdots \times \mathcal{M}_1(\Sigma_R). \end{aligned} \tag{3.12}$$

The extremal measure(s) are all contained in $\mathcal{F}$ since for a vector measure $\vec{\lambda} \notin \mathcal{F}$ we have

$$I_{A,\vec{V}}(\vec{\lambda}) \geq (1-c) \sum_{k=1}^{R} \int V_k(z) \mathrm{d}\lambda_k(z) - C \geq \mathcal{V}_{A,\vec{V}} + 1. \tag{3.13}$$

The function $\sum_k V_k(z)$ is non-negative, lower semi-continuous and goes to infinity as $|z| \to \infty$, and moreover

$$\frac{R}{(1-c)} (\mathcal{V}_{A,\vec{V}} + C + 1) > 0, \tag{3.14}$$

hence, by Lemma 3.1, all projections of $\mathcal{F}$ to the individual factors is a non-empty tight family of measures. Using Prokhorov's Theorem 3.1 we know that there exists a measure $\vec{\mu}^\star$ minimizing $I_{A,\vec{V}}(\cdot)$ such that $\frac{1}{R}\sum_{k=1}^{R}\mu_\star \in \mathcal{F}$. The existence of the (vector) equilibrium measure is therefore established.

Note that now statement (2) follows immediately: indeed from the condition 3 that $H_{j,k} \geq 0$ (and also $V_j \geq 0$) it follows that

$$\begin{aligned}\mathcal{V}_{A,\vec{V}} &= a_{11}\iint \ln\frac{1}{|z-t|}\mathrm{d}\mu_1^\star(z)\mathrm{d}\mu_1^\star(t) + \frac{2}{R}\int V_1(z)\mathrm{d}\mu_1^\star(z)\\ &\quad + \sum_{(j,k)\neq(1,1)}\iint H_{jk}(z,t)\mathrm{d}\mu_j^\star(z)\mathrm{d}\mu_k^\star(t)\\ &\geq a_{11}\iint \ln\frac{1}{|z-t|}\mathrm{d}\mu_1^\star(z)\mathrm{d}\mu_1^\star(t).\end{aligned} \tag{3.15}$$

Thus the logarithmic energy of $\mu_1^\star$ is bounded above by $\mathcal{V}_{A,\vec{V}}/a_{11}$. Repeating the argument for all $\mu_j^\star$'s we have that all the logarithmic energies of the $\mu_j^\star$'s are bounded above.

On the other hand, these log-energies are also bounded below using (2.6) with $j = k$:

$$a_{jj}\iint \ln\frac{1}{|z-t|}\mathrm{d}\mu_j^\star(z)\mathrm{d}\mu_j^\star(t) \geq -\frac{2c}{R}\int V_j(z)\mathrm{d}\mu_j^\star(z) - \frac{C}{R^2} \tag{3.16}$$

(boundedness from below follows since $\int V_j(z)\mathrm{d}\mu_j(z)$ is bounded above and appears with a negative coefficient in the formula).

Now, using the fact that the quantities $H_{jk}(z,t)$ are non-negative due to (2.5) and condition (3.12) it follows that

$$\varphi_j(z) = V_j(z) + \sum_{k\neq j} a_{jk}\int \ln\frac{1}{|z-t|}\mathrm{d}\mu_k^\star(t) \tag{3.17}$$

is finite wherever $V_j(z)$ is. Using condition [A5] it also follows that it is lower semi-continuous, bounded from below on Σ_j and hence admissible in the usual sense of minimizations of single measures [20]. We also claim that φ_j grows to infinity near all the contacts between Σ_j and any Σ_k for which $a_{jk} < 0$. Suppose $z_0 \in \Sigma_j \cap \Sigma_k$ (with $a_{jk} < 0$); then on a compact neighborhood K of z_0 we have

$$\varphi_j(z) \geq V_j(z) + \sum_{\substack{k\neq j\\ a_{jk}<0}} a_{jk}h_k(z) + M_K \tag{3.18}$$

for some finite constant M_K (which – of course – depends on K). From (5) then

$$V_j(z) + \sum_{\substack{k\neq j\\ a_{jk}<0}} a_{jk}h_k(z) + M_K \geq \frac{R-s_j}{R}V_j(z) + \widetilde{M}_K \tag{3.19}$$

where $s_j < R$ is the number of negative a_{jk} $(j \neq k)$. Since $V_j(z)$ tends to infinity at the contact points (from the same condition [A5]) then so must be for φ_j.

Note also that

$$\mathcal{V}_{A,\vec{V}} = \sum_j I_{\Sigma_j,\varphi_j}(\mu_{j,\star}), \tag{3.20}$$

and hence (as in [20]) each single $\mu_{j,\star}$ is the minimizer of the single variational problem on Σ_j under the effective potential φ_j. From the standard results it follows that the support of $\mu_j^\star$ is contained in the set where φ_j is bounded, which, due to our assumptions, are all compact and at finite non-zero distance from the common overlaps. This proves that the components of $\vec{\mu}^\star$ are actually compactly supported.

Uniqueness as well as the remaining properties are established essentially in the same way as in [20], Thm. 1 Chap. VIII using the positive definiteness of the interaction matrix A, which guarantees the convexity of the functional. □

4. The special case

We now specialize the above setting to the following collection of R conductors:

$$\Sigma_j := (-1)^{j-1}[0,\infty) \quad (j = 1, 2, \ldots, R), \tag{4.1}$$

and interaction matrix

$$A := \begin{bmatrix} 2q_1^2 & -q_1q_2 & 0 & \ldots & 0 \\ -q_1q_2 & 2q_2^2 & -q_2q_3 & \ldots & 0 \\ 0 & -q_2q_3 & 2q_3^2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2q_R^2 \end{bmatrix}. \tag{4.2}$$

Under the assumptions on the growth of the potentials $V_j(x)$ near the only common boundary point $x = 0$, Theorem 3.2 guarantees the existence of a unique vector minimizer.

We now investigate the regularity properties under the rather comfortable assumption that the potentials V_j are *real analytic* on $\Sigma_j \setminus \{0\}$ for all j; this is in addition to the host of assumptions specified in Definition 2.1.

In order to simplify slightly some algebraic manipulations to come we re-define the problem by rescaling the component of the vector of probability measures $\mu_j \mapsto q_j\mu_j$ so that now the interaction matrix becomes the simpler

$$A := \begin{bmatrix} 2 & -1 & 0 & \ldots & 0 \\ -1 & 2 & -1 & \ldots & 0 \\ 0 & -1 & 2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{bmatrix}. \tag{4.3}$$

The electrostatic energy can be rewritten as

$$I_{A,\vec{V}}(\vec{\mu}) = 2\sum_{j=1}^{R} \iint \ln \frac{1}{|x-y|} \mathrm{d}\mu_j(x)\mathrm{d}\mu_j(y)$$

$$- \sum_{j=1}^{R-1} \iint \ln \frac{1}{|x-y|} \mathrm{d}\mu_j(x)\mathrm{d}\mu_{j+1}(y) \tag{4.4}$$

$$+2\sum_{j=1}^{R} \int V_j(x)\mathrm{d}\mu_j(x). \tag{4.5}$$

As explained in the previous section, the above minimization problem has the interesting property that the same equilibrium measure is achieved by minimizing only one component of it in the mean field of the neighbors and, moreover, the supports of the minimizers satisfy

$$\mathrm{supp}(\rho_j) \cap \mathrm{supp}(\rho_{j+1}) = \emptyset. \tag{4.6}$$

Corollary 4.1. *Let $\vec{\mu}$ be the vector equilibrium measure for the above problem. For any $1 \le k \le R$ we have that*

$$I_{\widehat{V}_k}(\mu) := \int_{\Sigma_k}\int_{\Sigma_k} \ln \frac{1}{|z-t|} \mathrm{d}\mu(z)\mathrm{d}\mu(t) + 2\int_{\Sigma_k} \widehat{V}_k(z)\mathrm{d}\mu(z) \tag{4.7}$$

is minimized by the same μ_k, where the effective potentials $\widehat{V}_k$ are

$$\widehat{V}_1(z) := \frac{1}{2}V_1(z) - \frac{1}{2}\int_{\Sigma_2} \ln \frac{1}{|z-t|} \mathrm{d}\mu_2(t) \tag{4.8}$$

$$\widehat{V}_k(z) := \frac{1}{2}V_k(z) - \frac{1}{2}\int_{\Sigma_{k+1}} \ln \frac{1}{|z-t|} \mathrm{d}\mu_{k+1}(t) - \frac{1}{2}\int_{\Sigma_{k-1}} \ln \frac{1}{|z-t|} \mathrm{d}\mu_{k-1}(t) \tag{4.9}$$

$$\widehat{V}_R(z) := \frac{1}{2}V_R(z) - \frac{1}{2}\int_{\Sigma_{R-1}} \ln \frac{1}{|z-t|} \mathrm{d}\mu_{R-1}(t). \tag{4.10}$$

Note that the effective potential differs from the original potential by **harmonic** potentials because the supports of $\mu_{k\pm1}$ are disjoint from the support of μ_k.

We recall the following theorem:

Theorem 4.1 (*Thm. 1.34 in [6]*). *If the external potential belongs to the class C^k, $k \ge 3$ then the equilibrium measure is absolutely continuous and its density is Hölder continuous of order $\frac{1}{2}$.*

Combining Corollary 4.1 with Theorem 4.1 we have that the solution of our equilibrium problem consists of equilibrium measures which are absolutely continuous with respect to the Lebesgue measure with densities ρ_j at least Hölder-$\frac{1}{2}$ continuous as long as the external potentials are at least C^3. Moreover the supports of these equilibrium measures have a *finite positive distance* from the origin.

Our next goal is to prove that the supports of the ρ_j's consist of a finite union of disjoint compact intervals. For that we need a pseudo-algebraic curve given in the next section.

5. Spectral curve

Since the equilibrium measures have a smooth density we can now proceed with some manipulations using the variational equations.

For the remainder of the paper we will make the following additional **assumption** (besides those in Definition 2.1) on the nature of the potentials V_j:

Assumption. The derivative of the potential V_j' is the restriction to $\Sigma_j^o := (-1)^{j-1}(0, \infty)$ of a real analytic function defined in a neighborhood of the real axis possessing at most isolated *polar* singularities on $\mathbb{R} \setminus \Sigma_j$.

For a function f analytic on $\mathbb{C} \setminus \Gamma$, where Γ is an oriented smooth curve, we denote

$$\mathcal{S}(f)(x) := f_+(x) + f_-(x), \quad \Delta(f)(x) := f_+(x) - f_-(x), \quad x \in \Gamma \tag{5.1}$$

where the subscripts denote the boundary values. We remind the reader that under our assumptions, the equilibrium measures satisfy Eq. (4.6).

Definition 5.1. For the solution $\vec{\rho}$ of the variational problem, we define the **resolvents** as the expressions

$$W_j(z) := \int_{\Sigma_j} \frac{\rho_j(x)\mathrm{d}x}{z - x}, \quad z \in \mathbb{C} \setminus \mathrm{supp}(\rho_j). \tag{5.2}$$

The variational equations imply the following identities for $j = 1, \dots, R$:

$$\begin{aligned} &\mathcal{S}(W_j)(x) = V_j'(x) + W_{j+1} + W_{j-1} \\ &\Delta(W_j)(x) = -2i\pi\rho_j(x), \quad x \in \mathrm{supp}(\rho_j) \end{aligned} \tag{5.3}$$

where we have convened that $W_0 \equiv W_{R+1} \equiv 0$. Note that, under our assumptions for the growth of the potentials V_j at the contact points between conductors (in this case the origin), the support of ρ_j is disjoint from the supports of $\rho_{j\pm 1}$ and hence the resolvents on the rhs of the above equation are continuous on $\mathrm{supp}(\rho_j)$.

The following manipulations are purely algebraic: we first introduce the new vector of functions

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_R \end{bmatrix}^t := \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & (-1)^R \end{bmatrix} \left\{ A^{-1} \begin{bmatrix} V_1' \\ \vdots \\ V_R' \end{bmatrix} + \begin{bmatrix} W_1 \\ \vdots \\ W_R \end{bmatrix} \right\}. \tag{5.4}$$

Trivial linear algebra implies then the following relations for the newly defined functions Y_j:

$$\begin{array}{rclrcll} \mathcal{S}(Y_1) & = & -Y_2 & \Delta(Y_1) & = & 2i\pi\rho_1 & \text{on } \mathrm{supp}(\rho_1) \\ \mathcal{S}(Y_2) & = & -Y_1 - Y_3 & \Delta(Y_2) & = & -2i\pi\rho_2 & \text{on } \mathrm{supp}(\rho_2) \\ \mathcal{S}(Y_3) & = & -Y_2 - Y_4 & \Delta(Y_3) & = & 2i\pi\rho_3 & \text{on } \mathrm{supp}(\rho_3) \\ & \vdots & & & \vdots & & \vdots \\ \mathcal{S}(Y_{R-1}) & = & -Y_{R-2} - Y_R & \Delta(Y_{R-1}) & = & (-1)^R 2i\pi\rho_{R-1} & \text{on } \mathrm{supp}(\rho_{R-1}) \\ \mathcal{S}(Y_R) & = & -Y_{R-1} & \Delta(Y_R) & = & (-1)^{R+1} 2i\pi\rho_R & \text{on } \mathrm{supp}(\rho_R). \end{array} \tag{5.5}$$

The above relation should be understood at all points that do not coincide with some of the isolated singularities of some potential V_j (points of which type there are only finitely many within any compact set).

Define then the functions

$$\begin{aligned} &Z_0 := Y_1, \quad Z_1 := -Y_1 - Y_2, \quad Z_2 := Y_2 + Y_3, \dots, \\ &Z_{R-1} = (-1)^{R-1}(Y_{R-1} + Y_R), \quad Z_R := (-1)^R Y_R. \end{aligned} \tag{5.6}$$

Then:

Proposition 5.1. *All symmetric polynomials of* $\{Z_j\}_{0\le j\le R}$ *are real analytic in the common domain of analyticity of the potentials, namely they have no discontinuities on the supports of the measures* ρ_j.

Proof. A direct algebraic computation using the boundary values of the $\{Y_j\}$ functions gives the following boundary values of the functions Z_j:

$$2Z_{0_\pm} = -Y_2 \pm 2i\pi\rho_1 \tag{5.7}$$

$$2Z_{1_\pm} = \begin{cases} -Y_2 \mp 2i\pi\rho_1 = 2Z_{0\mp} & \text{on } \mathrm{supp}(\rho_1) \\ -Y_1 + Y_3 \pm 2i\pi\rho_2 & \text{on } \mathrm{supp}(\rho_2) \end{cases} \tag{5.8}$$

$$2Z_{2_\pm} = \begin{cases} -Y_1 + Y_3 \mp 2i\pi\rho_2 = 2Z_{1\mp} & \text{on } \mathrm{supp}(\rho_2) \\ Y_2 - Y_4 \pm 2i\pi\rho_3 & \text{on } \mathrm{supp}(\rho_3) \end{cases} \tag{5.9}$$

$$\vdots \tag{5.10}$$

$$2Z_{(R-1)\pm} = \begin{cases} (-1)^{R-1}(-Y_{R-2} + Y_R) \mp 2i\pi\rho_{R-1} = 2Z_{(R-2)_\mp} & \text{on } \mathrm{supp}(\rho_{R-1}) \\ (-1)^{R-1}Y_{R-1} \pm 2i\pi\rho_R & \text{on } \mathrm{supp}(\rho_R) \end{cases} \tag{5.11}$$

$$2Z_{R_\pm} = (-1)^{R-1}Y_{R-1} \mp 2i\pi\rho_R = 2Z_{(R-1)_\mp} \quad \text{on } \mathrm{supp}(\rho_R). \tag{5.12}$$

Consider a symmetric polynomial $P_K := 2^K\left({Z_0}^K + \cdots + {Z_R}^K\right)$ and its boundary values on, say, $\mathrm{supp}(\rho_1)$; we see above that $Z_{0_\pm} = Z_{1_\mp}$ and hence $Z_0^K + Z_1^K$ has no jump there. The support of ρ_2 has no intersection with Σ_1 and $\mathrm{supp}(\rho_1)$ (see (4.6)) due to our assumptions, and hence Z_2 may have a jump on $\mathrm{supp}(\rho_1)$ only if the support of ρ_3 has some intersection with it. In that case anyway $Z_{2_\pm} = Z_{3_\mp}$ and hence also $Z_2^K + Z_3^K$ has no jump on $\mathrm{supp}(\rho_3) \cap \mathrm{supp}(\rho_1)$.

In general on $\mathrm{supp}(\rho_k) \cap \mathrm{supp}(\rho_1)$ we have $Z_{k_\pm} = Z_{k_\mp}$ and so the same argument apply. In short one can thus check that all the jumps that may *a priori* occur in fact cancel out in a similar way.

Repeating the argument for all the other $\mathrm{supp}(\rho_j)$ instead of $\mathrm{supp}(\rho_1)$ proves that the expression has no jump on any of the supports, and since *a priori* it can have jumps only there, then it has no jumps at all. Invoking Morera's theorem, we see that the symmetric polynomials of the Z_k's can be extended analytically across the supports of the ρ_j's.

Finally, the statement that the symmetric polynomials are real analytic follows from the following reasoning: the Z_j's are linear expressions in the W_j's and the potentials. In particular they are analytic off the real axis (where all the W_j's are) and in the common domain of analyticity of the potentials. The same then applies to the symmetric polynomials in the Z_j's. Finally, on an open interval in $\mathbb{R}$, as long as it is outside of all the supports of the vector measure, the Z_j are all real analytic functions since W_j's are. This concludes the proof. □

A consequence of this proposition is that:

Theorem 5.1. *The functions* Z_k *are solution of a pseudo-algebraic equation of the form*

$$z^{R+1} + C_2(x)z^{R-1} + \cdots + C_{R+1}(x) = 0 \tag{5.13}$$

where $C_j(x) := (-1)^j \sum_{\ell_1,\ldots,\ell_j} Z_{\ell_1}\cdots Z_{\ell_j}$ *are (real) analytic functions on the common domain of analyticity of the potentials.*

Remark 5.1. This result is the direct analogue of the results about the existence of the spectral curve for the one-matrix model [18] which was established on a rigorous ground in [6]. In a different context of matrix models with external source Theorem 5.1 is conceptually similar to the result in [13].

Proof of Theorem 5.1. We set

$$E(z, x) := \prod_{j=0}^{R} (z - Z_j(x)), \tag{5.14}$$

and expand the polynomial in z. Clearly we have $Z_0 + Z_1 + \cdots + Z_R = 0$ and hence the coefficient C_1 vanishes identically. The other coefficients are polynomials in the elementary symmetric functions already shown to be real analytic and hence sharing the same property. □

Corollary 5.1. *The densities ρ_j are supported on a **finite union of compact intervals**. Moreover the supports of ρ_j and $\rho_{j\pm1}$ are disjoint.*

Proof. The supports of the measures are in correspondence with the jumps of the algebraic solutions of $E(z, x) = 0$; in particular the set of endpoints of the supports must be a subset of the zeroes or poles of the discriminant that belong to $\mathbb{R}$. Since the only singularities that these may have come from those of the derivatives of the potentials $V_j'(x)$ on the real axis, and these have been assumed to be meromorphic on $\mathbb{R}$ and be otherwise real analytic, then the discriminant of the pseudo-algebraic equation cannot have infinitely many zeroes on a compact set. We also know that the measures ρ_j are compactly supported *a priori* and hence there can be only finitely many intervals of support. □

Putting together Proposition 5.1 and Theorem 5.1 we can rephrase the properties of the functions $Z_j(x)$ by saying that they are the $R + 1$ branches of the polynomial equation (5.13), thus defining an $(R + 1)$-fold covering of (a neighborhood of) the real axis. The neighborhood is the maximal common neighborhood of joint analyticity of the potentials $V_j(x)$. The various *sheets* defined by the functions $Z_j(z)$ are glued together along the supports of the equilibrium measures ρ_j in a "chain" of sheets as the *Hurwitz* diagram in Fig. 1 shows.

Remark 5.2. In [1] a similar problem was considered in the context of multiple orthogonality for Nikishin systems on conductors without intersection and with fixed weights: this corresponds to the case of a minimization problem without external fields. It was shown that an algebraic curve similarly arises; in the formulation of [1] the algebraic curve involves, rather than the resolvents, their exponentiated antiderivatives Ψ_j's, namely

$$W_j = \frac{\mathrm{d}}{\mathrm{d}x} \ln \Psi_j(x) \tag{5.15}$$

and a mixture of algebraic geometry and geometric function theory was used to investigate their properties. In particular the functions Ψ_j figured in an algebraic equation (see Eq. 2.1 in [1]) as the various determinations of a polynomial relation

$$\Psi^{R+1} + r_1(x)\Psi^R + \cdots + r_R(x)\Psi + r_{R+1}(x) = 0\,, \quad r_j \in \mathbb{C}[x] \tag{5.16}$$

with the discriminant (w.r.t. Ψ) vanishing at the endpoints of the supports for the measures of the corresponding Nikishin problem. Along similar lines, examples of curves of algebraic type for Nikishin systems with special choices of external fields were recently obtained in [11].

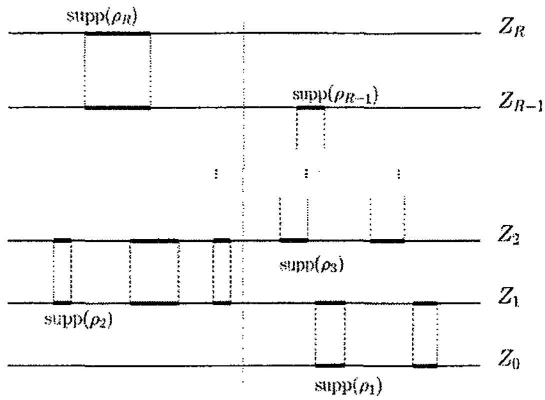


Fig. 1. The Hurwitz diagram of the spectral curve.

6. An explicit example

We consider the case with $R = 2$ and the two potentials are the same $V_1(x) = V_2(-x)$ and are of the simplest possible form that satisfies our requirements (see Fig. 2)

$$V_1(x) = bx - a\ln x, \quad x > 0; \qquad V_2(x) = -bx - a\ln(-x), \quad x < 0 \tag{6.1}$$

where both $a, b > 0$.

Quite clearly we can rescale the axis and set $b = 1$ without loss of generality.

Using the expressions for the coefficients of the spectral curve (Thm. (5.13)) in terms of the potentials $V_1 = V$ and $V_2 = V^\star = V(-x)$ we have

$$E(z, x) = z^3 - R(x)z - D(x) = 0 \tag{6.2}$$

where, on account of the fact that the derivative of the potentials have a simple pole at $x = 0$, the coefficients $R(x), D(x)$ have at most a double pole there. From the relationship between the three branches of Z and the resolvents W_1, W_2 (Eq. (5.4)) we have

$$Z^{(0)}(x) = -W_1 - \frac{a}{x} + \frac{1}{3} \tag{6.3}$$

$$Z^{(2)}(x) = W_2 + \frac{a}{x} + \frac{1}{3} \tag{6.4}$$

$$Z^{(1)}(x) = -Z^{(0)}(x) - Z^{(2)}(x) = W_1(x) - W_2(x) + \frac{2a}{x} \tag{6.5}$$

and hence the general forms that we can expect for the coefficients of the algebraic curve are

$$R(x) = \frac{a^2}{x^2} + \frac{1}{3} + \frac{C}{x}$$
$$D(x) = \frac{2a^2}{3x^2} - \frac{2}{27} + \frac{A}{x^2} + \frac{B}{x} \tag{6.6}$$

where the constants A, B, C have yet to be determined.

The spectral curve $z^3 - Rz - D = 0$ has in general 5 finite branchpoints (which is incompatible with the requirements of compactness of the support of the measures) and requiring that there are ≤ 4 branchpoints and symmetrically placed around the origin (by looking at the discriminant of the equation) imposes that $B = C = 0$.

The ensuing spectral curve is

$$z^3 - \left(\frac{1}{3} + \frac{a^2}{x^2}\right) z - \left(\frac{2a^2 + 3A}{3x^2} - \frac{2}{27}\right) = 0 \tag{6.7}$$

and a suitable rational uniformization of this curve is

$$X = \frac{\sqrt{a^2 + A}}{\lambda} - \frac{A}{2\sqrt{a^2 + A}} \left(\frac{1}{\lambda + 1} + \frac{1}{\lambda - 1}\right) \tag{6.8}$$

$$Z = -\frac{3A + 2a^2}{3a^2} - \frac{A(a^2 + A)}{\left(\lambda^2 - (1 + A/a^2)\right) a^4}. \tag{6.9}$$

The three points above $x = \infty$ are $\lambda = \pm 1, 0$ and Z is regular there.

We see that the condition that the measures ρ_1, ρ_2 have unit mass requires that

$$\operatorname*{res}_{x=\infty} Z^{(0)} \mathrm{d}x = 1 + a, \qquad \operatorname*{res}_{x=\infty} Z^{(2)} \mathrm{d}x = -1 - a. \tag{6.10}$$

We need only to decide which point $\lambda = \pm 1, 0$ correspond to the three points over infinity. But this is achieved by inspection of the behavior of $Y(\lambda)$ and $X(\lambda)$ near the three points $\lambda = 0, 1, -1. 0$.

By this inspection we have

$$\lambda = 1 \leftrightarrow \infty_1 \tag{6.11}$$

$$\lambda = -1 \leftrightarrow \infty_2 \tag{6.12}$$

$$\lambda = 0 \leftrightarrow \infty_0. \tag{6.13}$$

Computing the residues of $Z\mathrm{d}x = ZX'\mathrm{d}\lambda$ at these points we have

$$\operatorname*{res}_{x=\infty} Z^{(0)} \mathrm{d}x = \sqrt{a^2 + A} = 1 + a \tag{6.14}$$

$$\operatorname*{res}_{x=\infty} Z^{(2)} \mathrm{d}x = -\sqrt{a^2 + A} = -1 - a \tag{6.15}$$

which imply that $A = 2a + 1$.

Collecting the above, we have found that

$$X = \frac{a + 1}{\lambda} - \frac{2a + 1}{2a + 2} \left(\frac{1}{\lambda + 1} + \frac{1}{\lambda - 1}\right) \tag{6.16}$$

$$Z = -\frac{2a^2 + 6a + 3}{3a^2} - \frac{(2a + 1)(a + 1)}{\left(\lambda^2 - ((a + 1)^2/a^2)\right) a^4} \tag{6.17}$$

and the algebraic equation for $z = Z(\lambda)$ in terms of $x = X(\lambda)$ becomes

$$z^3 - \left(\frac{1}{3} + \frac{a^2}{x^2}\right) z - \left(\frac{2a^2 + 6a + 6}{3x^2} - \frac{2}{27}\right) = 0. \tag{6.18}$$

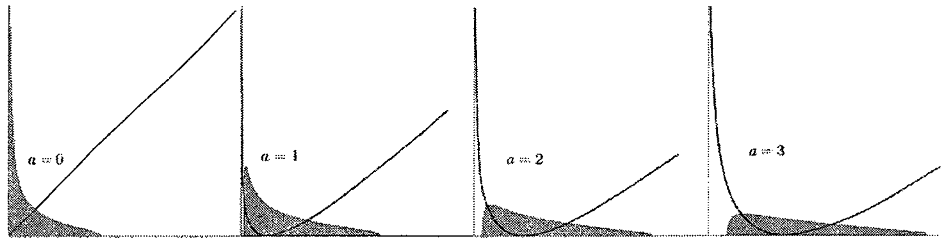
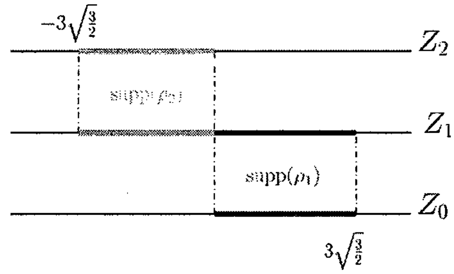


Fig. 2. Some examples for the equilibrium measure for the example worked out in the text, and $a = 0, 1, 2, 3$ respectively from left to right. In red is the graph of the potential V_1. The symmetry implies that the other equilibrium measure is simply the reflection of this around the ordinate axis. The units for the axes are the same in all cases. The growth of the density at $x = 0$ for $a = 0$ is $\mathcal{O}(x^{-2/3})$. Near the other edges the vanishing is of the form $\mathcal{O}((x - \alpha)^{\frac{1}{2}})$. (For interpretation of the references to colour in this figure legend. the reader is referred to the web version of this article.)

It is possible to write explicitly the expressions of the branchpoints in terms of a but it is not very interesting per se, except to discuss their behaviors in different regimes of a; we find that for $a > 0$ there are four symmetric branchpoints on the real axis and the inmost ones tend to zero as $a \to 0$, whereas they all tend to infinity as $a \to \infty$ according to $\pm(a \pm 2\sqrt{a}) + \mathcal{O}(1)$.



It is interesting to note that for $a = 0$ our general theorem does not apply: the potentials are *finite* on the common boundary of the condensers and hence cannot prevent accumulation of charge there. However the algebraic solution we have obtained is perfectly well-defined for $a = 0$ giving the algebraic relation

$$z^3 - \frac{z}{3} - \frac{2}{x^2} + \frac{2}{27} = 0. \tag{6.19}$$

A short exercise using Cardano's formulae shows that the origin is a branchpoint of order 3 and thus corresponding to the Hurwitz diagram on the side.

The behavior of the equilibrium densities ρ_j near the origin is (expectedly) $x^{-\frac{2}{3}}$.

7. Concluding remarks

We point out a few shortcomings and interesting open questions about the above problem.

The first problem would be to relax the growth condition of the potentials near common points of boundaries, if not in the general case at least in the specific example given in the second half of the paper, where we consider conductors being subsets of the real axis.

The importance of this setup is in relation to the asymptotic analysis of certain biorthogonal polynomials studied elsewhere [2] and their relationship with a random multi-matrix model [3].

In that setting, having *bounded* potentials near the origin $0 \in \mathbb{R}$ would allow the occurrence of new universality classes where new parametrices for the corresponding 3×3 (in the simplest case) Riemann–Hilbert problem would have to be constructed.

Based on heuristic considerations involving the analysis of the spectral curve of said RH problems, the density of eigenvalues should have a behavior of type $x^{-\frac{2}{3}}$ near the origin (to be compared with $x^{-\frac{1}{2}}$ for the usual hard-edge in the Hermitian matrix model). Generalization involving chain matrix model would allow arbitrary $-\frac{p}{q}$ behavior, $p < q$. However, for all these analyses to take place the corresponding equilibrium problem should be analyzed from the point of view of potential theory, allowing bounded potentials near the point of contact.

Acknowledgments

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Chapter 10

On the norms and roots of orthogonal polynomials in the plane and L^p-optimal polynomials with respect to varying weights

10.1 Summary

The present chapter is based on the preprint [10]. This work is concerned with the asymptotic behavior of weighted polynomials optimal with respect to the L^p-norm corresponding to a positive measure in the complex plane. The main results can be summarized as follows:

1. It is shown to be impossible that a positive proportion of zeroes of L^p-optimal weighted polynomials accumulates outside the polynomial convex hull of the corresponding equilibrium measure. (Sec. 2).

2. A lower and upper estimate for the L^p-norms of optimal weighted polynomials is presented (Sec. 3).

10.1.1 L^p-optimal weighted polynomials and their zeroes

A *weighted polynomial* with respect to the admissible varying weight w is of the form $w^n(z)P_n(z)$ where $P_n(z)$ is a monic polynomial of degree n [81]. The relevance of these polynomials for the weighted equilibrium problem is that the logarithm of the absolute value of such a weighted polynomial is of the form

$$\log|w^n(z)P_n(z)| = \sum_{k=1}^{n} \log|z - z_k| - nV(z) = -n\left(\sum_{k=1}^{n} \frac{1}{n} \log\frac{1}{|z - z_k|} + V(z)\right) , \tag{10.1}$$

where

$$P_n(z) = \prod_{k=1}^{n} (z - z_k) . \tag{10.2}$$

This can be thought of as a rescaled effective potential associated to the normalized counting measure of the zeroes of $P_n(z)$ in the presence of the background potential $V(z)$. Heuristically it plausible to expect that the absolute value of a weighted polynomial is small for polynomials whose zero distribution is close to the equilibrium measure of V and that the maximal value is close to $\exp(-nF_w)$ where F_w is the corresponding Robin constant (see Chap. 4). This motivates the following definition:

Definition 10.1.1 ([81]) *A sequence of weighted polynomials w^nP_n is called* asymptotically extremal *if*

$$\lim_{n\to\infty} (\|w^nP_n\|_\infty)^{\frac{1}{n}} = \exp(-F_w) . \tag{10.3}$$

It is important to note that there are asymptotically extremal sequences of weighted polynomials: for example, any sequence of *Fekete polynomials* (monic polynomials whose zeroes are given by a Fekete point combination) for a given weight w gives an asymptotically extremal sequence. Obviously, a sequence of optimal polynomials with respect to the supremum norm for fixed n, called the *generalized Chebishev polynomials*, is also an asymptotically optimal sequence [81].

Any limit point of the sequence of normalized counting measures of the zeroes of asymptotically optimal weighted polynomials is related to the equilibrium measure in the following way:

Theorem 10.1.1 **([81])** *Let Ω be the unbounded component of $\mathbb{C}P^1 \setminus S_w$ and assume that $\{P_n(z)\}$ a sequence of asymptotically extremal monic polynomials. Any weak-star limit point ν of the normalized counting measures associated with the zeroes of the polynomials P_n satisfies*

$$\int \log \frac{1}{|z-s|} d\nu(s) = \int \log \frac{1}{|z-s|} d\mu_w(s) \qquad z \in \Omega \ . \tag{10.4}$$

In particular, $\mathrm{supp}(\nu) \subseteq \mathbb{C} \setminus \Omega$.

In other words, it is not guaranteed that the limiting measure is μ_w (there may be more than one accumulation points) but any weak-star limit point satisfies the above *balayage* property outside the polynomial convex hull of the equilibrium measure.

Motivated by the asymptotic analysis of orthogonal polynomials and their zeroes, one may consider the following general notion of optimality with respect to the L^p-norm corresponding to a fixed measure σ:

Definition 10.1.2 **([81])** *The weighted polynomial $w^n P_n$ is called* L^p-optimal *with respect to the reference measure σ if*

$$\|P_n w^n\|_{L^p(\sigma)} = \min \left\{ \|Q_n w^n\|_{L^p(\sigma)} \ : \ Q_n \textit{ monic polynomial of degree } n \right\} \ . \tag{10.5}$$

Recall that the notation $\mathrm{Co}(S)$ and $\mathrm{Pc}(S)$ refers to the convex hull and the polynomial convex hull of the set S respectively. For L^p-optimal weighted polynomials we have the following result [10]:

Lemma 10.1.1 ([10], Prop. 2.1) *Let K be a closed subset in $\mathbb{C} \setminus \mathrm{Co}(S_w)$. Then there exists an $n_0 \in \mathbb{N}$ such that if $w^n P_n$ is L^p-optimal then $P_n(z)$ has no zeroes in K. In particular, for any $\varepsilon > 0$ there exists an $n_1 \in \mathbb{N}$ such that for all $n > n_1$ the zeroes of P_n are within a distance ε from the convex hull.*

Based on the proof of Thm. 10.1.1 in [81], an analogous statement can be proven for L^p-optimal weighted polynomials:

Theorem 10.1.2 (Cor. 2.1, [10]) *Let $\{w^n(z)P_n(z)\}$ be a sequence of L^p optimal polynomials with respect to a reference measure σ. Any weak-star limit point ν of the normalized counting measures associated with the zeroes of the polynomials P_n satisfies*

$$\int \log \frac{1}{|z-s|} d\nu(s) = \int \log \frac{1}{|z-s|} d\mu_w(s) \qquad z \in \mathbb{C} \setminus \mathrm{Pc}(S_w) . \tag{10.6}$$

In particular, $\mathrm{supp}(\nu) \subseteq \mathrm{Pc}(S_w)$.

10.2 On the norms and roots of orthogonal polynomials in the plane and L^p-optimal polynomials with respect to varying weights, arXiv:0910.4223, 2009.

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On the norms and roots of orthogonal polynomials in the plane and L^p-optimal polynomials with respect to varying weights

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Abstract

For a measure on a subset of the complex plane we consider L^p-optimal weighted polynomials, namely, monic polynomials of degree n with a varying weight of the form $w^n = e^{-nV}$ which minimize the L^p-norms, $1 \le p \le \infty$. It is shown that eventually all but a uniformly bounded number of the roots of the L^p-optimal polynomials lie within a small neighborhood of the support of a certain equilibrium measure; asymptotics for the nth roots of the L^p norms are also provided. The case $p = \infty$ is well known and corresponds to weighted Chebyshev polynomials; the case $p = 2$ corresponding to orthogonal polynomials as well as any other $1 \le p < \infty$ is our contribution.

1 Introduction, background and results

In approximation theory an important role is played by the so-called *Chebyshev polynomials* associated to a compact set $K \subseteq \mathbb{C}$, namely monic polynomials of degree n that minimize the supremum norm over K. As a natural generalization, one can consider *weighted Chebyshev polynomials* with respect to a varying weight of the form w^n on some $\Sigma \subseteq \mathbb{C}$ that are minimizing the supremum norm of weighted polynomials $Q_n w^n$ over Σ, where Q_n is a monic polynomial of degree n (the weight function w is assumed to satisfy certain standard admissibility conditions that make the extremal problem well-posed [1]).

Along the same lines, given a positive Borel measure σ on $\Sigma \subseteq \mathbb{C}$, one can consider optimal weighted polynomials in the $L^2(\sigma)$-sense; provided that the integrals below are finite, it is easy to see that there is a unique monic polynomial P_n for which the weighted polynomial $P_n w^n$ minimizes the $L^2(\sigma)$-norm

$$\|P_n w^n\|_{L^2(\sigma)} := \left(\int_\Sigma |P_n|^2 w^{2n} \mathrm{d}\sigma\right)^{\frac{1}{2}} \tag{1-1}$$

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among all monic weighted polynomials of degree n. This polynomial may be characterized as the nth monic *orthogonal polynomial* with respect to the varying measure $w^{2n}d\sigma$, satisfying

$$\int_\Sigma P_n(z)\bar{z}^k w^{2n}(z)d\sigma(z) = \delta_{kn}h_n \qquad 0 \le k \le n \;, \tag{1-2}$$

where

$$h_n = \inf\left\{\|Q_n w^n\|_{L^2(\sigma)} \;:\; Q_n(z) \text{ monic polynomial of degree } n\right\} \;. \tag{1-3}$$

Orthogonal polynomial sequences for varying measures of the form $w^n \mathrm{d}\sigma$ appear naturally in the context of random matrix models [2, 3]: on the space of $n \times n$ Hermitian matrices $\mathbb{H}_n$, probability distributions of the form

$$\rho_n(M)\mathrm{d}M = \frac{1}{\mathcal{Z}_n}\exp(-n\mathrm{Tr}(V(M)))\mathrm{d}M \;, \qquad \mathcal{Z}_n = \int_{\mathbb{H}_n}\exp(-n\mathrm{Tr}(V(M)))\mathrm{d}M \tag{1-4}$$

are considered where the potential function $V(x)$ grows sufficiently fast as $|x| \to \infty$ to make the integral in 1-4 finite ($\mathrm{d}M$ stands for the Lebesgue measure on $\mathbb{H}_n$). The apparent unitary invariance of 1-4 implies that the analysis of statistical observables of M may be reduced to that of the random eigenvalues $\lambda_1, \ldots, \lambda_n$ with probability distribution

$$\begin{aligned} p_n(\lambda_1, \ldots, \lambda_n) &= \frac{1}{Z_n}\prod_{1\le k<l\le n}(\lambda_k - \lambda_l)^2 e^{-n\sum_{k=1}^n V(\lambda_k)} \;, \\ Z_n &= \int\cdots\int_{\mathbb{R}^n}\prod_{1\le k<l\le n}(\lambda_k - \lambda_l)^2 e^{-n\sum_{k=1}^n V(\lambda_k)}\mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_n \;. \end{aligned} \tag{1-5}$$

The marginal distributions of p_n (referred to as *correlation functions*) are expressible as determinants of the weighted polynomials $p_n(x)e^{-nV(x)/2}$ where the p_n satisfies the orthogonality relation

$$\int_\mathbb{R} p_n(x)x^k e^{-nV(x)}dx = \delta_{kn}h_n \qquad k = 0, \ldots, n \;. \tag{1-6}$$

Therefore the asymptotic analysis of the correlation functions reduces to the study of the corresponding orthogonal polynomials. On the real line, the asymptotic analysis is done effectively by the so-called Riemann–Hilbert method [3]; however, for the so-called *normal matrix models* [4, 5], for which the eigenvalues may fill regions of the complex plane, much less is known in general. While random matrix theory was the original impetus behind our interest, the paper will not draw any conclusions on these important connections.

Following instead a more approximation-theoretical spirit, it is also natural to consider L^p**-optimal** weighted polynomials [6, 7, 8, 9] with respect to the varying weight w^n and the measure σ, i.e. to minimize the L^p-norm

$$\|P_n w^n\|_{L^p(\sigma)} := \left(\int |P_n|^p w^{np}\mathrm{d}\sigma\right)^{\frac{1}{p}} \tag{1-7}$$

over all monic polynomials of degree n. The paper addresses two questions; the first concerns the location of the roots of L^p-optimal polynomials or rather where the roots cannot be. We find that eventually (i.e. for sufficiently large n) all roots fall in an arbitrary neighborhood of the convex hull of the support S_w of the relevant equilibrium measure μ_w (whose definition is recalled in Sect. 1.1); this is accomplished in Prop. 2.1 (with a more precise statement).

If the support is not convex (possibly with holes and several disjoint connected components) then we can state that (Prop. 2.2) all but a finite (and uniformly bounded) number of roots falls within any arbitrary neighborhood of the polynomially convex hull of the support. A consequence of the above is that

$$\text{``}\lim_{n\to\infty}\frac{1}{n}\ln P_n(z) = \int \ln|z-t|\mathrm{d}\mu_w(t)\text{''} \tag{1-8}$$

where the quotation marks indicate that the statement is imprecise (see Thm. 2.1 for the precise one); the convergence is uniform over closed subsets of the unbounded component $\mathbb{C}\setminus S_w$. If K does not contain roots of P_n (eventually) then we can remove the quotations and the statement is correct (for example, if K is disjoint from the convex hull of S_w).

The second question deals with the leading order behaviour of the L^p norms of the p–optimal polynomials and we show that – in fact – they all have the exact same asymptotic behaviour

$$\lim_{n\to\infty}\left(\|P_n w^n\|_{L^p(\sigma)}\right)^{1/n} = \exp(-F_w) \tag{1-9}$$

where F_w is the *modified Robin's constant* of the equilibrium measure μ_w, and this limit is independent of $1\le p\le\infty$.

The case of $p=\infty$ of the above statements is known in the literature ([10] for the unweighted case, and [1] for the weighted one) even in the varying weight case. It seems to be new for $p\neq\infty$.

1.1 Potential-theoretic background

We will consider polynomials on a closed set $\Sigma\subsetneq\mathbb{C}$, called a **condenser**; on this set a reference measure σ is supposed to be given. Since we are not seeking the greatest generality (at cost of simplicity) we will restrict ourselves to the following situations:

- Σ is a finite collection of Jordan curves, with typical example the real axis or union of intervals thereof. In this case the measure σ is simply the arc-length,
- Σ is a finite union of regions of the plane, with the area measure $d\sigma = dA$,
- Σ is a finite union of elements of both types above.

The main focus will be $\Sigma=\mathbb{C}$ or $\Sigma=\mathbb{R}$ or $\Sigma=\gamma$ a smooth curve in $\mathbb{C}$.

The **weight function** $w:\ \Sigma\to[0,\infty)$ introduced above is assumed to satisfy the following standard **admissibility** conditions ([1]):

- w is upper semi-continuous,
- $\mathrm{cap}(\{z : w(z) > 0\})$ has positive capacity,
- $|z|w(z) \to 0$ as $|z| \to \infty$ in Σ.

The **potential** $V(z)$ is the function for which $w(z) = \exp(-V(z))$ and it inherits the corresponding admissibility conditions. The weighted energy functional is defined as follows; for a probability measure μ on Σ we define

$$\mathcal{I}_w[\mu] := \int\int \ln\frac{1}{|z-w|}\mathrm{d}\mu(z)\mathrm{d}\mu(w) + 2\int V(z)\mathrm{d}\mu(z)\ . \tag{1-10}$$

It is well known in potential theory [1] that there exists a unique measure μ_w that realizes the minimum of $\mathcal{I}_w$; such a measure is referred to as the **equilibrium measure**. Its support $S_w = \mathrm{supp}(\mu_w)$ is a compact set.

Although it will not be used directly we recall the following indirect characterization of μ_w: if we denote with

$$U^\mu(z) := \int \ln\frac{1}{|z-w|}\mathrm{d}\mu(w) \tag{1-11}$$

the **logarithmic potential** of a probability measure μ then μ_w is uniquely characterized as follows. There exists a constant F_w called the **modified Robin's constant** such that the **effective potential**

$$\Phi(z) := U^{\mu_w}(z) + V(z) - F_w \tag{1-12}$$

satisfies

$$\left.\begin{cases} \Phi(z) \le 0 & z \in S_w \\ \quad\text{and} & \\ \Phi(z) \ge 0 & z \in \Sigma \quad \text{q. e.}\end{cases}\right\} \Rightarrow \Phi(z) = 0 \quad z \in S_w \quad \text{q. e.} \tag{1-13}$$

where 'q. e.' stands for "quasi-everywhere", namely up to sets of zero logarithmic capacity.

2 Where the roots are not

Let $P_n(z)$ be any sequence of polynomials of degree $\le n$, $S_w = \mathrm{supp}(\mu_w)$ and let $\mathcal{N} \supset S_w$ be an open bounded set containing S_w.

In [1] III.6 (eq. 6.4) it is shown in general (under certain assumptions on Σ, w and σ) that if P_n is any sequence of polynomials of degree $\le n$ we have

$$\|P_n w^n\|^p_{L^p(\sigma)} = \int_\Sigma |P_n w^n|^p \mathrm{d}\sigma \le (1 + C\mathrm{e}^{-cn})\int_{\mathcal{N}} |P_n w^n|^p \mathrm{d}\sigma \tag{2-1}$$

where the constants $c > 0$ and C **do not depend** on the polynomial sequence under consideration (they depend –of course– on w, p and $\mathcal{N}$).

The inequality (2-1) can be rewritten or equivalently ($\chi_{\mathcal{N}}$ denotes the indicator function of the set $\mathcal{N}$)

$$1 \le \frac{\|P_n w^n\|^p_p}{\|P_n w^n \chi_{\mathcal{N}}\|^p_p} \le 1 + C\mathrm{e}^{-cn}\ . \tag{2-2}$$

The inequality (2-1) shows that the norm of $P_n w^n$ lives in a small neighborhood of S_w; this will be the main tool in what follows. The ideas follow very closely similar steps for the so-called *weighted Chebyshev polynomials* in III.3 of [1].

For any set $X \subset \mathbb{C}$ we will denote by $\mathrm{Co}(X)$ the **(closed) convex hull** of said set.

Let, as before $\mathcal{N} \supset S_w$ be an open, bounded neighborhood of S_w. We start from the

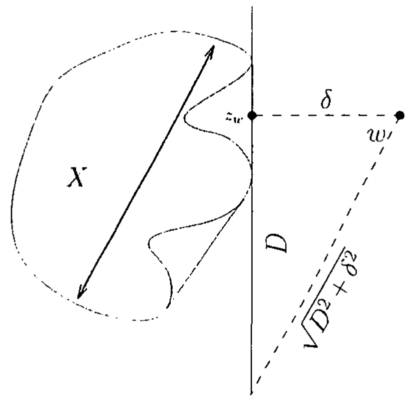

Figure 1: Figure for Lemma 2.1

Lemma 2.1 *Let $X \subset \mathbb{C}$ be compact that is not a singleton and $w \in \mathbb{C}$ be such that* $\mathrm{dist}(w, \mathrm{Co}(X)) = \delta > 0$. *Then*

$$\frac{|z - z_w|}{|z - w|} \leq \frac{D}{\sqrt{D^2 + \delta^2}} < 1\,, \qquad D := \mathrm{diam}(\mathrm{Co}(X)) \tag{2-3}$$

where $z_w \in \mathrm{Co}(X)$ is the (unique) closest point to w.

Proof. The set $\mathrm{Co}(X)$ lies entirely on one half-plane passing through z_w and perpendicular to the line segment $[z_w, w]$. Let θ_w the smallest angle such that $\mathrm{Co}(X)$ is entirely contained in a θ_w sector centered at w; by the convexity and compactness of $\mathrm{Co}(X)$, $\theta_w < \pi$. In fact we can estimate the upper bound on w of such θ_w as

$$\theta_w \leq \arctan\left(\frac{\delta}{D}\right)\,, \qquad D = \mathrm{diam}(\mathrm{Co}(X))\,. \tag{2-4}$$

from which (2-3) follows (see Fig. 1). **Q.E.D.**

Proposition 2.1 *Let K be a closed subset in $\mathbb{C} \setminus \mathrm{Co}(S_w)$. Then eventually there are no roots of P_n belonging to K. In particular, for any $\epsilon > 0$ there is a $n_0 \in \mathbb{N}$ such that $\forall n > n_0$ all roots of P_n are within distance ϵ from the convex hull.*

Proof. Since K is closed and has no intersection with $\mathrm{Co}(S_w)$ we have $\mathrm{dist}(K, \mathrm{Co}(S_w)) = 2\delta > 0$; Let $\mathcal{N}$ be the δ-fattening of $\mathrm{Co}(S_w)$, namely

$$\mathcal{N} := \{z \in \mathbb{C} \,:\, \mathrm{dist}(z, \mathrm{Co}(S_w)) \leq \delta\} \tag{2-5}$$

It is easy to see that $\mathcal{N}$ is convex as well.

Now consider the p-optimal polynomial $P_n(z)$ and let us decompose it as $P_n(z) = R_n(z)Q_n(z)$ where $R_n(z)$ is the factor of all roots within K; note that each of these roots is at distance $\geq \delta$ from $\mathcal{N}$.

For each root z_j of $R_n(z)$ we can find the closest point $\widetilde{z}_j \in \mathcal{N}$; hence we will define $\widetilde{R}_n(z)$ as the "proximal substitute" of R_n, where each root of R_n has been replaced by its proximal point in $\mathcal{N}$. Then for all $z \in \mathcal{N}$ we have $|\widetilde{R}_n(z)| \leq \rho^{r_n}|R_n(z)|$ where $r_n = \deg(R_n)$. Indeed, by Lemma 2.1,

$$|\widetilde{R}_n(z)| = \prod_{j=1}^{r_n} |z - \widetilde{z}_j| \leq \rho^{r_n} \prod_{j=1}^{r_n} |z - z_j| = \rho^{r_n}|R_n(z)| \tag{2-6}$$

Thus **pointwise**

$$|\widetilde{P}_n(z)| \leq \rho^{r_n}|P_n(z)| \quad z \in \mathcal{N}\ , \quad \rho := \frac{D}{\sqrt{D^2+\delta^2}} < 1\ , \quad D := \mathrm{diam}(\mathcal{N}). \tag{2-7}$$

We thus have

$$\left\|\widetilde{P}_n w^n \chi_{\mathcal{N}}\right\|_p^p \overset{\text{by (2-7)}}{\leq} \rho^{pr_n} \left\|P_n w^n \chi_{\mathcal{N}}\right\|_p^p \leq \rho^{pr_n} \left\|P_n w^n\right\|_p^p \tag{2-8}$$

By definition, the p–optimal polynomials P_n have the smallest L^p norm and hence

$$1 \leq \frac{\left\|\widetilde{P}_n w^n\right\|_p^p}{\|P_n w^n\|_p^p} \overset{(2\text{-}1)}{\leq} (1+C\mathrm{e}^{-cn})\frac{\left\|\widetilde{P}_n w^n \chi_{\mathcal{N}}\right\|_p^p}{\|P_n w^n\|_p^p} \overset{(2\text{-}7)}{\leq} (1+C\mathrm{e}^{-cn})\rho^{pr_n}. \tag{2-9}$$

where in the second inequality we have used (2-1) on the sequence of polynomials $\widetilde{P}_n$. Inequalities (2-9) amount to

$$1 \leq (1+C\mathrm{e}^{-cn})\rho^{pr_n} \tag{2-10}$$

and recall that $\rho < 1$. This inequality implies at once that $\limsup r_n = 0$, and hence the sequence of natural numbers r_n must eventually be identically zero. The second statement in the theorem is simply obtained by taking for K the complement of the ϵ–fattening of $\mathrm{Co}(S_w)$. **Q.E.D.**

Having established that there are no roots (eventually) "outside" the convex hull, we get some further information about what happens in general.

We borrow the following nice

Lemma 2.2 (Lemma III.3.5 in [1], originally in [11]) *If S and K are compact sets such that $Pc(S) \cap K = \emptyset$ then there is a positive integer $m = m(K)$ and a constant $0 < \alpha(K) < 1$ such that for all $(z_1, \ldots, z_m) \in K^m$ there are points $\widetilde{z}_1, \ldots, \widetilde{z}_m$ such that the rational function*

$$r(z) := \frac{\prod_{j=1}^{m(K)}(z-\widetilde{z}_j)}{\prod_{j=1}^{m(K)}(z-z_j)} \tag{2-11}$$

satisfies

$$\sup_{z\in S}|r(z)| \leq \alpha(K)\ . \tag{2-12}$$

Lemma 2.2 allows us to prove

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Proposition 2.2 *For any compact set K contained in the unbounded component of $\mathbb{C} \setminus S_w$ the number of roots of the p-optimal polynomials P_n within K is bounded. In particular $\forall \epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $\forall n > n_0$ all but a finite number (uniformly bounded) roots of P_n lie within distance ϵ from the polynomial convex hull of S_w (i.e. $\mathbb{C} \setminus \Omega$, where Ω is the unbounded component of $\mathbb{C} \setminus S_w$).*

Proof. In parallel with the proof of Prop. 2.1 let $2\delta = \mathrm{dist}(K, S_w)$ and let $\mathcal{N}$ be the δ-fattening of S_w. We decompose $P_n = R_n Q_n$ where R_n has r_n roots (counted with multiplicity) within K. We will prove that $r_n < m(K)$ eventually, where $m(K)$ is the number of poles in Lemma 2.2 for $S = \mathcal{N}$ and K. Proceeding by contradiction, there would be a subsequence where $r_n \geq m(K)$; but then we can use Lemma 2.2 to find a polynomial $\widetilde{R}_n$ such that

$$|\widetilde{R}_n(z)| \leq \alpha(K)|R_n(z)|\,, \qquad z \in \mathcal{N} \ \Rightarrow\ |\widetilde{P}_n(z)| \leq \alpha(K)|P_n(z)|\,,\ \ z \in \mathcal{N}. \tag{2-13}$$

At this point we proceed exactly as in the proof of Prop. 2.1 starting from (2-8) with $\rho^{pr_n} \mapsto \alpha(K)$, namely,

$$\left\|\widetilde{P}_n w^n \chi_{\mathcal{N}}\right\|_p^p \overset{\text{by (2-13)}}{\leq} \alpha(K)^p \left\|P_n w^n \chi_{\mathcal{N}}\right\|_p^p \leq \alpha(K)^p \left\|P_n w^n\right\|_p^p \ . \tag{2-14}$$

By the p–optimality of the polynomial P_n we must have

$$1 \leq \frac{\left\|\widetilde{P}_n w^n\right\|_p^p}{\|P_n w^n\|_p^p} \overset{(2\text{-}1)}{\leq} (1 + Ce^{-cn}) \frac{\left\|\widetilde{P}_n w^n \chi_{\mathcal{N}}\right\|_p^p}{\|P_n w^n\|_p^p} \overset{(2\text{-}13)}{\leq} (1 + Ce^{-cn})\alpha(K)^p. \tag{2-15}$$

It is clear that the last expression in (2-15) is eventually *less than one* (since $\alpha(K) < 1$), which leads to a contradiction with the assumption that there were $\geq m(K)$ roots in K. The last statement follows from the fact that there are no roots outside the convex hull by Prop. 2.1 together with the above. **Q.E.D.**

Example 2.1 *Suppose that the support of the equilibrium measure consists of intervals in the real axis, as in the case of ordinary orthogonal polynomials. It is an exercise to see that for any gap the number $m(K) = 2$ and hence there can be at most one zero within each gap.*

We next prove

Theorem 2.1 *Let Ω be the unbounded connected component of $\mathbb{C} \setminus S_w$ and $K \subset \Omega$ a compact subset. Let $z_{\ell,n}(K)$ be the roots of P_n belonging to K, $\ell = 1, \ldots, m_n(K)$. Then, uniformly in K we have*

$$\lim_{n\to\infty} \frac{1}{n} \ln|P_n(z)| + \frac{1}{n} \sum_{j=1}^{m_n(K)} G_\Omega(z, z_{\ell,n}) = \int \ln|z - t| \mathrm{d}\mu(t)\ , \tag{2-16}$$

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where $G_\Omega(z,w)$ is the Green's function of Ω, namely the function such that

$$\triangle_z G_\Omega(z,w) \equiv 0\ ,\quad z \in \Omega \setminus \{w\} \tag{2-17}$$

$$G_\Omega(z,w) = 0\ ,\quad z \in \partial\Omega \tag{2-18}$$

$$G(z,w) \geq 0\ .\quad z,w \in \Omega \tag{2-19}$$

$$G_\Omega(z,w) = \ln\frac{1}{|z-w|} + \mathcal{O}(1)\quad z \to w \tag{2-20}$$

Additionally, if K is closed and does not contain (eventually) any roots, then, uniformly,

$$\lim_{n\to\infty}\frac{1}{n}\ln|P_n(z)| = \int \ln|z-t|\mathrm{d}\mu(t) \tag{2-21}$$

Proof. We reason on the functions

$$f_n(z) := \frac{1}{n}\ln|P_n(z)| + \frac{1}{n}\sum_{j=1}^{m_n(K)} G_\Omega(z, z_{\ell,n}) - \int \ln|z-t|\mathrm{d}\mu(t) \tag{2-22}$$

We will see in Prop. 3.1 together with Corollary 3.1 that $\forall\epsilon > 0\ \exists n_0 :\ n \geq n_0$

$$\frac{1}{n}\ln|P_n(z)w^n(z)| \leq -F_w + \epsilon\ ,\qquad \forall z \in \mathbb{C} \tag{2-23}$$

Additionally, the $f_n(z)$'s are subharmonic in Ω and harmonic in a neighborhood of $z = \infty$: indeed all roots are uniformly bounded (from Prop. 2.1) and the Green's function $G_\Omega(z,w)$ is harmonic away from the singularity $z = w$ (in a neighborhood of which it is *super*harmonic) and in the neighborhoods of $z_{\ell,n}$ the f_n's are actually harmonic because the singularities coming from P_n's cancel out exactly those coming from the Green's functions.

For $z \in \partial\Omega$ and $\forall\epsilon > 0$ we have eventually (recall that $G_\Omega(z,w) = 0$ for $z \in \partial\Omega$)

$$f_n(z) \leq V(z) + U^{\mu_w}(z) - F_w + \epsilon \leq \epsilon\ ,\qquad z \in \partial\Omega. \tag{2-24}$$

Since $f_n(z)$ are subharmonic, they cannot have isolated maxima in the interior of Ω and hence we conclude that $f_n(z) \leq \epsilon$ throughout Ω (including $z = \infty$).

Let $f_\infty(z) = \limsup_{n\to\infty} f_n(z)$; then $\forall\epsilon > 0$

$$f_\infty(z) = \limsup_{n\to\infty} f_n(z) \leq \epsilon\ \Rightarrow f_\infty(z) \leq 0\ ,\qquad z \in \mathbb{C} \tag{2-25}$$

Let $\mathrm{Pc}(S_w) = \mathbb{C} \setminus \Omega$ be the *polynomial convex hull* of S_w and let now K be a **compact** set $K \subset \Omega$.

We next analyze the lim inf; let $z_0 \in K$ and set

$$L_{z_0} := \liminf f_n(z_0) \leq f_\infty(z_0) \leq 0\ . \tag{2-26}$$

where $z_0 \in K$ is some (arbitrary but fixed) point. There is a subsequence n_k of the numbers $f_n(z_0)$'s which converges to this limit; out of it, we can extract another subsequence (which we denote again n_k for brevity) such that the counting measures σ_{n_k} have a weak* limit (since they are all compactly supported) which we denote by σ_{z_0} (note that both the subsequence and this limiting distribution may depend on z_0). Prop. 2.2 implies that its support of σ_{z_0} lies in the polynomial convex hull of S_w; in particular the function $\ln|z - \bullet|$ is harmonic on $\mathrm{supp}(\sigma_{z_0})$ for any $z \in K$. Let $\widehat{\sigma}_{n_k}$ be the restriction of σ_{n_k} to those atoms outside of K; we know that it differs from σ_{n_k} by a finite number $m_n(K)$ of atoms (uniformly bounded in n) and hence it obviously has the same weak* limit. Now, for any $z \in K$ along the chosen subsequence we have

$$0 \geq f_\infty(z) \geq \lim_{k\to\infty} f_{n_k}(z) = \lim_{k\to\infty} \int \ln|z-t| \mathrm{d}\widehat{\sigma}_{n_k}(t) - \int \ln|z-t| \mathrm{d}\mu_w(t) + \\ + \frac{1}{n_k} \sum_{\ell=1}^{m_n(K)} \left(G_\Omega(z, z_{\ell,n_k}) + \ln|z - z_{\ell,n_k}|\right) \tag{2-27}$$

Since $G_\Omega(z, w) + \ln|z-w|$ is jointly continuous in z, w for $z, w \in \Omega$, it is also (jointly) bounded on compact sets; we know already that z_{ℓ,n_k} all are uniformly bounded, hence the last term in (2-27) tends to zero. We thus have

$$\lim_{k\to\infty} f_{n_k}(z) = \lim_{k\to\infty} \int \ln|z-t| \mathrm{d}\widehat{\sigma}_{n_k}(t) - \int \ln|z-t| \mathrm{d}\mu_w(t) = \\ = \int \ln|z-t| \mathrm{d}\widehat{\sigma}_{z_0}(t) - \int \ln|z-t| \mathrm{d}\mu_w(t) \tag{2-28}$$

The right hand side of (2-28) is harmonic in Ω (by inspection) and by (2-25) it is ≤ 0; on the other hand at $z = \infty$ it vanishes (since both measures are probability measures) and hence it must be identically zero. Evaluating it at $z = z_0$ yields that $L_{z_0} = \liminf_{n\to\infty} f_n(z_0) = 0$; since z_0 was arbitrary, this shows that $\lim_{n\to\infty} f_n(z) = 0$; the uniformity of the convergence follows from the fact that the sequence of functions

$$h_n(z) := \int \ln|z-t| \mathrm{d}\widehat{\sigma}_n(t) \tag{2-29}$$

are *equicontinuous* for $z \in K$ and hence the Arzela–Ascoli Theorem [12] guarantees uniform convergence. To see equicontinuity we compute

$$|h_n(z) - h_n(z')| = \int \ln\left|\frac{z-t}{z'-t}\right| \mathrm{d}\widehat{\sigma}_{n_k}(t) = \int \ln\left|1 + \frac{z-z'}{z'-t}\right| \mathrm{d}\widehat{\sigma}_{n_k}(t) \leq \int \frac{|z-z'|}{|z'-t|} \mathrm{d}\widehat{\sigma}_{n_k}(t) \leq \\ \leq \frac{|z-z'|}{\mathrm{dist}(K, S_w)} \tag{2-30}$$

Note that the above chain of inequalities applies more generally for any *closed* $K \subset \Omega$ and also says that the sequence is uniformly Lipschitz.

To prove (2-21) we note that we have used compactness only after (2-27), but if $m_n(K) \equiv 0$ (eventually) then the same arguments prove uniform convergence without having to use compactness. **Q.E.D.**

Theorem 2.1 says loosely speaking that $\frac{1}{n}\ln|P_n|$ converges to the logarithmic transform of the equilibrium measure as uniformly as it is possible on the "outside" of the support, given that there are possibly some stray roots; if we restrict to the outside of the convex hull of S_w, then this convergence is truly uniform (over closed subsets) because -eventually- there are no roots at all (Prop. 2.1).

Theorem 2.1 has an interesting corollary

Corollary 2.1 *Let $h(z)$ be any harmonic function on a neighborhood of $Pc(S_w)$ and let σ be a weak* limit point of the counting measures of the L^p-optimal polynomials. Then*

$$\int h(z)\mathrm{d}\sigma(z) = \int h(z)\mathrm{d}\mu_w(z) \ . \tag{2-31}$$

Proof. By Mergelyan's theorem it suffices to verify it for the monomials z^j; we have seen in the proof of Thm. 2.1 (2-27 and discussion thereafter) that

$$\int \ln|z-t|\mathrm{d}\sigma(t) - \int \ln|z-t|\mathrm{d}\mu_w(t) \equiv 0 \tag{2-32}$$

for $z \in \Omega$ (the complement of the polynomial convex hull of S_w). Taking the large z expansion we have easily the statement **Q.E.D.**

Remark 2.1 *The Theorem 2.1 and Corollary 2.1 assert that whatever limiting distribution the roots of the p-optimal polynomials may have, it must be a balayage of the equilibrium measure onto the support of this limiting distribution. In order not to swindle the reader, we should point out that it falls short of saying that there is a* unique *limiting distribution, and even further away from any statement about what distribution that should be.*

3 Norm estimates

3.1 Upper estimate for the norms

The aim of this section is twofold: first we will prove that if $P_n w^n$ are the p-optimal weighted polynomials then

$$\lim_{n\to\infty} \frac{1}{n} \ln \|P_n w^n\|_p = -F_w \quad \Leftrightarrow \quad \|P_n w^n\|_p = \mathrm{e}^{-nF_w + o(n)} \tag{3-1}$$

En route we will see that the L^p norms of the wave-functions $P_n w^n$ are asymptotically equal to the L^∞ ones. In particular this implies that the n-th root of the wave functions is uniformly bounded.

Proposition 3.1 *Let $P_n w^n$ be the p-optimal weighted polynomial; then*

$$\limsup_{n\to\infty} \frac{1}{n} \ln \|P_n w^n\|_p \leq -F_w \ , \tag{3-2}$$

where ℓ is the Robin constant for the equilibrium measure.

Proof. We compare the L^p norms of the $P_n w^n$'s with the weighted Fekete polynomials $F_n w^n$. Let $\mathcal{N}$ be a bounded open neighborhood of S_w. Then

$$\|P_n w^n\|_p \overset{\text{by optimality}}{\leq} \|F_n w^n\|_p \overset{(2\text{-}1)}{\leq} (1 + Ce^{-cn})\|F_n w^n \chi_{\mathcal{N}}\|_p \leq (1 + Ce^{-cn})\|F_n w^n\|_\infty \mathcal{A}rea(\mathcal{N})^{\frac{1}{p}} \quad (3\text{-}3)$$

Now taking $\frac{1}{n}\ln(\cdot)$ of both sides gives

$$\frac{1}{n}\ln(\|P_n w^n\|_p) \leq \frac{1}{n}\ln(\|F_n w^n\|_p) \leq \frac{1}{n}\ln(\|F_n w^n\|_\infty) + \mathcal{O}(n^{-1}) \quad (3\text{-}4)$$

Since

$$\lim_{n\to\infty} \frac{1}{n}\ln(\|F_n w^n\|_\infty) = -F_w \quad (3\text{-}5)$$

(see Thm. III.1.9 in [1]) we have

$$\limsup_{n\to\infty} \frac{1}{n}\ln(\|P_n w^n\|_p) \leq \limsup_{n\to\infty} \frac{1}{n}\ln(\|F_n w^n\|_p) \leq \limsup_{n\to\infty} \frac{1}{n}\ln(\|F_n w^n\|_\infty) \leq -F_w \,. \quad (3\text{-}6)$$

Q.E.D.

Remark 3.1 *It may be of some importance to note that the above proof can be used to show .*

$$\limsup_{n\to\infty} \ln \frac{\|P_n w^n\|_p}{\|F_n w^n\|_\infty} \leq \sqrt[p]{\mathcal{A}rea(S_w)} \quad (3\text{-}7)$$

3.2 Lower estimate for the norms

We follow the idea in [1], pp 182.

Lemma 3.1 *Let $P_n(z)$ be a sequence of polynomials of degree at most n. Assume further that the potential V is twice continuously differentiable. Then there is a constant $D > 0$ and d_Σ (the Hausdorff dimension of Σ, which for us is either 2 or 1) such that*

$$\frac{\|P_n w^n\|_p}{\|P_n w^n\|_\infty} \geq D n^{-\frac{d_\Sigma}{p}} \quad (3\text{-}8)$$

In particular

$$\liminf_{n\to\infty} \|P_n w^n\|_p^{\frac{1}{n}} \geq \liminf_{n\to\infty} \|P_n w^n\|_\infty^{\frac{1}{n}} \quad (3\text{-}9)$$

$$\limsup_{n\to\infty} \|P_n w^n\|_p^{\frac{1}{n}} \geq \limsup_{n\to\infty} \|P_n w^n\|_\infty^{\frac{1}{n}} \,. \quad (3\text{-}10)$$

Proof. We work with the normalized polynomials

$$Q_n(z) := \frac{1}{\|P_n w^n\|_\infty} P_n(z). \quad (3\text{-}11)$$

Let z_0 be a point where $|Q_n(z)w^n(z)|$ achieves its maximum value 1 (such a point exists by the assumed admissibility conditions on w). We claim that

$$\exists C > 0 : \ |z - z_0| \leq \frac{1}{2\mathrm{e}Cn} \implies |Q_n(z)|\mathrm{e}^{-nV(z)} \geq \frac{1}{2\mathrm{e}} \tag{3-12}$$

Since $|Q(z_0)|\mathrm{e}^{-nV(z_0)} = 1$ the inequality can be rewritten

$$|Q_n(z)| \leq |Q_n(z_0)|\mathrm{e}^{n(V(z)-V(z_0))} \ , \quad \forall z \in \mathbb{C}. \tag{3-13}$$

Let $\delta > 0$ and set $C_\delta(z_0) := \sup_{|z-z_0|=\delta} |V(z) - V(z_0)|$; since we are assuming $V(z)$ to be twice continuously differentiable, $z_0 \in S_w$ and S_w is compact, we see that a simple argument shows $C_\delta(z_0) < C\delta$ for some constant $C > 0$ (independent of $z_0 \in S_w$). Let $|z - z_0| < \frac{1}{2}\delta$; the formula of Cauchy for the derivative implies

$$|Q_n'(z)| \leq |Q_n(z_0)|\frac{2}{\delta}\mathrm{e}^{nC\delta} \ , \quad |z - z_0| \leq \frac{1}{2}\delta. \tag{3-14}$$

On the even smaller disk $|z - z_0| < \frac{1}{4\mathrm{e}}\delta$ we have

$$|Q_n(z) - Q_n(z_0)| \leq \int_{z_0}^{z} |Q_n'(t)||\mathrm{d}t| \leq |Q_n(z_0)|\frac{2\mathrm{e}^{nC\delta}|z - z_0|}{\delta} \leq \frac{1}{2}|Q_n(z_0)|\mathrm{e}^{nC\delta-1} \tag{3-15}$$

If we choose $\delta = \frac{1}{Cn}$ and hence $|z - z_0| < \frac{\delta}{4\mathrm{e}} = \frac{1}{4Cn\mathrm{e}}$ we have

$$|Q_n(z) - Q_n(z_0)| \leq \frac{1}{2}|Q_n(z_0)| \quad \Rightarrow \quad |Q_n(z)| \geq \frac{1}{2}|Q_n(z_0)| \,. \tag{3-16}$$

Multiplying both sides

$$|Q_n(z)|\mathrm{e}^{-n(V(z)-V(z_0))} \geq \frac{1}{2}|Q_n(z_0)|\mathrm{e}^{-n(V(z)-V(z_0))} \geq \frac{1}{2}|Q_n(z_0)|\mathrm{e}^{-nC\delta} = \frac{|Q_n(z_0)|}{2\mathrm{e}} \quad \Rightarrow$$

$$|Q_n(z)|\mathrm{e}^{-nV(z)} \geq \frac{1}{2\mathrm{e}}|Q_n(z_0)|\mathrm{e}^{-nV(z_0)} = \frac{1}{2\mathrm{e}} \,. \tag{3-17}$$

Integrating the inequality (3-12)

$$\left(\int_\Sigma |Q_n(z)w^n|^p \mathrm{d}_\Sigma z\right)^{\frac{1}{p}} \geq \left(\int_{|z-z_0|<\frac{\delta}{4\mathrm{e}}} |Q_n(z)w^n|^p \mathrm{d}_\Sigma z\right)^{\frac{1}{p}} \geq \tag{3-18}$$

$$\geq \frac{1}{2\mathrm{e}}\left[B_\Sigma\left(\frac{1}{4nC\mathrm{e}}\right)\right]^{\frac{1}{p}} \tag{3-19}$$

Here $B_\Sigma(\delta)$ is the $\mathrm{d}\sigma$ volume of the ball of radius δ centered at z_0 in Σ: in the case $\Sigma = \mathbb{C}$ this is simply $\pi\delta^2$. in the case Σ is a smooth curve then $B_\Sigma(\delta) \geq c\delta$ for some $c > 0$. The only important fact for us below is that $B_\Sigma(\delta)$ is bounded below by some positive power of δ. Therefore, recalling that $Q_n(z) = P_n(z)/\|P_n w^n\|_\infty$ the inequality (3-19) reads

$$(\|P_n(z)w^n\|_p) \geq \frac{1}{2\mathrm{e}}\left[B_\Sigma\left(\frac{1}{4nC\mathrm{e}}\right)\right]^{\frac{1}{p}} \|P_n w^n\|_\infty \tag{3-20}$$

Summarizing, there are constants $D > 0$ and d_Σ (the "dimension" of Σ. which for us is either 2 or 1) such that

$$\frac{\|P_n(z)w^n\|_p}{\|P_n w^n\|_\infty} \geq D n^{-\frac{d_\Sigma}{p}} . \tag{3-21}$$

Q.E.D.

Before proceeding we recall

Theorem 3.1 (Thm. I.3.6 in [1]) *Let P_n be any sequence of* monic *polynomials of degree n. Then*

$$\liminf_{n\to\infty} (\|P_n w^n\|_\infty)^{\frac{1}{n}} \geq \exp(-F_w) . \tag{3-22}$$

As a corollary of Thm. 3.1 and Prop. 3.1 we have

Corollary 3.1 *The norms of the p-optimal polynomials satisfy*

$$\sqrt[n]{\|P_n w^n\|_p} \to e^{-F_w} ; \qquad n \to \infty . \tag{3-23}$$

Proof. Using Lemma 3.1 and (3-9) together with Thm. 3.1 we have that the lim inf of the left hand side cannot be less than e^{-F_w}:

$$-F_w \overset{\text{Prop. 3.1}}{\geq} \limsup_{n\to\infty} \frac{1}{n} \ln \|P_n w^n\|_p \geq \liminf_{n\to\infty} \frac{1}{n} \ln \|P_n w^n\|_p \overset{\text{Prop. 3.1}}{\geq} \liminf_{n\to\infty} \frac{1}{n} \ln \|P_n w^n\|_\infty \overset{\text{Thm. 3.1}}{\geq} -F_w .$$

Q.E.D.

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Chapter 11

External potentials for two-point quadrature domains

11.1 Summary

This chapter is based on the manuscript presented in the following section. The goal of this study is to solve a forward potential problem corresponding to the simplest types of classical quadrature domains corresponding to two quadrature nodes. The paper is organized as follows:

1. Symmetric connected two point quadrature domains are considered and, based on the algebraic structure of the Schwarz function corresponding to the boundary, admissible quasi-harmonic external potentials are constructed (Sec. 1).

2. A confluent limit of two-point quadrature domains with a single second order node is studied and a corresponding admissible quasi-harmonic background potential is constructed (Sec. 2).

3. As a degenerate case, quadrature domains consisting of two disjoint congruent

disks are considered and a suitable admissible quasi-harmonic potential is found. The corresponding equilibrium measure is shown to coincide with the normalized area measure concentrated on the disks (Sec. 3).

4. As an appendix, the quasi-harmonic potentials are represented as superharmonic perturbations of the Gaussian potential for explicitly calculated positive perturbing measures along straight lines (Sec. 4).

11.1.1 Forward potential problem for external fields

For quasi-harmonic admissible potentials the equilibrium measure is often (not always) given by the normalized area measure restricted to a compact set K. Assuming that the boundary ∂K consists of a non-singular analytic Jordan curve, one of the methods to recover the support set K, as we have seen in Sec. 6.6, is to find the positive part $S^+(z)$ of the Schwarz function of the boundary and, by the singularity correspondence between the Schwarz function and the exterior conformal map, find a uniformizing map for $\mathbb{C}P^1 \setminus K$.

Motivated by applications in approximation theory [13], one may consider the *forward problem* of finding the quasi-harmonic potential associated to a given compact set K of positive area. For sets with simply connected exterior and an explicit exterior uniformizing map the Schwarz function and its decomposition into positive and negative parts can be calculated explicitly. This means that a corresponding quasi-harmonic potential can be obtained by integration according to Thm. 6.6.1.

It is easy to see that the exact same strategy can be followed if the set K is given by its interior conformal map instead: an explicitly given conformal map of the boundary allows the recovery of the Schwarz function and a corresponding background potential.

The special cases considered in the paper are illustrating the simplest cases in which K is given by a rational interior conformal map: two point quadrature domains with either two distinct or one confuent quadrature nodes, corresponding to rational interior conformal maps of degree two. As a consequence, the corresponding Schwarz functions satisfy quadratic equations of the form

$$A(z)S(z)^2 + B(z)S(z) + C(z) = 0 \tag{11.1}$$

where $A(z), B(z), C(z)$ are rational functions in z. The positive part $S^+(z)$ of the Schwarz function is therefore algebraic also (the negative part is rational because of the quadrature property). The integration of this algebraic function leads to a quasi-harmonic admissible potential $Q(z)$. A proper choice of the branch cut structure of $S^+(z)$ allows the external potential to be expressed in the form

$$Q(z) = \alpha|z|^2 + U^\nu(z) \tag{11.2}$$

where ν is a positive measure supported on branch cut of $S_+(z)$. The detailed formulae and the geometric construction of the background potentials can be found in Sec. 1 and 2.

The degenerate case of two disjoint disks is considered in Sec. 4. The corresponding external potential is of a much simpler form and therefore the variational inequalities may be checked easily, confirming that the equilibrium measure of the calculated potential is supported on the prescribed disks. A superharmonic perturbation representation for this potential is also obtained by finding a positive measure ν supported on the imaginary axis.

11.2 External potentials for two-point quadrature domains, manuscript

June 4, 2009

EXTERNAL POTENTIALS FOR TWO-POINT QUADRATURE DOMAINS

F. BALOGH

Abstract. Admissible background potentials are calculated for three different kinds of two-point quadrature domains: bicircular quartics, limacons and the union of two disjoint congruent disks. The equilibrium measures of the potentials are normalized area measures concentrated to the corresponding quadrature domains. Moreover, all three potentials may be represented as a sum of a dominant Gaussian quadratic term and a pure logarithmic potential of a positive measure on the plane.

Contents

1. Bicircular Quartics

1.1. The conformal map and the Schwarz function of the boundary. First of all, we start from a special class of rational conformal maps of degree two and show that the image of the unit disk under such mappings satisfies a two-point quadrature identity. We refer to [2], [4] and [5] as standard references on quadrature domains.

Let $r > 0$ and $E \in [1, \infty)$ be given and consider the following conformal mapping on $\mathbb{D}$:

$$f(\zeta) := \frac{r\zeta}{E^2 - \zeta^2} = \frac{r}{2}\left[\frac{1}{E-\zeta} - \frac{1}{E+\zeta}\right]. \tag{1}$$

This mapping is univalent for all considered values of the parameters: for a pair $\zeta_1 \neq \zeta_2$ the equality $f(\zeta_1) = f(\zeta_2)$ implies that

$$\zeta_1\zeta_2 = -E^2 \tag{2}$$

and $E \geq 1$ is equivalent to the fact that there are no such pairs $\zeta_1, \zeta_2 \in \mathbb{D}$ satisfying (2).

Let G denote the image of the open unit disk $\mathbb{D}$ of the ζ-plane under the mapping f in the z-plane:

$$G := f(\mathbb{D}). \tag{3}$$

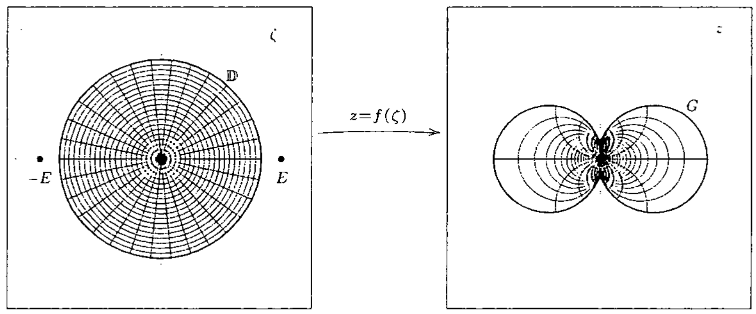

FIGURE 1. The conformal map

The boundary of G is a nonsingular analytic curve for $E > 1$ and therefore possesses a so-called *Schwarz function* analytic in a neighborhood of ∂G such that

$$S(z) = \bar{z} \quad z \in \partial G\ . \tag{4}$$

(see [2]). On the boundary ∂G,

$$\begin{cases} z = f(\zeta) = \dfrac{r\zeta}{E^2 - \zeta^2} \\ \bar{z} = \bar{f}\left(\dfrac{1}{\zeta}\right) = \dfrac{r\zeta}{E^2\zeta^2 - 1} \end{cases} \tag{5}$$

since the uniformizing parameter satisifies $\bar{\zeta} = \frac{1}{\zeta}$. This means that z and $y = \bar{z}$ are algebraically dependent on ∂G. Since, for fixed $z \in \partial G$, both polynomials

$$\begin{cases} p_1(\zeta) = z\zeta^2 + r\zeta - E^2 z \\ p_2(\zeta) = E^2 y\zeta^2 - r\zeta - y\ . \end{cases} \tag{6}$$

must vanish for some $\zeta \in \partial\mathbb{D}$, the resultant equation

$$P(z, y) = \left(z^2(1 - E^4)^2 - r^2E^2\right) y^2 - r^2(E^4 + 1)zy - r^2E^2z^2 \tag{7}$$

is satisfied by z and $\bar{z}$ on the boundary:

$$P(z, \bar{z}) = 0 \quad z \in \partial G\ . \tag{8}$$

It is important to note that

$$\partial G \subset \{z \colon P(z, \bar{z}) = 0\} \tag{9}$$

but the equation is also satisfied at $z = 0$ which is not on the boundary of G unless $E = 1$ (actually $\{z \colon P(z, \bar{z}) = 0\} = \partial G \cup \{0\}$).

1.2. **Quadrature identity.** The components of the standard inner-outer holomorphic decomposition of $S(z)$ with respect to ∂G are defined as follows:

$$\begin{cases} S_+(z) = \dfrac{1}{2\pi i}\displaystyle\int_{\partial G}\dfrac{S(t)dt}{t-z} & z \in G \\ S_-(z) = \dfrac{1}{2\pi i}\displaystyle\int_{\partial G}\dfrac{S(t)dt}{t-z} & z \in \mathbb{C}\setminus\overline{G} \end{cases} \tag{10}$$

Both functions are analytic in their domain of definition and have non-tangential limit functions on the boundary and they satisfy

$$S(z_0) = \lim_{z\to z_0} S_+(z) - \lim_{z\to z_0} S_-(z) \quad z_0 \in \partial G\ . \tag{11}$$

In our case, both functions admit simple analytic continuations to a large portion of the complex plane and the equality

$$S(z) = S_+(z) - S_-(z) \tag{12}$$

holds in every region containing ∂G into which all three functions above admit analytic continuations.

The explicit form of $S_-(z)$ can be calculated by using the conformal map f to parametrize the boundary of ∂G:

$$\begin{aligned} S_-(z) &= \frac{1}{2\pi i}\int_{\partial G}\frac{S(t)dt}{t-z} \\ &= \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{\overline{f(\zeta)}f'(\zeta)d\zeta}{f(\zeta)-z}\ . \end{aligned} \tag{13}$$

This can be rewritten as a contour integral of a rational function on the ζ-unit circle since $\overline{\zeta} = 1/\zeta$ there:

$$\begin{aligned} S_-(z) &= \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{\overline{f}\left(\frac{1}{\zeta}\right)f'(\zeta)d\zeta}{f(\zeta)-z} \\ &= \operatorname*{res}_{\zeta=\frac{1}{E}}\frac{\overline{f}\left(\frac{1}{\zeta}\right)f'(\zeta)d\zeta}{f(\zeta)-z} + \operatorname*{res}_{\zeta=-\frac{1}{E}}\frac{\overline{f}\left(\frac{1}{\zeta}\right)f'(\zeta)d\zeta}{f(\zeta)-z} \\ &= \frac{rf'\left(\frac{1}{E}\right)}{2E^2}\frac{1}{f\left(\frac{1}{E}\right)-z} + \frac{rf'\left(-\frac{1}{E}\right)}{2E^2}\frac{1}{f\left(-\frac{1}{E}\right)-z}\ . \end{aligned} \tag{14}$$

(For fixed $z \in \mathbb{C}\setminus\overline{G}$, this rational integrand has six poles, and only two of them, $\frac{1}{E}$ and $-\frac{1}{E}$, are inside the the unit circle.) In terms of the quantities

$$\begin{cases} m = \dfrac{rf'\left(\frac{1}{E}\right)}{2E^2} = \dfrac{r^2(E^4+1)}{2(E^4-1)^2} \\ a = -f\left(\dfrac{1}{E}\right) = \dfrac{rE}{E^4-1}\ , \end{cases} \tag{15}$$

the exterior part of the Schwarz function $S_-(z)$ is the rational function

$$S_-(z) = \frac{m}{a-z} - \frac{m}{a+z}\ , \tag{16}$$

with two simple poles. The polynomial equation $P(z,y)=0$ is equivalent to the rescaled equation

$$\tilde{P}(z,y) = \left(z^2 - a^2\right) y^2 - 2mzy - a^2 z^2 = 0 \,, \tag{17}$$

whose coefficients are simple functions of the *quadrature data* m, a.

Proposition 1. *The following statements hold:*

(1) *The domain G defined above is a classical holomorphic quadrature domain (in the sense of [4]) for the measure*

$$m\delta_a + m\delta_{-a} \,. \tag{18}$$

(2) *For all $m > 0, a > 0$ satisfying*

$$m \geq a^2 \qquad \text{(connectedness condition)} \tag{19}$$

there exists a unique choice of $E > 1$, $r > 0$ such that the corresponding interior conformal map f produces a quadrature domain with measure $m\delta_a + m\delta_{-a}$.

Proof. Let $h(z)$ be a test function holomorphic in a neighborhood of G and integrable on $\mathrm{cl}(G)$. Then

$$\begin{aligned}
\int_G h(z)dA(z) &= \frac{1}{2i}\int_\Gamma h(z)\bar{z}dz \\
&= \frac{1}{2i}\int_\Gamma h(z)S(z)dz \\
&= \underbrace{\frac{1}{2i}\int_\Gamma h(z)S_+(z)dz}_{0} - \frac{1}{2i}\int_\Gamma h(z)S_-(z)dz \\
&= \frac{1}{2\pi i}\int_\Gamma h(z)\left[\frac{\pi m}{z-a} + \frac{\pi m}{z+a}\right]dz \\
&= \pi m h(a) + \pi m h(-a).
\end{aligned} \tag{20}$$

This proves the first statement.

The system of equations

$$\begin{cases} a = \dfrac{rE}{E^4 - 1} \\ m = \dfrac{r^2}{2}\dfrac{E^4+1}{(E^4-1)^2} \end{cases} \tag{21}$$

for the unknowns r and E may be rewritten as

$$\begin{cases} E^4 - \dfrac{2m}{a^2}E^2 + 1 = 0 \\ r - a\dfrac{E^4 - 1}{E} = 0 \,. \end{cases} \tag{22}$$

The quadratic equation $x^2 - \frac{2m}{a^2}x + 1 = 0$ has a unique solution $x \geq 1$ because the quadratic expression has the negative value $2(1 - \frac{m}{a^2})$ at $x = 1$, and therefore both r and E are determined uniquely by m and a.

■

By taking $h(z) \equiv 1$ as test function we get the area of G:

$$\text{Area}(G) = \int_G 1 dA(z) = 2\pi m \ . \tag{23}$$

1.3. **Comparison with the formula of Davis.** In [2], Davis gives the equation of the so-called *bicircular quartic* in polar coordinates:

$$\Gamma = \left\{ z = \rho e^{i\theta} \ : \ \rho^2 = \alpha^2 + 4\varepsilon^2 \cos^2\theta \right\} , \tag{24}$$

where α and ε are positive real numbers. The Schwarz function is given by

$$S_\Gamma(z) = \frac{z(\alpha^2 + 2\varepsilon^2) + z\sqrt{\alpha^4 + 4\alpha^2\varepsilon^2 + 4\varepsilon^2 z^2}}{2(z^2 - \varepsilon^2)} \tag{25}$$

(see [2], p. 26, eq. (5.16)).

This means that $y = S_\Gamma(z)$ satisfies the quadratic equation

$$(\varepsilon^2 - z^2)y^2 + (\alpha^2 + 2\varepsilon^2)zy + \varepsilon^2 z^2 = 0. \tag{26}$$

After rescaling and equating the coefficients (note that a and therefore E has to be real) one gets

$$\begin{cases} \alpha = \dfrac{r}{E^2+1} \\ \varepsilon = \dfrac{rE}{E^4-1} \ . \end{cases} \tag{27}$$

1.4. **The Cauchy Transform and $S_-(z)$.** The Cauchy Transform of the area measure restricted to G is

$$C_G(z) := \int_G \frac{dA(t)}{t-z}. \tag{28}$$

By using Stokes' Theorem, $C_G(z)$ is expressible in terms of the inner and outer components of the Schwarz function of ∂G:

$$C_G(z) = \begin{cases} -\pi\bar{z} + \pi S_+(z) & z \in G \\ \pi S_-(z) & z \in \mathbb{C} \setminus \text{cl}(G). \end{cases} \tag{29}$$

Using the fact that

$$S_+(z) = S(z) + S_-(z) \ , \tag{30}$$

an argument shift in (17) implies that $w(z) = S_+(z)$ satisfies the equation

$$\left(z^2 - a^2\right) w^2 + 2mzw - a^2 z^2 = 0 \ . \tag{31}$$

Note that $-y$ and w satisfy the same quadratic equation (this is not a surprise since the identity

$$-y(z) + w(z) = -S(z) + S_+(z) = S_-(z) = -\frac{2mz}{z^2 - a^2} \tag{32}$$

is exactly Viète's Formula for the sum of the roots of the modified equation (31)). The branch $y(z)$ is the one which has two simple poles inside G and $w(z)$ is the other branch which is holomorphic inside G (by symmetry, $y(z)$ takes both poles at $z = a$ and $z = -a$).

1.5. **The external potential.** The aim now is to find an admissible background potential $Q(z)$ such that the equilibrium measure μ_w of the weighted energy problem for $w = e^{-Q}$ is the normalized area measure on $\mathrm{cl}(G)$.

Since the density is fixed, we look for the potential in the form

$$Q(z) = \frac{1}{4m}|z|^2 + h(z) \tag{33}$$

where $h(z)$ is a suitable perturbation of the Gaussian leading term and harmonic in a subdomain D of $\mathbb{C}$ including G.

The equilibrium condition on G reads as

$$Q(z) + \frac{1}{2\pi m}U^G(z) \equiv F \quad (z \in G) \tag{34}$$

for some constant F where

$$U^G(z) := \int_G \log \frac{1}{|z-w|} dA(w). \tag{35}$$

Then taking ∂_z formally, we have

$$\frac{1}{4m}\bar{z} + \partial_z h(z) + \frac{1}{4\pi m} C_G(z) = 0 \quad (z \in G). \tag{36}$$

Therefore

$$\partial_z h(z) = -\frac{1}{4m} S_+(z) \quad (z \in G). \tag{37}$$

To find a suitable function $h(z)$, the algebraic function $S_+(z)$ has to be integrated. The singularity structure of $S_+(z)$ is indicated by the discriminant of $\tilde{P}(z, y)$:

$$D(z) = 4z^2(a^2z^2 + m^2 - a^4) \ . \tag{38}$$

In terms of

$$\alpha := \sqrt{\frac{m^2 - a^4}{a^2}} \ , \tag{39}$$

(the positive square root is taken), the two simple branch points of $y(z)$ are

$$z_1 = i\alpha \text{ and } z_2 = -i\alpha \ , \tag{40}$$

and there is also a double point at $z = 0$. The antiderivative of $S_+(z)$ can be obtained by a standard Euler substitution:

$$\begin{cases} u = z - \sqrt{z^2 + \alpha^2} \\ z = \dfrac{u^2 - \alpha^2}{2u} \\ dz = \dfrac{u - z}{u} du = \dfrac{u^2 + \alpha^2}{2u^2} du \end{cases} \tag{41}$$

From now on,

$$\sqrt{z^2 + \alpha^2} \tag{42}$$

stands for the analytic function that lives on the complex z-plane with two infinite cuts along the imaginary axis:

$$\mathcal{S} := \mathbb{C} \setminus \left\{ z = it \colon t \in \mathbb{R},\ t^2 \geq \alpha^2 \right\} \ . \tag{43}$$

and takes the value α at $z = 0$.

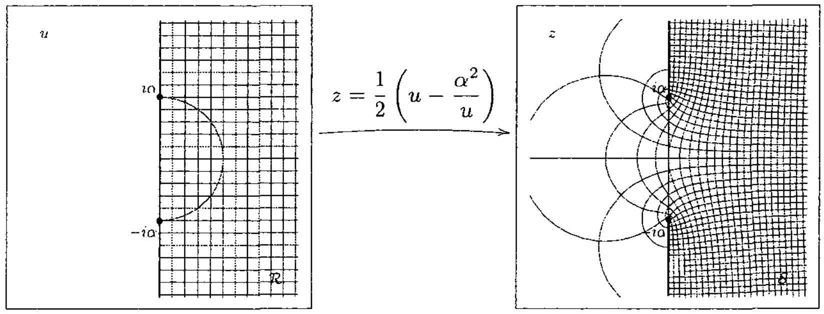

FIGURE 2. The mapping $z(u)$

The mapping

$$z(u) = \frac{u^2 - \alpha^2}{2u} = \frac{1}{2}\left(u - \frac{\alpha^2}{u}\right) \tag{44}$$

maps the open right halfplane $\mathcal{R}$ of the u-plane conformally onto the cut plane $\mathcal{S}$ on the z-plane. The points $u = i\alpha$ and $u = -i\alpha$ are fixed by this map. Moreover, if two points $u_1, u_2 \in \mathcal{R}$ are in inversion with respect to the circle $|u| = \alpha$, i.e. $u_1\overline{u_2} = \alpha^2$, then $z(u_1) + \overline{z(u_2)} = 0$, namely the inversion with respect to the circle $|u| = \alpha$ corresponds to the reflection to the imaginary axis of the z-plane.

By pulling back the one-form integrand to the u-plane, we get

$$\begin{aligned} S_+(z)dz &= \frac{-mz - az\sqrt{z^2+\alpha^2}}{z^2 - a^2}dz \\ &= \frac{(a^2u^2 + m^2 - a^4)(a^2u^2 - m^2 + a^4)}{2au^2(au + m - a^2)(au + m + a^2)}du \\ &= \left[\frac{a}{2} + \frac{m}{u} - \frac{m^2 - a^4}{2a}\frac{1}{u^2} - \frac{am}{au + m - a^2} - \frac{am}{au + m + a^2}\right] du\ . \end{aligned} \tag{45}$$

Therefore the real part of the antiderivative is

$$\begin{aligned} \mathrm{Re}\left(\int S_+(z)dz\right) &= \frac{a}{2}\mathrm{Re}(u) + m\ln|u| + \frac{m^2 - a^4}{2a}\mathrm{Re}\left(\frac{1}{u}\right) \\ &\quad - m\ln|au + m - a^2| - m\ln|au + m + a^2| \\ &= \frac{a}{2}\mathrm{Re}\left(u + \frac{\alpha^2}{u}\right) - m\ln\left|a^2\left(u + \frac{\alpha^2}{u}\right) + 2ma\right|\ . \end{aligned} \tag{46}$$

The function

$$F(u) := \frac{a}{2}\mathrm{Re}\left(u + \frac{\alpha^2}{u}\right) - m\ln\left|a^2\left(u + \frac{\alpha^2}{u}\right) + 2ma\right| \tag{47}$$

has no singularities in $\mathcal{R}$ and it is invariant under the inversion to the circle $|u| = \alpha$; in particular,

$$F(it) = F\left(\frac{\alpha^2}{\overline{it}}\right) = F\left(i\frac{\alpha^2}{t}\right)\ . \tag{48}$$

This means that $F(u(z))$ is smooth in $\mathcal{S}$ and continuous across the cuts of $\mathcal{S}$.

Therefore we find the external potential

$$
\begin{aligned}
Q(z) &= \frac{1}{4m}|z|^2 + h(z) \\
&= \frac{1}{4m}|z|^2 - \frac{1}{2m}F(u(z)) \\
&= \frac{1}{4m}|z|^2 - \frac{a}{4m}\mathrm{Re}\left(u(z) + \frac{\alpha^2}{u(z)}\right) + \frac{1}{2}\ln\left|a^2\left(u(z) + \frac{\alpha^2}{u(z)}\right) + 2ma\right| .
\end{aligned} \tag{49}
$$

Finally, the asymptotic behavior of $Q(z)$ has to be checked for large $|z|$ to conclude that Q is admissible in the sense of [3]. By pulling back Q to the u-plane, the difference

$$
\begin{aligned}
&Q(z(u)) - \ln|z(u)| = \\
&\frac{1}{4m}\left|\frac{u^2-\alpha^2}{2u}\right|^2 - \frac{a}{4m}\mathrm{Re}\left(u + \frac{\alpha^2}{u}\right) + \frac{1}{2}\ln\left|a^2\left(u + \frac{\alpha^2}{u}\right) + 2ma\right| - \ln\left|\frac{u^2-\alpha^2}{2u}\right|
\end{aligned} \tag{50}
$$

has to be investigated near $u = 0$ and $u = \infty$ on the halfplane $\mathcal{R}$. Now

$$
\begin{aligned}
Q(z(u)) &= \frac{\alpha^4}{16m}\frac{1}{|u|^2} - \frac{a\alpha^2}{4m}\frac{1}{|u|} - \frac{1}{2}\ln\frac{1}{|u|} + \mathcal{O}(1) \to \infty \quad |u| \to 0 \\
Q(z(u)) &= \frac{1}{16m}|u|^2 - \frac{a}{m}|u| - \frac{1}{2}\ln|u| + \mathcal{O}(1) \to \infty \quad |u| \to \infty
\end{aligned} \tag{51}
$$

shows that Q is sufficiently strong near $z = \infty$.

Remark. To conclude that μ_Q is the normalized area measure on G, the standard inequality for the effective potential has to be proven outside the support. At this point, it seems to be quite complicated.

2. Limacons

2.1. **Conformal map and Schwarz function.** To construct quadrature domains with a single second order quadrature node, we consider a family of mappings of the ζ-plane to the z-plane of the form

$$
f(\zeta) = r\zeta + b\zeta^2 \quad r > 0,\ b \in \mathbb{C}. \tag{52}
$$

The images of the unit circle under these mappings are called *limaçons* (see [2]).

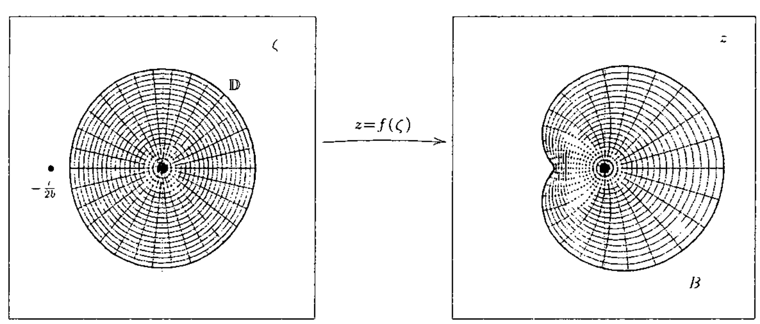

FIGURE 3. The conformal map

The mapping $z = f(\zeta)$ is univalent iff $|b| \leq \frac{r}{2}$: we have $f(\zeta_1) = f(\zeta_2)$ for two different points $\zeta_1 \neq \zeta_2$ when $r + b(\zeta_1 + \zeta_2) = 0$. The existence of such a pair is equivalent to having $-\frac{r}{b} \in 2\mathbb{D}$.

The boundary curve ∂B of the domain $B := f(\mathbb{D})$ is the zero locus of a polynomial equation $P(z, \overline{z}) = 0$ which can be calculated as above:

$$\begin{cases} z = f(\zeta) = r\zeta + b\zeta^2 \\ \overline{z} = \overline{f}\left(\dfrac{1}{\zeta}\right) = \dfrac{r}{\zeta} + \dfrac{\overline{b}}{\zeta^2} . \end{cases} \tag{53}$$

Therefore we are looking for the resultant of the two polynomials

$$\begin{cases} p_1(\zeta) = & b\zeta^2 + r\zeta - z \\ p_2(\zeta) = & y\zeta^2 - r\zeta + \overline{b}. \end{cases} \tag{54}$$

with $y = \overline{z}$ along the boundary of B. This gives the quadratic equation satisfied by $y = S(z)$:

$$P(z, y) = z^2y^2 - (2b\overline{b}z + r^2z + r^2\overline{b})y + b^2\overline{b}^2 - r^2b\overline{b} - r^2bz = 0. \tag{55}$$

The discriminant of $P(z, y)$, as a function of z, factorizes in the following way:

$$D(z) = r^2(4bz + r^2)(z + \overline{b})^2. \tag{56}$$

The exterior projection $S_-(z)$ of $S(z)$ is calculated using the conformal map parametrization:

$$\begin{aligned} S_-(z) &= \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\overline{f(\zeta)} f'(\zeta) d\zeta}{f(\zeta) - z} \\ &= \operatorname*{res}_{\zeta=0} \frac{\left(\frac{r}{\zeta} + \frac{\overline{b}}{\zeta^2}\right)(r + 2b\zeta)\, d\zeta}{r\zeta + b\zeta^2 - z} . \end{aligned} \tag{57}$$

Since

$$\begin{aligned} \frac{\left(\frac{r}{\zeta} + \frac{\overline{b}}{\zeta^2}\right)(r + 2b\zeta)}{r\zeta + b\zeta^2 - z} &= -\frac{1}{z}\left(\frac{r}{\zeta} + \frac{\overline{b}}{\zeta^2}\right)(r + 2b\zeta)\left(1 + \frac{r\zeta}{z} + \mathcal{O}(\zeta^2)\right) \\ &= -\frac{r\overline{b}}{z}\frac{1}{\zeta^2} - \frac{1}{z}\left(r^2 + 2b\overline{b} + \frac{r^2\overline{b}}{z}\right)\frac{1}{\zeta} + \mathcal{O}(1) \end{aligned} \tag{58}$$

near $\zeta = 0$, we have

$$S_-(z) = -\frac{r^2 + 2b\overline{b}}{z} - \frac{r^2\overline{b}}{z^2}. \tag{59}$$

The rational form of $S_-(z)$ implies the

Proposition 2. *The domain B is a holomorphic quadrature domain with a single second-order quadrature node: if $h(z)$ is a function analytic and integrable on B then*

$$\int_B h(z)dA(z) = \pi(r^2 + 2|b|^2)h(0) + \pi r^2\overline{b}h'(0) . \tag{60}$$

In particular, the area of B is

$$\text{Area}(B) = \pi(r^2 + 2|b|^2) . \tag{61}$$

The Schwarz function of ∂B is given by

$$S(z) = \frac{(2b\bar{b}z + r^2 z + r^2\bar{b}) + r(z+\bar{b})\sqrt{4bz+r^2}}{2z^2}\,, \tag{62}$$

where the square root

$$\sqrt{4bz+r^2} \tag{63}$$

means the branch that is defined on the cut plane

$$\mathcal{S} := \mathbb{C} \setminus \left\{ -\rho\frac{r^2}{4b} \;:\; \rho \geq 1 \right\} \tag{64}$$

that takes the value r at $z = 0$. Now,

$$\begin{aligned} S_+(z) &= S(z) + S_-(z) \\ &= \frac{-(2b\bar{b}z + r^2 z + r^2\bar{b}) + r(z+\bar{b})\sqrt{4bz+r^2}}{2z^2} \\ &= \mathcal{O}(1) \quad z \to 0\,. \end{aligned} \tag{65}$$

2.2. **The external potential.** Following the steps in the previous section, we need to find the antiderivative of $S_+(z)dz$ to obtain a suitable background potential which generates B. To this end, we use the Euler substitution again:

$$\begin{cases} u = \sqrt{4bz+r^2} \\ z = \dfrac{u^2 - r^2}{4b} \\ dz = \dfrac{u}{2b}du \end{cases} \tag{66}$$

The conformal mapping

$$z = \frac{u^2 - r^2}{4b} \tag{67}$$

takes the right halfplane $\mathcal{R}$ of the u-plane to $\mathcal{S}$ on the z-plane (the imaginary axis is mapped to the sides of the cut).

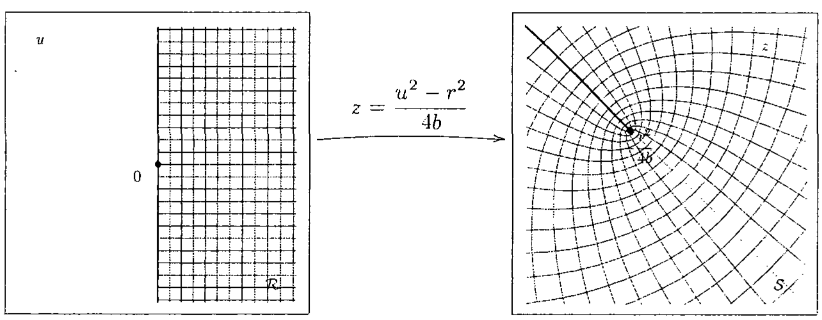

FIGURE 4. The mapping $z(u)$

The antiderivative of $S_+(z)$ can be calculated in terms of u:

$$\begin{aligned}\int S_+(z)dz &= \int \left[r - \frac{r^2+2b\bar{b}}{u+r} + \frac{2rb\bar{b}}{(u+r)^2}\right] du \\ &= ru - (r^2+2b\bar{b})\ln(u+r) - \frac{2rb\bar{b}}{u+r} \,.\end{aligned} \tag{68}$$

Let

$$\begin{aligned}F(u) :&= \mathrm{Re}\left(ru - (r^2+2b\bar{b})\ln(u+r) - \frac{2rb\bar{b}}{u+r}\right) \\ &= \mathrm{Re}\left(ru - \frac{2rb\bar{b}}{u+r}\right) - (r^2+2b\bar{b})\ln|u+r| \,.\end{aligned} \tag{69}$$

This function is symmetric under conjungation: $F(\bar{u}) = F(u)$.

By following the calculation in previous section, we get a suitable candidate for the background potential:

$$\begin{aligned}Q(z) &= \frac{1}{2(r^2+2|b|^2)}|z|^2 - \frac{1}{r^2+2|b|^2}F(u(z)) \\ &= \frac{1}{2(r^2+2|b|^2)}|z|^2 - \frac{1}{r^2+2|b|^2}\mathrm{Re}\left(ru(z) - \frac{2r|b|^2}{u(z)+r}\right) + \ln|u(z)+r| \,.\end{aligned} \tag{70}$$

This potential is smooth on S and continuous on the cut (by the conjugation symmetry of F). It is also easy to see that $Q(z)$ is sufficiently strong at infinity. Therefore Q is an admissible potential.

Remark. The variational inequality needs to be proved to conclude that B is the support of the equilibrium measure for Q.

3. Two disks

In this section we find an external potential whose equilibrium measure is a uniform measure supported on two disjoint congruent disks symmetrical to the origin. Recall that the logaritmic potential of the area measure restricted to a disk D of radius R centered at a is

$$U^D(z) = \begin{cases} -\frac{\pi}{2}|z-a|^2 + \frac{1}{2}R^2\pi + R^2\pi\log\frac{1}{R} & z \in D \\ R^2\pi\log\frac{1}{|z-a|} & z \notin D \,. \end{cases} \tag{71}$$

Without loss of generality, assume that $a \in \mathbb{R}^+$ and $R < a$ (to have disjoint disks). Take the following background potential of the form

$$Q(z) := \frac{1}{4R^2}|z|^2 + h(z) \,, \tag{72}$$

where

$$h(z) := \begin{cases} \frac{1}{2}\log|z+a| - \frac{a}{4R^2}(z+\bar{z}) & \mathrm{Re}(z) \geq 0 \\ \frac{1}{2}\log|z-a| + \frac{a}{4R^2}(z+\bar{z}) & \mathrm{Re}(z) < 0 \,. \end{cases} \tag{73}$$

This function is harmonic in the open left and right halfplanes and continuous on the imaginary axis.

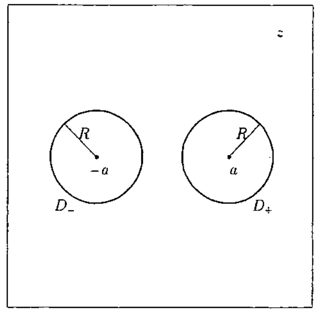

FIGURE 5. Disk configuration

Proposition 3. *The equilibrium measure is the normalized Lebesgue measure on the union of two disjoint disks* D_+ *and* D_-*:*

$$\begin{cases} D_+ &= \{z \;:\; |z-a| \le R\} \\ D_- &= \{z \;:\; |z+a| \le R\}\ . \end{cases} \tag{74}$$

Proof. The logarithmic potential of the measure ν above is

$$U^\nu(z) = \begin{cases} -\dfrac{1}{4R^2}|z-a|^2 + \dfrac{1}{2}\log\dfrac{1}{|z+a|} + \dfrac{1}{2}\log\dfrac{1}{R} + \dfrac{1}{4} & z \in D_+ \\ \dfrac{1}{2}\log\dfrac{1}{|z-a|} + \dfrac{1}{2}\log\dfrac{1}{|z+a|} & z \in \mathbb{C}\setminus(D_+\cup D_-) \\ -\dfrac{1}{4R^2}|z+a|^2 + \dfrac{1}{2}\log\dfrac{1}{|z-a|} + \dfrac{1}{2}\log\dfrac{1}{R} + \dfrac{1}{4} & z \in D_- \end{cases} \tag{75}$$

The effective potential for $z \in D_+$ is

$$Q(z) + U^\nu(z) = \frac{1}{2}\log\frac{1}{R} + \frac{1}{4} - \frac{a^2}{4R^2} \equiv \text{const}\ . \tag{76}$$

For $z \in \mathbb{C}_+ \setminus D_+$,

$$\begin{aligned} Q(z) + U^\nu(z) &= \frac{1}{4R^2}|z|^2 - \frac{a}{4R^2}(z+\bar{z}) + \frac{1}{2}\log\frac{1}{|z-a|} \\ &= \frac{1}{4R^2}|z-a|^2 + \frac{1}{2}\log\frac{1}{|z-a|} - \frac{a^2}{4R^2} \\ &\ge \frac{1}{2}\log\frac{1}{R} + \frac{1}{4} - \frac{a^2}{4R^2}\ . \end{aligned} \tag{77}$$

Since $Q(z)$ and $U^\nu(z)$ are both invariant under the sign change $z \mapsto -z$, the same equality and inequality holds for the corresponding reflected domains respectively. This is enough to conclude that $\mu_Q = \nu$.

∎

4. Appendix: Representations of External Fields as Pure Logarithmic Potentials

In this section we find positive charge distributions ν in terms of which the perturbing terms $h(z)$ in the potentials above are represented as 'pure' logarithmic potentials:

$$h(z) = \int \log \frac{1}{|z-t|} d\nu(t) \ . \tag{78}$$

These measures all have unbounded support (along the carefully chosen cuts of the algebraic functions used in the construction of the h's).

If ν admits a density $\rho(\tau)$ with respect to the one-form $d\tau$ along a contour Γ then $h(z)$ is differentiable at $z \in \mathbb{C} \setminus \Gamma$ and

$$\partial_z h(z) = \frac{1}{2} \int \frac{d\nu(\tau)}{\tau - z} = \frac{1}{2} \int \frac{\rho(\tau) d\tau}{\tau - z} \ , \tag{79}$$

By the Plemelj formulae [1],

$$\rho(\tau) = \frac{1}{\pi i} \left[\lim_{z \to \tau^+} (\partial_z h)(z) - \lim_{z \to \tau^-} (\partial_z h)(z) \right] \ , \tag{80}$$

where the limits above are referring to the non-tangential limiting values along Γ from its positive or negative side respectively.

4.1. **Charge density for two-point quadrature domains.** The notations of the first section are used. Consider the contour

$$\Gamma := \{z = it \ : \ t \in (\alpha, \infty)\} \tag{81}$$

emanating from the branchpoint $z = \alpha$ oriented upwards. Then

$$\begin{cases} \lim\limits_{z \to (it)^+} u(z) = i\left(t - \sqrt{t^2 - \alpha^2}\right) \\ \lim\limits_{z \to (it)^-} u(z) = i\left(t + \sqrt{t^2 - \alpha^2}\right) \ . \end{cases} \tag{82}$$

Consequently,

$$\begin{cases} \lim\limits_{z \to (it)^+} S_+(z) = \dfrac{imt - at\sqrt{t^2 - \alpha^2}}{a^2 + t^2} \\ \lim\limits_{z \to (it)^-} S_+(z) = \dfrac{imt + at\sqrt{t^2 - \alpha^2}}{a^2 + t^2} \ . \end{cases} \tag{83}$$

As

$$\partial_z h(z) = -\frac{1}{4m} S_+(z) \ , \tag{84}$$

$$\begin{aligned} \rho(it) &= \frac{1}{\pi i} \left[\lim_{z \to (it)^+} (\partial_z h)(z) - \lim_{z \to (it)^-} (\partial_z h)(z) \right] \\ &= -\frac{1}{4m\pi i} \left[\lim_{z \to (it)^+} S_+(z) - \lim_{z \to (it)^-} S_+(z) \right] \\ &= \frac{1}{2m\pi i} \frac{at\sqrt{t^2 - \alpha^2}}{a^2 + t^2} \ . \end{aligned} \tag{85}$$

Since $\rho(\tau)d\tau = \rho(it)idt$, the non-negative density in terms of the real parameter t is given by

$$\rho(\tau)d\tau = \frac{1}{2m\pi}\frac{at\sqrt{t^2-\alpha^2}}{a^2+t^2}dt \ . \tag{86}$$

By the reflection symmetry of h with respect to the x axis of the z-plane, the density along the opposite cut is obtained by reflection to the x axis.

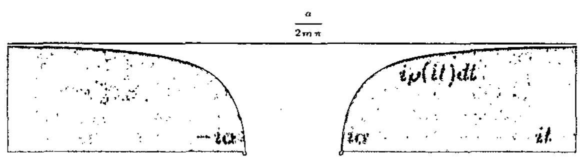

FIGURE 6. The density profile along the imaginary axis

4.2. **Charge density for limacons.** Let

$$\beta := -\frac{r^2}{4b} \ . \tag{87}$$

Let Γ be the contour

$$\Gamma := \left\{t\frac{\beta}{|\beta|} \ : \ t \geq |\beta|\right\} \tag{88}$$

oriented outwards (parametrized by t). Then

$$\begin{cases} \lim\limits_{z\to(t\beta/|\beta|)^+} u(z) = -i\dfrac{r}{\sqrt{|\beta|}}\sqrt{t-|\beta|} \\ \lim\limits_{z\to(t\beta/|\beta|)^-} u(z) = i\dfrac{r}{\sqrt{|\beta|}}\sqrt{t-|\beta|} \ . \end{cases} \tag{89}$$

Consequently,

$$\begin{cases} \lim\limits_{z\to(t\beta/|\beta|)^+} S_+(z) = \dfrac{-(2b\bar{b}+r^2)t\beta/|\beta| - r^2\bar{b} - i\frac{r^2}{\sqrt{|\beta|}}(t\beta/|\beta|+\bar{b})\sqrt{t-|\beta|}}{2(t\beta/|\beta|)^2} \\ \lim\limits_{z\to(t\beta/|\beta|)^+} S_-(z) = \dfrac{-(2b\bar{b}+r^2)t\beta/|\beta| - r^2\bar{b} + i\frac{r^2}{\sqrt{|\beta|}}(t\beta/|\beta|+\bar{b})\sqrt{t-|\beta|}}{2(t\beta/|\beta|)^2} \end{cases} \tag{90}$$

As

$$\partial_z h(z) = -\frac{1}{2(r^2+2|b|^2)}S_+(z) \ , \tag{91}$$

$$
\begin{aligned}
\rho\left(\frac{t\beta}{|\beta|}\right) &= \frac{1}{\pi i}\left[\lim_{z\to(t\beta/|\beta|)^+}(\partial_z h)(z) - \lim_{z\to(t\beta/|\beta|)^-}(\partial_z h)(z)\right] \\
&= -\frac{1}{2(r^2+2|b|^2)\pi i}\left[\lim_{z\to(t\beta/|\beta|)^+} S_+(z) - \lim_{z\to(t\beta/|\beta|)^-} S_+(z)\right] \\
&= \frac{1}{(r^2+2|b|^2)\pi}\,\frac{\frac{r^2}{\sqrt{|\beta|}}(t\beta/|\beta|+\bar{b})\sqrt{t-|\beta|}}{2(t\beta/|\beta|)^2}
\end{aligned} \tag{92}
$$

Since $\rho(\tau)d\tau = \rho\left(\frac{\beta}{|\beta|}t\right)\frac{\beta}{|\beta|}dt$, the non-negative density in terms of the real parameter t is given by

$$
\begin{aligned}
\rho(\tau)d\tau &= \frac{r^2}{2(r^2+2|b|^2)\sqrt{|\beta|}\pi}\,\frac{\left(t+\bar{b}\frac{|\beta|}{\beta}\right)\sqrt{t-|\beta|}}{t^2}dt \\
&= \frac{r^2}{2(r^2+2|b|^2)\sqrt{|\beta|}\pi}\,\frac{(t-|b|)\sqrt{t-|\beta|}}{t^2}dt\ .
\end{aligned} \tag{93}
$$

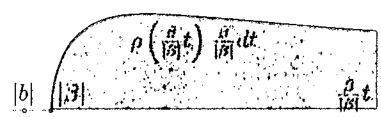

FIGURE 7. The density profile along Γ

4.3. **Charge density for two disks.** Since

$$
\partial_z h(z) = \begin{cases} \dfrac{1}{4}\dfrac{1}{z+a} - \dfrac{a}{4R^2} & \mathrm{Re}(z) > 0 \\[2ex] \dfrac{1}{4}\dfrac{1}{z-a} + \dfrac{a}{4R^2} & \mathrm{Re}(z) < 0\ . \end{cases} \tag{94}
$$

$$
\begin{aligned}
\rho(it) &= \frac{1}{\pi i}\left[\lim_{z\to(it)^+}(\partial_z h)(z) - \lim_{z\to(it)^-}(\partial_z h)(z)\right] \\
&= \frac{a}{2\pi i}\left(\frac{1}{R^2} - \frac{1}{a^2+t^2}\right)\ .
\end{aligned} \tag{95}
$$

The density is

$$
\rho(it)idt = \frac{a}{2\pi}\left(\frac{1}{R^2} - \frac{1}{a^2+t^2}\right)dt\ . \tag{96}
$$

This is a positive density since $a > R$.

FIGURE 8. The density profile along the imaginary axis

If $a = R$ this perturbing charge configuration is the same as the singular limit corresponding to a bicircular quartic with quadrature data $m = a^2$:

$$\rho(it)idt = \frac{1}{2a^2\pi}\frac{at^2}{a^2+t^2}dt\ . \tag{97}$$

FIGURE 9. The density profile along the imaginary axis

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Chapter 12

Riemann-Hilbert analysis of the Bratwurst orthogonal polynomials

12.1 Summary

This chapter is concerned with the asymptotic analysis of orthogonal polynomials with respect to quasi-harmonic weights corresponding to the family of semiclassical potentials of the form

$$Q(z) = |z|^2 + 2c \log \frac{1}{|z-a|} , \tag{12.1}$$

where $\alpha, c > 0$ and $a \in \mathbb{C}$. The following results are proved in the manuscript presented in the next section that forms the basis of the forthcoming publication [11]:

1. It is shown that the system of orthogonality relations

$$\int_{\mathbb{C}} P_{n,N}(z) \overline{z}^k |z-a|^{2Nc} e^{-N|z|^2} dA(z) = 0 \qquad k = 0, \ldots, n-1 \tag{12.2}$$

can be reduced to an equivalent system of contour integral orthogonality conditions for an analytic weight function (Sec. 3).

2. The associated Fokas-Its-Kitaev Riemann-Hilbert problem for the non-Hermitian orthogonal polynomials is shown to have a unique solution (Sec. 4).

3. A g-function is constructed for the Deift–Zhou nonlinear steepest descent method in terms of a suitable quadratic differential (Sec. 5).

4. Complete strong asymptotics for the orthogonal polynomials is obtained by applying the Deift–Zhou nonlinear steepest descent method. As an application, Conjecture 8.1.1 is confirmed for this special case.

From now on, without loss of generality, we assume that $\alpha = 1$ and $a \in \mathbb{R}^+$. For the asymptotic analysis of the monic orthogonal polynomials $P_{n,N}(z)$ in the scaling limit

$$n \to \infty, \qquad N \to \infty , \qquad \frac{n}{N} \to t , \tag{12.3}$$

the relevant equilibrium measure corresponds to the rescaled potential

$$Q_t(z) := \frac{1}{2t} Q(z) = \frac{1}{2t}|z|^2 + \frac{t}{c} \log \frac{1}{|z-a|} . \tag{12.4}$$

12.1.1 Time evolution of the equilibrium support

The support sets of the one-parameter family of the equilibrium measures

$$t \mapsto \mu_{Q_t} \qquad t > 0 \tag{12.5}$$

gives an increasing one-parameter family of compact sets

$$t \mapsto S_{Q_t} \qquad t > 0 . \tag{12.6}$$

Lemma 8.1.1 can be used to find the support of the equilibrium measure. By matching the data of the t-dependent potential, the radii $R(t)$ and $r(t)$ depend on t in the following way:

$$R(t) = \sqrt{t+c} \qquad r(t) \equiv \sqrt{c} . \tag{12.7}$$

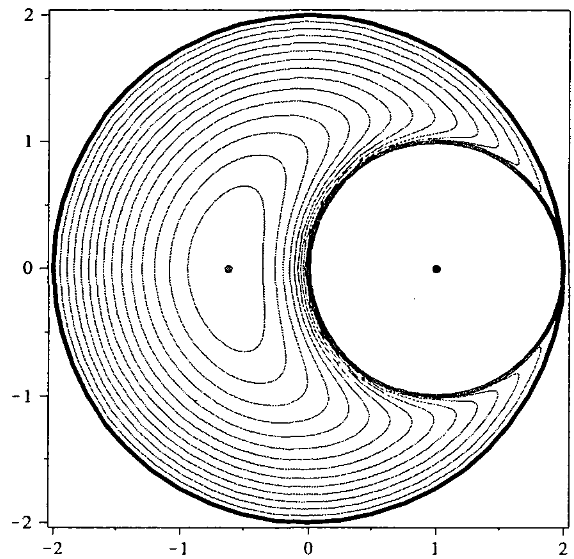


Figure 12.1: The boundary evolution for $a = 1$, $c = 1$ before the critical time

There is a *critical time*

$$t_c = a(a + 2\sqrt{c}) \tag{12.8}$$

when the disks $D(0, R(t))$ and $D(a, r(t))$ are in a critical position:

$$r(t_c) + a = R(t_c) \ . \tag{12.9}$$

Hence the topology of the support changes from simply connected to doubly connected as t passes through t_c. The time evolution of the domains S_{Q_t} can be described in the following way:

- As $t \longrightarrow 0$, the domain shrinks to the point z_0 of absolute minimum of $Q(z)$:

$$z_0 = \frac{a - \sqrt{a^2 + 4c}}{2} \ . \tag{12.10}$$

  This is the equilibrium position of one simple point charge in the presence of $Q(z)$.

- For the *pre-critical* regime $0 < t < t_c$ the support is given by a t-dependent Joukowski-type exterior uniformizing map (see Lemma 8.1.1 and Fig. 12.1).

- At the *critical case* $t = t_c$ the support of the equilibrium measure becomes doubly connected:

$$S_{Q_{t_c}} = \overline{D}(0, R(t_c)) \setminus D(a, r(t_c)) \ . \tag{12.11}$$

- For the *post-critical* regime $t \geq t_c$ the equilibrium measure is supported on the doubly connected set

$$S_{Q_t} = \overline{D}(0, R(t)) \setminus D(a, r(t)) \ . \tag{12.12}$$

12.1.2 The orthogonal polynomials

Since

$$\int_{\mathbb{C}} |z|^n e^{-NQ(z)} dA(z) = \int_{\mathbb{C}} |z|^n |z-a|^{2Nc} e^{-N|z|^2} dA(z) < \infty \tag{12.13}$$

for all $n, N > 0$, the monic orthogonal polynomials

$$P_{n,N}(z) = P_n(e^{-NV} dA; z) \tag{12.14}$$

are uniquely determined by the two-dimensional orthogonality conditions

$$\int_{\mathbb{C}} P_{n,N}(z) \overline{z}^k |z-a|^{2Nc} e^{-N|z|^2} dA(z) = \delta_{kn} h_{n,N} \qquad k = 0, 1, \ldots, n \ . \tag{12.15}$$

These polynomials are referred to as *Bratwurst polynomials*: the term was coined in [61, 66] for the Joukowski mapping that corresponds to the equilibrium measure.

There are no known formulae expressing the polynomials $P_{n,N}(z)$ explicitly or in terms of classical orthogonal polynomials. A numerical Gram-Schmidt orthogonalization procedure may be used to investigate the behavior of the zeroes of $P_{n,N}(z)$ for fixed values of the parameters a, c and t. The value scaling parameter N depends on

n and, for simplicity, it is set to be

$$N = \frac{n}{t} \, . \tag{12.16}$$

Numerical plots of the zeroes and the boundary of the corresponding equilibrium support are shown in Figs. 12.2, 12.3.

The zeroes apparently accumulate along certain arcs of the complex plane: the numerical plots of the zeroes indicate that the arc appears to connect the branch points of the uniformizing map F for $t < t_c$, and it becomes a closed curve for $t \geq t_c$.

12.1.3 Reduction to contour integrals

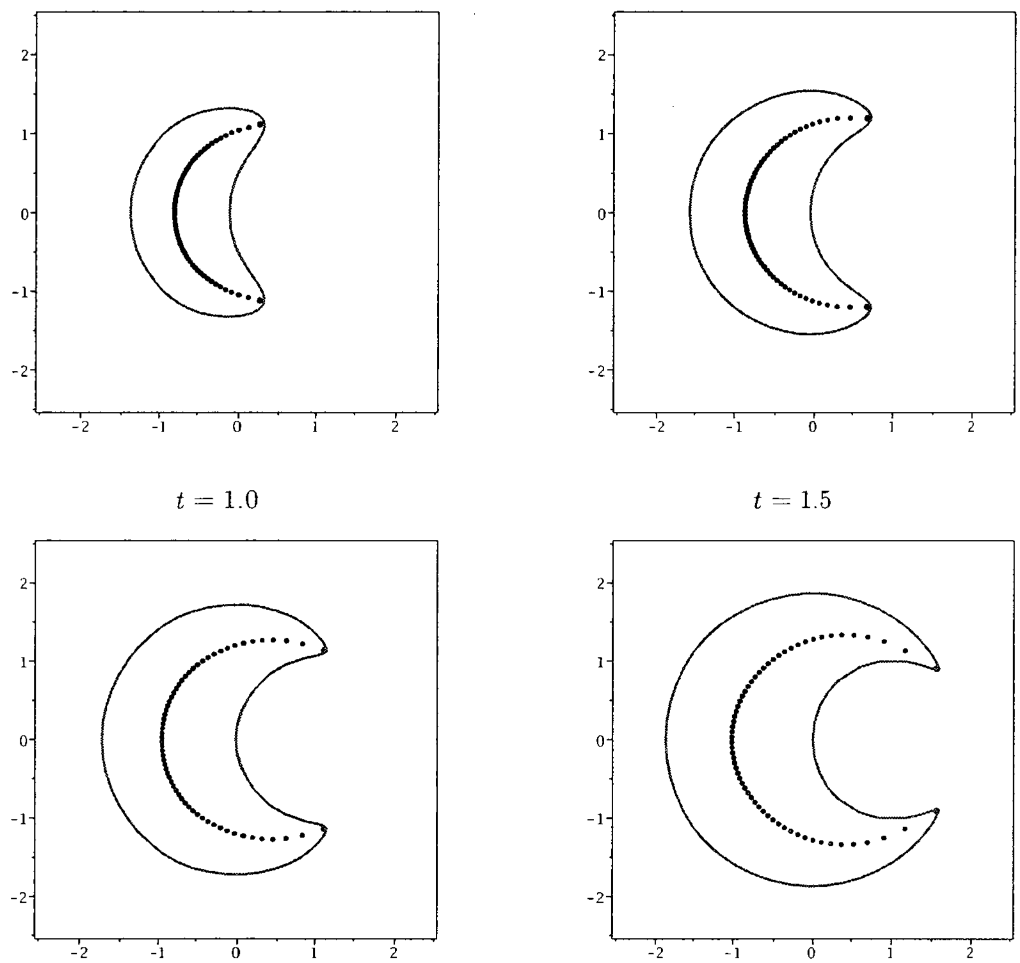
To identify the supporting arcs and densities of these asymptotic zero distributions one has to investigate the asymptotic behavior of the polynomials of $P_{n,N}(z)$ in the limit $n, N \to \infty$, $n/N \to t$. However, the asymptotic theory of general orthogonal polynomials in the complex plane still lacks a method comparable the powerful Riemann-Hilbert approach for orthogonal polynomials on the real line. Below it is shown how to reduce the problem involving two-dimensional integrals to an equivalent problem expressed in terms of contour integrals. The method developed to prove Thm. 3.5.1 can be adapted to this special case which implies the following quadrature-type identity for the measure

$$|z-a|^{Nc} e^{-N|z|^2} dA(z) \tag{12.17}$$

on polynomial test functions:

Theorem 12.1.1 (Lemma 3.1, [11]) *For any polynomial $p(z)$, the following integral identity holds:*

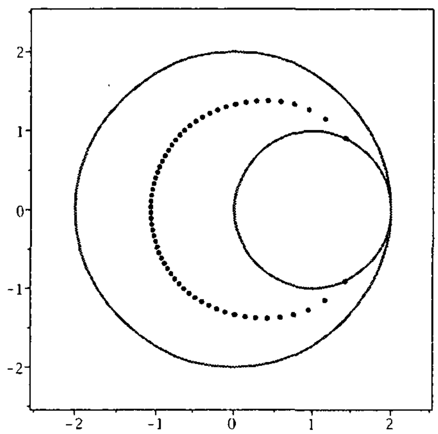
$$\begin{aligned} &\int_{\mathbb{C}} p(z) \bar{z}^k |z-a|^{2Nc} e^{-Nz\bar{z}} dA(z) = \\ &\quad \frac{\mathrm{e}^{-2\pi i Nc}}{2i} \sum_{l=0}^{k} \binom{k}{l} a^{k-l} \frac{\Gamma(l+Nc+1)}{N^{l+Nc+1}} \oint_{\Gamma} p(z) \frac{(z-a)^{Nc} e^{-az}}{z^{Nc}} \frac{dz}{z^{l+1}} \, . \end{aligned} \tag{12.18}$$



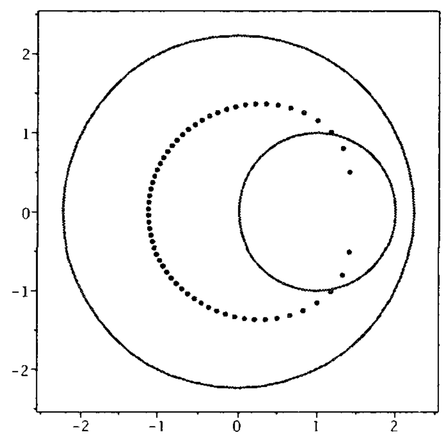
$t = 1.0$ $t = 1.5$

$t = 2.0$ $t = 2.5$

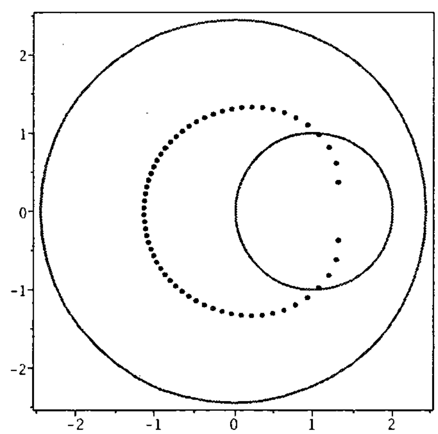
Figure 12.2: The zeroes of $P_{n,N}(z)$ for $n = 40$ with $N = n/t$ and $a = 1$ $c = 1$ for different values of $t < t_c$ and the corresponding equilibrium supports



$t = t_c = 3.0$



$t = 4.0$



$t = 5.0$

Figure 12.3: The zeroes of $P_{n,N}(z)$ for $n = 40$ with $N = n/t$ and $a = 1$ $c = 1$ for different values of $t \geq t_c$ and the corresponding equilibrium supports

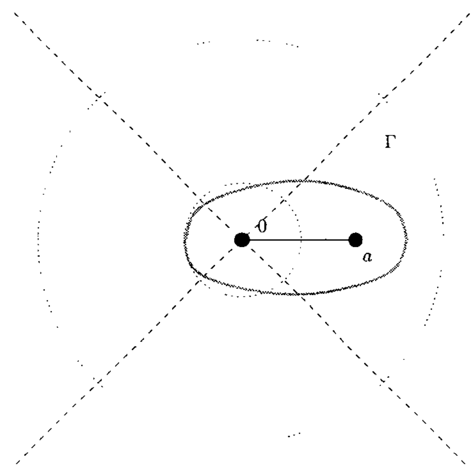


Figure 12.4: The contour Γ

where the term

$$\left(\frac{z-a}{z}\right)^{Nc} \sim 1 \qquad z \to \infty \tag{12.19}$$

in the integrand has a branch cut along the segment $[0, a]$ *and* Γ *is a simple positively oriented closed contour encircling* $[0, a]$ *(Fig. 12.4).*

Note that if $Nc \in \mathbb{Z}$ then (12.18) simplifies considerably: $w_{n,N}(z)$ becomes meromorphic and the contour Γ can be deformed into a contour enclosing $z = 0$ only. All the Γ-integrals are expressible explicitly in terms of residues. This is also clear by looking at the finite expansion of the perturbative term

$$|z-a|^{2Nc} = \sum_{k,l=0}^{Nc} \binom{Nc}{k}\binom{Nc}{l}(-1)^{k+l} a^{Nc-k}\overline{a}^{Nc-l} z^k \overline{z}^l \ . \tag{12.20}$$

valid only for $Nc \in \mathbb{Z}$. In terms of the non-Hermitian weight function

$$w_{n,N}(z) := \frac{(z-a)^{Nc} e^{-az}}{z^{Nc+n}} \qquad z \in \mathbb{C} \setminus [0, a] \tag{12.21}$$

the integral identities (12.18) can be written as

$$
2ie^{2\pi iNc}\begin{bmatrix} \displaystyle\int_{\mathbb{C}} P_{n,N}(z)e^{-NQ(z)}dA(z) \\ \displaystyle\int_{\mathbb{C}} P_{n,N}(z)\bar{z}e^{-NQ(z)}dA(z) \\ \vdots \\ \displaystyle\int_{\mathbb{C}} P_{n,N}(z)\bar{z}^{n-1}e^{-NQ(z)}dA(z) \end{bmatrix}
$$

$$
= \begin{bmatrix} \binom{0}{0}\frac{\Gamma(Nc+1)}{N^{Nc+1}} & 0 & \cdots & 0 \\ \binom{1}{0}a\frac{\Gamma(Nc+1)}{N^{Nc+1}} & \binom{1}{1}\frac{\Gamma(Nc+2)}{N^{Nc+2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n-1}{0}a^{n-1}\frac{\Gamma(Nc+1)}{N^{Nc+1}} & \binom{n-1}{1}a^{n-2}\frac{\Gamma(Nc+2)}{N^{Nc+2}} & \cdots & \binom{n-1}{n-1}\frac{\Gamma(Nc+n)}{N^{Nc+n}} \end{bmatrix}\begin{bmatrix} \displaystyle\oint_\Gamma p(z)z^{n-1}w_{n,N}(z)dz \\ \displaystyle\oint_\Gamma p(z)z^{n-2}w_{n,N}(z)dz \\ \vdots \\ \displaystyle\oint_\Gamma p(z)w_{n,N}(z)dz\ . \end{bmatrix}
\tag{12.22}
$$

This means that the system of orthogonality relations (12.15) is equivalent to the system of non-Hermitian orthogonality relations

$$
\oint_\Gamma P_{n,N}(z)z^k\frac{(z-a)^{Nc}e^{-az}}{z^{Nc+n}}dz = 0 \qquad k = 0,1,\ldots,n-1\ . \tag{12.23}
$$

It is important to note that the moment matrices $M^{(n-1)}(e^{-NQ(z)})$ and $M^{(n-1)}(w_{n,N},\Gamma)$ are related by

$$
M^{(n-1)}(e^{-NQ(z)}) = P^{(n-1)}A^{(n-1)}M^{(n-1)}(w_{n,N},\Gamma) \tag{12.24}
$$

where

$$
\begin{aligned} P^{(n-1)} &= (\delta_{k\ n-1-l})_{0\le k,l\le n-1} \\ A^{(n-1)} &= \frac{e^{-2\pi iNc}}{2i}\left(\binom{l}{k}a^{l-k}\frac{\Gamma(k+Nc+1)}{N^{k+Nc+1}}\right)_{0\le k,l\le n-1}\ . \end{aligned} \tag{12.25}
$$

Therefore the moment determinants are connected by the equation

$$
\begin{aligned} &\det\left(M^{(n-1)}(e^{-NQ(z)})\right) \\ &= \det\left(M^{(n-1)}(w_{n,N},\Gamma)\right)(-1)^{\lfloor\frac{n}{2}\rfloor}\frac{e^{-2n\pi iNc}}{(2i)^n N^{nNc+\frac{n(n+1)}{2}}}\prod_{k=0}^{n-1}\Gamma(k+Nc+1)\ . \end{aligned} \tag{12.26}
$$

Since the Gamma function $\Gamma(z)$ has no zeroes and $M^{(n-1)}$ is non-singular, we obtain

$$\det(M^{(n)}(w_{n,N},\Gamma)) \neq 0 . \tag{12.27}$$

12.1.4 The associated Riemann–Hilbert problem

Since $P_{n,N}(z)$ satisfies the non-Hermitian orthogonality relations (12.23) the standard 2×2 Fokas-Its-Kitaev Riemann–Hilbert problem [55] can be formulated on the contour Γ defined above:

Theorem 12.1.2 (Lemma 4.1, [11]) *For fixed n, N, the Riemann–Hilbert problem*

(Y.1) $Y(z)$ *is holomorphic in* $\mathbb{C}\setminus\Gamma$,

(Y.2) $Y_+(z) = Y_-(z)\begin{bmatrix} 1 & w_{n,N}(z) \\ 0 & 1 \end{bmatrix}$ *for* $z \in \Gamma$,

(Y.3) $Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) z^{-n\sigma_3}$ *as* $z \to \infty$

is solvable as a consequence of (12.27) *and there is a unique solution of the form*

$$Y(z) = \begin{bmatrix} p_n(z) & \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \frac{p_n(t)(t)dt}{t-z} \\ q_{n-1}(z) & \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \frac{q_{n-1}(t)w(t)dt}{t-z} \end{bmatrix} \tag{12.28}$$

where

$$Y_{11}(z) = p_n(z) = P_{n,N}(z) \tag{12.29}$$

and $q_{n-1}(z)$ *is a polynomial of degree at most* $n-1$.

12.1.5 The g-function

For what follows we fix the dependence of the parameter N on the degree n:

$$N := \frac{n}{t} . \tag{12.30}$$

The non-Hermitian weight function is of the form

$$w_{n,N}(z) = e^{-NV(z)} \tag{12.31}$$

where the contour potential function $V(z)$ is of the semiclassical type [68] in the usual sense:

$$V(z) = az - c\log(z-a) + (c+t)\log z\ , \qquad V'(z) = a - \frac{c}{z-a} + \frac{c+t}{z}\ . \tag{12.32}$$

We seek a function $g(z)$ in $\mathbb{C} \setminus \mathcal{L}$ where $\mathcal{L}$ is a system of oriented contours (also to be determined) and a suitable homotopic deformation of Γ in $\mathbb{C} \setminus [0,a]$ such that the following conditions are satisfied (generalizing the g-function conditions in Chap. 7).

(g.1) $g(z)$ is holomorphic in $\mathbb{C}\setminus\mathcal{L}$ and has continuous boundary values $g_+(z)$ and $g_-(z)$ along $\mathcal{L}$,

(g.2) $g(z) = \log z + \mathcal{O}\left(\frac{1}{z}\right)$ as $z \to \infty$,

(g.3) There exists an arc $\mathcal{B} \subset \mathcal{L} \cap \Gamma$ such that

$$g_+(z) + g_-(z) - \frac{1}{t}V(z) \equiv \ell \qquad z \in \mathcal{B}\ , \tag{12.33}$$

for some constant ℓ and

$$\mathrm{Re}(g_+(z) - g_-(z)) \equiv 0 \qquad z \in \mathcal{B}\ . \tag{12.34}$$

(g.4) The inequality

$$\mathrm{Re}\left(g_+(z) + g_-(z) - \frac{1}{t}V(z)\right) < 0 \qquad z \in \Gamma \setminus \mathcal{B}\ , \tag{12.35}$$

holds and

$$\frac{1}{2\pi i}\left(g_+(z) - g_-(z)\right) \in \mathbb{Z} \tag{12.36}$$

for each part of $\mathcal{L} \setminus \mathcal{B}$, i.e., $e^{ng(z)}$ is holomorphic in $z \in \mathbb{C} \setminus \mathcal{B}$.

(g.5) The function

$$h(z) := g_+(z) - g_-(z) \tag{12.37}$$

has an analytic continuation in a thin *lens-shaped region* L around $\mathcal{B}$ such that

$$\begin{cases} \mathrm{Re}(h(z)) > 0 & z \in L \text{ on the positive side of } \mathcal{B} , \\ \\ \mathrm{Re}(h(z)) < 0 & z \in L \text{ on the negative side of } \mathcal{B} . \end{cases} \tag{12.38}$$

To construct the actual g-function for the nonlinear steepest descent analysis, we follow the constructive approach detailed in [16]. The central object in the construction of a suitable g-function is a meromorphic quadratic differential [89] of the form

$$R(z)dz^2 := -\frac{a^2 J(z)}{z^2(z-a)^2} dz^2 \tag{12.39}$$

on the Riemann sphere, where $J(z)$ is a monic polynomial of degree four:

$$J(z) = \prod_{i=1}^{4}(z - z_i) = z^4 + j_3 z^3 + j_2 z^2 + j_1 z + j_0 \ . \tag{12.40}$$

$J(z)$ is assumed to have real coefficients because of the symmetry of the problem with respect to the real axis. Consider the one-form

$$y(z)dz := \frac{1}{i}\sqrt{R(z)}dz = \frac{a\sqrt{J(z)}}{z(z-a)} dz \ . \tag{12.41}$$

on a double cover of the z-plane. To meet the g-function conditions, we impose the equations

$$\begin{aligned} &\operatorname*{res}_{z=0} y(z)dz = \pm(t+c) \ , \\ &\operatorname*{res}_{z=a} y(z)dz = \mp c \ , \\ &\operatorname*{res}_{z=\infty} y(z)dz = t \ , \end{aligned} \tag{12.42}$$

i.e.,

$$R(z) = \begin{cases} -\dfrac{(t+c)^2}{z^2}\left(1+\mathcal{O}(z)\right) & z \to 0 \ , \\ -\dfrac{c^2}{(z-a)^2}\left(1+\mathcal{O}(z-a)\right) & z \to a \ , \\ -a^2 + \dfrac{2at}{z} + \mathcal{O}\left(\dfrac{1}{z^2}\right) & z \to \infty \ . \end{cases} \tag{12.43}$$

This gives the equations

$$J(0) = (t+c)^2 \qquad J(a) = c^2 \qquad -a^2(2a + j_3) = 2at \ , \tag{12.44}$$

$$J(z) = J(t,x;z) = z^4 - \frac{2(t+a^2)}{a} z^3 + xz^2 + \left(a^3 + (2t-x)a - \frac{t(t+2c)}{a}\right) z + (t+c)^2 \ , \tag{12.45}$$

where the value of the real parameter x has to be chosen appropriately. The numerical results suggest that the zeroes condense along a connected arc which motivates the

Genus zero ansatz. *We seek a one-form $y(z)dz$ whose associated algebraic curve*

$$y(z)^2 + R(z) = 0 \tag{12.46}$$

is of genus zero, in other words, we assume that the polynomial $J(z)$ has at least one double root, i.e., the discriminant of $J(z)$ vanishes. Of course, this ansatz has to be justified by proving that the resulting g-function satisfies the required conditions.

This imposes an algebraic equation of degree five on the possible values of x and, depending on the value of the parameter t, there is a unique choice that is compatible with the g-function conditions, as shown below.

Lemma 12.1.1 (Lemma 5.1, [11]) *There exists a continuous function $x\colon (0,\infty) \to \mathbb{R}$ such that the genus zero ansatz is satisfied for all $t > 0$:*

$$\operatorname{discrim}(J(x(t),t;z)) \equiv 0 \quad t > 0 \ , \tag{12.47}$$

with the following properties:

- $$\lim_{t\to 0} J(x(t),t;z) = (z-z_1)^2(z-z_2)^2 \ , \tag{12.48}$$

- *The double roots z_1 and z_2 split into a pair of complex conjugate and a pair of real roots respectively for small positive t:*

$$J(z) = (z-b)^2(z-\beta)(z-\overline{\beta}) \text{ for some } b=b(t) \ , \beta = \beta(t) \ , \tag{12.49}$$

with $b > a$ and $\beta \notin \mathbb{R}$,

- *At the critical value $t = t_c$ all of the four roots collide:*

$$J(z) = (z-a-\sqrt{c})^4 \tag{12.50}$$

- *For $t > t_c$, there are two real double roots:*

$$J(z) = \left(z^2 - \frac{a^2+t}{a}z + (c+t)\right)^2 = (z-b_-)^2(z-b_+)^2 \tag{12.51}$$

Pre-critical case. The analysis of the quadratic differential $R(z)dz^2$ shows that there exists a critical trajectory $\mathcal{B}$ that connects β and $\overline{\beta}$ whose intersection with the real line is negative (goes on the left of $z=0$). The orientation is chosen on $\mathcal{B}$ from β to $\overline{\beta}$. Now $y(z)dz$ can be defined unambiguously on $\mathbb{C}\setminus\mathcal{B}$ and the equations (12.43) correspond to the residue relations

$$\begin{aligned} &\operatorname*{res}_{z=0} y(z)dz = t + c \ , \\ &\operatorname*{res}_{z=a} y(z)dz = -c \ , \\ &\operatorname*{res}_{z=\infty} y(z)dz = t \end{aligned} \tag{12.52}$$

as stated above (12.42). Consider the normalized integral function of $y(z)dz$:

$$\phi(z) := \frac{1}{2t}\int_{\beta}^{z} y(s)ds \ . \tag{12.53}$$

The real part of $\phi(z)$ is smooth and harmonic on $\mathbb{C} \setminus (\mathcal{B} \cup \{0\} \cup \{a\})$ with

$$\mathrm{Re}(\phi(z)) \sim \begin{cases} \dfrac{t+c}{2t} \log|z| & z \longrightarrow 0 \, , \\ \\ -\dfrac{c}{2t} \log|z-a| & z \longrightarrow a \, . \end{cases} \tag{12.54}$$

Moreover $\mathrm{Re}(\phi(z))$ vanishes along all critical trajectories emanating from β or $\overline{\beta}$. Also

$$\frac{1}{2t} V(z) - \phi(z) \sim \log z - \frac{\ell}{2} + \mathcal{O}\left(\frac{1}{z}\right) \qquad z \longrightarrow \infty \tag{12.55}$$

for some $\ell \in \mathbb{R}$ (the reality of ℓ follows from the symmetry of $y(z)$).

The g-function is constructed as follows:

Lemma 12.1.2 (Lemma 5.2, [11]) *The function*

$$g(z) := \frac{1}{2t} V(z) - \phi(z) + \frac{\ell}{2} \tag{12.56}$$

for $z \in \mathbb{C} \setminus \mathcal{L}$ *satisfies the conditions of the* g*-function for* $0 < t < t_c$ *with the choice of the integration contour*

$$\Gamma := \mathcal{B} \cup \mathcal{R} \, , \tag{12.57}$$

where $\mathcal{R}$ *is chosen so that* Γ *is a simple positively oriented contour around the branch cut* $\mathbb{C} \setminus [0, a]$.

Proof.

There is a probability measure given by the one-form $y(z)dz$:

Lemma 12.1.3 (Lemma 5.3, [11]) *The measure*

$$d\mu(s) := \left. \frac{1}{2\pi i t} y_+(s) ds \right|_{\mathcal{B}} \tag{12.58}$$

supported on $\mathcal{B}$ *is a probability measure. Moreover*

$$Re(g(z)) = \int_{\mathcal{B}} \log|z-u| d\mu(u) \, . \tag{12.59}$$

Post-critical case. Past the critical time $t > t_c$ the numerator of $R(z)$ has two double real roots:

$$J(z) = \left(z^2 - \frac{a^2+t}{a}z + (c+t)\right)^2 = (z-b_-)^2(z-b_+)^2 \,, \tag{12.60}$$

where

$$b_\pm = \frac{a^2 + t \pm \sqrt{(t-t_c)(t-\tau_c)}}{2a} \,. \tag{12.61}$$

The corresponding one-form becomes rational and therefore single-valued:

$$y(z) = \left(a - \frac{t+c}{z} + \frac{c}{z-a}\right) dz \,. \tag{12.62}$$

The corresponding critical trajectory structure changes: there is a critical trajectory $\mathcal{B}$ emanating from and returning to b_- encircling 0 and a.

Now the integral function has the explicit form

$$\phi(z) := \frac{1}{2t}\int_{b_-}^{z} y(s)ds = \frac{a}{2t}(z-b_-) + \frac{c}{2t}\log\left(\frac{z-a}{b_- - a}\right) - \frac{t+c}{2t}\log\frac{z}{b_-} \,. \tag{12.63}$$

for $z \in \mathbb{C} \setminus [0,\infty)$. Thus

$$\phi(z) \sim \begin{cases} \dfrac{a}{2t}z - \dfrac{1}{2}\log z - \dfrac{\ell}{2} + \mathcal{O}\left(\dfrac{1}{z}\right) & z \to \infty \\[2em] -\dfrac{t+c}{2t}\log z + \mathcal{O}(1) & z \to 0 \\[2em] \dfrac{c}{2t}\log(z-a) + \mathcal{O}(1) & z \to a \,, \end{cases} \tag{12.64}$$

with

$$\ell = -\frac{b_-}{t} - \frac{c}{t}\log(b_- - a) + \frac{t+c}{t}\log b_- \in \mathbb{R} \,. \tag{12.65}$$

Now the real part of $\phi(z)$ is smooth and harmonic on $\mathbb{C} \setminus (\{0\} \cup \{a\})$.

Lemma 12.1.4 (Lemma 5.4, [11]) *The function*

$$g(z) := \begin{cases} \frac{1}{2t}V(z) + \phi(z) + \frac{\ell}{2} & z \text{ inside } \mathcal{B} \\ & \\ \frac{1}{2t}V(z) - \phi(z) + \frac{\ell}{2} & z \text{ outside } \mathcal{B} \end{cases} \qquad z \in \mathbb{C} \setminus [0, \infty) \tag{12.66}$$

satisfies the conditions of the g-function along the integration contour $\Gamma = \mathcal{B}$.

Note that the simple form of $y(z)$ makes $g(z)$ quite explicit:

$$g(z) = \begin{cases} -\frac{c}{t}\log(z-a) + \frac{t+c}{t}\log z & \text{outside } \mathcal{B} \\ \frac{a}{t}z + \ell & \text{inside } \mathcal{B}\ . \end{cases} \tag{12.67}$$

Lemma 12.1.5 (Lemma 5.5, [11]) *The measure*

$$d\mu(s) := -\frac{1}{2\pi i t} y(s)ds\bigg|_{\mathcal{B}} \tag{12.68}$$

supported on $\mathcal{B}$ *is a probability measure. Moreover*

$$Re(g(z)) = \int_{\mathcal{B}} \log|z-u|d\mu(u)\ . \tag{12.69}$$

The conclusion of the asymptotic analysis. The steps of the nonlinear steepest descent method give the strong asymptotics of the polynomials:

Theorem 12.1.3 ([11]) *The following strong asymptotic results hold:*

- *For* $\mathbb{C} \setminus \mathcal{B}$,

  $$P_{n,N}(z) = \left(\frac{B(z) + B(z)^{-1}}{2} + \mathcal{O}\left(\frac{1}{n}\right)\right) e^{ng(z)} \tag{12.70}$$

  holds uniformly on compact subsets of $\mathbb{C} \setminus \mathcal{B}$.

- *For* $z \in \mathcal{B} \setminus \{\beta, \overline{\beta}\}$:

  $$P_{n,N}(z) = \frac{1}{2}\left((B(z) + B(z)^{-1})e^{-in\varphi(z)} - i(B(z) - B(z)^{-1})e^{in\varphi(z)} + \mathcal{O}\left(\frac{1}{n}\right)\right) e^{\frac{1}{2t}V(z) + \frac{\ell}{2}}\ ; \tag{12.71}$$

with the notation

$$\varphi(z) := \pi \int_{\beta}^{z} d\mu(s) \quad z \in \mathcal{B} . \tag{12.72}$$

In particular, since $B(z) + B(z)^{-1}$ has no zeroes in $\mathbb{C} \setminus \mathcal{B}$, by Hurwitz theorem the zeroes of $P_{n,N}(z)$ may accumulate only on $\mathcal{B}$. The asymptotic result above gives that

$$\lim_{n\to\infty} \left[\frac{1}{n} \log |P_n(z)| - \mathrm{Re}(g(z)) \right] = 0 \qquad z \in \mathbb{C} \setminus \mathcal{B} . \tag{12.73}$$

on $\mathbb{C} \setminus \mathcal{B}$. Since

$$\mathrm{Re}(g(z)) = \int_{\mathcal{B}} \log |z - u| d\mu(u) , \tag{12.74}$$

this confirms the conjecture 8.1.1 in this special case:

Theorem 12.1.4 *The (full) sequence of normalized counting measures*

$$\nu_{n,N} := \frac{1}{n} \sum_{k=1}^{n} \delta_{z_{k,n,N}} \qquad n = 0, 1, 2, \ldots \tag{12.75}$$

of the zeroes of the nth orthogonal polynomial

$$P_{n,N}(z) = \prod_{k=1}^{n} (z - z_{k,n,N}) \qquad n = 0, 1, 2, \ldots \tag{12.76}$$

converges to μ_t in the weak-star sense.

12.1.6 Post-critical case

The study of the post-critical case follows essentially the same steps as in the pre-critical situation but the

Theorem 12.1.5 *The following strong asymptotics holds for the polynomials as $n \longrightarrow \infty$:*

- *For z outside $\mathcal{B}$:*

$$P_n(z) = \left(1 + \mathcal{O}\left(\frac{1}{n^{\frac{3}{2}}} \right) \right) e^{ng(z)} \tag{12.77}$$

uniformly on compact subsets in the exterior of $\mathcal{B}$.

- *For z inside $\mathcal{B}$*

$$P_n(z) = \left(-\frac{1}{\sqrt{4\pi n\kappa}}\frac{1}{z-b_-} + \mathcal{O}\left(\frac{1}{n^{\frac{3}{2}}}\right)\right) e^{ng(z)} . \qquad (12.78)$$

where $\kappa = -y'(b_-) > 0$. Again, this result is valid also uniformly in every compact subset inside $\mathcal{B}$.

- *For $z \in \mathcal{B} \setminus \{b_-\}$:*

$$P_{n,N}(z) = \left(-\frac{\sqrt{4\pi n\kappa}}{z-b_-}e^{in\varphi(z)} + e^{-in\varphi(z)} + \mathcal{O}\left(\frac{1}{n}\right)\right) e^{\frac{n}{2t}(V(z)+t\ell)} , \qquad (12.79)$$

again with the notation

$$\varphi(z) := \pi \int_{\beta}^{z} d\mu(s) \quad z \in \mathcal{B} . \qquad (12.80)$$

The Hurwitz theorem implies that the zeroes of $P_{n,N}(z)$ may accumulate only on $\mathcal{B}$ and therefore we get

Theorem 12.1.6 *The (full) sequence of normalized counting measures*

$$\nu_{n,N} := \frac{1}{n}\sum_{k=1}^{n} \delta_{z_{k,n,N}} \qquad n = 0, 1, 2, \ldots \qquad (12.81)$$

of the zeroes of the nth orthogonal polynomial

$$P_{n,N}(z) = \prod_{k=1}^{n}(z - z_{k,n,N}) \qquad n = 0, 1, 2, \ldots \qquad (12.82)$$

converges to μ_t in the weak-star sense.

12.1.7 The g-function and the equilibrium measure

Figs. 12.2 and 12.3 suggest that for every value of the parameter t the support of the asymptotic zero distribution encoded into the g-function is closely related to the geometry of support of the equilibrium measure. The following very important result says that the support of the equilibrium measure and the asymptotic distribution of the zeroes of the orthogonal polynomials connected by the same quadrature property observed for the Gaussian weights above:

Theorem 12.1.7 *The derivative of the g-function is equal to the negative of Cauchy transform of the equilibrium measure μ_{Q_t} outside the polynomial convex hull of S_{Q_t}. In other words, $int(S_{Q_t})$ is a holomorphic quadrature domain with respect to the measure μ_t.*

Proof. Since $\mathrm{supp}(\mu_t) \subset S_{Q_t}$ by Lemma 10.1.1, $g'(z)$ is holomorphic in the unbounded component of $\mathbb{C} \setminus S_{Q_t}$.

For $t \geq t_c$, the derivative of g and the negative of the normalized Cauchy transform of the support $S_{Q_t} = D(0, \sqrt{t+c}) \setminus D(a, \sqrt{c})$ are both given by the same rational function outside $D(0, \sqrt{t+c})$:

$$g'(z) = -\frac{1}{\pi t} C_{Q_t}(z) = \frac{1}{t}\left(\frac{t+c}{z} - \frac{c}{z-a}\right) . \tag{12.83}$$

For $0 < t < t_c$, as shown in the appendix to this chapter in Sec. 12.3, the derivative of the g-function satisfies the same algebraic equation and, as and they have the same asymptotic behavior as $z \longrightarrow \infty$, they are equal in the unbounded component of $\mathbb{C} \setminus S_{Q_t}$.

■

12.2 Riemann–Hilbert analysis for the Bratwurst orthogonal polynomials, manuscript

Riemann-Hilbert analysis for the Bratwurst orthogonal polynomials

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June 13, 2010

Contents

1 The potential and the orthogonal polynomials

We consider the function

$$Q(z) := |z|^2 + 2c \log \frac{1}{|z-a|} \tag{1}$$

in the complex plane where $c > 0$ and, without loss of generality, we assume that $a \in \mathbb{R}^+$.

The only critical points of the potential are

$$z_1 = \frac{a - \sqrt{a^2+4c}}{2} \qquad z_2 = \frac{a + \sqrt{a^2+4c}}{2} . \tag{2}$$

These are both local minima, the absolute minimum of Q is attained at $z = z_1$. The corresponding weight function in the context of normal matrix models is given by

$$e^{-Q(z)} = |z-a|^{2c} e^{-|z|^2} . \tag{3}$$

By introducing the scaling parameter $N > 0$ we get a one-parameter family of weight functions. For any $N > 0$ the absolute moments are all finite:

$$\int_{\mathbb{C}} |z|^k e^{-NQ(z)} dA(z) < \infty, \qquad k = 0, 1, \dots , \tag{4}$$

where dA denotes the area measure in $\mathbb{C}$. Therefore the *monic orthogonal polynomials* $P_{n,N}(z)$ satisfying

$$\int_{\mathbb{C}} P_{n,N}(z)\overline{P_{m,N}(z)} e^{-NQ(z)} dA(z) = h_{n,N}\delta_{nm} \qquad n, m = 0, 1, \dots . \tag{5}$$

exist and they are unique. The *normalization constants* $h_{n,N}$ are all positive. The nth monic orthogonal polynomial is characterized by the orthogonality relations

$$\int_{\mathbb{C}} P_{n,N}(z)\bar{z}^k e^{-NQ(z)} dA(z) = 0 \qquad k = 0, 1, \dots, n-1 . \tag{6}$$

In the degenerate case $a = 0$ the monic orthogonal polynomials are the pure monomials $P_{n,N}(z) = z^n$; from now on we assume that $a > 0$.

2 Results

Motivated by random matrix models [5], we consider the obtain asymptotics of the orthogonal polynomials $P_{n,N}(z)$ in the *scaling limit*

$$n \to \infty , \quad N \to \infty , \quad \frac{n}{N} \to t . \tag{7}$$

By this we mean that for every compact subset K of $\mathbb{C}$ we get asymptotic formulae for $P_{n,N}(z)$ that holds uniformly in K.

For simplicity, we will assume that $N = \frac{n}{t}$.

As a consequence of the Deift–Zhou nonlinear steepest descent or Riemann–Hilbert method applied to the $P_{n,N}$'s we obtain the following main result of this paper:

Theorem 2.1 *There exists a family of probability measures*

$$t \to \mu_t \qquad t > 0 \tag{8}$$

given by (64) *and* (87) *such that the sequence of normalized counting measures*

$$\nu_{n,N} := \frac{1}{n}\sum_{k=1}^{n} \delta_{z_{k,n,N}} \qquad n = 0, 1, 2, \dots \tag{9}$$

of the zeroes of the nth orthogonal polynomial

$$P_{n,N}(z) = \prod_{k=1}^{n} (z - z_{k,n,N}) \qquad n = 0, 1, 2, \dots \tag{10}$$

converges to μ_t *in the weak-star sense.*

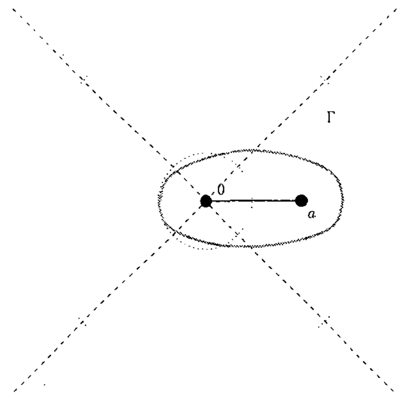


Figure 1: The contour Γ

3 Non-Hermitian orthogonality

First we show that the polynomials $P_{n,N}(z)$ satisfy a *non-Hermitian orthogonality* with respect to an analytic weight function on a closed contour in the complex plane. This is a consequence of the following

Lemma 3.1 *For any polynomial $p(z)$, the following integral identity holds:*

$$\int_{\mathbb{C}} p(z)\bar{z}^k |z-a|^{2Nc} e^{-Nz\bar{z}} dA(z) = \frac{e^{-2\pi i Nc}}{2i} \sum_{l=0}^{k} \binom{k}{l} a^{k-l} \frac{\Gamma(l+Nc+1)}{N^{l+Nc+1}} \oint_{\Gamma} p(z) \frac{(z-a)^{Nc} e^{-az}}{z^{Nc}} \frac{dz}{z^{l+1}} . \tag{11}$$

where the integrand has a branch cut along the segment $[0,a]$ and Γ is a simple positively oriented closed contour encircling $[0,a]$ (Fig. 3).

The proof of Lemma 3.1 can be found in the Appendix.

The triangular structure of the above identities implies that the system of orthogonality conditions (6) is equivalent to

$$\oint_{\Gamma} P_{n,N}(z) \frac{(z-a)^{Nc} e^{-az}}{z^{Nc}} \frac{dz}{z^{l+1}} = 0 \qquad l = 0, 1, \ldots, n-1 \tag{12}$$

(the diagonals are all different from zero in the above triangular linear system). By relabeling the monomials above, we get the following

Corollary 3.1 *The system of Hermitian orthogonality relations (6) is equivalent to the system of non-Hermitian orthogonality relations*

$$\oint_{\Gamma} P_{n,N}(z) z^k \frac{(z-a)^{Nc} e^{-az}}{z^{Nc+n}} dz = 0 \qquad k = 0, 1, \ldots, n-1 \tag{13}$$

with respect to the varying weight

$$w_{n,N}(z) := \frac{(z-a)^{Nc}e^{-az}}{z^{Nc+n}} \tag{14}$$

with branch cut along $[0, a]$.

Corollary 3.2 *The complex moments of the weight* (3) *are given by*

$$\begin{aligned}&\int_{\mathbb{C}} z^j \bar{z}^k |z-a|^{2Nc} e^{-Nz\bar{z}} dA(z)\\ &\quad = \frac{\mathrm{e}^{-2\pi i Nc}}{2i} \sum_{l=0}^{k} \binom{k}{l} a^{k-l} \frac{\Gamma(l+Nc+1)}{N^{l+Nc+1}} \oint_{\Gamma} z^{j-l} \frac{(z-a)^{Nc}e^{-az}}{z^{Nc}} \frac{dz}{z} .\end{aligned} \tag{15}$$

The area integral moment matrices

$$M^{(n)} = \left(\int_{\mathbb{C}} z^k \bar{z}^l |z-a|^{2Nc} e^{-Nz\bar{z}} dA(z)\right)_{0\le k,l\le n-1} \tag{16}$$

and the Hänkel matrix associated to the contour integral moments

$$m^{(n)} = \left(\oint_{\Gamma} z^{k+l} \frac{(z-a)^{Nc}e^{-az}}{z^{Nc+n}} dz\right)_{0\le k,l\le n-1} \tag{17}$$

satisfy the relation

$$M^{(n)} = m^{(n)} P^{(n)} A^{(n)} \tag{18}$$

where

$$\begin{aligned}P^{(n)} &= (\delta_{k\ n-1-l})_{0\le k,l\le n-1}\\ A^{(n)} &= \frac{\mathrm{e}^{-2\pi i Nc}}{2i}\left(\binom{l}{k} a^{l-k} \frac{\Gamma(k+Nc+1)}{N^{k+Nc+1}}\right)_{0\le k,l\le n-1} ,\end{aligned} \tag{19}$$

and $\Gamma(z)$ is the Euler gamma function.

Therefore the moment determinants are connected by the equation

$$\det\left(M^{(n)}\right) = \det\left(m^{(n)}\right)(-1)^{\lfloor \frac{n}{2} \rfloor} \frac{\mathrm{e}^{-2n\pi i Nc}}{(2i)^n N^{nNc+\frac{n(n+1)}{2}}} \prod_{k=0}^{n-1} \Gamma(k+Nc+1) . \tag{20}$$

Since $\Gamma(z)$ has no zeroes and $M^{(n)}$ is non-singular, $\det(m^{(n)}) \neq 0$.

4 The Riemann–Hilbert problem

Since $P_{n,N}(z)$ satisfies the non-hermitian orthogonality relations, we may consider the standard 2×2 Riemann–Hilbert problem [6] on the contour Γ defined above.

Lemma 4.1 *For fixed* n, N, *the Riemann Hilbert problem*

(Y.1) $Y(z)$ *is holomorphic in* $\mathbb{C} \setminus \Gamma$,

(Y.2) $Y_+(z) = Y_-(z)\begin{bmatrix} 1 & w_n(z) \\ 0 & 1 \end{bmatrix}$ *for* $z \in \Gamma$,

(Y.3) $Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) z^{-n\sigma_3}$ *as* $z \to \infty$

has a unique solution and $Y_{11}(z) = P_{n,N}(z)$.

The proof can be found in the Appendix.

5 The g-function

The Deift–Zhou *nonlinear steepest descent method* [4] will be applied to obtain the asymptotics of the solution of the Riemann–Hilbert problem in Lemma 4.1. To perform the standard steps of the method, an appropriate auxiliary function $g(z)$ (the so-called *g-function*) has to be constructed. Finding the g-function is not as straightforward as in the real line case; however, it is still related to the logarithmic equilibrium problem in the complex plane corresponding to the potential (1). The equilibrium measure is studied in detail in [1]; instead of using the equilibrium measure, we construct a suitable function $g(z)$ from an ansatz satisfying based on the specific conditions expected from the g-function.

The weight function is of the form

$$w_{n,N}(z) = \mathrm{e}^{-NV(z)} \tag{21}$$

where the *potential function* $V(z)$ is of the semiclassical type [7]:

$$V(z) = az - c\log(z-a) + (c+t)\log z \ , \tag{22}$$

for $z \in \mathbb{C} \setminus [0,\infty)$ with jumps

$$V_+(x) - V_-(x) = \begin{cases} -2(t+c)\pi i & x \in (0,a) \\ -2t\pi i & x \in (a,\infty) \ . \end{cases} \tag{23}$$

Notice that the function $\exp(-V(z)/t)$ is holomorphic in $\mathbb{C} \setminus [0,a]$.

The g-function conditions. We seek a function $g(z)$ in $\mathbb{C} \setminus \mathcal{L}$ where $\mathcal{L}$ is a system of oriented contours (also to be determined) and a suitable homotopic deformation of Γ in $\mathbb{C} \setminus [0,a]$ such that the the following conditions are satisfied:

(g.1) $g(z)$ is holomorphic in $\mathbb{C} \setminus \mathcal{L}$ and has continuous boundary values $g_+(z)$ and $g_-(z)$ along $\mathcal{L}$,

(g.2) $g(z) = \log z + \mathcal{O}\left(\frac{1}{z}\right)$ as $z \to \infty$,

(g.3) There exists an arc $\mathcal{B} \subset \mathcal{L} \cap \Gamma$ such that

$$g_+(z) + g_-(z) - \frac{1}{t}V(z) \equiv \ell \qquad z \in \mathcal{B} \ , \tag{24}$$

for some constant ℓ and

$$\mathrm{Re}(g_+(z) - g_-(z)) \equiv 0 \qquad z \in \mathcal{B} \ . \tag{25}$$

(g.4) The inequality

$$\mathrm{Re}\left(g_+(z) + g_-(z) - \frac{1}{t}V(z)\right) < 0 \qquad z \in \Gamma \setminus \mathcal{B}\ , \tag{26}$$

holds and

$$\frac{1}{2\pi i}\left(g_+(z) - g_-(z)\right) \in \mathbb{Z} \tag{27}$$

for each part of $\mathcal{L} \setminus \mathcal{B}$, i.e., $e^{ng(z)}$ is holomorphic in $z \in \mathbb{C} \setminus \mathcal{B}$.

(g.5) The function

$$h(z) := g_+(z) - g_-(z) \tag{28}$$

has an analytic continuation in a thin *lens-shaped region* L around $\mathcal{B}$ such that

$$\begin{cases} \mathrm{Re}(h(z)) > 0 & z \in L \text{ on the positive side of } \mathcal{B}\ , \\ \mathrm{Re}(h(z)) < 0 & z \in L \text{ on the negative side of } \mathcal{B}\ . \end{cases} \tag{29}$$

To central object in the construction of a suitable g-function is a meromorphic quadratic differential [9] of the form

$$R(z)dz^2 := -\frac{a^2 J(z)}{z^2(z-a)^2}dz^2 \tag{30}$$

on the Riemann sphere, where $J(z)$ is a monic polynomial of degree four:

$$J(z) = \prod_{i=1}^{4}(z - z_i) = z^4 + j_3 z^3 + j_2 z^2 + j_1 z + j_0\ . \tag{31}$$

$J(z)$ is assumed to have real coefficients because of the symmetry of the problem with respect to the real axis. Consider the one-form

$$y(z)dz := \frac{1}{i}\sqrt{R(z)}dz = \frac{a\sqrt{J(z)}}{z(z-a)}dz\ . \tag{32}$$

on a double cover of the z-plane. To make it single-valued on the z-plane we need to introduce a suitable branch cut structure between the roots of $J(z)$ that accomodates the g-function conditions and it will depend on the geometry of the *critical horizontal trajectories* (see [9]) of the quadratic differential $R(z)dz^2$. The integral function of this differential form is the essential building block of the g-function: this integral has to cancel the jumps of of $V(z)$ and its logarithmic singularities at $z = 0$ and $z = a$ (the fixed poles of $y(z)$) which gives residue conditions of the form

$$\begin{aligned} \operatorname*{res}_{z=0} y(z)dz &= \pm(t + c)\ , \\ \operatorname*{res}_{z=a} y(z)dz &= \mp c\ , \\ \operatorname*{res}_{z=\infty} y(z)dz &= t\ , \end{aligned} \tag{33}$$

where the sign ambiguity comes from the the fact that the branch cuts are not specified yet and and the overall sign of the y-term in the definition of $g(z)$ may vary depending on the region in the

z-plane. In terms of the quadratic differential there is no sign ambiguity: the conditions (33) are expressed as

$$R(z) = \begin{cases} -\dfrac{(t+c)^2}{z^2}\left(1+\mathcal{O}(z)\right) & z \to 0 \ , \\ -\dfrac{c^2}{(z-a)^2}\left(1+\mathcal{O}(z-a)\right) & z \to a \ , \\ -a^2 + \dfrac{2at}{z} + \mathcal{O}\left(\dfrac{1}{z^2}\right) & z \to \infty \ . \end{cases} \tag{34}$$

This gives the equations

$$J(0) = (t+c)^2 \qquad J(a) = c^2 \qquad -a^2(2a + j_3) = 2at \ , \tag{35}$$

that are linear in the coefficients of $J(z)$. Therefore

$$J(z) = J(t,x;z) = z^4 - \frac{2(t+a^2)}{a} z^3 + xz^2 + \left(a^3 + (2t-x)a - \frac{t(t+2c)}{a}\right) z + (t+c)^2 \ , \tag{36}$$

where the value of the real parameter x has to be chosen appropriately. Based on the calculations in [1] concerning the potential $Q(z)$, we make the following

Genus zero ansatz. *We seek a one-form $y(z)dz$ whose associated algebraic curve*

$$y(z)^2 + R(z) = 0 \tag{37}$$

is of genus zero, in other words, we assume that the polynomial $J(z)$ has at least one double root, i.e., the discriminant of $J(z)$ vanishes.

This imposes an algebraic equation of degree five on the possible values of x and, depending on the value of the parameter t, there is a unique choice that is compatible with the g-function conditions, as shown below.

Lemma 5.1 *There exists a continuous function $x\colon (0,\infty) \to \mathbb{R}$ such that the genus zero ansatz is satisfied for all $t > 0$:*

$$\mathrm{discrim}(J(x(t), t; z)) \equiv 0 \quad t > 0 \ , \tag{38}$$

with the following properties:

- $$\lim_{t\to 0} J(x(t), t; z) = (z - z_1)^2 (z - z_2)^2 \ , \tag{39}$$
- *The double roots z_1 and z_2 split into a pair of complex conjugate and a pair of real roots respectively for small positive t:*

  $$J(z) = (z-b)^2 (z-\beta)(z-\overline{\beta}) \text{ for some } b = b(t) \ , \beta = \beta(t) \ , \tag{40}$$

  with $b > a$ and $\beta \notin \mathbb{R}$,
- *There is a critical value*

  $$t_c := a(a + 2\sqrt{c}) \tag{41}$$

  of the parameter t for which all four roots collide:

  $$J(z) = (z - a - \sqrt{c})^4 \tag{42}$$

- *For $t > t_c$, there are two real double roots:*

$$J(z) = \left(z^2 - \frac{a^2+t}{a}z + (c+t)\right)^2 = (z-b_-)^2(z-b_+)^2 \tag{43}$$

The analysis of the discriminant of $J(t,x;z)$ and the construction of $x(t)$ can be found in the Appendix.

The parameter ranges $0 < t < t_c$ and $t_c < t < \infty$ are referred to as *pre-critical* and *post-critical* respectively. There is a transition in the behavior at $t = t_c$; in the vicinity of t_c a double scaling limit will be considered.

5.1 Pre-critical case

Consider the quadratic differential $R(z)dz^2$ fixed by the conditions above for $t < t_c$. It has five critical points: $z = 0$ and $z = a$ are poles of order two, $z = \infty$ is a pole of order four, $z = \beta$, $z = \overline{\beta}$ are simple zeroes and $z = b$ is a zero of order two. The local behavior of the horizontal trajectories is governed by the following rules [9]:

- In a neighborhood of the double poles with negative leading coefficient the horizontal trajectories are conformally equivalent to concentric circles.
- In the vicinity of the single zeroes there are three distinguished directions of incoming critical horizontal trajectories (with asymptotic angles $2\pi/3$).
- In a neighborhood of a double zero there are four distinguished directions of incoming critical horizontal trajectories (with asymptotic angles $\pi/2$).
- Out of the four outgoing critical trajectories from b there are at most two converging to ∞.

The two double poles are surrounded by conformal punctured disk domains that are separated by critical trajectories. Therefore on the boundary of both disks there has to be at least one zero of $R(z)$. Since $R(z)$ has real coefficients, all horizontal trajectories are symmetric with respect to the real axis. It follows that β is on the boundary if and only if $\overline{\beta}$ is on the boundary. The topological possibilities for the global arrangement of the critical trajectories are shown in table 5.1. Using the theory of quadratic differentials and the conditions of the ansatz we conclude (see Appendix) that the only possible trajectory structure is given by the $(3,1)$ entry of the above table.

The contour $\mathcal{B}$ is chosen to be the critical trajectory that connects β and $\overline{\beta}$ whose intersection with the real line is negative (goes on the left of $z = 0$). The orientation is chosen on $\mathcal{B}$ from β to $\overline{\beta}$.

Since the choice of $\mathcal{B}$ and the asymptotic behavior of $y(z)$ is fixed there are no sign ambiguities for the one-form $y(z)dz$; in particular, the equations (34) translate into the residue relations

$$\begin{aligned} \operatorname*{res}_{z=0} y(z)dz &= t + c\ , \\ \operatorname*{res}_{z=a} y(z)dz &= -c\ , \\ \operatorname*{res}_{z=\infty} y(z)dz &= t \end{aligned} \tag{44}$$

as stated above (33). Consider the normalized integral function of $y(z)dz$:

$$\phi(z) := \frac{1}{2t}\int_{\beta}^{z} y(s)ds \tag{45}$$

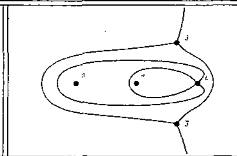
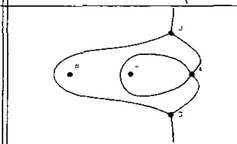
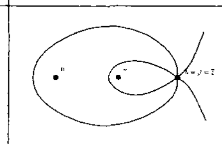
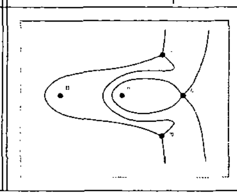
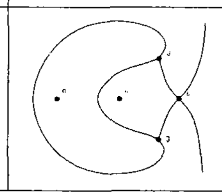
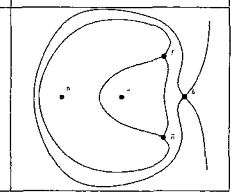
$0 \backslash a$	b	b and β	β
b		impossible	impossible
b and β			impossible
β			

Table 1: Possible trajectory topologies.

on $\mathbb{C} \setminus \mathcal{L}$ where

$$\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3 \cup \mathcal{B} \tag{46}$$

with $\mathcal{L}_1 = (0, a)$, $\mathcal{L}_2 = (a, \infty)$ and $\mathcal{L}_3 = \{\overline{\beta} - iv \ : \ v \geq 0\}$. The poles of $y(z)$ give logarithmic singularities at $z = 0$ and $z = a$. Since

$$y(z) \sim a - \frac{t}{z} + \mathcal{O}\left(\frac{1}{z^2}\right) \qquad z \to \infty \ , \tag{47}$$

integration yields

$$\phi(z) \sim \frac{a}{2t} z - \frac{1}{2} \log z + \mathcal{O}\left(1\right) \qquad z \to \infty \ . \tag{48}$$

Along the different parts of $\mathcal{L}$ the function $\phi(z)$ has the following jumps:

$$\phi_+(z) - \phi_-(z) = \begin{cases} -\dfrac{2\pi i}{2t} \operatorname*{res}_{s=0} y(s)ds = -\dfrac{t+c}{t}\pi i & z \in \mathcal{L}_1 \\[2ex] -\dfrac{2\pi i}{2t} \left(\operatorname*{res}_{s=0} y(s)ds + \operatorname*{res}_{s=a} y(s)ds \right) = -\pi i & z \in \mathcal{L}_2 \\[2ex] \dfrac{2\pi i}{2t} \left(\operatorname*{res}_{s=0} y(s)ds + \operatorname*{res}_{s=a} y(s)ds + \operatorname*{res}_{s=\infty} y(s)ds \right) = 2\pi i & z \in \mathcal{L}_3 \ . \end{cases} \tag{49}$$

Since $\mathcal{B}$ is chosen along a critical horizontal trajectory of $R(z)dz^2$, we have [9]

$$\int_{\beta}^{z} \sqrt{R(s)} ds = i \int_{\beta}^{z} y(s) ds \in \mathbb{R} \qquad z \in \mathcal{B} \ , \tag{50}$$

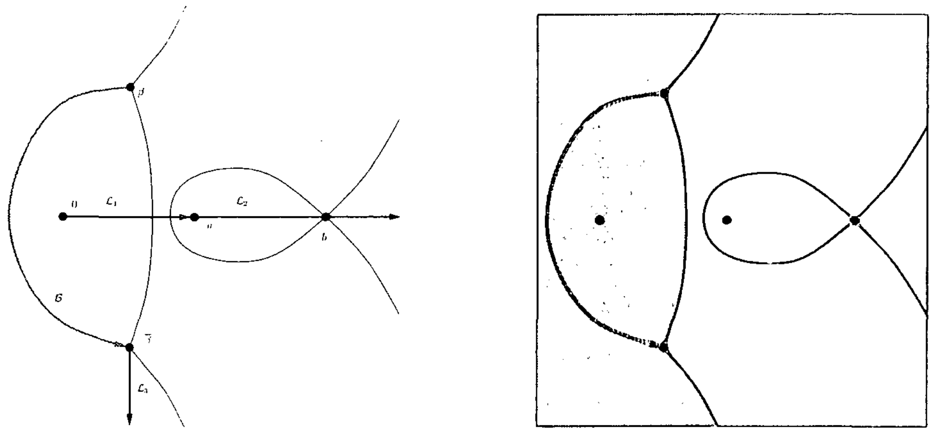


Figure 2: The contour system $\mathcal{L}$ and the sign of $\mathrm{Re}(\phi(z))$ (dark gray=positive)

and therefore

$$\phi_+(z) - \phi_-(z) = \frac{1}{2t}\int_\beta^z (y_+(s) - y_-(s))ds = \frac{1}{t}\int_\beta^z y_+(s)ds \in i\mathbb{R} \qquad z \in \mathcal{B}\,. \tag{51}$$

Obviously,

$$\phi_+(z) + \phi_-(z) = \frac{1}{2t}\int_\beta^z (y_+(s) + y_-(s))ds = 0\,. \tag{52}$$

The real part of $\phi(z)$ is smooth and harmonic on $\mathbb{C} \setminus (\mathcal{B} \cup \{0\} \cup \{a\})$ with

$$\mathrm{Re}(\phi(z)) \sim \left\{ \begin{array}{ll} \dfrac{t+c}{2t}\log|z| & z \to 0\,, \\ -\dfrac{c}{2t}\log|z-a| & z \to a\,. \end{array} \right. \tag{53}$$

Moreover $\mathrm{Re}(\phi(z))$ vanishes along all critical trajectories emanating from β or $\overline{\beta}$.

$$\frac{1}{2t}V(z) - \phi(z) \sim \log z - \frac{\ell}{2} + \mathcal{O}\left(\frac{1}{z}\right) \qquad z \to \infty \tag{54}$$

for some $\ell \in \mathbb{R}$ (the reality of ℓ follows from the symmetry of $y(z)$).

Lemma 5.2 *The function*

$$g(z) := \frac{1}{2t}V(z) - \phi(z) + \frac{\ell}{2} \tag{55}$$

for $z \in \mathbb{C}\setminus\mathcal{L}$ satisfies the conditions of the g-function for $0 < t < t_c$ with the choice of the integration contour

$$\Gamma := \mathcal{B} \cup \mathcal{R}\,, \tag{56}$$

where $\mathcal{R}$ is chosen so that Γ is a simple positively oriented contour around the branch cut $\mathbb{C} \setminus [0, a]$.

Proof. The above function is holomorphic on $\mathbb{C} \setminus \mathcal{L}$ and the choice of ℓ gives that

$$g(z) = \log z + \mathcal{O}\left(\frac{1}{z}\right) \qquad z \to \infty . \tag{57}$$

Moreover, the logarithmic singularities of V and ϕ at 0 and a are cancelled and we have $g_+(x) = g_-(x)$ along $\mathcal{L}_1 \cup \mathcal{L}_2$. On $\mathcal{L}_3$ the jump is given by

$$g_+(z) - g_-(z) = \phi_+(z) - \phi_-(z) = 2\pi i . \tag{58}$$

Therefore we conclude that $g(z)$ is holomorphic on $\mathbb{C} \setminus \mathcal{B} \cup \mathcal{L}_3$ and $e^{ng(z)}$ is holomorphic in $\mathbb{C} \setminus \mathcal{B}$.

Also for $z \in \mathcal{B}$

$$g_+(z) + g_-(z) - \frac{1}{t}V(z) = \phi_+(z) + \phi_-(z) + \ell = \ell . \tag{59}$$

Along $\mathcal{R}$ we have

$$\mathrm{Re}\left(2g(z) - \frac{1}{t}V(z) - \ell\right) = 2\mathrm{Re}(\phi(z)) < 0 . \tag{60}$$

Along the cut $\mathcal{B}$ the jump of $g(z)$ is given by

$$g_+(z) - g_-(z) = \phi_-(z) - \phi_+(z) = 2\phi_-(z) , \tag{61}$$

and therefore the function

$$h(z) := \begin{cases} -2\phi(z) & \text{on the positive side of } \mathcal{B} \\ 2\phi(z) & \text{on the negative side of } \mathcal{B} \end{cases} \tag{62}$$

provides the analytic continuation of $g_+(z) - g_-(z)$ to a lens-shaped region surrounding $\mathcal{B}$. The sign of $\mathrm{Re}(\phi(z))$ implies that

$$\begin{cases} \mathrm{Re}(h(z)) > 0 & \text{on the positive side of } \mathcal{B} \\ \mathrm{Re}(h(z)) < 0 & \text{on the negative side of } \mathcal{B} , \end{cases} \tag{63}$$

as requested.

Q.E.D.

Lemma 5.3 *The measure*

$$d\mu(s) := \left.\frac{1}{2\pi i t} y_+(s)ds\right|_{\mathcal{B}} \tag{64}$$

supported on $\mathcal{B}$ is a probability measure. Moreover

$$Re(g(z)) = \int_{\mathcal{B}} \log|z-u| d\mu(u) . \tag{65}$$

Proof. First

$$\int_{\mathcal{B}} d\mu(s) = \frac{1}{2\pi i t}\int_{\mathcal{B}} y_+(s)ds = \frac{1}{2\pi i}(\phi_+(z) - \phi_-(z)) = 1 , \tag{66}$$

where $z \in \mathcal{L}_3$ an arbitrary point. In terms of the arclength parametrization $z(v)$ of $\mathcal{B}$ we have

$$\frac{d}{dv}\int_{\beta}^{z(v)} d\mu = \frac{d}{dv}\frac{1}{2\pi i t}\int_{\beta}^{z(v)} y_+(s)ds = \frac{1}{2\pi i t} y_+(z(v))z'(v) \tag{67}$$

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which is real by the choice of $\mathcal{B}$ and nonzero since $y_+(s)$ is nonzero along $\mathcal{B}$. Therefore the integral function $\int_{\mathcal{B}}^{z(v)} d\mu$ can only increase and therefore μ is a probability measure supported on $\mathcal{B}$.

To show that $\mathrm{Re}(g(z))$ is the logarithmic potential of μ we note that the Cauchy transform of μ is

$$\begin{aligned}\int_{\mathcal{B}} \frac{d\mu(s)}{s-z} &= \frac{1}{2\pi i t}\int_{\mathcal{B}} \frac{y(s)ds}{s-z} \\ &= \frac{1}{2t}\left(\operatorname*{res}_{s=0} \frac{y(s)ds}{s-z} + \operatorname*{res}_{s=a} \frac{y(s)ds}{s-z} + \operatorname*{res}_{s=z} \frac{y(s)ds}{s-z} + \operatorname*{res}_{s=\infty} \frac{y(s)ds}{s-z}\right) \\ &= \frac{1}{2t}\left(-\frac{c+t}{z} + \frac{c}{z-a} + y(z) - a\right) = -\frac{1}{2t}V'(z) + y(z) = -g'(z) .\end{aligned} \tag{68}$$

This means that

$$\mathrm{Re}(g(z)) = \int_{\mathcal{B}} \log|z-u| d\mu(u) + K \tag{69}$$

for some constant, but $g(z) \sim \log z + \mathcal{O}(1/z)$ gives that $K = 0$.

Q.E.D.

5.2 Post-critical case

Past the critical time $t > t_c$ the numerator of $R(z)$ has two double real roots:

$$J(z) = \left(z^2 - \frac{a^2+t}{a}z + (c+t)\right)^2 = (z-b_-)^2(z-b_+)^2 , \tag{70}$$

where

$$b_\pm = \frac{a^2 + t \pm \sqrt{(t-t_c)(t-\tau_c)}}{2a} . \tag{71}$$

The corresponding one-form becomes rational and therefore single-valued:

$$y(z) = \left(a - \frac{t+c}{z} + \frac{c}{z-a}\right) dz . \tag{72}$$

The corresponding critical trajectory structure evolves through the degenerate $(2,2)$ case as t passes through the critical time t_c. This is the topological structure of the level set $\mathrm{Re}(\phi(z)) = 0$ which contains all the critical trajectories of $R(z)dz^2$ in this degenerate case. The critical trajectory emanating from b_- encircling both $z = 0$ and $z = a$ will be denoted by $\mathcal{B}$.

Now the integral function has the explicit form

$$\phi(z) := \frac{1}{2t}\int_{b_-}^{z} y(s)ds = \frac{a}{2t}(z-b_-) + \frac{c}{2t}\log\left(\frac{z-a}{b_- - a}\right) - \frac{t+c}{2t}\log\frac{z}{b_-} . \tag{73}$$

for $z \in \mathbb{C} \setminus [0, \infty)$. Thus

$$\phi(z) \sim \begin{cases} \frac{a}{2t}z - \frac{1}{2}\log z - \frac{\ell}{2} + \mathcal{O}\left(\frac{1}{z}\right) & z \to \infty \\ -\frac{t+c}{2t}\log z + \mathcal{O}(1) & z \to 0 \\ \frac{c}{2t}\log(z-a) + \mathcal{O}(1) & z \to a , \end{cases} \tag{74}$$

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with

$$\ell = -\frac{b_-}{t} - \frac{c}{t}\log(b_- - a) + \frac{t+c}{t}\log b_- \in \mathbb{R}\ . \tag{75}$$

There are jumps only along $(0,\infty)$:

$$\phi_+(z) - \phi_-(z) = \begin{cases} \dfrac{t+c}{t}\pi i & z \in (0,a) \\ \pi i & z \in (a, b_-) \\ \pi i & z \in (b_-, \infty)\ . \end{cases} \tag{76}$$

Therefore the real part of $\phi(z)$ is smooth and harmonic on $\mathbb{C} \setminus (\{0\} \cup \{a\})$. The behavior around $z = b_-$:

$$\phi(z) = \begin{cases} -\dfrac{\kappa}{2}(z-b_-)^2(1+\mathcal{O}(z-b_-)) & z \to b_-\ , \mathrm{Re}(z-b_-) > 0 \\ -\pi i - \dfrac{\kappa}{2}(z-b_-)^2(1+\mathcal{O}(z-b_-)) & z \to b_-\ , \mathrm{Re}(z-b_-) < 0 \end{cases} \tag{77}$$

where

$$\kappa := \frac{c}{(b_- - a)^2} - \frac{c+t}{b_-^2} = -y'(b_-) > 0\ . \tag{78}$$

(The function $y(z)$ changes sign from $+$ to $-$ at its zero $z = b_-$.)

Lemma 5.4 *The function*

$$g(z) := \begin{cases} \dfrac{1}{2t}V(z) + \phi(z) + \dfrac{\ell}{2} & z \textit{ inside } \mathcal{B} \\ \dfrac{1}{2t}V(z) - \phi(z) + \dfrac{\ell}{2} & z \textit{ outside } \mathcal{B} \end{cases} \qquad z \in \mathbb{C} \setminus [0,\infty) \tag{79}$$

satisfies the conditions of the g-function along the integration contour $\Gamma = \mathcal{B}$.

Proof. The function $g(z)$ is holomorphic in $\mathbb{C} \setminus \mathcal{L}$ where

$$\mathcal{L} := \mathcal{B} \cup \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3 \tag{80}$$

with $\mathcal{L}_1 = (0,a)$, $\mathcal{L}_2 = (a, b_-)$ and $\mathcal{L}_3 = (b_-, \infty)$. By the choice of ℓ,

$$g(z) = \log z + \mathcal{O}\left(\frac{1}{z}\right) \qquad z \to \infty\ . \tag{81}$$

Combining the singularities of $V(z)$ and $\phi(z)$ the logarithmic singularities are cancelled, and the jumps of $g(z)$ are:

$$g_+(z) - g_-(z) = \begin{cases} 0 & z \in \mathcal{L}_1 \cup \mathcal{L}_2 \\ -2\pi i & z \in \mathcal{L}_3\ . \end{cases} \tag{82}$$

Therefore $e^{ng(z)}$ is holomorphic on $\mathbb{C} \setminus \mathcal{B}$. Obviously,

$$g_+(z) + g_-(z) - \frac{1}{t}V(z) = \ell \qquad z \in \mathcal{B}\ . \tag{83}$$

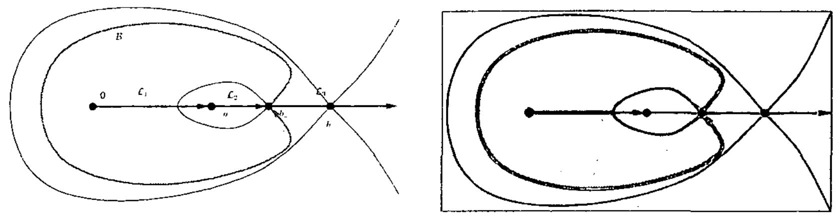

Figure 3: The contour system $\mathcal{L}$ and the sign of $\mathrm{Re}(\phi(z))$

The function

$$h(z) := 2\phi(z) \tag{84}$$

provides the analytic continuation of $g_+(z) - g_-(z)$ in a lens-shaped region around $\mathcal{B} \setminus \{b_-\}$ with

$$\begin{cases} \mathrm{Re}(h(z)) > 0 & \text{inside } \mathcal{B} \\ \mathrm{Re}(h(z)) < 0 & \text{outside } \mathcal{B} \, . \end{cases} \tag{85}$$

Q.E.D.

Note that the simple form of $y(z)$ makes $g(z)$ quite explicit:

$$g(z) = \begin{cases} -\frac{c}{t}\log(z-a) + \frac{t+c}{t}\log z & \text{outside } \mathcal{B} \\ \frac{a}{t}z + \ell & \text{inside } \mathcal{B} \, . \end{cases} \tag{86}$$

Lemma 5.5 *The measure*

$$d\mu(s) := -\frac{1}{2\pi i t} y(s) ds \Big|_{\mathcal{B}} \tag{87}$$

supported on $\mathcal{B}$ is a probability measure. Moreover

$$Re(g(z)) = \int_{\mathcal{B}} \log|z-u| d\mu(u) \, . \tag{88}$$

Proof. By the residue theorem,

$$\int_{\mathcal{B}} d\mu(s) = -\frac{1}{2\pi i t}\int_{\mathcal{B}} y(s)ds = -\frac{1}{t}\left(\operatorname*{res}_{z=0} y(z)dz + \operatorname*{res}_{z=a} y(z)dz\right) = 1 \, . \tag{89}$$

The rest is proven exactly the same way as above.

Q.E.D.

6 Riemann–Hilbert analysis

We have all the ingredients to perform the *nonlinear steepest descent method* of Deift and Zhou [4]. Since the structure of the g-function is very different for the pre-critical and the post-critical values of the parameter t the Riemann-Hilbert analysis is done separately for each case. It is particularly interesting to look at the transition around $t \sim t_c$ where a standard double scaling limit procedure due to Bleher and Its [3] is used.

6.1 Pre-critical case

We describe a sequence of invertible transformations $Y \mapsto U \mapsto T \mapsto S$ to relate the original matrix Y to the solution of a model problem Ψ.

First transformation (undressing). To normalize the Riemann–Hilbert problem at infinity, we consider the *undressed* 2×2 matrix

$$U(z) := \mathrm{e}^{-\frac{1}{2}n\ell\sigma_3} Y(z) \mathrm{e}^{-ng(z)\sigma_3} \mathrm{e}^{\frac{1}{2}n\ell\sigma_3} . \tag{90}$$

This characterized by the Riemann–Hilbert problem

(U.1) $U(z)$ is holomorphic in $\mathbb{C} \setminus \Gamma$ and has continuous boundary values along Γ,

(U.2) $U_+(z) = U_-(z) V_U(z)$, where

$$V_U(z) = \begin{cases} \begin{bmatrix} \mathrm{e}^{-nh(z)} & 1 \\ 0 & \mathrm{e}^{nh(z)} \end{bmatrix} & z \in \mathcal{B} , \\ \\ \begin{bmatrix} 1 & \mathrm{e}^{-2n\phi(z)} \\ 0 & 1 \end{bmatrix} & z \in \Gamma \setminus \mathcal{B} . \end{cases} \tag{91}$$

(U.3) $U(z) = I + \mathcal{O}\left(\frac{1}{z}\right)$ as $z \to \infty$.

Second transformation (lens opening). Let L be a lens-shaped domain in which $h(z)$ is holomorphic and take two contours $\mathcal{S}_+$ and $\mathcal{S}_-$ on the positive and on the negative side of $\mathcal{B}$ respectively, contained entirely in L. Let the domains enclosed by $\mathcal{S}_+$ and $\mathcal{B}$ and by $\mathcal{S}_-$ and $\mathcal{B}$ denoted by Ω_+ and Ω_- respectively. Ω_∞ and Ω_0 are the remaining domains bounded by $\mathcal{B}$ and $\mathcal{R}$ (unbounded) and by $\mathcal{S}_+$ and $\mathcal{R}$ respectively.

Then the second transformation amounts to

$$T(z) := \begin{cases} U(z) \begin{bmatrix} 1 & 0 \\ -e^{-nh(z)} & 1 \end{bmatrix} & z \in \Omega_+ \\ \\ U(z) \begin{bmatrix} 1 & 0 \\ e^{nh(z)} & 1 \end{bmatrix} & z \in \Omega_- \\ \\ U(z) & z \in \Omega_0 \cup \Omega_\infty . \end{cases} \tag{92}$$

The matrix-valued function T is the unique solution of the RH problem:

(T.1) $T(z)$ is holomorphic in $\mathbb{C} \setminus \hat{\Gamma}$ with continuous boundary values on $\mathcal{B} \cup \mathcal{R} \cup \mathcal{S}_+ \cup \mathcal{S}_-$,

(T.2) $T_+(z) = T_-(z) V_T(z)$, where

$$V_T(z) = \begin{cases} \begin{bmatrix} 1 & 0 \\ e^{-nh(z)} & 1 \end{bmatrix} & z \in \mathcal{S}_+ , \\ \\ \begin{bmatrix} 1 & 0 \\ e^{nh(z)} & 1 \end{bmatrix} & z \in \mathcal{S}_- , \\ \\ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & z \in \mathcal{B} , \\ \\ \begin{bmatrix} 1 & \mathrm{e}^{-2n\phi(z)} \\ 0 & 1 \end{bmatrix} & z \in \mathcal{R} . \end{cases} \tag{93}$$

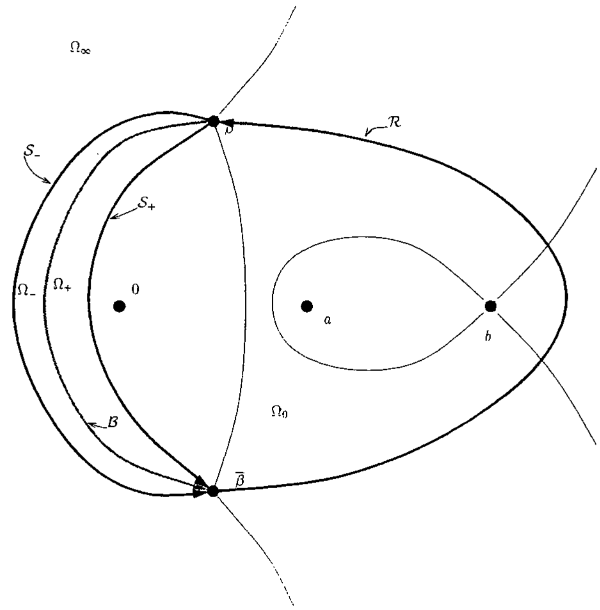

Figure 4: The contours of the Riemann–Hilbert problem for T

(T.3) $T(z) = I + \mathcal{O}\left(\frac{1}{z}\right)$ as $z \to \infty$.

Model problem. Because of the properties

$$\operatorname{Re}(h(z)) > 0 \quad z \in \mathcal{S}_+ , \qquad \operatorname{Re}(h(z)) < 0 \quad z \in \mathcal{S}_- , \qquad \operatorname{Re}(\phi(z)) < 0 \quad z \in \mathbb{R} , \tag{94}$$

the corresponding off-diagonal elements of $V_T(z)$ are exponentially small pointwise (but not uniformly) along the contours $\mathcal{S}_+, \mathcal{S}_-$ and $\mathcal{R}$. By neglecting the jumps that are suppressed as $n \to \infty$, we get the following model problem:

(T.1) $\Psi(z)$ is holomorphic in $\mathbb{C} \setminus \mathcal{B}$,

(T.2)

$$\Psi_+(z) = \Psi_-(z) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad z \in \mathcal{B} \tag{95}$$

(T.3) $\Psi(z) = I + \mathcal{O}\left(\frac{1}{z}\right)$ as $z \to \infty$.

The unique solution is provided by the standard construction [5]

$$\Psi(z) = \begin{bmatrix} \dfrac{B(z) + B(z)^{-1}}{2} & \dfrac{B(z) - B(z)^{-1}}{2i} \\ \dfrac{-B(z) + B(z)^{-1}}{2i} & \dfrac{B(z) + B(z)^{-1}}{2} \end{bmatrix} \tag{96}$$

where

$$B(z) = \left(\frac{z-\overline{\beta}}{z-\beta}\right)^{\frac{1}{4}}, \qquad B(z) \sim 1 + \mathcal{O}\left(\frac{1}{z}\right) \quad z \to \infty . \tag{97}$$

Note that $B(z)$ is bounded and bounded away from zero on every compact subset of $\mathbb{C} \setminus \mathcal{B}$.

Local parametrices. The estimate

$$V_T(z) = \mathcal{O}\left(e^{-cn}\right) \qquad n \to \infty \tag{98}$$

is valid only pointwise and cannot be made uniform in the vicinity of the branch points β and $\overline{\beta}$. The standard method [4, 5] to deal with an end point like β is to try to solve the Riemann-Hilbert problem $T(z)$ *exactly* in small neighborhood of β using a conformal change of variables $\zeta = \zeta(z)$, $\zeta(0) = \beta$. The original set of jumps simplifies in terms of the new coordinate and an exact solution $p(\zeta)$ is found whose asymptotic behavior as $\zeta \to \infty$ is matched by the local asymptotics of $\Psi(z)$ near $z = \beta$. Then the original jumps of $T(z)$ around β can be traded off for the jumps of $P(z) := p(\zeta(z))$.

For $\varepsilon > 0$ consider two disks around the branch points β and $\overline{\beta}$:

$$D_1(\varepsilon) := \{z \in \mathbb{C} : |z-\beta| < \varepsilon\}, \quad D_2(\varepsilon) := \{z \in \mathbb{C} : |z-\overline{\beta}| < \varepsilon\} . \tag{99}$$

In the disks $D_j(\varepsilon)$ we construct *local parametrices*, i.e., 2×2 matrix-valued functions $P_j(z)$ such that

(P.1) $P_j(z)$ is holomorphic in $D_j(\varepsilon) \setminus \mathcal{B} \cup \mathcal{S}_+ \cup \mathcal{S}_- \cup \mathcal{R}$ with continuous boundary values along the contours,

(P.2) $P_{j+}(z) = P_{j-}(z) V_T(z)$ for $z \in \mathcal{B} \cup \mathcal{S}_+ \cup \mathcal{S}_- \cup \mathcal{R}$,

(P.3) as $n \to \infty$,

$$P_j(z) = \left(I + \mathcal{O}\left(\frac{1}{n}\right)\right) \Psi(z) . \tag{100}$$

uniformly on ∂D_j .

We give the construction of the local parametrix $P_1(z)$ only; the construction of $P_2(z)$ follows by symmetry.

By construction, $h(z) = -2\phi(z)$ along $\mathcal{S}_+$ and $h(z) = 2\phi(z)$ along $\mathcal{S}_-$ and

$$\phi(z) \sim \frac{d}{3t}(z-\beta)^{\frac{3}{2}}(1 + \mathcal{O}(z-\beta)) \qquad z \in \mathbb{C} \setminus \mathcal{B}, z \to \beta , \tag{101}$$

where d is given by

$$y(z) \sim d(z-\beta)^{\frac{1}{2}}(1 + \mathcal{O}(z-\beta)) \qquad z \in \mathbb{C} \setminus \mathcal{B}, z \to \beta . \tag{102}$$

So we introduce a new coordinate

$$\zeta := \left(\frac{3}{2} n \phi(z)\right)^{\frac{2}{3}} . \tag{103}$$

This is a holomorphic change of coordinate because

$$\left(\frac{3}{2} n \phi(z)\right)^{\frac{2}{3}} \sim n^{\frac{2}{3}} \left(\frac{d}{2t}\right)^{\frac{2}{3}} (z-\beta) \qquad z \to \beta . \tag{104}$$

We assume that ε is small enough to ensure that ζ is one-to-one in D_j.

In the ζ-plane, we need need to solve the following Riemann-Hilbert problem:

$(p.1)$ $p(\zeta)$ is holomorphic on $\mathbb{C} \setminus \Sigma$,

$(p.2)$ $p_+(z) = p_-(z) v_p(z)$ along Σ, where

$$v_p(z) = \begin{cases} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & \zeta \in \Sigma_1 \\ \\ \begin{bmatrix} 1 & 0 \\ e^{\frac{4}{3}\zeta^{\frac{3}{2}}} & 1 \end{bmatrix} & \zeta \in \Sigma_2 \\ \\ \begin{bmatrix} 1 & e^{-\frac{4}{3}\zeta^{\frac{3}{2}}} \\ 0 & 1 \end{bmatrix} & \zeta \in \Sigma_3 \\ \\ \begin{bmatrix} 1 & 0 \\ e^{\frac{4}{3}\zeta^{\frac{3}{2}}} & 1 \end{bmatrix} & \zeta \in \Sigma_4 \,, \end{cases} \tag{105}$$

$(p.3)$

$$p(\zeta) = \zeta^{-\frac{1}{4}\sigma_3} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \left(I + \mathcal{O}\left(\frac{1}{\zeta}\right)\right) \,, \tag{106}$$

where $\zeta^{\frac{1}{4}}$ has a branch cut along $[0, \infty)$ and the branch is fixed by $\zeta^{\frac{1}{4}}|_{\zeta=-1} = \frac{1+i}{\sqrt{2}}$.

The last asymptotic equation is motivated by the fact that

$$\Psi(z) = E(z)(\zeta(z))^{-\frac{1}{4}\sigma_3} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \qquad \text{near } z = \beta \,, \tag{107}$$

where $E(z)$ is a holomorphic function in $D_1(\varepsilon)$ (assuming that ε is small enough to ensure this). The solution to this problem is standard [5], it is given in the Appendix in terms of the Airy function. Actually by the known asymptotic properties of the Airy function (106) improves to

$$p(\zeta) = \zeta^{-\frac{1}{4}\sigma_3} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \left(I + \mathcal{O}\left(\frac{1}{\zeta^{3/2}}\right)\right) \,, \tag{108}$$

Therefore the local parametrix

$$P_1(z) := E(z) p(\zeta(z)) \tag{109}$$

satisfies the requirements.

The final transformation (error matrix). Consider now

$$S(z) = \begin{cases} T(z)\Psi(z)^{-1} & z \in \mathbb{C} \setminus (\mathcal{B} \cup \mathcal{S}_+ \cup \mathcal{S}_- \cup \mathcal{R} \cup D_1(\varepsilon) \cup D_2(\varepsilon)) \\ \\ T(z)P_j(z)^{-1} & z \in D_j(\varepsilon) \setminus (\mathcal{B} \cup \mathcal{S}_+ \cup \mathcal{S}_- \cup \mathcal{R}) \qquad (j = 1, 2) \,. \end{cases} \tag{110}$$

Because of the matching jumps of T and P_j inside the disk $D_j(\varepsilon)$, $S(z)$ is holomorphic in $D_j(\varepsilon)$. Let Σ_S denote the contours along which S has non-identical jumps. So $S(z)$ solves the Riemann-Hilbert problem

(S.1) $S(z)$ is holomorphic in $\mathbb{C} \setminus \Sigma_S$, (S.2) $S_+(z) = S_-(z)V_S(z)$, where

$$V_S(z) = \begin{cases} P_j(z)\Psi^{-1}(z) & z \in \partial D_j(\varepsilon) \ , \\ \Psi(z)V_T(z)\Psi^{-1}(z) & z \in \Sigma_S \setminus (D_1(\varepsilon) \cup D_2(\varepsilon)) \ . \end{cases} \tag{111}$$

(S.3) $S(z) = I + \mathcal{O}\left(\frac{1}{z}\right)$ as $z \to \infty$.

The jump matrix $V_S(z) \sim I$ as $n \to \infty$ for every $z \in \Sigma_S$. We have that

$$V_S(z) = \begin{cases} I + \mathcal{O}\left(\frac{1}{n}\right) & z \in \partial D_j(\varepsilon) \ , \\ I + \mathcal{O}\left(e^{-cn}\right) & z \in \Sigma_S \setminus (D_1(\varepsilon) \cup D_2(\varepsilon)) \text{ uniformly for some } c > 0 \end{cases} \tag{112}$$

as $n \to \infty$. Since $S(z)$ is a *small-norm Riemann-Hilbert problem*, $V_S(z)$ is close to the identity in both the L^1 and the L^∞-norms, we have that [5]

$$S(z) = I + \mathcal{O}\left(\frac{1}{n}\right) \qquad n \to \infty \tag{113}$$

uniformly for every compact subset of $\mathbb{C} \setminus \Sigma_S$.

The conclusion of the asymptotic analysis. The consecutive transformations $Y \mapsto U \mapsto T \mapsto S$ are all invertible in the domains in question which allows to get from $\Psi(z)$ to $Y(z)$ and therefore to extract asymptotic information on the polynomial $P_n(z)$ as $n \to \infty$. This provides strong asymptotics uniform on every compact subset on $\mathbb{C}$; the particular form of the asymptotic result depends on the region in question.

- $z \in \Omega_\infty \cup \Omega_0 \setminus (D_1(\varepsilon) \cup D_2(\varepsilon))$: here $T(z) = U(z)$ and therefore

$$\begin{aligned} P_n(z) &= Y_{11}(z) \\ &= U_{11}(z)e^{ng(z)} \\ &= T_{11}(z)e^{ng(z)} \\ &= \left(S_{11}(z)\Psi_{11}(z) + S_{12}(z)\Psi_{21}(z)\right) e^{ng(z)} \\ &= \left(\frac{B(z) + B(z)^{-1}}{2} + \mathcal{O}\left(\frac{1}{n}\right)\right) e^{ng(z)} \end{aligned} \tag{114}$$

  uniformly on compact subsets of $\Omega_\infty \cup \Omega_0 \setminus (D_1(\varepsilon) \cup D_2(\varepsilon))$. By deforming the contour $\mathcal{R}$ we obtain the same asymptotic behavior for the points on $\mathcal{R}$. Since $\varepsilon > 0$ can be arbitrarily small, the same asymptotic result holds for compact subsets of int $\left(\overline{\Omega_\infty \cup \Omega_0}\right)$. Also, by deforming the contours $\mathcal{S}_+$ and $\mathcal{S}_-$, the validity of this asymptotic formula is extendible to the whole cut plane $\mathbb{C} \setminus \mathcal{B}$.

- $z \in \Omega_\pm \setminus (D_1(\varepsilon) \cup D_2(\varepsilon))$: here $U(z) = T(z) \begin{bmatrix} 1 & 0 \\ \pm e^{\mp nh} & 1 \end{bmatrix}$ and therefore

$$\begin{aligned} P_n(z) &= Y_{11}(z) \\ &= U_{11}(z) e^{ng(z)} \\ &= (T_{11}(z) \pm e^{\mp nh(z)} T_{12}(z)) e^{ng(z)} \\ &= \left(\Psi_{11}(z) \pm e^{\mp nh(z)} \Psi_{12}(z) + \mathcal{O}\left(\frac{1}{n}\right)\right) e^{ng(z)} \\ &= \left(\frac{B(z) + B(z)^{-1}}{2} \pm e^{\mp nh(z)} \frac{B(z) - B(z)^{-1}}{2i} + \mathcal{O}\left(\frac{1}{n}\right)\right) e^{ng(z)} \end{aligned} \tag{115}$$

uniformly on compact subsets of $\Omega_\pm \setminus (D_1(\varepsilon) \cup D_2(\varepsilon))$.

- $z \in \mathcal{B} \setminus \{\beta, \overline{\beta}\}$:

$$\begin{aligned} P_{n,N}(z) &= (Y_+)_{11}(z) \\ &= (U_+)_{11}(z) e^{ng_+(z)} \\ &= ((T_+)_{11}(z) + e^{-nh(z)} (T_+)_{12}(z)) e^{ng_+(z)} \\ &= (T_+)_{11}(z) e^{ng_+(z)} + (T_+)_{12}(z) e^{ng_-(z)} \\ &= \left[\left((\Psi_+)_{11}(z) + \mathcal{O}\left(\frac{1}{n}\right)\right) e^{-n\phi_+(z)} + \left((\Psi_+)_{12}(z) + \mathcal{O}\left(\frac{1}{n}\right)\right) e^{-n\phi_-(z)}\right] e^{\frac{1}{2t} V(z) + \frac{\ell}{2}} . \end{aligned} \tag{116}$$

With the notation

$$\varphi(z) := \pi \int_\beta^z d\mu(s) \quad z \in \mathcal{B} , \tag{117}$$

Since

$$\phi_\pm(z) = \pm \frac{1}{2t} \int_\beta^z y_+(s) ds = \pm i\varphi(z) , \tag{118}$$

and we have

$$P_{n,N}(z) = \frac{1}{2}\left((B(z) + B(z)^{-1}) e^{-in\varphi(z)} - i(B(z) - B(z)^{-1}) e^{in\varphi(z)} + \mathcal{O}\left(\frac{1}{n}\right)\right) e^{\frac{1}{2t} V(z) + \frac{\ell}{2}} . \tag{119}$$

Since $B(z) + B(z)^{-1}$ has no zeroes in $\mathbb{C} \setminus \mathcal{B}$, by Hurwitz theorem we get the following:

Corollary 6.1 *The zeroes of $P_{n,N}(z)$ may accumulate only on $\mathcal{B}$ as $n, N \to \infty, n/N \to t$, $t < t_c$.*

This is enough to complete the proof of our first main theorem:
Proof of Theorem 2.1. The asymptotic formulae gives that

$$\lim_{n\to\infty} \left[\frac{1}{n} \log |P_n(z)| - \mathrm{Re}(g(z))\right] = 0 \qquad z \in \mathbb{C} \setminus \mathcal{B} . \tag{120}$$

on $\mathbb{C} \setminus \mathcal{B}$. Since

$$\mathrm{Re}(g(z)) = \int_{\mathcal{B}} \log|z - u| d\mu(u) , \tag{121}$$

this means that

$$\int_{\mathcal{B}} \log|z-u|d\nu_{n,N}(u) \to \int_{\mathcal{B}} \log|z-u|d\mu(u) \qquad n \to \infty \tag{122}$$

pointwise in $\mathbb{C} \setminus \mathcal{B}$. Since $\mathcal{B}$ is a simple arc, this is enough to conclude that $\nu_{n,N}$ converges to μ in the weak-star sense [8].

Q.E.D.

6.2 Post-critical case

We follow essentially the same steps as in the pre-critical case; the notations $g(z), y(z), h(z)$ and ℓ refer to the construction relevant in the post-critical case.
First transformation (undressing). Let

$$U(z) := \mathrm{e}^{-\frac{1}{2}n\ell\sigma_3} Y(z) \mathrm{e}^{-ng(z)\sigma_3} \mathrm{e}^{\frac{1}{2}n\ell\sigma_3} . \tag{123}$$

This satisfies the RH problem

(U.1) $U(z)$ is holomorphic in $\mathbb{C} \setminus \mathcal{B}$.

(U.2) $U_+(z) = U_-(z)V_U(z)$, where

$$V_U(z) = \begin{bmatrix} \mathrm{e}^{-nh(z)} & 1 \\ 0 & \mathrm{e}^{nh(z)} \end{bmatrix} \qquad z \in \mathcal{B} . \tag{124}$$

(U.3) $U(z) = I + \mathcal{O}\left(\frac{1}{z}\right)$ as $z \to \infty$.

Second transformation (lens opening).
Let S_+ be a closed contour inside $\mathcal{B}$ such that $h(z)$ is defined and positive there (it has to pass through the point $z = b_-$ because it is a saddle point of $\mathrm{Re}(h(z))$ at the zero level). We may choose $\mathcal{S}_-$ to be the critical trajectory emanating from $z = b_+$ surrounding $\mathcal{B}$. Let Ω_+ and Ω_- be the domains enclosed by $\mathcal{S}_+$ and $\mathcal{B}$ and by $\mathcal{S}_-$ and $\mathcal{B}$ respectively. Ω_0 and Ω_∞ denotes the component $\mathbb{C}_\infty \setminus (\Omega_+ \cup \Omega_- \cup \mathcal{B} \cup \mathcal{S}_1 \cup \mathcal{S}_2)$ that contains 0 and ∞ respectively.

$$T(z) := \begin{cases} U(z) \begin{bmatrix} 1 & 0 \\ -e^{-nh(z)} & 1 \end{bmatrix} & z \in \Omega_+ \\ U(z) \begin{bmatrix} 1 & 0 \\ e^{nh(z)} & 1 \end{bmatrix} & z \in \Omega_- \\ U(z) & z \in \Omega_0 \cup \Omega_\infty . \end{cases} \tag{125}$$

Therefore T is the unique solution of the RH problem:

(T.1) $T(z)$ is holomorphic in $\mathbb{C} \setminus (\mathcal{B} \cup \mathcal{S}_1 \cup \mathcal{S}_2)$,

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(T.2) $T_+(z) = T_-(z)V_T(z)$, where

$$V_T(z) = \begin{cases} \begin{bmatrix} 1 & 0 \\ e^{-nh(z)} & 1 \end{bmatrix} & z \in \mathcal{S}_+ \ , \\ \begin{bmatrix} 1 & 0 \\ e^{nh(z)} & 1 \end{bmatrix} & z \in \mathcal{S}_- \ , \\ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & z \in \mathcal{B} \ . \end{cases} \tag{126}$$

(T.3) $T(z) = I + \mathcal{O}\left(\frac{1}{z}\right)$ as $z \to \infty$.

Model problem. The asymptotic behavior of the entries of $V_T(z)$ suggests that we have to look at the model problem

(Ψ.1) $\Psi(z)$ is holomorphic in $\mathbb{C} \setminus \mathcal{B}$,

(Ψ.2) $\Psi(z)$ has continuous boundary values *everywhere* on $\mathcal{B}$,

$$\Psi_+(z) = \Psi_-(z) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad z \in \mathcal{B} \tag{127}$$

(Ψ.3) $\Psi(z) = I + \mathcal{O}\left(\frac{1}{z}\right)$ as $z \to \infty$.

The solution is simple:

$$\Psi(z) = \begin{cases} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & z \in \mathcal{B}_+ \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & z \in \mathcal{B}_- \ . \end{cases} \tag{128}$$

Local parametrices. Now $z - b_-$ is the only point where the jump matrix $V_T(z)$ is not close to the identity in the L^∞-sense. We build a suitable parametrix that solves the Riemann–Hilbert problem explicitly and it is uniformly close to Ψ in a circle around b_-. For a fixed $\varepsilon > 0$ consider the disk

$$D(\varepsilon) := \{z \in \mathbb{C} : \ |z - b_-| < \varepsilon\} \ , \tag{129}$$

where we assume that ε is small enough that a and b_+ are not in D. We seek a function $P(z)$ that satisfies

(P.1) $P(z)$ is holomorphic in $D \setminus (\mathcal{B} \cup \mathcal{S}_1 \cup \mathcal{S}_2)$,

(P.2) $P_+(z) = P_-(z)V_T(z)$

(P.3) as $n \to \infty$,

$$P(z)\Psi^{-1}(z) \sim I \qquad n \to \infty \tag{130}$$

on the circle $\partial D(\varepsilon)$.

Introduce the coordinate

$$\zeta^2 := \begin{cases} -4n\phi(z) & \mathrm{Re}(z) > 0 \\ -4n(\phi(z) + \pi i) & \mathrm{Re}(z) > 0 \end{cases} \tag{131}$$

in the vicinity of $z = b_-$. This change of variables is holomorphic in a neighborhood of b and univalent for $\varepsilon > 0$ small enough since

$$\zeta \sim \sqrt{2n\kappa}(z - b_-)(1 + \mathcal{O}(z - b_-)) \qquad z \to b_- \ . \tag{132}$$

In the ζ-plane, we look for the solution of the RH-problem

$(p.1)$ $p(\zeta)$ is holomorphic in $\mathbb{C} \setminus \Gamma$,

$(p.2)$ $p_+(\zeta) = p_-(\zeta)v_p(\zeta)$ on $\Gamma \setminus \{0\}$, where

$$v_p(z) = \begin{cases} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & \zeta \in \Sigma_1 \cup \Sigma_4 \\ \\ \begin{bmatrix} 1 & 0 \\ e^{\zeta^2/2} & 1 \end{bmatrix} & \zeta \in \Sigma_2 \cup \Sigma_3 \ , \end{cases} \tag{133}$$

$(p.3)$ as $\zeta \to \infty$,

$$p(\zeta) = \left(I + \mathcal{O}\left(\frac{1}{\zeta}\right)\right)\psi(\zeta) \ , \tag{134}$$

where $\psi(\zeta)$ is defined by the local behavior of the model solution $\Psi(z)$:

$$\psi(\zeta) := \begin{cases} I & \zeta \in \Delta_1 \\ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & \zeta \in \Delta_2 \cup \Delta_2 \cup \Delta_3 \ . \end{cases} \tag{135}$$

This is solved explicitly in the Appendix; the local parametrix $P(z)$ in terms of z is now simply

$$P(z) = p(\zeta(z)) \qquad z \in D(\varepsilon) \ . \tag{136}$$

If we proceed in the same way as above, the error matrix

$$S(z) = \begin{cases} T(z)\Psi(z)^{-1} & z \in \mathbb{C} \setminus (\mathcal{B} \cup \mathcal{S}_1 \cup \mathcal{S}_2 \cup D(\varepsilon)) \\ \\ T(z)P(z)^{-1} & z \in D(\varepsilon) \setminus (\mathcal{B} \cup \mathcal{S}_1 \cup \mathcal{S}_2) \ . \end{cases} \tag{137}$$

satisfies a small-norm Riemann-Hilbert problem and it can be proven that

$$S(z) = I + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \qquad n \to \infty \tag{138}$$

But the fact that $\Psi_{11}(z) = 0$ for $z \in \Omega_0$ results in the asymptotics

$$P_n(z) = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) e^{ng(z)} \tag{139}$$

which is not strong enough to exclude the presence of zeroes there asymptotically as $n \to \infty$. This means that we have to obtain more terms in the asymptotic expansion of $P_n(z)$; one possible way to do this is to construct a modified model problem $\hat{\Psi}(z)$ and an improved local parametrix $\hat{P}(z)$.

Modified model problem and improved local parametrix. Following [2], we seek $\hat{\Psi}(z)$ and $\hat{P}(z)$ such that

($\hat{P}$.1) $\hat{P}(z)$ is holomorphic in $D(\varepsilon) \setminus (\mathcal{B} \cup \mathcal{S}_1 \cup \mathcal{S}_2)$,

($\hat{P}$.2) $\hat{P}_+(z) = \hat{P}_-(z) V_T(z)$ on $\mathcal{B} \cup \mathcal{S}_1 \cup \mathcal{S}_2 \setminus \{b_-\}$

($\hat{P}$.3) as $n \to \infty$,

$$\hat{P}(z) = \left(I + \mathcal{O}\left(\frac{1}{n}\right)\right) \hat{\Psi}(z) \tag{140}$$

uniformly on the circumference of $D(\varepsilon)$.

The function $\hat{\Psi}(z)$ is a (possibly n-dependent) modification of $\Psi(z)$ such that $\hat{P}(z)\hat{\Psi}(z)^{-1}$ is bounded as $z \to b_-$.

Consider therefore the following problem that allows a singularity at $\zeta = 0$ in order to improve the asymptotic behavior as $\zeta \to \infty$:

($\hat{p}$.1) $\hat{p}(\zeta)$ is holomorphic in $\mathbb{C} \setminus \Gamma$,

($\hat{p}$.2) $\hat{p}_+(\zeta) = \hat{p}_-(\zeta) v_p(\zeta)$ on $\Gamma \setminus \{0\}$

($\hat{p}$.3)

$$\hat{p}(\zeta) = \left(I + \mathcal{O}\left(\frac{1}{\zeta^2}\right)\right) \psi(\zeta) \qquad \zeta \to \infty \,. \tag{141}$$

Note that the matrix function

$$m(\zeta) := \hat{p}(\zeta) p^{-1}(\zeta) \tag{142}$$

that has the same jumps and therefore holomorphic in the punctured ζ-plane $\mathbb{C} \setminus \{0\}$ with asymptotic behavior $m(\zeta) = I + \mathcal{O}\left(\frac{1}{\zeta}\right)$ as $\zeta \to \infty$. Therefore

$$m(\zeta) = I + \sum_{k=1}^{\infty} \frac{m_k}{\zeta^k} \,. \tag{143}$$

for some 2×2 matrix-valued coefficents m_k. The asymptotic condition on $\hat{p}$

$$\left(I + \sum_{k=1}^{\infty} \frac{m_k}{\zeta^k}\right)\left(I + \begin{bmatrix} 0 & \frac{1}{\sqrt{2\pi}} \\ 0 & 0 \end{bmatrix} \frac{1}{\zeta} + \mathcal{O}\left(\frac{1}{\zeta^2}\right)\right) = I + \mathcal{O}\left(\frac{1}{\zeta^2}\right) \tag{144}$$

gives that the Schlesinger transformation

$$m(\zeta) = \begin{bmatrix} 1 & -\frac{1}{\sqrt{2\pi}} \frac{1}{\zeta} \\ 0 & 1 \end{bmatrix} \tag{145}$$

is suitable for this purpose. Since

$$\frac{1}{\zeta(z)} = \frac{1}{\sqrt{2n\kappa}} \frac{1}{z - b_-} (1 + \mathcal{O}(z - b_-)) \qquad z \to b_- \,, \tag{146}$$

the matrix

$$M(z) := \begin{bmatrix} 1 & \frac{1}{\sqrt{4\pi n\kappa}}\frac{1}{z-b_-} \\ 0 & 1 \end{bmatrix} \tag{147}$$

satisfies

$$M(z)m(\zeta(z)) = \mathcal{O}(1) \qquad z \to b_- \ . \tag{148}$$

The improved error matrix. With the definitions

$$\hat{P}(z) := M(z)\hat{p}(\zeta(z)) \quad z \in D(\varepsilon)\ , \qquad \hat{\Psi}(z) := M(z)\Psi(z)\ , \tag{149}$$

we define the following modified error matrix:

$$\hat{S}(z) = \begin{cases} T(z)\hat{\Psi}(z)^{-1} & z \in \mathbb{C}\setminus(\mathcal{B}\cup\mathcal{S}_1\cup\mathcal{S}_2\cup D(\varepsilon)) \\ T(z)\hat{P}(z)^{-1} & z \in D(\varepsilon)\setminus(\mathcal{B}\cup\mathcal{S}_1\cup\mathcal{S}_2)\ . \end{cases} \tag{150}$$

Since

$$\hat{P}(z)\hat{\Psi}(z)^{-1} = M(z)\hat{p}(\zeta(z))\psi(\zeta(z))\Psi(z)^{-1}M(z)^{-1} = M(z)\left(I+\mathcal{O}\left(\frac{1}{n^{\frac{3}{2}}}\right)\right)M(z)^{-1} = I+\mathcal{O}\left(\frac{1}{n^{\frac{3}{2}}}\right) \tag{151}$$

(note that

$$\hat{p}(\zeta) = \left(I+\mathcal{O}\left(\frac{1}{\zeta^3}\right)\right)\psi(\zeta) \qquad \zeta\to\infty \tag{152}$$

is valid) uniformly on $\partial D(\varepsilon)$, the function $\hat{S}(z)$ satisfies a small-norm Riemann-Hilbert problem and it can be proven that

$$\hat{S}(z) = I+\mathcal{O}\left(\frac{1}{n^{\frac{3}{2}}}\right) \qquad n\to\infty\ . \tag{153}$$

The conclusion of the asymptotic analysis.

- For $z \in \Omega_\infty$:

$$\begin{aligned} P_n(z) &= Y_{11}(z) \\ &= U_{11}(z)e^{ng(z)} \\ &= T_{11}(z)e^{ng(z)} \\ &= \left(\hat{S}_{11}(z)\hat{\Psi}_{11}(z)+\hat{S}_{12}(z)\hat{\Psi}_{21}(z)\right)e^{ng(z)} \\ &= \left(1+\mathcal{O}\left(\frac{1}{n^{\frac{3}{2}}}\right)\right)e^{ng(z)} \end{aligned} \tag{154}$$

  uniformly on compact subsets of Ω_∞. By deforming the contour $\mathcal{S}_+$ we obtain the same asymptotic behavior for compact subsets outside $\mathcal{B}$ as before.

- For $z \in \Omega_0 \setminus D(\varepsilon)$:

$$\begin{aligned} P_n(z) &= Y_{11}(z) \\ &= U_{11}(z) e^{ng(z)} \\ &= T_{11}(z) e^{ng(z)} \\ &= \left(\hat{S}_{11}(z)\hat{\Psi}_{11}(z) + \hat{S}_{12}(z)\hat{\Psi}_{21}(z)\right) e^{ng(z)} \\ &= \left(-\frac{1}{\sqrt{4\pi n\kappa}}\frac{1}{z-b_-} + \mathcal{O}\left(\frac{1}{n^{\frac{3}{2}}}\right)\right) e^{ng(z)} \end{aligned} \tag{155}$$

Again, this result is valid also uniformly in every compact subset inside $\mathcal{B}$.

- For $z \in \Omega_\pm \setminus D(\varepsilon)$:

$$\begin{aligned} P_n(z) &= Y_{11}(z) \\ &= U_{11}(z) e^{ng(z)} \\ &= (T_{11}(z) \pm e^{\mp nh(z)} T_{12}(z)) e^{ng(z)} \\ &= (T_{11}(z) \pm e^{\mp nh(z)} T_{12}(z)) e^{ng(z)} \\ &= \left((\hat{S}_{11}(z)\hat{\Psi}_{11}(z) + \hat{S}_{12}(z)\hat{\Psi}_{21}(z)) \pm e^{\mp nh(z)}(\hat{S}_{11}(z)\hat{\Psi}_{12}(z) + \hat{S}_{12}(z)\hat{\Psi}_{22}(z))\right) e^{ng(z)} \\ &= \begin{cases} \left(1 + e^{-nh(z)} + \mathcal{O}\left(\frac{1}{n}\right)\right) e^{ng(z)} & z \in \Omega_+ \\ \left(-\frac{\sqrt{4\pi n\kappa}}{z-b_-} - e^{nh(z)} + \mathcal{O}\left(\frac{1}{n}\right)\right) e^{ng(z)} & z \in \Omega_- \end{cases} \end{aligned} \tag{156}$$

uniformly on compact subsets of $\Omega_\pm$. The last term indicates that there are no zeroes asymptotically in Ω_+: $\mathrm{Re}(h(z)) > 0$ in Ω_+ and therefore $1 + e^{-nh(z)}$ may have no zero in Ω_+. In Ω_- the zeroes appear along the curve where the magnitude of two terms balances, i.e., where

$$\mathrm{Re}(h(z)) = \frac{1}{n}\log\left|\frac{\sqrt{4\pi n\kappa}}{z-b_-}\right| . \tag{157}$$

This implies that the zeroes lie in the region where $\mathrm{Re}(h) = \mathcal{O}\left(\frac{\log n}{n}\right)$ as $n \to \infty$.

- For $z \in \mathcal{B} \setminus \{b_-\}$:

$$\begin{aligned} P_{n,N}(z) &= (Y_+)_{11}(z) \\ &= (U_+)_{11}(z) e^{ng_+(z)} \\ &= ((T_+)_{11}(z) + e^{-nh(z)}(T_+)_{12}(z)) e^{ng_+(z)} \\ &= \left(-\frac{\sqrt{4\pi n\kappa}}{z-b_-} e^{in\varphi(z)} + e^{-in\varphi(z)} + \mathcal{O}\left(\frac{1}{n}\right)\right) e^{\frac{n}{2t}(V(z)+t\ell)} . \end{aligned} \tag{158}$$

with the notation

$$\varphi(z) := \pi \int_\beta^z d\mu(s) \quad z \in \mathcal{B} . \tag{159}$$

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The Hurwitz theorem applied to the asymptotic formulae obtained above implies the following

Corollary 6.2 *The zeroes of $P_{n,N}(z)$ may accumulate only on $\mathcal{B}$ as $n, N \to \infty, n/N \to t,\ t > t_c$.*

Proof of Theorem 2.1. The asymptotic formulae gives that

$$\lim_{n\to\infty}\left[\frac{1}{n}\log|P_n(z)| - \mathrm{Re}(g(z))\right] = 0 \qquad z \in \mathbb{C}\setminus\mathcal{B}\,. \tag{160}$$

on $\mathbb{C}\setminus\mathcal{B}$. Since

$$\mathrm{Re}(g(z)) = \int_{\mathcal{B}} \log|z-u|d\mu(u)\ , \tag{161}$$

this means that

$$\int_{\mathcal{B}} \log|z-u|d\nu_{n,N}(u) \to \int_{\mathcal{B}} \log|z-u|d\mu(u) \qquad n\to\infty \tag{162}$$

pointwise in $\mathbb{C}\setminus\mathcal{B}$. Since $\mathcal{B}$ is a simple arc, this is enough to conclude that $\nu_{n,N}$ converges to μ in the weak-star sense [8].

Q.E.D.

A Proofs

A.1 Proof of Lemma 3.1.

For a given polynomial $p(z)$ and fixed $N, c > 0$ we seek a one-form ω_k on a possibly infinite sheeted covering of the complex plane such that

$$d\omega_k = \frac{1}{2i}p(z)\bar{z}^k|z-a|^{2Nc}\mathrm{e}^{-Nz\bar{z}}d\bar{z}\wedge dz\ . \tag{A.1}$$

This is equivalent to solving the d-bar problem

$$\partial_{\bar{z}}f_k(z,\bar{z}) = -\bar{z}^k|z-a|^{2Nc}\mathrm{e}^{-Nz\bar{z}}\ , \tag{A.2}$$

since the one-form

$$\omega_k := \frac{i}{2}p(z)f_k(z,\bar{z})dz \tag{A.3}$$

satisfies

$$d\omega_k = \partial_{\bar{z}}\left(\frac{i}{2}p(z)f_k(z,\bar{z})\right)d\bar{z}\wedge dz = \frac{1}{2i}p(z)\bar{z}^k|z-a|^{2Nc}\mathrm{e}^{-Nz\bar{z}}d\bar{z}\wedge dz\ . \tag{A.4}$$

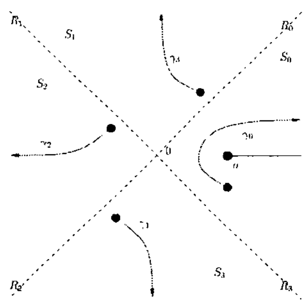

Figure 5: The sectors S_i and the integration contours γ_i, $i = 0, 1, 2, 3$

A particular piecewise solution to the d-bar problem (A.2) may be constructed using contour integration in the following way: the complex plane is divided into four sectors

$$S_j := \left\{z\in\mathbb{C}\ :\ \frac{2j-1}{4}\pi < \arg(z) < \frac{2j+1}{4}\pi\right\} \tag{A.5}$$

where $j = 0, 1, 2, 3$. The boundary ray R_j separates the sectors S_j and S_{j+1} respectively (the sectorial indices are always understood mod 4).

The sectorial solutions are defined by the formulae

$$f_k^{(j)}(z,\bar{z}) := \mathrm{e}^{-2\pi iNc}(z-a)^{Nc}\int_{\gamma_j(\bar{z})} s^k(s-a)^{Nc}\mathrm{e}^{-Nzs}ds \tag{A.6}$$

where $\gamma_j(\bar{z})$ is a contour from $\bar{z}$ to ∞ in the sector $S_{(-1)^j j}$ (to ensure the convergence of the integral, $\mathrm{Re}(zs) > 0$ is needed for a fixed value of z). The branch of algebraic factor $(z-a)^c$ is fixed by placing a cut along the half line

$$C_a := [a, \infty) \tag{A.7}$$

and taking $(z-a)^{Nc} := |z-a|^{Nc}e^{iNc\phi}$ where $0 \le \phi < 2\pi$. This explains the presence of the $\mathrm{e}^{-2\pi iNc}$ factor in the definition of $f_k(z,\bar{z})$:

$$(z-a)^{Nc}(\bar{z}-a)^{Nc} = \mathrm{e}^{2\pi iNc}|z-a|^{2Nc}\ . \tag{A.8}$$

The integration contour $\gamma_0(\bar{z})$ should not cross the cut C_a in the s-plane; by convention, $\gamma_0(\bar{z})$ goes to infinity above C_a (in the halfplane $\mathrm{Im}(z) > 0$).

The function $f_k^{(0)}(z,\overline{z})$ has a jump along C_a:

$$\begin{cases} f_k^{(j)}(z,\overline{z})_+ = \mathrm{e}^{-2\pi i Nc}(z-a)_+^{Nc}\left[\int_a^\infty s^k(s-a)_+^{Nc}\mathrm{e}^{-Nzs}ds \right. \\ \qquad\qquad\qquad\qquad\qquad\qquad \left. -\int_a^z s^k(s-a)_-^{Nc}\mathrm{e}^{-Nzs}ds\right] \\ f_k^{(j)}(z,\overline{z})_- = \mathrm{e}^{-2\pi i Nc}(z-a)_-^{Nc}\int_z^\infty s^k(s-a)_+^{Nc}\mathrm{e}^{-Nzs}ds\ . \end{cases} \tag{A.9}$$

Since $(z-a)_-^{Nc} = (z-a)_+^{Nc}\mathrm{e}^{2\pi i Nc}$, we find that the jump itself is independent of $\overline{z}$:

$$f_k^{(0)}(z,\overline{z})_+ - f_k^{(0)}(z,\overline{z})_- = (\mathrm{e}^{-2\pi i Nc}-1)(z-a)_+^{Nc}\int_a^\infty s^k(s-a)_+^{Nc}\mathrm{e}^{-Nzs}ds\ . \tag{A.10}$$

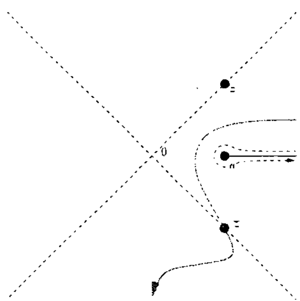

Figure 6: The jump on R_0

It is easy to see that

$$f^{(j+1)}(z,\overline{z}) - f^{(j)}(z,\overline{z}) = 0 \qquad z\in R_j \tag{A.11}$$

for $j=1,2,3$ (all these contour integrals vanish). For $j=0$, however, we have to take the into account the jump of the integrand that is given by a holomorphic contour integral that can be deformed to the cut C_a (see Figure A.1):

$$\begin{aligned} &f^{(1)}(z,\overline{z}) - f^{(0)}(z,\overline{z}) = \\ &\quad -(\mathrm{e}^{-2\pi i Nc}-1)(z-a)^{Nc}\int_a^\infty s^k(s-a)_+^{Nc}\mathrm{e}^{-Nzs}ds\ . \end{aligned} \tag{A.12}$$

As $z\to\infty$, the functions $f_k^{(j)}(z,\overline{z})$ decay exponentially; we prove that for $f_k^{(2)}(z,\overline{z})$; the proof for other three is essentially the same. To this end, let $z=-x_0+iy_0$; the integration contour is chosen to be the half line $-(x_0+t)+iy_0$ with $t\geq 0$:

$$\begin{aligned} |f_k^{(2)}(z,\overline{z})| &= \left|\mathrm{e}^{-2\pi i Nc}(z-a)^{Nc}\int_{\gamma_j(\overline{z})} s^k(s-a)^{Nc}\mathrm{e}^{-Nzs}ds\right| \\ &\leq \left((x_0+a)^2+y_0^2\right)^{Nc/2}\times \\ &\quad\times\left|\int_0^\infty(-x_0-t-iy_0)^k(x_0-t-iy_0-a)^{Nc}\mathrm{e}^{N(-x_0+iy_0)(x_0+t+iy_0)}dt\right| \\ &\leq \left((x_0+a)^2+y_0^2\right)^{Nc/2}\mathrm{e}^{-N(x_0^2+y_0^2)}\times \\ &\quad\times\int_0^\infty((x_0+t)^2+y_0^2)^{k/2}((x_0+t+a)^2+y_0^2)^{Nc/2}\mathrm{e}^{-Nx_0t}dt \\ &\leq \left((x_0+a)^2+y_0^2\right)^{Nc/2}\mathrm{e}^{-N(x_0^2+y_0^2)}\int_0^\infty C\mathrm{e}^{-Nx_0t/2}dt \\ &\leq C\frac{\mathrm{e}^{-N(1-\varepsilon)(x_0^2+y_0^2)}}{x_0}\ . \end{aligned} \tag{A.13}$$

As $z\to 0$, $|f_k^{(j)}(z,\overline{z})|\to\infty$. To overcome this difficulty, we introduce a local solution to the d-bar problem (A.2) in the vicinity of the origin:

$$\tilde{f}_k(z,\overline{z}) := -\mathrm{e}^{-2\pi i Nc}(z-a)^{Nc}\int_a^{\overline{z}} s^k(s-a)^{Nc}\mathrm{e}^{-Nzs}ds \qquad |z|=\delta<\frac{a}{2}\ . \tag{A.14}$$

(Of course, the contour of integration must not cross the branch cut C_a.) We define the functions

$$F_k^{(j)}(z) := \mathrm{e}^{-2\pi i Nc}(z-a)^{Nc}\int_a^\infty s^k(s-a)^{Nc}\mathrm{e}^{-Nzs}ds \qquad z\in S_j\ , \tag{A.15}$$

where the homotopy class for each sector is chosen as above. The function $F_k^{(0)}(z)$ is holomorphic in $S_0\setminus C_a$ and the other $F_k^{(j)}(z)$ functions provide an analytic continuation to the cut plane $\mathbb{C}\setminus(C_a\cup R_0)$ (if we compare the boundary values on the other R_j's we find that there is no jump on either of them and therefore, by Morera's theorem, they are analytic continuations of each other). We may suppress the upper index and use $F_k(z)$ to denote this analytic continuation. The jump of $F_k(z)$ on C_a is

$$\begin{aligned} F_k(z)_+ - F_k(z)_- &= (1-\mathrm{e}^{2\pi i Nc})\mathrm{e}^{-2\pi i Nc}(z-a)_+^{Nc}\int_a^\infty s^k(s-a)_+^{Nc}\mathrm{e}^{-Nzs}ds \\ &= (1-\mathrm{e}^{2\pi i Nc})F_k(z)_+ \end{aligned} \tag{A.16}$$

and therefore

$$F_k(z)_- = \mathrm{e}^{2\pi i Nc}F_k(z)_+\ . \tag{A.17}$$

The jump on R_0 is

$$\begin{aligned} F_k(z)_+ - F_k(z)_- &= -(1-\mathrm{e}^{2\pi i Nc})\mathrm{e}^{-2\pi i Nc}(z-a)^{Nc}\int_a^\infty s^k(s-a)_+^{Nc}\mathrm{e}^{-Nzs}ds \\ &= -(1-\mathrm{e}^{2\pi i Nc})F_k(z)_- \end{aligned} \tag{A.18}$$

and hence

$$F_k(z)_- = \mathrm{e}^{-2\pi i Nc}F_k(z)_+\ . \tag{A.19}$$

The jumps of the solutions of the d-bar problem are expressed in terms of $F_k(z)$:

$$\begin{array}{rcll} f_k^{(0)}(z,\overline{z})_+ - f_k^{(0)}(z,\overline{z})_- &=& F_k(z)_+ - F_k(z)_- & z\in C_a \\ f_k^{(1)}(z,\overline{z}) - f_k^{(0)}(z,\overline{z}) &=& F_k(z)_+ - F_k(z)_- & z\in R_0 \\ f_k^{(j)}(z,\overline{z}) - \tilde{f}_k(z,\overline{z}) &=& F_k(z) & z\in S_j,\ |z|=\delta\ . \end{array} \tag{A.20}$$

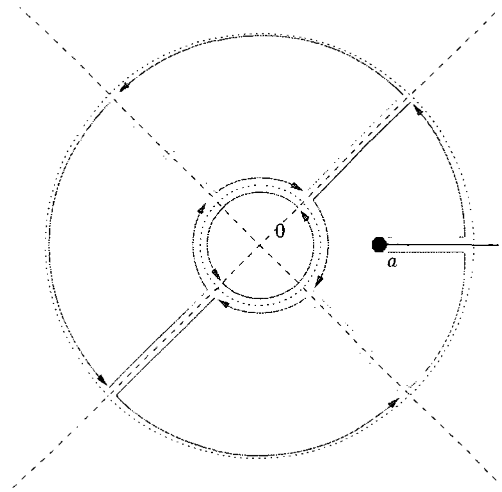

Figure 7: The approximate domains of integration

Let $\tilde{R}_0$ denote the part of the ray R_0 that is outside the disk $|z| < \delta$. Therefore

$$
\begin{aligned}
&\int_{\mathbb{C}} p(z)\bar{z}^k |z-a|^{2Nc} e^{-Nz\bar{z}} dA(z) \\
&= \frac{1}{2i}\int_{\mathbb{C}} p(z)\bar{z}^k |z-a|^{2Nc} e^{-Nz\bar{z}} d\bar{z}\wedge dz \\
&= \lim_{R\to\infty}\left[\sum_{j=0}^{3}\int_{D_R^{(j)}} d\omega_k^{(j)} + \int_{|z|<\delta} d\tilde{\omega}_k\right] \\
&= \lim_{R\to\infty}\left[\sum_{j=0}^{3}\oint_{\partial D_R^{(j)}} \omega_k^{(j)} + \oint_{|z|=\delta} \tilde{\omega}_k\right] \qquad \text{(A.21)} \\
&= \frac{i}{2}\int_{\tilde{R}_0} p(z)(F_k(z)_+ - F_k(z)_-)dz \\
&+ \frac{i}{2}\int_{\tilde{C}_a} p(z)(F_k(z)_+ - F_k(z)_-)dz \\
&- \frac{i}{2}\oint_{|z|=\delta} p(z)F_k(z)dz \,.
\end{aligned}
$$

This means that

$$\int_{\mathbb{C}} p(z)\bar{z}^k |z-a|^{2Nc} e^{-Nz\bar{z}} dA(z) = \frac{1}{2i}\int_{\Gamma} p(z)F_k(z)dz \,, \qquad \text{(A.22)}$$

where Γ is the union of a keyhole contour around R_0 and a simple contour around C_a oriented as shown on Figure A.1.

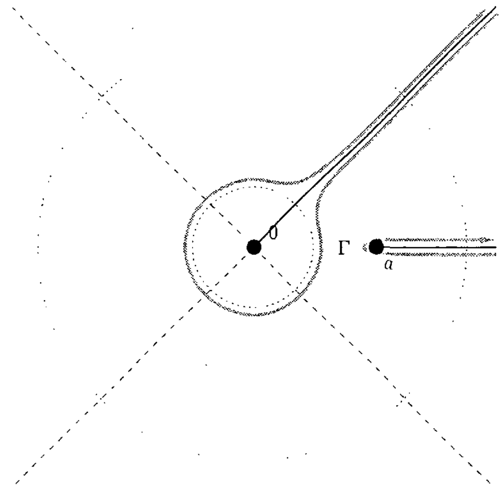

Figure 8: The integration contour Γ

The function $F_k(z)$ can be calculated explicitly (the integral is taken along the real line for simplicity, on the positive side of C_a):

$$\begin{aligned}
F_k(z) &= e^{-2\pi i N c}(z-a)^{Nc}\int_a^\infty s^k (s-a)_+^{Nc} e^{-Nzs} ds \\
&= e^{-2\pi i N c}(z-a)^{Nc} e^{-az}\int_0^\infty (u+a)^k u^{Nc} e^{-Nzu} du \\
&= e^{-2\pi i N c}(z-a)^{Nc} e^{-az}\sum_{l=0}^{k}\binom{k}{l} a^{k-l}\int_0^\infty u^{l+Nc} e^{-Nzu} du \\
&= e^{-2\pi i N c}\frac{(z-a)^{Nc} e^{-az}}{z^{Nc}}\sum_{l=0}^{k}\binom{k}{l} a^{k-l}\frac{\Gamma(l+Nc+1)}{N^{l+Nc+1}}\frac{1}{z^{l+1}} .
\end{aligned} \tag{A.23}$$

Therefore

$$\begin{aligned}
&\int_{\mathbb{C}} p(z)\bar{z}^k |z-a|^{2Nc} e^{-Nz\bar{z}} dA(z) = \\
&\frac{e^{-2\pi i N c}}{2i}\sum_{l=0}^{k}\binom{k}{l} a^{k-l}\frac{\Gamma(l+Nc+1)}{N^{l+Nc+1}}\int_\Gamma p(z)\frac{(z-a)^{Nc} e^{-az}}{z^{Nc}}\frac{dz}{z^{l+1}} .
\end{aligned} \tag{A.24}$$

The matching phase terms in the multiplicative jumps $F_k(z)$ allows a different analytic continuation consisting of a single cut along the segment $[0, a]$. The presence of the factor e^{-az} permits a deformation of the integration contour into a closed loop around $[0, a]$.

Q.E.D.

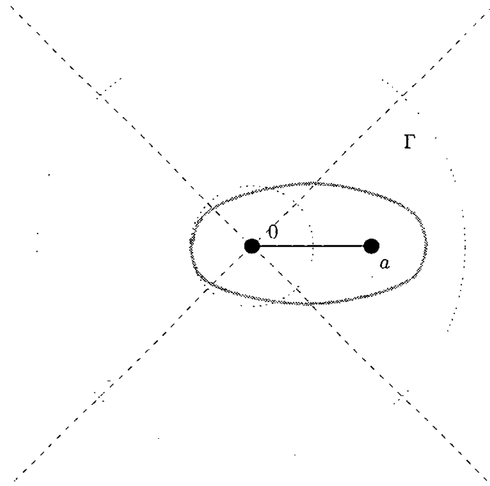

Figure 9: The final integration contour Γ

A.2 Proof of Lemma 4.1.

The proof of the uniqueness is standard [5]. The analyticity, the jump condition and the asymptotics for large z implies that

$$Y(z) = \begin{bmatrix} p_n(z) & \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \frac{p_n(t)w_{n,N}(t)dt}{t-z} \\ q_{n-1}(z) & \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \frac{q_{n-1}(t)w_{n,N}(t)dt}{t-z} \end{bmatrix} . \tag{A.25}$$

where $p_n(z)$ and $q_{n-1}(z)$ are polynomials of degree at most n and $n-1$ respectively. $p_n(z)$ satisfies

$$p_n(z) = z^n + \mathcal{O}\left(z^{n-1}\right) \text{ and } \int_\Gamma p_n(z) z^k w_{n,N}(z)dz = 0 \quad k = 0, 1, \ldots, n-1 . \tag{A.26}$$

This implies that $p_n(z) = P_{n.N}(z)$ (and this settles the existence of the first row). The polynomial $q_{n-1}(z)$ satisfies

$$\int_\Gamma q_{n-1}(z) z^k w_{n,N}(z)dz = 0 \quad k = 0, 1, \ldots, n-2 \text{ and } \int_\Gamma q_{n-1}(z) z^{n-1} w_{n,N}(z)dz = 1. \tag{A.27}$$

The necessary and sufficient condition to find such $q_{n-1}(z)$ is that $\det(m^{(n)}) \neq 0$ which is true because of the moment matrix equation.

Q.E.D.

A.3 The analysis of the discriminant of $J(z)$

Consider the polynomial

$$J(z) = J(t, x; z) . \tag{A.28}$$

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The discriminant is of the form

$$D(x) = \frac{1}{a^6} A(x)^2 B(x) \ , \tag{A.29}$$

where

$$\begin{aligned}
A(x) &= a^2x - 2\,a^2c - a^4 - 4\,a^2t - t^2 \\
B(x) &= -4\,a^4x^3 + \left(-16\,a^4c + 28\,a^4t + 4\,a^2t^2 + 16\,a^2tc + 16\,a^2c^2 + 13\,a^6\right)x^2 \\
&\quad + \left(-48\,a^4c^2 - 14\,a^8 - 54\,a^4t^2 + 64\,a^2c^3 + 24\,t^2ca^2 - 60\,a^6t + 48\,a^4tc + 36\,a^6c + 20\,t^3a^2\right)x \\
&\quad - 232\,t^3ca^2 - 384\,t^3c^2 - 64\,a^2c^3t - 132\,a^4t^2c - 32\,t^5 - 64\,a^4c^3 - 192\,t^4c + 50\,a^6t^2 \\
&\quad + 48\,a^4c^2t + 36\,a^6c^2 - 72\,a^6tc - 96\,a^2c^2t^2 - 256\,t^2c^3 - 32\,a^4t^3 \\
&\quad + 64\,c^4a^2 + 32\,a^8t + 5\,a^{10} - 131\,a^2t^4 - 20\,ca^8 \ .
\end{aligned} \tag{A.30}$$

By picking the double root coming from $A(x)$

$$x_A(t) := 2c + a^2 + 4t + \frac{t^2}{a^2} \tag{A.31}$$

we get two pairs of coalescent roots in $J(z)$:

$$J(z) = \frac{\left(az^2 - (a^2+t)z + a(c+t)\right)^2}{a^2} \ . \tag{A.32}$$

Note that $x_A(t)$ is strictly increasing for $t > 0$. The discriminant of the quadratic polynomial $az^2 - (a^2+t)z + a(c+t)$ is

$$(a^2+t)^2 - 4a^2(c+t) = (t - \tau_c)(t - t_c) \tag{A.33}$$

where the critical times are

$$\tau_c := a(a - 2\sqrt{c}) \quad t_c := a(a + 2\sqrt{c}) \qquad \tau_c < t_c \ . \tag{A.34}$$

For $t = 0$ we have

$$\begin{aligned}
A(x) &= a^2(x - 2c - a^2) \\
B(x) &= a^2(5a^2 - 4xa^2 + 16c^2)(x + 2c - a^2)^2 \ .
\end{aligned} \tag{A.35}$$

We claim that for $t \in \mathbb{R}^+ \setminus \{\tau_c, t_c\}$ we have exactly one root of $B(x)$ in each of the intervals

$$(-\infty, a^2 - 2c) \ , \ (a^2 - 2c, x_A(t)) \ , \ (x_A(t), \infty) \ . \tag{A.36}$$

This follows from

$$\begin{aligned}
&\lim_{x\to-\infty} B(x) = \infty \ , \\
&B(a^2 - 2c) = -t^2\left(272\,a^2tc + 32\,t^3 + 128\,a^2c^2 + 16\,a^4c + 192\,t^2c + 131\,a^2t^2 + 12\,a^4t + 384\,tc^2 + 256\,c^3\right) < 0 \ , \\
&B(x_A(t)) = 16\,\frac{c\,(t+c)\left((t-\tau_c)(t-t_c)\right)^2}{a^2} > 0 \ ; \\
&\lim_{x\to\infty} B(x) = -\infty \ .
\end{aligned} \tag{A.37}$$

As $t \to 0$ we expect that

$$J(z) = (z - z_1)^2 (z - z_2)^2 = (z^2 - az - c)^2 \tag{A.38}$$

and the branch cut opens at the minimum of the potential $z = z_1$ for very small t. This happens for root $x = a^2 - 2c$ of $B(x)$; let $x_-(t)$ and $x_+(t)$ the solutions in the intervals $(\infty, a^2 - 2c)$ and $(a^2 - 2c, x_A(t))$ respectively.

By elementary perturbation theory,

$$\frac{d}{dt} x_\pm(t) = 2 \frac{a \pm \sqrt{a^2 + 4c}}{a} = \frac{4z_\pm}{a} \; . \tag{A.39}$$

Now, we are interested in the following:

$$\begin{aligned} \frac{d}{dt} J(t, x_\pm(t); z)\bigg|_{t=0} &= \left[\frac{\partial}{\partial t} J(t, x_\pm(t); z) + \frac{\partial}{\partial x} J(t, x_\pm(t); z) \frac{dx_\pm}{dt}(t) \right]\bigg|_{t=0} \\ &= -\frac{2(z - a)(z^2 - 2zz_\pm + az + c)}{a} \end{aligned} \tag{A.40}$$

This implies that

$$\begin{aligned} \frac{d}{dt} J(t, x_+(t); z_+)\bigg|_{t=0} &= 0 \\ \frac{d}{dt} J(t, x_+(t); z_-)\bigg|_{t=0} &= -\frac{2(z_- - a)(z_-^2 - 2z_- z_+ + az_- + c)}{a} = -\frac{4(z_- - a) z_- (z_- - z_+)}{a} > 0 \; , \end{aligned} \tag{A.41}$$

$$\begin{aligned} \frac{d}{dt} J(t, x_-(t); z_+)\bigg|_{t=0} &= -\frac{2(z_+ - a)(z_+^2 - 2z_- z_+ + az_+ + c)}{a} = -\frac{4(z_+ - a) z_+ (z_+ - z_-)}{a} < 0 \\ \frac{d}{dt} J(t, x_-(t); z_-)\bigg|_{t=0} &= 0 \; . \end{aligned} \tag{A.42}$$

This means the following:

- Along the solution $x_-(t)$ the double root $z = z_+$ splits into a pair of real roots while the solution $z = z_-$ evolves further as a double root along the real line.
- Along the solution $x_-(t)$ the double root $z = z_-$ leaves the real line by splitting into a pair of complex conjugate roots while the solution $z = z_+$ evolves further as a double root along the real line.

Since the resultant of the two factors $A(x)$ and $B(x)$ is

$$\text{resultant}(A(x), B(x)) = 16a^4 c(t + c)((t - \tau_c)(t - t_c))^2 \; , \tag{A.43}$$

the two critical times $t = \tau_c$ and $t = t_c$ are the only possibilities for them to have a common root. The two critical times are different: if $t = \tau_c$ then $x_A(\tau_c)$ coincides with the largest root of $B(x)$ while for $t = t_c$ the root $x_A(t_c)$ hits the solution $x_+(t)$, as it can be seen easily from the factorization of $B(x)$ for these critical times.

The following solution is chosen to fix $J(z)$ for every $t > 0$:

$$x(t) := \begin{cases} x_+(t) & 0 < t \le t_c \\ x_A(t) & t_c < t \; . \end{cases} \tag{A.44}$$

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A.4 The trajectory structure in the pre-critical case

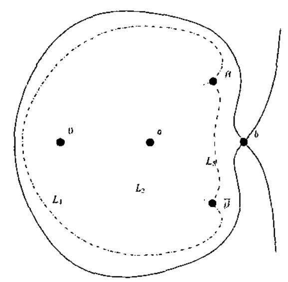

Figure 10: The (3,3) entry with marked trajectories

The quadratic differential $R(z)dz^2$ gives rise to a flat metric $|R(z)||dz|^2$ with singularities at the poles and zeroes at $R(z)$ [9]. In this metric the trajectories have an associated length, the so-called *Strebel length.* The length L of a horizontal trajectory h is given by

$$L = \left| \int_h \sqrt{R(z)}dz \right| = \left| \int_h y(z)dz \right| \tag{A.45}$$

since the form $y(z)dz$ is real along a horizontal trajectory.

For the $(3,3)$ configuration simple residue calculus gives

$$\begin{cases} L_1 + L_2 = 2\pi(t+c) \\ L_2 + L_3 = 2\pi c \\ L_1 + L_3 = 2\pi t \ . \end{cases} \tag{A.46}$$

where the L_i's are the Strebel lengths of the critical trajectories shown on Figure A.4. This implies $L_3 = 0$ that is impossible. This means that the $(3,3)$ configuration is not compatible with the conditions (??).

The same argument can be used to eliminate the configuration $(3,2)$.

A.5 Local parametrices

Airy parametrix for the pre-critical case. It is easy to see that

$$\hat{p}(\zeta) := \sigma_3 p(-\zeta)\sigma_3 \tag{A.47}$$

satisfies the following R-H problem:

($\hat{p}$.1) $\hat{p}$ is holomorphic on $\mathbb{C} \setminus \hat{\Sigma}$,

($\hat{p}$.2) $p_+(\zeta) = \hat{p}_-(z)v_{\hat{p}}(z)$ along $\hat{\Sigma}$ where

$$v_{\hat{p}}(z) = \begin{cases} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & \zeta \in \Sigma_1 \\ \begin{bmatrix} 1 & 0 \\ e^{\frac{4}{3}\zeta^{\frac{3}{2}}} & 1 \end{bmatrix} & \zeta \in \Sigma_2 \\ \begin{bmatrix} 1 & e^{-\frac{4}{3}\zeta^{\frac{3}{2}}} \\ 0 & 1 \end{bmatrix} & \zeta \in \Sigma_3 \\ \begin{bmatrix} 1 & 0 \\ e^{\frac{4}{3}\zeta^{\frac{3}{2}}} & 1 \end{bmatrix} & \zeta \in \Sigma_4 \ , \end{cases} \tag{A.48}$$

($\hat{p}$.3) As $\zeta \to \infty$,

$$\hat{p}(\zeta) = \frac{1}{2}\left(\frac{1+i}{\sqrt{2}}\right)^{-\sigma_3} \zeta^{-\frac{1}{4}\sigma_3} \begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix} \left(I + \mathcal{O}\left(\frac{1}{\zeta}\right)\right) , \tag{A.49}$$

and $\zeta^{1/4}$ has a cut along the negative real axis and it is positive for positive real ζ.

This can be written in terms of the *Airy function*

$$\mathrm{Ai}(\zeta) := \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\frac{1}{3}q^3 - \zeta q} dq \,. \tag{A.50}$$

Namely,

$$\hat{p}(\zeta) = \left(\frac{1+i}{\sqrt{2}}\right)^{-\sigma_3} \begin{bmatrix} f_0(\zeta) & if_1(\zeta) \\ f_0'(\zeta) & if_1'(\zeta) \end{bmatrix} C(\zeta) e^{\frac{2}{3}\zeta^{\frac{3}{2}}\sigma_3} , \tag{A.51}$$

where

$$f_0(\zeta) = \mathrm{Ai}(\zeta) \ , f_1(\zeta) = e^{-\frac{\pi i}{6}} \mathrm{Ai}(e^{-\frac{2\pi i}{3}}\zeta) \tag{A.52}$$

and

$$C(\zeta) = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \zeta \in \Omega_1 \\[2ex] \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} & \zeta \in \Omega_2 \\[2ex] \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} & \zeta \in \Omega_3 \\[2ex] \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} & \zeta \in \Omega_4 \,. \end{cases} \tag{A.53}$$

Parametrix for the post-critical case.

Let us define

$$\tilde{p}(z) := p(z)\psi(\zeta)^{-1} \,. \tag{A.54}$$

This satisfies

($\tilde{p}$.1) $\tilde{p}(\zeta)$ is holomorphic on Γ;

($\tilde{p}$.2) $\tilde{p}_+(\zeta) = \tilde{p}_-(\zeta)\tilde{v}_{\tilde{p}}(z)$ along Γ where

$$v_{\tilde{p}}(z) = \begin{cases} I & \zeta \in \Gamma_1 \cup \Gamma_4 \\[2ex] \begin{bmatrix} 1 & -e^{\zeta^2/2} \\ 0 & 1 \end{bmatrix} & \zeta \in \Gamma_2 \cup \Gamma_3 \,. \end{cases} \tag{A.55}$$

($\tilde{p}$.3)

$$\tilde{p}(\zeta) = I + \mathcal{O}\left(\frac{1}{\zeta}\right) \qquad \zeta \to \infty \,. \tag{A.56}$$

This has a simple solution:

$$\tilde{p}(\zeta) = \begin{bmatrix} 1 & -f(-i\zeta) \\ 0 & 1 \end{bmatrix}, \qquad f(u) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{e^{-s^2/2}}{s-u} ds \ , \quad u \notin \mathbb{R} \ . \tag{A.57}$$

Since

$$f(u) \sim -\frac{1}{\sqrt{2\pi} i} \sum_{k=0}^{\infty} \frac{(2k)!}{2^k k!} \frac{1}{u^{2k+1}} \quad u \to \infty \ , \tag{A.58}$$

we have

$$\tilde{p}(\zeta) \sim I + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{2^k k!} \frac{1}{\zeta^{2k+1}} = I + \begin{bmatrix} 0 & \frac{1}{\sqrt{2\pi}} \\ 0 & 0 \end{bmatrix} \frac{1}{\zeta} + \mathcal{O}\left(\frac{1}{\zeta^3}\right) \quad \zeta \to \infty \ . \tag{A.59}$$

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12.3 Appendix

In this section it is shown that the Cauchy transform of S_{Q_t} satisfies the same equation as $g'(z)$. To this end, consider a general Joukowski-type conformal mapping

$$F(u) := ru + a_0 + \frac{v}{u - A} \qquad r > 0\ , |A| < 1\ . \tag{12.84}$$

that maps the exterior of the unit disk to the exterior of a domain B. The condition $|A| < 1$ is needed to ensure that F is holomorphic in $\{u\colon |u| > 1\}$. From now on we assume that F the following symmetry property:

$$F(\overline{u}) = \overline{F(u)} \tag{12.85}$$

which is equivalent to

$$\overline{F}(u) = F(u)\ . \tag{12.86}$$

This implies that every parameter of F is real. Since F is a rational function of degree two, $F\colon \mathbb{C}P^1 \to \mathbb{C}P^1$ is a double covering. The branch points of this covering map are given by the quadratic equation

$$(z - rA - a_0)^2 - 4rv = 0\ . \tag{12.87}$$

The mapping F is univalent if and only if

$$(1 - |A|)^2 > \left|\frac{v}{r}\right|\ . \tag{12.88}$$

For such mappings the image G of the unit disk is given by a nonsingular analytic Jordan curve. The function $S^+(z)$ is given by

$$\begin{aligned} S^+(z) &= \frac{1}{2\pi i}\int_G \frac{\overline{w}dw}{w - z} \\ &= \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{\overline{f}\left(\frac{1}{\zeta}\right) f'(\zeta)}{f(\zeta) - z} d\zeta\ . \end{aligned} \tag{12.89}$$

The rational one-form

$$\frac{\overline{f}\left(\frac{1}{\zeta}\right) f'(\zeta)}{f(\zeta) - z} d\zeta \tag{12.90}$$

has six poles counted with multiplicities:

$$u = 0\ , \infty\ , A\ , \frac{1}{A}\ , F^{-1}(z)_1\ , F^{-1}(z)_2\ . \tag{12.91}$$

If $z \in G_+$ then only the two poles

$$u = \frac{1}{A} \qquad u = \infty \tag{12.92}$$

are outside the unit circle of the u-plane. Simple residue calculus gives

$$S^+(z) = u_0 - \frac{v}{A} - \frac{\frac{v}{A^2}\left(r - \frac{vA^2}{(1-A^2)^2}\right)}{z - \frac{r}{A} - u - \frac{vA}{1-A^2}}\ . \tag{12.93}$$

This gives that $u_0 = \frac{v}{A}$ and

$$\begin{cases} c = -\dfrac{v}{A^2}\left(r - \dfrac{vA^2}{(1-A^2)^2}\right) \\ a = \dfrac{r}{A} + \dfrac{v}{A(1-A^2)}\ . \end{cases} \tag{12.94}$$

The area equation gives

$$t = r^2 - \frac{v^2}{(1-A^2)^2}\ . \tag{12.95}$$

The Cauchy transform of the domain B is given by

$$C(z) = \frac{1}{2i}\int_B \frac{d\overline{w} \wedge dw}{w - z} = \begin{cases} -\pi\overline{z} + \pi S^+(z) & z \in B \\ \pi S^-(z) & z \in B^c \end{cases} \tag{12.96}$$

The algebraic equation satisfied by the Schwarz function is obtained from the parametrization of z and $w = \overline{z}$ on ∂B in terms of the uniformizing coordinate u:

$$\begin{aligned} P_1(u) &:= Aru^2 + (v - A^2 r - Az)u + A^2 z \\ P_2(u) &:= A^2 wu^2 + (v - A^2 r - Aw)u + rA \end{aligned} \tag{12.97}$$

The resultant of the above polynomials provide the algebraic equation

$$E(z,w) := \text{resultant}(P_1(u), P_2(u)) = C_2(z)w^2 + C_1(z)w + C_0(z) \tag{12.98}$$

satisfied by $w = S(z)$, where the functions $C_k(z)$ are explicit polynomial expressions in z. By completion of the square, it can be shown that $E(z,w) = 0$ is equivalent to the equation

$$\left(w - \frac{a}{2} - \frac{1}{2}\frac{c}{z-a} - \frac{1}{2}\frac{c+t}{z}\right)^2 = \frac{a^2(z-b)^2((z-d)^2-h)}{4z^2(z-a)^2} \tag{12.99}$$

where

$$\begin{aligned} b &= \frac{r}{A} \\ d &= rA + \frac{v}{A} \\ h &= 4rv \ . \end{aligned} \tag{12.100}$$

The normalized Cauchy transform in the exterior of B is given by

$$\tilde{w} = -\frac{1}{A(B)}C_B(z) = -\frac{1}{t}S^-(z) = \frac{1}{t}\left(w - \frac{c}{z-a}\right) \ . \tag{12.101}$$

that satisfies the equation

$$\left(t\tilde{w} - \frac{a}{2} + \frac{1}{2}\frac{c}{z-a} - \frac{1}{2}\frac{c+t}{z}\right)^2 = \frac{a^2(z-b)^2((z-d)^2-h)}{4z^2(z-a)^2} \tag{12.102}$$

which is equivalent to

$$(2t\tilde{w} - V'(z))^2 = \frac{a^2(z-b)^2((z-d)^2-h)}{z^2(z-a)^2} \ . \tag{12.103}$$

It follows from (12.100) that the rational expression on the left hand side is the same as $R(z)$ and therefore $\tilde{w} = g'(z)$ solves the same equation as the normalized Cauchy transform of $B = S_{Q_t}$ outside S_{Q_t}. This is enough to conclude that these two functions are the same.

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Appendix A

Statement on collaborations

[1] F. Balogh, J. Harnad, Superharmonic perturbations of a Gaussian measure, equilibrium measures and orthogonal polynomials

I found and proved the results Thm. 3.4 and Thm. 3.5 on the structure of the support of the equilibrium measure for superharmonic perturbation potentials. Most of the calculations in the proof Prop. 3.3 and the proofs of the estimates needed for Prop. 5.1 are based on my work. I have performed extensive numerical experiments that support the conjecture of the zeroes of the orthogonal polynomials detailed in Sec. 6.

[2] F. Balogh, M. Bertola, Regularity of a vector potential problem and its spectral curve

My work on this paper contributed to the suitable generalization of the definition of admissibility for vector potential problems (Def. 2.1) and also to the detailed proof of the existence and uniqueness of the vector equilibrium measure (Thm. 3.2) based on [81]. In the second algebraic part I did the linear algebra on which the existence of the prseudo-algebraic curve (Prop. 5.1, Thm. 5.1) relies upon.

[3] **F. Balogh, M. Bertola, On the norms and roots of orthogonal polynomials in the plane and L^p-optimal polynomials with respect to varying weights**

I was responsible for the potential theoretic background of the proofs of the theorems, most of which were worked out jointly.

[4] **F. Balogh, External potentials for two-point quadrature domains**

This manuscript is based solely on my work.

[5] **F. Balogh, M. Bertola, K. T-R. McLaughlin, S. Y. Lee, Riemann-Hilbert analysis of the Bratwurst orthogonal polynomials**

I found the integral identity used to reformulate the orthogonality in terms of contour integrals (Lemma 4.1), constructed a rigorous proof based on a piecewise solution of a corresponding scalar d-bar problem. I calculated the formula for the determinant of the moment matrices. The construction of the g-function relies on a finding a suitable quadratic differential $R(z)dz^2$; based on an earlier draft manuscript I reformulated the conditions in terms of the numerator polynomial $J(z)$ of $R(z)$ that reduces the problem to a quintic equation for a free parameter x and analyzed the root behavior of the discriminant of $J(z)$ as a polynomial of x depending on the continuous parameter t. I cleaned up the construction of the g-function and the limiting zero distributions in the pre-critical and the post-critical cases. I carefully checked and completed the calculations leading to the strong asymptotic formulae provided by the nonlinear steepest descent analysis.

Appendix B

Non-Hermitian orthogonality for the rotated Hermite polynomials

Proof of Thm. 3.5.1. Our goal now is to rewrite the standard orthogonality relations

$$\int_{\mathbb{C}} P_n(z)\overline{z}^k e^{-|z|^2+\frac{T^2z^2}{2}+\frac{\overline{T}^2\overline{z}^2}{2}} dA(z) = 0 \qquad k = 0, 1, \ldots, n-1 \tag{B.1}$$

as non-Hermitian orthogonality relations with respect to a weight function supported along a contour in the complex plane. In terms of the complex coordinates $z, \overline{z}$ the area integral is written as

$$\int_{\mathbb{C}} P_n(z)\overline{z}^k e^{-|z|^2+\frac{T^2z^2}{2}+\frac{\overline{T}^2\overline{z}^2}{2}} dA(z) = \frac{1}{2i}\int_{\mathbb{C}} P_n(z)\overline{z}^k e^{-z\overline{z}+\frac{T^2z^2}{2}+\frac{\overline{T}^2\overline{z}^2}{2}} d\overline{z}\wedge dz\ . \tag{B.2}$$

The basic idea [69] is to use the Stokes' Theorem to reduce the area integral to integrals on the boundary. To this end, we are looking for a one-form ω_{nk} in the complex plane such that

$$d\omega_{nk} = \frac{1}{2i}P_n(z)\overline{z}^k e^{-z\overline{z}+\frac{T^2z^2}{2}+\frac{\overline{T}^2\overline{z}^2}{2}} d\overline{z}\wedge dz\ . \tag{B.3}$$

We seek ω_{nk} of the form

$$\omega_{nk} = -\frac{1}{2i} P_n(z) f_k(z,\overline{z}) dz \tag{B.4}$$

where f_k solves the d-bar problem

$$\partial_{\overline{z}} f_k(z,\overline{z}) = -\overline{z}^k e^{-z\overline{z}+\frac{\overline{T}^2}{2}\overline{z}^2+\frac{T^2}{2}z^2} . \tag{B.5}$$

A solution to (B.5) can be easily constructed in a contour integral form:

$$f_k(z,\overline{z}) := e^{\frac{T^2}{2}z^2} \int_{\overline{z}}^{\infty} s^k e^{-zs+\frac{\overline{T}^2}{2}s^2} ds \qquad k = 0, 1, \ldots , \tag{B.6}$$

where the contour of integration is asymptotic to the ray

$$\{s = iTu \ : \ u > 0\} . \tag{B.7}$$

This choice ensures that the integral in (B.6) is finite for all values of z. Obviously, $f_k(z,\overline{z})$ is a smooth function for which the one-form ω_{nk} defined above satisfies

$$\begin{aligned} d\omega_{nk} &= -d\left(\frac{1}{2i} P_n(z) f_k(z,\overline{z}) dz\right) \\ &= -\frac{1}{2i}\partial_{\overline{z}}\left(P_n(z) f_k(z,\overline{z})\right) d\overline{z} \wedge dz \\ &= \frac{1}{2i} P_n(z) \overline{z}^k e^{-z\overline{z}+\frac{T^2z^2}{2}+\frac{\overline{T}^2\overline{z}^2}{2}} d\overline{z} \wedge dz . \end{aligned} \tag{B.8}$$

To use Stokes' Theorem, the integral (B.2) in the complex plane has to be approximated by integrals of the same function restricted to some fixed bounded domains:

$$\begin{aligned} \int_{\mathbb{C}} P_n(z)\overline{z}^k e^{-|z|^2+\frac{T^2z^2}{2}+\frac{\overline{T}^2\overline{z}^2}{2}} dA(z) &= \lim_{R\to\infty} \int_{|z|\le R} P_n(z)\overline{z}^k e^{-|z|^2+\frac{T^2z^2}{2}+\frac{\overline{T}^2\overline{z}^2}{2}} dA(z) \\ &= \lim_{R\to\infty} \oint_{|z|=R} \omega_{nk} . \end{aligned} \tag{B.9}$$

To understand the boundary behavior of ω_{nk} as $|z| \to \infty$ one has to investigate the asymptotics of $f_k(z,\overline{z})$ as $|z| \to \infty$.

A linear substitution of the form

$$s = \frac{iTu}{|T|^2} \qquad (u \in \mathbb{R}^+) \tag{B.10}$$

gives

$$
\begin{aligned}
f_k(z,\bar{z}) &= \frac{i^{k+1} e^{\frac{T^2}{2} z^2}}{\overline{T}^{k+1}} \int_{-i\overline{T}\bar{z}}^{\infty} u^k e^{-i\frac{z}{T}u - \frac{u^2}{2}}\, du \\
&= \frac{i^{k+1} e^{\frac{z^2}{2}\left(T^2 - \frac{1}{T^2}\right)}}{\overline{T}^{k+1}} \int_{-i\overline{T}\bar{z}}^{\infty} u^k e^{-\frac{1}{2}\left(u + i\frac{z}{T}\right)^2} du \\
&= \frac{i^{k+1} e^{\frac{z^2}{2}\left(T^2 - \frac{1}{T^2}\right)}}{\overline{T}^{k+1}} \int_{i\left(\frac{z}{T} - \overline{T}\bar{z}\right)}^{\infty} \left(v - i\frac{z}{T}\right)^k e^{-\frac{v^2}{2}} dv \\
&= \frac{i^{k+1} e^{\frac{z^2}{2}\left(T^2 - \frac{1}{T^2}\right)}}{\overline{T}^{k+1}} \sum_{l=0}^{k} \binom{k}{l} \left(\frac{z}{iT}\right)^{k-l} \int_{i\left(\frac{z}{T} - \overline{T}\bar{z}\right)}^{\infty} v^l e^{-\frac{v^2}{2}} dv\,.
\end{aligned}
\tag{B.11}
$$

The contour of integration above on the v-plane is asymptotic to the line

$$
\left\{ v = u + i\frac{z}{T} \;:\; u \in \mathbb{R}^+ \right\}\,. \tag{B.12}
$$

Since the integrand is exponentially decaying in the sector

$$
\left\{ v \;:\; |\arg(v)| < \frac{\pi}{4} \right\}\,, \tag{B.13}
$$

the integration can be performed along a contour that is asymptotic to the positive real line on the v-plane. For this standard choice of the contour of integration, it is well-known (see [2] 7.1.23) that the asymptotic series expansion

$$
\begin{aligned}
\int_z^{\infty} e^{-\frac{v^2}{2}} dv &= \sqrt{\frac{\pi}{2}}\operatorname{erfc}\left(\frac{z}{\sqrt{2}}\right) \\
&\sim \frac{e^{-\frac{z^2}{2}}}{z}\left(1 + \sum_{m=1}^{\infty} (-1)^m \frac{(2m-1)!!}{z^{2m}}\right)
\end{aligned}
\tag{B.14}
$$

holds as $z \longrightarrow \infty$ in the sector

$$
\left\{ z \;:\; |\arg(z)| < \frac{3\pi}{4} \right\}\,. \tag{B.15}
$$

Successive integration by parts gives that for $k = 0, 1, \ldots$ the asymptotic expansion

$$
\int_z^{\infty} v^{2k} e^{-\frac{v^2}{2}} dv \sim e^{-\frac{z^2}{2}} z^{2k-1}\left(1 + \mathcal{O}\left(\frac{1}{z^2}\right)\right) \tag{B.16}
$$

is valid as $z \longrightarrow \infty$ in the sector

$$\left\{ z \, : \, |\arg(z)| < \frac{3\pi}{4} \right\} \, . \tag{B.17}$$

Since

$$c_n := \int_{-\infty}^{\infty} v^n e^{-\frac{v^2}{2}} dv = \begin{cases} \sqrt{2\pi}\dfrac{(2k)!}{2^k k!} & n = 2k \\ 0 & n = 2k+1 \end{cases} , \tag{B.18}$$

simple symmetry considerations ([1], 6.6.3) imply that

$$\int_z^{\infty} v^{2k} e^{-\frac{v^2}{2}} dv \sim c_{2k} + e^{-\frac{z^2}{2}} z^{2k-1} \left(1 + \mathcal{O}\left(\frac{1}{z^2}\right)\right) \tag{B.19}$$

is valid as $z \longrightarrow \infty$ in the sector

$$\left\{ z \, : \, \frac{\pi}{4} < \arg(z) < \frac{7\pi}{4} \right\} \, . \tag{B.20}$$

In other words, there is a *Stokes phenomenon* occurring on the overlap of the two different sectors.

The case of odd powers is much simpler: we have the explicit formula

$$\int_z^{\infty} v^{2k+1} e^{-\frac{v^2}{2}} dv = 2^n n! e^{-\frac{z^2}{2}} \sum_{k=0}^{n} \frac{1}{2^k k!} z^{2k} \tag{B.21}$$

using successive integration by parts. Therefore the expansion

$$\int_z^{\infty} v^{2k+1} e^{-\frac{v^2}{2}} dv \sim e^{-\frac{z^2}{2}} z^{2k} \left(1 + \mathcal{O}\left(\frac{1}{z^2}\right)\right) \tag{B.22}$$

is valid as $|z| \longrightarrow \infty$ without restriction. To sum up, we have

$$\int_z^{\infty} v^n e^{-\frac{v^2}{2}} dv \sim \begin{cases} e^{-\frac{z^2}{2}} z^{n-1} \left(1 + \mathcal{O}\left(\frac{1}{z^2}\right)\right) & -\frac{3\pi}{4} < \arg(z) < \frac{3\pi}{4} \\ c_n + e^{-\frac{z^2}{2}} z^{n-1} \left(1 + \mathcal{O}\left(\frac{1}{z^2}\right)\right) & \frac{\pi}{4} < \arg(z) < \frac{7\pi}{4} \end{cases} . \tag{B.23}$$

The argument of the integral above is

$$i\left(\frac{z}{\overline{T}} - \overline{T}\overline{z}\right) = -\frac{1+|T|^2}{|T|^2}\mathrm{Im}(Tz) + i\frac{1-|T|^2}{|T|^2}\mathrm{Re}(Tz) \, . \tag{B.24}$$

Therefore

$$f_k(z,\overline{z}) \sim \frac{i^{k+1} e^{-z\overline{z}+\frac{T^2 z^2}{2}+\frac{\overline{T}^2\overline{z}^2}{2}}}{\overline{T}^{k+1}} \sum_{l=0}^{k} \binom{k}{l} \left(\frac{z}{i\overline{T}}\right)^{k-l} \left(i\left(\frac{z}{\overline{T}} - \overline{T}\overline{z}\right)\right)^{l-1} \left(1+\mathcal{O}\left(\frac{1}{|z|^2}\right)\right) \tag{B.25}$$

as $z \to \infty$ in the half-plane $\mathrm{Im}(Tz) \le 0$ and

$$\begin{aligned} f_k(z,\overline{z}) \sim\ & \frac{i^{k+1} e^{\frac{z^2}{2}\left(T^2-\frac{1}{\overline{T}^2}\right)}}{\overline{T}^{k+1}} \sum_{l=0}^{k} \binom{k}{l} \left(\frac{z}{i\overline{T}}\right)^{k-l} c_l \\ &+ \frac{i^{k+1} e^{-z\overline{z}+\frac{T^2 z^2}{2}+\frac{\overline{T}^2\overline{z}^2}{2}}}{\overline{T}^{k+1}} \sum_{l=0}^{k} \binom{k}{l} \left(\frac{z}{i\overline{T}}\right)^{k-l} \left(i\left(\frac{z}{\overline{T}} - \overline{T}\overline{z}\right)\right)^{l-1} \left(1+\mathcal{O}\left(\frac{1}{|z|^2}\right)\right) \end{aligned} \tag{B.26}$$

as $z \to \infty$ in the other half-plane $\mathrm{Im}(Tz) \ge 0$. So let

$$\tilde{f}_k(z,\overline{z}) := f_k(z,\overline{z}) - \frac{i^{k+1} e^{\frac{z^2}{2}\left(T^2-\frac{1}{\overline{T}^2}\right)}}{\overline{T}^{k+1}} \sum_{l=0}^{k} \binom{k}{l} \left(\frac{z}{i\overline{T}}\right)^{k-l} c_l \tag{B.27}$$

The difference $f_k(z,\overline{z}) - \tilde{f}_k(z,\overline{z})$ is analytic and therefore

$$\partial_{\overline{z}} \tilde{f}_k(z,\overline{z}) = -\overline{z}^k e^{-z\overline{z}+\frac{\overline{T}^2}{2}\overline{z}^2+\frac{T^2}{2}z^2} \tag{B.28}$$

also.

Let

$$\begin{aligned} D_R^+ &:= \{z \colon |z| < R,\ \mathrm{Im}(Tz) > 0\} \\ D_R^- &:= \{z \colon |z| < R,\ \mathrm{Im}(Tz) < 0\}\ . \end{aligned} \tag{B.29}$$

$$\begin{aligned} \int_{\mathbb{C}} P_n(z)\overline{z}^k e^{-|z|^2+\frac{T^2z^2}{2}+\frac{\overline{T}^2\overline{z}^2}{2}} dA(z) &= \lim_{R\to\infty} \int_{D_R^+\cup D_R^-} P_n(z)\overline{z}^k e^{-|z|^2+\frac{T^2z^2}{2}+\frac{\overline{T}^2\overline{z}^2}{2}} dA(z) \\ &= \lim_{R\to\infty} \frac{i}{2}\left[\oint_{\partial D_R^+} P_n(z)\tilde{f}_k(z,\overline{z})dz + \oint_{\partial D_R^-} P_n(z) f_k(z,\overline{z})dz\right]\ . \end{aligned} \tag{B.30}$$

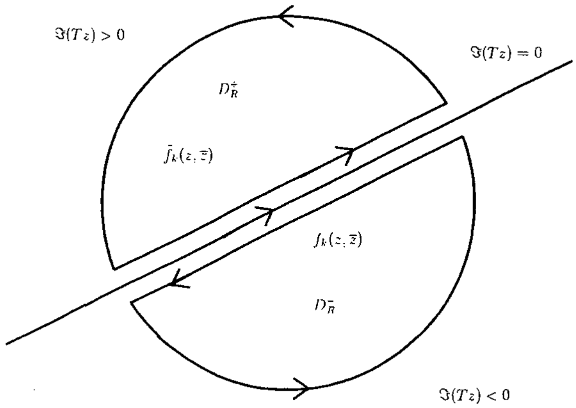

Figure B.1: Integration contours

The asymptotic behavior of f_k and $\tilde{f}_k$ imply that the contributions from integrals on the circular contours along $|z| = R$ tend to 0 as $R \to \infty$. The dominant contribution comes from the difference of the integrals on the line segment along $\mathrm{Im}(Tz) = 0$. Therefore

$$\begin{aligned}
&\int_{\mathbb{C}} P_n(z)\overline{z}^k e^{-|z|^2+\frac{T^2z^2}{2}+\frac{\overline{T}^2\overline{z}^2}{2}}\, dA(z) \\
&= \frac{i}{2}\int_{\mathrm{Im}(Tz)=0} P_n(z)\left[\tilde{f}_k(z,\overline{z}) - f_k(z,\overline{z})\right] dz \\
&= -\frac{i}{2}\int_{\mathrm{Im}(Tz)=0} P_n(z)\left[\frac{i^{k+1}e^{\frac{z^2}{2}\left(T^2-\frac{1}{\overline{T}^2}\right)}}{\overline{T}^{k+1}}\sum_{l=0}^{k}\binom{k}{l}\left(\frac{z}{i\overline{T}}\right)^{k-l} c_l\right] dz \\
&= \sum_{l=0}^{k}\frac{1}{2}\binom{k}{l}\frac{i^{k-l}}{\overline{T}^{k+l+1}}c_{k-l}\int_{\mathrm{Im}(Tz)=0} P_n(z)z^l e^{\frac{z^2}{2}\left(T^2-\frac{1}{\overline{T}^2}\right)}dz
\end{aligned} \tag{B.31}$$

where the orientation of the line $\mathrm{Im}(Tz) = 0$ is chosen in the direction of $\overline{T}$.

Since the equations above are triangular and $c_0 \neq 0$, the system of linear constraints

$$\int_{\mathbb{C}} P_n(z)\overline{z}^k e^{-|z|^2+\frac{T^2z^2}{2}+\frac{\overline{T}^2\overline{z}^2}{2}} dA(z) = 0 \qquad k = 0, 1, \ldots, n-1 \tag{B.32}$$

is equivalent to the system of contour integral conditions

$$\int_{\mathrm{Im}(Tz)=0} P_n(z) z^k e^{\frac{z^2}{2}\left(T^2-\frac{1}{T^2}\right)} dz = 0 \qquad k = 0, 1, \ldots, n-1 \ . \tag{B.33}$$

A simple change of coordinate

$$z = \frac{\sqrt{2}}{\sqrt{1-|T|^4}}\overline{T}u \qquad u \in \mathbb{R} \tag{B.34}$$

gives

$$\int_{-\infty}^{\infty} P_n\left(\frac{\sqrt{2}}{\sqrt{1-|T|^4}}\overline{T}u\right) u^k e^{-u^2} du = 0 \qquad k = 0, 1, \ldots, n-1 \ . \tag{B.35}$$

The unique monic polynomial that solves the above system is

$$\frac{1}{2^n} H_n(u) \ , \tag{B.36}$$

where $H_n(u)$ is the nth Hermite polynomial (using the conventions of [2]). Hence

$$P_n(z) = \frac{\overline{T}^n}{2^{\frac{n}{2}}(1-|T|^4)^{\frac{n}{2}}} H_n\left(\sqrt{\frac{1-|T|^4}{2|T|^4}}Tz\right) \ . \tag{B.37}$$

■

We note that the truncated moment matrix for the two-dimensional problem can be obtained from the truncated moment matrix of the contour integral setup:

$$\int_{\mathbb{C}} z^n \overline{z}^k e^{-|z|^2+\frac{T^2z^2}{2}+\frac{\overline{T}^2\overline{z}^2}{2}} dA(z) = \sum_{l=0}^{k} \frac{1}{2}\binom{k}{l} \frac{i^{k-l}}{\overline{T}^{k+l+1}} c_{k-l} \int_{\mathrm{Im}(Tz)=0} z^n z^l e^{\frac{z^2}{2}\left(T^2-\frac{1}{T^2}\right)} dz \ , \tag{B.38}$$

i.e.

$$M_{nk}^{(N)} = \sum_{l=0}^{N} m_{nl}^{(N)} S_{lk}^{(N)} \tag{B.39}$$

where

$$S_{lk}^{(N)} = \begin{cases} \dfrac{1}{2}\dbinom{k}{l}\dfrac{i^{k-l}}{\overline{T}^{k+l+1}}c_{k-l} & k \geq l \\ 0 & k > l\ . \end{cases} \tag{B.40}$$

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