Some Families of Complex Biorthogonal Polynomials in a Model of Non-commutative Quantum Mechanics

Nurisya Mohd Shah

A Thesis

In The Department

of

Physics

Presented in Partial Fulfillment of the Requirements

For the Degree of DOCTOR OF PHILOSOPHY at

Concordia University

Montréal, Québec, Canada

May 2014

CONCORDIA UNIVERSITY

School of Graduate Studies

This is to certify that the thesis prepared

By: **Nurisya Mohd Shah**

Entitled: **Some Families of Complex Biorthogonal Polynomials in a Model of Non-commutative Quantum Mechanics**

and submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY (Physics)

complies with the regulations of the University and meets the accepted standards with respect to originality and quality.

Signed by the final Examining Committee:

______________________________ Chair
Dr. A.K.W. Ahmed

______________________________ External Examiner
Dr. N. Saad

______________________________ External to Program
Dr. M. Bertola

______________________________ Examiner
Dr. M. Frank

______________________________ Examiner
Dr. P. Vasilopoulos

______________________________ Thesis Supervisor
Dr. S.T. Ali

Approved by ______________________________
Dr. L. Kalman, Graduate Program Director

May 12, 2014 ______________________________
Dr. J. Locke, Dean, Faculty of Arts & Science

Abstract

Some Families of Complex Biorthogonal Polynomials in A Model of Non-commutative Quantum Mechanics

Nurisya Mohd Shah, Ph.D

Concordia University, 2014

The main objective of this thesis is to study some new families of complex biorthogonal polynomials which arise in a model of non-commutative quantum mechanics (NCQM) in two dimensions. The NCQM model in which we are interested is a system with an extended Landau Hamiltonian, such as the one which arises when an electron is placed in a constant magnetic field. The new polynomials are deformed versions of the well-known complex Hermite polynomials, generated by the non-commutative raising and lowering operators. We work out in detail the construction of the polynomials and compute some of their useful properties, e.g., the associated generating functions and three-term recurrence relations. It is shown that relative to a non-commutative scaling factor, these polynomials form a class of biorthogonal complex polynomials and in a certain well-defined limit, they reduce to the standard complex Hermite polynomials.

The second objective of this thesis is to study the group theoretical properties of certain bilinear combinations of the non-commutative ladder operators. This is done following the well-known manner in which the angular momentum generators of standard quantum mechanics can be obtained from bilinear combinations of the raising and lowering operators of the harmonic oscillator. We study the Lie group structures that are generated by these operators, which again depend on a parameter characterizing the non-commutativity and which, in an appropriate limit, reduce to the standard quantum mechanical operators.

Acknowledgements

My special thanks go to my supervisor Professor S. Twareque Ali, for accepting me as his student and guiding me all the way from the very beginning.

I am also indebted to my co-supervisors, Professor Mariana Frank and Professor Panagiotis Vasilopoulos, for their comments and advice towards the completion of my thesis.

I would like to express my gratitude to Dr. F. Balogh for our discussions on the orthogonal polynomials. Also I would like to thank my fellow friends and the staff of both the Department of Mathematics and Statistics, and the Department of Physics for being very helpful and welcoming during my years of study at Concordia University.

I am also grateful to Professor P. Winternitz from the CRM and the Université de Montréal, for giving me some insight and additional knowledge on the topic of Lie algebras with a lively short discussion.

Also I would like to recognize the Ministry of Higher Education (MOHE) of Malaysia for the support I received under the Skim Latihan Akademik Institut Pengajian Tinggi (SLAI) Universiti Putra Malaysia (UPM) scholarship.

And finally, I would like to show my love and gratitude to my other half, my daughter and the rest of my family for their support and understanding during the past few years.

Faizal and Ain,

Contents

Introduction

The mathematical transition from classical mechanics to quantum mechanics is a well-known procedure in which the commutative algebra of classical observables is transformed to the *non-commutative* (NC) algebra of quantum mechanical observables. The quantum observables are represented by self-adjoint operators acting on a Hilbert space [60], which is most often taken to be some convenient function space.

As is well known, the main ingredient in the quantum formulation consists of the *canonical commutation relations*, between the operators of position $\hat{x}$ and momentum $\hat{p}$[1] [35]:

$$[\hat{x}_j, \hat{p}_k] = \hat{x}_j\hat{p}_k - \hat{p}_k\hat{x}_j = i\hbar\,\delta_{jk}, \qquad \text{for} \quad j,k = 1,2,\ldots,n\ , \tag{1}$$

where n is the number of degrees of freedom of the system, $\hbar$ the reduced Planck's constant and δ_{jk} the Kronecker delta (equal to 1 if $j = k$ and zero otherwise). The algebra generated by these operators, together with the identity operator I on the Hilbert space, is known as the Heisenberg algebra [36], which integrates to the group of the canonical commutation relations, the Weyl-Heisenberg group.

Our aim in this thesis is to work within a more general framework than that of standard quantum mechanics. In doing so we shall modify the canonical commutation relations, so that both the position and momentum operators will satisfy non-trivial commutation relations among themselves. We shall use the term non-commutative quantum mechanics (NCQM) for the resulting formalism, as is customary in the

[1] Later we shall drop the *hat* notation when there is no risk of confusion about the x, p denoting operators.

literature [68]. Using capital letters for the non-commuting position and momentum operators, the modified commutation relations are written as

$$[X_j, X_k] = i\Theta_{jk}, \quad [P_j, P_k] = i\widetilde{\Theta}_{jk}, \quad [X_j, P_k] = i\hbar\,\delta_{jk} \tag{2}$$

where Θ_{jk} and $\widetilde{\Theta}_{jk}$ are two constant, anti-symmetric tensors, $\Theta_{ij} = -\Theta_{ji}$ ($\widetilde{\Theta}_{ij} = -\widetilde{\Theta}_{ji}$) [55]. In the literature, the components of the constant tensors $(\Theta, \widetilde{\Theta})$ are called the non-commutative (NC) parameters. For a system with two degrees of freedom ($n = 2$), the two tensors are characterized by two constants, marking the non-commutativity of the two position and the two momentum operators, respectively. The non-commutativity of the two position operators is taken to reflect a non-standard structure of space at very short distances. The non-commutativity of the two momentum operators is analogous to having a constant magnetic field in the system with the associated Landau problem [21, 59, 73].

It is worthwhile highlighting the motivation for exploring non-commutativity in quantum systems at this level. To begin with, the route towards a NC theory may have come from purely mathematical considerations, noted in the work of Connes in the early 1980s on non-commutative geometry (NCG) [16, 17]. We refer to [47] for a very basic introduction to NCG, including historical facts, development of the subject and its applications, along with references cited there.

The question now is, how one can establish a model of NCQM. The key answers come from different and extremely rich mathematical formulations, which can all be traced back to the three basic formulations of quantum mechanics.

First we have the method of phase space quantization and the related deformation quantization, transforming the commutative product operation for classical observables to a non-commutative *star-product* to arrive at the NC level [23] . Other closely related formulations are the Weyl-Wigner formulation [8, 20], the Seiberg-Witten map formula [51, 66] and also the Moyal product (again a star product type of) formulation [11, 22, 23, 25, 28, 29, 46, 53]. Next there is also the method of path-integrals for arriving at a NCQM [1, 12]. This method has been widely used for field theoretic formulations of the theory [26].

Next, there are the low-dimensional quantum mechanical models which, in a way, are the most actively studied models within the standard operator approach to quantization. A seminal paper in this direction is [68]. Here the authors introduce a systematic formulation of an NCQM model for systems with two degrees of freedom, in which the configutation space is identified with a Hilbert space and the state space of the quantum system with the space of all Hilbert-Schmidt operators on it. It also includes immediate applications such as a non-commutative version of the harmonic oscillator [30, 31, 44, 58, 63, 66, 67, 72], and an electron in a constant magnetic field, leading to problems reminiscent of the well-known Landau levels in quantum mechanics [34, 45, 62].

Yet another direction of research in NCQM concerns itself with studying its group theoretical structure. In [13], the authors define the underlying group of two dimensional non-commutative quantum mechanics to be the triply centrally extended group of translations in $\mathbb{R}^4$, which they denote by G_{NC}. The different unitary irreducible representations (UIR) of G_{NC} then describe all the possible types of non-commutativities that may arise. The authors also show that that the invariance group of the Hilbert space representation of the Lie algebra, in the case where the two operators of position and those of momentum are non-commuting, is isomorphic to the symplectic group. In the case where only the two position operators are taken to be non-commuting, the underlying group can be shown [14] to be the (2 +1)-Galilei group (of two spatial plus one time dimensions) with two central extensions. Using the appropriate representations of this group coherent states can be constructed by deciding which coherent state quantization of the underlying phase space may be carried out. Such a quantization has been shown to yield the correct operators of position and momentum of non-commutative quantum mechanics.

However, our aim here is not so much to elucidate the basic structure of NCQM but to use the above mentioned operator theoretic formulation to study some associated families of generalized complex Hermite polynomials. In particular, we shall use the results from Kang *et. al* in [44] to construct NC raising and lowering operators. We will then use this type of *deformed operators*, on an abstract Hilbert space $\mathcal{H}$, and find the associated vectors which, adopting a specific representation on a Hilbert space of complex functions $\mathcal{H}(\mathbb{C})$, will turn out to be the above mentioned generalized Hermite polynomials. It will turn out that these polynomials, as vectors in this Hilbert space, will also be a basis for the space.

Interestingly, an analysis of the quantum mechanical problem of the Landau levels [3, 76], when formulated in the complex representation space mentioned above, shows that mathematically the problem is intimately connected with non-commutative quantum mechanics, in which the two operators of position and those of momentum are no longer taken to be commutative. Physically, the non-commutativity of the two position operators is taken to model a situation in which the geometry of space itself is considered to be non-commutative, while the non-commutativity of the two momentum operators is reminiscent of a situation where there is an external magnetic field in the problem. Thus, introducing the harmonic oscillator Hamiltonian, with the position and momentum operators replaced by their NC counterparts, has the appearance of an ordinary quantum mechanical oscillator coupled to a constant magnetic field (see, e.g. [21]).

The solution to the problem of the Landau levels in above mentioned complex setting is well known and involves the so-called complex Hermite polynomials [32, 40]. These polynomials, which have also been studied in [82], and have been called Laguerre 2D-polynomials, form an orthonormal basis of $\mathcal{H}(\mathbb{C})$ and are obtainable in a standard manner, using powers of the usual creation operators acting on the ground state of the harmonic oscillator. By an entirely analogous method, replacing the two creation operators by their NC counterparts, the *deformed Hermite polynomials* are then obtained.

A simple group theoretical analysis then allows us to write these deformed polynomials as linear combinations of the undeformed ones, the linear transformation operator being the matrix of a certain group representation. While the undeformed complex Hermite polynomials constitute an orthonormal basis of $\mathcal{H}(\mathbb{C})$, their deformed counterparts no longer have this property. However, as we show here, it is possible to find a second family of such polynomials which are *biorthogonal* with respect to the first [6], and indeed, the computation and properties of this dual pair of biorthogonal polynomials will be the main contribution of our thesis. Some similar results, using a different construction have also been discussed in [48, 82].

The deformed polynomials of NC quantum mechanics will be shown to depend on a parameter α which ranges between 0 and 1. In the limit of $\alpha \to 0$ one recovers the undeformed polynomials, while at the other limit, $\alpha \to 1$, one obtains their complex conjugates.

As the final result in NCQM obtained in this thesis, we look at the Lie algebra generated by bilinear combinations of the deformed creation and annihilation operators. The Lie algebra generated by the undeformed bilinear operators is just the $\mathfrak{so}(3)$ [2] algebra. The deformed bilinear combinations lead to more general group algebras, which are different for different ranges of values of α.

Problem Highlights

Most of literature mentioned above deals with the fundamental features that have to be incorporated in formulating the idea of non-commutative quantum mechanics. We have identified here three different approaches to the problem. Most often, the key feature is to introduce a modification of the canonical commutation relations of standard quantum mechanics. It is this feature that also motivates the present study. However, here we go further than in the current literature, in that we look at yet another aspect of the problem namely, that of certain associated classes of complex biorthogonal polynomials. These are seen to arise directly from the altered commutation relations. What we gain in the process is a deeper understanding of noncommutativity, but this time in terms of the appearance of this polynomial class. One could try to go further using, for instance, results from the operator theoretic formulation discussed in [44]. These should lead to polynomials in more than two variables. Our approach is interesting and useful since it links the study of NCQM to the field of orthogonal polynomials in an intrinsic manner. We ought to mention however, that the association of quantum mechanics to orthogonal polynomials is not new. For example, the Hermite and Laguerre polynomials appear in standard quantum mechanics when one studies the harmonic oscillator and the hydrogen atom Hamiltonians, and there are many others. Similarly, the complex orthogonal polynomials appear when one studies the Landau problem of an electron in a constant magnetic field, formulated on a Hilbert space over the complexes [32, 40]. The appearance of our biorthogonal polynomials, however, is linked directly to the commutation relations, in other words to the very structure of NCQM. This we feel is a novel aspect of the general problem.

Outline of Thesis

The thesis is divided into two parts.

Part I

In the first part, the main results we obtain are families of biorthogonal polynomials, which we call *deformed generalized* Hermite polynomials, which are polynomials in a complex variable z and its conjugate $\overline{z}$ and live in a complex Hilbert space $\mathcal{H}(\mathbb{C})$. In doing this we use the results from [44] on the NC raising and lowering operators together with [3] on the complex Hermite polynomials as a starting point.

Next, we systematically generate the deformed polynomials, this involves a matrix transform operation on the basis of undeformed polynomials. We also present the matrix transform explicitly.

An important feature of these generalized families of polynomials is their biorthogonality property. We prove that while a given family of these polynomials does not constitute an orthogonal set, there is a dual set of such polynomials which are biorthogonal with respect to the first.

In addition, we also obtain other standard properties of these polynomials, such as their generating function and three-term recurrence relations.

Part II

In the second part of the work, we look at bilinear combinations of the deformed raising and lowering operators and write them in terms of their undeformed counterparts. While the undeformed bilinear combinations generate the Lie algebra of the $SU(2) \times U(1)$ group, the undeformed operators also give rise to second and different Lie algebra, for one particular value of the NC deformation parameter, in the range of values considered.

To summarize:

1. In Chapter 1, we present the mathematical tools that will be essential to our work.

2. In Chapter 2, we give an overview of the formulation of the non-commutative (NC) algebra following Kang *et.al.*, where a parameter denoted by Θ was used to indicate the strength of the non-commutativity. The NC creation and annihilation (raising and lowering) operators are also written in terms of the deformed or NC position and momentum operators and these are then used to define an NC harmonic oscillator model.

3. Chapter 3 contains the first set of main results of the thesis. We obtain a generalized basis of deformed vectors in an abstract Hilbert space. The basis is denoted by $|k, \ell\rangle_g$ parameterized by a group element $g \in GL(2, \mathbb{C})$. Using a unitary map introduced in [3] we next construct the associated deformed complex vectors in $\mathcal{H}(\mathbb{C})$ which we denote by $H^g_{k,\ell}(z, \overline{z})$ and thus define our deformed generalized Hermite polynomials. We present a formula giving the associated generating function for these polynomials and work out their three term recurrence relations. Information on the orthogonality properties are also given before we end this chapter.

4. In Chapter 4, we develop a complete procedure to compute the transformation matrix of the deformed generalized Hermite polynomials. We discuss some intertwining relations and interesting characteristic properties of these polynomials with regard to this matrix.

5. In Chapter 5, we give the definitions of biorthogonal bases of polynomials and further show the biorthogonality that arises within families of the deformed generalized Hermite polynomials.

6. In Chapter 6, we discuss the second part of results of this thesis. We analyze the group structure arising from the Lie algebra formed by taking bilinear combinations of the deformed raising and lowering operators. This leads to modified rotation-like operators.

7. In Chapter 7, ends with some concluding remarks.

We also include an appendix detailing the steps of some of the computations appearing in Chapter 6.

Chapter 1

Mathematical Preliminaries

In this chapter, we review some mathematical concepts and tools that play an essential role in the development of this work. References to this material are among them we mention in particular we mention Griffiths [35], Tannoudji, *et.al* [76], Messiah [56], Prugovečki [60], Weber-Arfken [80] and Reed & Simon [61].

1.1 Conventions

1.1.1 Notation

This subsection is intended for quick reference only. The individual notations will be properly introduced where they appear in the text for the first time.

- The spaces of the real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. Generically we shall denote them by $\mathbb{F}$.
- Imaginary unit $i = \sqrt{-1}$.
- Vector spaces are denoted by $U, V, W, \ldots$. Elements in a space such as U are denoted by $u_1, u_2, \ldots u_n, \ldots$.

- The Hermitian adjoint of an operator A will be denoted by i.e $A^\dagger$; if $a_{ij} \in \mathbb{C}$ are the matrix elements of A then $\overline{a}_{ji}$ are the elements of $A^\dagger$.

- The "bra-ket" notation. We follow the usual physics convention when writing the inner product in the bra and ket notation. The quantity $\langle f|g\rangle$ is conjugate linear in the first entry and linear in the second.

- The Kronecker delta function δ_{jk} is defined as

$$\delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}.$$

- Following the usual convention, we denote a Lie group in general by G and its Lie algebra by $\mathfrak{g}$, i.e., using the corresponding small Gothic letter.

- In most cases, we denote operators, polynomials, and physical observables which relate to standard (as opposed to non-commutative) quantum mechanics using a lower case symbol. Thus, $a, a^\dagger, h(z, \overline{z}), x_j, p_j$ correspond to the lowering and raising operators, complex Hermite polynomials, position and momentum coordinates, respectively.

1.1.2 Layout

The numbering of definitions, theorems, remarks, etc., are according to their section. For example **Definition 1.1.1** comes just after section 1.1. Proofs of some theorems, where applicable, will follow just after the statement of the theorem. The end of a proof will be signalled by $\blacksquare$. Equations however are numbered sequentially within each Chapter. For example, Chapter 2 starts with equation (2.1), followed by (2.2), etc. Subequations are written as (2.1a), (2.1b) etc.

1.2 Abstract basis in $\mathcal{H}$ generated by the bosonic operators

Let $\mathcal{H}$ be an abstract Hilbert space equipped with the inner product $\langle *|*\rangle_{\mathcal{H}}$, which is conjugate linear in the first entry and linear in the second entry. Also let a_j and $a_k^\dagger,\ j,k = 1,2$ be a pair of the usual lowering and raising operators, (also known as the bosonic operators) [78], satisfying the commutation relations

$$[a_j, a_k^\dagger] = \delta_{jk}\ , \quad j,k = 1,2\ . \tag{1.1}$$

Let $|0,0\rangle$ denote the vacuum state vector, for which

$$a_j\,|0,0\rangle = 0, \quad \text{for} \quad j = 1,2 \tag{1.2}$$

An orthonormal basis of vectors, $|k,\ell\rangle\ ,\ k,\ell = 0,1,2,\ldots\infty$, for $\mathcal{H}$ can be built in the manner [76]

$$|k,\ell\rangle = \frac{1}{\sqrt{k!\ell!}}(a_1^\dagger)^k(a_2^\dagger)^\ell\,|0,0\rangle\ , \quad k,\ell = 0,1,2,\ldots \tag{1.3}$$

The operators a_j and $a_j^\dagger$ are also called the bosonic operators. The algebra generated by them and the identity operator on $\mathcal{H}$, under the commutation relations (1.1), is called the bosonic algebra [78].

The action of the raising operator on the vector $|k,\ell\rangle$ is given by

$$a_1^\dagger\,|k,\ell\rangle = \sqrt{k+1}\,|k+1,\ell\rangle\ ,\ a_2^\dagger\,|k,\ell\rangle = \sqrt{\ell+1}\,|k,\ell+1\rangle\ . \tag{1.4}$$

Further, the number operators is define

$$N_j = a_j^\dagger a_j, \qquad \text{for} \quad j = 1,2 \tag{1.5}$$

which act on the vector $|k,\ell\rangle \in \mathcal{H}$ as follows

$$N_1|k,\ell\rangle = k|k,\ell\rangle, \tag{1.6a}$$

$$N_2|k,\ell\rangle = \ell|k,\ell\rangle. \tag{1.6b}$$

To support (1.4) and (1.6) we state the following Lemma [76]

Lemma 1.2.1. *Let $|k,\ell\rangle$ be a non-zero eigenvector of N_j with the eigenvalue k,ℓ. Then for $j = 1,2$ [76] (page 492, of Vol. I):*

i. Properties of the vector $a_j|k,\ell\rangle$

a. If $|k,\ell\rangle = |0,0\rangle$, the ket $a_j\,|0,0\rangle$ is zero.

b. If $k \neq 0$ and $\ell \neq 0$, the kets $a_1|k,\ell\rangle$, $a_2|k,\ell\rangle$ are non-zero eigenvectors of N_1, N_2 with eigenvalues $k-1$ and $\ell-1$ respectively.

ii. Properties of the vector $a_j^\dagger|k,\ell\rangle$

a. $a_j^\dagger\,|0,0\rangle$ is always non-zero.

b. $a_1^\dagger|k,\ell\rangle$, $a_2^\dagger|k,\ell\rangle$ is an eigenvector of N_1, N_2 with eigenvalue $k+1$ and $\ell+1$ respectively.

1.3 Realization of the complex Hilbert space $\mathcal{H}(\mathbb{C})$

Let $\mathbb{R}$ and $\mathbb{C}$ denote the set of all real and complex numbers, respectively. In generical, while referring to them both together we shall use the symbol $\mathbb{F}$. We shall mainly be working with the complex Hilbert space $\mathcal{H}(\mathbb{C}) = \mathcal{L}^2(\mathbb{C}, d\nu(z,\overline{z}))$, where $d\nu$ is the measure

$$d\nu(z,\overline{z}) = \frac{e^{-|z|^2}}{2\pi}\frac{dz\wedge d\overline{z}}{i} = \frac{e^{-|z|^2}}{2\pi}dx_1\,dx_2, \tag{1.7}$$

and by convention we write $z = (x_1 - ix_2)/\sqrt{2}$, in terms of its real and imaginary parts. Vectors in this Hilbert space are functions $f(z, \overline{z})$ of the complex variable z and its complex conjugate $\overline{z}$, satisfying

$$\|f\| := \int_{\mathbb{C}} |f(z, \overline{z})|^2 \, d\nu(z, \overline{z}) < \infty.$$

Elements in $\mathcal{H}(\mathbb{C})$ can also be thought of as complex valued functions of the real variables x_1 and x_2 and it is easy to see that the map $W : \mathcal{H}(\mathbb{C}) \longrightarrow \mathcal{L}^2(\mathbb{R}^2, dx_1\, dx_2)$ given by

$$(Wf)(z, \overline{z}) = \sqrt{2\pi} e^{\frac{|z|^2}{2}} f(x_1, x_2), \tag{1.8}$$

is a Hilbert space isometry. We also introduce below a transform $\mathcal{V}$ [3, 5] which will unitarily map the abstract Hilbert space $\mathcal{H}$ to $\mathcal{H}(\mathbb{C})$.

1.3.1 The Wigner transform

Let $\mathcal{K}$ be an abstract Hilbert space on which the two operators $a, a^\dagger$, satisfying $[a, a^\dagger] = I$ are irreducibly realized. Let $\mathcal{B}_2(\mathcal{K})$ be the Hilbert space of all Hilbert-Schmidt operators on it, under the trace norm. The scalar product of two elements $X, Y \in \mathcal{B}_2(\mathcal{K})$ is given by

$$\langle X \mid Y \rangle := \mathrm{Tr}[X^\dagger Y]. \tag{1.9}$$

Defining an orthonormal basis $|n\rangle,\ n = 0, 1, 2, \ldots$, in $\mathcal{K}$ in the manner,

$$a|0\rangle = 0, \qquad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle,$$

we may define an orthonormal basis in $\mathcal{B}_2(\mathcal{K})$ by the vectors

$$X_{k,\ell} = |k\rangle\langle\ell|, \quad k, \ell = 0, 1, 2, \ldots.$$

Using the orthonormal basis (1.3) of the Hilbert space $\mathcal{H}$ we define an isometric linear map $\mathcal{V} : \mathcal{H} \longrightarrow \mathcal{B}_2(\mathcal{K})$ by setting

$$\mathcal{V}|k, \ell\rangle = X_{k,\ell}, \quad k, \ell = 0, 1, 2, \dots . \tag{1.10}$$

Next we introduce the well-known Wigner transform $\mathcal{W}$ which isometrically maps $\mathcal{B}_2(\mathcal{K})$ to $\mathcal{H}(\mathbb{C})$. This map is defined as [5]

$$(\mathcal{W}X)(z, \overline{z}) = \mathrm{Tr}[e^{-za^\dagger} X e^{\overline{z}a}]\, e^{z\overline{z}} = \mathrm{Tr}[e^{\overline{z}a} X e^{-za^\dagger}], \quad X \in \mathcal{B}_2(\mathcal{K}). \tag{1.11}$$

The composite transform $\widetilde{\mathcal{W}} = \mathcal{W}\mathcal{V}$, which we shall refer to as the Wigner map, is a linear isometry between $\mathcal{H}$ and $\mathcal{H}(\mathbb{C})$.

1.3.2 The bosonic algebra on $\mathcal{H}(\mathbb{C})$

It is well known [32, 40] that the bosonic operators a_j and $a_j^\dagger$, defined on the abstract Hilbert space $\mathcal{H}$ have the following realization on $\mathcal{H}(\mathbb{C})$ via differential operators:

$$a_1 = \frac{\partial}{\partial z}, \qquad a_2 = \frac{\partial}{\partial \overline{z}}, \tag{1.12a}$$

$$a_1^\dagger = z - \frac{\partial}{\partial \overline{z}}, \qquad a_2^\dagger = \overline{z} - \frac{\partial}{\partial z}, \tag{1.12b}$$

which are in fact the transforms of the abstract operators on $\mathcal{H}$ under the map Wigner transform $\widetilde{\mathcal{W}}$. Similarly, the basis vectors $|k, \ell\rangle$ in (1.3) transform to the functions $\Psi_{k,\ell}(z, \overline{z}) \in \mathcal{H}(\mathbb{C})$

$$|k, \ell\rangle \mapsto \Psi_{k,\ell} = \widetilde{\mathcal{W}}|k, \ell\rangle. \tag{1.13}$$

Thus the space $\mathcal{H}(\mathbb{C})$ is spanned by the vectors

$$\begin{aligned}\Psi_{k,\ell}(z,\overline{z}) &= \frac{1}{\sqrt{k!\ell!}}(a_1^\dagger)^k(a_2^\dagger)^\ell\Psi_{0,0}(z,\overline{z}) \ , \\ &= \frac{1}{\sqrt{k!\ell!}}\Big(z-\frac{\partial}{\partial\overline{z}}\Big)^k\Big(\overline{z}-\frac{\partial}{\partial z}\Big)^\ell\Psi_{0,0}(z,\overline{z}),\end{aligned} \tag{1.14}$$

where

$$\Psi_{0,0}(z,\overline{z}) = 1, \qquad \forall\,(z,\overline{z}) \tag{1.15}$$

is a unit vector in $\mathcal{H}(\mathbb{C})$.

It is not hard to see that the functions $\Psi_{k,\ell}$ are polynomials, of order $k+\ell$ in the variables z and $\overline{z}$, which we shall identify below with the complex Hermite polynomials (up to normalization).

1.3.3 Holomorphic functions

We shall be identifying two special subspaces of $\mathcal{H}(\mathbb{C})$, consisting of holomorphic and antiholomorphic functions. For this we shall use the following holomorphic and anti-holomorphic differential operators [3]

$$\begin{aligned}\frac{\partial}{\partial z} &= \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x_1}+i\frac{\partial}{\partial x_2}\Big), \\ \frac{\partial}{\partial \overline{z}} &= \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x_1}-i\frac{\partial}{\partial x_2}\Big).\end{aligned} \tag{1.16}$$

For a function f that is continuously differentiable to be holomorphic, the derivative of f with respect to $\overline{z}$ must equal to zero, i.e.,

$$\frac{\partial f}{\partial\overline{z}}(z) = 0 \ , \tag{1.17}$$

with a similar definition for antiholomorphic functions.

1.3.4 The Bargmann space

Consider the two sets of vectors $\Psi_{0,\ell}, \Psi_{k,0},\ \ell, k = 0, 1, 2, \ldots$ with

$$\Psi_{0,\ell}(z,\overline{z}) = \frac{1}{\sqrt{\ell!}}\overline{z}^{\ell} \quad \text{and} \quad \Psi_{k,0}(z,\overline{z}) = \frac{1}{\sqrt{k!}}z^{k}. \tag{1.18}$$

Taking the linear span of the vectors $\Psi_{k,0},\ k = 0, 1, 2, \ldots, \infty$, and closing it in the norm of $\mathcal{H}(\mathbb{C})$ gives us a subspace of analytic functions [7]. This space, which we shall denote by $\mathcal{B}$, is generally referred to as the Bargmann space. Elements of this space are functions $f(z)$ of z alone. Similarly, the set of vectors $\Psi_{0,\ell}, \ell = 0, 1, 2, \ \ldots, \infty$, spans a subspace of $\mathcal{H}(\mathbb{C})$ which consists of antiholomorphic functions. The vector $\Psi_{0,0}$ belongs to both subspaces, however its orthogonal complements in the holomorphic and antiholomorphic subspaces are mutually orthogonal.

1.4 The complex Hermite polynomials

As mentioned earlier, the complex Hermite polynomials, in the variables $z, \overline{z}$ are just the basis vectors $\Psi_{k,\ell}(z,\overline{z})$, up to normalization. We now look at them more closely.

1.4.1 Definition of classical complex Hermite polynomials

Consider the Gaussian function

$$F(z,\overline{z}) = e^{-|z|^2}. \tag{1.19}$$

Computing the successive derivatives of F for $k, \ell = 0, 1, 2, \ldots,$ we find

$$\begin{aligned} &\frac{\partial}{\partial z}F(z,\overline{z}) = -2\overline{z}F(z,\overline{z}), \qquad \frac{\partial}{\partial \overline{z}}F(z,\overline{z}) = -2zF(z,\overline{z}),\\ &\frac{\partial^2}{\partial z^2}F(z,\overline{z}) = 4\overline{z}^2F(z,\overline{z}), \quad \frac{\partial^2}{\partial \overline{z}^2}f(z,\overline{z}) = 4z^2F(z,\overline{z}),\\ &\frac{\partial}{\partial z\partial \overline{z}}F(z,\overline{z}) = (4|z|^2-2)F(z,\overline{z}), \quad \text{etc.} \end{aligned}$$

The (k,ℓ)-th order derivative, $F^{(k,\ell)}(z,\overline{z})$ can be written as

$$F^{(k,\ell)}(z,\overline{z}) = \frac{(-1)^{k+\ell}}{\sqrt{k!\,l!}}h^c_{k,\ell}(z,\overline{z})e^{-|z|^2}. \tag{1.20}$$

where $h^c_{k,\ell}(z,\overline{z})$ is a $(k+\ell)$-th degree polynomial in $(z,\overline{z})$.

Definition 1.4.1. *The polynomials $h^c_{k,\ell}(z,\overline{z})$ are called the classical (or, sometimes, commutative) complex Hermite polynomials. They are explicitly derivable using the Rodrigues formula*

$$h^c_{k,\ell}(z,\overline{z}) = \frac{(-1)^{k+\ell}}{\sqrt{k!\,l!}}e^{|z|^2}\frac{\partial^k}{\partial z^k}\frac{\partial^\ell}{\partial \overline{z}^\ell}e^{-|z|^2}, \tag{1.21}$$

for all $k, \ell = 0, 1, 2, \ldots$.

The word commutative in the above definition does not refer to any mathematical commutativity property. Rather it is an unfortunate usage which refers to the fact these polynomials are associated to standard quantum mechanics, as opposed to non-commutative quantum mechanics.

It is also customary to use the *unnormalized* polynomials

$$H_{m,n}(z,\overline{z}) = \sqrt{m!n!}\; h^c_{m,n}(z,\overline{z}) = (-1)^{k+\ell}e^{|z|^2}\frac{\partial^k}{\partial z^k}\frac{\partial^\ell}{\partial \overline{z}^\ell}e^{-|z|^2}, \tag{1.22}$$

and in the sequel we shall be using them both. The $H_{m,n}$ can explicitly be written as [57]

$$H_{m,n}(z,\overline{z}) = \sum_{k=0}^{m\wedge n}(-1)^k k!\binom{m}{k}\binom{n}{k} z^{m-k}\overline{z}^{n-k}. \tag{1.23}$$

where $m \wedge n$ denotes the smaller of the two integers m and n.

1.4.2 Some properties of $\{H_{m,n}(z,\overline{z})\}$

In this section, we collect some useful known properties of the complex Hermite polynomials, mainly for later comparison with the case of the deformed complex Hermite polynomials. Several of these properties appear in [57].

Let

$$F(z+u,\overline{z}+\overline{u}) = e^{-(z+u)(\overline{z}+\overline{u})}. \tag{1.24}$$

According to [76], we may write

$$\begin{aligned} F(z+u,\overline{z}+\overline{u}) &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{u^k\overline{u}^\ell}{\sqrt{k!\ell!}}F^{(k,\ell)}(z,\overline{z}), \\ &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{u^k\overline{u}^\ell}{k!\ell!}(-1)^{k+\ell}h^c_{k,\ell}(z,\overline{z})e^{-|z|^2}. \end{aligned}$$

Multiplying this relation by $e^{|z|^2}$ and replacing $(u,\overline{u})$ with $(-u,-\overline{u})$ we obtain

$$e^{|z|^2}F(z-u,\overline{z}-\overline{u}) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{u^k\overline{u}^\ell}{k!\,\ell!}h^c_{k,\ell}(z,\overline{z}), \tag{1.25}$$

Replacing $F(z-u,\overline{z}-\overline{u})$ by its explicit value we get,

$$e^{|z|^2}e^{-(z-u)(\overline{z}-\overline{u})} = e^{-u\overline{u}+u\overline{z}+\overline{u}z}. \tag{1.26}$$

Thus the classical complex Hermite polynomials $h^c_{k,\ell}(z,\overline{z})$ are given in terms of their generating function by the expression

$$F(u,\overline{u};z,\overline{z}) := e^{-u\overline{u}+uz+\overline{u}\overline{z}} = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{u^k\overline{u}^\ell}{k!\ell!}(-1)^{k+\ell} h^c_{k,\ell}(z,\overline{z})e^{-|z|^2} \tag{1.27}$$

The following is a generalization of the Kibble-Slepian formula [57] for real Hermite polynomials to the (classical) complex Hermite polynomials $\{H_{m,n}(z,\overline{z})\}$.

Theorem 1.4.1. *Let $Z = (z_1, z_2, \ldots, z_N)$, and H be an $N \times N$ Hermitian matrix with matrix elements $h_{j,k}$, $|H| < 1$, and I_N is an $N \times N$ identity matrix. Then*

$$\begin{aligned} &Det[(1_N - H)^{-1}]\,exp(ZH(1_N - H)^{-1}Z^*) \\ &\quad = \sum_K \Pi_{1\le j,k\le N} \frac{(h_{j,k})^{n_{j,k}}}{n_{j,k}!} H_{r_1,c_1}(z_1,\overline{z}_1)\ldots H_{r_N,c_N(z_N,\overline{z}_N)}, \end{aligned} \tag{1.28}$$

where $K = (n_{j,k} : 1 \le j,k \le N)$ is a general matrix with nonnegative integer entries, c_k is the sum of the elements of K in column k and r_j is the sum of the elements of K in row j, that is

$$c_k = \sum_{k=1}^{N} n_{j,k}\,; \qquad r_k = \sum_{j=1}^{N} n_{j,k}. \tag{1.29}$$

The complex Hermite polynomials satisfy three term recurrence relations of the first and second kind, namely [3, 57],

First kind

$$h^c_{k+1,\ell}(z,\overline{z}) = zh^c_{k,\ell}(z,\overline{z}) - \ell h^c_{k,\ell-1}(z,\overline{z}), \qquad \text{or} \tag{1.30a}$$

$$h^c_{k+1,\ell+1}(z,\overline{z}) = zh^c_{k,\ell+1}(z,\overline{z}) - (\ell+1)h^c_{k,\ell}(z,\overline{z}), \tag{1.30b}$$

Second kind

$$h^c_{k,\ell+1}(z,\overline{z}) = \overline{z}h^c_{k,\ell}(z,\overline{z}) - kh^c_{k-1,\ell}(z,\overline{z}), \qquad \text{or} \tag{1.31a}$$

$$h^c_{k+1,\ell+1}(z,\overline{z}) = \overline{z}h^c_{k+1,\ell}(z,\overline{z}) - (k+1)h^c_{k,\ell}(z,\overline{z}), \tag{1.31b}$$

thus also, subtracting (1.31b) from (1.30b) yields

$$(\ell - k)h^c_{k,\ell}(z,\overline{z}) = zh^c_{k,\ell+1}(z,\overline{z}) - \overline{z}h^c_{k+1,\ell}(z,\overline{z}). \tag{1.32}$$

There is a relation connecting the complex Hermite polynomials to the real Hermite polynomials $H_n(x)$, given by [5]

$$H_{m,n}(z,\overline{z}) = m\,!n\,!\Big(\frac{i}{2}\Big)^{m+n}\sum_{j=0}^{m}\sum_{k=0}^{n}\frac{i^{j+k}}{j\,!k\,!}(-1)^{j+n}\frac{H_{j+k}(x)H_{m+n-j-k}(y)}{(m-j)!(n-k)!}. \tag{1.33}$$

The polynomials $h^c_{k,\ell}(z,\overline{z})$ satisfy the following orthogonality relations: $h^c_{k,\ell}(z,\overline{z})$. They satisfy the orthogonality relation [32] and [40]

$$\int_{\mathbb{C}} \overline{h^c_{m,n}(z,\overline{z})}h^c_{k,\ell}(z,\overline{z})d\nu(z,\overline{z}) = \delta_{mk}\delta_{nl}. \tag{1.34}$$

Finally, there is the following useful relation which can be used to explicitly compute the complex Hermite polynomials [57].

Theorem 1.4.2. *The following operational representation holds:*

$$H_{m,n}(z,\overline{z}) = exp(-\partial_z\partial_{\overline{z}})(z^m\overline{z}^n). \tag{1.35}$$

Chapter 2

General Discussion of Non-commutative Algebras

Introduction

This chapter presents an overview on the subject of non-commutative algebras in quantum mechanics. To reiterate, by non-commutativity we mean here a problem in which the standard N-bosonic algebra is replaced by one involving additional non-zero commutators, in a specific manner.

We first discuss the formalism of the non-commutative algebra following the lines of Kang *et.al* [44, 45, 46] and detail out the coordinate representation with respect to the properties of the algebra. The last part will involve the implementation of the new set of NC coordinates in a simple quantum harmonic oscillator system.

Remark 2.0.1. ***Notations used for non-commutative configuration space.*** *We will denote the coordinates of the configuration space and operators in the non-commutative system by an upper case letter. For example (X, P) for the position and momentum coordinate, respectively. Coordinates and operators in the commutative case will be written using lower case letters (x, p). However, for certain types of* *operators, for example the Hamiltonian, we will use a "hat" notation to signal the NC case.*

2.1 Formulation of the non-commutative algebra

As just mentioned, our discussion of the NC algebra will follow [44] and we shall only mention the main results. This, of course, is not the only possible formulation of the NC problem. For example, there is also the setup using complex coordinates as in [31, 68].

2.1.1 The algebra

We begin with the well-known abstract Heisenberg algebra involving the position and momentum coordinates of a quantum system with d degrees of freedom, obeying the following commutation rules

$$[x_j, x_k] = 0, \tag{2.1a}$$

$$[p_j, p_k] = 0, \tag{2.1b}$$

$$[x_j, p_k] = i\hbar\delta_{jk}, \qquad \text{for} \quad j, k = 1, 2, \ldots, d. \tag{2.1c}$$

In case the configuration is only two-dimensional ($d = 2$), the commutation relations of this algebra can be summarized in the table below

	x_1	p_1	x_2	p_2
x_1	0	$i\hbar$	0	0
p_1	$-i\hbar$	0	0	0
x_2	0	0	0	$i\hbar$
p_2	0	0	$-i\hbar$	0

The set up of the non-commutative algebra is in fact a straightforward mathematical construction. Following (2.1), we write down a *deformed Heisenberg algebra* by introducing a non-commutative parameter Θ which is sort of a Planck's constant.

Let X_j and P_j for $j = 1, 2, \ldots, N$, represent the position and momentum coordinates in a NC phase space. Their commutation rules are given by

$$[X_j, X_k] = i\hbar\Theta_{jk}, \tag{2.2a}$$

$$[P_j, P_k] = i\hbar\widetilde{\Theta}_{jk}, \tag{2.2b}$$

$$[X_j, P_k] = i\hbar\,\delta_{jk}, \tag{2.2c}$$

where $\Theta_{ij} = -\Theta_{ji}$, $\widetilde{\Theta}_{ij} = -\widetilde{\Theta}_{ji}$ and $\Theta_{ii} = \widetilde{\Theta}_{ii} = 0$ for $i, j = 1, 2, \ldots$. In the two dimensional case this deformed Heisenberg algebra (NC algebra) is summarized in the table below

	X_1	P_1	X_2	P_2
X_1	0	$i\hbar$	$i\hbar\Theta$	0
P_1	$-i\hbar$	0	0	$i\hbar\widetilde{\Theta}$
X_2	$-i\hbar\Theta$	0	0	$i\hbar$
P_2	0	$-i\hbar\widetilde{\Theta}$	$-i\hbar$	0

2.2 Linear transformation

It is possible to write the non-commutative coordinates as linear combinations of the canonical or standard coordinates. Conversely, it should also be possible to deduce the canonical set of coordinates from the non-commutative ones. In general, the relations are taken to have the form

$$X_j = a_{jk}x_l + b_{jk}p_l, \tag{2.3a}$$

$$P_j = c_{jk}x_l + d_{jk}p_l, \tag{2.3b}$$

parameterized by a $N \times N$ block matrix, consisting of four blocks A, B, C, D, having elements $\{a_{jk}\}, \{b_{jk}\}, \{c_{jk}\}$ and $\{d_{jk}\}$, respectively, with $j, k = 1, 2, \ldots, N$. Obviously, equations (2.3) define a matrix transformation which can be written in the matrix form

$$\begin{pmatrix} X \\ P \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}, \tag{2.4}$$

or explicitly, in the two dimensional case,

$$\begin{pmatrix} X_1 \\ X_2 \\ P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ c_{11} & c_{12} & d_{11} & d_{12} \\ c_{21} & c_{22} & d_{21} & d_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ p_1 \\ p_2 \end{pmatrix}.$$

We now state a result giving the characteristic forms of the representation matrices.

Theorem 2.2.1. *Let (x_j, p_j) and (X_j, P_j) satisfy equations (2.1) and (2.2). Then* *the matrices A, B, C, D satisfy*

$$AB^{tr} - BA^{tr} = \Theta, \tag{2.5}$$

$$CD^{tr} - DC^{tr} = \widetilde{\Theta}, \tag{2.6}$$

$$AD^{tr} - BC^{tr} = \mathbf{1} \tag{2.7}$$

where A^{tr} denotes the transpose of the matrix A and $\mathbf{1}$ is the unit (or identity) matrix.

Proof. All we need to show is the computation of the commutation relations between the NC coordinates (X_j, P_j), written in terms of the canonical ones (x_j, p_j)

$$\begin{aligned}
[X_j, X_k] &= [a_{jk}x_l + b_{jk}p_l, a_{kj}x_l + b_{kj}p_l] = i\hbar\Theta, \\
&= [a_{jk}x_l, b_{kj}p_l] + [b_{jk}p_l, a_{kj}x_l], \\
&= [a_{jk}x_l, b^{tr}_{jk}p_l] + [b_{jk}p_l, a^{tr}_{jk}x_l], \\
&= a_{jk}b^{tr}_{jk}[x_l, p_l] + b_{jk}a^{tr}_{jk}[p_l, x_l], \\
&= i\hbar\, a_{jk}b^{tr}_{jk} - i\hbar\, b_{jk}a^{tr}_{jk}, \\
&= i\hbar(a_{jk}b^{tr}_{jk} - b_{jk}a^{tr}_{jk}).
\end{aligned}$$

Immediately, (2.5) follows. The same method can be applied to prove (2.6) and (2.7).

■

Theorem 2.2.2. *Suppose, A and D are proportional to identity matrix by scaling factors, which denote by α and β, respectively, which cannot both be zero. Then, (2.5), (2.6) and (2.7) can be reduced to*

i. $\alpha(B^{tr} - B) = \Theta$,

ii. $\beta(C - C^{tr}) = \widetilde{\Theta}$,

iii. $BC^{tr} = \alpha\beta - 1$.

Theorem 2.2.3. *Let B and C be symmetric matrices. Then, $\Theta = \tilde{\Theta} = 0$,which give us back the canonical (or commutative) phase space.*

i. If B and C both be antisymmetric matrices, then

$$B = -\frac{1}{2\alpha}\Theta, \qquad C = \frac{1}{2\beta}\widetilde{\Theta}. \tag{2.8}$$

ii. Therefore, together with Theorem 2.2.2 (iii), the non-commutative parameter Θ and $\tilde{\Theta}$ are related by

$$\Theta\widetilde{\Theta} = 4\alpha\beta(\alpha\beta - 1)\mathbf{1}. \tag{2.9}$$

Proof.

i. Since B and C are antisymmetric matrices, then $B = -B^{tr}$ ($C = -C^{tr}$). Therefore from (2.8)

$$\alpha(-B^{tr} + B) = 2\alpha B = \Theta \implies B = \frac{1}{2\alpha}\Theta.$$

ii. This is a straightforward proof:

$$BC^{tr} = \Big(-\frac{1}{2\alpha}\Theta\Big)\Big(-\frac{1}{2\beta}\tilde{\Theta}\Big) = \frac{\Theta\widetilde{\Theta}}{4\alpha\beta}.$$

Equating with the right hand side of Theorem 2.2.2 (iii) gives us the answer. ■

Combining the results from Theorem 2.2.2 and Theorem 2.2.3, we arrive at the general representation matrix of (X, P) in the non-commutative space:

$$X_j = \alpha x_j - \frac{1}{2\alpha}\Theta_{jk}p_k, \tag{2.10a}$$

$$P_j = \beta p_j + \frac{1}{2\beta}\tilde{\Theta}_{jk}x_k. \tag{2.10b}$$

Again, in the two dimensional case it is convenient to write above equations in the explicit matrix form:

$$\begin{pmatrix} X_1 \\ P_1 \\ X_2 \\ P_2 \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 & -\frac{1}{2\alpha}\Theta \\ 0 & \alpha & \frac{1}{2\alpha}\Theta & 0 \\ 0 & \frac{1}{2\beta}\widetilde{\Theta} & \beta & 0 \\ -\frac{1}{2\beta}\widetilde{\Theta} & 0 & 0 & \beta \end{pmatrix} \begin{pmatrix} x_1 \\ p_1 \\ x_2 \\ p_2 \end{pmatrix}. \tag{2.11}$$

2.3 Non-commutative coordinates for a quantum mechanical Hamiltonian

In order to realize these sets of NC coordinates, we investigate the simple problem in the quantum mechanical system, namely the harmonic oscillator for the two-dimensional case. Since the NC coordinates have been formulated as a linear combinations of the commutative coordinates, the construction of a NC harmonic oscillator is easy to derive.

We start with the time-independent Schrodinger equation involving the Hamiltonian written in the non-commutative space:

$$\hat{H}(X, P)\psi = E\psi. \tag{2.12}$$

The N-dimensional Hamiltonian is written as

$$\hat{H} = \frac{1}{2m}P_j^2 + \frac{1}{2}m\omega^2 X_j^2, \quad j = 1, 2, \ldots, N, \tag{2.13}$$

where m and ω represent the mass and the angular frequency of the oscillator. Using (2.10), we arrive at

$$\begin{aligned}\hat{H}(X_j, P_j) &= \hat{H}(\alpha x_j - \frac{1}{2\alpha}\Theta_{jk}p_k, \beta p_j + \frac{1}{2\beta}\widetilde{\Theta}_{jk}x_k), \\ &= \frac{1}{2m}(\beta p_j + \frac{1}{2\beta}\widetilde{\Theta}_{jk}x_k)^2 + \frac{1}{2}m\omega^2(\alpha x_j - \frac{1}{2\alpha}\Theta_{jk}p_k)^2.\end{aligned}$$

For convenience, the standard operator method algebra is introduced and we state another results from [44].

Theorem 2.3.1 (Kang, 2005). *Let A and $A^\dagger$ be the creation and annihilation operators defined as*

$$A_j = \sqrt{\frac{m\omega}{2\hbar}}(X_j + \frac{i}{m\omega}P_j); \qquad A_j^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(X_j - \frac{i}{m\omega}P_j), \tag{2.14}$$

in the NC space, then

$$\widetilde{\Theta}_{jk} = m^2\omega^2\Theta_{jk}, \tag{2.15}$$

so that the commutators

$$[A_j, A_k] = [A_j^\dagger, A_k^\dagger] = 0, \qquad \textit{for} \quad j, k = 1, 2. \tag{2.16}$$

In addition,

$$[A_j, A_k^\dagger] = \delta_{jk} + im\omega\Theta_{jk}. \tag{2.17}$$

Theorem 2.3.2 (Kang, 2005). *Suppose A_i is an operator that annihilates a particle. Then for this to be consistent in the commutative space, A_i should be a linear* *combination of only lowering operators in the commutative space i.e.* a_i. *The same applies for* $A_i^\dagger$ *which can be written in terms of the* $a_i^\dagger$. *Therefore the choice of the scalar constants becomes*

$$\alpha = \beta \equiv \alpha. \tag{2.18}$$

The coordinate commutator matrix Θ_{jk} *and the scalar factor* α *are related by*

$$\Theta_{jk}\Theta_{kl} = -\frac{4\alpha^2}{m^2\omega^2}(1-\alpha^2). \tag{2.19}$$

Proof. We check the commutation rules between X_i and P_i which should equal to $i\hbar$

$$\begin{aligned}
[X_j, P_j] &= [\alpha x_j - \frac{1}{2\alpha}\Theta_{jk}p_k, \alpha p_j - \frac{1}{2\alpha}m^2\omega^2\Theta_{jk}x_k] = i\hbar, \\
&= \alpha^2[x_j, p_j] + \frac{1}{4\alpha^2}m^2\omega^2\Theta_{jk}\Theta_{kl}[p_l, x_l], \\
&= \alpha^2(i\hbar) + \frac{1}{4\alpha^2}m^2\omega^2\Theta_{jk}\Theta_{kl}(-i\hbar),
\end{aligned}$$

$$\Longrightarrow$$

$$\begin{aligned}
&i\hbar(\alpha^2 - \frac{1}{4\alpha^2}m^2\omega^2\Theta_{ik}\Theta_{kj}) = i\hbar, \\
&\Theta_{jk}\Theta_{kl} = -\frac{4\alpha^2}{m^2\omega^2}(1-\alpha^2). \qquad \blacksquare
\end{aligned}$$

Substituting (2.15) and (2.18) in the representation matrix (2.10) yields

$$X_j = \alpha x_j - \frac{1}{2\alpha}\Theta_{jk}p_k, \tag{2.20a}$$

$$P_j = \alpha p_j + \frac{1}{2\alpha}m^2\omega^2\Theta_{jk}x_k. \tag{2.20b}$$

We analyze further the antisymmetric properties of matrix parameter Θ_{ij} from (2.19). The type of matrix reads

$$\begin{pmatrix} 0 & \theta_{12} \\ \theta_{21} & 0 \end{pmatrix}\begin{pmatrix} 0 & \theta'_{12} \\ \theta'_{21} & 0 \end{pmatrix} = -\frac{4\alpha^2}{m^2\omega^2}(1-\alpha^2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.21}$$

where θ_{ij} and θ'_{ij} are any two elements in Θ. Upon matrix multiplication

$$\begin{pmatrix} \theta_{12}\theta'_{21} & 0 \\ 0 & \theta_{21}\theta'_{12}, \end{pmatrix} = \begin{pmatrix} -\frac{4\alpha^2}{m^2\omega^2}(1-\alpha^2) & 0 \\ 0 & -\frac{4\alpha^2}{m^2\omega^2}(1-\alpha^2) \end{pmatrix}$$

which provides an explicit solution of the element:

$$\theta = \frac{2\alpha}{m\omega}\sqrt{1-\alpha^2}, \tag{2.22}$$

satisfying (2.21) with

$$\Theta_{2\times 2} = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix},$$

since $\Theta_{ij} = -\Theta_{ji}$.

The Hamiltonian in NC coordinates therefore reads

$$\hat{H}(X_j, P_j) = \hat{H}(\alpha x_j - \frac{1}{2\alpha}\Theta_{jk}p_k, \alpha p_j + \frac{1}{2\alpha}m^2\omega^2\Theta_{jk}x_k). \tag{2.23}$$

The final form of the NC coordinates parameterized by the scalar constant α are given in the following matrix form in two dimensions

$$\begin{pmatrix} X_1 \\ P_1 \\ X_2 \\ P_2 \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 & -\frac{\sqrt{1-\alpha^2}}{m\omega} \\ 0 & \alpha & m\omega\sqrt{1-\alpha^2} & 0 \\ 0 & \frac{\sqrt{1-\alpha^2}}{m\omega} & \alpha & 0 \\ -m\omega\sqrt{1-\alpha^2} & 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} x_1 \\ p_1 \\ x_2 \\ p_2 \end{pmatrix}. \tag{2.24}$$

Chapter 3

Computation of The Deformed Generalized Hermite Polynomials

Introduction

In this chapter, we write down the formula for the non-commutative raising and lowering operators and give their representations in the abstract Hilbert space $\mathcal{H}$, introducing a new notation to distinguish them from the standard raising and lowering operators.

Next, we write down the transforms of the deformed vectors in $\mathcal{H}$ to the complex coordinates representation space $\mathcal{H}(\mathbb{C})$. They appear as a linear combinations of the standard vectors. In this way, we go on to generate the family of the *deformed or generalized* Hermite polynomials.

Some interesting properties of these *new*, deformed polynomials, namely their generating functions and recurrence relations are also given. Before we end, we raise some issues regarding the orthogonality of these new polynomials.

3.1 Non-commutative raising and lowering operators in $\mathcal{H}$

Using the formulae for the NC algebra given in Chapter 2, we write down the formula of the non-commutative raising and lowering operators acting on the abstract space $\mathcal{H}$. These are obtainable using the results of [44].

Definition 3.1.1. *Let a_j^α and $a_j^{\alpha\dagger}$ for $j = 1, 2$ be operators of the NC algebra satisfying*

$$[a_j^\alpha, a_k^{\alpha\dagger}] = \delta_{jk} + \varepsilon_{jk} i2\alpha\sqrt{1-\alpha^2}, \tag{3.1}$$

where,

$$\begin{aligned} a_j^\alpha &= \alpha a_j + i\sqrt{1-\alpha^2}\,\varepsilon_{jk}\,a_k, \\ a_j^{\alpha\dagger} &= \alpha a_j^\dagger - i\sqrt{1-\alpha^2}\,\varepsilon_{jk}\,a_k^\dagger \end{aligned} \tag{3.2}$$

with ε_{jk} being the antisymmetric tensor, $\varepsilon_{jk} = -\varepsilon_{kj}$.

The commutation relations (3.1) are summarized in the table below

	a_1^α	$a_1^{\alpha\dagger}$	a_2^α	$a_2^{\alpha\dagger}$
a_1^α	0	1	0	$i2\alpha\sqrt{1-\alpha^2}$
$a_1^{\alpha\dagger}$	-1	0	$-i2\alpha\sqrt{1-\alpha^2}$	0
a_2^α	0	$-i2\alpha\sqrt{1-\alpha^2}$	0	1
$a_2^{\alpha\dagger}$	$i2\alpha\sqrt{1-\alpha^2}$	0	-1	0

and the matrix transforming the commutative algebra to the non-commutative algebra is

$$\begin{pmatrix} a_1^{\alpha} \\ a_1^{\alpha\dagger} \\ a_2^{\alpha} \\ a_2^{\alpha\dagger} \end{pmatrix} = \begin{pmatrix} \alpha & 0 & i\sqrt{1-\alpha^2} & 0 \\ 0 & \alpha & 0 & -i\sqrt{1-\alpha^2} \\ -i\sqrt{1-\alpha^2} & 0 & \alpha & 0 \\ 0 & i\sqrt{1-\alpha^2} & 0 & \alpha \end{pmatrix} \begin{pmatrix} a_1 \\ a_1^{\dagger} \\ a_2 \\ a_2^{\dagger} \end{pmatrix}. \tag{3.3}$$

The parameter α is restricted in the range $0 < \alpha < 1$. There are the two limiting cases:

Case 1: $\alpha \to 1$

$$\begin{pmatrix} a_1^{\alpha} \\ a_1^{\alpha\dagger} \\ a_2^{\alpha} \\ a_2^{\alpha\dagger} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_1^{\dagger} \\ a_2 \\ a_2^{\dagger} \end{pmatrix}. \tag{3.4}$$

Case 2: $\alpha \to 0$

$$\begin{pmatrix} a_1^{\alpha} \\ a_1^{\alpha\dagger} \\ a_2^{\alpha} \\ a_2^{\alpha\dagger} \end{pmatrix} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_1^{\dagger} \\ a_2 \\ a_2^{\dagger} \end{pmatrix}, \tag{3.5}$$

In the first case, the limit of $\alpha = 1$, gives commutation relations which are directly proportional to the standard ones, while the second case gives its adjoint relation.

3.2 The $\{|k,\ell\rangle_g\}$ representation

It is best however, to take a more abstract route, and therefore we first consider a generalized version of the non-commutative coordinates, before specializing to the situation above, involving α. We start with an important definition.

Definition 3.2.1. ***General linear group over the complexes*** $GL(N,\mathbb{C})$*. The general linear group* $GL(N,\mathbb{C})$ *consists of nonsingular* $N\times N$ *matrices with complex entries.*

The group properties of $GL(N,\mathbb{C})$ are easy to verify. We shall mainly be concerned with the case in which $N=2$ and we write a general element of the resulting group as the 2×2 matrix

$$g=\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}, \tag{3.6}$$

where the elements $g \in GL(2,\mathbb{C})$, for $j,k=1,2$, are in general complex numbers. Moreover $\det[g]=g_{11}g_{22}-g_{12}g_{21}\neq 0$.

The NC raising and lowering operators, now parameterized by g, are defined as

$$A_1^g=g_{11}a_1+g_{21}a_2, \qquad A_2^g=g_{12}a_1+g_{22}a_2, \tag{3.7a}$$

$$A_1^{g\dagger}=g_{11}a_1^\dagger+g_{21}a_2^\dagger\ , \qquad A^{g\dagger}_{\ 2}=g_{12}a_1^\dagger+g_{22}a_2^\dagger\ . \tag{3.7b}$$

Equivalently we may write

$$\begin{pmatrix} A_1^{g\dagger} \\ A_2^{g\dagger} \end{pmatrix}=\begin{pmatrix} \overline{g}_{11} & \overline{g}_{21} \\ \overline{g}_{12} & \overline{g}_{22} \end{pmatrix}\begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix}. \tag{3.7c}$$

The basic action of A_j^g on a vacuum state $|0,0\rangle_g$ is just

$$A_j^g\,|0,0\rangle_g = 0, \qquad \text{for} \quad j=1,2$$

because it is easy to see that

$$A_1^g = (g_{11}a_1 + g_{21}a_2)\,|0,0\rangle = 0, \qquad A_2^g = (g_{12}a_1 + g_{22}a_2)\,|0,0\rangle = 0.$$

Note that from (3.7), operator A_j^g is just a linear combination of the a_j's; therefore the vacuum state is unchanged, $|0,0\rangle_g = |0,0\rangle \equiv |0,0\rangle$.

Using g we construct the new (in general, non-orthogonal) basis for the Hilbert space $\mathcal{H}$, whose vectors are

$$\begin{aligned}
|k,\ell\rangle_g &= \frac{1}{\sqrt{k!\ell!}}(g_{11}a_1^\dagger + g_{21}a_2^\dagger)^k(g_{12}a_1^\dagger + g_{22}a_2^\dagger)^\ell|0,0\rangle, \\
&= \sum_{p=0}^{k}\sum_{j=0}^{\ell}\frac{1}{\sqrt{(p+j)!\sqrt{k+\ell-j-p}!}}\binom{k}{p}\binom{\ell}{j}g_{11}^p g_{21}^{k-p} g_{12}^j g_{22}^{\ell-j}(a_1^\dagger)^{p+j}(a_2^\dagger)^{k+\ell-j-p}|0,0\rangle, \\
&= \sum_{p=0}^{k}\sum_{j=0}^{\ell}\frac{1}{\sqrt{(p+j)!\sqrt{k+\ell-j-p}!}}\binom{k}{p}\binom{\ell}{j}g_{11}^p g_{21}^{k-p} g_{12}^j g_{22}^{\ell-j}|p+j,k+\ell-j-p\rangle,
\end{aligned} \tag{3.8}$$

where the binomial coefficients are the usual ones,

$$\binom{k}{p} = \frac{k!}{p!(k-p)!}\ .$$

In particular, we will call $\{|k,\ell\rangle_g\}$ the *g-deformed basis* that spans the state space of a particle in a two-dimensional non-commutative system.

The actions of the NC operators on the above basis vectors $\mathcal{H}(\mathbb{C})$ are easily obtained. We have,

$$A_1^g|k,\ell\rangle_g = \sqrt{k}|k-1,\ell\rangle_g, \qquad A_1^{g\dagger}|k,\ell\rangle_g = \sqrt{k+1}|k+1,\ell\rangle_g, \tag{3.9a}$$

$$A_2^g|k,\ell\rangle_g = \sqrt{l}|k,\ell-1\rangle_g, \qquad A_2^{g\dagger}|k,\ell\rangle_g = \sqrt{l+1}|k,\ell+1\rangle_g, \tag{3.9b}$$

and we can always rewrite them in terms of linear combinations of the standard operators $a_1, a_2, a_1^\dagger, a_2^\dagger$.

3.3 The deformed complex coordinates representation

We will now transform the vectors $|k,\ell\rangle_g$ to the Hilbert space $\mathcal{H}(\mathbb{C})$ of complex functions and write the resulting functions as $H_{k,\ell}^g(z,\overline{z})$. The transformation from $|k,\ell\rangle_g$ to $H_{k,\ell}^g(z,\overline{z})$ follows through a simple manipulation.

We first look at

$$\begin{aligned} |k,\ell\rangle_g &= \frac{1}{\sqrt{k!\,\ell!}}(A_1^{g\dagger})^k(A_2^{g\dagger})^\ell\,|0,0\rangle\,, \\ &= \frac{1}{\sqrt{k!\,\ell!}}(g_{11}a_1^\dagger + g_{21}a_2^\dagger)^k(g_{12}a_1^\dagger + g_{22}a_2^\dagger)^\ell\,|0,0\rangle\,, \end{aligned} \tag{3.10}$$

for $k,\ell = 0,1,\ldots$. Substituting the complex forms of the operators $a_1^\dagger, a_2^\dagger$, as in equation (1.12), gives us the deformed complex vectors in $\mathcal{H}(\mathbb{C})$

$$h_{k,l}^g(z,\overline{z}) = \frac{1}{\sqrt{k!\,\ell!}}(A_1^{g\dagger})^k(A_2^{g\dagger})^\ell h_{0,0}^c(z,\overline{z}) = \frac{H_{k,\ell}^g(z,\overline{z})}{\sqrt{k!\,\ell!}},$$

which explicitly looks like

$$
\begin{aligned}
H^g_{k,\ell}(z,\overline{z}) &= \frac{1}{\sqrt{k!\ell!}}\Big(g_{11}(z-\partial_{\overline{z}})+g_{21}(\overline{z}-\partial_z)\Big)^k\Big(g_{12}(z-\partial_{\overline{z}})+g_{22}(\overline{z}-\partial_z)\Big)^\ell H^g_{0,0}(z,\overline{z}),\\
&= \sum_{p=0}^{k}\sum_{j=0}^{\ell}\frac{1}{\sqrt{k!\ell!}}\binom{k}{p}\binom{\ell}{j} g_{11}^p g_{21}^{k-p} g_{12}^j g_{22}^{\ell-j}(z-\partial_{\overline{z}})^{p+j}(\overline{z}-\partial_z)^{k+\ell-j-p}h^c_{0,0}(z,\overline{z}),\\
&= \sum_{p=0}^{k}\sum_{j=0}^{\ell}\frac{1}{\sqrt{k!\ell!}}\binom{k}{p}\binom{\ell}{j} g_{11}^p g_{21}^{k-p} g_{12}^j g_{22}^{\ell-j} H_{p+j,k+\ell-j-p}(z,\overline{z}),
\end{aligned}
$$

and this set of vectors spans the space $\mathcal{H}(\mathbb{C})$. Again it is evident that $H^g_{0,0}(z,\overline{z}) = h^c_{0,0}(z,\overline{z}) = 1$ for all $z,\overline{z}$.[1]

In the special case where g is the identity matrix, it is clear that the polynomials $h^g_{k,\ell}(z,\overline{z})$ are just the normalized complex Hermite polynomials defined in (1.21), which are also the same polynomials as the $\Psi_{k,\ell}(z,\overline{z})$ defined in (1.14).

Lets look at what happens if we fix the value of either k,ℓ to be equal to zero. Setting $k=0$ we obtain

$$
\begin{aligned}
H^g_{0,\ell}(z,\overline{z}) &= \sum_{j=0}^{\ell}\frac{\sqrt{j!(\ell-j)!}}{\sqrt{\ell}}\binom{\ell}{j} g_{12}^j g_{22}^{\ell-j} H_{j,\ell-j}(z,\overline{z}),\\
&= \sum_{j=0}^{\ell}\frac{\sqrt{j!(\ell-j)!}}{\sqrt{\ell}}\binom{\ell}{j} g_{12}^j g_{22}^{\ell-j} z^j\overline{z}^{\ell-j}.
\end{aligned} \tag{3.11}
$$

Setting $\ell=0$, we obtain

$$
\begin{aligned}
H^g_{k,0}(z,\overline{z}) &= \sum_{p=0}^{k}\frac{\sqrt{p!(k-p)!}}{\sqrt{k!}}\binom{k}{p} g_{11}^p g_{21}^{k-p} H_{p,k-p}(z,\overline{z}),\\
&= \sum_{p=0}^{k}\frac{\sqrt{p!(k-p)!}}{\sqrt{k!}}\binom{k}{p} g_{11}^p g_{21}^{k} z^p\overline{z}^{k-p}.
\end{aligned} \tag{3.12}
$$

We see that $H^g_{0,\ell}(z,\overline{z})$ and $H^g_{k,0}(z,\overline{z})$ do not span the subspaces of $\mathcal{H}_{\text{hol}}$ and $\mathcal{H}_{\text{a-hol}}$, of holomorphic and anti-holomorphic functions ($\mathcal{H}_{\text{hol}}$ and $\mathcal{H}_{\text{a-hol}}$) as did the unde-

[1] $h^c_{k,\ell}(z,\overline{z})$ is the classical (undeformed) complex Hermite polynomials

formed functions $h^c_{0,\ell}(z,\overline{z})$ and $h^c_{k,0}(z,\overline{z})$.

3.4 Deformed generalized generating function

The deformed generalized Hermite polynomials are obtainable via a more convenient formulation, namely the *generating function*. We will later use this formulation in order to construct the $H^g_{k,\ell}(z,\overline{z})$.

To compute the generalized generating function, we first consider a vector-valued function

$$F_g(u,\overline{u};z,\overline{z}) = \sum_{k,\ell=0}^{\infty} \frac{u^k \overline{u}^\ell}{\sqrt{k!\ell!}} \left|k,\ell\right\rangle_g . \tag{3.13}$$

This function will serve as a useful book-keeping device in the subsequent calculations based on the identities

$$\frac{\partial F_g}{\partial u}(u,\overline{u};z,\overline{z}) = A_1^{g\dagger} F_g(u,\overline{u};z,\overline{z}) \quad \text{and} \quad \frac{\partial F_g}{\partial \overline{u}}(u,\overline{u};z,\overline{z}) = A_2^{g\dagger} F_g(u,\overline{u};z,\overline{z}) . \tag{3.14}$$

Obviously, using the deformed state in (3.10) we obtain

$$\begin{aligned} F_g(u,\overline{u};z,\overline{z}) &= \sum_{k,\ell=0}^{\infty} \frac{(uA_1^{g\dagger})^k}{k!} \frac{(\overline{u}A_2^{g\dagger})^\ell}{\ell!} \left|0,0\right\rangle , \\ &= e^{uA_1^{g\dagger} + \overline{u}A_2^{g\dagger}} \left|0,0\right\rangle , \end{aligned}$$

where on the second line, we used the exponential formula.

Expanding $A_1^{g\dagger}$ and $A_2^{g\dagger}$ gives the equality

$$\begin{aligned}
F_g(u,\overline{u};z,\overline{z}) &= e^{uA^g{}_1^\dagger+\overline{u}A^g{}_2^\dagger}\,|0,0\rangle \\
&= e^{u(g_{11}a_1^\dagger+g_{21}a_2^\dagger)+\overline{u}(g_{12}a_1^\dagger+g_{22}a_2^\dagger)}\,|0,0\rangle \\
&= e^{(g_{11}u+g_{21}\overline{u})a_1^\dagger+(g_{12}u+g_{22}\overline{u})a_2^\dagger}\,|0,0\rangle \\
&= F(g_{11}u+g_{21}\overline{u},g_{12}u+g_{22}\overline{u};z,\overline{z}),
\end{aligned} \tag{3.15}$$

which is easily seen to be the *deformed* version of the function $F(u,\overline{u};z,\overline{z})$, now written as$F(U_1,U_2;z,\overline{z})$, with

$$U_1 = g_{11}u+g_{21}\overline{u} \qquad \text{and} \qquad U_2 = g_{12}u+g_{22}\overline{u}.$$

Then from (3.15) and (1.27) we get

$$F(U_1,U_2;z,\overline{z}) = e^{U_1z+U_2z-U_1U_2}. \tag{3.16}$$

In particular, we have the formula for the *g-deformed generalized generating function* for Hermite polynomials

$$\begin{aligned}
F_g(u,\overline{u};z,\overline{z}) &= \exp\left((g_{11}u+g_{12}\overline{u})z+(g_{21}u+g_{22}\overline{u})\overline{z}-(g_{11}u+g_{12}\overline{u})(g_{21}u+g_{22}\overline{u})\right) \\
&= \exp(-g_{11}g_{21}u^2-g_{12}g_{22}\overline{u}^2-(g_{11}g_{22}+g_{12}g_{21})u\overline{u}+(g_{11}z+g_{21}\overline{z})u \\
&\quad +(g_{12}z+g_{22}\overline{z})\overline{u}).
\end{aligned} \tag{3.17}$$

According to (3.10), all we must do to find the eigenfunctions $|k,\ell\rangle_g$ is expand this expression in powers of $u,\overline{u}$. Therefore, we can rewrite (3.13) as

$$F_g(u,\overline{u};z,\overline{z}) = \sum_{k,\ell=0}^{\infty} H^g_{k,\ell}(z,\overline{z})\frac{u^k\overline{u}^\ell}{k!\ell!}, \tag{3.18}$$

which is to be compared with (1.27). The polynomials

$$H^g_{k,\ell}(z,\overline{z}), \qquad k,\ell = 0,1,\dots \tag{3.19}$$

are the *g-deformed generalized complex Hermite polynomials* in two-variables. Going back to (3.2) and using the particular form for the matrix g, corresponding to the creation and annihilation operators of NC quantum mechanics obtained there, i.e.,

$$g = \begin{pmatrix} \alpha & -i\sqrt{1-\alpha^2} \\ i\sqrt{1-\alpha^2} & \alpha \end{pmatrix}, \tag{3.20}$$

we have for $0 < \alpha < 1$,

$$\begin{aligned} F_g(u,\overline{u};z,\overline{z}) = \exp\{&-i\alpha\sqrt{1-\alpha^2}u^2 + i\alpha\sqrt{1-\alpha^2}\overline{u}^2 - (1)u\overline{u} \\ &+ (\alpha z + i\sqrt{1-\alpha^2}\overline{z})u + (-i\sqrt{1-\alpha^2}z + \alpha\overline{z})\overline{u}\}. \end{aligned} \tag{3.21}$$

while in the two extreme limiting situations,

i. For $\alpha = 1$. $F_g^{\alpha=1}(u,\overline{u};z,\overline{z}) = -u\overline{u} + uz + \overline{uz}$.

ii. For $\alpha = 0$. $F_g^{\alpha=0}(u,\overline{u};z,\overline{z}) = -u\overline{u} + iu\overline{z} - i\overline{u}z$.

3.5 Some properties of $\{H^g_{k,l}(z,\overline{z})\}$

We collect below some properties of g-deformed generalized Hermite polynomials $\{H^g_{k,l}(z,\overline{z})\}$.

3.5.1 Three term recurrence relations

Differentiating both sides of (3.18) with respect to z and $\bar{z}$:

$$\partial_z H^g_{k,\ell}(z,\overline{z}) = g_{11} k H^g_{k-1,\ell}(z,\overline{z}) + g_{12}\,\ell H^g_{k,\ell-1}(z,\overline{z}), \tag{3.22}$$

$$\partial_{\overline{z}} H^g_{k,\ell}(z,\overline{z}) = g_{21} k H^g_{k-1,\ell}(z.\overline{z}) + g_{22}\,\ell H^g_{k,\ell-1}(z,\overline{z}), \tag{3.23}$$

Subtracting Equation (3.23) from (3.22) yields

$$\Big(\partial_z - \partial_{\overline{z}}\Big) H^g_{k,\ell}(z,\overline{z}) = (g_{11} - g_{21}) k H^g_{k-1,\ell}(z,\overline{z}) + (g_{12} - g_{22}) \ell H^g_{k,\ell-1}(z,\overline{z}). \tag{3.24}$$

Next, differentiating (3.18) with respect to u and $\overline{u}$ once yields

$$\begin{aligned}
\frac{\partial}{\partial u} F_g(u,\overline{u};z,\overline{z}) &= \sum_{k,\ell=0}^{\infty} \frac{\partial}{\partial u} \frac{u^k \overline{u}^\ell}{k!\,\ell!} H^g_{k,\ell}(z,\overline{z}),\\
\frac{\partial}{\partial \overline{u}} F_g(u,\overline{u};z,\overline{z}) &= \sum_{k,\ell=0}^{\infty} \frac{\partial}{\partial \overline{u}} \frac{u^k \overline{u}^\ell}{k!\,\ell!} H^g_{k,\ell}(z,\overline{z}).
\end{aligned}$$

Equating coefficients of equal powers of u and $\overline{u}$ we obtain the recurrence relation for the first kind

$$\begin{aligned}
H^g_{k+1,\ell}(z,\overline{z}) = (g_{11} z + g_{12}\overline{z}) H^g_{k,\ell}(z,\overline{z}) - 2 g_{11} g_{12} (k+1) H^g_{k-1,\ell}(z,\overline{z})\\
- (g_{11} g_{22} + g_{21} g_{12})\,\ell H^g_{k,\ell-1}(z,\overline{z})
\end{aligned} \tag{3.25}$$

and for the second kind;

$$\begin{aligned}
H^g_{k,\ell+1}(z,\overline{z}) = (g_{21} z + g_{22}\overline{z}) H^g_{k,\ell}(z,\overline{z}) - 2 g_{21} g_{22} (\ell+1) H^g_{k,\ell-1}(z,\overline{z})\\
- (g_{11} g_{22} + g_{21} g_{12})\,k H^g_{k-1,\ell}(z,\overline{z})
\end{aligned} \tag{3.26}$$

Also, subtracting (3.26) from (3.25) yields

$$
\begin{aligned}
&(g_{11}g_{22} + g_{12}g_{21})(\ell - k)H^g_{k,\ell}(z,\overline{z}) \\
&= (g_{11}z + g_{12}\overline{z})H^g_{k,\ell+1}(z,\overline{z}) + 2g_{21}g_{22}(\ell+1)H^g_{k+1,\ell-1}(z,\overline{z}) \\
&\quad - 2g_{11}g_{12}(k+1)H^g_{k-1,\ell+1}(z,\overline{z}) - (g_{21}z + g_{22}\overline{z})H^g_{k+1,\ell}(z,\overline{z}).
\end{aligned} \tag{3.27}
$$

This result is consistent with equation (1.32) and, as discussed in [3], in the limit in which g goes to the identity matrix.

Some additional properties of the $\{H^g_{k,l}(z,\overline{z})\}$ as worked out in [5] are stated in the following:

Theorem 3.5.1. *Let $S^{(g)}$ be the operator defined by*

$$
(S^{(g)}f)(z,\overline{z}) = f(g_{11}z + g_{21}\overline{z}, g_{12}z + g_{22}\overline{z}). \tag{3.28}
$$

We have

$$
H^g_{m,n}(z,\overline{z}) = e^{-\partial_z\partial_{\overline{z}}}S^{(g)}e^{\partial_z\partial_{\overline{z}}}H_{m,n}(z,\overline{z}), \tag{3.29}
$$

$$
H^g_{m,n}(z,\overline{z}) = e^{-\partial_z\partial_{\overline{z}}}(g_{11}z + g_{21}\overline{z})^m(g_{12}z + g_{22}\overline{z})^n, \tag{3.30}
$$

$$
H^g_{m,n}(z,\overline{z}) = \sum_{j=0}^{m}\sum_{k=0}^{n}\binom{m}{j}\binom{m}{k}g_{11}^j g_{21}^{m-j} g_{12}^k g_{22}^{n-k} H_{j+k,n+m-j-k}(z,\overline{z}). \tag{3.31}
$$

Note that the relation (3.30) is the Rodrigues formula for $\{H^g_{k,l}(z,\overline{z})\}$.

3.5.2 Orthogonality relation of $\{H^g_{k,\ell}(z,\overline{z})\}$

Considered as elements in $\mathcal{H}(\mathbb{C})$, the deformed polynomials $\{H^g_{k,\ell}(z,\overline{z})\}$ cannot be mutually orthogonal, for different values of k,ℓ, unless the matrix g has a very specific form. This issue will be considered in detail in Chapter 5. Here we only note that since the undeformed vectors $|k,\ell\rangle$ form an orthonormal basis of $\mathcal{H}$, in order for the deformed vectors $|k,\ell\rangle_g$ to be orthonormal, the mapping $|k,\ell\rangle \longmapsto |k,\ell\rangle_g$ has to be unitary. This of course is not true for an arbitrary nonsingular matrix g. In fact, even in the special case of a matrix of the type (3.20), this will not be true. However, we shall arrive at a more interesting *biorthogonality* property for the polynomials $\{H^g_{k,\ell}(z,\overline{z})\}$ as one of the main results of this thesis.

Chapter 4

Matrix Representation

Introduction

We present in this chapter the procedure for explicitly computing the transformation $T_g : |k, \ell\rangle \longmapsto |k, \ell\rangle_g$, of the standard basis to the non-commutative or deformed basis of vectors, on the Hilbert space $\mathcal{H}$, which will then yield the linear map on the Hilbert space $\mathcal{H}(\mathbb{C})$, connecting the standard complex Hermite polynomials $H_{k,\ell}(z, \overline{z})$ to the deformed polynomials $H^g_{k,\ell}(z, \overline{z})$.

In the first section we introduce a notion of intertwining relations which holds between two sets of operators namely T_g and $R_g,\ g \in GL(2, \mathbb{C})$. The operator T_g is a mapping on the space $\mathcal{H}$ while R_g is a mapping on a space of complex polynomials in two-variables, $\mathbb{C}[*, *]$.

Using the intertwining rule it is easy to explicitly determine the coefficients representing the elements of the matrix of the linear transformation T_g or R_g. An interesting property regarding the characteristic polynomials associated to this matrix is given in the last section.

4.1 Intertwining relations

Our aim is to describe the operator T_g, satisfying

$$T_g|k,\ell\rangle = |k,\ell\rangle_g, \qquad \text{for} \quad k,\ell = 0,1,\ldots \tag{4.1}$$

and parameterized by the group element $g \in GL(2,\mathbb{C})$. But before going further into the calculations, we need first to give some insight regarding the intertwining operators and discuss its role in the construction of matrix of the operator T_g.

Definition 4.1.1. ***Intertwining operator***. *[74]*
*Let U_g and V_g, $g \in GL(2,\mathbb{C})$ be representations of the group $GL(2,\mathbb{C})$ in the spaces Hilbert spaces X and Y, respectively. Assume further that $P : X \to Y$ is a linear operator. We say that P is an **intertwining operator** for U_g and V_g if*

$$PU_g = V_g P, \quad \textit{for all} \quad g \in GL(2,\mathbb{C}). \tag{4.2}$$

Let us note some properties of the map (4.1). First recall that T_g maps the vector $|k,\ell\rangle$ in $\mathcal{H}$ to $T_g(|k,\ell\rangle)$ which also in $\mathcal{H}$:

$$T_g\colon \mathcal{H} \to \mathcal{H}, \tag{4.3}$$

and moreover, $g \longmapsto T_g$ is a representation of the group $GL(2,\mathbb{C})$ on the Hilbert space $\mathcal{H}$.

If we consider a map

$$\begin{aligned} P\colon \mathcal{H} &\to \mathbb{C}[s,t], \\ |k,\ell\rangle &\mapsto s^k t^\ell, \end{aligned} \tag{4.4}$$

where $\mathbb{C}[s,t]$ denotes the set of all polynomials in the two complex variables s and t, with complex coefficients, then P satisfies the following intertwining relation between the operators $a_1^\dagger$ and $a_2^\dagger$:

$$Pa_1^\dagger = M_s P, \qquad Pa_2^\dagger = M_t P, \tag{4.5}$$

where M_s and M_t stand for the operators of multiplication by s and t respectively. It means that for a vector $|k,\ell\rangle \in \mathcal{H}$, the vector $P(|k,\ell\rangle)$ is in $\mathbb{C}[s,t]$.

Substituting $a_1^\dagger$ by $A_1^{g\dagger}$ and $a_2^\dagger$ by $A_2^{g\dagger}$ gives

$$PA_1^{g\dagger} = P(g_{11}a_1^\dagger + g_{21}a_2^\dagger) = g_{11}M_sP + g_{21}M_tP, \tag{4.6a}$$

$$PA_2^{g\dagger} = P(g_{12}a_1^\dagger + g_{22}a_2^\dagger) = g_{12}M_sP + g_{22}M_tP. \tag{4.6b}$$

Furthermore, under the intertwining operator P, the action of T_g on $|k,\ell\rangle$ is given by

$$PT_g|k,\ell\rangle = (g_{11}s + g_{21}t)^k(g_{12}s + g_{22}t)^\ell. \tag{4.7}$$

Let us define the representation R_g of the group $GL(2,\mathbb{C})$ on the space $\mathbb{C}[s,t]$ by

$$[R_g f](s,t) = f(g_{11}s + g_{21}t, g_{12}s + g_{22}t). \tag{4.8}$$

Comparing (4.7) with (4.8), it is easy to show that the following intertwining relation holds with respect to $P : \mathcal{H} \to \mathbb{C}[s,t]$:

$$PT_g = R_gP\ . \tag{4.9}$$

Equivalently, there is a map P such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{H} \arrow[r, "P"] \arrow[d, "T_g"] & \mathbb{C}[s,t] \arrow[d, "R_g"] \\ \mathcal{H} \arrow[r, "P"] & \mathbb{C}[s,t] \end{tikzcd}$$

Thus P intertwines the two representations T_g and R_g, of the group $GL(2,\mathbb{C})$, on the two spaces $\mathcal{H}$ and $\mathbb{C}[s,t]$, respectively. We proceed to construct the matrix of the representation R_g on its irreducible subspaces.

4.2 Matrix representation R_g

Definition 4.2.1. ***Matrix representation of a linear operator****. [81]*
Let T be a linear operator on a (finite dimensional) vector space V and suppose $e = \{e_1, e_2, \ldots, e_n\}$ is a basis of V. Now $T(e_1), T_g(e_2), \ldots, T(e_n)$ are vectors in V and so each is a linear combination of the elements of the basis $\{e_j\}$: Symbolically we write

$$T(e_1) = t_{11}e_1 + t_{12}e_2 + \ldots + t_{1n}e_n$$

$$T(e_2) = t_{21}e_1 + t_{22}e_2 + \ldots + t_{2n}e_n$$

....

$$T(e_n) = t_{n1}e_1 + t_{n2}e_2 + \cdots + t_{nn}e_n.$$

The transpose of the above matrix of coefficients, denoted by $M(T)$, is called the matrix representation of T relative to the basis $\{e_j\}$ that is

$$M(T) = \begin{pmatrix} t_{11} & t_{21} & \dots & t_{n1} \\ t_{12} & t_{22} & \dots & t_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ t_{1n} & t_{2n} & \dots & t_{nn} \end{pmatrix}. \tag{4.10}$$

Following the above intertwining relation P, we now proceed to compute the matrix of R_g (equivalently of T_g) relative to the basis vectors $s^k t^\ell$ in $\mathbb{C}[s,t]$ (equivalently, the basis vectors $|k,\ell\rangle$ in $\mathcal{H}$). It is easily seen, from the defining relation (4.8) that R_g maps a homogeneous polynomial of degree L (i.e., a polynomial of the type $\sum_{k=0}^{L} a_k s^k t^{L-k},\ a_k \in \mathbb{C}$), to a polynomial of the same type. Thus, under the action of R_g the space $\mathbb{C}[s,t]$ splits into the infinite direct sum

$$\mathbb{C}[s,t] = \bigoplus_{L=0}^{\infty} \mathbb{C}_L[s,t], \tag{4.11}$$

where $\mathbb{C}_L[s,t]$ stands for the subspace of homogeneous polynomials of degree L:

$$\mathbb{C}_L[s,t] = \mathrm{span}\{s^k t^{L-k} \colon k = 0, \dots, L\}\ . \tag{4.12}$$

On each $\mathbb{C}_L[s,t]$ the representation R_g acts irreducibly. Also, it is not hard to see that $\mathbb{C}_L[s,t]$ is vector space isomorphic to the L-fold symmetric tensor product of

$$V \equiv \mathbb{C}_1[s,t] = \mathrm{span}\{s,t\}, \tag{4.13}$$

i.e., $\mathbb{C}_L[s,t] \simeq \mathrm{Sym}^L V$.

For V we know that $R_g|_V$ is the standard representation of $GL(2,\mathbb{C})$ and that the subspaces $\mathbb{C}_L[s,t]$ is isomorphic to the symmetric tensor of order L of the vector space

V which we denote by

$$\mathbb{C}_L[s,t] \simeq \mathrm{Sym}^L V \ . \tag{4.14}$$

The next process will enable us to fully construct the matrix representation of R_g.

We expand by a straightforward calculation using the binomial theorem,

$$\begin{aligned}
&(g_{11}s + g_{21}t)^k (g_{12}s + g_{22}t)^\ell \\
&= \sum_{p=0}^{k} \sum_{j=0}^{\ell} \binom{k}{p} \binom{\ell}{j} g_{11}^p g_{21}^{k-p} g_{12}^j g_{22}^{\ell-j} s^{p+j} t^{k+\ell-j-p}, \\
&= \sum_{r=0}^{k+\ell} \left[\sum_{q=\max\{0,r-\ell\}}^{\min\{r,k\}} \binom{k}{q} \binom{\ell}{r-q} g_{11}^q g_{21}^{k-q} g_{12}^{r-q} g_{22}^{\ell+q-r} \right] s^r t^{k+\ell-r} \ ,
\end{aligned}$$

where we make a substitution of $r = p + j$ and $q = p$.

If we choose the basis in $\mathbb{C}_L[s,t]$ as

$$f_k(s,t) = s^k t^{L-k}, \qquad k = 0, 1, \ldots, L \tag{4.15}$$

and let $L = k + \ell$ plus substituting for $\ell = L - k$, then

$$R_g f_k = \sum_{r=0}^{L} \left[\sum_{q=\max\{0,r+k-L\}}^{\min\{r,k\}} \binom{k}{q} \binom{L-k}{r-q} g_{11}^q g_{21}^{k-q} g_{12}^{r-q} g_{22}^{L-k+q-r} \right] f_r \ . \tag{4.16}$$

So the matrix $M(g,L)$ of $R_g|_{C_L[s,t]}$ in the basis $\{f_k\}_{k=0}^{L}$ is given by the matrix elements

$$M(g,L)_{rk} = \sum_{q=\max\{0,r+k-L\}}^{\min\{r,k\}} \binom{k}{q} \binom{L-k}{r-q} g_{11}^q g_{21}^{k-q} g_{12}^{r-q} g_{22}^{L-k+q-r}, \quad 0 \le r,k \le L, \tag{4.17}$$

with the matrix size being equal to $(L+1) \times (L+1)$. Also note that

$$M(g,L)^* = M(g^*,L), \tag{4.18}$$

with the star denoting the adjoint.

From the intertwining relation (4.9) it now follows that the operator T_g, with $T_g|k,\ell\rangle_g = |k,\ell\rangle$, leaves invariant the subspace of $\mathcal{H}$ spanned by the vectors $|k,\ell\rangle,\ k+\ell = L$. This also means that

$$|L-k,k\rangle_g = \sum_{r=0}^{N} M(g,L)_{rk}|L-r,r\rangle. \tag{4.19}$$

Next, it is clear that when $\mathcal{H} = \mathcal{H}(\mathbb{C})$, if we again denote by T_g the operator $T_g H_{m,n} = H^g_{m,n}$, acting on the complex Hermite polynomials, then T_g leaves invariant the $(L+1)$-dimensional subspace of $\mathcal{H}(\mathbb{C})$ spanned by the vectors

$$\mathfrak{S}(L) = \{H_{L,0},\ H_{L-1,1},\ H_{L-2,2},\ \ldots,\ H_{0,L}\} \tag{4.20}$$

Let $T(g,L)$ denote the restriction of T_g to this subspace. Then the matrix elements of $T(g,L)$ in the $\mathfrak{S}(L)$-basis are just the $M(g,L)_{rk}$ in (4.17):

$$H^g_{L-k,k} = T_g H_{L-k,k} = \sum_{r=0}^{N} M(g,L)_{rk} H_{L-r,r} \tag{4.21}$$

In the special case of non-commutative quantum mechanics, when

$$g = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \equiv \begin{bmatrix} \alpha & -i\sqrt{1-\alpha^2} \\ i\sqrt{1-\alpha^2} & \alpha \end{bmatrix}, \tag{4.22}$$

writing $M(\alpha, L)$ for the corresponding $M(g, L)$, we get

$$M(\alpha, L)_{rk} = \sum_{q=\max\{0,r+k-L\}}^{\min\{r,k\}} \binom{k}{q} \binom{L-k}{r-q} (-1)^{r-q} \alpha^{L-k+2q-r} (i\sqrt{1-\alpha^2})^{k+r-2q} \ . \tag{4.23}$$

This completes the representation of the deformed complex Hermite polynomials in terms of the standard (or undeformed) complex Hermite polynomials.

4.3 Characteristic polynomials of $M(g, L)$

It is not hard to compute the characteristic polynomial of the matrix $M(g, L)$, using the eigenvalues of $M(g, 1)$. Indeed, let us denote by λ_1, λ_2 the two eigenvalues of $M(g, 1)$, corresponding to two non-zero eigenvectors f_1 and f_2. Then it can be seen, in view of (4.15), that the set

$$\Lambda_L = \{\lambda_1^k \lambda_2^{L-k}\, ;\, k = 0, 1, \ldots, L\}, \tag{4.24}$$

gives all possible eigenvalues of the matrix $M(g, L)$. Thus, the characteristic polynomial for the matrices $M(g, L),\ L = 1, 2, 3, \ldots$, would be

$$\begin{aligned}
&(\lambda - \lambda_1)(\lambda - \lambda_2) = p_1(\lambda), \\
&(\lambda - \lambda_1^2)(\lambda - \lambda_1\lambda_2)(\lambda - \lambda_2^2) = p_2(\lambda), \\
&(\lambda - \lambda_1^3)(\lambda - \lambda_1\lambda_2^2)(\lambda - \lambda_1^2\lambda_2)(\lambda - \lambda_2^3) = p_3(\lambda), \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

Chapter 5

Biorthogonal Families of Polynomials

In this chapter we give results which concern biorthogonal families of polynomials arising as dual pairs, from our deformed generalized complex Hermite polynomials $H^{\alpha}_{m,n}(z,\overline{z})$ and $H^{g}_{m,n}(z,\overline{z})$, computed in the previous chapter (see (3.18) and (3.21)). The main result is stated in Theorem 5.2.1 together with its proof.

The understanding of the concept of biorthogonal polynomials emerges from two different mathematical approaches. The first can be found in the seminal work of Iserles and Nørsett in 1988 (see, for example, [41, 42]). They define biorthogonal polynomials with respect to a sequence of measures.

We follow here a second approach, originally proposed by Szego [75], introducing the concept of two sequences of polynomials that are biorthogonal with respect to each other. Our definition of biorthogonality is also similar to that of biorthogonal bases as given, for example, in [15].

5.1 Biorthogonal and biorthonormal bases

Biorthogonal and biorthonormal basis sets generalize the notion of orthogonal and orthonormal basis sets of vectors. Typically they involve a pair of bases [52], one in a vector space and a second one in its dual space. In the case where the vector space has an inner product (such as in a Hilbert space) the two sets of bases vectors can be considered as elements of the same space.

Let V be a vector space and V^* its dual. The dimension of V could be finite or infinite and we denote it by N. Let $\{v_k\}_{k=1}^N$ be a basis of V and $\{\tilde{v}_k\}_{k=1}^N$ a basis of V^*. The two bases are said to be **biorthogonal** if

$$\langle \tilde{v}_j; v_k \rangle = \lambda_k \delta_{jk} = \begin{cases} \lambda_k & \text{if } j = k \\ 0 & \text{if } j \neq k. \end{cases}, \tag{5.1}$$

where the λ_k are non-zero real numbers and $\langle \; ; \; \rangle$ denotes the dual pairing between V and V^*. In the case where $\lambda_k = 1$, for all k, the two bases are said to be **biorthonormal**. If $\{v_k\}_{k=1}^N$ and $\{\tilde{v}_k\}_{k=1}^N$ are biorthonormal basis then any element $x \in V$ can be expanded in terms of these vectors as $x = \sum_{k=0}^N \langle \tilde{v}_k; x \rangle \tilde{v}_k$. If V is an inner product space, such as a Hilbert space, so that V^* can be identified with V, then the $\tilde{v}_k$ may also be thought of as being elements of V. In that case one has a *resolution of the identity* on V:

$$\sum_{k=0}^N |v_k\rangle\langle \tilde{v}_k| = I_V.$$

5.2 Biorthogonal families from g-deformed polynomials

As already mentioned, our g-deformed complex Hermite polynomials $H_{m,n}^g(z, \overline{z})$ cannot be expected to form an orthogonal set, except in very special cases. However, as we show below, there always exists a dual basis $\widetilde{h}^g_{m,n}(z,\overline{z})$, $m+n=L$, of polynomials which are biorthonormal to the normalized polynomials $h^g_{m,n}(z,\overline{z}) = \dfrac{1}{\sqrt{m!n!}} H^g_{m,n}(z,\overline{z})$, $m+n=L$. Recall that when g is the identity matrix, the polynomials $h^g_{m,n}(z,\overline{z})$ are the normalized (classical) complex Hermite polynomials $h^c_{m,n}(z,\overline{z})$ given in (1.21). We have the result

Theorem 5.2.1. *Let*

$$\widetilde{h}^g_{m,n} = [T(g,L)^*]^{-1} h^c_{m,n} = h^{(g^*)^{-1}}_{m,n}, \qquad m+n=L. \tag{5.2}$$

The polynomials $\widetilde{h}^g_{m,n}$, $m+n=L$, are biorthonormal with respect to the polynomials $h^g_{m,n}(z,\overline{z})$, $m+n=L$, i.e.,

$$\int_{\mathbb{C}} \overline{\widetilde{h}^g_{L-n,n}(z,\overline{z})}\; h^g_{M-k,k}(z,\overline{z})\; d\nu(z,\overline{z}) = \delta_{LM}\,\delta_{nk}, \tag{5.3}$$

where, $n=0,1,2,\ldots,L$, $k=0,1,2,\ldots,M$.

Proof. Since the matrix $T(g,L)$, with matrix elements $M(g,L)_{rk}$, constitutes a representation of $GL(2,\mathbb{C})$ on the subspace $\mathcal{H}_L(\mathbb{C})$ of $\mathcal{H}(\mathbb{C})$, generated by the basis $\mathfrak{S}(L)$, we know that $T(g,L)^{-1} = T(g^{-1},L)$. From (4.18) it also follows that $T(g,L)^* = T(g^*,L)$, which implies the second equality in (5.2). The integral in (5.3), giving the biorthonormality, is just the scalar product between the vectors $\widetilde{h}^g_{m,n}$ and $h^g_{m,n}$, so that the equality in that equation follows from (4.18) and the orthonormality relation (1.34) satisfied by the $h^c_{m,n}(z,\overline{z})$. $\square$

To summarize, the Hilbert space $\mathcal{H}(\mathbb{C})$ decomposes into the orthogonal direct sum

$$\mathcal{H}(\mathbb{C}) = \bigoplus_{L=0}^{\infty} \mathcal{H}_L(\mathbb{C}),$$

of $(L+1)$-dimensional subspaces $\mathcal{H}_L(\mathbb{C})$, spanned by the orthonormal basis vectors

$\mathfrak{S}(L)$, consisting of the complex Hermite polynomials $h^c_{L-k.k}$, $k = 0, 1, 2, \ldots, L$ (see (4.20)). On each such subspace the operators $T(g, L)$, $g \in GL(2, \mathbb{C})$ define an $(L+1)\times(L+1)$-matrix representation of $GL(2, \mathbb{C})$. For each $g \in GL(2, \mathbb{C})$ one obtains a set of g-deformed complex Hermite polynomials $h^g_{L-k.k} = T(g, L)h^c_{L-k,k}$, $k = 0, 1, 2, \ldots, L$, in $\mathcal{H}_L(\mathbb{C})$ and a biorthonormal set $\widetilde{h}^{g'}_{L-k.k}$, $k = 0, 1, 2, \ldots, L$, which constitutes a family of g'-deformed complex Hermite polynomials, with $g' = (g^{-1})^\dagger$. It is also clear that the deformed polynomials $h^c_{L-k,k}$, $k = 0, 1, 2, \ldots, L$, form a basis of $\mathcal{H}_L(\mathbb{C})$, for each $g \in GL(2, \mathbb{C})$. The above result has been reported in [6]. A somewhat similar result, though less general and using different techniques, has also been given in [82].

A somewhat more general result, regarding the biorthogonality of the polynomials $H^g_{m,n}(z, \bar{z})$ has been proved in [5], which we state below.

Theorem 5.2.2. *The biorthogonality relation*

$$\int_{\mathbb{C}} H^g_{m,n}(z, \bar{z})\overline{H^{g'}_{k,l}(z, \bar{z})}\, d\nu(z, \overline{z}) = 0,$$

for $(m, n) \neq (k, l)$*, holds if and only if*

$$g'^{*}g = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{5.4}$$

When (5.4) *holds then*

$$\int_{\mathbb{C}} H^g_{m,n}(z, \bar{z})\overline{H^{g'}_{k,l}(z, \bar{z})}\, d\nu(z, \overline{z}) = m!\, n!\, \lambda_1^m\, \lambda_2^n\, \delta_{mk}\, \delta_{nl}. \tag{5.5}$$

5.3 Biorthogonality in non-commutative quantum mechanics

Specializing now to the case of non-commutative quantum mechanics, the matrix g has the form

$$g = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \alpha \end{pmatrix}, \quad \alpha \in \mathbb{R}, \;\; 0 < \alpha < 1, \quad \beta = i\sqrt{1-\alpha^2}\,. \tag{5.6}$$

which is a Hermitian matrix and the corresponding $M(\alpha, L)$ operator has the matrix elements given in (4.23). Let us denote the corresponding deformed polynomials in by $H^{\alpha}_{m,n}(z,\overline{z})$. However, for a matrix of the above type, it is clear that g^{-1} is not of the same type (except in the trivial case where $\alpha = 1$) and hence does not relate to non-commutative quantum mechanics. This means that one cannot have orthonormal families, satisfying Theorem 5.2.1, with both bases coming from matrices of the type (5.6). However, biorthogonal families, of the type appearing in Theorem 5.2.2, do exist with both families of polynomials coming from matrices of this type. Indeed, with g as above, taking

$$g' = \begin{pmatrix} \alpha & -\beta \\ -\overline{\beta} & \alpha \end{pmatrix}, \quad \beta = i\sqrt{1-\alpha^2},$$

we easily see that (5.4) is satisfied, with $\lambda_1 = \lambda_2 = 2\alpha^2 - 1$. We then have the biorthogonality relation

$$\int_{\mathbb{C}} \overline{\widetilde{H}^{-\alpha}_{k,l}(z,\overline{z})} H^{\alpha}_{m,n}(z,\overline{z})\, d\nu(z,\overline{z}) = (2\alpha^2-1)^{m+n}\delta_{mk}\,\delta_{nl}.$$

We end this chapter by explicitly giving the forms of the operators $M(g, L)$ and $M(g', L)$, when g is of the form (5.6) and g' is related to g through $g' = (g^{\dagger})^{-1}$ in a few simple cases.

Case $L = 1$

We have

$$M(g, 1) = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \alpha \end{pmatrix}, \tag{5.7}$$

Since this matrix is Hermitian,

$$M(g', 1) = \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \alpha \end{pmatrix}$$

and clearly $M(g', 1)M(g, 1) = \Delta_1 \mathbb{I}_2$.

Case $L = 2$

In this case the matrix $M(g, L)$ is no longer Hermitian:

$$M(g, 2) = \begin{pmatrix} \alpha^2 & 2\alpha\beta & \beta^2 \\ \alpha\overline{\beta} & \alpha^2 + \beta^2 & \alpha\beta \\ \beta^2 & 2\alpha\overline{\beta} & \alpha^2 \end{pmatrix}. \tag{5.8}$$

One finds that

$$M(g', 2) = \begin{pmatrix} \alpha^2 & 2\alpha\overline{\beta} & \beta^2 \\ \alpha\beta & \alpha^2 + \beta^2 & \alpha\overline{\beta} \\ \beta^2 & 2\alpha\beta & \alpha^2 \end{pmatrix}, \tag{5.9}$$

and again $M(g', 2)M(g, 2) = \Delta_2 \mathbb{I}_3$.

5.4 Hermite polynomials in physics

In this section we draw a connecting line between various types of Hermite polynomials appearing in physics and the ones we discuss in this work.

1. We start with the well known real Hermite polynomials, $H_n(x)$ of a single real variable x. These appear as eigenfunctions of the harmonic oscillator Hamiltonian,

$$H_{osc} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2.$$

The eigenfunctions $|n\rangle,\ \ n = 0, 1, 2, \ldots,$ are obtained by writing

$$a = -i\hbar\frac{d}{dx} - im\omega x, \quad a^\dagger = -i\hbar\frac{d}{dx} + im\omega x, \quad \text{with} \quad [a, a^\dagger] = 1,$$

and defining

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle, \quad \text{and} \quad a|0\rangle = 0.$$

The coordinate space representation of these vectors, by functions on $L^2(\mathbb{R}, dx)$, is

$$\langle x \mid n\rangle := \phi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} H_n(x')e^{-\frac{x'^2}{2}}, \qquad x' = \sqrt{\frac{m\omega}{2\hbar}}x.$$

2. Moving to a two-dimensional oscillator, one has the Hamiltonian

$$H_{2D} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{2}m\omega^2(x^2 + y^2).$$

This time, defining the operators

$$a_1 = -i\hbar\frac{\partial}{\partial x} - im\omega x, \quad a_2 = -i\hbar\frac{\partial}{\partial y} - im\omega y,$$

and their adjoints $a_1^\dagger,\ a_2^\dagger$, on $L^2(\mathbb{R}^2, dx\, dy)$, one obtains the eigenfunctions

$$|n,m\rangle = \frac{(a_1^\dagger)^n (a_2^\dagger)^m}{\sqrt{n!\, m!}} |0,0\rangle, \quad \text{and} \quad a_i|0,0\rangle = 0 \;\; i = 1,2,$$

Again, in the coordinate space representation, these appear as products of two Hermite polynomials,

$$\langle x, y \mid n\rangle := \phi_{n,m}(x,y) = \frac{m\omega}{\pi\hbar} H_n(x') H_m(y') e^{-\frac{x'^2 + y'^2}{2}},$$

with

$$x' = \sqrt{\frac{m\omega}{2\hbar}}\, x, \quad y' = \sqrt{\frac{m\omega}{2\hbar}}\, y.$$

These polynomials, and also some of their variants, have also been used in quantum optical studies using beam-splitters.

3. Next going to the Landau problem of an electron in a constant magnetic field, the Hamiltonian (in some conveniently chosen units) can be written as:

$$H_{\text{elec}} = \frac{1}{2}(\vec{p} - \vec{A})^2 = \frac{1}{2}\left(p_x + \frac{y}{2}\right)^2 + \frac{1}{2}\left(p_y - \frac{x}{2}\right)^2, \tag{5.10}$$

On the Hilbert space $L^2(\mathbb{R}^2, dxdy)$ of our problem, we make the replacements

$$p_x + \frac{y}{2} \longrightarrow Q_1 = -i\frac{\partial}{\partial x} + \frac{y}{2}, \qquad p_y - \frac{x}{2} \longrightarrow P_1 = -i\frac{\partial}{\partial y} - \frac{x}{2}.$$

Then, $[Q_1, P_1] = iI$ and the Hamiltonian becomes

$$H_{\text{elec}} = \frac{1}{2}\left(P_1^2 + Q_1^2\right). \tag{5.11}$$

The eigenvalues of this Hamiltonian, the so-called *Landau levels,* are $E_\ell = (\ell + \frac{1}{2}),\ \ell = 0, 1, 2, \ldots \infty$, each level being infinitely degenerate.

There is a second set of operators that one can define,

$$Q_2 = -i\frac{\partial}{\partial y} + \frac{x}{2} \, , \qquad P_2 = -i\frac{\partial}{\partial x} - \frac{y}{2} \, ,$$

and $[Q_2, P_2] = iI$. The two sets of operators $\{Q_1, P_1\}$ and $\{Q_2, P_2\}$ mutually commute:

$$[Q_2, Q_1] = [P_2, Q_1] = [Q_2, P_1] = [P_2, P_1] = 0 \, .$$

Defining the operators,

$$A_1 = \frac{1}{\sqrt{2}}(iQ_1 - P_1), \qquad A_2 = \frac{1}{\sqrt{2}}(Q_2 + iP_2)$$

and their adjoints, which then satisfy the commutation relations,

$$[A_i, A_i^\dagger] = 1, \qquad i = 1, 2,$$

the eigenstates of the Hamiltonian are

$$\Psi_{\ell,n} := \frac{1}{\sqrt{n!\ell!}} \left(A_1^\dagger\right)^n \left(A_2^\dagger\right)^\ell \Psi_{00},$$

where $n, \ell = 0, 1, 2, \ldots$, where $A_1\Psi_{00} = A_2\Psi_{00} = 0$.

Of course, these eigenstates can be written in terms of the real Hermite polynomials, $H_\ell(x)$, $H_n(y)$, but for the present problem it is often useful to go over to a complex representation of the operators A_i, $A_i^\dagger$ on the Hilbert space $\mathcal{H}(\mathbb{C}) = \mathcal{L}^2(\mathbb{C}, d\nu(z, \overline{z}))$, defined earlier. We do this by applying the composite Wigner transform $\widetilde{\mathcal{W}}$ to the vectors $\Psi_{\ell,n}$ and then we obtain the normalized complex Hermite polynomials $h_{\ell,n}^c$

$$h_{\ell,n}^c = \widetilde{\mathcal{W}}\Psi_{\ell,n}, \qquad \ell, n = 0, 1, 2, \ldots, \infty.$$

4. Finally one arrives at the deformed complex Hermite polynomials $h^g_{\ell,n}$ by using the operator $T(g)$:

$$h^g_{\ell,n} = T(g) h^c_{\ell,n} = T(g)\widetilde{\mathcal{W}}\Psi_{\ell,n} = T(g)\widetilde{\mathcal{W}}\left(\frac{1}{\sqrt{n!\ell!}}\left(A_1^\dagger\right)^n\left(A_2^\dagger\right)^\ell\Psi_{00}\right),$$

with $\ell, n = 0, 1, 2, \ldots, \infty$. These polynomials form the basis of our study of noncommutative quantum mechanics in the present thesis. As mentioned earlier, these polynomials also feature in quantum optical studies using polarized optical modes in beam splitters.

5. We should also mention one other family of Hermite polynomials, which have been used in the study of squeezed states in quantum optics, however which have not been dealt with in this thesis. These Hermite polynomials are obtained by taking the real Hermite polynomials $H_n(x)$ and replacing the real variable x by a complex variable z. The resulting polynomials can also be shown to be orthogonal with respect to a particular measure on the complex plane.

5.5 The Landau problem in NCQM

We end this part of the thesis with a concrete physical example of a situation where the introduction of non-commutativity makes a difference in the observed energy spectrum of a quantum system. We go back to the problem of the electron in a constant magnetic field, for which we had earlier obtained the energy levels using the Hamiltonian (5.10). We now rewrite this Hamiltonian, by replacing the standard position operators by the non-commutative ones. We denote the noncommutative operators with a hat. Thus, we get the Hamiltonian

$$H^{nc}_{elec} = \frac{1}{2}(\vec{P} - \vec{\widehat{A}})^2 = \frac{1}{2}\left(\widehat{P}_1 + \frac{\widehat{Q}_2}{2}\right)^2 + \frac{1}{2}\left(\widehat{P}_2 - \frac{\widehat{Q}_1}{2}\right)^2, \tag{5.12}$$

where

$$\widehat{Q}_1 = q_x - \frac{\Theta}{2}p_y, \quad \widehat{Q}_2 = q_y + \frac{\Theta}{2}p_x, \quad \widehat{P}_1 = p_x, \;\; \widehat{P}_2 = p_y.$$

Substituting in (5.12)

$$\begin{aligned} H_{elec}^{nc} &= \frac{1}{2}\Big(p_x + \frac{q_y}{2} + \frac{\Theta}{4}p_x\Big)^2 + \frac{1}{2}\Big(p_y - \frac{q_x}{2} + \frac{\Theta}{2}p_y\Big)^2, \\ &= \frac{1}{2}\Big[(1+\frac{\Theta}{4})p_x + \frac{q_y}{2}\Big]^2 + \frac{1}{2}\Big[(1+\frac{\Theta}{4})p_y - \frac{q_x}{2}\Big]^2. \end{aligned}$$

Let $\gamma = (1 + \frac{\Theta}{4})$ and introduce the operators

$$\widetilde{Q}_1 = \sqrt{\gamma}p_x + \frac{1}{\sqrt{\gamma}}\frac{q_y}{2}, \quad \widetilde{P}_1 = \sqrt{\gamma}p_y - \frac{1}{\sqrt{\gamma}}\frac{q_x}{2}, \tag{5.13}$$

then

$$[\widetilde{Q}_1, \widetilde{P}_1] = -\frac{1}{2}[p_x, q_x] + \frac{1}{2}[q_y, p_y] = i.$$

Finally we get the Hamiltonian:

$$\begin{aligned} H_{elec}^{nc} &= \frac{1}{2}\gamma\Big[\Big(\sqrt{\gamma}p_x + \frac{1}{\sqrt{\gamma}}\frac{q_y}{2}\Big)^2 + \Big(\sqrt{\gamma}p_y - \frac{1}{\sqrt{\gamma}}\frac{q_x}{2}\Big)^2\Big] \\ &= \frac{1}{2}(1+\frac{\Theta}{4})(\widetilde{P}_1^2 + \widetilde{Q}_1^2). \end{aligned}$$

We conclude that the effect of non-commutativity is to shift all the energy levels by the amount $\frac{\Theta}{8}$:

$$\Delta E = \frac{\Theta}{8}.$$

As can be easily seen, introducing an additional non-commutativity, where the two observables of momentum also do not commute, would just shift the energies by an additional amount.

Chapter 6

Some Associated Lie Groups and Deformed Lie Algebra

Introduction

This chapter constitutes the second part of this thesis. We present here some group structures that emerge from the Lie algebras built using bilinear combinations of the noncommutative raising and lowering operators in (3.1). These algebras and groups depend on the non-commutative parameter α, and we explore the situations as α ranges through its allowed values: $0 < \alpha < 1$. The results obtained in this chapter are independent of the first part of the thesis and represents work in progress.

6.1 The associated rotation-like operators

Definition 6.1.1. ***Rotation operators***.

Let $a_j,\ a_k^\dagger,\ j,k = 1,2$ be a pair of the standard raising and lowering operators. Then, the angular momentum operators can be expressed in terms of these as

$$J_1 = \frac{1}{2}\Big(a_1^\dagger a_2 + a_2^\dagger a_1\Big), \tag{6.1a}$$

$$J_2 = \frac{1}{2i}\Big(a_1^\dagger a_2 - a_2^\dagger a_1\Big), \tag{6.1b}$$

$$J_3 = \frac{1}{2}\Big(a_1^\dagger a_1 - a_2^\dagger a_2\Big). \tag{6.1c}$$

They satisfy the commutation relations

$$[J_j, J_k] = i\varepsilon_{jkl}J_l. \tag{6.2}$$

where ε_{jkl} is the completely antisymmetric tensor. In addition, we have the two number operators

$$N_1 = a_1^\dagger a_1, \qquad N_2 = a_2^\dagger a_2. \tag{6.3}$$

Equation (6.2) defines the well-known Lie algebra $\mathfrak{su}(2)$ of the rotation group $SU(2)$ in three dimensions, the generators J_i being given in the Schwinger representation [58].

Remark 6.1.1. *Note that the representation of the rotation generators in (6.1) is not unique. For example, one can refer to [10] or to [29] for other representations of the angular momentum operators in terms of ladder operators.*

The transformation to the non-commutative space is straightforward. We use the operators of noncommutative quantum mechanics in (3.1) and write

$$J_1^\alpha = \frac{1}{2}\Big(a_1^{\alpha\dagger} a_2^\alpha + a_2^{\alpha\dagger} a_1^\alpha\Big), \tag{6.4a}$$

$$J_2^\alpha = \frac{1}{i2}\Big(a_1^{\alpha\dagger} a_2^\alpha - a_2^{\alpha\dagger} a_1^\alpha\Big), \tag{6.4b}$$

$$J_3^\alpha = \frac{1}{2}\Big(a_1^{\alpha\dagger} a_1^\alpha - a_2^{\alpha\dagger} a_2^\alpha\Big). \tag{6.4c}$$

Also, the deformed number operators take the form

$$N_1^\alpha = a_1^{\alpha\dagger} a_1^\alpha, \qquad N_2^\alpha = a_2^{\alpha\dagger} a_2^\alpha. \tag{6.5}$$

Further, let $J_4 := \frac{1}{2}\Big(N_1 + N_2\Big)$ and $J_4^\alpha := \frac{1}{2}\Big(N_1^\alpha + N_2^\alpha\Big)$, then the J_j^α can be written as a linear combinations of the J_j:

$$J_1^\alpha = (2\alpha^2 - 1)J_1, \tag{6.6a}$$

$$J_2^\alpha = J_2 - 2\alpha\sqrt{1-\alpha^2}J_4, \tag{6.6b}$$

$$J_3^\alpha = (2\alpha^2 - 1)J_3, \tag{6.6c}$$

$$J_4^\alpha = -2\alpha\sqrt{1-\alpha^2}J_2 + J_4. \tag{6.6d}$$

To summarize we have a matrix transformation:

$$\begin{pmatrix} J_1^\alpha \\ J_2^\alpha \\ J_3^\alpha \\ J_4^\alpha \end{pmatrix} = \begin{pmatrix} 2\alpha^2 - 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2\alpha\sqrt{1-\alpha^2} \\ 0 & 0 & 2\alpha^2 - 1 & 0 \\ 0 & -2\alpha\sqrt{1-\alpha^2} & 0 & 1 \end{pmatrix} \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \end{pmatrix}. \tag{6.7}$$

6.2 The associated deformed Lie algebra

We know that a_j^α and $a_k^{\alpha\dagger}$ satisfy

$$[a_j^\alpha, a_k^{\alpha\dagger}] = \delta_{jk} + \epsilon_{jk} i2\alpha\sqrt{1-\alpha^2}. \tag{6.8}$$

Therefore, we can compute the commutation relations for J_j^α which give us new sets of commutation relations (the deformed Lie algebra):

$$\begin{aligned}
&[J_1^\alpha, J_2^\alpha] = iJ_3^\alpha, && [J_2^\alpha, J_3^\alpha] = iJ_1^\alpha, \\
&[J_3^\alpha, J_4^\alpha] = i2\alpha\sqrt{1-\alpha^2}J_1^\alpha, && [J_4^\alpha, J_1^\alpha] = i2\alpha\sqrt{1-\alpha^2}J_3^\alpha, \\
&[J_3^\alpha, J_1^\alpha] = iJ_2^\alpha + i2\alpha\sqrt{1-\alpha^2}J_4^\alpha, && [J_2^\alpha, J_4^\alpha] = 0,
\end{aligned} \tag{6.9}$$

Remark 6.2.1. *Details of the computations leading to (6.9) can be found in Appendix. Since the commutation relations close, they define a Lie algebra.*

For convenience, we define a scale factor

$$\vartheta \equiv \alpha\sqrt{1-\alpha^2}, \tag{6.10}$$

and will point out the range of values of this parameter, considering only positive values for ϑ. Since

$$\vartheta^2 = 4\alpha^2(1-\alpha^2), \tag{6.11}$$

we get two (non-independent) NC parameters obeying the following constraints

$$0 < \vartheta < 1, \qquad 0 < \alpha < 1. \tag{6.12}$$

Also

$$\vartheta = 1 \implies \alpha = \frac{1}{\sqrt{2}},$$

and

$$\vartheta = 0 \implies \alpha = 0, \text{ or } \alpha = 1.$$

These limits are essential for the identification of the associated Lie algebras (Lie groups).

Using (6.10) we get the revised commutation relations

$$\begin{aligned} &[J_1^\alpha, J_2^\alpha] = iJ_3^\alpha, && [J_2^\alpha, J_3^\alpha] = iJ_1^\alpha, \\ &[J_4^\alpha, J_1^\alpha] = i\vartheta J_3^\alpha, && [J_3^\alpha, J_4^\alpha] = i\vartheta J_1^\alpha, \\ &[J_3^\alpha, J_1^\alpha] = iJ_2^\alpha + i\vartheta J_4^\alpha, && [J_2^\alpha, J_4^\alpha] = 0. \end{aligned} \tag{6.13}$$

We now have a Lie algebra that is generated by four elements $J_j^\alpha, j = 1, 2, 3, 4$.

The commutation relations of J_j^α is summarized in the table below:

	J_1^α	J_2^α	J_3^α	J_4^α
J_1^α	0	iJ_3^α	$-i(J_2^\alpha + \vartheta J_4^\alpha)$	$-i\vartheta J_3^\alpha$
J_2^α	$-iJ_3^\alpha$	0	iJ_1^α	0
J_3^α	$i(J_2^\alpha + \vartheta J_4^\alpha)$	$-iJ_1^\alpha$	0	$i\vartheta J_1^\alpha$
J_4^α	$i\vartheta J_3^\alpha$	0	$-i\vartheta J_1^\alpha$	0

6.3 Analysis of the associated algebras of deformed generators

In this section, we will analyze the associated Lie algebra $\mathfrak{g}$ generated by the commutation relations (6.13). It is convenient to first make a change of basis. The idea is to find an isomorphic set of commutation relations, which would render the identification of the algebras simple. Following [39], we define two mutually commuting Lie algebras, $\mathfrak{g}_1$ and $\mathfrak{g}_2$, generated by the elements of the transformed basis as follows.

6.3.1 Type I

By the change of basis

$$X_1^\vartheta = iJ_1^\alpha, \qquad X_2^\vartheta = iJ_3^\alpha, \qquad X_3^\vartheta = i(J_2^\alpha + \vartheta J_4^\alpha), \qquad X_j^\vartheta \in \mathfrak{g}_1 \tag{6.14a}$$

$$Y^\vartheta = \{\vartheta J_2^\alpha + J_4^\alpha\} \in \mathfrak{g}_2, \tag{6.14b}$$

equation (6.13) will transform to

$$[X_1^\vartheta, X_2^\vartheta] = X_3^\vartheta, \qquad [X_2^\vartheta, X_3^\vartheta] = (1-\vartheta^2)X_1^\vartheta, \qquad [X_3^\vartheta, X_1^\vartheta] = (1-\vartheta^2)X_2^\vartheta,$$
$$[Y^\vartheta, X_j^\vartheta] = 0. \tag{6.15}$$

6.3.2 Type II

With the re-scaled generators

$$Z_1^\vartheta = \sqrt{1-\alpha^2}X_1^\vartheta, \qquad Z_2^\vartheta = X_2^\vartheta, \qquad Z_3^\vartheta = \sqrt{1-\alpha^2}X_3^\vartheta, \tag{6.16}$$

together with the Y^ϑ above, we get the commutators:

$$[Z_j^\vartheta, Z_k^\vartheta] = \epsilon_{jkl}Z_l^\vartheta, \qquad [Y^\vartheta, Z_j^\vartheta] = 0. \tag{6.17}$$

Our next set of results are obtained upon substituting the limits of the parameter ϑ. In the Type I situation we take the limit $\vartheta \to 0$, which leads to

$$[X_j^0, X_k^0] = \varepsilon_{jkl}X_l^0, \quad [Y^0, X_j^0] = 0, \quad j,k = 1,2,3. \tag{6.18}$$

Thus $\mathfrak{g}_1 = \mathfrak{su}(2)$ and as expected,

$$\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{u}(1), \qquad \vartheta = 0. \tag{6.19}$$

Note that in this case $\alpha = 0$, so that we are in the undeformed (so-called classical) situation.

For $\vartheta \to 1$, and again for the Type I basis, we have

$$[X_1^1, X_3^1] = [X_2^1, X_3^1] = 0, \quad [X_1^1, X_2^1] = X_3^1, \quad [Y^1, X_j^1] = 0, \tag{6.20}$$

$i = 1, 2, 3$, which is a nilradical basis, isomorphic to the Heisenberg algebra $\mathfrak{h}$ [36]. Thus,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{u}(1), \qquad \vartheta = 1. \tag{6.21}$$

Finally for $0 < \vartheta < 1$, using the Type II basis, we again get

$$[Z_j^\vartheta, Z_k^\vartheta] = \varepsilon_{jkl} Z_l^\vartheta, \quad [Y^\vartheta, Z_j^\vartheta] = 0, \quad j, k = 1, 2, 3, \tag{6.22}$$

as in (6.18) and once again

$$\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{u}(1), \qquad 0 < \vartheta < 1. \tag{6.23}$$

Summarizing, according to our analysis, except in the case where $\vartheta = 1$, the algebra generated by the deformed generators is that of $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$, exactly as in the undeformed case. The corresponding Lie group is $SU(2) \times U(1)$. In the other limiting case of $\vartheta = 1$, $(\alpha^2 = \frac{1}{2})$, the Lie nilgebra is the direct product of the Weyl-Heisenberg group with $U(1)$. Also, in this case the commutation relation $[a_1^\alpha, a_2^{\alpha\dagger}] = 2i\alpha\sqrt{1-\alpha^2}$ in (6.8) becomes $[a_1^\alpha, a_2^{\alpha\dagger}] = i$.

Chapter 7

Conclusion

We end this thesis with summaries of the results obtained in its two parts and indication of possible future work.

Part I

In the first part (Chapters 1 - 5) of the thesis, we began by describing the basic structure of non-commutative quantum mechanics and then obtaining complex Hermite polynomials associated to the oscillator algebra of two degrees of freedom. These were termed the standard or classical complex Hermite polynomials; they form an orthonormal basis for the Hilbert space $\mathcal{H}(\mathbb{C})$. From there we went on to obtain the first group of main results of the thesis. This involved defining the families of deformed complex Hermite polynomials and obtaining some of their properties. Each family is characterized by a 2×2 complex, nonsingular matrix g and we derived the operator T_g, mapping the undeformed polynomials to the deformed ones. The deformed polynomials form a basis for $\mathcal{H}(\mathbb{C})$ but are not mutually orthogonal. However for each such family we constructed a dual basis which is biorthogonal to the previous one.

Part II

The second part of the work (Chapter 6) moves in a different direction. Using the deformed raising and lowering operators constructed earlier, we form bilinear combinations of them, to get deformed versions of the Schwinger type of rotation generators. We next study the Lie algebras formed by these generators for different ranges of values of the deformation parameter. Interestingly, except in one particular case, the deformed algebra turns out to be identical to the undeformed one.

Result highlights

Two papers, [5] and [6], one completed and already posted on the arXiv ([6]) and to be submitted for publication, and the other one ([5]) nearing completion, are expected from the thesis. These report on the results detailed in Chapters 3 to 5. The main results in these papers are the contents of Theorems 3.5.1 and 5.2.1. It is expected that a third publication, dealing with certain functional analytic properties of such biorthogonal sets will be worked out soon.

Standard Hermite polynomials are considered to be functions of a real variable. The extension of these polynomials to a complex variable has been considered in the literature before (cf., for example, [4] and references contained therein). These polynomials have been found useful in the theory of squeezed coherent states in quantum optics. Our concern in this thesis has been with a different set of complex Hermite polynomials, namely those of two complex variables and they find applications to NCQM. Moreover, a related set of polynomials have also been shown to have applications to quantum optics [82].

Comments and outlook

There are at least two future directions for the work started in this thesis. The first would be to extend the analysis of the deformed polynomials to higher dimensions, i.e., to polynomials in several complex variables. Secondly we would like to look at the problem of deformation quantization for our model and study its possible connections with the deformed polynomials.

Bibliography

[1] C. Acatrinei, *Path integral formulation of noncommutative quantum mechanics*, J. High Energy Phys. 09:007 (2001).

[2] B.G. Adam, J. Cizek and J. Paldus. *Lie algebraic methods and their applications to simple quantum systems*, Advances in Quantum Chemistry, 19:1-85 (1987) (reproduced in [9]).

[3] S.T. Ali, F. Bagarello and G. Honnouvo, *Modular Structures on trace class operators and applications to Landau Levels*, J. Phys. A: Math. Theor. 43:105202 (2010).

[4] S.T. Ali, K. Górska, A. Horzela, and F. H. Szafraniec, *Squeezed states and Hermite polynomials in a complex variable*, J. Math. Phys. 55:012107 (2014).

[5] S.T. Ali, Mourad E.H. Ismail and Nurisya M. Shah, *Deformed complex Hermite polynomials*, *Concordia University ppt. 2014*.

[6] F. Balogh, Nurisya M. Shah and S.T. Ali, *Some biorthogonal families of polynomials arising in noncommutative quantum mechanics*, (2013). [arXiv: math-ph/1309.4163]

[7] V. Bargmann, *On a Hilbert space of analytic functions and associated integral transform, Part II. A family of related function spaces application to distribution theory*, Comm. on Pure and Applied Math. 20:1-101 (1967).

[8] C. Bastos, O. Bertolami, N.C. Dias and J.N. Prata, *Weyl-Wigner formulation of noncommutative quantum mechanics*, J. Math. Phys. 49:072101 (2008). *Deformation quantization of noncommutative quantum mechanics and dissipation*, J. of Phys.: Conf. Series 67:012058 (2007).

[9] A. Bohm, Y. Ne'eman and A.O. Barut, *Dynamical Groups and Spectrum Generating Algebras*, Vol. 1, World Scientific Pub. (1988).

[10] S.K. Bose, *Dynamical algebra of spin waves*, J. Phys. A: Math. Gen., Vol. 18:903-922 (1985) (reproduced in [9]).

[11] M. Chaichian, M.M. Sheikh-Jabbari and A. Tureanu, *Hydrogen atom spectrum and the Lamb shift in non-commutative QED*, Phys. Rev. Lett. 86: (2001).

[12] I. Chepelve and C. Ciorcarlie, *A path integral approach to noncommutative superspace*, J. High Energy Phys. 06: (2003).

[13] S.H.H. Chowdhury and S.T. Ali, *Triply extended group of translations of $\mathbb{R}^4$ as defining group of NCQM: relation to various gauges*, J. Phys. A: Vol. 47:085301 (2014).

[14] S.H.H. Chowdhury and S. T. Ali, *The symmetry groups of noncommutative quantum mechanics and coherent state quantization*, Journal of Mathematical Physics, 54:032101 (2013).

[15] O. Christensen, *An introduction to frames and Riesz bases*, Applied and numerical harmonic analysis, Birkhausar Boston (2003).

[16] A. Connes, *Noncommutative Geometry*, Academic Press, San Diego, CA, (1994).

[17] A. Connes and M. Marcolli, *Noncommutative Geometry: Quantum Fields and Motives*, AMS Colloquium Publications, 55, (2007).

[18] J.F. Cornwell, *Group theory in physics*, Academic Press, Vol. 1 and 2 (1984).

[19] I. Daubechies and A. Grossmann, *Frames in the Bargmann space of entire functions*, Commun. Pure Appl. Math. 41:151-164 (1988).

[20] O.F. Dayi and L.T. Kelleyane, *Wigner Functions for the Landau Problem in Noncommutative Spaces*, Mod.Phys.Lett. A17:1937-1944 (2002).

[21] F. Delduc, Q. Duret, F. Gieres and M. Lefrancois, *Magnetic fields in noncommutative quantum mechanics*, J. Phys.: Conf. Ser., 103:012020 (2008).

[22] M Demetrian and D. Kochan, *Quantum mechanics on non-commutative plane*, Acta Physica Slovaca 52, 1:1-9, (2002).

[23] N.C. Dias, M. Gosson, F. Luef and J.N. Prata, *A deformation quantization theory for noncommutative quantum mechanics*, J. Maths. Phys., 51:072101 (2010).

[24] M.R. Douglas and N.A. Nekrasov, *Noncommutative Field Theory*, Rev. Mod. Phys. 73:977-1029, (2001).

[25] S. Dulat, Kang Li and Jianhua Wang, *Wigner Functions for the Landau Problem in Noncommutative Spaces*, Theoretical and Mathematical Physics, 167:628-635 (2011).

[26] K. Fujikawa, *Path integral for space-time noncommutative field theory*, Phys. Rev. D 70:085006 (2004).

[27] W. Fulton and J. Harris, *Representation Theory: A first course*, Springer-Verlag (1991).

[28] J. Gamboa, M. Loewe, F. Mendez and J.C. Rojas, *The Landau problem and noncommutative Quantum Mechanics*, Mod. Phys. Lett. A16:2075-2078 (2001).

[29] J. Gamboa, M. Loewe and J.C. Rojas, *Non-commutative Quantum Mechanics*, Phys. Rev. D 64:067901 (2001).

[30] J. B. Geloun, J. Govaerts and M.N. Hounkonnou, *A (p, q)-deformed Landau problem in a spherical harmonic well: spectrum and non-commutative coordinates*, EPL 80:30001 (2007).

[31] J. B. Geloun, S. Gangopadhyay and F.G. Scholtz, *Harmonic oscillator in a background magnetic field in noncommutative quantum phase space*, EPL, 86:51001 (2009).

[32] A. Ghanmi, *A class of generalized complex Hermite polynomials*, J. Math. Anal. and App., 340:13951406 (2008).

[33] R. Gilmore, *Lie groups, physics and geometry* 2nd Edition, Cambridge University Press (2008).

[34] P.R. Giri and P. Roy, *Noncommutative oscillator, symmetry and Landau problem*, Eur.Phys.J.C57:835-839, (2008).

[35] D.J. Griffiths, *Introduction to Quantum Mechanics*, Addison-Wesley (2004).

[36] B.C. Hall, *Lie Groups, Lie algebras and Representations: An Elementary Introduction*, Springer-Verlag, New York Inc. (2003).

[37] A. Hatzinikitas and I. Smyrnakis, *The noncommutative harmonic oscillator in more than one dimension*, J. Math. Physics, 43:113-125 (2002).

[38] M.N. Houkonnou and D.O. Samary, *Spectrum of the harmonic oscillator in a general noncommutative phase space*, (2011). Pre-print[math-ph1108.1585].

[39] F. Iachello, *Lie algebras and applications*, Springer (2006).

[40] A. Intissar and A. Intissar. Spectral properties of the Cauchy transform on $l_2(\mathbb{C}, e^{-|z|^2}\lambda(z))$. J. Math. Anal. and Applications., 313:400-418 (2006).

[41] A. Iserles and S. P. Nørsett, *On the theory of biorthogonal polynomials*, Transactions of the American Mathematical Society, 306:455-474 (1988).

[42] A. Iserles, *Biorthogonal polynomials: Recent developments*, Numerical Algorithms, 11:215-228 (1996).

[43] Jian-Zu Zhang, *Constraint on Quantum Gravitational Well and Bose - Einstein Statistics in Noncommutative Space* (2006). Pre-print [hep-th/0508164]

[44] Kang Li, Jianhua Wang and Chiyi Chen *Representation of Noncommutative Phase Space*, Modern Physics Letters A, 20:2165-2174 (2005).

[45] Kang Li, Ciao Xiao-Hua and Wnag Dong-Yan, *Heisenberg algebra for noncommutative Landau problem*, Chin. Phys. Soc. **15** 2236 (2006).

[46] Kang Li and S. Dulat, *Non-commutative phase space and its space-time symmetry*, Chin. Phys. C 34:944948 (2012).

[47] M. Khalkhali, *Very basic noncommutative geometry*, (2004). Pre-print [math/0408416]

[48] S. Khan and M.A. Pathan, *Lie-theoretic generating relations of 2D-Hermite polynomials*, J. Comput. Appl. Math **160** (2003) 139-146.

[49] A.A. Kirillov, *Elements of the Theory of Representations*, Springer-Verlag (1976).

[50] J.R. Klauder and B.S Skagerstam, *Coherent States: Applications in Physics and Mathematical Physics*, Singapore, Philadelphia, World Scientific, (1985).

[51] A. Kokado, T. Okamura and T. Saito, *Noncommutative quantum mechanics and Seiberg-Witten map*, Phys. Rev. D69:125007 (2004).

[52] J. Kovačević and A. Chebira, *Life Beyond Bases: The Advent of Frames*, Signal Processing Magazine, IEEE, 24:86-104 (2007).

[53] Lin Bing-Sheng and Heng Tai-Hua, *Energy spectra of the harmonic oscillator in a generalized noncommutative phase space of arbitrary dimension*, Chin. Phys. Lett. 28:070303 (2011).

[54] S.D. Lindenbaum, *Mathematical methods in physics*, World Scientific (1996).

[55] G. Magro, *Noncommuting coordinates in the Landau problem*,(2003). Pre-print [quant-ph/0302001]

[56] A. Messiah, *Quantum mechanics*, Dover Publications Inc. (1999).

[57] Mourad E.H. Ismail, *Analytic Properties of Complex Hermite Polynomials*, to appear in Transaction of AMS.

[58] B. Muthukumar and P. Mitra, *Non-commutative oscillators and the commutative limit*, Phys. Rev. D66:027701 (2002).

[59] V.P. Nair and A.P. Polychronakos, *Quantum mechanics on the noncommutative plane and sphere*, Phys. Lett. B505:267-274 (2001).

[60] E. Prugovečki, *Quantum Mechanics in Hilbert space*, 2nd Ed. Dover Publications (2006).

[61] M. Reed and B. Simon, *Methods of modern mathematical physics* Vol. 1, Elsevier (1980).

[62] M. Riccardi, *Physical observables for noncommutative Landau levels*, J. Phys. A: Math. Gen. 39:4257 (2006).

[63] C.M. Rohwer, K.G. Zloshchastiev, L. Gouba and F.G. Scholtz, *Noncommutative quantum mechanics-a perspective on structure and spatial extent*, J. Phys. A: Math. Theor. 43:345302 (2010).

[64] J.L. Rubin and P. Winternitz, *Solvable Lie algebras with Heisenberg ideals*, J. Phys. A: Math. Gen. 26:1123 (1993).

[65] L.I. Schiff, *Quantum Mechanics*, McGraw Hill (1968).

[66] F.G. Scholtz, B. Chakraborty, S. Gangopadhyay and A.G. Hazra, *Dual families of Non-commutative quantum system*, Phys. Rev. D71:085005 (2005).

[67] F.G. Scholtz, B. Chakraborty, J. Govaerts and S. Vaidya, *Spectrum of the non-commutative spherical well*, J. Phys. A: Math. Theor. 40:14581-14592 (2007).

[68] F.G. Scholtz, L. Gouba, A. Hafver and C.M. Rohwer, *Formulation, interpretation and application of non-commutative quantum mechanics* J. Phys. A42:175303, (2009).

[69] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, J. High Energy Physics 9909:032 (1999).

[70] Sicong Jing and Bingsheng Lin, *A new kind of representations on noncommutative phase space*, Phys. Lett. A 372:7109-7116 (2008).

[71] R.A. Silverman, *Introductory Complex Analysis*, Dover Publications Inc. (1972).

[72] A. Smailagic, *Isotropic representation of noncommutative 2D harmonic oscillator*, Phys. Rev. D65:107701 (2002).

[73] A. Smailagic and E. Spallucci, *Noncommutative 3D harmonic oscillator*, J. Phys. A: Math. Gen. **35**, L363-L368 (2002). [hep-th/0205232].

[74] M. Sugiura, *Unitary Representations and Harmonic Analysis*, 2nd Ed. Elsevier Science Publishing Company, INC. (1990).

[75] G. Szegö, *Orthogonal Polynomials*, AMS Colloq. Publications (1939), reprinted (2003).

[76] C.C. Tannoudji, B. Diu and F. Laloe, *Quantum Mechanics*, Vol. 1 and 2, John Wiley and Sons (1977).

[77] C.L. Thibaut, *Holomorphic Function Theory in Several Variables*, Springer. (2011).

[78] Torben T. Nielsen, *Bose Algebra: The complex and real wave representations*, Lecture notes in Mathematics, Springer-Verlag (1991).

[79] B. Van Der Waerden, *Sources of quantum mechanics*, Classics of Science Vol. 5, Dover (1967).

[80] H.J. Weber, H.J. and G.B. Arfken, *Essential Mathematical Methods for Physicists*, Elsevier Science (2004).

[81] E.P. Wigner, *Group theory and its application to the quantum mechanics of atomic spectra*, Academic Press (translated by Griffin J.J.) (1959).

[82] A. Wuënsche, *General Hermite and Laguerre two-dimensional polynomials*, J. Phys. A: Math. Gen. **33**:1603-1629 (2000).

[83] B.G. Wybourne, *Classical Groups for Physicists*, John Wiley and Sons, (1974).

[84] K. Zhu, *An Introduction to Operator Algebras*, CRC Press (1993).

Appendix

The Form of J_j^α in terms of J_j

We write the equations for J_j^α, $j = 1, 2, 3$ with respect to J_j. (Note: here we are denoting by A_1, A_2 and $A_1^\dagger, A_2^\dagger$ the generic deformed raising and lowering operators.)

$$\begin{aligned}
J_1^\alpha &= \frac{1}{2}\Big(A_1^\dagger A_2 + A_2^\dagger A_1\Big),\\
&= \frac{1}{2}\Big((\alpha a1^\dagger - i\sqrt{1-\alpha^2}a_2^\dagger)(\alpha a_2 - i\sqrt{1-\alpha^2}a_1)\\
&\quad + (\alpha a_2^\dagger + i\sqrt{1-\alpha^2}a_1^\dagger)(\alpha a_1 + i\sqrt{1-\alpha^2}a_2)\Big),\\
&= \frac{1}{2}(\alpha^2 + (i\sqrt{1-\alpha^2})^2)\Big(a_1^\dagger a_2 + a_2^\dagger a_1\Big),\\
&= (\alpha^2 + (i\sqrt{1-\alpha^2})^2)J_1,\\
&= (2\alpha^2 - 1)J_1.
\end{aligned} \tag{1}$$

$$\begin{aligned}
J_2^\alpha &= \frac{1}{i2}\Big(A_1^\dagger A_2 - A_2^\dagger A_1\Big),\\
&= \frac{1}{i2}\Big((\alpha a_1^\dagger - i\sqrt{1-\alpha^2}a_2^\dagger)(\alpha a_2 - i\sqrt{1-\alpha^2}a_1)\\
&\quad - (\alpha a_2^\dagger + i\sqrt{1-\alpha^2}a_1^\dagger)(\alpha a_1 + i\sqrt{1-\alpha^2}a_2)\Big),\\
&= \frac{1}{i2}(\alpha^2 + i\sqrt{1-\alpha^2}^2)\Big(a_1^\dagger a_2 - a_2^\dagger a_1\Big) + i\alpha\sqrt{1-\alpha^2}(a_1^\dagger a_1 + a_2^\dagger a_2),\\
&= J_2 - \alpha\sqrt{1-\alpha^2}J_4.
\end{aligned} \tag{2}$$

$$
\begin{aligned}
J_3^\alpha &= \frac{1}{2}\Big(A_1^\dagger A_1 - A_2^\dagger A_2\Big),\\
&= \frac{1}{2}\Big((\alpha a_1^\dagger - i\sqrt{1-\alpha^2}a_2^\dagger)(\alpha a_1 + i\sqrt{1-\alpha^2}a_2)\\
&\quad - (\alpha a_2^\dagger + i\sqrt{1-\alpha^2}a_1^\dagger)(\alpha a_2 - i\sqrt{1-\alpha^2}a_1)\Big),\\
&= \frac{1}{2}(\alpha^2 + (i\sqrt{1-\alpha^2})^2)\Big(a_1^\dagger a_1 - a_2^\dagger a_2\Big),\\
&= (\alpha^2 + (i\sqrt{1-\alpha^2})^2)J_3,\\
&= (2\alpha^2 - 1)J_3.
\end{aligned}
$$

The deformed number operator is (3)

$$
\begin{aligned}
J_4^\alpha &= \frac{1}{2}\Big(A_1^\dagger A_1 + A_2^\dagger A_2\Big),\\
&= \frac{1}{2}\Big((\alpha a_1^\dagger - i\sqrt{1-\alpha^2}a_2^\dagger)(\alpha a_1 + i\sqrt{1-\alpha^2}a_2)\\
&\quad + (\alpha a_2^\dagger + i\sqrt{1-\alpha^2}a_1^\dagger)(\alpha a_2 - i\sqrt{1-\alpha^2}a_1)\Big),\\
&= \frac{1}{2}\Big(a_1^\dagger a_1 + a_2^\dagger a_2 + i\alpha\sqrt{1-\alpha^2}a_1^\dagger a_2 - i\sqrt{1-\alpha^2}\alpha a_2^\dagger a_1 - i\alpha\sqrt{1-\alpha^2}a_2^\dagger a_1\\
&\quad + i\sqrt{1-\alpha^2}\alpha a_1^\dagger a_2\Big),\\
&= J_4 + i2\alpha\sqrt{1-\alpha^2}(a_1^\dagger a_2 - a_2^\dagger a_1),\\
&= J_4 - 2\alpha\sqrt{1-\alpha^2}J_2.
\end{aligned}
\tag{4}
$$

Details on commutation relations of J_j^α:

$$
\begin{aligned}
[J_1^\alpha, J_2^\alpha] &= \Big[\frac{1}{2}\Big(A_1^\dagger A_2 + A_2^\dagger A_1\Big), \frac{1}{i2}\Big(A_1^\dagger A_2 - A_2^\dagger A_1^\dagger A_2\Big)\Big], \\
&= \frac{1}{i4}[A_1^\dagger, A_2]A_1^\dagger A_2 + \frac{1}{i4}[A_2, A_1^\dagger]A_1^\dagger A_2 + \frac{1}{i4}[A_1, A_1^\dagger]A_2^\dagger A_2 + \frac{1}{i4}[A_2^\dagger, A_2]A_1^\dagger A_1 \\
&\quad - \frac{1}{i4}[A_1^\dagger, A_1]A_2^\dagger A_2 - \frac{1}{i4}[A_2, A_2^\dagger]A_1^\dagger A_1 - \frac{1}{i4}[A_2^\dagger, A_1]A_2^\dagger A_1 - \frac{1}{i4}[A_1, A_2^\dagger]A_2^\dagger A_1, \\
&= (2\alpha^2 - 1)J_3^\alpha. \qquad (5)
\end{aligned}
$$

$$
\begin{aligned}
[J_1^\alpha, J_4^\alpha] &= \Big[\frac{1}{2}\Big(A_1^\dagger A_2 + A_2^\dagger A_1\Big), A_1^\dagger A_1 + A_2^\dagger A_2\Big], \\
&= \frac{1}{2}[A_1^\dagger, A_1]A_1^\dagger A_2 + \frac{1}{2}[A_2, A_1^\dagger]A_1^\dagger A_1 + \frac{1}{2}[A_2^\dagger, A_1]A_1^\dagger A_1 + \frac{1}{2}[A_1, A_1^\dagger]A_2^\dagger A_1 \\
&\quad + \frac{1}{2}[A_2, A_2^\dagger]A_1^\dagger A_2 + \frac{1}{2}[A_1^\dagger, A_2]A_2^\dagger A_2 + \frac{1}{2}[A_1, A_2^\dagger]A_2^\dagger A_2 + \frac{1}{2}[A_2^\dagger, A_2]A_2^\dagger A_1, \\
&= i4\alpha\sqrt{1-\alpha^2}J_3^\alpha. \qquad (6)
\end{aligned}
$$

$$
\begin{aligned}
[J_2^\alpha, J_4^\alpha] &= \Big[\frac{1}{i2}\Big(A_1^\dagger A_2 - A_2^\dagger A_1\Big), A_1^\dagger A_1 + A_2^\dagger A_2\Big], \\
&= \frac{1}{i2}[A_1^\dagger, A_1]A_1^\dagger A_2 + \frac{1}{i2}[A_2, A_1^\dagger]A_1^\dagger A_1 - \frac{1}{i2}[A_2^\dagger, A_1]A_1^\dagger A_1 - \frac{1}{i2}[A_1, A_1^\dagger]A_2^\dagger A_1 \\
&\quad + \frac{1}{i2}[A_2, A_2^\dagger]A_1^\dagger A_2 + \frac{1}{i2}[A_1^\dagger, A_2]A_2^\dagger A_2 - \frac{1}{i2}[A_1, A_2^\dagger]A_2^\dagger A_2 - \frac{1}{i2}[A_2^\dagger, A_2]A_2^\dagger A_1, \\
&= 0. \qquad (7)
\end{aligned}
$$

$$
\begin{aligned}
[J_3^\alpha, J_4^\alpha] &= \frac{1}{4}\Big[A_1^\dagger A_1 - A_2^\dagger A_2, A_1^\dagger A_1 + A_2^\dagger A_2\Big], \\
&= -\frac{1}{4}\Big([A_2^\dagger, A_1]A_1^\dagger A_2 - [A_2, A_1^\dagger]A_2^\dagger A_1 + [A_1^\dagger, A_2]A_2^\dagger A_1 + [A_1, A_2^\dagger]A_1^\dagger A_2\Big), \\
&= i2\alpha\sqrt{1-\alpha^2}\hat{J}_1. \qquad (8)
\end{aligned}
$$

$$
\begin{aligned}
[J_3^\alpha, J_1^\alpha] &= \Big[\frac{1}{2}(A_1^\dagger A_1 - A_2^\dagger A_2), \frac{1}{2}(A_1^\dagger A_2 + A_2^\dagger A_1)\Big], \\
&= \frac{1}{4}[A_1^\dagger A_1, A_1^\dagger A_2] - \frac{1}{4}[A_2^\dagger A_2, A_1^\dagger A_2] + \frac{1}{4}[A_1^\dagger A_1, A_2^\dagger A_1] - \frac{1}{4}[A_2^\dagger A_2, A_2^\dagger A_1], \\
&= iJ_2^\alpha + i\alpha\sqrt{1-\alpha^2}J_4^\alpha, \\
&= iJ_2^\alpha - i2\alpha\beta J_4^\alpha. \qquad (9)
\end{aligned}
$$

J^α as a Lie algebra

Here, we show the *new* rotation operator $J_j^\alpha, j = 1,2,3,4$ with the NC parameter which we denote by $\vartheta = \alpha\sqrt{1-\alpha^2}$ satisfying the properties of Lie algebra. Using

$$J_1^\alpha = (2\alpha^2 - 1)J_1, \qquad (10a)$$

$$J_2^\alpha = J_2 - 2\alpha\sqrt{1-\alpha^2}J_4, \qquad (10b)$$

$$J_3^\alpha = (2\alpha^2 - 1)J_3, \qquad (10c)$$

$$J_4^\alpha = -2\alpha\sqrt{1-\alpha^2}J_2 + J_4 \qquad (10d)$$

we can easily show that J_j^α satisfy the properties of $[J_j^\alpha, J_j^\alpha] = 0$ and antisymmetric ($[J_j^\alpha, J_k^\alpha] = -[J_k^\alpha, J_j^\alpha]$). Next, we confirm also for J_j^α satisfying the Jacobi identity:

$$
\begin{aligned}
&[J_1^\alpha,[J_2^\alpha, J_3^\alpha]] + [J_2^\alpha,[J_3^\alpha, J_1^\alpha]] + [J_3^\alpha,[J_1^\alpha, J_2^\alpha]] \\
&= [J_1^\alpha, iJ_1^\alpha] + [J_2^\alpha, iJ_2^\alpha + i\vartheta J_4^\alpha] + [J_3^\alpha, iJ_3^\alpha], \\
&= 0. \qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
&[J_2^\alpha,[J_3^\alpha,J_4^\alpha]]+[J_3^\alpha,[J_4^\alpha,J_2^\alpha]]+[J_4^\alpha,[J_2^\alpha,J_3^\alpha]]\\
&=[J_2^\alpha,i4\vartheta J_1^\alpha]+[J_3^\alpha,0]+[J_4^\alpha,iJ_1^\alpha],\\
&=i4\vartheta(-iJ_3^\alpha)+i4\vartheta(iJ_3^\alpha),\\
&=0.
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&[J_1^\alpha,[J_3^\alpha,J_4^\alpha]]+[J_3^\alpha,[J_4^\alpha,J_1^\alpha]]+[J_4^\alpha,[J_1^\alpha,J_3^\alpha]]\\
&=[J_1^\alpha,i4\vartheta J_1^\alpha]+[J_3^\alpha,i4\vartheta J_3^\alpha]+[J_4^\alpha,-iJ_2^\alpha-i\vartheta J_4^\alpha]\\
&=0
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
&[J_1^\alpha,[J_2^\alpha,J_4^\alpha]]+[J_2^\alpha,[J_4^\alpha,J_1^\alpha]]+[J_4^\alpha,[J_1^\alpha,J_2^\alpha]]\\
&=[J_1^\alpha,0]+[J_2^\alpha,i4\vartheta J_3^\alpha]+[J_4^\alpha,iJ_3^\alpha]\\
&=i4\vartheta(iJ_1^\alpha)+i(-i4\vartheta J_1^\alpha),\\
&=0.
\end{aligned}
\tag{14}
$$