On Modular Forms, Hecke Operators, Replication and Sporadic Groups

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ABSTRACT

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In the first part of this thesis we find all congruence subgroups of $PSL_2(\mathbb{R})$ and respective weights for which the corresponding space of cusp forms is one-dimensional. We compute generators for those spaces.

In the second part we establish a connection between the Hecke Algebra of $\Gamma_0(2)+$ and the group $2 \cdot \mathbb{B}$, the double cover of the Baby Monster group. Namely, we find a new form of replication, $2A$-replication, that is reflected in the power map structure of $2 \cdot \mathbb{B}$. This is very similar to the fact that usual replication reflects the power map structure in the Monster group. We use a vertex operator algebra and a Lie algebra that were constructed by Höhn and see that the McKay-Thompson series for $2 \cdot \mathbb{B}$ satisfy $2A$-replication identities. This also simplifies the computations made by Höhn to identify every McKay-Thompson series as a Hauptmodul by using generalized Mahler recurrence relations. This strategy follows in spirit Borcherd's proof of the original Moonshine Conjectures.

We also extend these ideas to $\Gamma_0(3)+$ and $3 \cdot F_{3+}$. However, even though the generalization is straightforward there are McKay-Thompson series that have irrational coefficients for which our replication formulas don't work.

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Introduction

For long time, the groups $\Gamma_0(N)$, $\Gamma_1(N)$ and $\Gamma(N)$ were, among all subgroups of $PSL_2(\mathbb{R})$, the ones number theorists were most interested in. However, since the beginning of Moonshine some interest has arised in genus zero, congruence subgroups of $PSL_2(\mathbb{R})$ that are commensurable with $PSL_2(\mathbb{Z})$ as they play an important role in this theory. Moonshine has its roots in the initial observation by McKay, in 1978, that the first coefficients of the modular j-function:

$$j(q) = \frac{1}{q} + 744 + 196884 \cdot q + 21493760 \cdot q^2 + 864299970 \cdot q^3 + \ldots, \;\; q = e^{2\pi i z}$$

are linear combinations with positive integer coefficients of dimensions of irreducible representations of the Monster group, $\mathbb{M}$, the biggest sporadic simple group. This suggest the existence of a natural graded representation $V^\natural = \bigoplus_{n=-1}^{\infty} V^\natural_{(n)}$ such that $j(q) = \sum_{n=-1}^{\infty} \dim\left(V^\natural_{(n)}\right) q^n$. Thompson later suggested that the functions $T_g(q) = \sum_{n=-1}^{\infty} \mathrm{Tr}\left(g, V^\natural_{(n)}\right) q^n$ would be worth investigating. These are called McKay-Thompson series. Actually, from their power series expansion, Conway and Norton ([12]) realized that each McKay-Thompson series seemed to be the Hauptmodul (canonical generator for the function field of modular functions) for a subgroup of $PSL_2(\mathbb{R})$ between some $\Gamma_0(N)$ and its normalizer in $PSL_2(\mathbb{R})$. These groups are also genus zero (a Hauptmodul only exists in that case), whence our interest in the groups of type mentioned above. All such groups were classified by Cummins ([14]) who also gave information about the number fields where the coefficients lie in. It turns out that there are 616 groups whose Hauptmodul has rational coefficients (which include all Monstrous McKay-Thompson series) and 3870 irrational ones. The first problem addressed in this thesis is to find among all such groups the ones that have a one dimensional space of cusp forms of even weight and to compute the generators of those spaces. The interest on this classification is related to the fact that such forms have appeared in some contexts directly related or not to Moonshine, for example [47] and [23].

An important concept in Moonshine is that of replicability. We say that a function $f(q) = \frac{1}{q} + a_1 q + a_2 q^2 + \ldots$ is replicable if there is, for every $n \in \mathbb{N}$, a function

$$f^{(n)}(q) = \frac{1}{q} + a_1^{(n)} q + a_2^{(n)} q^2 + \ldots$$

such that

$$\sum_{\substack{ad=n \\ 0 \leq b < d}} f^{(a)} \left(\frac{az+b}{d} \right) = P_n(f(q))$$

where $P_n(f(q))$ is the Faber polynomial of f. This is closely related to the definition of the Hecke operators for the modular group. A function is completely replicable if, in addition, it satisifes $\left(f^{(m)}\right)^{(n)} = f^{(mn)}$. It was observed by Conway and Norton ([12]) that the McKay-Thompson series T_g are replicable and satisfy $T_g^{(n)} = T_{g^n}$. In particular, they are completely replicable. The proof of the existence of $V^\natural$ was obtained by Frenkel, Lepowsky and Meurman ([28]). It is a vertex operator algebra and from that Borcherds constructed a $\mathbb{Z} \times \mathbb{Z}$ graded generalized Kac-Moody algebra, the Monster Lie algebra, whose (m, n) piece is isomorphic to $V^\natural_{(mn)}$. This Lie algebra has a twisted denominator formula which says essentialy that the McKay-Thompson series are replicable and satisfy $T_g^{(n)} = T_{g^n}$ as stated above. It was known at that time that the Hauptmoduls corresponding to each McKay-Thomson series were also completely replicable. The fact that a function is completely replicable implies recurrence relations among the coefficients of a function and its replicates that allows us to compute all coefficients of a function knowing just the first five coefficients of its replicates. It is then enough to compare the first 5 coefficients of every McKay-Thompson series and those of the corresponding Hauptmodul to be sure that they are actually the same functions. This was what Borcherds did ([7]) proving all the original Moonshine Conjectures.

Norton ([48]) generalized the Moonshine Conjectures in a way that it implies the existence of Moonshine properties for other groups. Höhn proved this ([34]) for the group $2 \cdot \mathbb{B}$, where $\mathbb{B}$ is the Baby Monster group. $2 \cdot \mathbb{B}$ is the centralizer of an element of class $2A$ in $\mathbb{M}$. In the second part of this thesis we use his results to prove that there is a form of replicability coming from the Hecke Algebra of $\Gamma_0(2)+$ that respects the power map structure in $2 \cdot \mathbb{B}$, in the same way that usual replicability respects the power map structure in $\mathbb{M}$. In this case, we also find recurrence relations that allow us to compute all coefficients of a function from the first 5 coefficients of the fuction and its replicates. This simplifies the argument Höhn uses to match every McKay-Thompson with a Hauptmodul and is closer in spirit to Borcherds proof of the original Moonshine Conjectures. We also make computations that show the same property holding for the group $3 \cdot F_{3+}$, at least for the cases where the Hauptmodul has rational coefficients.

Table of Notations

$\mathbb{Z}$	-	Set of integers
$\mathbb{Q}$	-	Set of rational numbers
$\mathbb{R}$	-	Set of real numbers
$\mathbb{C}$	-	Set of complex numbers
η	-	Dedekind eta-function
μ	-	Multiplier system for the Dedekind eta-function
$\left(\frac{\cdot}{\cdot}\right)$	-	Jacobi symbol
π	-	Partition
η_π	-	Eta-product associated to the partition π
ϕ	-	Euler totient function
$g(\chi)$	-	Gauss sum of χ
μ_N	-	$e^{\frac{2\pi i}{N}}$
$L(s,\chi)$	-	L-function of χ
$\Gamma(s)$	-	Gamma function
$\Re$	-	Real part of a complex number
$\Im$	-	Imaginary part of a complex number
B_k	-	k-th Bernoulli Polynomial
$B_{k\,\varphi}$	-	Bernoulli numbers of the character φ
$E_k^{\psi,\varphi}$	-	Eisenstein series
$[\ \ ]$	-	Integer part of a real number
$\mathbf{T}_n$	-	Hecke Operator
$\mathcal{H}$	-	Complex upper half-plane

Chapter 1

One-dimensional spaces of cusp forms

1.1 Introduction

In this chapter we find all genus zero congruence subgroups of $PSL_2(\mathbb{R})$ for which there is a one-dimensional space of cusp forms and compute the expressions of those forms. We can see these forms as analogs of the Δ function, a cusp form of weigh 12 for the modular group.

In Section 1.2 we introduce some definitions and notation about the groups we are going to work with.

Sections 1.3 and 1.4 are devoted to the Dedekind eta-function, Eisenstein Series and some of their properties which will be used to find expressions to our forms. All the results in section 1.4 can be found in [22] except 1.4.3.

In section 1.5 we find all possible signatures for a group to have a one-dimensional space of cusp forms. Using these signatures we extract from the tables given in [14] all genus zero congruence groups that we are interested in.

In the last section we compute these forms using the results from sections 1.3 and 1.4. For groups of type $n|h + e_1, e_2 \dots$ we found that they can be expressed as product of a multiplicative eta-product and an Eisenstein series. All the other groups will be analysed individually using *ad hoc* methods.

1.2 Some subgroups of $PSL_2(\mathbb{R})$

We define, as usual,

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} (\bmod N) \right\}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} (\bmod N) \right\}$$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\bmod N) \right\}$$

The normaliser of $\Gamma_0(N)$ in $PSL_2(\mathbb{R})$ was described in [12] as the set of matrices of the form $\begin{pmatrix} ae & \frac{b}{h} \\ cn & de \end{pmatrix}$ where a, b, c, d, e, h and n are integers satisfying:

1. the determinant $ade^2 - \frac{bcn}{h}$ equals e.

2. h is the largest divisor of 24 such that h^2 divides N and $N = nh$.

3. e is an exact divisor of $\frac{n}{h}$ (we write $e||\frac{n}{h}$), i.e. e divides $\frac{n}{h}$ and $\left(e, \frac{n}{he}\right) = 1$.

The set W_e of all matrices of the form $\begin{pmatrix} ae & b \\ cN & de \end{pmatrix}$ of determinant e where $e||N$ is a coset of $\Gamma_0(N)$ and is called an Atkin-Lehner "involution".

We define $\Gamma_0(n|h)$ to be the subgroup of the normalizer of $\Gamma_0(N)$ of matrices of determinant 1, i.e., it is the set of matrices of the form $\begin{pmatrix} a & \frac{b}{h} \\ cn & d \end{pmatrix}$. This group is a conjugate of $\Gamma_0(\frac{n}{h})$, whence the notation. The matrices $\begin{pmatrix} ae & \frac{b}{h} \\ cn & de \end{pmatrix}$ with a fixed e form a coset of this subgroup and we denote them by w_e since they are conjugates of the Atkin-Lehner involutions of $\Gamma_0(\frac{n}{h})$.

The normalizer of $\Gamma_0(N)$ in $PSL_2(\mathbb{R})$ can thus be described as the union of $\Gamma_0(n|h)$ with all its Atkin-Lehner involutions.

Following [12], we use the notation $\Gamma_0(n|h) + e, f \ldots$ to denote the group obtained from $\Gamma_0(n|h)$ by adjoining the Atkin-Lehner involutions w_e, w_f,.... Also, $\Gamma_0(n|h)+$ will mean that all Atkin-Lehner involutions are present.

We are interested in a certain subgroup of index h in $\Gamma_0(n|h) + e, f \ldots$ which we will denote by $n|h + e, f \ldots$. These groups are important in the context of Monstrous Moonshine and were first introduced by Conway and Norton ([12]). They defined them as the kernel of the homomorphism $\lambda : \Gamma_0(n|h) + e, f, \ldots \longrightarrow \mathbb{C}^\times$ defined in the following way:

1. $\lambda = 1$ for elements in $\Gamma_0(N)$,

2. $\lambda = 1$ for every Atkin-Lehner involution W_E of $\Gamma_0(N)$ in $\Gamma_0(n|h) + e, f \ldots$ such that every prime dividing E also divides n/h,

3. $\lambda = e^{-\frac{2\pi i}{h}}$, for the coset containing $\begin{pmatrix} 1 & \frac{1}{h} \\ 0 & 1 \end{pmatrix}$,

4. $\lambda = e^{\pm\frac{2\pi i}{h}}$ for the coset containing $\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$, where the sign is $+$ if $\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ is present, otherwise it is $-$.

These homomorphisms are well defined in the cases Conway and Norton considered but it is not obvious when this gives a well defined homomorphism, in general. This point was considered in [13] and [24] and it happens that the homomorphism λ is unique and well-defined, and hence $n|h + e, f, \ldots$ exists, if the following conditions are satisfied:

- if $h = 3$ then either $9 \mid n$ or $\frac{n}{h} \equiv \pm 1 \mod h$. The sign is $+$ (resp. $-$) if $\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ is (resp. not) present.

- if $h = 4$ then $8 \mid n$.

- if $h = 8$ then $32 \mid n$.

- if h is a prime power and $+e$ is present then either $e \equiv 1 \mod h$ or $\frac{n}{eh} \equiv \pm 1 \mod h$. The sign is $+$ (resp. $-$) if $\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ is (resp. not) present.

- if $h = 6, 12, 24$ then either both groups $\Gamma_0(3n|j)$ and $\Gamma_0(jn|3)$ or both groups $\Gamma_0(\frac{n}{3}|j)$ and $\Gamma_0(\frac{n}{j}|3)$, where $h = 3j$, satisfy all conditions above.

We shall use this result later when we consider such groups.

1.3 Eta-Products

The Dedekind eta-function is the function defined in the upper half-plane by:

$$\eta(z) = q^{\frac{1}{24}} \cdot \prod_{n \geq 1} (1 - q^n), \quad q = e^{2\pi i z}$$

It satisfies the following transformation formulas:

1) $\eta(z+1) = e^{\frac{\pi i}{12}} \cdot \eta(z)$

2) $\eta\left(-\frac{1}{z}\right) = \sqrt{-iz} \cdot \eta(z)$

where $z^r = |z|^r e^{ir\arg(z)}$, $-\pi < \arg(z) \leq \pi$. We assume this convention for the rest of this chapter.

In fact, if we consider the Jacobi symbol $\left(\frac{\cdot}{\cdot}\right)$ and define $\left(\frac{c}{d}\right)^* = \left(\frac{c}{|d|}\right)$ and $\left(\frac{c}{d}\right)_* = \left(\frac{c}{|d|}\right)(-1)^{\frac{sgn(c)-1}{2}\frac{sgn(d)-1}{2}}$ we have, more generally (Theorem 2.18 in [33]).

Theorem 1.3.1. *For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$:*

$$\eta(Mz) = \nu(M)(cz+d)^{\frac{1}{2}}\eta(z)$$

where

$$\nu(M) = \begin{cases} \left(\frac{d}{c}\right)^{*} e^{\frac{\pi i}{12}\left((a+d)c-bd(c^2-1)-3c\right)} & \text{, if } c \text{ is odd,} \\ \left(\frac{c}{d}\right)_{*} e^{\frac{\pi i}{12}\left((a+d)c-bd(c^2-1)+3d-3-3cd\right)} & \text{, if } c \text{ is even.} \end{cases} \tag{1.3.1}$$

Theorem 1.3.2. *Let* $\pi = \prod_{n\geq 1} n^{r_n}$ *be a partition. If*

1. $\sum_{n\geq 1} r_n = 2k,\ k \in \mathbb{Z}$,
2. $\sum_{n\geq 1} nr_n \equiv 0 (mod\ 24)$,
3. $\prod_{n\geq 1} n^{|r_n|} = m^2 f$ *where* f *is a square free integer,*

   then for a natural number N *satisfying:*

4. $r_n = 0$ *if* $n \nmid N$,
5. $\sum_{n\geq 1} \frac{N}{n} r_n \equiv 0 (mod\ 24)$,
6. $N \equiv 0 (mod\ 4)$ *if* $f \equiv (-1)^k\ (mod\ 4)$,

   $N \equiv 0 (mod\ 8)$ *if* $f \equiv 2\ (mod\ 4)$.

7. $\sum_{n\geq 1} \frac{(n,c)^2}{n} r_n \geq 0$ *(resp.* > 0*) for every* $c \mid N$.

we have that η_π *is a modular form (resp. cusp form) for* $\Gamma_0(N)$ *with character equal to the quadratic character of* $\mathbb{Q}\left(\sqrt{(-1)^k f}\right)$.

Proof. See theorem 3.6 in [33]

An eta-product is a function of the form $\prod_{k\geq 1} \eta(kz)^{r_k}$ where the r_k are non-negative integers. We will abbreviate, as usual, $\prod_k k^{r_k}$ for this product. For example, $1^3 7^3$ represents

Partition	Character (if not trivial)	Level	Partition	Character (if not trivial)	Level
24^1	-	-	$4^2 8^2$		32
8^3	-	-	6^4		36
$1^1 23^1$	$\left(\frac{-23}{\cdot}\right)$	23	$2^2 4^2$		8
$2^1 22^1$	$\left(\frac{-11}{\cdot}\right)$	44	$1^2 2^1 4^1 8^2$	$\left(\frac{-2}{\cdot}\right)$	8
$3^1 21^1$	$\left(\frac{-7}{\cdot}\right)$	63	$1^3 7^3$	$\left(\frac{-7}{\cdot}\right)$	7
$4^1 20^1$	$\left(\frac{-20}{\cdot}\right)$	80	$2^3 6^3$	$\left(\frac{-3}{\cdot}\right)$	12
$6^1 18^1$	$\left(\frac{-3}{\cdot}\right)$	108	4^6	$\left(\frac{-1}{\cdot}\right)$	16
$8^1 16^1$	$\left(\frac{-2}{\cdot}\right)$	128	$1^2 2^2 3^2 6^2$		6
12^2	$\left(\frac{-1}{\cdot}\right)$	144	$1^4 5^4$		5
$1^1 3^1 5^1 15^1$		15	3^8		9
$1^1 2^1 7^1 14^1$		14	$1^4 2^2 4^4$	$\left(\frac{-1}{\cdot}\right)$	4
$2^1 4^1 6^1 12^1$		24	$1^6 3^6$		3
$1^2 11^2$		11	2^{12}		4
$2^2 10^2$		20	$1^8 2^8$		2
$3^2 9^2$		27	1^{24}		1

Table 1.1: Multiplicative eta-products

$\eta(z)^3\eta(7z)^3$. A multiplicative eta-product is one with the property that if a_m are the coefficients of its power series expansions then $a_{mn} = a_m a_n$ whenever m and n are prime to each other (a necessary condition for this is $\sum_k k r_k = 24$).

All multiplicative eta-products have been classified in [23] and are listed in table 1.1

We note that all partitions associated to multiplicative eta-products have balanced cycle shapes, i.e. $\prod_{k\geq 1} k^{r_k} = \prod_{k\geq 1} \left(\frac{N}{k}\right)^{r_k}$ for some N.

We will be interested in the action of the Atkin-Lehner involutions on these eta-products. For this, we first note that

$$\begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} ae & b \\ cN & de \end{pmatrix} = \begin{pmatrix} a(e,f) & b\frac{f}{(e,f)} \\ cN\frac{(e,f)}{ef} & d\frac{e}{(e,f)} \end{pmatrix} \begin{pmatrix} \frac{ef}{(e,f)} & 0 \\ 0 & (e,f) \end{pmatrix} \tag{1.3.2}$$

and define on the set of divisors of N, $e * f := \frac{ef}{(e,f)^2}$.

We define the weight-k operator $_{|k}[\gamma]$, for $\gamma \in GL_2(\mathbb{C})$, in the following way

$$\left(f_{|k}[\gamma]\right)(z) = \det(\gamma)^{\frac{k}{2}} (cz+d)^{-k} f(\gamma z)$$

If we assume that $e||N$ and $e*$ fixes the partition $\pi = \prod_{k\geq 1} k^{r_k}$ then we have

$$\eta_{\pi|_k}\left[W_e\right] = e^{\frac{k}{2}}\left(cNz+de\right)^{-k}\prod_{n\geq 1}\eta\left(n\begin{pmatrix} ae & b \\ cN & de\end{pmatrix}z\right)^{r_n}$$
$$= e^{\frac{k}{2}}\left(cNz+de\right)^{-k}\prod_{n\geq 1}\eta\left(\begin{pmatrix} a(e,n) & b\frac{n}{(e,n)} \\ cN\frac{(e,n)}{en} & d\frac{e}{(e,n)}\end{pmatrix}\frac{en}{(e,n)^2}z\right)^{r_n}$$
$$= e^{\frac{k}{2}}\left(cNz+de\right)^{-k}\left(\prod_{n\geq 1}\nu\begin{pmatrix} a(e,n) & b\frac{n}{(e,n)} \\ cN\frac{(e,n)}{en} & d\frac{e}{(e,n)}\end{pmatrix}^{r_n}\left(\frac{cNz+de}{(e,n)}\right)^{\frac{r_n}{2}}\right)\prod_{n\geq 1}\eta\left(\frac{en}{(e,n)^2}z\right)^{r_n}$$
$$= \left(\prod_{n\geq 1}\nu\begin{pmatrix} a(e,n) & b\frac{n}{(e,n)} \\ cN\frac{(e,n)}{en} & d\frac{e}{(e,n)}\end{pmatrix}^{r_n}\right)\eta_{\pi}\left(z\right).$$

We denote this root of unity by ω_{π}^{e}.

1.4 Eisenstein Series

The aim of this section is to give the necessary background on Eisenstein series and give the action of the Atkin-Lehner involutions on them. All the facts that are stated but not proven here can be found in [22], except the ones in section 1.4.3 for which we provide a proof. We use the notation $\mathcal{E}_k(N,\chi)$ for the orthogonal complement of $\mathcal{S}_k(N,\chi)$ in $\mathcal{M}_k(N,\chi)$ with respect to the Peterson inner product.

1.4.1 Dirichlet Characters, Gauss sums, Gamma-function and L-functions

A Dirichlet character modulo N is a homomorphism of multiplicative groups

$$\chi : \left(\mathbb{Z}/N\mathbb{Z}\right)^{*} \longrightarrow \mathbb{C}^{*}$$

Any Dirichlet character χ modulo N lifts to a Dirichlet character χ_M modulo M for any M with $N|M$ setting $\chi_M(x) = \chi(x \bmod N)$.

A Dirichlet character module M is said to be primitive if it is not obtained by lifting another Dirichlet character module N, with $N|M$, in this way.

Every Dirichlet character χ modulo N can be extended to a function $\chi : \mathbb{Z} \longrightarrow \mathbb{C}$ setting

$$\chi(n) = \begin{cases} \chi(n \bmod N) & , \text{if } (n, N) = 1, \\ 0 & , \text{if } (n, N) > 1. \end{cases}$$

The trivial character modulo N is the character 1_N defined by

$$1_N(x) = \begin{cases} 1 & , \text{if } (x, N) = 1, \\ 0 & , \text{if } (x, N) > 1. \end{cases}$$

We note that $1_N(0) = 1$ if and only if $N = 1$.

If χ and ψ are Dirichlet characters modulo N then

$$\sum_{n=0}^{N-1} \chi(n)\overline{\psi}(n) = \begin{cases} \phi(N) & , \text{if } \chi = \psi \\ 0 & , \text{if } \chi \neq \psi \end{cases}$$

In particular,

$$\sum_{n=0}^{N-1} \chi(n) = \begin{cases} \phi(N) & , \text{if } \chi = 1_N \\ 0 & , \text{if } \chi \neq 1_N \end{cases} \tag{1.4.1}$$

The Gauss sum of a Dirichlet character modulo N is defined by

$$g(\chi) = \sum_{n=0}^{N-1} \chi(n)\mu_N^n, \quad \mu_N = e^{\frac{2\pi i}{N}}$$

We have for a primitive Dirichlet character χ modulo N the following

$$\sum_{n=0}^{N-1} \chi(n)\mu_N^{nm} = \overline{\chi(m)}g(\chi) \tag{1.4.2}$$

Every Dirichlet character modulo N has an L-function attached to it:

$$L(s, \chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s}, \; \Re(s) \geq 1$$

This function converges for $\Re(s) > 1$ and can be extended meromorphically to all $\mathbb{C}$. This extension is always entire, except when $\chi = 1_N$ in which case it has a simple pole at $s = 1$.

We introduce now the Gamma function

$$\Gamma(s) = \int_{t=0}^{+\infty} e^{-t} t^{s-1} dt,\ s \in \mathbb{C}$$

This function is defined for $\Re(s) > 0$. However, it satisfies the following functional equation

$$\Gamma(s+1) = s\Gamma(s)$$

and can thus be extended meromorphically to all $\mathbb{C}$.

It satisfies the following formula for every positive even integer k:

$$\frac{\pi^{-\frac{1-k}{2}}\Gamma(\frac{1-k}{2})}{\pi^{-\frac{k}{2}}\Gamma(\frac{k}{2})} = \frac{1}{2}\frac{(-2\pi i)^k}{(k-1)!}$$

We have for a Dirichlet character χ modulo N:

$$\begin{aligned}
\pi^{-\frac{s}{2}}\Gamma\left(\tfrac{s}{2}\right) N^s L(s,\chi) &= \pi^{-\frac{1-s}{2}}\Gamma\left(\tfrac{1-s}{2}\right) g(\chi) L(1-s,\overline{\chi}) && \text{, if } \chi(-1) = 1 \\
\pi^{-\frac{s+1}{2}}\Gamma\left(\tfrac{s+1}{2}\right) N^s L(s,\chi) &= -i\pi^{-\frac{2-s}{2}}\Gamma\left(\tfrac{2-s}{2}\right) g(\chi) L(1-s,\overline{\chi}) && \text{, if } \chi(-1) = -1
\end{aligned}$$

In any of these cases we have, using the properties above,

$$L(k,\chi) = \frac{1}{2}\frac{(-2\pi i)^k}{N^k(k-1)!} g(\chi) L(1-k,\overline{\chi}) \tag{1.4.3}$$

We now extend the notion of Bernoulli numbers in two ways.

First, we define the Bernoulli polynomials $B_k(X)$ by

$$\frac{te^{tX}}{e^t - 1} = \sum_{k=0}^{+\infty} B_k(X)\frac{t^k}{k!}$$

and then we define the Bernoulli numbers of ψ, where ψ is a Dirichlet character modulo

u by

$$\sum_{c=0}^{u-1} \psi(c) \frac{te^{ct}}{e^{ut}-1} = \sum_{k=0}^{+\infty} B_{k,\psi} \frac{t^k}{k!}$$

From this definitions we can see that

$$B_{k,\psi} = u^{k-1} \sum_{c=0}^{u-1} \psi(c) B_k \left(\frac{c}{u}\right)$$

and the important fact that we will need later is that for $k = 1$ and $\psi \neq 1_1$

$$\sum_{c=0}^{u-1} \psi(c) \left(\frac{c}{u} - \frac{1}{2}\right) = B_{1,\psi} = -L(0, \psi). \tag{1.4.4}$$

1.4.2 Eisenstein Series

Fix $N \in \mathbb{N}$ and $k \geq 3$ and consider for each $\overline{v} = (c_v, d_v) \in (\mathbb{Z}/N\mathbb{Z})^2$ of order N:

$$G_k^{\overline{v}}(\tau) = \sum_{(c,d) \equiv v(N)} \frac{1}{(c\tau + d)^k}.$$

where v is any lift of (c_v, d_v) to $\mathbb{Z}^2$.

This is a modular form of weight k on $\Gamma(N)$ and has a power series expansion

$$G_k^{\overline{v}}(\tau) = \delta(\overline{c_v})\zeta^{\overline{d_v}}(k) + \frac{(-2\pi i)^k}{(k-1)!N^k} \sum_{\substack{n,m \in \mathbb{Z} \\ mn>0 \\ n \equiv c_v(N)}} sgn(m) m^{k-1} \mu_N^{d_v m} q_N^{nm}, \ \ q_N = e^{\frac{2\pi i \tau}{N}}. \tag{1.4.5}$$

where $\delta(\overline{c_v}) = \begin{cases} 1 & \text{, if } \overline{c_v} = 0, \\ 0 & \text{, otherwise.} \end{cases}$ and $\zeta^{\overline{d_v}}(k) = \sum_{d \equiv d_v(N)}' \frac{1}{d^k}$.

For $k = 1, 2$ we define $G_k^{\overline{v}}$ to be the function given by (1.4.5) and set also

$$g_k^{\overline{v}}(\tau) = \begin{cases} G_k^{\overline{v}}(\tau) & \text{, if } k \geq 3 \\ G_2^{\overline{v}}(\tau) - \frac{\pi}{N^2 \Im(\tau)} & \text{, if } k = 2 \\ G_1^{\overline{v}}(\tau) + \frac{2\pi i}{N}\left(\frac{c_v}{N} - \frac{1}{2}\right) & \text{, if } k = 1 \end{cases}$$

These are weight-k invariant under $\Gamma(N)$, even though $g_2(\tau)$ happens to be non-holomorphic. They satisfy

$$g_k^{\overline{v}}\ [\gamma] = g_k^{\overline{v\gamma}},\ \text{for } \gamma \in SL_2(\mathbb{Z}). \tag{1.4.6}$$

Consider also for each ψ, φ primitive characters modulo u and v, respectively,

$$G_k^{\psi,\varphi}(\tau) = \sum_{c=0}^{u-1} \sum_{d=0}^{v-1} \sum_{e=0}^{u-1} \psi(c)\overline{\varphi(d)} g_k^{\overline{(cv,d+ev)}}(\tau) \tag{1.4.7}$$

$$E_k^{\psi,\varphi}(\tau) = \begin{cases} \delta(\psi)L(1-k,\varphi) + 2\sum_{n=1}^{+\infty} \sigma_{k-1}^{\psi,\varphi}(n) q^n & \text{, if } k \geq 2 \\ \delta(\varphi)L(0,\psi) + \delta(\psi)L(0,\varphi) + 2\sum_{n=1}^{+\infty} \sigma_0^{\psi,\varphi}(n) q^n & \text{, if } k = 1 \end{cases} \tag{1.4.8}$$

where $\sigma_k^{\psi,\varphi}(n) = \sum_{m|n} \psi(n/m)\varphi(m)m^k$ and $\delta(\theta) = \begin{cases} 1, & \text{if } \theta = 1_1 \\ 0, & \text{otherwise} \end{cases}$

Using (1.4.5) we can show that $G_k^{\psi,\varphi}(\tau) = \frac{(-2\pi i)^k}{(k-1)!}\frac{g(\overline{\varphi})}{v^k} E_k^{\psi,\varphi}(\tau)$.

We describe now a basis for the space $\mathcal{E}_k(N,\chi)$ (recall definition given at the beginning of this section).

Define $E_k^{\psi,\varphi,t}(\tau) = \begin{cases} E_k^{1_1,1_1}(\tau) - tE_k^{1_1,1_1}(t\tau) & \text{, if } \psi = \varphi = 1_1 \text{ and } k = 2 \\ E_k^{\psi,\varphi}(t\tau) & \text{, otherwise} \end{cases}$

For $k \geq 3$ the set

$$\left\{ E_k^{\psi,\varphi,t} | \, (\psi,\varphi,t) \in A_{N,k},\ \ \psi\varphi = \chi \right\} \tag{1.4.9}$$

form a basis for $\mathcal{E}_k(N,\chi)$, where $A_{N,k}$ is the set of triples (ψ,φ,t) such that ψ, φ are primitive Dirichlet characters modulo u and v with $(\psi\varphi)(-1) = (-1)^k$ and t is a positive integer with $tuv|N$.

For $k = 2$ the set

$$\left\{ E_2^{\psi,\varphi,t} | \, (\psi, \varphi, t) \in A_{N,2}, \;\; \psi\varphi = \chi \right\} \tag{1.4.10}$$

form a basis for $\mathcal{E}_2(N, \chi)$ where $A_{N,2}$ is the set of triples (ψ, φ, t) such that ψ, φ are primitive Dirichlet characters modulo u and v with $(\psi\varphi)(-1) = 1$ and t is a positive integer with $1 < tuv | N$.

For $k = 1$ the set

$$\left\{ E_1^{\psi,\varphi,t} | \, (\{\psi, \varphi\}, t) \in A_{N,1}, \;\; \psi\varphi = \chi \right\} \tag{1.4.11}$$

form a basis for $\mathcal{E}_1(N, \chi)$ where $A_{N,1}$ is the set of triples $(\{\psi, \varphi\}, t)$ such that ψ, φ are primitive Dirichlet characters modulo u and v with $(\psi\varphi)(-1) = -1$ and t is a positive integer with $tuv|N$.

1.4.3 Action of the Atkin-Lehner involutions on Eisenstein Series

When both ψ, φ are trivial, $E_k = E_k^{1_1,1_1}$ is a modular form on $SL_2(\mathbb{Z})$ for even $k \geq 3$ and using (1.3.2) for any Atkin-Lehner involution W_e we have

$$E_k(nz)_{|_k} [W_e] = \left(\frac{e}{(n,e)^2} \right)^{\frac{k}{2}} E_k \left(\frac{ne}{(n,e)^2} z \right) \tag{1.4.12}$$

When $k = 2$ we note that the vector space generated by $\left\{ E_2^{1_1,1_1,d} | \; d \text{ divides } N \right\}$ is

$$\left\{ \sum_{d|N} \beta_d d E_2(d\tau) | \sum_{d|N} \beta_d = 0 \right\} \tag{1.4.13}$$

and a simple calculation shows that

$$E_2^{1_1,1_1,d}{}_{|_k} [W_e] = E_2^{1_1,1_1,e} - E_2^{1_1,1_1,e*d} \tag{1.4.14}$$

Now we find what the action of the Fricke Involution $\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ on Eisenstein series $E_k^{\psi,\varphi}$ is for ψ, φ not both trivial.

For $k \geq 3$,

$$G_k^{\psi,\varphi}{}_{|_k} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (\tau) = \sum_{c=0}^{u-1} \sum_{d=0}^{v-1} \sum_{e=0}^{u-1} \psi(c)\overline{\varphi(d)} g_k^{\overline{(cv,d+ev)}}{}_{|_k} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (\tau)$$

Using (1.4.6), this is

$$\sum_{c=0}^{u-1} \sum_{d=0}^{v-1} \sum_{e=0}^{u-1} \psi(c)\overline{\varphi(d)} g_k^{\overline{(d+ev,-cv)}}(\tau)$$

$$= \sum_{c=0}^{u-1} \sum_{d=0}^{v-1} \sum_{e=0}^{u-1} \psi(c)\overline{\varphi(d)}\delta(\overline{d+ev})\zeta^{\overline{-cv}}(k) +$$

$$\sum_{c=0}^{u-1} \sum_{d=0}^{v-1} \sum_{e=0}^{u-1} \psi(c)\overline{\varphi(d)} \frac{(-2\pi i)^k}{(k-1)!N^k} \sum_{\substack{n,m\in\mathbb{Z} \\ mn>0 \\ n\equiv d+ev(N)}} sgn(m)m^{k-1}\mu_N^{-cvm}q_N^{nm}$$

$$= \varphi(0)\sum_{c=0}^{u-1} \psi(c)\zeta^{\overline{-cv}}(k) + \frac{(-2\pi i)^k}{(k-1)!N^k} \sum_{c=0}^{u-1} \sum_{d=0}^{v-1} \sum_{\substack{n,m\in\mathbb{Z} \\ mn>0 \\ n\equiv d(v)}} \psi(c)\overline{\varphi(d)} sgn(m)m^{k-1}\mu_u^{-cm}q_N^{nm}$$

$$= 2\delta(\varphi)\psi(-1)L(k,\psi) + \tfrac{(-2\pi i)^k}{(k-1)!N^k} \sum_{d=0}^{v-1} \sum_{\substack{n,m\in\mathbb{Z} \\ mn>0 \\ n\equiv d(v)}} \overline{\varphi(n)} sgn(m)m^{k-1} \left(\sum_{c=0}^{u-1} \psi(c)\mu_u^{-cm}\right) q_N^{nm}$$

which from (1.4.2) and (1.4.3) gives

$$\frac{(-2\pi i)^k}{N^k(k-1)!}\delta(\varphi)\psi(-1)g(\psi)L(1-k,\overline{\psi}) +$$

$$\frac{(-2\pi i)^k}{N^k(k-1)!}\psi(-1)g(\psi) \sum_{\substack{n,m\in\mathbb{Z} \\ mn>0}} \overline{\psi(m)\varphi(n)} sgn(m)m^{k-1}q_N^{nm}$$

$$= \tfrac{(-2\pi i)^k}{N^k(k-1)!}\psi(-1)g(\psi)\left(\delta(\varphi)L(1-k,\overline{\psi}) + 2\sum_{\substack{n=1\\m=1}}^{+\infty}\overline{\psi(m)\varphi(n)}m^{k-1}q_N^{nm}\right)$$
$$= \tfrac{(-2\pi i)^k}{N^k(k-1)!}\psi(-1)g(\psi)\left(\delta(\varphi)L(1-k,\overline{\psi}) + 2\sum_{n=1}^{+\infty}\left(\sum_{m|n}\overline{\psi(m)\varphi(n/m)}m^{k-1}\right)q_N^{n}\right)$$
$$= \tfrac{(-2\pi i)^k}{N^k(k-1)!}\psi(-1)g(\psi)\left(\delta(\varphi)L(1-k,\overline{\psi}) + 2\sum_{n=1}^{+\infty}\sigma_{k-1}^{\overline{\varphi},\overline{\psi}}q_N^{n}\right)$$
$$= \tfrac{(-2\pi i)^k}{N^k(k-1)!}\psi(-1)g(\psi)E_k^{\overline{\varphi},\overline{\psi}}(\tfrac{\tau}{N}).$$

From this we conclude that

$$G_k^{\psi,\varphi}{}_{|_k}\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}(\tau) = N^{\frac{k}{2}}\frac{(-2\pi i)^k}{N^k(k-1)!}\psi(-1)g(\psi)E_k^{\overline{\varphi},\overline{\psi}}(\tau)$$

or equivalently

$$E_k^{\psi,\varphi}{}_{|_k}\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}(\tau) = \psi(-1)\left(\frac{v}{u}\right)^{\frac{k}{2}}\frac{g(\psi)}{g(\overline{\varphi})}E_k^{\overline{\varphi},\overline{\psi}}(\tau) \tag{1.4.15}$$

For $k=2$ the proof is essentially the same since the non-holomorphic parts of all the $g_2^{\overline{v}}$ in the linear combination (1.4.7) cancel out if at least one of the characters is non-trivial.

To prove (1.4.15) for $k=1$ we just have to show that

$$-N^{\frac{1}{2}}\frac{1}{2\pi i}\frac{v}{g(\overline{\varphi})}\sum_{c=0}^{u-1}\sum_{d=0}^{v-1}\sum_{e=0}^{u-1}\psi(c)\overline{\varphi(d)}\frac{2\pi i}{N}\left(\frac{d+ev}{N}-\frac{1}{2}\right) = \psi(-1)\left(\frac{v}{u}\right)^{\frac{1}{2}}\frac{g(\psi)}{g(\overline{\varphi})}\delta(\overline{\psi})L(0,\overline{\varphi})$$

In fact,

$$-N^{\frac{1}{2}}\tfrac{1}{2\pi i}\tfrac{v}{g(\overline{\varphi})}\sum_{c=0}^{u-1}\sum_{d=0}^{v-1}\sum_{e=0}^{u-1}\psi(c)\overline{\varphi(d)}\frac{2\pi i}{N}\left(\frac{d+ev}{N}-\frac{1}{2}\right) =$$
$$= -\left(\tfrac{v}{u}\right)^{\frac{1}{2}}\tfrac{1}{g(\overline{\varphi})}\psi(0)\sum_{d=0}^{v-1}\sum_{e=0}^{u-1}\overline{\varphi(d)}\left(\frac{d+ev}{N}-\frac{1}{2}\right) =$$

$= -\left(\frac{v}{u}\right)^{\frac{1}{2}} \frac{1}{g(\overline{\varphi})}\delta(\psi)\sum_{d=0}^{N-1}\overline{\varphi(d)}\left(\frac{d}{N}-\frac{1}{2}\right)$

Using (1.4.4) this equals

$-\left(\frac{v}{u}\right)^{\frac{1}{2}} \frac{1}{g(\overline{\varphi})}\delta(\psi)B_{1,\overline{\varphi}} = \left(\frac{v}{u}\right)^{\frac{1}{2}} \frac{1}{g(\overline{\varphi})}\delta(\psi)L(0,\overline{\varphi}) = \psi(-1)\left(\frac{v}{u}\right)^{\frac{1}{2}} \frac{g(\psi)}{g(\overline{\varphi})}\delta(\psi)L(0,\overline{\varphi})$

From this we conclude that:

- for even $k \geq 2$, (1.4.12) implies that the operator ${}_{|k}[W_e]$ acts on $\sum_{d|N}\alpha_d d^{\frac{k}{2}}E_k(d\tau)$ as multiplication by λ if and only if the α_d satisfy

$$\alpha_{\frac{de}{(d,e)^2}} = \lambda\alpha_d, \text{ for all } d|N. \tag{1.4.16}$$

  and (1.4.15) gives that

- if k is even

$$\left(E_k^{\psi,\varphi} \pm \psi(-1)\left(\frac{v}{u}\right)^{\frac{k}{2}}\frac{g(\psi)}{g(\overline{\varphi})}E_k^{\overline{\varphi},\overline{\psi}}\right)_{|k}\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} = \pm\left(E_k^{\psi,\varphi} \pm \psi(-1)\left(\frac{v}{u}\right)^{\frac{k}{2}}\frac{g(\psi)}{g(\overline{\varphi})}E_k^{\overline{\varphi},\overline{\psi}}\right) \tag{1.4.17}$$

- if k is odd

$$\left(E_k^{\psi,\varphi} \mp i\psi(-1)\left(\frac{v}{u}\right)^{\frac{k}{2}}\frac{g(\psi)}{g(\overline{\varphi})}E_k^{\overline{\varphi},\overline{\psi}}\right)_{|k}\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} = \pm i\cdot\left(E_k^{\psi,\varphi} \mp i\psi(-1)\left(\frac{v}{u}\right)^{\frac{k}{2}}\frac{g(\psi)}{g(\overline{\varphi})}E_k^{\overline{\varphi},\overline{\psi}}\right) \tag{1.4.18}$$

1.5 Genus Zero Congruence Subgroups with one dimensional spaces of Cusp Forms

In [55] a formula for the dimension of the space of cusp forms of weight $k \geq 2$ for any subgroup of $SL_2(\mathbb{R})$ is given.

This formula says that the dimension of the space of cusp forms of even weight k for a certain group is

$$\begin{cases} (k-1)(g-1) + \left(\frac{k}{2} - 1\right) m + \sum_{i=1}^{r} \left[\frac{k}{2}\left(1 - \frac{1}{e_i}\right)\right] & \text{if } k \geq 4, \\ g & \text{if } k = 2, \\ 1 & \text{if } k = 0,\ m = 0, \\ 0 & \text{if } k = 0,\ m > 0, \\ 0 & \text{if } k < 0. \end{cases} \tag{1.5.1}$$

where g is the genus of the group, m the number of non-equivalent cusps, r the number of non-equivalent elliptic fixed points and e_i their order.

From that formula we can find all the possible signatures for groups with 1-dimensional spaces of cusp forms. For this we just establish a bound on all the possible values of k, g, m, r and the e_i's (we note that for $e \geq \frac{k}{2}$, $\left[\frac{k}{2}\left(1 - \frac{1}{e}\right)\right]$ remains constant) and it is easy to make a computer program that gives all the possible values for these quantities. We present the results in table 1.2 (since we are working with congruence subgroups we omit the case where there are no cusps).

We can find in [14] a complete list of all congruence subgroups (up to conjugacy) of $SL_2(\mathbb{R})$ of genus 0 and 1.

From that list we can extract the genus zero groups which have signature in table 1.2. The list of groups with the respective spaces of cusp forms of dimension 1 are listed in table 1.3.

Signature			Weight												
g	m	T	2	4	6	8	10	12	14	16	18	20	22	24	26
0	1	2^4			•										
0	1	2^3		•			•								
0	1	$2^2 M^1$, $3 \leq M$		•	•										
0	1	$2^1 M^1 N^1$, $3 \leq M \leq N$		•											
0	1	$M^1 N^1 P^1$, $3 \leq M \leq N \leq P$		•											
0	1	$2^1 3^1$						•		•	•	•	•		•
0	1	$2^1 4^1$				•		•	•		•				
0	1	$2^1 5^1$				•	•	•	•						
0	1	$2^1 6^1$				•	•		•						
0	1	$2^1 M^1$, $7 \leq M$				•	•								
0	1	3^2			•		•		•						
0	1	$3^1 4^1$			•	•	•								
0	1	$3^1 M^1$, $5 \leq M$			•	•									
0	1	4^2			•		•								
0	1	$4^1 M^1$, $5 \leq M$			•										
0	1	$M^1 N^1$, $5 \leq M \leq N$			•										
0	2	2^2		•	•										
0	2	$2^1 M^1$, $3 \leq M$		•											
0	2	$M^1 N^1$, $3 \leq M \leq N$		•											
0	2	2^1				•	•								
0	2	3^1			•	•									
0	2	M^1, $4 \leq M$			•										
0	3	Torsion Free			•										
0	3	M^1, $2 \leq M$		•											
0	4	Torsion Free		•											
1	1	Torsion Free	•	•											
1	Any	Any	•												

Table 1.2: List of signatures with corresponding weights for which the space of cusp forms is one dimensional.
g=genus, m=number of cusps, T=torsion.

Group		Weights												
		2	4	6	8	10	12	14	16	18	20	22	24	26
$1A_1^0$	$1+$						•		•	•	•	•		•
$2A_1^0$	$2\|2$			•		•		•						
$2B_1^0/1B_2^0$	$2-$				•	•								
$2C_1^0/4B_1^0/2D_2^0$	$4-$			•										
$3A_1^0$	$3\|3$		•			•								
$3B_1^0/1B_3^0$	$3-$			•	•									
$3C_1^0$	$9+$		•	•										
$3D_1^0/9B_1^0/3D_3^0$	$9-$		•											
$4A_1^0$			•	•										
$4C_1^0/2C_2^0$	$4\|2$		•											
$4D_1^0$			•											
$4E_1^0/8D_1^0/2F_2^0/4F_2^0$	$8-$		•											
$5A_1^0$			•											
$5B_1^0/1B_5^0$	$5-$		•	•										
$5D_1^0/5C_5^0$			•											
$6A_1^0$			•											
$6B_1^0$				•										
$6C_1^0/2D_3^0$	$6\|2$		•											
$6F_1^0/3F_2^0/2F_3^0/1E_6^0$	$6-$		•											
$7C_1^0/1B_7^0$	$7-$		•											
$1A_2^0$	$2+$				•		•	•		•				
$2A_2^0$	$4\|2+$		•			•								
$2B_2^0$	$4\|2+2'$			•		•								
$2E_2^0$	$8+$		•	•										
$3A_2^0$			•	•										
$3B_2^0/1C_6^0$	$6+2$		•	•										
$3E_2^0$			•											
$4A_2^0$	$8\|4+$			•										
$4B_2^0$				•										
$4C_2^0$			•											
$5B_2^0/1D_{10}^0$	$10+2$		•											
$1A_3^0$	$3+$				•	•		•						
$2A_3^0$	$6\|2+3'$		•	•										
$2B_3^0$	$6\|2+3$			•										
$2C_3^0/1D_6^0$	$6+3$		•	•										
$3A_3^0$	$9\|3+$			•										
$3B_3^0$			•											
$12B_3^0$			•											
$1A_5^0$	$5+$		•			•								
$2A_5^0$	$10\|2+5'$			•										
$5A_5^0$			•	•										
$5B_5^0$			•	•										
$10A_5^0$				•										
$10B_5^0$				•										
$1A_6^0$	$6+$		•			•								
$1B_6^0$	$6+6$		•	•										
$2A_6^0$	$12\|2+$			•										
$2B_6^0$	$12\|2+2'$			•										
$2C_6^0$	$12\|2+2$			•										
$1A_7^0$	$7+$		•	•										
$1A_{10}^0$	$10+$		•	•										
$1A_{11}^0$	$11+$			•										
$1A_{14}^0$	$14+$			•										
$1A_{15}^0$	$15+$			•										

Table 1.3: Genus zero groups with respective weights for which the space of cusp forms is one dimensional

1.6 Computation of the Cusp Forms

In this section we compute the cusp forms for each of the groups in table 1.3. It turns out that we can write explicitly our forms using only the Dedekind eta-function and Eisenstein series. We will make use of the properties of these functions stated in the previous sections. We will also make use of formula (1.3.1) when we want to find the action of a particular Atkin-Lehner involution on an eta-product, so we don't mention it.

For all the forms except three, namely the weight 6 forms for $4A_1^0$ and $3A_2^0$ and the weight 4 for $12B_3^0$, we can prove they are the forms we want using some description of the group. Either we know it is of the form $n|h+e, f, \ldots$ or we use a set of matrices that generate the group. For the other three forms we find their power series expansions by a method to be described below. With the first few coefficients we can, with the aid of a computer, find a candidate expression for the form. Having this candidate we use the Sturm bound to guarantee that in fact this is the form we want.

Theorem 1.6.1. *If Γ is a congruence subgroup of $SL_2(\mathbb{Z})$ of index μ and $f \in M_k(\Gamma)$ is a modular then if the first $\frac{\mu k}{12}$ coefficients of f are zero then f is identically zero.*

Proof. See Theorem 3.13 in [38].

The method we use to find a power series expansion of our forms is the following. If G is a group of genus 0 we know that the field of modular functions of G is generated by a unique (after normalization) element t_G, i.e. every modular function for G is a rational function of t_G. We say that t_G is a Hauptmodul for G. If we differentiate t_G we get an automorphic function of weight 2 which may fail to be a cusp form because it is not necessarily holomorphic. The same applies to the weight $2k$ form $\left(\frac{dt_G}{d\tau}\right)^k$ however, we can multiply this form by a suitable rational function of t_G in order to obtain a cusp form. Thus, the problem of finding a q-expansion for an even weight cusp form for a group G is reduced to finding a q-expansion for the Hauptmodul of G.

For several groups in our list the expression for the Hauptmodul is known, for example [12]. When we don't know t_H for a group H but we know t_G for some G containing H we can find t_H as follows.

Since t_G is invariant under G it will also be invariant under H and then it is a rational function of t_H. To find the expression of this rational function we just have to consider the following map:

$$\begin{array}{rcccc} \pi & : & H\backslash\mathcal{H}^* & \longrightarrow & G\backslash\mathcal{H}^* \\ & & H\tau & \longmapsto & G\tau \end{array}$$

where $\mathcal{H}^*$ denotes $\mathcal{H}\cup\mathbb{Q}$ and $H\backslash\mathcal{H}^*$ (resp. $G\backslash\mathcal{H}^*$) is the Riemann surface associated to H (resp. G). Since a modular function is completely determined, up to multiplication by a constant, by its divisor, we can write for any $P\in G\backslash\mathcal{H}^*$, $P\neq G\infty$:

$$t_G(\tau)-t_G(P)=\frac{\displaystyle\prod_{Q\in\pi^{-1}(P)}\big(t_H(\tau)-t_H(Q)\big)^{r_Q}}{\displaystyle\prod_{Q\in\pi^{-1}(G\infty)\backslash\{H\infty\}}\big(t_H(\tau)-t_H(Q)\big)^{r_Q}} \tag{1.6.1}$$

where r_Q is the ramification index of π at Q. Using this equation for various P we can express t_G as a rational function of t_H which allows us to write a q-expansion of t_H.

We illustrate this method with an example. We take $G=1A_1^0$, $H=4A_1^0$, $P=G\cdot i$ and $Q=G\cdot\rho$ where $\rho=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$.

There are in $H\backslash\mathcal{H}^*$ three points A, B and C lying above P, with ramification indexes 1, 1 and 2 respectively, and two points D and E lying above Q with ramification indexes 1 and 3 respectively.

Using (1.6.1) for P and Q we get:

$$t_G(\tau)-1728=\big(t_H(\tau)-t_H(A)\big)\big(t_H(\tau)-t_H(B)\big)\big(t_H(\tau)-t_H(C)\big)^2$$

$$t_G(\tau)=\big(t_H(\tau)-t_H(D)\big)\big(t_H(\tau)-t_H(E)\big)^3$$

We can set $t_H(E) = 0$ since we can add a constant to a Hauptmodul and it will still be a generator for the function field of modular functions.

From the first equation we see that the polynomial $X^4 - t_H(D)X^3 - 1728$ has to have a double root and this implies that $t_H(D)$ has only four possible values: $\pm 8 \pm 8i$.

These four values are the values of $t_H(D)$ for the four different conjugates of $4A_1^0$ under powers of $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and so we fix, for example, $t_H(D) = 8 + 8i$.

Now, we state our theorem:

Theorem 1.6.2. *The following list gives generators for the* 1*-dimensional spaces of cusp forms of table 1.3:*

$1A_1^0$	-	$S_{12} = 1^{24},\ S_{16} = 1^{24} \cdot E_4(\tau),\ S_{18} = 1^{24} \cdot E_6(\tau),\ S_{20} = 1^{24} \cdot E_8(\tau)$
		$S_{22} = 1^{24} \cdot E_{10}(\tau),\ S_{26} = 1^{24} \cdot E_{14}(\tau)$
$2A_1^0$	-	$S_6 = 1^{12},\ S_{10} = 1^{12} \cdot E_4(\tau),\ S_{14} = 1^{12} \cdot E_8(\tau)$
$2B_1^0/1B_2^0$	-	$S_8 = 1^8 2^8,\ S_{10} = 1^8 2^8 \cdot (E_2(\tau) - 2 \cdot E_2(2\tau))$
$2C_1^0/4B_1^0/2D_2^0$	-	$S_6 = 2^{12}$
$3A_1^0$	-	$S_4 = 1^8,\ S_{10} = 1^8 \cdot E_6(\tau)$
$3B_1^0/1B_3^0$	-	$S_6 = 1^6 3^6,\ S_8 = 1^6 3^6 \cdot (E_2(\tau) - 3 \cdot E_2(3\tau))$
$3C_1^0$	-	$S_4 = 3^8,\ S_6 = 3^8 \cdot (E_2(\tau) - 6 \cdot E_2(3\tau) + 9 \cdot E_2(9\tau))$
$3D_1^0/9B_1^0/3D_3^0$	-	$S_4 = 3^8$
$4A_1^0$	-	$S_4 = \eta\left(\frac{z}{4}\right)^2 \eta\left(\frac{z}{2}\right)^{-1} \eta(z)^4 \eta(2z)^5 \eta(4z)^{-2} + 2i \cdot \eta\left(\frac{z}{4}\right)^{-2} \eta\left(\frac{z}{2}\right)^5 \eta(z)^4 \eta(2z)^{-1} \eta(4z)^2$
		$S_6 = 1^6 \cdot \left(E_3^{1_1,\chi}\left(\frac{\tau}{4}\right) + E_3^{\chi,1_1}\left(\frac{\tau}{4}\right) + iE_3^{1_1,\chi}\left(\frac{\tau}{2}\right) + 4iE_3^{\chi,1_1}\left(\frac{\tau}{2}\right) - 2iE_3^{1_1,\chi}(\tau) - 32E_3^{\chi,1_1}(\tau)\right)$
$4C_1^0/2C_2^0$	-	$S_4 = 1^4 2^4,\ S_6 = 1^4 2^4 \cdot (E_2(\tau) - 2 \cdot E_2(2\tau))$
$4D_1^0$	-	$S_4 = \eta\left(\frac{z}{4}\right)^2 \eta\left(\frac{z}{2}\right)^{-1} \eta(z)^4 \eta(2z)^5 \eta(4z)^{-2} + 2i \cdot \eta\left(\frac{z}{4}\right)^{-2} \eta\left(\frac{z}{2}\right)^5 \eta(z)^4 \eta(2z)^{-1} \eta(4z)^2$
$4E_1^0/8D_1^0/2F_2^0/4F_2^0$	-	$S_4 = 2^4 4^4$
$5A_1^0$	-	$S_4 = 1^{-1} 5^{10} 25^{-1} + 5 \cdot 1^2 5^4 25^2$
$5B_1^0/1B_5^0$	-	$S_4 = 1^4 5^4,\ S_6 = 1^4 5^5 \cdot (E_2(\tau) - 5 \cdot E_2(5\tau))$
$5D_1^0/5C_5^0$	-	$S_4 = 1^4 5^4$
$6A_1^0$	-	$S_4 = \eta\left(\frac{z}{2}\right)^8 + \left(8 + 8\sqrt{3}i\right) \cdot \eta(2z)^8$
$6B_1^0$	-	$S_6 = \eta\left(\frac{z}{3}\right)^{12} + 18 \cdot \eta(z)^{12} + 729 \cdot \eta(3z)^{12}$
$6C_1^0/2D_3^0$	-	$S_4 = 2^3 6^3 \cdot E_1^{1_1,\chi}(2\tau)$, *where* $\chi(\cdot) = \left(\frac{\cdot}{3}\right)$
$6F_1^0/3F_2^0/2F_3^0/1E_6^0$	-	$S_4 = 1^2 2^2 3^2 6^2$
$7C_1^0/1B_7^0$	-	$S_4 = 1^3 7^3 \cdot E_1^{1_1,\chi}(\tau)$, *where* $\chi(\cdot) = \left(\frac{\cdot}{7}\right)$
$1A_2^0$	-	$S_8 = 1^8 2^8,\ S_{12} = 1^8 2^8 \cdot (E_4(\tau) + 4 \cdot E_4(2\tau)),\ S_{14} = 1^8 2^8 \cdot (E_6(\tau) + 8 \cdot E_6(2\tau))$
		$S_{18} = 1^8 2^8 \cdot (E_{10}(\tau) + 32 \cdot E_{10}(2\tau))$
$2A_2^0$	-	$S_4 = 1^4 2^4,\ S_{10} = 1^4 2^4 \cdot (E_6(\tau) + 8 \cdot E_6(2\tau))$
$2B_2^0$	-	$S_6 = 1^4 2^4 \cdot (E_2(\tau) - 2 \cdot E_2(2\tau))$
$2E_2^0$	-	$S_4 = 2^4 4^4,\ S_6 = 2^4 4^4 \cdot (E_2(2\tau) - 2 \cdot E_2(4\tau))$
$3A_2^0$	-	$S_4 = 3^8 + 4 \cdot 6^8,\ S_6 = 2^4 \cdot \left(4\left(E_4(\tau) - 18E_4(2\tau) + 32E_4(4\tau)\right) - \left(1^8 + 32 \cdot 4^8\right)\right)$
$3B_2^0/1C_6^0$	-	$S_4 = 1^2 2^2 3^2 6^2,\ S_6 = 1^2 2^2 3^2 6^2 \cdot (E_2(\tau) + 2 \cdot E_2(2\tau) - 3 \cdot E_3(3\tau) - 6 \cdot E_6(6\tau))$
$3E_2^0$	-	$S_4 = 1^8 + 4 \cdot 2^8$
$4A_2^0$	-	$S_6 = 1^2 2^2 \cdot (E_4(\tau) - 4 \cdot E_4(2\tau))$
$4B_2^0$	-	$S_6 = 1^2 2^2 \cdot \left(E_4\left(\frac{\tau}{2}\right) - 7 \cdot E_4(\tau) - 28 \cdot E_4(2\tau) + 64 \cdot E_4(4\tau)\right)$
$4C_2^0$	-	$S_4 = 4^{10} 8^{-2} - 8i \cdot 4^{-2} 8^{10}$
$5B_2^0/1D_{10}^0$	-	$S_4 = 1^4 5^4 + 4 \cdot 2^4 10^4$

$1A_3^0$	-	$S_8 = 1^63^6 \cdot (E_2(\tau) - 3 \cdot E_2(3\tau))$, $S_{10} = 1^63^6 \cdot (E_4(\tau) - 9 \cdot E_4(3\tau))$
		$S_{14} = 1^63^6 \cdot (E_8(\tau) - 81 \cdot E_8(3\tau))$
$2A_3^0$	-	$S_4 = 1^33^3 \cdot E_1^{1_1,\chi}(\tau)$, $S_6 = 1^33^3 \cdot \left(E_3^{1_1,\chi}(\tau) + 3 \cdot E_3^{\chi,1_1}(\tau)\right)$, *where* $\chi(\cdot) = \left(\frac{\cdot}{3}\right)$
$2B_3^0$	-	$S_6 = 1^33^3 \cdot \left(E_3^{1_1,\chi}(\tau) - 3 \cdot E_3^{\chi,1_1}(\tau)\right)$, *where* $\chi(\cdot) = \left(\frac{\cdot}{3}\right)$
$2C_3^0/1D_6^0$	-	$S_4 = 1^22^23^26^2$, $S_6 = 1^22^23^26^2 \cdot (E_2(\tau) - 2 \cdot E_2(2\tau) + 3 \cdot E_3(3\tau) - 6 \cdot E_6(6\tau))$
$3A_3^0$	-	$S_6 = 1^23^2 \cdot (E_4(\tau) - 9 \cdot E_4(9\tau))$
$3B_3^0$	-	$S_4 = 1^33^49^1 + 6 \cdot 3^49^4 + 27 \cdot 3^19^427^3$
$12B_3^0$	-	$S_4 = \left(3 \cdot 2^{-2}4^76^612^{-3} - 2 \cdot 2^{-3}4^{12}6^112^{-2}\right) +$
		$+ 3\sqrt{3}i \cdot \left(5 \cdot 2^{-6}4^96^{10}12^{-5} - 2 \cdot 2^{-7}4^{14}6^512^{-4} - 3 \cdot 2^{-5}4^46^{15}12^{-6}\right)$
$1A_5^0$	-	$S_4 = 1^45^4$, $S_{10} = 1^45^4 \cdot (E_6(\tau) + 125 \cdot E_6(5\tau))$
$2A_5^0$	-	$S_6 = 1^25^2 \cdot (E_4(\tau) - 25 \cdot E_4(5\tau))$
$5A_5^0$	-	$S_4 = 1^45^4$, $S_6 = 1^95^3 + 5\sqrt{5} \cdot 1^35^9$
$5B_5^0$	-	$S_4 = 1^45^4$, $S_6 = 1^95^3 - 5\sqrt{5} \cdot 1^35^9$
$10A_5^0$	-	$S_6 = 1^25^2 \cdot \left(E_4^{1_1,\chi}(\tau) - 5\sqrt{5}E_4^{\chi,1_1}(\tau)\right)$, *where* $\chi(\cdot) = \left(\frac{\cdot}{5}\right)$
$10B_5^0$	-	$S_6 = 1^25^2 \cdot \left(E_4^{1_1,\chi}(\tau) + 5\sqrt{5}E_4^{\chi,1_1}(\tau)\right)$, *where* $\chi(\cdot) = \left(\frac{\cdot}{5}\right)$
$1A_6^0$	-	$S_4 = 1^22^23^26^2$, $S_10 = 1^22^23^26^2 \cdot (E_6(\tau) + 8 \cdot E_6(2\tau) + 27 \cdot E_6(3\tau) + 216 \cdot E_6(6\tau))$
$1B_6^0$	-	$S_4 = 1^22^23^26^2$, $S_6 = 1^22^23^26^2 \cdot (E_2(\tau) - 2 \cdot E_2(2\tau) - 3 \cdot E_3(3\tau) + 6 \cdot E_6(6\tau))$
$2A_6^0$	-	$S_6 = 1^12^13^16^1 \cdot (E_4(\tau) + 4 \cdot E_4(2\tau) - 9 \cdot E_4(3\tau) - 36 \cdot E_4(6\tau))$
$2B_6^0$	-	$S_6 = 1^12^13^16^1 \cdot (E_4(\tau) - 4 \cdot E_4(2\tau) + 9 \cdot E_4(3\tau) - 36 \cdot E_4(6\tau))$
$2C_6^0$	-	$S_6 = 1^12^13^16^1 \cdot (E_4(\tau) - 4 \cdot E_4(2\tau) - 9 \cdot E_4(3\tau) + 36 \cdot E_4(6\tau))$
$1A_7^0$	-	$S_4 = 1^37^3 \cdot E_1^{\chi,1}(\tau)$, $S_6 = 1^37^3 \cdot \left(E_3^{1_1,\chi}(\tau) + 7E_3^{\chi,1_1}(\tau)\right)$, *where* $\chi(\cdot) = \left(\frac{\cdot}{7}\right)$
$1A_{10}^0$	-	$S_4 = 1^45^4 + 4 \cdot 2^410^4$
		$S_6 = \left(1^45^4 - 4 \cdot 2^410^4\right) \cdot (E_2(\tau) - 2 \cdot E_2(2\tau) + 5 \cdot E_2(5\tau) - 10 \cdot E_2(10\tau))$
$1A_{11}^0$	-	$S_6 = 1^211^2 \cdot (E_4(\tau) - 121 \cdot E_4(11\tau))$
$1A_{14}^0$	-	$S_6 = 1^12^17^114^1 \cdot (E_4(\tau) + 4 \cdot E_4(2\tau) - 49 \cdot E_4(7\tau) - 196 \cdot E_4(14\tau))$
$1A_{15}^0$	-	$S_6 = 1^13^15^115^1 \cdot (E_4(\tau) + 9 \cdot E_4(3\tau) - 25 \cdot E_4(5\tau) - 225 \cdot E_4(15\tau))$

Proof. We split the argument in two subcases. The main division is $n|h+e, f\ldots$ and others. We will show that for these groups (with two exceptions $10+$ and $10+2$ which we discuss in more detail below) we can always express the cusp forms as a product of a multiplicative eta-product and an Eisenstein series.

- Groups of type $n|h+e, f$ (except $10+$ and $10+2$).

  For every group of this form in our list we can always find a multiplicative eta-product on $\Gamma_0(N)$, where $N = nh$. This is unique if we always choose, when possible, one of even weight. The only cases for which the eta-product has odd weight is $N = 7, 12$ which correspond to 1^37^3, 2^36^3, respectively.

  So, for each N we fix a η_π of level N.

  We note that for our choice of the eta-product every part of π is divisible by h and find it more convenient to work with the conjugate of $n|h+e, f\ldots$ obtained from conjugation by $M_h = \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$.

We deal first with the cases where η_π has even weight. In order to obtain a cusp form for this group we consider the product

$$\eta_\pi(\tau/h)\cdot\left(\sum_{d|\frac{n}{h}}\alpha_d d^{\frac{k}{2}}\cdot E_k(d\tau)\right) \tag{1.6.2}$$

The fact that η_π is a cusp form guarantees that this is a cusp form as well. In order to have cusp forms on that conjugate of $n|h+e,f\ldots$ we start by choosing the α_d so that that product is invariant (resp. negated) under the appropriate Atkin-Lehner involutions. To make it invariant (resp. negated) under the action of an Atkin-Lhener involution W_e we just have to restrict to those $\sum_{d|\frac{n}{h}}\alpha_d d^{\frac{k}{2}}\cdot E_k(d\tau)$ that satisfy condition (1.4.16) with $\lambda=\overline{\omega_\pi^e}$ (resp. $-\overline{\omega_\pi^e}$).

For $k=2$ we also have to add the condition $\sum_{d|n}\alpha_d=0$ from (1.4.13). It happens that it is always possible to choose the α_d in this way and this choice is unique for the groups and weights in table 1.3 (this was already necessarily true for those groups with $h=1$).

For example, if we consider 6+2 and try to find its weight 6 form, we take $\eta_\pi=1^2 2^2 3^2 6^2$ and since this has weight 4 and is fixed by W_2 we need to find α_d such that

$$\alpha_1 E_2(\tau)+2\alpha_2 E_2(2\tau)+3\alpha_3(3\tau)+6\alpha(6\tau)$$

is invariant under W_2. But, from (1.4.13) and (1.4.16) we see that, up to multiplication by a scalar, the only possibility is $\alpha_1=\alpha_2=1$, $\alpha_3=\alpha_6=-1$.

It turns out that after doing this, the matrices $M_1=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and $M_2=\begin{pmatrix}1&0\\\frac{n}{h}&1\end{pmatrix}$ have the right action on these forms, i.e., they act as multiplication by $\lambda\left(M_h^{-1}M_1M_h\right)$, $\lambda\left(M_h^{-1}M_2M_h\right)$ respectively, where λ is the homomorphism from section 1.2.

This proves that these forms are invariant under the kernel of λ - or, more precisely, its conjugate by $\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$ - which is exactly the definition of $n|h+e, f, \ldots$.

The case where $h = 1$ is trivial so we give an example with $h > 1$.

For $G = 2|2$ the form to take is $1^{12}E_k(\tau)$.

Since E_k is invariant under $SL_2(\mathbb{Z})$ we just have to check the action on 1^{12}. Using (1.3.1)

$\eta(M_1 z)^{12} = \mu(M_1)^{12}\eta(z)^{12} = e^{\pi i}\eta(z)^{12}$

$\eta(M_2 z)^{12} = \mu(M_2)^{12}\eta(z)^{12} = e^{\pi i}\eta(z)^{12}$.

which agrees with the given definition of λ.

When the weight k is odd we have to consider Eisenstein series with character. For $N = n = 7$, we can work only with $7+$ because the weight 4 for $7-$ must be the same for $7+$. In this case $\eta_\pi = 1^3 7^3$ has as a character $\chi = \left(\frac{-7}{\cdot}\right)$ and we multiply by an Eisenstein series with character χ in order to obtain something without character. So, $1^3 7^3 \cdot E_1^{1_1,\chi}$ is clearly the weight 4 cusp form on $7-$ with trivial character.

Using (1.3.2) and (1.3.1) we see that the Fricke involution $\begin{pmatrix} 0 & -1 \\ 7 & 0 \end{pmatrix}$ acts on $1^3 7^3$ as multiplication by i, and knowing that $g(\chi) = \sqrt{7}i$ (see Theorem 1 of Chapter 6 in [35]), (1.4.15) applied to $E_1^{1_1,\chi}$ gives that it acts on $E_1^{1_1,\chi}$ as multiplication by $-i$. This proves that $1^3 7^3 \cdot E_1^{1_1,\chi}$ is the weight 4 form for $7+$.

For the weight 6 form we consider $E_3^{1_1,\chi} + 7E_3^{\chi,1_1}$ instead of $E_1^{1_1,\chi}$ and the invariance under the Fricke involution follows from (1.4.18).

The other odd weight case which corresponds to the groups $6|2$, $6|2+3'$ and $6|2+3$, all with $\eta_\pi = 2^3 6^3$, (or, actually, $1^3 3^3$ after conjugation by M_h) are done similarly. In this case the Atkin-Lehner involution $\begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}$ acts as multiplication by $-i$ and, again, we can see that the matrices M_1 and M_2 have the right action on these forms.

- $10+$ and $10+2$. We first note that we only need to work with $10+$ because the weight 4 form for $10+2$ must necessarily be the same for $10+$.

  In this case we don't have a multiplicative eta-product of level 10. However we have $1^4 5^4$ which is of level 5 and then it is a form on $\Gamma_0(10)$. Since it is invariant under the Atkin-Lehner involution W_5 we only need to symmetrize under W_2 to get the weight 4 form for $10+$. This form is $1^4 5^5 + 4 \cdot 2^4 10^4$.

  To obtain the weight 6 form we note that $E_2(q) - 2 \cdot E_2(q^2) + 5 \cdot E_2(q^5) - 10 \cdot E_2(q^{10})$ is a form on $\Gamma_0(10)$ that is fixed by W_5 and negated by W_2. So if we take the product of this form not with $1^4 5^5 + 4 \cdot 2^4 10^4$ but with $1^4 5^5 - 4 \cdot 2^4 10^4$ we get a form that is invariant under $\Gamma_0(10)$ and also all the Atkin-Lehner involutions.

We are left with the groups $4A_1^0$, $4D_1^0$, $5A_1^0$, $5D_1^0(5C_5^0)$, $6A_1^0$, $6B_1^0$, $3A_2^0$, $3E_2^0$, $4B_2^0$, $4C_2^0$, $3B_3^0$, $12B_3^0$, $5A_5^0$, $5B_5^0$, $10A_5^0$ and $10B_5^0$, that will be analyzed individually.

We will repeatedly make statements about inclusion and normality of subgroups in larger groups as well as claims that a set of matrices generates a group. All this information can be found in [14] and [15].

Also, when we state the dimension of the space of cusp forms for a certain group we are using formula (1.5.1) and the signature of the group can be found in [14].

Statements about linear independence of modular forms can also easily be checked by analysing the first coefficients of their power series expansion.

- $4A_1^0$

  This groups is $\left\langle \Gamma_0(4) \cap \Gamma^0(4), \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \right\rangle$ and the weight 4 form for $4A_1^0$ is the same for $4D_1^0 = \left\langle 4G_1^0, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \right\rangle$.

  We have a basis for the space of weight 4 cusp forms on $4G_1^0$ consisting only of eta-quotients. We can take for example $\eta(\frac{z}{2})^{-4}\eta(z)^{16}\eta(2z)^{-4}$, $\eta(\frac{z}{2})^4\eta(z)^4$ and $\eta(z)^4\eta(2z)^4$.

We see what the action of $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ is on this basis.

$$\left(\eta(\tfrac{z}{2})^{-4}\eta(z)^{16}\eta(2z)^{-4}\right)_{|4}\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} =$$

$$= z^{-4}\eta\left(\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} z\right)^{-4}\eta\left(\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} z\right)^{16}\eta\left(\begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix} z\right)^{-4} =$$

$$= z^{-4}\eta\left(\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}\left(\tfrac{z+1}{2}\right)\right)^{-4}\eta\left(\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} z\right)^{16}\eta\left(\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}\left(\tfrac{z}{2}\right)\right)^{-4}$$

$$= z^{-4}(-1)z^{-2}\eta\left(\tfrac{z+1}{2}\right)^{-4}e^{-\frac{2\pi i}{3}}z^{8}\eta\left(z\right)^{16}e^{\frac{\pi i}{3}}\left(\tfrac{z}{2}\right)^{-2}\eta\left(\tfrac{z}{2}\right)^{-4} =$$

$$= 4e^{\frac{\pi i}{3}}\eta\left(\tfrac{z+1}{2}\right)^{-4}\eta\left(z\right)^{16}\eta\left(\tfrac{z}{2}\right)^{-4} = 4i\left(q_2 - 4q_2^3 - 2q_2^5 + 24q_2^7\ldots\right) =$$

$$= 4i\eta(z)^4\eta(2z)^4.$$

$$\left(\eta(\tfrac{z}{2})^{4}\eta(z)^{4}\right)_{|4}\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = z^{-4}\eta\left(\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} z\right)^{4}\eta\left(\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} z\right)^{4} =$$

$$= z^{-4}\eta\left(\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} z\right)^{4}\eta\left(\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} z\right)^{4} =$$

$$= z^{-4}(-1)z^{2}\eta\left(\tfrac{z+1}{2}\right)^{4}e^{-\frac{2\pi i}{3}}z^{2}\eta(z)^4 = e^{\frac{\pi i}{3}}\eta\left(\tfrac{z+1}{2}\right)^{4}\eta(z)^4 =$$

$$= i\left(q_4 + 4q_4^3 - 2q_4^5 - 24q_4^7 + \ldots\right) = i\eta(\tfrac{z}{2})^{-4}\eta(z)^{16}\eta(2z)^{-4}.$$

$$\left(\eta(z)^{4}\eta(2z)^{4}\right)_{|4}\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = z^{-4}\eta\left(\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} z\right)^{4}\eta\left(\begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix} z\right)^{4} =$$

$$= z^{-4}\eta\left(\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} z\right)^{4}\eta\left(\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} z\right)^{4} =$$

$$= z^{-4}e^{-\frac{2\pi i}{3}}z^{2}\eta(z)^4e^{-\frac{\pi i}{3}}\left(\tfrac{z}{2}\right)^{2}\eta\left(\tfrac{z}{2}\right)^{4} = -\tfrac{1}{4}\eta(z)^4\eta\left(\tfrac{z}{2}\right)^{4}$$

The action of $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ on the basis is thus given by the following matrix:

$$\begin{bmatrix} 0 & i & 0 \\ 0 & 0 & -\frac{1}{4} \\ 4i & 0 & 0 \end{bmatrix}$$

and the only eigenvector associated to the eigenvalue 1 is

$\eta\left(\frac{z}{2}\right)^4\eta(z)^4 - 4\cdot\eta(z)^4\eta(2z)^4 + i\cdot\eta\left(\frac{z}{2}\right)^{-4}\eta(z)^{16}\eta(2z)^{-4}$.

Knowing that

$$\eta\left(\frac{z}{4}\right)^2\eta\left(\frac{z}{2}\right)^{-1}\eta\left(z\right)^4\eta\left(2z\right)^5\eta\left(4z\right)^{-2}$$

and

$$\eta\left(\frac{z}{4}\right)^{-2}\eta\left(\frac{z}{2}\right)^{5}\eta\left(z\right)^4\eta\left(2z\right)^{-1}\eta\left(4z\right)^{2}$$

are forms on $4G_1^0$ it is a matter of writing these as a linear combination of the elements in the basis to verify that the weigth 4 form for $4A_1^0$ can also be expressed as

$$\eta\left(\frac{z}{4}\right)^2\eta\left(\frac{z}{2}\right)^{-1}\eta\left(z\right)^4\eta\left(2z\right)^5\eta\left(4z\right)^{-2} + 2i\cdot\eta\left(\frac{z}{4}\right)^{-2}\eta\left(\frac{z}{2}\right)^{5}\eta\left(z\right)^4\eta\left(2z\right)^{-1}\eta\left(4z\right)^{2}$$

We find now the weight 6 form for this group. In order to obtain the its power series expansion and following what has been done at the beginning of this section we first observe that the divisors of $\frac{dt_H}{d\tau}$ and $\left(\frac{dt_H}{d\tau}\right)^3$ are $\frac{1}{2}A+\frac{1}{2}B+\frac{2}{3}D-\infty$ and $\frac{3}{2}A+\frac{3}{2}B+2D-3\infty$, respectively.

From this we conclude that

$$\frac{\left(\frac{dt_H}{d\tau}\right)^3}{\big(t_H(\tau)-t_H(A)\big)\big(t_H(\tau)-t_H(B)\big)\big(t_H(\tau)-t_H(D)\big)^2}$$

which has divisor $\frac{1}{2}A+\frac{1}{2}B+\infty$ and consequently is our weight 6 form.

From the factorization $X^4-(8+8i)X^3-1728 = \big(X^2+(4+4i)X+24i\big)\big(X-(6+6i)\big)^2$ we conclude that $\big(t_H(\tau)-t_H(A)\big)\big(t_H(\tau)-t_H(B)\big) = t_H(\tau)^2+(4+4i)t_H(\tau)+24i$ and the expression above becomes

$$\frac{\left(\frac{dt_H}{d\tau}\right)^3}{\left(t_H(\tau)^2+(4+4i)t_H(\tau)+24i\right)\left(t_H(\tau)-(8+8i)\right)^2}$$

This gives a power series expansion for this form. If we conjugate this group by $\begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix}$ we get a group with cusp width 1 at infinity that contains $\Gamma_0(16)$. This has the effect of having integer powers of q in the power series expansion.

Using (1.4.8) we can find a power series expansion for

$$(2+2i)4^6(E_3^{1_1,\chi}(\tau)+E_3^{\chi,1_1}(\tau)+iE_3^{1_1,\chi}(2\tau)+4iE_3^{\chi,1_1}(2\tau)-2iE_3^{1_1,\chi}(4\tau)-32E_3^{\chi,1_1}(4\tau))$$

where $\chi=\left(\frac{-1}{\cdot}\right)$, and then check if sufficiently many coefficients agree with the power series expansion above.

Since this has weight 6 and is a modular form on $\Gamma_0(16)$ which has index 24 in $SL_2(\mathbb{Z})$ we just have to compare the first $\frac{6\cdot 24}{12}=12$ coefficients of these two modular forms to guarantee that they are the equal.

With the help of a computer we have confirmed so and the form on $4A_1^0$ is then

$$1^6\cdot\left(E_3^{1_1,\chi}\left(\frac{\tau}{4}\right)+E_3^{\chi,1_1}\left(\frac{\tau}{4}\right)+iE_3^{1_1,\chi}\left(\frac{\tau}{2}\right)+4iE_3^{\chi,1_1}\left(\frac{\tau}{2}\right)-2iE_3^{1_1,\chi}(\tau)-32E_3^{\chi,1_1}(\tau)\right)$$

- $4D_1^0$

  The weight 4 form for this group is the same for $4A_1^0$.

- $5A_1^0$

  This group contains $5G_1^0=\Gamma_0(5)\cap\Gamma^0(5)$ but not normally. However, we have a normal series $5G_1^0\lhd 5E_1^0\lhd 5A_1^0$.

  The space of weight 4 cusp forms for $5G_1^0$ has dimension 5 and a basis for this space is:

$$\eta(\frac{z}{5})^4\eta(z)^4,\eta(z)^4\eta(5z)^4,\eta(\frac{z}{5})^3\eta(z)^4\eta(5z),\eta(\frac{z}{5})\eta(z)^4\eta(5z)^3,\eta(\frac{z}{5})^2\eta(z)^4\eta(5z)^2$$

There is one more weight 4 eta-product on $5G_1^0$. It is

$$\eta(\frac{z}{5})^{-1}\eta(z)^{10}\eta(5z)^{-1}$$

We will use the fact that

$$\eta(\frac{z}{5})^{-1}\eta(z)^{10}\eta(5z)^{-1} = \eta(\frac{z}{5})^{4}\eta(z)^{4} + 25\cdot\eta(z)^{4}\eta(5z)^{4}+$$
$$+ 5\cdot\eta(\frac{z}{5})^{3}\eta(z)^{4}\eta(5z) + 25\cdot\eta(\frac{z}{5})\eta(z)^{4}\eta(5z)^{3} + 15\cdot\eta(\frac{z}{5})^{2}\eta(z)^{4}\eta(5z)^{2}$$

The group $5E_1^0$ is generated over $5G_1^0$ by $\begin{bmatrix} 5 & -1 \\ 1 & 0 \end{bmatrix}$

Symmetrization by this matrix gives a basis for the space of weight 4 forms on $5E_1^0$. This basis is

$$\eta(\frac{z}{5})^{4}\eta(z)^{4} + 25\cdot\eta(z)^{4}\eta(5z)^{4}, \eta(\frac{z}{5})^{3}\eta(z)^{4}\eta(5z) + 5\eta(\frac{z}{5})\eta(z)^{4}\eta(5z)^{3}, \eta(\frac{z}{5})^{2}\eta(z)^{4}\eta(5z)^{2}$$

Now, $5A_1^0$ is generated over $5E_1^0$ by $\begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix}$ and the action of this matrix in the basis above is given by:

$$\begin{bmatrix} \frac{-1+\sqrt{5}}{2} & -\frac{5+\sqrt{5}}{10} & \frac{1}{5} \\ 4\sqrt{5} & -3 & \frac{5-\sqrt{5}}{5} \\ 10\sqrt{5} & -10 & \frac{7-\sqrt{5}}{2} \end{bmatrix}$$

The eigenvector associated to the eigenvalue 1 is

$$\eta(\frac{z}{5})^{4}\eta(z)^{4} + 25\cdot\eta(z)^{4}\eta(5z)^{4} + 5\eta(\frac{z}{5})^{3}\eta(z)^{4}\eta(5z)+$$
$$+ 25\eta(\frac{z}{5})\eta(z)^{4}\eta(5z)^{3} + 20\cdot\eta(\frac{z}{5})^{2}\eta(z)^{4}\eta(5z)^{2}$$

Using the linear relation above we conclude that the weight 4 form for $5A_1^0$ is

$$\eta(\frac{z}{5})^{-1}\eta(z)^{10}\eta(5z)^{-1}+5\cdot\eta(\frac{z}{5})^{2}\eta(z)^{4}\eta(5z)^{2}$$

- $5D_1^0(5C_5^0)$

  This group is $\Gamma_1(5)$ and its weight 4 form is $\eta(z)^4\eta(5z)^4$.

- $6A_1^0$

  This group is $\left\langle \Gamma_0(6)\cap\Gamma^0(6), \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 2 & -5 \end{bmatrix}, \begin{bmatrix} -7 & 4 \\ -2 & 1 \end{bmatrix} \right\rangle$.

  The following are weight 4 forms on $6F_1^1=\Gamma_0(6)\cap\Gamma^0(6)$: $\eta\left(\frac{z}{2}\right)^8$, $\eta\left(z\right)^8$ and $\eta\left(2z\right)^8$.

  Since they are also invariant under $\begin{bmatrix} -1 & 2 \\ 2 & -5 \end{bmatrix}$ and $\begin{bmatrix} -7 & 4 \\ -2 & 1 \end{bmatrix}$ they are forms on $6C_1^1=\left\langle \begin{bmatrix} -1 & 2 \\ 2 & -5 \end{bmatrix}, \begin{bmatrix} -7 & 4 \\ -2 & 1 \end{bmatrix}, 6F_1^1 \right\rangle$. They also form a base because the space of weight 4 cusp forms has dimension 3.

  The group $6C_1^1$ is normal in $6A_1^0$ and the action of $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ on this basis is given by the matrix:

$$\begin{bmatrix} e^{\frac{\pi i}{3}} & 0 & \frac{1}{16}e^{-\frac{2\pi i}{3}} \\ 0 & e^{\frac{2\pi i}{3}} & 0 \\ 16e^{\frac{\pi i}{3}} & 0 & 0 \end{bmatrix}$$

  The eigenvector associated to the eigenvalue 1 is

$$\eta\left(\frac{z}{2}\right)^8+\left(8+8\sqrt{3}i\right)\cdot\eta(2z)^8.$$

  This is the weight 4 form for $6A_1^0$.

- $6B_1^0$

This group is $\left\langle \Gamma_0(6) \cap \Gamma^0(6), \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} \right\rangle$ and contains the group $6F_1^1 = \Gamma_0(6) \cap \Gamma^0(6)$ normally.

It is not hard to see that $\eta\left(\frac{z}{3}\right)^{12}$, $\eta(z)^{12}$, $\eta(3z)^{12}$, $\eta\left(\frac{z}{3}\right)^{6} \cdot \eta(z)^6$, $\eta\left(\frac{z}{3}\right)^{6} \cdot \eta(3z)^6$ and $\eta(z)^6 \cdot \eta(3z)^6$ are forms on $6F_1^1$.

Symmetrizing by $\begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}$ we get that

$$\eta(\frac{z}{3})^{12} + 729\eta(3z)^{12}, \eta(z)^{12}, \eta(\frac{z}{3})^6\eta(3z)^6, \eta(\frac{z}{3})^6\eta(z)^6 - 27\eta(z)^6\eta(3z)^6$$

are forms on $6B_1^1 = \left\langle 6F_1^1, \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} \right\rangle$

The space of weight 4 cusp forms for $6B_1^1$ has dimension 4 and then they form a basis for this space.

Now, $6B_1^0$ normalizes $6B_1^1$ and is generated by $M = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$ over $6B_1^1$. The matrix that gives the action of $_{|4}[M]$ on that basis is:

$$\begin{bmatrix} 1 & 0 & \frac{1}{27} & 0 \\ 36 & -1 & \frac{2}{3} & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

and the only eigenvector associated to the eigenvalue 1 is

$$\eta\left(\frac{z}{3}\right)^{12} + 18 \cdot \eta(z)^{12} + 729 \cdot \eta(3z)^{12}.$$

This is the weight 6 form for $6B_1^0$.

- $3A_2^0$

This groups is $\left\langle \Gamma_0(6)\cap\Gamma^0(3), \begin{bmatrix} 6 & -1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 4 & -5 \\ 2 & -2 \end{bmatrix} \right\rangle$ and we claim that the weight 4 form for this group is $\eta(z)^8 + 4\cdot\eta(2z)^8$.

It is clear that both $\eta(z)^8$ and $\eta(2z)^8$ are fixed by $\Gamma_0(6)\cap\Gamma^0(3)$ and interchanged by the other two matrices.

For the weight 6 form we proceed as for $4A_1^0$. In this case we have $t_{1A_2^0} = t_{3A_2^0}^3 + 12t_{3A_2^0} - 256$ and the expression for the weight 6 form is $\frac{\left(\frac{dt_{3A_2^0}}{d\tau}\right)^3}{(t_{3A_2^0}+12)^2(t_{3A_2^0}+4)t_{3A_2^0}}$.

If we conjugate this group by $\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$ we get a group with cusp width 1 at infinity that contains $\Gamma_0(36)$. This has the effect of having integer powers of q in the power series expansion.

Since $6^4\cdot\left(16\left(E_4(3\tau) - 18E_4(6\tau) + 32E_4(12\tau)\right) - 4\left(3^8 + 32\cdot 12^8\right)\right)$ is also a weight 6 form on $\Gamma_0(36)$ which has index 72 in $SL_2(\mathbb{Z})$ we just have to compare the first $\frac{72\cdot 6}{12} = 36$ coefficients of both modular forms to be sure that they are equal.

This is in fact true and the weight 6 form on $3A_2^0$ is then

$$2^4\cdot\left(4\left(E_4(\tau) - 18E_4(2\tau) + 32E_4(4\tau)\right) - \left(1^8 + 32\cdot 4^8\right)\right)$$

- $3E_2^0$

This group is $\langle M_1, M_2, M_3\rangle$ where $M_1 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$, $M_2 = \begin{bmatrix} 17 & -3 \\ 6 & -1 \end{bmatrix}$ and $M_3 = \begin{bmatrix} 4 & -5 \\ 2 & -2 \end{bmatrix}$.

Now, the weight 4 form for this group is $\eta(z)^8 + 4\cdot\eta(2z)^8$. This follows from the action of the matrices above on $\eta(z)^8$ and $\eta(2z)^8$:

$\eta(z)^8\ _{|4}\,[M_1] = \eta(z)^8,\ \ \eta(2z)^8\ _{|4}\,[M_1] = \eta(2z)^8$

$\eta(z)^8\ _{|4}\,[M_2] = \eta(z)^8,\ \ \eta(2z)^8\ _{|4}\,[M_2] = \eta(2z)^8$

$\eta(z)^8 \,_{|4}\, [M_3] = 4 \cdot \eta(2z)^8, \;\; \eta(2z)^8 \,_{|4}\, [M_3] = \frac{1}{4} \cdot \eta(z)^8$

- $4B_2^0$

This group is $\langle 4F_2^1, A, B\rangle$ where $4F_2^1 = \Gamma_0(8) \cap \Gamma^0(4)$, $A = \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ 2 & -2 \end{bmatrix}$.

We claim that $\eta(z)^2\eta(2z)^2 \cdot \big(E_4(\frac{z}{2}) - 7 \cdot E_4(z) - 28 \cdot E_4(2z) + 64 \cdot E_4(4z)\big)$ is the weight 6 form for $4B_2^0$.

It is easy to see that it is invariant under the action of $4F_2^1$ and the matrix A.

We now prove it is invariant under the action B.

Since B negates $\eta(z)^2\eta(2z)^2$ we have to prove that it also negates $E_4(\frac{z}{2}) - 7 \cdot E_4(z) - 28 \cdot E_4(2z) + 64 \cdot E_4(4z)$.

But this is equivalent to $E_4(\frac{z}{2}) + E_4(\frac{z+1}{2}) - 14 \cdot E_4(z) - 56 \cdot E_4(2z) + 4 \cdot E_4(z + \frac{1}{2}) + 64 \cdot E_4(4z) = 0$.

We know that E_4 is an eigenform for the Hecke Operators. In particular, for $\mathbf{T}_2$ we have $\mathbf{T}_2(E_4) = 9 \cdot E_4$ which is equivalent to $E_4(\frac{z}{2}) + E_4(\frac{z+1}{2}) + 16 \cdot E_4(2z) = 18 \cdot E_4(z)$.

With this, we can prove that B negates $E_4(\frac{z}{2}) - 7 \cdot E_4(z) - 28 \cdot E_4(2z) + 64 \cdot E_4(4z)$:

$E_4(\frac{z}{2}) + E_4(\frac{z+1}{2}) - 14 \cdot E_4(z) - 56 \cdot E_4(2z) + 4 \cdot E_4(z + \frac{1}{2}) + 64 \cdot E_4(4z) =$

$18 \cdot E_4(z) - 16 \cdot E_4(2z) - 14 \cdot E_4(z) - 56 \cdot E_4(2z) + 4 \cdot E_4(z + \frac{1}{2}) + 64 \cdot E_4(4z) =$

$4 \cdot E_4(z) + 4 \cdot E_4(z + \frac{1}{2}) - 72 \cdot E_4(2z) + 64 \cdot E_4(4z)$

$72 \cdot E_4(2z) - 64 \cdot E_4(4z) - 72 \cdot E_4(4z) + 64 \cdot E_4(4z) = 0$

- $4C_2^0$

This group is $\langle M_1, M_2, M_3, M_4\rangle$, where $M_1 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$, $M_2 = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, $M_3 = \begin{bmatrix} 2 & -5 \\ 2 & -4 \end{bmatrix}$ and $M_4 = \begin{bmatrix} 6 & -25 \\ 2 & -8 \end{bmatrix}$.

Now, the weight 4 form for this group is $\eta(z)^{10}\eta(2z)^{-2} - 8i \cdot \eta(z)^{-2}\eta(2z)^{10}$. This follows from the action of the matrices above on $\eta(z)^{10}\eta(2z)^{-2}$ and $\eta(z)^{-1}\eta(2z)^{10}$:

$$\eta(z)^{10}\eta(2z)^{-2}\,_{|4}\,[M_1] = \eta(z)^{10}\eta(2z)^{-2},\ \ \eta(z)^{-2}\eta(2z)^{10}\,_{|4}\,[M_1] = \eta(z)^{-2}\eta(2z)^{10}$$

$$\eta(z)^{10}\eta(2z)^{-2}\,_{|4}\,[M_2] = \eta(z)^{10}\eta(2z)^{-2},\ \ \eta(z)^{-2}\eta(2z)^{10}\,_{|4}\,[M_2] = \eta(z)^{-2}\eta(2z)^{10}$$

$$\eta(z)^{10}\eta(2z)^{-2}\,_{|4}\,[M_3] = -8i \cdot \eta(z)^{-2}\eta(2z)^{10},\ \ \eta(z)^{-2}\eta(2z)^{10}\,_{|4}\,[M_3] = \frac{i}{8} \cdot \eta(z)^{10}\eta(2z)^{-2}$$

$$\eta(z)^{10}\eta(2z)^{-2}\,_{|4}\,[M_4] = -8i \cdot \eta(z)^{-2}\eta(2z)^{10},\ \ \eta(z)^{-2}\eta(2z)^{10}\,_{|4}\,[M_4] = \frac{i}{8} \cdot \eta(z)^{10}\eta(2z)^{-2}$$

- $3B_3^0$

This group is $\left\langle \Gamma_0(9) \cap \Gamma^0(3), \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix}, \begin{bmatrix} 9 & -7 \\ 12 & -9 \end{bmatrix} \right\rangle$ and it contains the subgroup $\Gamma_0(9) \cap \Gamma^0(3) = 3B_3^1$ normally.

We can see that the following eta-products

$$\eta(\frac{z}{3})^3\eta(z)^4\eta(3z), \eta(z)\eta(3z)^4\eta(9z)^3, \eta(\frac{z}{3})^3\eta(z)^1\eta(3z)^1\eta(9z)^3, \eta(z)^8, \eta(3z)^8, \eta(z)^4\eta(3z)^4$$

are forms on $3B_3^1$. Symmetrizing under $\begin{bmatrix} 9 & -7 \\ 12 & -9 \end{bmatrix}$ gives that $\eta(\frac{z}{3})^3\eta(z)^4\eta(3z) + 27 \cdot \eta(z)\eta(3z)^4\eta(9z)^3$, $\eta(z)^8 + 9 \cdot \eta(3z)^8$, $\eta(\frac{z}{3})^3\eta(z)^1\eta(3z)^1\eta(9z)^3$ and $\eta(z)^4\eta(3z)^4$ are forms on $3E_3^0 = \left\langle 3B_3^1, \begin{bmatrix} 9 & -7 \\ 12 & -9 \end{bmatrix} \right\rangle$.

Since these forms are linearly independent and the space of weight 4 cusp forms for $3E_3^0$ has dimension 4 they form a basis.

Now the action of $\begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix}$ on this basis is given by the matrix

$$\begin{bmatrix} 1 & 0 & \frac{3-\sqrt{3}i}{18} & 0 \\ 0 & e^{\frac{2\pi i}{3}} & 0 & 0 \\ 0 & 0 & e^{-\frac{2\pi i}{3}} & 0 \\ 9-\sqrt{3}i & 0 & -1 & e^{\frac{2\pi i}{3}} \end{bmatrix}$$

The eigenvector associated to the eigenvalue 1 is

$$\eta\left(\frac{z}{3}\right)^3 \eta(z)^4\eta(3z) + 6\cdot\eta(z)^4\eta(3z)^4 + 27\cdot\eta(z)\eta(3z)^4\eta(9z)^3$$

This is the weight 4 form for $3B_3^0$.

- $12B_3^0$

For the weight 4 form for this group we proceed as for $4A_1^0$ and $3A_2^0$. In this case we have $t_{12B_3^0} = t_{2B_3^0}^2 + 6\sqrt{3}i$ and $t_{1A_3^0} = t_{2B_3^0}^2 + 54$ and an expression for $t_{1A_3^0}$ can be found in [12]. The expression for the weight 6 form is $\frac{\left(\frac{dt_{12B_3^0}}{d\tau}\right)^2}{t_{12B_3^0}\left(t_{12B_3^0}^2+12\sqrt{3}i\right)}$

If we conjugate $12B_3^0$ by $\begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix}$ we get a group with cusp width 1 at infinity that contains $\Gamma_0(432)\bigcap\Gamma_1(12)$. This has the effect of having integer powers of q in the power series expansion.

Since the form given in the table is also a weight 4 form on $\Gamma_0(432)\bigcap\Gamma_1(12)$ (because of theorem 1.3.2 it is actually a form on $\Gamma_0(48)$ with a character mod 12), which has index 3456 in $SL_2(\mathbb{Z})$, we just have to compare the first $\frac{3456\cdot 4}{12} = 1152$ coefficients of both modular forms to be sure that they are equal.

This is in fact true and the weight 4 form on $12B_3^0$ is thus the one given in the table.

- $5A_5^0$ and $5B_5^0$

Both groups contain $5C_5^0 = \Gamma_1(5)$ with index two. $5A_5^0$ and $5B_5^0$ are generated over $5C_5^0$

by $M_1 = \begin{bmatrix} 5 & -3 \\ 10 & -5 \end{bmatrix}$ and $M_2 = \begin{bmatrix} 0 & -1 \\ 5 & 0 \end{bmatrix}$, respectively.

The weight 4 form for these groups must be the same for $5C_5^0$ which is $\eta(z)^4\eta(5z)^4$.

For the weight 6 form we note that $\eta(z)^5\eta(5z)^{-1}$ and $\eta(z)^{-1}\eta(5z)^5$ are forms of weight 2 on $5C_5^0$ and that

$\eta(z)^5\eta(5z)^{-1}\ {}_{|2}\,[M_1] = 5\sqrt{5}\cdot\eta(z)^{-1}\eta(5z)^5$

$\eta(z)^5\eta(5z)^{-1}\ {}_{|2}\,[M_2] = -5\sqrt{5}\cdot\eta(z)^{-1}\eta(5z)^5$

Then, the weight 6 forms for $5A_5^0$ and $5B_5^0$ are, respectively, $\eta(z)^9\eta(5z)^3 + 5\sqrt{5}\cdot\eta(z)^3\eta(5z)^9$ and $\eta(z)^9\eta(5z)^3 - 5\sqrt{5}\cdot\eta(z)^3\eta(5z)^9$.

- $10A_5^0$

This group is $\left\langle \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 10 & -1 \\ 5 & 0 \end{bmatrix}, \begin{bmatrix} 15 & -8 \\ 10 & -5 \end{bmatrix}, \begin{bmatrix} 5 & -6 \\ 5 & -5 \end{bmatrix} \right\rangle$ and we can write

$$\begin{bmatrix} 10 & -1 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -10 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 15 & -8 \\ 10 & -5 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -15 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -6 \\ 5 & -5 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -5 & 6 \end{bmatrix}$$

Now, $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ fixes both $1^2 5^2$ and $E_4^{1_1,\chi}(\tau) - 5\sqrt{5}E_4^{\chi,1_1}(\tau)$ and using (1.3.1) and (1.4.17) we see that both these forms are negated by the Fricke involution, proving that the product is fixed.

The action of the matrices $\begin{bmatrix} 1 & 0 \\ -10 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & -1 \\ -15 & 8 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 \\ -5 & 6 \end{bmatrix}$ can be seen, using (1.3.1) again and (1.4.9), to be multiplication by 1, -1 and 1, respectively, guaranteeing that they fix the product again.

This proves that $1^2 5^2 \cdot \left(E_4^{1_1,\chi}(\tau) - 5\sqrt{5}E_4^{\chi,1_1}(\tau)\right)$ is the weight 6 form for this group.

- $10B_5^0$

This group is $\left\langle \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 5 & -1 \\ 5 & 0 \end{bmatrix}, \begin{bmatrix} 5 & -3 \\ 10 & -5 \end{bmatrix}, \begin{bmatrix} 10 & -11 \\ 5 & -5 \end{bmatrix}, \begin{bmatrix} 15 & -23 \\ 10 & -15 \end{bmatrix} \right\rangle$

and a proof can be carried out in an exactly similar manner as for $10A_5^0$

Chapter 2

2A-replication and the Baby-Monster

2.1 Introduction

In [12], a great amount of data relating $\mathbb{M}$, the Monster group, to modular objects is given. Namely, we can attach a Hauptmodul for some genus zero congruence subgroup of $PSL_2(\mathbb{R})$ to every conjugacy class in the $\mathbb{M}$. This correspondence the has following property. If we denote by $T_g(q) = \frac{1}{q} + \sum_{k\geq 1} a_k(g)q^k$ the Hauptmodul attached to the element $g \in \mathbb{M}$ then, for every $n \in \mathbb{N}$, the map $g \to a_n(g)$ is the character of a representation of $\mathbb{M}$. Also, the notion of replicability was defined in the same paper.

A function $f(q) = \frac{1}{q} + \sum_{k\geq 1} a_k q^k$ is said to be replicable if, for each $n \in \mathbb{N}$, there exists $f^{(n)}(q) = \frac{1}{q} + \sum_{k\geq 1} a_k^{(n)} q^k$ such that

$$\sum_{\substack{ad=n \\ 0\leq b<d}} f^{(a)}\left(\frac{a\tau + b}{d}\right) = P_{n,f}(f(q)),$$

where $P_{n,f}(X)$ is the n-th Faber polynomial of f. This is the unique polynomial such that $P_{n,f}(f(q)) - \frac{1}{q^n}$ has only positive powers of q.

It was proved by Borcherds ([7]) that the McKay-Thompson series associated to conjugacy classes of elements in $\mathbb{M}$ are replicable and their replicates are given by the power map structure in the group, i.e., $T_g^{(n)} = T_{g^n}$. His proof uses the theory of generalized Kac-Moody algebras and vertex operator algebras, namely $V^\natural$ which has the Monster group as group of automorphisms. The vertez operator algebra $V^\natural$ was first constructed by Frenkel, Lepowsky and Meurman ([28]) and shown to have graded dimension equal to the J-function, the Hauptmodul (for the modular group) associated to the identity element in $\mathbb{M}$.

Norton ([48]) generalized the conjectures from [12] a bit further stating that, for every pair (g, h) of commuting elements in the Monster, there is a function $F_{g,h}(\tau)$ such that:

- $F_{g,h}(\tau)$ is invariant under simultaneous conjugation of g and h.
- For any $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$ there exists a root of unity γ such that

$$F_{g^a h^c, g^b h^d}(\tau) = \gamma F_{g,h}(\frac{a\tau + b}{c\tau + d})$$

- The coefficients of the q-expansion of $F_{g,h}(\tau)$ for a fixed g form characters of a central extension of $C_{\mathbb{M}}(g)$, the centralizer of g in $\mathbb{M}$.
- Unless $F_{g,h}(\tau)$ is constant, its invariance group will be genus-zero and commensurable with $SL_2(\mathbb{Z})$.
- $F_{g,h}(\tau) = j(q) - 744$ if and only if $g = h = 1$.

In [34], Höhn takes g equal to an element in class $2A$ in the Monster. Similarly to what Borcherds did for the Monster, Höhn constructs a vertex operator algebra which has $2 \cdot \mathbb{B}$ symmetry. From that vertex operator algebra he obtains a generalized Kac-Moody algebra, the Baby-Monster Lie algebra, whose denominator identity gives enough information to conclude that the McKay-Thompson series of the vertex operator algebra are replicable functions. Knowing that replicable functions are determined by the first few coefficients, Höhn just computes enough of them and compares the results to the coefficients of known Hauptmoduls for some genus zero groups, which are also known to be replicable. This comparison is enough to identify every McKay-Thompson series as a Hauptmodul for some genus zero group.

In this chapter we find a different form of replicability and prove that it reflects the power map structure in $2 \cdot \mathbb{B}$. We call it $2A$-replicability as it is motivated by the Hecke Operators of $\Gamma_0(2)+$, the group associated to an element of class $2A$ in $\mathbb{M}$.

The identities coming from $2A$-replication identities are implicit in the work of Carnahan ([9], [10]) and Borcherds ([7], section 10). However, we have shown that these identities make more sense if we look at them as $2A$-replication identities. This is what will allow us to mimic Borcherds' proof of the Moonshine Conjectures in order to prove that the McKay-Thompson series for the vertex operator algebra with $2 \cdot \mathbb{B}$ symmetry are the ones stated by Höhn in [34].

In section 2.2 we make an introduction to Hecke Operators, find the Hecke Operators for $\Gamma_0(2)+$ and from that define $2A$-replicability. In section 2.3 we prove that a completely $2A$-replicable function is completely determined by the first 5 coeffcients of the function and their replicates. Section 2.4 is the final and most important section of this chapter as we prove that the McKay-Thompson series are completely $2A$-replicable with $2A$-replicability respecting the power map structure in $2 \cdot \mathbb{B}$.

2.2 Hecke operators and replication

Some references for the background material needed for this section are: Chapter 5 in [22] for Hecke Operators and [12] for the basics of Moonshine and the concept of replicability.

For G a subgroup of $PSL_2(\mathbb{R})$ we consider $\mathcal{F}(G)$ the set of functions defined on the upper half-plane that are invariant under the action G. Given two subgroups G_1, G_2 of $PSL_2(\mathbb{R})$ and an element $\alpha \in PSL_2(\mathbb{R})$ we define the Hecke operator

$$[G_1 \alpha G_2] \; : \; \mathcal{F}(G_1) \longrightarrow \mathcal{F}(G_2)$$

in the following way. Consider a decomposition $G_1\alpha G_2 = \bigcup_{j=1}^{n} G_1\gamma_j$ as a disjoint union and define

$$f\left[G_1\alpha G_2\right](z) = \sum_{j=1}^{n} f(\gamma_j z)$$

If f is invariant under G_1 then $f\left[G_1\alpha G_2\right]$ is invariant under G_2.

For example, if $G = PSL_2(\mathbb{R})$ the Hecke operators $\mathbf{T}_n$ are given by

$$\mathbf{T}_n(f)(z) = \sum_{\substack{ad=n \\ 0\leq b<d}} f\left(\frac{az+b}{d}\right) \tag{2.2.1}$$

where $\mathbf{T}_p$, for p prime, is the Hecke operator associated with the matrix $\alpha = \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}$ and the $\mathbf{T}_n$, for composite n, are defined in the following way. $\mathbf{T}_{p^{n+1}} = \mathbf{T}_{p^n}\mathbf{T}_p - p\mathbf{T}_{p^{n-1}}$, for p prime and $n \in \mathbb{N}$ and $\mathbf{T}_m\mathbf{T}_n = \mathbf{T}_{mn}$ when m and n are natural numbers with $(m,n) = 1$.

We compare 2.2.1 with the definition of replicability. A function $f(z)$ is replicable if for every $n \in \mathbb{N}$ there exists $f^{(n)}(z)$ such that

$$P_{n,f}(f)(z) = \sum_{\substack{ad=n \\ 0\leq b<d}} f^{(a)}\left(\frac{az+b}{d}\right) \tag{2.2.2}$$

where $P_{n,f}$ is the n-th Faber polynomial of f.

The Hauptmoduls for genus zero groups with rational integer coefficients are known to be replicable functions ([19]). The genus zero congruence subgroups of $PSL_2(\mathbb{R})$ have been classified in [14] and it is conjectured that their Hauptmoduls are all replicable functions with rational integer coefficients. The set of completely replicable functions is a distinguished subset of the set of replicable functions. These are the functions that besides satisfying 2.2.2 also satisfy $\left(f^{(m)}\right)^{(n)} = f^{(mn)}$ for every positive integers m, n. They were classified in [1] and include the set of Monstrous functions because, for $g \in \mathbb{M}$, $T_g^{(n)} = T_{g^n}$. The purpose of the next section is to define a sort of replicability, that we call $2A$-replicability, based on the

Hecke Operators for $\Gamma_0(2)+$. Also in this case we can define complete $2A$-replicability and we will see that the set of completely $2A$-replicable functions also include the McKay-Thompson series for the Baby Monster group.

2.2.1 Hecke Operators for $\Gamma_0(2)+$

We start by seeing what the Hecke operators are for $\Gamma_0(2)+$. We want to find for every $m \in \mathbb{N}$ a disjoint union decomposition

$$(\Gamma_0(2)+) \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix} (\Gamma_0(2)+) = \bigcup_{j=1}^{4} (\Gamma_0(2)+)\, \gamma_j$$

Since $\Gamma_0(2)+$ has only one cusp we can choose the representatives γ_j to have lower left entry equal to zero. We start with an element $\begin{bmatrix} a & b \\ 2c & d \end{bmatrix} \in \Gamma_0(2)$ and see what the representative of this type is for

$$(\Gamma_0(2)+) \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} a & b \\ 2c & d \end{bmatrix} = (\Gamma_0(2)+) \begin{bmatrix} a & b \\ 2mc & md \end{bmatrix}$$

Since this matrix sends ∞ to $\frac{a}{2mc} = \frac{\frac{a}{(a,m)}}{2\frac{m}{(a,m)}c}$ we can take some matrix of the form

$$\begin{bmatrix} x & y \\ -2\frac{m}{(a,m)}c & \frac{a}{(a,m)} \end{bmatrix} \in \Gamma_0(2)$$

and now

$$\begin{bmatrix} x & y \\ -2\frac{m}{(a,m)}c & \frac{a}{(a,m)} \end{bmatrix} \begin{bmatrix} a & b \\ 2mc & md \end{bmatrix} =$$

$$= \begin{bmatrix} ax + 2mcy & bx + ymd \\ 0 & \frac{m}{(a,m)}(ad - 2bc) \end{bmatrix} = \begin{bmatrix} (a,m) & b' \\ 0 & \frac{m}{(a,m)} \end{bmatrix}$$

is a representative for the same right coset. If need be we can multiply this identity on the left by a power of $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, which is obviously in $\Gamma_0(2)+$, to have $0 \leq b' < \frac{m}{(a,m)}$. We have proved that every right coset

$$(\Gamma_0(2)+) \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} a & b \\ 2c & d \end{bmatrix}$$

has a representative of the form $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$, with $xz = m$, x odd and $0 \leq y < z$, i.e., we proved one inclusion of the following equality

$$(\Gamma_0(2)+) \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix} \Gamma_0(2) = \bigcup_{\substack{xz=m \\ x \text{ odd}}} \bigcup_{0 \leq y < z} (\Gamma_0(2)+) \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$$

the other inclusion being obvious.

We consider now an element $\begin{bmatrix} 2a & b \\ 2c & 2d \end{bmatrix} \in \Gamma_0(2)+$ of determinant 2 and see what the representative looks like in this case. Since

$$\begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} 2a & b \\ 2c & 2d \end{bmatrix} = \begin{bmatrix} 2a & b \\ 2mc & 2md \end{bmatrix}$$

sends ∞ to $\frac{a}{mc} = \frac{\frac{a}{(a,m)}}{\frac{m}{(a,m)}c}$ we have to consider two cases, depending on whether $\frac{m}{(a,m)}$ is even or odd. If $\frac{m}{(a,m)}$ is even we take a matrix of the form

$$\begin{bmatrix} x & y \\ -\frac{m}{(a,m)}c & \frac{a}{(a,m)} \end{bmatrix} \in \Gamma_0(2)$$

and

$$\begin{bmatrix} x & y \\ -\frac{m}{(a,m)}c & \frac{a}{(a,m)} \end{bmatrix} \begin{bmatrix} 2a & b \\ 2mc & 2md \end{bmatrix} =$$

$$= \begin{bmatrix} 2(ax+ymc) & bx+2ymd \\ 0 & \frac{m}{(a,m)}(2ad-bc) \end{bmatrix} = \begin{bmatrix} 2(a,m) & b' \\ 0 & \frac{m}{(a,m)} \end{bmatrix}.$$

Again, we can multiply on the left by a power of $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ to consider only $0 \leq b' < \frac{m}{(a,m)}$. We also note that b' also is an odd number. This shows that the right coset

$$(\Gamma_0(2)+) \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} 2a & b \\ 2c & 2d \end{bmatrix} = (\Gamma_0(2)+) \begin{bmatrix} 2a & b \\ 2mc & 2md \end{bmatrix}$$

has a representative of the form $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$, where $xz = 2m$, x and z are even and b is odd with $0 \leq y < z$. Conversely, it is easy to see that every such matrix is a representative of some coset $(\Gamma_0(2)+) \begin{bmatrix} 2a & b \\ 2mc & 2md \end{bmatrix}$ with $\frac{m}{(a,m)}$ even.

If $\frac{m}{(a,m)}$ is odd we now take a matrix of the form

$$\begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} -\frac{m}{(a,m)}c & \frac{a}{(a,m)} \\ 2x & y \end{bmatrix} \in \Gamma_0(2)+$$

and

$$\begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} -\frac{m}{(a,m)}c & \frac{a}{(a,m)} \\ 2x & y \end{bmatrix} \begin{bmatrix} 2a & b \\ 2mc & 2md \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{m}{(a,m)}(2ad-bc) \\ 2(2ax+ymc) & 2(bx+ymd) \end{bmatrix} = \begin{bmatrix} 2(a,m) & 2b' \\ 0 & 2\frac{m}{(a,m)} \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & \frac{m}{(a,m)}(2ad-bc) \\ 2(2ax+ymc) & 2(bx+ymd) \end{bmatrix} = \begin{bmatrix} (a,m) & b' \\ 0 & \frac{m}{(a,m)} \end{bmatrix}$$

Once again, we can multiply on the left by a power of $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ to consider only $0 \leq b' < \frac{m}{(a,m)}$. This shows that the right coset

$$(\Gamma_0(2)+) \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} 2a & b \\ 2c & 2d \end{bmatrix} = (\Gamma_0(2)+) \begin{bmatrix} 2a & b \\ 2mc & 2md \end{bmatrix}$$

has a representative of the form $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$, where $xz = m$, z is odd and $0 \leq y < z$. Conversely, it is easy to see that every such matrix is a representative of some coset $(\Gamma_0(2)+) \begin{bmatrix} 2a & b \\ 2mc & 2md \end{bmatrix}$ with $\frac{m}{(a,m)}$ odd.

We define the following sets

$$M_m^1 = \left\{ \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \mid xz = m,\ 0 \leq y < z,\ x \text{ odd} \right\}$$

$$M_m^2 = \left\{ \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \mid xz = 2m,\ 0 \leq y < z,\ x, z \text{ even},\ y \text{ odd} \right\}$$

$$M_m^3 = \left\{ \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \mid xz = m,\ 0 \le y < z,\ z \text{ odd} \right\}$$

We have proved that

$$(\Gamma_0(2)+) \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix} (\Gamma_0(2)+) = \bigcup_{i=1}^{3} \bigcup_{\gamma \in M_m^i} (\Gamma_0(2)+)\,\gamma$$

However this is not a disjoint union if m is odd. In this case M_m^2 is empty and $M_m^1 = M_m^3$. If m is even it is not hard to see it is a disjoint union.

We have then

$$(\Gamma_0(2)+) \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix} (\Gamma_0(2)+) = \begin{cases} \displaystyle\bigcup_{i=1}^{3} \bigcup_{\gamma \in M_m^i} (\Gamma_0(2)+)\,\gamma & \text{, if } m \text{ is even} \\ \displaystyle\bigcup_{\gamma \in M_m^1} (\Gamma_0(2)+)\,\gamma & \text{, if } m \text{ is odd} \end{cases}$$

as a disjoint union in both cases.

We define

$$D_m = \begin{cases} M_m^1 \cup M_m^2 \cup M_m^3 & \text{, if } m \text{ is even} \\ M_m^1 & \text{, if } m \text{ is odd} \end{cases}$$

and

$$\tilde{\mathbf{T}}_m(f)(z) = \sum_{\gamma \in D_m} f(\gamma z)$$

for any function f.

Remark 2.2.1. *If m is odd, $\tilde{\mathbf{T}}_m$ is a Hecke operator for the modular group.*

Remark 2.2.2. *If $m = 2$, $D_m = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ and $\tilde{\mathbf{T}}_2$ has one extra term when compared to the corresponding Hecke operator for the modular group.*

We now define the operators $\mathbf{T}_m$, for $m \in \mathbb{N}$, in the following way. If $m = 2^k$ then $\mathbf{T}_m(f) = \sum_{i=0}^{k} \tilde{\mathbf{T}}_{2^i}(f)$, if m is odd $\mathbf{T}_m(f) = \tilde{\mathbf{T}}_m(f)$ and finally if $m = 2^k n$ with $(2, n) = 1$ then

$\mathbf{T}_m(f) = \mathbf{T}_{2^k}\mathbf{T}_n(f)$. As we will see below, the operators $\mathbf{T}_m$ defined this way give the Faber polynomial of T_{2A} when apllied to T_{2A} (see Proposition 2.2.1 and 2.2.3 below).

Proposition 2.2.1. $\mathbf{T}_m(f)(\tau) = \sum_{\substack{ad=m \\ 0\le b<d}} f\left(\frac{a\tau+b}{d}\right) + \sum_{\substack{ad=m \\ d \text{ even} \\ 0\le b<d}} f\left(\frac{2a\tau+b}{d}\right)$

Proof. The result is trivial for m odd. We prove the result for powers of 2 by induction. Assume that

$$\mathbf{T}_{2^n}(f)(\tau) = \sum_{\substack{ad=2^n \\ 0\le b<d}} f\left(\frac{a\tau+b}{d}\right) + \sum_{\substack{ad=2^n \\ d \text{ even} \\ 0\le b<d}} f\left(\frac{2a\tau+b}{d}\right)$$

then

$$\mathbf{T}_{2^{n+1}}(f)(\tau) = \tilde{\mathbf{T}}_{2^{n+1}}(f)(\tau) + \mathbf{T}_{2^n}(f)(\tau) =$$

$$\sum_{\substack{ad=2^{n+1} \\ a \text{ odd} \\ 0\le b<d}} f\left(\frac{a\tau+b}{d}\right) + \sum_{\substack{ad=2^{n+2} \\ a,d \text{ even} \\ 0\le b<d \\ b \text{ odd}}} f\left(\frac{a\tau+b}{d}\right) + \sum_{\substack{ad=2^{n+1} \\ d \text{ odd} \\ 0\le b<d}} f\left(\frac{a\tau+b}{d}\right) + \mathbf{T}_{2^n}(f)(\tau) =$$

$$\sum_{0\le b<2^{n+1}} f\left(\frac{\tau+b}{2^{n+1}}\right) + \sum_{\substack{ad=2^{n+1} \\ d \text{ even} \\ 0\le b<d \\ b \text{ odd}}} f\left(\frac{2a\tau+b}{d}\right) + f\left(2^{n+1}\tau\right) +$$

$$+ \sum_{\substack{ad=2^{n+1} \\ 0\le b<d \\ b,d \text{ even}}} f\left(\frac{2a\tau+b}{d}\right) + \sum_{\substack{ad=2^{n+1} \\ a,d \text{ even} \\ 0\le b<d}} f\left(\frac{a\tau+b}{d}\right)$$

But now

$$\sum_{0\le b<2^{n+1}} f\left(\frac{\tau+b}{2^{n+1}}\right) + f\left(2^{n+1}\tau\right) + \sum_{\substack{ad=2^{n+1} \\ a,d \text{ even} \\ 0\le b<d}} f\left(\frac{a\tau+b}{d}\right) = \sum_{\substack{ad=2^{n+1} \\ 0\le b<d}} f\left(\frac{a\tau+b}{d}\right)$$

and

$$\sum_{\substack{ad=2^{n+1}\\ d\text{ even}\\ 0\le b<d\\ b\text{ odd}}} f\left(\frac{2a\tau+b}{d}\right)+\sum_{\substack{ad=2^{n+1}\\ 0\le b<d\\ b,d\text{ even}}} f\left(\frac{2a\tau+b}{d}\right)=\sum_{\substack{ad=2^{n+1}\\ d\text{ even}\\ 0\le b<d}} f\left(\frac{2a\tau+b}{d}\right)$$

which proves the result for the case when m is a power of 2.

The general result comes as a consequence of the two cases proved above and the fact that

$$\left\{\begin{bmatrix} a & b\\ 0 & d\end{bmatrix}\begin{bmatrix} a' & b'\\ 0 & d'\end{bmatrix}|ad=2^k, a'd'=n, 0\le b<d, 0\le b'<d', (d\text{ even})\right\}=$$

$$=\left\{\begin{bmatrix} a'' & b''\\ 0 & d''\end{bmatrix}|a''d''=2^k n, 0\le b''<d'', (d''\text{ even})\right\}$$

2.2.2 2A-replicability

The operators $\mathbf{T}_m$ and $\tilde{\mathbf{T}}_m$ just defined map the field of modular functions for $\Gamma_0(2)+$ to itself. This means that $\mathbf{T}_m(T_{2A})(z)$ is a rational function of $T_{2A}(z)$ and since $\mathbf{T}_m(T_{2A})(z)$ has no poles in the upper half-plane this rational function is actually a polynomial. From the power series expansion we can see that it has to be the m-th Faber polynomial of T_{2A}. We have just said that

$$P_m(T_{2A}(\tau))=\sum_{\substack{ad=m\\ 0\le b<d}} T_{2A}\left(\frac{a\tau+b}{d}\right)+\sum_{\substack{ad=m\\ d\text{ even}\\ 0\le b<d}} T_{2A}\left(\frac{2a\tau+b}{d}\right) \tag{2.2.3}$$

Definition 2.2.1. *A function f is 2A-replicable if there are $f^{[n]}$ and $f^{[n\sqrt{2}]}$, for $n\in\mathbb{N}$, such that*

$$P_{n,f}(f)=\sum_{\substack{ad=n\\ 0\le b<d}} f^{[a]}\left(\frac{a\tau+b}{d}\right)+\sum_{\substack{ad=n\\ d\text{ even}\\ 0\le b<d}} f^{[a\sqrt{2}]}\left(\frac{2a\tau+b}{d}\right) \tag{2.2.4}$$

Equation 2.2.3 is the $2A$-self-replication property of T_{2A}.

We make a few remarks on this definition.

Remark 2.2.3. *Given a 2A-replicable function f, its 2A-replicates are not uniquely determined. For example, for $m = 2$, equation 2.2.4 becomes*

$$f^{[2]}(2\tau) + f^{[\sqrt{2}]}(\tau) + f^{[\sqrt{2}]}\left(\tau + \frac{1}{2}\right) + f\left(\frac{\tau}{2}\right) + f\left(\frac{\tau+1}{2}\right) = P_{2,f}(f)$$

and we can see that $f^{[2]}$ is known when f and $f^{[\sqrt{2}]}$ are known. Also, the odd-power coefficients of the replicates $f^{[\sqrt{2}n]}$ can be changed freely and identity 2.2.4 is still true. We will see instances of 2A-replicable functions that are 2A-replicables in different ways, i.e. have different 2A-replicates.

Remark 2.2.4. *If a function is 2A-replicable then it is replicable with*

$$f^{(n)}(z) = \begin{cases} f^{[n]}(z), & n \text{ odd} \\ f^{[n]}(z) + f^{[\frac{n}{\sqrt{2}}]}(\frac{z}{2}) + f^{[\frac{n}{\sqrt{2}}]}(\frac{z+1}{2}), & n \text{ even} \end{cases} \tag{2.2.5}$$

Also, if f is replicable then f is 2A-replicable by taking, for example, $f^{\left[n\sqrt{2}\right]} = 0$. Conversely, we can also say that if f is replicable and if, for every n even, we can write $f^{(n)}(z) = f^{[n]}(z) + f^{[\frac{n}{\sqrt{2}}]}(\frac{z}{2}) + f^{[\frac{n}{\sqrt{2}}]}(\frac{z+1}{2})$, for some $f^{[n]}, f^{[\frac{n}{\sqrt{2}}]}$, then f is 2A-replicable. For n odd we obvioulsy have $f^{[n]} = f^{(n)}$. This is shown by the following simple manipulation.

$$\sum_{\substack{ad=n \\ 0\le b<d}} f^{(a)}\left(\frac{az+b}{d}\right) = \sum_{\substack{ad=n \\ a \text{ odd} \\ 0\le b<d}} f^{(a)}\left(\frac{az+b}{d}\right) + \sum_{\substack{ad=n \\ a \text{ even} \\ 0\le b<d}} f^{(a)}\left(\frac{az+b}{d}\right) =$$

$$\sum_{\substack{ad=n \\ a \text{ odd} \\ 0\le b<d}} f^{[a]}\left(\frac{az+b}{d}\right) + \sum_{\substack{ad=n \\ a \text{ even} \\ 0\le b<d}} f^{[a]}\left(\frac{az+b}{d}\right) +$$

$$+ \sum_{\substack{ad=n \\ a \text{ even} \\ 0\le b<d}} f^{[\sqrt{2}\frac{a}{2}]}\left(\frac{az+b}{2d}\right) + \sum_{\substack{ad=n \\ a \text{ even} \\ 0\le b<d}} f^{[\sqrt{2}\frac{a}{2}]}\left(\frac{az+b+d}{2d}\right) =$$

$$= \sum_{\substack{ad=n\\0\le b<d}} f^{[a]}\left(\frac{az+b}{d}\right) + \sum_{\substack{ad=n\\a\ even\\0\le b<2d}} f^{[\sqrt{2}\frac{a}{2}]}\left(\frac{2\left(\frac{a}{2}\right)z+b}{2d}\right) =$$

$$= \sum_{\substack{ad=n\\0\le b<d}} f^{[a]}\left(\frac{az+b}{d}\right) + \sum_{\substack{ad=n\\d\ even\\0\le b<d}} f^{[\sqrt{2}a]}\left(\frac{2az+b}{d}\right).$$

Remark 2.2.5. *Because of remark 2.2.4 what we are actually interested in is finding the possible $2A$-replicables for a given $2A$-replicable function. For example, we could ask if a $2A$-replicable function is completely $2A$-replicable in the sense given below. There are examples of Hauptmoduls that are not complete replicable functions but are completely $2A$-replicable. As we will see, some examples of these are $T_{4\sim b}$, $T_{12\sim d}$ among many others (see [50] for the notation)*

Definition 2.2.2. *We say that a function f is completely $2A$-replicable if*

- *it is $2A$-replicable, with replicates $f^{[n]}$, $n \in \mathbb{N} \cup \sqrt{2}\mathbb{N}$, and*
- *for every $n \in \mathbb{N} \cup \sqrt{2}\mathbb{N}$ the function $f^{[n]}$ is $2A$-replicable with $2A$-replicates $\left(f^{[n]}\right)^{[m]} = f^{[mn]}$, for any $m \in \mathbb{N} \cup \sqrt{2}\mathbb{N}$.*

The following result is Theorem 5.15 in Ferenbaugh's Ph.D thesis [24] and will be useful in proving that certain Hauptmoduls that are not completely replicable are completely $2A$-replicable.

Theorem 2.2.1. *Let f be the Hauptmodul for some group $np|h + e_1, e_2 \ldots$ with $p \nmid h$. Then*

- *If p is one of the e_1, $e_2, \ldots$ then $f^{(p)}(z) = f(z) + f\left(\frac{z}{p}\right) + \ldots + f\left(\frac{z+p-1}{p}\right)$.*
- *if p is not one of the e_1, $e_2, \ldots$ but ep is, for some e with $p \nmid e$, then $f^{(p)}(W_e z) = f(W_p z) + f\left(\frac{z}{p}\right) + \ldots + f\left(\frac{z+p-1}{p}\right)$.*
- *if ep is one of the e_1, $e_2, \ldots$ with $p \mid e$ then $f^{(p)}(W_e z) = f\left(\frac{z}{p}\right) + \ldots + f\left(\frac{z+p-1}{p}\right)$.*

Remark 2.2.4 gives motivation to find formulas of type $A(z) = B(z) + C\left(\frac{z}{2}\right) + C\left(\frac{z+1}{2}\right)$. This will help us finding $2A$-replicates for certain Hauptmoduls. Before we proceed, we need some notation. The group $n|h+\{e\}_O+\{2^k e\}_E$, with $2^k \parallel N$, is just a splitting of the Atkin-Lehner involutions in two sets. O for the odd ones and E for the even ones. For example, the group $30+6,10,15 = 30+\{e\}_O+\{2e\}_E$, with $O=\{15\}$ and $E=\{3,5\}$. Also, G^α denotes $\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} G \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}^{-1}$.

Lemma 2.2.1. *We have the following identities:*

$$(1)\quad T_{N+\{e\}_O}(z) = T_{2N+\{e,2e\}_O}(z) + T_{2N+\{e,2e\}_O}\left(\frac{z}{2}\right) + T_{2N+\{e,2e\}_O}\left(\frac{z+1}{2}\right)$$

or, equivalently,

$$T_{2N+\{e,2e\}_O}(z) = T_{N+\{e\}_O}(z) + T_{(2N+\{e,2e\}_O)^{\frac{1}{2}}}\left(\frac{z}{2}\right) + T_{(2N+\{e,2e\}_O)^{\frac{1}{2}}}\left(\frac{z+1}{2}\right)$$

if $(2,N)=1$.

$$(2)\quad T_{N+\{e\}_O}(z) = T_{N+\{e\}_O+\{2^k e\}_E}(z) + \\ + T_{(2N+\{e\}_O+\{2^{k+1}e\}_E)^{\frac{1}{2}}}\left(\frac{z}{2}\right) + T_{(2N+\{e\}_O+\{2^{k+1}e\}_E)^{\frac{1}{2}}}\left(\frac{z+1}{2}\right)$$

if $2^k \parallel N$, $k \geq 1$.

This is true for any E *as long as it is not empty. In particular, if* $2 \parallel N$ *and* $O=E$ *we have*

$$T_{N+\{e\}_O}(z) = T_{N+\{e,2e\}_O}(z) + T_{N+\{e\}_O}\left(\frac{z}{2}\right) + T_{N+\{e\}_O}\left(\frac{z+1}{2}\right)$$

$$(3)\quad T_{N+\{e\}_O+\{2^k e\}_E}(z) = T_{N+\{e\}_O}(z) + \\ + T_{2N+\{e\}_O+\{2^{k+1}e\}_E}\left(\frac{z}{2}\right) + T_{2N+\{e\}_O+\{2^{k+1}e\}_E}\left(\frac{z+1}{2}\right)$$

if $2^k \parallel N$, $k \geq 1$ *and* E *is not empty.*

Proof. The first identity is Theorem 2.2.1 applied to $f = T_{2N+\{e,2e\}_O}$ and $p = 2$. The third identity is also a consequence of Theorem 2.2.1 applied to $f = T_{2N+\{e\}_O+\{2^{k+1}e\}_E}$ and $p = 2$. Since $2^k \parallel N$, for $k \geq 1$, we have

$$T_{2N+\{e\}_O+\{2^{k+1}e\}_E}\left(\frac{z}{2}\right) + T_{2N+\{e\}_O+\{2^{k+1}e\}_E}\left(\frac{z+1}{2}\right) = T_{N+\{e\}_O}\left(W_{e'}z\right)$$

where $e' = 2^k e$ for some $e \in E$. But

$$T_{N+\{e\}_O}(z) + T_{N+\{e\}_O}(W_{e'}z) = T_{N+\{e\}_O+\{2^k e\}_E}(z)$$

and this proves the third identity. The second identity is the third one written in a different way.

Remark 2.2.6. *If the matrix* $\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$ *normalizes the group* $n|h + \{e\}_O + \{e\}_E$ *then* $T_{n|h+\{e\}_O+\{e\}_E}(z + \frac{1}{2}) = -T_{n|h+\{e\}_O+\{e\}_E}(z)$. *This happens exactly when either* h *is even or* $h = 1$, 4 *divides* n *and* E *is empty.*

Also, this matrix conjugates the groups $2N + \{e\}_O$ *and* $4N + \{4e\}_O$ *onto each other, when* $(2, N) = 1$.

Theorem 2.2.2. *Every function of type as given in tables 2.1 and 2.2 is* $2A$*-replicable with* $f^{[2^{\frac{n}{2}}]}$, $n \in \mathbb{N}$, *as given in the tables. The replicates* $f^{[2^{\frac{n}{2}}m]}$, *for* n *a non-negative integer and* m *odd are given by the formula* $f^{[2^{\frac{n}{2}}m]}(z) = \left(f^{[2^{\frac{n}{2}}]}\right)^{(m)}(z)$.

Proof. For the cases where the function is completely replicable we know how replication works. We just use remarks 2.2.4, 2.2.6 and Lemma 2.2.1 to show that the $2A$-replicates are given as stated. This also shows that in tables 2.1 and 2.2 we can take $n \parallel h + \{e\}_O + \{2^k e\}_E$ instead of $n|h + \{e\}_O + \{2^k e\}_E$ and the result is still valid.

If the function is not completely replicable, but its invariance group is conjugate by $\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$, with $\alpha = \frac{1}{2}, \frac{1}{4}$ or $\frac{1}{8}$, to some group whose Hauptmodul is completely replicable, let's say $g = T_G$ and $f = T_{G^\alpha}$ with T_{G^α} completely replicable, then we prove it in the following way. We do the proof for $\alpha = \frac{1}{2}$, the other cases being similar.

We start by arranging the terms in the right hand side of

$$P_{n,f}(f(z)) = \sum_{\substack{ad=n \\ 0\le b<d}} f^{[a]}\left(\frac{az+b}{d}\right) + \sum_{\substack{ad=n \\ d \text{ even} \\ 0\le b<d}} f^{[\sqrt{2}a]}\left(\frac{2az+b}{d}\right)$$

by the highest power of 2 that divides a. This becomes:

$$P_{n,f}(f(z)) = \sum_{i=0}^{\infty}\left(\sum_{\substack{ad=\frac{n}{2^i} \\ a \text{ odd} \\ 0\le b<d}} f^{[2^i a]}\left(\frac{2^i az+b}{d}\right) + \sum_{\substack{2^i ad=n \\ a \text{ odd} \\ d \text{ even} \\ 0\le b<d}} f^{[2^i\sqrt{2}a]}\left(\frac{2^{i+1}az+b}{d}\right)\right) \tag{2.2.6}$$

We now substitute z by $z + \frac{1}{2}$ in this identity and make the following remarks. Firstly, $P_{n,f}(f(z+\frac{1}{2})) = (-1)^n P_{n,f^{\frac{1}{2}}}(f^{\frac{1}{2}}(z))$, where $h^{\frac{1}{2}}(z) = -h(z+\frac{1}{2})$ for any h in general. Secondly, all the sumands in the right-hand side of 2.2.6 remain unchanged except for $\sum_{\substack{ad=n \\ a \text{ odd} \\ 0\le b<d}} f^{[a]}\left(\frac{az+b}{d}\right)$. This will imply that $g^{[2^{\frac{n}{2}}m]} = f^{[2^{\frac{n}{2}}m]}$ for $n \ge 2$.

We consider first $g = T_{(2N+\{e,2e\}_O)^{\frac{1}{2}}}$, (thus $f = T_{2N+\{e,2e\}_O}$), which requires a different proof.

If n is odd, from the remarks above we see that

$$P_{n,g}(g(z)) = -\sum_{\substack{ad=n \\ 0\le b<d}} f^{[a]}\left(\frac{az+b}{d} + \frac{a}{2d}\right) = \sum_{\substack{ad=n \\ 0\le b<d}} (f^{[a]})^{\frac{1}{2}}\left(\frac{az+b+\frac{a-d}{2}}{d}\right) =$$

$$= \sum_{\substack{ad=n \\ 0\le b<d}} (f^{[a]})^{\frac{1}{2}}\left(\frac{az+b}{d}\right)$$

If n is even, then

$$P_{n,g}(g(z)) = \sum_{i=0}^{\infty} \left(\sum_{\substack{ad=\frac{n}{2^i} \\ a \text{ odd} \\ 0\le b<d}} f^{[2^i a]} \left(\frac{2^i az + b}{d} \right) + \sum_{\substack{2^i ad=n \\ a \text{ odd} \\ d \text{ even} \\ 0\le b<d}} f^{[2^i\sqrt{2}a]} \left(\frac{2^{i+1}az + b}{d} \right) \right) \tag{2.2.7}$$

we analyse the part corresponding to $i = 0$, i.e.

$$\sum_{\substack{ad=n \\ a \text{ odd} \\ 0\le b<d}} f^{[a]} \left(\frac{az + b}{d} + \frac{a}{2d} \right) + \sum_{\substack{ad=n \\ a \text{ odd} \\ d \text{ even} \\ 0\le b<d}} f^{[\sqrt{2}a]} \left(\frac{2az + b}{d} \right)$$

in the last equation.

Each $f^{[a]}$ in this sum is of type $T_{2M+\{e,2e\}_O}$, with $(M, 2) = 1$, and we know from 2.2.1 that in this case

$$T_{2M+\{e,2e\}_O} \left(\frac{z}{2} \right) + T_{2M+\{e,2e\}_O} \left(\frac{z+1}{2} \right) + T_{2M+\{e,2e\}_O} (z) = T_{M+\{e\}_O} (z)$$

In particular, substituting z by $\frac{2az+b}{d}$ and summing over b we have

$$\sum_{0\le b<2d} T_{2M+\{e,2e\}_O} \left(\frac{az}{d} + \frac{b}{2d} \right) + \sum_{0\le b<d} T_{2M+\{e,2e\}_O} \left(\frac{2az + b}{d} \right) = \sum_{0\le b<d} T_{M+\{e\}_O} \left(\frac{2az + b}{d} \right)$$

or, equivalently,

$$\begin{aligned} &\sum_{0\le b<d} T_{2M+\{e,2e\}_O} \left(\frac{az + b}{d} + \frac{1}{2d} \right) + \sum_{0\le b<d} T_{2M+\{e,2e\}_O} \left(\frac{2az + b}{d} \right) = \\ &\qquad = -\sum_{0\le b<d} T_{2M+\{e,2e\}_O} \left(\frac{az + b}{d} \right) + \sum_{0\le b<d} T_{M+\{e\}_O} \left(\frac{2az + b}{d} \right) = \\ &\qquad = \sum_{0\le b<d} T_{(2M+\{e,2e\}_O)^{\frac{1}{2}}} \left(\frac{az + b}{d} + \frac{1}{2} \right) + \sum_{0\le b<d} T_{M+\{e\}_O} \left(\frac{2az + b}{d} \right) = \\ &\qquad = \sum_{0\le b<d} T_{(2M+\{e,2e\}_O)^{\frac{1}{2}}} \left(\frac{az + b}{d} \right) + \sum_{0\le b<d} T_{M+\{e\}_O} \left(\frac{2az + b}{d} \right) \end{aligned}$$

We make the observation that we haven't made any choice about the $2A$-replicates of f. We now choose the column in the tables that have $f^{[\sqrt{2}]}$ of the form $T_{2M+\{e,2e\}_O}$ and use the last identity for every $f^{[a]}$ in 2.2.7. This finishes the proof for functions of the form $T_{(2N+\{e,2e\}_O)^{\frac{1}{2}}}$.

We now consider f of the form $T_{(2^kN+\{e\}_O+\{2^ke\}_E)^{\frac{1}{2}}}$, with $k \geq 2$. The case where n is odd is similar to the previous one. If n is even, we analyse again the part corresponding to $i = 0$ in the sum in 2.2.7. In this case, the $f^{[a]}$ are equal to something of the form $T_{2^kM+\{e\}_O+\{2^ke\}_E}$ and applying 2.2.1 to this function we have

$$T_{2^kM+\{e\}_O+\{2^ke\}_E}\left(\frac{z}{2}\right) + T_{2^kM+\{e\}_O+\{2^ke\}_E}\left(\frac{z+1}{2}\right) = T_{2^{k-1}M+\{e\}_O}(W_{2^{k-1}e'}(z))$$

for some $e' \in E$. This gives

$$T_{2^kM+\{e\}_O+\{2^ke\}_E}\left(\frac{z}{2}\right) + T_{2^kM+\{e\}_O+\{2^ke\}_E}\left(\frac{z+1}{2}\right) + T_{2^{k-1}M+\{e\}_O}(z) =$$
$$= T_{2^{k-1}M+\{e\}_O}(W_{2^{k-1}e'}(z)) + T_{2^{k-1}M+\{e\}_O}(z) = T_{2^{k-1}M+\{e\}_O+\{2^{k-1}e\}_E}(z)$$

Substituting z by $\frac{2az+b}{d}$ and summing over b we obtain

$$\sum_{0\leq b<2d} T_{2^kM+\{e\}_O+\{2^ke\}_E}\left(\frac{az}{d}+\frac{b}{2d}\right) + \sum_{0\leq b<d} T_{2^{k-1}M+\{e\}_O}\left(\frac{2az+b}{d}\right) =$$
$$= \sum_{0\leq b<d} T_{2^{k-1}M+\{e\}_O+\{2e\}_E}\left(\frac{2az+b}{d}\right)$$

and, in particular,

$$\sum_{0\leq b<d} T_{2^kM+\{e\}_O+\{2^ke\}_E}\left(\frac{az+b}{d}+\frac{1}{2d}\right) + \sum_{0\leq b<d} T_{2^{k-1}M+\{e\}_O}\left(\frac{2az+b}{d}\right) =$$
$$= -\sum_{0\leq b<d} T_{2^kM+\{e\}_O+\{2^ke\}_E}\left(\frac{az+b}{d}\right) + \sum_{0\leq b<d} T_{2^{k-1}M+\{e\}_O+\{2^{k-1}e\}_E}\left(\frac{2az+b}{d}\right) =$$

$$= \sum_{0\le b<d} T_{(2^kM+\{e\}_O+\{2^ke\}_E)^{\frac{1}{2}}}\left(\frac{az+b}{d}+\frac{1}{2}\right) + \sum_{0\le b<d} T_{2^{k-1}M+\{e\}_O+\{2^{k-1}e\}_E}\left(\frac{2az+b}{d}\right) =$$

$$= \sum_{0\le b<d} T_{(2^kM+\{e\}_O+\{2^ke\}_E)^{\frac{1}{2}}}\left(\frac{az+b}{d}\right) + \sum_{0\le b<d} T_{2^{k-1}M+\{e\}_O+\{2^{k-1}e\}_E}\left(\frac{2az+b}{d}\right)$$

Again, we haven't made any choice about the $2A$-replicates of f and we can choose the column in the tables that have $f^{[\sqrt{2}]}$ of the form $T_{2M+\{2^{k-1}e\}_O}$. We can then use the last identity for every $f^{[a]}$ in 2.2.7 and this finishes the proof for functions of the form $T_{(2N+\{e,2e\}_O)^{\frac{1}{2}}}$.

We use this identity for every $f^{[a]}$ in 2.2.7 and this finishes the proof for functions of the form $T_{(2^kN+\{e\}_O+\{2^ke\}_E)^{\frac{1}{2}}}$.

The remaining cases can be done similarly. For Hauptmoduls of groups of the form $G^{\frac{1}{4}}$ (resp.$G^{\frac{1}{8}}$) it is the summand corresponding to $i=1$ (resp. $i=2$) that matters. We just note that the replicates $f^{[a\sqrt{2}]}$ (resp. $f^{[\sqrt{2}a]}$ and $f^{[2\sqrt{2}a]}$) with a odd, have a power series expansion with coefficients of even powers of q equal to zero. This means that any other function with the same property will work as well. We just chose those particular ones because they give complete $2A$-replicability.

Corollary 2.2.1. *The functions on tables 2.1 and 2.2 are completely $2A$-replicable.*

Proof. We need to prove that for any column in the tables, which corresponds to a Hauptmodul f together with a set of $2A$-replicates, f, $f^{[\sqrt{2}]}$, $f^{[2]}, \ldots$ the function $f^{[2^{\frac{m}{2}}n]}$ (which is $\left(f^{[2^{\frac{m}{2}}]}\right)^{(n)}$ by definition), is $2A$-replicable with $2A$-replicates given by $\left(f^{[2^{\frac{m}{2}}n]}\right)^{[2^{\frac{m'}{2}}n']} = f^{[2^{\frac{m+m'}{2}}nn']}$ for any m, m', n, n' non-negative integers, with n, n' odd. The idea of the proof is essentially the following. We show that we have completely $2A$-replicability for powers of $\sqrt{2}$ and that odd $2A$-replicability (or odd replicability, which is the same) "commutes" with $2A$-replication by powers of $\sqrt{2}$.

We first note that $\left(g^{(n)}\right)^{(n')} = g^{(nn')}$ for any function g in the tables and n, n' odd positive integers. This is obviously true if g is a completely replicable function and if $g(z) = h^{\alpha}(z)$, for $\alpha = \frac{1}{2}, \frac{1}{4}$ or $\frac{1}{8}$, it comes from the fact that $g^{(n)} = \left(h^{(n)}\right)^{\alpha}$, for n odd. As a consequence we obtain the fact that taking an odd replicate of all the elements of a column in the tables gives a column of the same type, with N, O and E possibly different from original ones. This is the "commutation" result mentioned above.

Then, we also note that the $2A$-replicate $f^{[2^{\frac{m}{2}}n]}$, for any m and n, together with the sequence $f^{[2^{\frac{m+m'}{2}}n]}$, $m' = 0, 1, 2, \ldots$ corresponds to some column in the tables. This is so because:

- of the fact that if we eliminate the first few entries of a column in the tables we obtain a column that is in the tables (case by case check), and
- of the remark made above on taking odd-replicates of full columns. Recall that, by definition, $f^{[2^{\frac{m}{2}}n]} = \left(f^{\left[2^{\frac{m}{2}}\right]}\right)^{(n)}$.

The proof is now easy:

$$f^{[2^{\frac{m+m'}{2}}nn']} = \left(f^{[2^{\frac{m+m'}{2}}]}\right)^{(nn')} = \left(\left(f^{[2^{\frac{m+m'}{2}}]}\right)^{(n)}\right)^{(n')} = \left(f^{[2^{\frac{m+m'}{2}}n]}\right)^{(n')} =$$
$$\left(\left(f^{[2^{\frac{m}{2}}n]}\right)^{[2^{\frac{m'}{2}}]}\right)^{(n')} = \left(f^{[2^{\frac{m}{2}}n]}\right)^{[2^{\frac{m'}{2}}n']}$$

f	$2N+\{e\}_O+\{2e\}_E$	$4N+\{e\}_O+\{4e\}_E$	$4N+\{e\}_O+\{4e\}_E$	$8N+\{e\}_O+\{8e\}_E$
$f^{[\sqrt{2}]}$	$2N+\{e,2e\}_O$	$2N+\{e\}_O$	$4N+\{e\}_O$	$4N+\{e\}_O$
$f^{[2]}$	$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$	$2N+\{e\}_O$	$4N+\{e\}_O$
$f^{[2\sqrt{2}]}$	$\cdots$	$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$	$2N+\{e\}_O$
$f^{[4]}$		$\cdots$	$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$
$f^{[4\sqrt{2}]}$			$\cdots$	$2N+\{e,2e\}_O$
$\vdots$				$\cdots$

f	$8N+\{e\}_O+\{8e\}_E$	$16N+\{e\}_O+\{16e\}_E$	$16N+\{e\}_O+\{16e\}_E$	$32N+\{e\}_O+\{4e\}_E$
$f^{[\sqrt{2}]}$	$8N+\{e\}_O$	$8N+\{e\}_O$	$16N+\{e\}_O$	$16N+\{e\}_O$
$f^{[2]}$	$4N+\{e\}_O$	$8N+\{e\}_O$	$8N+\{e\}_O$	$16N+\{e\}_O$
$f^{[2\sqrt{2}]}$	$4N+\{e\}_O$	$4N+\{e\}_O$	$8N+\{e\}_O$	$8N+\{e\}_O$
$f^{[4]}$	$2N+\{e\}_O$	$4N+\{e\}_O$	$4N+\{e\}_O$	$8N+\{e\}_O$
$f^{[4\sqrt{2}]}$	$2N+\{e,2e\}_O$	$2N+\{e\}_O$	$4N+\{e\}_O$	$4N+\{e\}_O$
$f^{[8]}$	$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$	$2N+\{e\}_O$	$4N+\{e\}_O$
$f^{[8\sqrt{2}]}$	$\cdots$	$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$	$2N+\{e\}_O$
$f^{[16]}$		$\cdots$	$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$
$f^{[16\sqrt{2}]}$			$\cdots$	$2N+\{e,2e\}_O$
$\vdots$				$\cdots$

f	$2N\|2+\{e\}_O$	$4N\|2+\{e\}_O+\{2e\}_E$	$8N\|2+\{e\}_O+\{4e\}_E$	$8N\|2+\{e\}_O+\{4e\}_E$
$f^{[\sqrt{2}]}$	$2N+\{e,2e\}_O$	$4N+\{e\}_O+\{4e\}_E$	$4N\|2+\{e\}_O+\{2e\}_E$	$8N+\{e\}_O+\{8e\}_E$
$f^{[2]}$	$2N+\{e,2e\}_O$	$2N+\{e\}_O$	$4N+\{e\}_O+\{4e\}_E$	$4N+\{e\}_O$
$f^{[2\sqrt{2}]}$	$\cdots$	$2N+\{e,2e\}_O$	$2N+\{e\}_O$	$4N+\{e\}_O$
$f^{[4]}$		$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$	$2N+\{e\}_O$
$f^{[4\sqrt{2}]}$		$\cdots$	$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$
$f^{[8]}$			$\cdots$	$2N+\{e,2e\}_O$
$\vdots$				$\cdots$

f	$16N\|2+\{e\}_O+\{8e\}_E$	$16N\|2+\{e\}_O+\{8e\}_E$	$32N\|2+\{e\}_O+\{16e\}_E$	$32N\|2+\{e\}_O+\{16e\}_E$
$f^{[\sqrt{2}]}$	$8N\|2+\{e\}_O+\{4e\}_E$	$16N+\{e\}_O+\{16e\}_E$	$16N\|2+\{e\}_O+\{8e\}_E$	$32N+\{e\}_O+\{32e\}_E$
$f^{[2]}$	$8N+\{e\}_O+\{8e\}_E$	$8N+\{e\}_O$	$16N+\{e\}_O+\{16e\}_E$	$16N+\{e\}_O$
$f^{[2\sqrt{2}]}$	$4N+\{e\}_O$	$8N+\{e\}_O$	$8N+\{e\}_O$	$16N+\{e\}_O$
$f^{[4]}$	$4N+\{e\}_O$	$4N+\{e\}_O$	$8N+\{e\}_O$	$8N+\{e\}_O$
$f^{[4\sqrt{2}]}$	$2N+\{e\}_O$	$4N+\{e\}_O$	$4N+\{e\}_O$	$8N+\{e\}_O$
$f^{[8]}$	$2N+\{e,2e\}_O$	$2N+\{e\}_O$	$4N+\{e\}_O$	$4N+\{e\}_O$
$f^{[8\sqrt{2}]}$	$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$	$2N+\{e\}_O$	$4N+\{e\}_O$
$f^{[16]}$	$\cdots$	$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$	$2N+\{e\}_O$
$f^{[16\sqrt{2}]}$		$\cdots$	$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$
$f^{[32]}$			$\cdots$	$2N+\{e,2e\}_O$
$\vdots$				$\cdots$

f	$8N\|4+\{e\}_O+\{2e\}_E$	$16N\|4+\{e\}_O+\{4e\}_E$	$16N\|4+\{e\}_O+\{4e\}_E$	$32N\|4+\{e\}_O+\{8e\}_E$
$f^{[\sqrt{2}]}$	$8N\|2+\{e\}_O+\{4e\}_E$	$16N\|2+\{e\}_O+\{8e\}_E$	$8N\|4+\{e\}_O+\{2e\}_E$	$32N\|2+\{e\}_O+\{16e\}_E$
$f^{[2]}$	$4N\|2+\{e\}_O+\{2e\}_E$	$8N\|2+\{e\}_O+\{4e\}_E$	$8N\|2+\{e\}_O+\{4e\}_E$	$16N\|2+\{e\}_O+\{8e\}_E$
$f^{[2\sqrt{2}]}$	$4N+\{e,4e\}_O$	$8N+\{e\}_O+\{8e\}_E$	$4N\|2+\{e\}_O+\{2e\}_E$	$16N+\{e\}_O+\{16e\}_E$
$f^{[4]}$	$2N+\{e\}_O$	$4N+\{e\}_O$	$4N+\{e\}_O+\{4e\}_E$	$8N+\{e\}_O$
$f^{[4\sqrt{2}]}$	$2N+\{e,2e\}_O$	$4N+\{e\}_O$	$2N+\{e\}_O$	$8N+\{e\}_O$
$f^{[8]}$	$2N+\{e,2e\}_O$	$2N+\{e\}_O$	$2N+\{e,2e\}_O$	$4N+\{e\}_O$
$f^{[8\sqrt{2}]}$	$\cdots$	$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$	$4N+\{e\}_O$
$f^{[16]}$		$2N+\{e,2e\}_O$	$\cdots$	$2N+\{e\}_O$
$f^{[16\sqrt{2}]}$		$\cdots$		$2N+\{e,2e\}_O$
$f^{[32]}$				$2N+\{e,2e\}_O$
$\vdots$				$\cdots$

f	$4N+\{e\}_O$	$4N\|2+\{e\}_O+\{2e\}_E$	$4N\|2+\{e\}_O$	$8N+\{e\}_O$
$f^{[\sqrt{2}]}$	$(4N+\{e\}_O+\{4e\}_E)^{\frac{1}{2}}$	$(2N+\{e,2e\}_O)^{\frac{1}{2}}$	$(4N+\{e\}_O+\{4e\}_E)^{\frac{1}{2}}$	$(8N\|2+\{e\}_O+\{4e\}_E)^{\frac{1}{4}}$
$f^{[2]}$	$2N+\{e\}_O+2e_E$	$N+\{e\}_O$	$2N+\{e\}_O+2e_E$	$4N+\{e\}_O$
$f^{[2\sqrt{2}]}$	$2N+\{e,2e\}_O$	$2N+\{e\}_O$	$2N+\{e,2e\}_O$	$(4N+\{e\}_O+\{4e\}_E)^{\frac{1}{2}}$
$f^{[4]}$	$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$	$2N+\{e\}_O+\{2e\}_E$
$f^{[4\sqrt{2}]}$	$\cdots$	$2N+\{e,2e\}_O$	$\cdots$	$2N+\{e,2e\}_O$
$f^{[8]}$		$\cdots$		$2N+\{e,2e\}_O$
$\vdots$				$\cdots$

Table 2.1: $2A$-replicates of Hauptmoduls

f	$8N\|2+\{e\}_O$	$8N\|4+\{e,2e\}_O$	$8N\|2+\{e\}_O$	$8N\|4+\{e\}_O$
$f^{[\sqrt{2}]}$	$(8N+\{e\}_O+\{8e\}_E)^{\frac{1}{2}}$	$(4N\|2+\{e,2e\}_O)^{\frac{1}{4}}$	$(8N\|2+\{e\}_O+\{4e\}_E)^{\frac{1}{4}}$	$(8N\|2+\{e\}_O+\{4e\}_E)^{\frac{1}{4}}$
$f^{[2]}$	$4N+\{e\}_O+\{4e\}_E$	$4N\|2+\{e,2e\}_O$	$4N+\{e\}_O$	$4N\|2+\{e\}_O$
$f^{[2\sqrt{2}]}$	$4N+\{e\}_O$	$(2N+\{e,2e\}_O)^{\frac{1}{2}}$	$(4N+\{e\}+\{4e\}_O)^{\frac{1}{2}}$	$(4N+\{e\}+\{4e\}_O)^{\frac{1}{2}}$
$f^{[4]}$	$2N+\{e\}_O$	$N+\{e\}_O$	$2N+\{e\}_O+\{2e\}_E$	$2N+\{e\}_O+\{2e\}_E$
$f^{[4\sqrt{2}]}$	$2N+\{e.2e\}_O$	$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$
$f^{[8]}$	$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$
$\vdots$	$\dots$	$\dots$	$\dots$	$\dots$

f	$(2N+\{e,2e\}_O)^{\frac{1}{2}}$	$(4N+\{e\}_O+\{4e\}_E)^{\frac{1}{2}}$	$(8N+\{e\}_O+\{4e\}_E)^{\frac{1}{2}}$	$(16N+\{e\}_O+\{8e\}_E)^{\frac{1}{2}}$
$f^{[\sqrt{2}]}$	$N+\{e\}_O$	$2N+\{e\}_O+\{2e\}_E$	$4N+\{e\}_O+\{4e\}_E$	$8N+\{e\}_O+8e_E$
$f^{[2]}$	$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$	$4N+\{e\}_O$	$8N+\{e\}_O$
$f^{[2\sqrt{2}]}$	$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$	$2N+\{e\}_O$	$4N+\{e\}_O$
$f^{[4]}$	$\cdots$	$\cdots$	$2N+\{e,2e\}_O$	$4N+\{e\}_O$
$f^{[4\sqrt{2}]}$			$2N+\{e,2e\}_O$	$2N+\{e\}_O$
$f^{[8]}$			$\cdots$	$2N+\{e,2e\}_O$
$\vdots$				$\dots$

f	$(32N+\{e\}_O+\{32e\}_E)^{\frac{1}{2}}$	$(4N\|2+\{e,2e\}_O)^{\frac{1}{4}}$	$(8N\|2+\{e\}_E+\{4e\}_E)^{\frac{1}{4}}$	$(8N\|2+\{e\}_E+\{4e\}_E)^{\frac{1}{4}}$
$f^{[\sqrt{2}]}$	$16N+\{e,16e\}_O$	$4N\|2+\{e,2e\}_O$	$4N+\{e\}_O$	$4N\|2+\{e\}_O$
$f^{[2]}$	$16N+\{e\}_O$	$(2N+\{e,2e\}_O)^{\frac{1}{2}}$	$(4N+\{e\}_E+\{4e\}_E)^{\frac{1}{2}}$	$(4N+\{e\}_E+\{4e\}_E)^{\frac{1}{2}}$
$f^{[2\sqrt{2}]}$	$8N+\{e\}_O$	$N+\{e\}_O$	$2N+\{e\}_O+\{2e\}_O$	$2N+\{e\}_O+\{2e\}_O$
$f^{[4]}$	$8N+\{e\}_O$	$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$
$f^{[4\sqrt{2}]}$	$4N+\{e\}_O$	$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$
$f^{[8]}$	$4N+\{e\}_O$	$\cdots$	$\cdots$	$\cdots$
$f^{[8\sqrt{2}]}$	$2N+\{e\}_O$			
$f^{[16]}$	$2N+\{e,2e\}_O$			
$f^{[16\sqrt{2}]}$	$2N+\{e,2e\}_O$			
$\vdots$	$\dots$			

f	$(16N\|2+\{e\}_O+\{8e\}_E)^{\frac{1}{4}}$	$(8N\|4+\{e\}_O+\{2e\}_E)^{\frac{1}{8}}$	$(16N\|4+\{e\}_O+\{2e\}_E)^{\frac{1}{8}}$	$(16N\|4+\{e\}_O+\{2e\}_E)^{\frac{1}{8}}$
$f^{[\sqrt{2}]}$	$8N\|2+\{e\}_O+\{8e\}_O$	$8N\|4+\{e,2e\}_O$	$8N+\{e\}_O$	$8N\|2+\{e\}_O$
$f^{[2]}$	$(8N+\{e\}_O+\{8e\}_E)^{\frac{1}{2}}$	$(4N\|2+\{e,2e\}_O)^{\frac{1}{4}}$	$(8N\|2+\{e\}_O+\{4e\}_E)^{\frac{1}{4}}$	$(8N\|2+\{e\}_O+\{4e\}_E)^{\frac{1}{4}}$
$f^{[2\sqrt{2}]}$	$4N+\{e\}_O+\{4e\}_E$	$4N\|2+\{e,2e\}_O$	$4N+\{e\}_O$	$4N+\{e\}_O$
$f^{[4]}$	$4N+\{e\}_O$	$(2N+\{e,2e\}_O)^{\frac{1}{2}}$	$(4N+\{e\}_O+\{4e\}_E)^{\frac{1}{2}}$	$(4N+\{e\}_O+\{4e\}_E)^{\frac{1}{2}}$
$f^{[4\sqrt{2}]}$	$2N+\{e\}_O$	$N+\{e\}_O$	$2N+\{e\}_O+\{2e\}_E$	$2N+\{e\}_O+\{2e\}_E$
$f^{[8]}$	$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$
$f^{[8\sqrt{2}]}$	$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$	$2N+\{e,2e\}_O$
$\vdots$	$\dots$	$\dots$	$\dots$	$\dots$

f	$(16N\|4+\{e\}_O+\{4e\}_E)^{\frac{1}{8}}$
$f^{[\sqrt{2}]}$	$8N\|4+\{e\}_O$
$f^{[2]}$	$(8N\|2+\{e\}_O+\{4e\}_E)^{\frac{1}{4}}$
$f^{[2\sqrt{2}]}$	$4N\|2+\{e\}_O$
$f^{[4]}$	$(4N+\{e\}_O+\{4e\}_E)^{\frac{1}{2}}$
$f^{[4\sqrt{2}]}$	$2N+\{e\}_O+\{2e\}_E$
$f^{[8]}$	$2N+\{e,2e\}_O$
$f^{[8\sqrt{2}]}$	$2N+\{e,2e\}_O$
$\vdots$	$\dots$

Table 2.2: $2A$-replicates of Hauptmoduls (continued)

2.3 Complete $2A$-replicability and generalized Mahler recurrence relations

This section is *mutatis mutandis* what we can find in [43].

We consider $\mathcal{L}$ a field extension of $\mathbb{Q}$ containing all roots of unity, the ring $K = \mathcal{L}[\ldots, x_m^{[n]}, \ldots]$, $m \in \mathbb{N}$, $n \in \mathbb{N} \cup \sqrt{2}\mathbb{N}$, the series

$$h^{[r]}(q) = \frac{1}{q} + \sum_{m=1}^{\infty} x_m^{[r]} q^m$$

for $r \in \mathbb{N} \cup \sqrt{2}\mathbb{N}$, and the polynomials $P_{k,r}(t)$ defined inductively by $P_{1,r}(t) = t$ and $P_{k,r}(t) = tP_{k-1,r}(t) - \sum_{s=1}^{k-2} x_s^{[r]} P_{k-s-1,r}(t) - kx_{k-1}^{[k]}$.

We fix $r \in \mathbb{N} \cup \sqrt{2}\mathbb{N}$ and consider the set of equations indexed by $k \geq 1$

$$\sum_{\substack{ad=k \\ 0 \leq b < d}} h^{[ra]}\left(e^{2\pi i \frac{b}{d}} q^{\frac{a}{d}}\right) + \sum_{\substack{ad=k \\ d \text{ even} \\ 0 \leq b < d}} h^{[ra\sqrt{2}]}\left(e^{2\pi i \frac{b}{d}} q^{\frac{2a}{d}}\right) = P_{k,r}(h^{[r]}(q)) \tag{2.3.1}$$

These equations give an infinite set of identities in K by equating the coefficiens of equal powers of q in both sides of the each equation. We denote by $I^{[r]}$ the ideal in K generated by them and write $I = \bigcup_{r \in \mathbb{N} \cup \sqrt{2}\mathbb{N}} I^{[r]}$.

If $f(q) = \frac{1}{q} + \sum_{k=1}^{\infty} a_k q^k$ is completely $2A$-replicable with replicates $f^{[n]}(q) = \frac{1}{q} + \sum_{k=1}^{\infty} a_k^{[n]} q^k$, $n \in \mathbb{N} \cup \sqrt{2}\mathbb{N}$ then they satisfy equations 2.3.1 with $h^{[n]}(q)$ and $h^{[\sqrt{2}n]}(q)$ replaced by $f^{[n]}(q)$ and $f^{[\sqrt{2}n]}(q)$, respectively. This means that every completely $2A$-replicable function f induces a non-trivial homomorphism $E_f : K \longrightarrow \mathbb{C}$ with $E_f(x_k^{[n]}) = a_k^{[n]}$, whose kernel contains I.

For $u \in \mathbb{N} \cup \sqrt{2}\mathbb{N}$ we define a $\mathcal{L}$-algebra endomorphism ψ_u of K letting it fix every element of $\mathcal{L}$ and mapping $x_m^{[n]}$ to $x_m^{[nu]}$. Since the equations defining $I^{[r]}$ and $I^{[ru]}$ have the same form, it is clear that $\psi_u(I^{[r]}) = I^{[ru]}$ for any r. Consequently, $\psi_u(I) \subseteq I$ and we think of ψ_u as an $\mathcal{L}$-algebra endomorphism of the quotient K/I.

For each $M \geq 1$ we set $R_M = K/I[[q^{\frac{1}{2M}}]]$ and also

$$\Delta = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} | a, b, d \in \mathbb{N} \right\} \cup \left\{ \begin{bmatrix} \sqrt{2}a & b/\sqrt{2} \\ 0 & \sqrt{2}d \end{bmatrix} | a, b, d \in \mathbb{N} \right\}$$

We define an action of Δ in R_M in the following way. For $\alpha \in \Delta$ we set $e \parallel \alpha = e$, $x_m^{[r]} \parallel \alpha = \psi_u(x_m^{[r]}) = x_m^{[ru]}$ and $q^{\frac{1}{M}} \parallel \alpha = e^{2\pi i \frac{v}{My}} q^{\frac{u}{My}}$. Then we extend this map to an $\mathcal{L}$-homomorphism from $K[[q^{\frac{1}{M}}]]$ to $K[[q^{\frac{1}{My}}]]$.

Remark 2.3.1. *The ideal I of K is stable under $\parallel \alpha$ for every $\alpha \in \Delta$.*

Remark 2.3.2. *For every $\alpha, \beta \in \Delta$ and $h(q) \in R_M$ we have*

$$(h(q) \parallel \alpha) \parallel \beta = h(q) \parallel \alpha\beta.$$

Remark 2.3.3. *If* $\alpha = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in \Delta$ *then* $h^{[r]}(q) \parallel \alpha = h^{[ru]}(e^{2\pi i \frac{x}{y}} q^{\frac{u}{y}})$

Remark 2.3.4. *For every positive integer j, $(h^{[r]}(q))^j \parallel \alpha = (h^{[r]}(q) \parallel \alpha)^j$ as $\parallel \alpha$ is a ring homomorphism.*

Definition 2.3.1. *Let n be a positive integer and $h(q) \in R_1$. We define $\mathbf{T}_n$ by*

$$\mathbf{T}_n(h(q)) = \sum_{\substack{uy=n \\ 0 \leq v < y}} h(q) \parallel \begin{bmatrix} u & v \\ 0 & y \end{bmatrix} + \sum_{\substack{uy=\frac{n}{2} \\ 0 \leq v < 2y}} h(q) \parallel \begin{bmatrix} \sqrt{2}u & \frac{v}{\sqrt{2}} \\ 0 & \sqrt{2}y \end{bmatrix}$$

and call $\mathbf{T}_n$ a generalized Hecke Operator. Both sums are over positive integer numbers which makes the second sum be zero if n is odd.

With this definition equation 2.3.1 becomes

$$\mathbf{T}_k(h^{[r]}(q)) = P_{k,r}(h^{[r]}(q)) \tag{2.3.2}$$

Definition 2.3.2. *Let* $n \in \mathbb{N} \cup \sqrt{2}\mathbb{N}$ *and* $h(q) \in R_1$. *We denote by* Ψ_n *the mapping* $\Psi_n h(q) = h(q) \parallel \begin{bmatrix} n & 0 \\ 0 & n \end{bmatrix}$.

Proposition 2.3.1. *Let* l_1 *and* l_2 *be relatively prime positive integers and* $h(q) \in R_1$. *Then*

$$\mathbf{T}_{l_1}\mathbf{T}_{l_2}h(q) = \mathbf{T}_{l_1 l_2}h(q)$$

In particular, $\mathbf{T}_{l_1}$ *and* $\mathbf{T}_{l_2}$ *commute.*

Proof. This comes from the fact that

$$\left\{ \begin{bmatrix} u & v \\ 0 & y \end{bmatrix} \begin{bmatrix} u' & v' \\ 0 & y' \end{bmatrix} | uy = l_1, u'y' = l_2, 0 \leq v < y, 0 \leq v' < y', (yy' \text{ even}) \right\} =$$
$$= \left\{ \begin{bmatrix} u'' & v'' \\ 0 & y'' \end{bmatrix} | u''v'' = l_1 l_2, 0 \leq v'' < y'', (y'' \text{ even}) \right\}$$

Proposition 2.3.2. *If* p *is an odd prime then*

$$\mathbf{T}_{p^n}\mathbf{T}_p(h(q)) = \mathbf{T}_{p^{n+1}}(h(q)) + p\mathbf{T}_{p^{n-1}}\Psi_p(h(q)).$$

If $p = 2$ *then*

$$\mathbf{T}_{2^n}\mathbf{T}_2(h(q)) = \mathbf{T}_{2^{n+1}}(h(q)) + 2\mathbf{T}_{2^n}\Psi_{\sqrt{2}}(h(q)) + 2\mathbf{T}_{2^{n-1}}\Psi_2(h(q)).$$

Proof. The case where p is odd is true as the operators $\mathbf{T}_{p^n}$ agree with the Hecke operator for $SL_2(\mathbb{Z})$. For the case $p = 2$ we note that $\mathbf{T}_{2^n}\mathbf{T}_2(h(q))$ equals

$$h(q) \parallel \left(\sum_{\substack{i=0,1 \\ 0 \leq b < 2^i}} \begin{bmatrix} 2^{1-i} & b \\ 0 & 2^i \end{bmatrix} + \sum_{0 \leq b < 2} \begin{bmatrix} \sqrt{2} & \frac{b}{\sqrt{2}} \\ 0 & \sqrt{2} \end{bmatrix} \right) \left(\sum_{\substack{i=0,\ldots,n \\ 0 \leq b < 2^i}} \begin{bmatrix} 2^{n-i} & b \\ 0 & 2^i \end{bmatrix} \right) +$$

$$+h(q)\parallel\left(\sum_{\substack{i=0,1\\0\le b<2^i}}\begin{bmatrix}2^{1-i} & b\\ 0 & 2^i\end{bmatrix}+\sum_{0\le b<2}\begin{bmatrix}\sqrt{2} & \frac{b}{\sqrt{2}}\\ 0 & \sqrt{2}\end{bmatrix}\right)\left(\sum_{\substack{i=0,\dots,n-1\\0\le b<2^{i+1}}}\begin{bmatrix}2^{n-1-i}\sqrt{2} & \frac{b}{\sqrt{2}}\\ 0 & 2^i\sqrt{2}\end{bmatrix}\right)$$

$$\begin{aligned}=h(q)\parallel\Bigg(&\sum_{\substack{i=0,\dots,n\\0\le b<2^i}}\begin{bmatrix}2^{n+1-i} & 2b\\ 0 & 2^i\end{bmatrix}+\sum_{\substack{i=0,\dots,n\\0\le b<2^i}}\begin{bmatrix}2^{n-i}\sqrt{2} & \sqrt{2}b\\ 0 & 2^i\sqrt{2}\end{bmatrix}+\\
&+\sum_{\substack{i=0,\dots,n\\0\le b<2^i}}\begin{bmatrix}2^{n-i}\sqrt{2} & \sqrt{2}b+\frac{2^i}{\sqrt{2}}\\ 0 & 2^i\sqrt{2}\end{bmatrix}+\sum_{\substack{i=0,\dots,n\\0\le b<2^i}}\begin{bmatrix}2^{n-i} & b\\ 0 & 2^{i+1}\end{bmatrix}+\sum_{\substack{i=0,\dots,n\\0\le b<2^i}}\begin{bmatrix}2^{n-i} & b+2^i\\ 0 & 2^{i+1}\end{bmatrix}+\\
&+\sum_{\substack{i=0,\dots,n-1\\0\le b<2^{i+1}}}\begin{bmatrix}2^{n-i}\sqrt{2} & \sqrt{2}b\\ 0 & 2^i\sqrt{2}\end{bmatrix}+\sum_{\substack{i=0,\dots,n-1\\0\le b<2^{i+1}}}\begin{bmatrix}2^{n-i} & b\\ 0 & 2^{i+1}\end{bmatrix}+\sum_{\substack{i=0,\dots,n-1\\0\le b<2^{i+1}}}\begin{bmatrix}2^{n-i} & b+2^i\\ 0 & 2^{i+1}\end{bmatrix}+\\
&+\sum_{\substack{i=0,\dots,n-1\\0\le b<2^{i+1}}}\begin{bmatrix}2^{n-1-i}\sqrt{2} & \frac{b}{\sqrt{2}}\\ 0 & 2^{i+1}\sqrt{2}\end{bmatrix}+\sum_{\substack{i=0,\dots,n-1\\0\le b<2^{i+1}}}\begin{bmatrix}2^{n-1-i}\sqrt{2} & \frac{b}{\sqrt{2}}+2^ib\\ 0 & 2^{i+1}\sqrt{2}\end{bmatrix}\Bigg)=\end{aligned}$$

$$\begin{aligned}=h(q)\parallel\Bigg(&\sum_{\substack{i=0,\dots,n\\0\le b<2^i}}\begin{bmatrix}2^{n+1-i} & 2b\\ 0 & 2^i\end{bmatrix}+\sum_{\substack{i=0,\dots,n\\0\le b<2^i}}\begin{bmatrix}2^{n-i}\sqrt{2} & \sqrt{2}b\\ 0 & 2^i\sqrt{2}\end{bmatrix}+\\
&+\sum_{\substack{i=0,\dots,n\\0\le b<2^i}}\begin{bmatrix}2^{n-i}\sqrt{2} & \sqrt{2}b+\frac{2^i}{\sqrt{2}}\\ 0 & 2^i\sqrt{2}\end{bmatrix}+\sum_{\substack{i=0,\dots,n\\0\le b<2^{i+1}}}\begin{bmatrix}2^{n-i} & b\\ 0 & 2^{i+1}\end{bmatrix}+\\
&+2\cdot\sum_{\substack{i=0,\dots,n-1\\0\le b<2^i}}\begin{bmatrix}2^{n-i}\sqrt{2} & \sqrt{2}b\\ 0 & 2^i\sqrt{2}\end{bmatrix}+2\cdot\sum_{\substack{i=0,\dots,n-1\\0\le b<2^{i+1}}}\begin{bmatrix}2^{n-i} & b\\ 0 & 2^{i+1}\end{bmatrix}+\\
&+\sum_{\substack{i=0,\dots,n-1\\0\le b<2^{i+2}}}\begin{bmatrix}2^{n-1-i}\sqrt{2} & \frac{b}{\sqrt{2}}\\ 0 & 2^{i+1}\sqrt{2}\end{bmatrix}\Bigg)\end{aligned}$$

Now, the first and fourth summands equal

$$h(q) \parallel \left(\sum_{\substack{i=0,\ldots,n+1 \\ 0\le b<2^i}} \begin{bmatrix} 2^{n+1-i} & b \\ 0 & 2^i \end{bmatrix} + 2 \cdot \sum_{\substack{i=0,\ldots,n-1 \\ 0\le b<2^i}} \begin{bmatrix} 2^{n-i} & 2b \\ 0 & 2^i \end{bmatrix} \right)$$

the second, third and last summands equal

$$h(q) \parallel \left(\sum_{\substack{i=0,\ldots,n \\ 0\le b<2^{i+1}}} \begin{bmatrix} 2^{n-i}\sqrt{2} & \frac{b}{\sqrt{2}} \\ 0 & 2^i\sqrt{2} \end{bmatrix} + 2 \cdot \sum_{\substack{i=0,\ldots,n-2 \\ 0\le b<2^{i+1}}} \begin{bmatrix} 2^{n-2-i} & \sqrt{2}b \\ 0 & 2^{i+1}\sqrt{2} \end{bmatrix} \right)$$

and this shows that

$$\begin{aligned} \mathbf{T}_{2^n}\mathbf{T}_2(h(q)) = h(q) \parallel & \left(\sum_{\substack{i=0,\ldots,n+1 \\ 0\le b<2^i}} \begin{bmatrix} 2^{n+1-i} & b \\ 0 & 2^i \end{bmatrix} + \sum_{\substack{i=0,\ldots,n \\ 0\le b<2^{i+1}}} \begin{bmatrix} 2^{n-i}\sqrt{2} & \frac{b}{\sqrt{2}} \\ 0 & 2^i\sqrt{2} \end{bmatrix} + \right. \\ & + 2 \cdot \sum_{\substack{i=0,\ldots,n-1 \\ 0\le b<2^{i+1}}} \begin{bmatrix} 2^{n-i}\sqrt{2} & \sqrt{2}b \\ 0 & 2^i\sqrt{2} \end{bmatrix} + 2 \cdot \sum_{\substack{i=0,\ldots,n-1 \\ 0\le b<2^{i+1}}} \begin{bmatrix} 2^{n-i} & b \\ 0 & 2^{i+1} \end{bmatrix} + \\ & \left. +2 \cdot \sum_{\substack{i=0,\ldots,n-1 \\ 0\le b<2^i}} \begin{bmatrix} 2^{n-i} & 2b \\ 0 & 2^{i+1} \end{bmatrix} + 2 \cdot \sum_{\substack{i=0,\ldots,n-2 \\ 0\le b<2^{i+1}}} \begin{bmatrix} 2^{n-2-i} & \sqrt{2}b \\ 0 & 2^{i+1}\sqrt{2} \end{bmatrix} \right) = \\ & = \mathbf{T}_{2^{n+1}}(h(q)) + 2\mathbf{T}_{2^n}\Psi_{\sqrt{2}}(h(q)) + 2\mathbf{T}_{2^{n-1}}\Psi_2(h(q)) \end{aligned}$$

and the theorem is proven.

Corollary 2.3.1. *The algebra generated by the operators $\mathbf{T}_n$, for $n \in \mathbb{N}$, is commutative.*

Let l be a fixed prime. We set $Q_k = \mathbf{T}_l(h(q)^k)$ for $k \geq 1$, and for $b \in \frac{K}{I}$ we use $b^{[l]}$ to denote $b \parallel \begin{bmatrix} l & * \\ 0 & * \end{bmatrix}$. We set also $Q_0 = \begin{cases} 2l+1, & \text{if } l = 2 \\ l+1, & \text{if } l \neq 2 \end{cases}$ and define $\mathbf{T}_{\frac{k}{l}}$ as the operator that sends R_1 to zero if l does not divide k. We use the same notation to denote both $P_{k,r}(t)$

and its image in $\frac{K}{I}[t]$.

Proposition 2.3.3. *For $k \in \mathbb{N}$ write $P_{k,r}(t) = t^k + \sum_{i=1}^{k} b_{k,i} t^{k-i} \in \dfrac{K}{I}[t]$. If l is odd then*

$$Q_k + \sum_{i=1}^{k} b_{k,i} Q_{k-i} + \sum_{i=1}^{k} (b_{k,i}^{[l]} - b_{k,i}) h^{[l]}(q^l)^{k-i} = P_{kl}(h(q)) + k\mathbf{T}_{\frac{k}{l}} \Psi_l(h(q))$$

and if $l = 2$ we have

$$Q_k + \sum_{i=1}^{k} b_{k,i} Q_{k-i} + \sum_{i=1}^{k} (b_{k,i}^{[\sqrt{2}]} - b_{k,i})(h^{[\sqrt{2}]}(q)^{k-i} + h^{[\sqrt{2}]}(-q)^{k-i}) + \sum_{i=1}^{k} (b_{k,i}^{[2]} - b_{k,i}) h^{[2]}(q^2)^{k-i}$$
$$= \begin{cases} P_{2k,1}(h(q)), & \text{if } 2 \nmid k \\ P_{2k,1}(h(q)) + 2\mathbf{T}_k \Psi_{\sqrt{2}}(h(q)) + 2\mathbf{T}_{\frac{k}{2}} \Psi_2(h(q)), & \text{if } 2 \mid k \end{cases}$$

Proof. The case where l is odd can be found in [43]. When $l = 2$ we apply $\mathbf{T}_2$ to both sides of equation 2.3.2

$$\mathbf{T}_2 \mathbf{T}_k(h(q)) = \mathbf{T}_2 \left(h(q)^k + \sum_{i=1}^{k} b_{k,i} h(q)^{k-i} \right)$$

and the following manipulation

$$\mathbf{T}_2(b_{k,i} h(q)^{k-i}) = b_{k,i} h(q)^{k-i} \parallel \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} + b_{k,i} h(q)^{k-i} \parallel \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} +$$
$$+ b_{k,i} h(q)^{k-i} \parallel \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{2} \end{bmatrix} + b_{k,i} h(q)^{k-i} \parallel \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + b_{k,i} h(q)^{k-i} \parallel \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} =$$

$$= (b_{k,i}^{[2]} - b_{k,i}) \left(h(q)^{k-i} \parallel \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right) + (b_{k,i}^{[\sqrt{2}]} - b_{k,i}) \left(h(q)^{k-i} \parallel \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \right) +$$
$$+ (b_{k,i}^{[\sqrt{2}]} - b_{k,i}) \left(h(q)^{k-i} \parallel \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{2} \end{bmatrix} \right) + b_{k,i} \mathbf{T}_2(h(q)^{k-i}) =$$

$$= (b_{k,i}^{[2]} - b_{k,i})h^{[2]}(q^2)^{k-i} + (b_{k,i}^{[\sqrt{2}]} - b_{k,i}) \left(h^{[\sqrt{2}]}(q)^{k-i} + h^{[\sqrt{2}]}(-q)^{k-i} \right) + b_{k,i}Q_{k-i}$$

shows that $\mathbf{T}_2\mathbf{T}_k(h(q))$ equals

$$Q_k + \sum_{i=1}^{k} b_{k,i}Q_{k-i} + \sum_{i=1}^{k} \left((b_{k,i}^{[2]} - b_{k,i})h^{[2]}(q^2)^{k-i} + (b_{k,i}^{[\sqrt{2}]} - b_{k,i}) \left(h^{[\sqrt{2}]}(q)^{k-i} + h^{[\sqrt{2}]}(-q)^{k-i} \right) \right)$$

If $(2,k) = 1$ then $\mathbf{T}_2\mathbf{T}_k(h(q)) = \mathbf{T}_{2k}(h(q)) = P_{2k,1}(h(q))$ and if 2^r is the exact power of 2 that divides k we have that

$$\mathbf{T}_2\mathbf{T}_k(h(q)) = \mathbf{T}_{\frac{k}{2^r}}\mathbf{T}_{2^r}\mathbf{T}_2 = \mathbf{T}_{\frac{k}{2^r}} \left(\mathbf{T}_{2^{r+1}}(h(q)) + 2\mathbf{T}_{2^r}\Psi_{\sqrt{2}}(h(q)) + 2\mathbf{T}_{2^{r-1}}\Psi_2(h(q)) \right) =$$

$$= \mathbf{T}_{2k}(h(q)) + 2\mathbf{T}_k\Psi_{\sqrt{2}}(h(q)) + 2\mathbf{T}_{\frac{k}{2}}\Psi_2(h(q))$$

This proves the assertion of the theorem.

When $l = 2$ the Q_k are simply the power sum symmetric functions on $h^{[2]}(q^2)$, $h^{[\sqrt{2}]}(q)$, $h^{[\sqrt{2}]}(-q)$, $h(q)$, $h(-q)$. By induction, using the previous result, one shows that every Q_j is a polynomial in $h^{[2]}(q)$, $h^{[2]}(q^2)$, $h^{[\sqrt{2}]}(q)$, $h^{[\sqrt{2}]}(-q)$ and $h(q)$. Let $\sigma_n = \sum_{1 \le i_1 < i_2 < \ldots < i_n \le 5} x_{i_1} \ldots x_{i_n}$ be the elementary symmetric functions in the indeterminates x_1, x_2, x_3, x_4, x_5. We know that the elementary symmetric function are polynomials in the power sum symmetric functions and from this we conclude that the elementary symmetric functions on $h^{[2]}(q^2)$, $h^{[\sqrt{2}]}(q)$, $h^{[\sqrt{2}]}(-q)$, $h(q)$, $h(-q)$ are polynomials in $h^{[2]}(q)$, $h^{[2]}(q^2)$, $h^{[\sqrt{2}]}(q)$, $h^{[\sqrt{2}]}(-q)$ and $h(q)$. We use these facts in the proof of the following proposition.

Proposition 2.3.4. *If $f(q) = \frac{1}{q} + \sum_{k=1}^{\infty} a_k q^k$ is completely $2A$-replicable with replicates $f^{[n]}(q) = \frac{1}{q} + \sum_{k=1}^{\infty} a_k^{[n]} q^k$, for $n \in \mathbb{N} \cup \sqrt{2}\mathbb{N}$, then*

$$\sigma_2 \left(f^{[2]}(2z), f^{[\sqrt{2}]}(z), f^{[\sqrt{2}]}\left(z + \tfrac{1}{2}\right), f\left(\tfrac{z}{2}\right), f\left(\tfrac{z+1}{2}\right) \right) =$$

$$= 2a_2 f(z) - f^{[2]}(z) + 2\left(a_4 - a_1\right) + 2a_2^{[\sqrt{2}]} - \left(f^{[\sqrt{2}]}(z)\right)^2$$

Proof. We take $h^{[n]}(q)$, for $n \in \mathbb{N} \cup \sqrt{2}\mathbb{N}$ as before and start by noticing that

$$\sigma_2\left(h^{[2]}(2z), h^{[\sqrt{2}]}(z), h^{[\sqrt{2}]}\left(z+\tfrac{1}{2}\right), h\left(\tfrac{z}{2}\right), h\left(\tfrac{z+1}{2}\right)\right) = \tfrac{1}{2}\left(Q_1^2 - Q_2\right)$$

Then we use Proposition 2.3.3 to express Q_1 and Q_2 as polynomials in $h^{[2]}(q)$, $h^{[2]}(q^2)$, $h^{[\sqrt{2}]}(q)$, $h^{[\sqrt{2}]}(-q)$ and $h(q)$.

Now,

$$Q_1 = \mathbf{T}_1(h(q)) = P_1(h(q)) = h^2(q) - 2x_1$$

and from Proposition 2.3.3

$$Q_2 = P_4(h(q)) + 2P_2(h^{[\sqrt{2}]}(q)) + 2h^{[2]}(q) - (-10x_1 + 2(-2x_1^{[\sqrt{2}]} + 2x_1) + (-2x_1^{[2]} + 2x_1))$$

because $b_{2,1} = b_{2,1}^{[\sqrt{2}]} = b_{2,1}^{[2]} = 0$ and $b_{2,1} = -2x_1$, $b_{2,1}^{[\sqrt{2}]} = -2x_1^{[\sqrt{2}]}$, $b_{2,1}^{[2]} = -2x_1^{[2]}$, and $\frac{1}{2}\left(Q_1^2 - Q_2\right)$ becomes

$$\begin{aligned}\frac{1}{2}(h^4(q) - 4x_1h^2(q) + 4x_1^2 - \big(h^4(q) - 4x_1h^2(q) - 4x_2h(q) - 4x_3 + 2x_1^2 + \\ + 2h^{[\sqrt{2}]}(q)^2 - 4x_1^{[\sqrt{2}]} + 2h^{[2]}(q) + 10x_1 + 4x_1^{[\sqrt{2}]} - 4x_1 + 2x_1^{[2]} - 2x_1)\big) = \\ = 2x_2h(q) - \left(h^{[\sqrt{2}]}(q)\right)^2 - h^{[2]}(q) + x_1^2 + 2x_3 - 2x_1 - x_1^{[2]}\end{aligned}$$

Applying the homomorphism E_f defined at the beginning of this section we get

$$\begin{aligned}\sigma_2\left(f^{[2]}(2z), f^{[\sqrt{2}]}(z), f^{[\sqrt{2}]}\left(z+\frac{1}{2}\right), f\left(\frac{z}{2}\right), f\left(\frac{z+1}{2}\right)\right) = \\ = 2a_2f(q) - \left(f^{[\sqrt{2}]}(q)\right)^2 - f^{[2]}(q) + a_1^2 + 2a_3 - 2a_1 - a_1^{[2]}\end{aligned}$$

Equating the coefficient of q^2 in both sides of the equation

$$f^{[2]}(q^2) + f^{[\sqrt{2}]}(q) + f^{[\sqrt{2}]}(-q) + f(q^{\frac{1}{2}}) + f(-q^{\frac{1}{2}}) = P_2(f(q))$$

we see that $a_1^2 + 2a_3 - a_1^{[2]} = 2a_4 + 2a_2^{[\sqrt{2}]}$ and this concludes the proof.

Proposition 2.3.5. *From the following two identities*

- $\sigma_1\left(f^{[2]}(2z), f^{[\sqrt{2}]}(z), f^{[\sqrt{2}]}\left(z+\frac{1}{2}\right), f\left(\frac{z}{2}\right), f\left(\frac{z+1}{2}\right)\right) = P_{2,f}(f(z))$
- $\sigma_2\left(f^{[2]}(2z), f^{[\sqrt{2}]}(z), f^{[\sqrt{2}]}\left(z+\frac{1}{2}\right), f\left(\frac{z}{2}\right), f\left(\frac{z+1}{2}\right)\right) =$

$$= 2a_2 f(z) - f^{[2]}(z) + 2\left(a_4 - a_1\right) + 2a_2^{[\sqrt{2}]} - \left(f^{[\sqrt{2}]}(z)\right)^2$$

we obtain the recurrence relations:

1) $a_{4k} = a_{2k+1} + \sum_{j=1}^{k-1} a_j a_{2k-j} + \frac{1}{2}\left(a_k^2 - a_k^{[2]}\right) - a_{2k}^{[\sqrt{2}]}$

2) $a_{4k+1} = a_{2k+3} - a_2 a_{2k} + \sum_{j=1}^{k} a_j a_{2k+2-j} + \sum_{j=1}^{k-1} a_j^{[2]} a_{2k-2j}^{[\sqrt{2}]} + 2\sum_{j=1}^{k-1} a_{4j} a_{2k-2j}^{[\sqrt{2}]} +$
$+ \sum_{j=1}^{k-1} a_{4j} a_{k-j}^{[2]} + \sum_{j=1}^{2k-1} (-1)^j a_j a_{4k-j} + \sum_{j=1}^{k-1} a_{2j}^{[\sqrt{2}]} a_{2k-2j}^{[\sqrt{2}]} + \frac{1}{2}\left(a_{k+1}^2 - a_{k+1}^{[2]} + a_{2k}^2 + a_{2k}^{[2]}\right)$

3) $a_{4k+2} = \sum_{j=1}^{k} a_j a_{2k+1-j} + a_{2k+2}$

4) $a_{4k+3} = a_{2k+4} - a_2 a_{2k+1} - \frac{1}{2}\left(a_{2k+1}^2 - a_{2k+1}^{[2]}\right) + \sum_{j=1}^{2k} (-1)^j a_j a_{4k+2-j} + a_{2k+2}^{[\sqrt{2}]} +$
$+ \sum_{j=1}^{k} a_{4j-2} a_{k+1-j}^{[2]} + \sum_{j=1}^{k+1} a_j a_{2k+3-j} + 2\sum_{j=1}^{2k} a_{2j} a_{2k+1-j}^{[\sqrt{2}]} + \sum_{j=1}^{k} a_j^{[\sqrt{2}]} a_{2k+1-j}^{[\sqrt{2}]}$

2.4 The Baby-Monster Lie Algebra and $2A$-replication

The moonshine module $V^\natural$ was constructed in [28] by Frenkel, Lepwosky and Meurman. This is a vertex operator algebra that has $\mathbb{M}$, the Monster group, as symmetry group. A vertex operator algebra is an intricate algebraic structure and we refer to [40] for the definition and the basics of its theory. The moonshine module has a grading $V^\natural = \bigoplus_{n\geq -1} V^\natural_{(n)}$ and its graded dimension $\sum_{n\geq -1}\left(\dim V^\natural_{(n)}\right) q^n$ is the J-function. Borcherds ([7]) uses this vertex operator algebra to prove the Moonshine Conjectures in the following way. First, he shows that the McKay-Thompson series for $V^\natural$, i.e. $T_g = \sum_{n\geq -1} Tr(g|V^\natural_{(n)})q^n$ are completely replicable functions. To do this, he builds a generalized Kac-Moody algebra, the Monster Lie algebra, whose denominator identity is essentialy the statement that the n-th replicate of a T_g is T_{g^n}. Knowing that Hauptmoduls for genus-zero congruence groups are also completely replicable functions and that completely replicable functions satisfy some recurrence relations that determine a function from the first 5 coefficients of the function and its replicates, he was able to show that every McKay-Thompson series is indeed a Hauptmodul for some genus-zero congruence groups by just comparing the first few coefficients of the functions involved.

In [34], Höhn shows that there is a vertex operator algebra W that has $2\cdot\mathbb{B}$ as symmetry group. This group is a central extension of $\mathbb{B}$, the Baby Monster group, and it arises as the centralizer of an element of class $2A$ in $\mathbb{M}$. This vertex operator algebra plays for $2\cdot\mathbb{B}$ the role that $V^\natural$ plays for the $\mathbb{M}$ and it was used to prove the generalized Moonshine conjectures for the case of the Baby Monster, i.e., when g (see introduction to this chapter) is an involution of type $2A$ in $\mathbb{M}$. In this section, we use t to represent a (fixed) element in class $2A$ in $\mathbb{M}$. More precisely, what Höhn states in [34] is the following. If

$$V^\natural(t) = \bigoplus_{n\geq -1} V^\natural_{(\frac{n}{2})}(t)$$

is the t-twisted module and h is an element in the centralizer of t in $\mathbb{M}$ then the McKay-

Thompson series

$$T_{t,g} = \sum_{n \geq -1} Tr(g|V^{\natural}_{(\frac{n}{2})}(t))q^n$$

is the Hauptmodul for some genus zero congruence subgroup.

We give a very brief sketch of the results from [34]. From the decompositions of $V^{\natural} = V^{00} \bigoplus V^{01}$ and $V^{\natural}(t) = V^{10} \bigoplus V^{11}$ of $+1$ and -1 eigenspaces for t, Höhn builds the vertex algebra W mentioned above on which $2 \cdot \mathbb{B}$ acts. Using this vertex algebra W a Lie algebra $g^{\natural}_{\mathbb{B}}$, the Baby Monster vertex algebra, is constructed too. This is a $\frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z}$-graded Lie algebra that has an action of $2 \cdot \mathbb{B}$ in it that respects the grading and its $\left(\frac{m}{2}, \frac{n}{2}\right)$ piece is isomorphic to $V^{[m,n]}_{(2mn)}$ ($[\cdot,\cdot]$ represents reduction $\mod 2$). Höhn also shows that this is a generalized Kac-Moody algebra with twisted denominator formula:

$$\sum_{m \in \mathbb{Z}} \mathrm{Tr}\left(g|V^{[m,1]}_{(\frac{m}{2})}\right) p^{\frac{m}{2}} - \sum_{n \in \mathbb{Z}} \mathrm{Tr}\left(g|V^{[1,n]}_{(\frac{n}{2})}\right) q^{\frac{n}{2}} = p^{-\frac{1}{2}} \exp\left(-\sum_{i>0} \sum_{\substack{m \in \mathbb{Z}^+ \\ n \in \mathbb{Z}}} \mathrm{Tr}\left(g^i|V^{[m,n]}_{(\frac{mn}{2})}\right) \frac{p^{\frac{im}{2}} q^{\frac{in}{2}}}{i}\right) \tag{2.4.1}$$

We can now state our main result.

Theorem 2.4.1. *The McKay-Thompson series $T_{t,g}(z)$, for $g \in 2 \cdot \mathbb{B}$, are completely 2A-replicable with replicates $T^{[n]}_{t,g} = T_{t,g^n}$ and $T^{[\sqrt{2}n]}_{t,g} = T_{1,g^n t}$.*

Proof. From the twisted denominator identity for the Baby Monster Lie algebra, 2.4.1, we have, after substituting $p^{\frac{1}{2}}$ for p and $q^{\frac{1}{2}}$ for q:

$$\sum_{m \in \mathbb{Z}} \mathrm{Tr}\left(g|V^{[m,1]}_{(\frac{m}{2})}\right) p^{m} - \sum_{n \in \mathbb{Z}} \mathrm{Tr}\left(g|V^{[1,n]}_{(\frac{n}{2})}\right) q^{n} = p^{-1} \exp\left(-\sum_{i>0} \sum_{\substack{m \in \mathbb{Z}^+ \\ n \in \mathbb{Z}}} \mathrm{Tr}\left(g^i|V^{[m,n]}_{(\frac{mn}{2})}\right) \frac{p^{im} q^{in}}{i}\right)$$

The right hand side of this equation is

$$p^{-1} \exp\left(-\left(\sum_{i>0} \sum_{\substack{m \in \mathbb{Z}^+ \\ n \in \mathbb{Z} \\ m \text{ or } n \text{ odd}}} \mathrm{Tr}\left(g^i|V^{[1,mn]}_{(\frac{mn}{2})}\right) \frac{p^{im} q^{in}}{i} + \sum_{i>0} \sum_{\substack{m \in \mathbb{Z}^+ \\ n \in \mathbb{Z} \\ m \text{ and } n \text{ even}}} \mathrm{Tr}\left(g^i t|V^{00}_{(\frac{mn}{2})}\right) \frac{p^{im} q^{in}}{i}\right)\right)$$

$$= p^{-1}\exp\left(-\left(\sum_{i>0}\sum_{\substack{m\in\mathbb{Z}^+\\ n\in\mathbb{Z}\\ m \text{ or } n \text{ odd}}} \mathrm{Tr}\left(g^i|V^{[1,mn]}_{\left(\frac{mn}{2}\right)}\right)\frac{p^{im}q^{in}}{i}+\right.\right.$$

$$\left.\left.+\sum_{i>0}\sum_{\substack{m\in\mathbb{Z}^+\\ n\in\mathbb{Z}\\ m \text{ and } n \text{ even}}}\left(\mathrm{Tr}\left(g^it|V^{\natural}_{\left(\frac{mn}{2}\right)}\right)-\mathrm{Tr}\left(g^it|V^{01}_{\left(\frac{mn}{2}\right)}\right)\right)\frac{p^{im}q^{in}}{i}\right)\right)$$

$$p^{-1}\exp\left(-\left(\sum_{i>0}\sum_{\substack{m\in\mathbb{Z}^+\\ n\in\mathbb{Z}\\ m \text{ or } n \text{ odd}}} \mathrm{Tr}\left(g^i|V^{[1,mn]}_{\left(\frac{mn}{2}\right)}\right)\frac{p^{im}q^{in}}{i}+\right.\right.$$

$$\left.\left.+\sum_{i>0}\sum_{\substack{m\in\mathbb{Z}^+\\ n\in\mathbb{Z}\\ m \text{ and } n \text{ even}}}\left(\mathrm{Tr}\left(g^it|V^{\natural}_{\left(\frac{mn}{2}\right)}\right)+\mathrm{Tr}\left(g^i|V^{01}_{\left(\frac{mn}{2}\right)}\right)\right)\frac{p^{im}q^{in}}{i}\right)\right)$$

$$= p^{-1}\exp\left(-\left(\sum_{i>0}\sum_{\substack{m\in\mathbb{Z}^+\\ n\in\mathbb{Z}}}\mathrm{Tr}\left(g^i|V^{[1,mn]}_{\left(\frac{mn}{2}\right)}\right)\frac{p^{im}q^{in}}{i}+\sum_{i>0}\sum_{\substack{m\in\mathbb{Z}^+\\ n\in\mathbb{Z}}}\mathrm{Tr}\left(g^it|V^{\natural}_{(2mn)}\right)\frac{p^{2im}q^{2in}}{i}\right)\right)$$

$$= p^{-1}\exp\left(-\left(\sum_{n=1}^{+\infty}\frac{1}{n}\left(\sum_{ad=n}d\cdot\sum_{k\in\mathbb{Z}}\mathrm{Tr}\left(g^a|V^{[1,kd]}_{\left(\frac{kd}{2}\right)}\right)q^{ak}+\sum_{\substack{ad=n\\ d\text{ even}}}d\cdot\sum_{k\in\mathbb{Z}}\mathrm{Tr}\left(g^at|V^{\natural}_{\left(\frac{kd}{2}\right)}\right)q^{2ak}\right)p^n\right)\right)$$

$$= p^{-1}\exp\left(-\sum_{n=1}^{+\infty}\frac{1}{n}\left(\sum_{\substack{ad=n\\ 0\le b<d}}T_{t,g^a}\left(\frac{a\tau+b}{d}\right)+\sum_{\substack{ad=n\\ d\text{ even}\\ 0\le b<d}}T_{1,g^at}\left(\frac{2a\tau+b}{d}\right)\right)p^n\right)$$

Since

$$T_{t,g}(p)-T_{t,g}(q)=p^{-1}\exp\left(-\sum_{n=1}^{+\infty}\frac{1}{n}P_n(T_{t,g}(q))p^n\right)$$

we conclude that, for all $n \in \mathbb{N}$,

$$\sum_{\substack{ad=n \\ 0\le b<d}} T_{t,g^a}\left(\frac{a\tau+b}{d}\right) + \sum_{\substack{ad=n \\ d \text{ even} \\ 0\le b<d}} T_{1,g^a t}\left(\frac{2a\tau+b}{d}\right) = P_n(T_{t,g}(q))$$

and we get that $T_{t,g}$ is $2A$-replicable with $2A$-replicates given as stated in the theorem.

By substituting g by g^n we see that T_{t,g^n} is $2A$-replicable with replicates $T_{t,g^n}^{[m]} = T_{t,g^{mn}} = T_{t,g}^{[mn]}$ and $T_{t,g^n}^{[\sqrt{2}m]} = T_{1,g^{mn}t} = T_{t,g}^{[\sqrt{2}nm]}$, i.e. $(T_{t,g}^{[n]})^{[m]} = T_{t,g}^{[mn]}$ and $(T_{t,g}^{[n]})^{[\sqrt{2}m]} = T_{t,g}^{[\sqrt{2}nm]}$. To complete the proof it remains to see that $T_{t,g}^{[\sqrt{2}n]} = T_{1,g^n t}$ is $2A$-replicable with replicates $(T_{t,g}^{[\sqrt{2}n]})^{[m]} = T_{1,g^{nm}t}$ and $(T_{t,g}^{[\sqrt{2}n]})^{[\sqrt{2}m]} = T_{t,g^{2nm}}$, for $m \in \mathbb{N}$. Equivalently, what we need to prove is that, for every $m \in \mathbb{N}$,

$$\sum_{\substack{ad=n \\ 0\le b<d}} T_{1,g^{ma}t}\left(\frac{a\tau+b}{d}\right) + \sum_{\substack{ad=n \\ d \text{ even} \\ 0\le b<d}} T_{t,g^{2ma}}\left(\frac{2a\tau+b}{d}\right) = P_n(T_{1,g^m t}(q))$$

But since $T_{1,g^m t}$ is a monstrous function we know that

$$\sum_{\substack{ad=n \\ 0\le b<d}} T_{1,(g^m t)^a}\left(\frac{a\tau+b}{d}\right) = P_n(T_{1,g^m t}(q))$$

But now,

$$\sum_{\substack{ad=n \\ 0\le b<d}} T_{1,g^{ma}t}\left(\frac{a\tau+b}{d}\right) = n\sum_{k\in\mathbb{N}}\left(\sum_{a|(n,k)} \frac{1}{a} Tr\left(g^{ma}t|V^{\natural}_{\left(\frac{nk}{a^2}\right)}\right)\right) q^k$$

$$\sum_{\substack{ad=n \\ d \text{ even} \\ 0\le b<d}} T_{t,g^{2ma}}\left(\frac{2a\tau+b}{d}\right) = n\sum_{k\in\mathbb{N}}\left(\sum_{\substack{a|(n,k) \\ \frac{n}{a}\text{even}}} \frac{1}{a} Tr\left(g^{2ma}|V^{[1,\frac{kn}{a^2}]}_{\left(\frac{nk}{2a^2}\right)}\right)\right) q^{2k}$$

$$\sum_{\substack{ad=n \\ 0\le b<d}} T_{1,(g^m t)^a}\left(\frac{a\tau+b}{d}\right) = n\sum_{k\in\mathbb{N}}\left(\sum_{a|(n,k)} \frac{1}{a} Tr\left((g^m t)^a|V^{\natural}_{\left(\frac{nk}{a^2}\right)}\right)\right) q^k$$

and what we have to show is

$$\sum_{k\in\mathbb{N}}\left(\sum_{a|(n,k)}\frac{1}{a}Tr\left(g^{ma}t|V^{\natural}_{\left(\frac{nk}{a^2}\right)}\right)\right)q^k+\sum_{k\in\mathbb{N}}\left(\sum_{\substack{a|(n,k)\\ \frac{n}{a}\text{even}}}\frac{1}{a}Tr\left(g^{2ma}|V^{[1,\frac{kn}{a^2}]}_{\left(\frac{nk}{2a^2}\right)}\right)\right)q^{2k}=$$

$$=\sum_{k\in\mathbb{N}}\left(\sum_{a|(n,k)}\frac{1}{a}Tr\left((g^mt)^a|V^{\natural}_{\left(\frac{nk}{a^2}\right)}\right)\right)q^k$$

This means that for k odd we have to show that

$$\sum_{a|(n,k)}\frac{1}{a}Tr\left(g^{ma}t|V^{\natural}_{\left(\frac{nk}{a^2}\right)}\right)=\sum_{a|(n,k)}\frac{1}{a}Tr\left((g^mt)^a|V^{\natural}_{\left(\frac{nk}{a^2}\right)}\right)$$

which is true because t and g commute and every a in the sum, being a divisor of k, is odd too.

For $k=2k'$ even we have to show that

$$\sum_{a|(n,2k')}\frac{1}{a}Tr\left(g^{ma}t|V^{\natural}_{\left(\frac{2nk'}{a^2}\right)}\right)+\sum_{\substack{a|(n,k')\\ \frac{n}{a}\text{even}}}\frac{1}{a}Tr\left(g^{2ma}|V^{[1,\frac{k'n}{a^2}]}_{\left(\frac{nk'}{2a^2}\right)}\right)=\sum_{a|(n,2k')}\frac{1}{a}Tr\left((g^mt)^a|V^{\natural}_{\left(\frac{2nk'}{a^2}\right)}\right)$$

This identity is clearly true for n odd and for $n=2n'$ even it becomes

$$\sum_{a|2(n',k')}\frac{1}{a}Tr\left(g^{ma}t|V^{\natural}_{\left(\frac{4n'k'}{a^2}\right)}\right)+\sum_{\substack{a|(2n',k')\\ \frac{2n'}{a}\text{even}}}\frac{1}{a}Tr\left(g^{2ma}|V^{01}_{\left(\frac{n'k'}{a^2}\right)}\right)=\sum_{a|2(n',k')}\frac{1}{a}Tr\left((g^mt)^a|V^{\natural}_{\left(\frac{4n'k'}{a^2}\right)}\right)$$

and this is now easy to prove

$$\sum_{a|2(n',k')}\frac{1}{a}Tr\left(g^{ma}t|V^{\natural}_{\left(\frac{4n'k'}{a^2}\right)}\right)+\sum_{\substack{a|(2n',k')\\ \frac{2n'}{a}\text{even}}}\frac{1}{a}Tr\left(g^{2ma}|V^{01}_{\left(\frac{n'k'}{a^2}\right)}\right)=$$

$$= \sum_{a|2(n',k')} \frac{1}{a} Tr\left(g^{ma}t|V^{\natural}_{\left(\frac{4n'k'}{a^2}\right)}\right) + \sum_{a|(n',k')} \frac{1}{a} Tr\left(g^{2ma}|V^{01}_{\left(\frac{4n'k'}{(2a)^2}\right)}\right) =$$

$$= \sum_{a|2(n',k')} \frac{1}{a} Tr\left(g^{ma}t|V^{\natural}_{\left(\frac{4n'k'}{a^2}\right)}\right) - \sum_{\substack{a|2(n',k') \\ a \text{ even}}} \frac{2}{a} Tr\left(g^{ma}t|V^{01}_{\left(\frac{4n'k'}{a^2}\right)}\right) =$$

$$= \sum_{a|2(n',k')} \frac{1}{a} Tr\left(g^{ma}t|V^{00}_{\left(\frac{4n'k'}{a^2}\right)}\right) + \sum_{\substack{a|2(n',k') \\ a \text{ even}}} \frac{1}{a} Tr\left(g^{ma}t|V^{01}_{\left(\frac{4n'k'}{a^2}\right)}\right) +$$

$$+ \sum_{\substack{a|2(n',k') \\ a \text{ odd}}} \frac{1}{a} Tr\left(g^{ma}t|V^{01}_{\left(\frac{4n'k'}{a^2}\right)}\right) - \sum_{\substack{a|2(n',k') \\ a \text{ even}}} \frac{2}{a} Tr\left(g^{ma}t|V^{01}_{\left(\frac{4n'k'}{a^2}\right)}\right) =$$

$$= \sum_{a|2(n',k')} \frac{1}{a} Tr\left((g^{m}t)^a|V^{00}_{\left(\frac{4n'k'}{a^2}\right)}\right) + \sum_{\substack{a|2(n',k') \\ a \text{ even}}} \frac{1}{a} Tr\left((g^{m}t)^a|V^{01}_{\left(\frac{4n'k'}{a^2}\right)}\right) +$$

$$+ \sum_{\substack{a|2(n',k') \\ a \text{ odd}}} \frac{1}{a} Tr\left((g^{m}t)^a|V^{01}_{\left(\frac{4n'k'}{a^2}\right)}\right) = \sum_{a|2(n',k')} \frac{1}{a} Tr\left((g^{m}t)^a|V^{\natural}_{\left(\frac{4n'k'}{a^2}\right)}\right)$$

And the theorem is proven.

At this point we know that the McKay-Thompson series $T_{t,g}$ are $2A$-completely replicable and consequently satisfy the recurrence relations from Proposition 2.3.5. We know that $T_{t,g}^{[n\sqrt{2}]} = T_{1,tg^n}$ is a Monstrous function and therefore its coefficients are known once we know in what class in the Monster the element tg is, for every $g \in 2 \cdot \mathbb{B}$. This can be done in GAP. Hence, the first five coefficients of every $T_{t,g}$ determine the coefficients of all $T_{t,g}$ completely. From subsection 2.2.2 we also have some completely $2A$-replicability results for some Hauptmoduls and thus these Hauptmoduls satisfy the same recurrence relations from Proposition 2.3.5. To prove that every McKay-Thompson series is a Hauptmodul it is enough to compare, for every $g \in 2 \cdot \mathbb{B}$, the first five coefficients of $T_{t,g}$, $T_{1,tg}$, T_{t,g^2}, $T_{1,tg^2}, \ldots$ with those of f, $f^{[\sqrt{2}]}$, $f^{[2]}$, $f^{[2\sqrt{2}]}, \ldots$, respectively, for some Hauptmodul f in tables 2.1 and 2.2. This is analogous to Borcherds proof of the original Moonshine Conjectures.

The decomposition of the first five head characters is given in [34]:

- $H_1 = \chi_1 + \chi_2$

- $H_2 = \chi_{185}$

- $H_3 = 2\chi_1 + \chi_2 + \chi_3 + \chi_4$

- $H_4 = 2\chi_{185} + \chi_{186}$

- $H_5 = 3\chi_1 + 3\chi_2 + 2\chi_3 + \chi_4 + \chi_6 + \chi_7$

However, this works well for all 247 classes in $2 \cdot \mathbb{B}$ but 13 of them. This happens because the Hauptmodul associated to each of these 13 classes is neither a completely replicable function nor a dash (see [26] for the definition of the dash operator) of a completely replicable function and our tables 2.1 and 2.2 only contain such functions. The names of these 13 classes are, using GAP notation: $12h$, $12i$, $20h$, $20i$, $24i$, $24m$, $24n$, $36d$, $36e$, $40f$, $40g$, $60d$ and $60e$. To solve this problem we use again the same recurrence relations from Proposition 2.3.5 to find the first 23 coefficients. Since we know that a replicable function (we recall from remark 2.2.4 that a $2A$-replicable function is replicable) is completely determined by its first 23 coefficients, a simple check among the power series expansions of the 616 Hauptmoduls for genus zero groups with rational integer coefficients allows us to match every such class with some Hauptmodul.

The correspondence is given in table 2.3.

Class (Square)	Function	Sq. R. Func.	Class (Square)	Function	Sq. R. Func.
$1a$ $(1a)$	$2A$	$2A$	$2a$ $(1a)$	$4\sim b$	$1A$
$2b$ $(1a)$	$2a$	$2A$	$2c$ $(1a)$	$4A$	$2B$
$2d$ $(1a)$	$2B$	$2A$	$2e$ $(1a)$	$4C$	$2B$
$3a$ $(3a)$	$6A$	$6A$	$3b$ $(3b)$	$6D$	$6D$
$4a$ $(2a)$	$8\sim b$	$4B$	$4b$ $(2d)$	$4a$	$4A$
$4c$ $(2d)$	$4B$	$4A$	$4d$ $(2d)$	$4C$	$4C$
$4e$ $(2c)$	$8a$	$4B$	$4f$ $(2e)$	$8A$	$4C$
$4g$ $(2e)$	$8\sim d$	$4A$	$4h$ $(2d)$	$4D$	$4C$
$4i$ $(2c)$	$8B$	$4B$	$4j$ $(2e)$	$8E$	$4C$
$4k$ $(2e)$	$8D$	$4D$	$5a$ $(5a)$	$10A$	$10A$
$5b$ $(5b)$	$10C$	$10C$	$6a$ $(3a)$	$12\sim d$	$3A$
$6b$ $(3b)$	$12\sim f$	$3B$	$6c$ $(3a)$	$6a$	$6A$
$6d$ $(3a)$	$6b$	$6A$	$6e$ $(3a)$	$12A$	$6C$
$6f$ $(3a)$	$6C$	$6A$	$6g$ $(3b)$	$6c$	$6D$
$6h$ $(3a)$	$12c$	$6C$	$6i$ $(3b)$	$12B$	$6E$
$6j$ $(3b)$	$6E$	$6D$	$6k$ $(3a)$	$12E$	$6C$
$6l$ $(3b)$	$12H$	$6E$	$6m$ $(3b)$	$12\sim h$	$6B$
$6n$ $(3b)$	$12I$	$6E$	$7a$ $(7a)$	$14A$	$14A$
$8a$ $(4a)$	$16\sim a$	$8C$	$8b$ $(4d)$	$8a$	$8A$
$8c$ $(4c)$	$8b$	$8B$	$8d$ $(4c)$	$8c$	$8B$
$8e$ $(4d)$	$8B$	$8A$	$8f$ $(4c)$	$8C$	$8B$
$8g$ $(4d)$	$8D$	$8E$	$8h$ $(4g)$	$16\sim d$	$8D$
$8i$ $(4d)$	$8E$	$8E$	$8j$ $(4h)$	$8F$	$8D$
$8k$ $(4f)$	$16A$	$8B$	$8l$ $(4j)$	$16C$	$8E$
$8m$ $(4j)$	$16\sim e$	$8A$	$8n$ $(4i)$	$16a$	$8C$
$8o$ $(4j)$	$16B$	$8E$	$8p$ $(4k)$	$16d$	$8F$
$9a$ $(9a)$	$18A$	$18A$	$9b$ $(9b)$	$18B$	$18B$
$10a$ $(5a)$	$20\sim c$	$5A$	$10b$ $(5b)$	$20\sim d$	$5B$
$10c$ $(5a)$	$10a$	$10A$	$10d$ $(5a)$	$20A$	$10B$
$10e$ $(5a)$	$10B$	$10A$	$10f$ $(5b)$	$20C$	$10E$
$10g$ $(5b)$	$10E$	$10C$	$10h$ $(5a)$	$20d$	$10B$
$10i$ $(5b)$	$20F$	$10E$	$10j$ $(5b)$	$20\sim g$	$10D$
$11a$ $(11a)$	$22A$	$22A$	$12a$ $(6a)$	$24\sim f$	$12C$
$12b$ $(6b)$	$24\sim h$	$12G$	$12c$ $(6f)$	$12a$	$12A$
$12d$ $(6j)$	$12G$	$12B$	$12e$ $(6f)$	$12b$	$12A$
$12f$ $(6f)$	$12C$	$12A$	$12g$ $(6e)$	$24a$	$12C$
$12h$ $(6m)$	$24\sim j$	$12H$	$12i$ $(6m)$	$24\sim k$	$12B$
$12j$ $(6f)$	$12E$	$12E$	$12k$ $(6e)$	$24b$	$12C$
$12l$ $(6f)$	$12d$	$12E$	$12m$ $(6k)$	$24B$	$12E$
$12n$ $(6k)$	$24\sim m$	$12A$	$12o$ $(6i)$	$24c$	$12G$
$12p$ $(6e)$	$24E$	$12C$	$12q$ $(6j)$	$12I$	$12I$
$12r$ $(6n)$	$24C$	$12I$	$12s$ $(6n)$	$24\sim o$	$12B$

Class (Square)	Function	Sq. R. Func.	Class (Square)	Function	Sq. R. Func.
$12t\,(6j)$	$12F$	$12H$	$12u\,(6k)$	$24h$	$12E$
$12v\,(6m)$	$24\sim q$	$12I$	$12w\,(6l)$	$24H$	$12F$
$12x\,(6n)$	$24I$	$12I$	$12y\,(6n)$	$24\sim r$	$12H$
$13a\,(13a)$	$26A$	$26A$	$14a\,(7a)$	$28\sim c$	$7A$
$14b\,(7a)$	$14a$	$14A$	$14c\,(7a)$	$14c$	$14A$
$14d\,(7a)$	$28B$	$14B$	$14e\,(7a)$	$14B$	$14A$
$14f\,(7a)$	$28C$	$14B$	$15a\,(15a)$	$30B$	$30B$
$15b\,(15b)$	$30F$	$30F$	$16a\,(8e)$	$16b$	$16A$
$16b\,(8e)$	$16c$	$16A$	$16c\,(8i)$	$16B$	$16B$
$16d\,(8i)$	$16A$	$16C$	$16e\,(8e)$	$16a$	$16A$
$16f\,(8i)$	$16A$	$16C$	$16g\,(8l)$	$32B$	$16A$
$16h\,(8o)$	$32A$	$16B$	$16i\,(8o)$	$32\sim e$	$16C$
$17a\,(17a)$	$34A$	$34A$	$18a\,(9a)$	$36\sim h$	$9B$
$18b\,(9b)$	$36\sim e$	$9A$	$18c\,(9b)$	$18c$	$18B$
$18d\,(9b)$	$18c$	$18B$	$18e\,(9b)$	$36A$	$18C$
$18f\,(9b)$	$18C$	$18B$	$18g\,(9b)$	$36f$	$18C$
$18h\,(9a)$	$36D$	$18D$	$18i\,(9a)$	$36\sim q$	$18E$
$19a\,(19a)$	$38A$	$38A$	$20a\,(10a)$	$40\sim c$	$20B$
$20b\,(10e)$	$20a$	$20A$	$20c\,(10g)$	$20c$	$20C$
$20d\,(10e)$	$20b$	$20A$	$20e\,(10e)$	$20B$	$20A$
$20f\,(10d)$	$40a$	$20B$	$20g\,(10d)$	$40B$	$20B$
$20h\,(10j)$	$40\sim h$	$20F$	$20i\,(10j)$	$40\sim i$	$20C$
$20j\,(10g)$	$20E$	$20F$	$20k\,(10i)$	$40C$	$20E$
$21a\,(21a)$	$42A$	$42A$	$22a\,(11a)$	$44\sim b$	$11A$
$22b\,(11a)$	$22a$	$22A$	$22c\,(11a)$	$22a$	$22A$
$22d\,(11a)$	$44A$	$22B$	$22e\,(11a)$	$22B$	$22A$
$23a\,(23a)$	$46C$	$46C$	$23b\,(23b)$	$46C$	$46C$
$24a\,(12b)$	$48\sim c$	$24G$	$24b\,(12f)$	$24d$	$24A$
$24c\,(12f)$	$24e$	$24A$	$24d\,(12f)$	$24g$	$24A$
$24e\,(12f)$	$24f$	$24A$	$24f\,(12j)$	$24b$	$24B$
$24g\,(12q)$	$24c$	$24C$	$24h\,(12j)$	$24A$	$24B$
$24i\,(12i)$	$48\sim h$	$24H$	$24j\,(12n)$	$48\sim i$	$24D$
$24k\,(12m)$	$48A$	$24A$	$24l\,(12q)$	$24H$	$24I$
$24m\,(12v)$	$48\sim j$	$24I$	$24n\,(12v)$	$48\sim k$	$24C$
$24o\,(12t)$	$24F$	$24H$	$24p\,(12w)$	$48h$	$24F$
$25a\,(25a)$	$50A$	$50A$	$26a\,(13a)$	$52\sim c$	$13A$
$26b\,(13a)$	$26a$	$26A$	$27a\,(27a)$	$54A$	$54A$
$28a\,(14a)$	$56\sim d$	$28A$	$28b\,(14e)$	$28A$	$28B$
$28c\,(14e)$	$28C$	$28C$	$28d\,(14e)$	$28a$	$28B$
$28e\,(14d)$	$56a$	$28A$	$28f\,(14f)$	$56A$	$28C$
$28g\,(14f)$	$56\sim g$	$28B$	$30a\,(15a)$	$60\sim c$	$15A$
$30b\,(15b)$	$60\sim l$	$15C$	$30c\,(15a)$	$30a$	$30B$

Class (Square)	Function	Sq. R. Func.	Class (Square)	Function	Sq. R. Func.
$30d\,(15a)$	$30d$	$30B$	$30e\,(15a)$	$60B$	$30C$
$30f\,(15a)$	$30C$	$30B$	$30g\,(15a)$	$60a$	$30C$
$30h\,(15b)$	$60D$	$30G$	$30i\,(15b)$	$60\sim m$	$30A$
$30j\,(15b)$	$60C$	$30G$	$30k\,(15b)$	$30G$	$30F$
$30l\,(15b)$	$60C$	$30G$	$30m\,(15b)$	$30G$	$30F$
$31a\,(31a)$	$62A$	$62A$	$31b\,(31b)$	$62A$	$62A$
$32a\,(16c)$	$32B$	$32A$	$32b\,(16c)$	$32B$	$32A$
$32c\,(16d)$	$32b$	$32B$	$32d\,(16d)$	$32b$	$32B$
$33a\,(33a)$	$66A$	$66A$	$34a\,(17a)$	$68\sim b$	$17A$
$34b\,(17a)$	$34a$	$34A$	$34c\,(17a)$	$34a$	$34A$
$35a\,(35a)$	$70A$	$70A$	$36a\,(18b)$	$72\sim c$	$36C$
$36b\,(18f)$	$36C$	$36A$	$36c\,(18e)$	$72a$	$36C$
$36d\,(18i)$	$72\sim p$	$36D$	$36e\,(18i)$	$72\sim q$	$36B$
$38a\,(19a)$	$76\sim b$	$19A$	$38b\,(19a)$	$38a$	$38A$
$38c\,(19a)$	$38a$	$38A$	$39a\,(39a)$	$78A$	$78A$
$40a\,(20a)$	$80\sim a$	$40A$	$40b\,(20e)$	$40b$	$40B$
$40c\,(20e)$	$40c$	$40B$	$40d\,(20e)$	$40A$	$40B$
$40e\,(20g)$	$80e$	$40A$	$40f\,(20i)$	$80\sim e$	$40C$
$40g\,(20i)$	$80\sim e$	$40C$	$42a\,(21a)$	$84\sim e$	$21A$
$42b\,(21a)$	$42a$	$42A$	$42c\,(21a)$	$42b$	$42A$
$44a\,(22e)$	$44c$	$44A$	$44b\,(22e)$	$44c$	$44A$
$46a\,(23a)$	$92\sim b$	$23A$	$46b\,(23b)$	$92\sim b$	$23A$
$46c\,(23c)$	$92A$	$46A$	$46d\,(23a)$	$46A$	$46C$
$46e\,(23b)$	$92A$	$46A$	$46f\,(23b)$	$46A$	$46C$
$47a\,(47a)$	$94A$	$94A$	$47b\,(47b)$	$94A$	$94A$
$48a\,(24h)$	$48a$	$48A$	$48b\,(24h)$	$48b$	$48A$
$50a\,(25a)$	$100\sim c$	$25A$	$52a\,(26a)$	$104\sim c$	$52A$
$54a\,(27a)$	$108\sim g$	$27A$	$55a\,(55a)$	$110A$	$110A$
$56a\,(28c)$	$56a$	$56A$	$56b\,(28c)$	$56a$	$56A$
$60a\,(30a)$	$120\sim d$	$60A$	$60b\,(30f)$	$60b$	$60B$
$60c\,(30e)$	$120a$	$60A$	$60d\,(30i)$	$120\sim g$	$60D$
$60e\,(30i)$	$120\sim h$	$60C$	$62a\,(31a)$	$124\sim b$	$31A$
$62b\,(31b)$	$124\sim b$	$31A$	$66a\,(33a)$	$132\sim c$	$33B$
$66b\,(33a)$	$66a$	$66A$	$66c\,(33a)$	$66a$	$66A$
$68a\,(34a)$	$136\sim b$	$68A$	$70a\,(35a)$	$140\sim b$	$35A$
$70b\,(35a)$	$70a$	$70A$	$70c\,(35a)$	$70a$	$70A$
$78a\,(39a)$	$156\sim d$	$39A$	$84a\,(42a)$	$168\sim c$	$84A$
$94a\,(47a)$	$188\sim b$	$47A$	$94b\,(47a)$	$188\sim b$	$47A$
$104a\,(52a)$	$208\sim a$	$104A$	$104b\,(52a)$	$208\sim a$	$104A$
$110a\,(55a)$	$220\sim b$	$55A$			

Table 2.3: Classes in $2\cdot\mathbb{B}$ and corresponding McKay-Thompson series together with their $\sqrt{2}$-replicates

2.5 Conclusion

In this chapter we have proved that the McKay-Thompson series for $V^\natural(t)$ are the Hauptmoduls given in [34]. The approach taken here resembles Borcherds proof of the Moonshine conjectures except for the last part where we had to compute the first 23 coefficients of certain 13 functions that are neither completely replicable nor the dash of a completely replicable function. For those classes we applied the same strategy Höhn used in [34] to identify the McKay-Thompson series. However, the computations are now much simpler because we have generalized Mahler recurrence relations coming from $2A$-replication. It also shows that the action of the Hecke Operators for $\Gamma_0(2)+$ are, through $2A$-replication, somehow reflected in the structure of the vertex algebra W. As it happens for the Monster group and ordinary replication, $2A$-replication respects the power map structure in $2 \cdot \mathbb{B}$.

It is also interesting to see that some Hauptmoduls that are not completely replicable, are completely $2A$-replicable. This gives an expression for their 2-replicates using remark 2.2.4.

A natural question to ask is whether we can extend these ideas to other groups arising from centralizers of Fricke elements in $\mathbb{M}$. This is the subject of the next chapter.

Chapter 3

3A-replication and $3 \cdot F_{3+}$

3.1 Introduction

In this chapter we apply the ideas from the preceding chapter, namely defining $3A$-replicability, and we show that in this case it is possible to associate to every conjugacy class of $3 \cdot F_{3+}$, the centralizer of an element of class $3A$ in $\mathbb{M}$, a Hauptmodul for some genus-zero congruence subgroup of $PSL_2(\mathbb{R})$. This correspondence has the property that $3A$-replicability respects the power map structure of $3 \cdot F_{3+}$.

3.2 3A-replicability

As in the previous section we define $3A$ replicability by looking at the algebra of Hecke Operators for $\Gamma_0(3)+$. It is easy to see that most of the results from subsection 2.2.1 carry over to this situation and we make the following definition:

Definition 3.2.1. *A function f is $3A$-replicable if there are $f^{[n]}$ and $f^{[n\sqrt{3}]}$, for $n \in \mathbb{N}$, such that*

$$P_{n,f}(f) = \sum_{\substack{ad=n \\ 0 \le b < d}} f^{[a]}\left(\frac{a\tau + b}{d}\right) + \sum_{\substack{ad=n \\ 3|d \\ 0 \le b < d}} f^{\left[a\sqrt{3}\right]}\left(\frac{3a\tau + b}{d}\right) \tag{3.2.1}$$

As in the case of $2A$-replication we can make the following remarks.

Remark 3.2.1. *Given a 3A-replicable function f, its 3A-replicates are not uniquely determined. For example, for $m = 3$, equation 3.2.1 becomes*

$$f^{[3]}(3\tau) + f^{[\sqrt{3}]}(\tau) + f^{[\sqrt{3}]}\left(\tau + \frac{1}{3}\right) + f^{[\sqrt{3}]}\left(\tau + \frac{2}{3}\right) + \\ + f\left(\frac{\tau}{3}\right) + f\left(\frac{\tau+1}{3}\right) + f\left(\frac{\tau+2}{3}\right) = P_{3,f}(f)$$

and we can see that $f^{[3]}$ is known when f and $f^{[\sqrt{3}]}$ are known.

Remark 3.2.2. *If a function is 3A-replicable then it is replicable with*

$$f^{(n)}(z) = \begin{cases} f^{[n]}(z), & 3 \nmid n \\ f^{[n]}(z) + f^{[\frac{n}{\sqrt{3}}]}(\frac{z}{3}) + f^{[\frac{n}{\sqrt{3}}]}(\frac{z+1}{3}) + f^{[\frac{n}{\sqrt{3}}]}(\frac{z+2}{3}), & 3 \mid n \end{cases} \tag{3.2.2}$$

Also, if f is replicable then f is 3A-replicable by taking, for example, $f^{\left[n\sqrt{3}\right]} = 0$. Conversely, we can also say that if f is replicable and if, for every n multiple of 3, we can write $f^{(n)}(z) = f^{[n]}(z) + f^{[\frac{n}{\sqrt{3}}]}(\frac{z}{3}) + f^{[\frac{n}{\sqrt{3}}]}(\frac{z+1}{3}) + f^{[\frac{n}{\sqrt{3}}]}(\frac{z+2}{3})$, for some $f^{[n]}, f^{[\frac{n}{\sqrt{3}}]}$, then f is 3A-replicable. For n not multiple of 3 we obvioulsy have $f^{[n]} = f^{(n)}$.

Definition 3.2.2. *We say that a function f is completely 3A-replicable if*

- *it is replicable, with replicates $f^{[n]}$, $n \in \mathbb{N} \cup \sqrt{3}\mathbb{N}$, and*
- *for every $n \in \mathbb{N} \cup \sqrt{3}\mathbb{N}$ the function $f^{[n]}$ is replicable with replicates $\left(f^{[n]}\right)^{[m]} = f^{[mn]}$, for any $m \in \mathbb{N} \cup \sqrt{3}\mathbb{N}$.*

To prove 3A-replication formulas for Hauptmodul we need an analog of Theorem 2.2.1 when the prime p does not divide any of the Atkin-Lehner involutions.

We believe the following is true:

Conjecture 3.2.1. *Let f be the Hauptmodul for some group $n|h + e_1, e_2 \dots$ with $p \nmid h$ and $p \nmid e_i$ for any i*

- *If $p \parallel n$, $f(W_p z) + f\left(\frac{z}{p}\right) + \ldots + f\left(\frac{z+p-1}{p}\right) = 0$*

- *If* $p^2 \mid n$, $f\left(\frac{z}{p}\right) + \ldots + f\left(\frac{z+p-1}{p}\right) = 0$

We use the same notation as in the previous Chapter to make a distinction between those Atkin-Lehner involutions that are divisible by three and those that are not. For example, now the group $30+6, 10, 15 = 30+\{e\}_O+\{3e\}_E$ with $O = \{10\}$ and $E = \{2, 5\}$. Combining Theorems 2.2.1 and Conjecture 3.2.1 we obtain an analog of Lemma 2.2.1.

Lemma 3.2.1. *We have the following identities:*

$$(1) \quad T_{3N+\{e\}_O}(z) = T_{3N+\{e,3e\}_O}(z) + \sum_{b=0}^{2} T_{3N+\{e\}_O}\left(\frac{z+b}{3}\right)$$

if $(3, N) = 1$.

$$(2) \quad T_{N+\{e\}_O}(z) = T_{3N+\{e,3e\}_O}(z) + \sum_{b=0}^{2} T_{3N+\{e,3e\}_O}\left(\frac{z+b}{3}\right)$$

if $(3, N) = 1$.

$$(3) \quad T_{N+\{e\}_O+\{3^k e\}_E}(z) = T_{N+\{e\}_O}(z) + \sum_{b=0}^{2} T_{3N+\{e\}_O+\{3^{k+1}e\}_E}\left(\frac{z+b}{3}\right)$$

if $3^k \parallel N$, $k \geq 1$ *and* E *is not empty.*

Theorem 3.2.1. *Every function of type as given in table 3.1 is* $3A$*-replicable with* $f^{[3^{\frac{n}{3}}]}$, $n \in \mathbb{N}$, *as given in the tables. The replicates* $f^{[3^{\frac{n}{3}}m]}$, *for* n *a non-negative integer and* m *not a multiple of* 3 *are given by the formula* $f^{[3^{\frac{n}{3}}m]}(z) = \left(f^{[3^{\frac{n}{3}}]}\right)^{(m)}(z)$.

Proof. The proof is similar to that Theorem 2.2.2. We use Conjecture 3.2.1 and Lemma 3.2.1 in this case.

Corollary 3.2.1. *The functions in table 3.1 are completely* $3A$*-replicable.*

Proof. Same argument as in Corollary 2.2.1.

f	$N + \{e\}_O + \{3e\}_E$	$3N + \{e\}_O + \{3e\}_E$	$9N + \{e\}_O + \{9e\}_E$	$9N + \{e\}_O + \{9e\}_E$
$f^{[\sqrt{3}]}$	$3N + \{e, 3e\}_O$	$3N + \{e, 3e\}_O$	$3N + \{e\}_O$	$9N + \{e\}_O$
$f^{[3]}$	$3N + \{e, 3e\}_O$	$3N + \{e, 3e\}_O$	$3N + \{e, 3e\}_O$	$3N + \{e\}_O$
$f^{[3\sqrt{3}]}$	$\cdots$	$\cdots$	$3N + \{e, 3e\}_O$	$3N + \{e, 3e\}_O$
$f^{[9]}$			$\cdots$	$3N + \{e, 3e\}_O$
$\vdots$				$\cdots$

f	$27N + \{e\}_O + \{27e\}_E$	$27N + \{e\}_O + \{27e\}_E$	$3N\mid 3 + \{e\}_O$	$9N\mid 3 + \{e\}_O + \{3e\}_E$
$f^{[\sqrt{3}]}$	$27N + \{e\}_O$	$9N + \{e\}_O$	$3N + \{e, 3e\}_O$	$9N + \{e\}_O + \{9e\}_E$
$f^{[3]}$	$9N + \{e\}_O$	$9N + \{e\}_O$	$3N + \{e, 3e\}_O$	$3N + \{e\}_O$
$f^{[3\sqrt{3}]}$	$9N + \{e\}_O$	$3N + \{e\}_O$	$\cdots$	$3N + \{e, 3e\}_O$
$f^{[9]}$	$3N + \{e\}_O$	$3N + \{e, 3e\}_O$		$3N + \{e, 3e\}_O$
$f^{[9\sqrt{3}]}$	$3N + \{e, 3e\}_O$	$3N + \{e, 3e\}_O$		$\cdots$
$f^{[27]}$	$3N + \{e, 3e\}_O$	$\cdots$		
$\vdots$	$\cdots$			

f	$9N\mid 3 + \{e, 3e\}_O$	$27N\mid 3 + \{e\}_O + \{9e\}_E$	$27N\mid 3 + \{e\}_O + \{9e\}_E$
$f^{[\sqrt{3}]}$	$3N\mid 3 + \{e\}_O$	$27N + \{e\}_O + \{27e\}_E$	$9N\mid 3 + \{e\}_O + \{3e\}_E$
$f^{[3]}$	$3N + \{e, 3e\}_O$	$9N + \{e\}_O$	$9N + \{e\}_O + \{9e\}_E$
$f^{[3\sqrt{3}]}$	$3N + \{e, 3e\}_O$	$9N + \{e\}_O$	$3N + \{e\}_O$
$f^{[9]}$	$\cdots$	$3N + \{e\}_O$	$3N + \{e, 3e\}_O$
$f^{[9\sqrt{3}]}$		$3N + \{e, 3e\}_O$	$3N + \{e, 3e\}_O$
$f^{[27]}$		$3N + \{e, 3e\}_O$	$\cdots$
$\vdots$		$\cdots$	

Table 3.1: $3A$-replicates of Hauptmoduls.

3.3 $3A$-replication and $3 \cdot F_{3+}$

In this section we will be using GAP notation for the names of classes and characters of $3 \cdot F_{3+}$.

We make a few observations about the classes and characters of $3 \cdot F_{3+}$. The characters χ_i, for $1 \leq i \leq 108$ are characters of F_{3+} and characters χ_i, for $109 \leq i \leq 256$ are characters of $3 \cdot F_{3+}$ that are not characters of F_{3+}. The last ones occur as algebraic conjugate pairs. If a class has at least one non-zero character value for some χ_i for $109 \leq i \leq 256$ then there are two other distinct classes whose corresponding character values are obtained from the first ones by multiplication by cube roots of unity, in an obvious way. We will refer to the first ones as the "essential" classes, as the Hauptmoduls for the other two classes are obtained by translations $z \to z + \frac{1}{3}$ and $z \to z + \frac{2}{3}$. Considering only the essencial classes reduces the number of classes from 256 to 108. Now, we try to find the Hauptmoduls for these essential classes.

From the fact that the function associated to the identity element in $3 \cdot F_{3+}$ should be

T_{3+} and its power series expansions starts off:

$$T_{3+}(q) = \frac{1}{q} + 783 \cdot q + 8672 \cdot q^2 + 65367 \cdot q^3 \ldots$$

we conclude that the decomposition of the first Head character should be:

$H_{-1} = \chi_1$ / $H_1 = \chi_{109}$ or χ_{110} / $H_2 = \chi_1 + \chi_2$

This decomposition determines the Hauptmodul we should associate to some other classes in $3{\cdot}F_{3+}$. The more classes we have identified the more decompositions of the Head characters we can determine accurately.

We can attach to every class a Hauptmodul with rational integer coefficients except for classes $18m$ and $18p$. Their functions have coefficients in the field $\mathbb{Q}(\sqrt{5})$ and seem to correspond to $54 \sim c$ and $54 \sim b$ (see [50]), respectively, if we consider $H_1 = \chi_{109}$. Choosing $H_1 = \chi_{110}$ instead only has the effect of switching these two essential classes, the other essential ones remaining unchanged.

For this choice of H_1 the decomposition of the Head characters is:

$H_{-1} = \chi_1$ / $H_1 = \chi_{109}$ / $H_2 = \chi_1 + \chi_2$ / $H_3 = \chi_{110} + \chi_{112}$ / $H_4 = \chi_{109} + \chi_{111} + \chi_{115}$

$H_5 = 3\chi_1 + 2\chi_2 + \chi_3 + \chi_8$ / $H_6 = 2\chi_{110} + 2\chi_{112} + \chi_{116} + \chi_{118}$

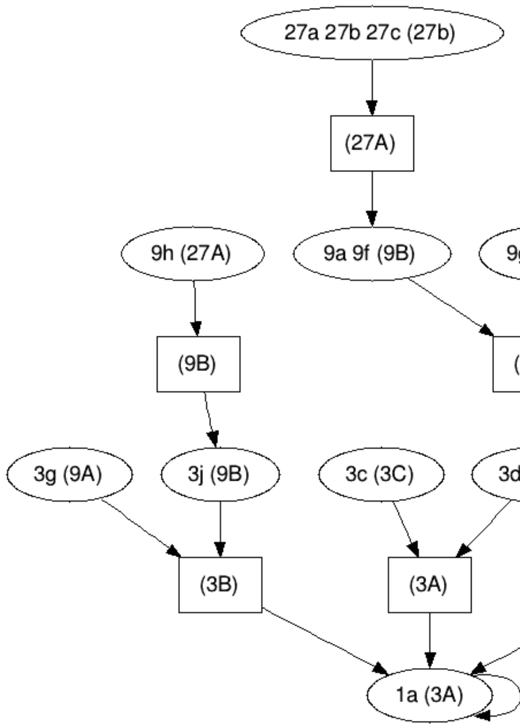
$H_7 = 3\chi_{109} + 3\chi_{111} + \chi_{113} + \chi_{115} + \chi_{117} + \chi_{119}$ / $H_8 = 4\chi_1 + 4\chi_2 + 2\chi_3 + \chi_5 + 2\chi_8 + \chi_9 + \chi_{10} + \chi_{13}$

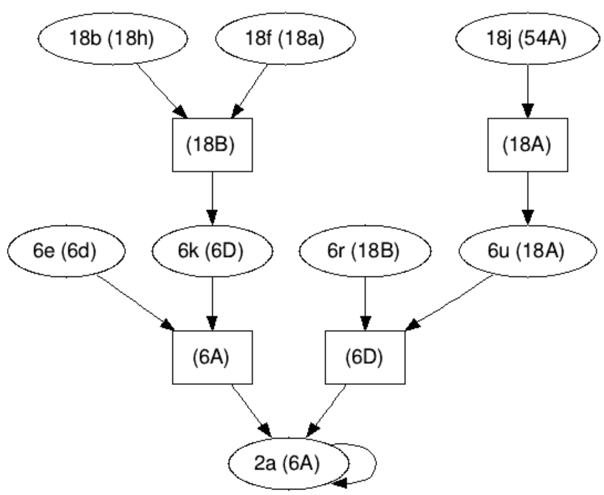
$H_{11} = 9\chi_1 + 10\chi_2 + 6\chi_3 + 3\chi_5 + 6\chi_7 + 2\chi_8 + 2\chi_9 + 2\chi_{10} + 4\chi_{12} + \chi_{16} + \chi_{18} + \chi_{24}$

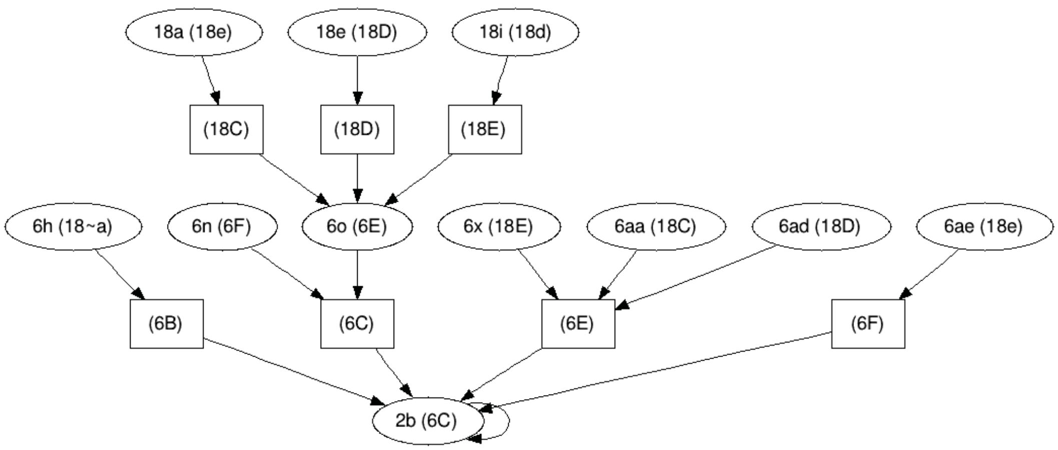
This decomposition of the Head characters doesn't determine what Hauptmodul we should attach to classes $24i$ and $24j$ since there are two possibilities: $72b$ and $216 \sim c$. We would have to find the decomposition of H_{17} to distinguish between those two functions. However, not only the more natural choice is $72b$ because $72 = 3 \cdot 24$ but also this choice makes $3A$-replicability respect the power map structure in $3 \cdot F_{3+}$.

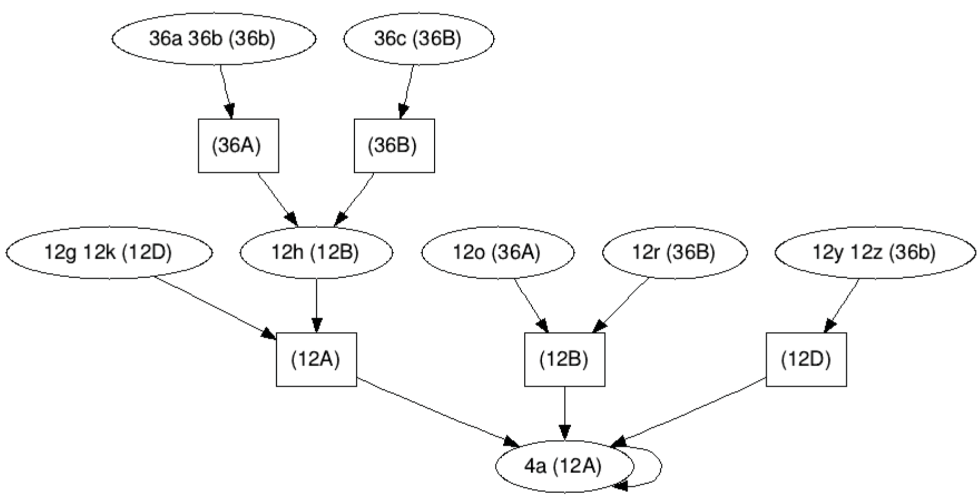
In fact, we have made computations that show that $3A$-replicability respects the power map structure in $3 \cdot F_{3+}$ for all the essential classes except $18m$ and $18p$. It is a curious fact that these are the only essential classes whose cubes are not essential. The cubes of $18m$ and $18p$ are $6y$ and $6z$, respectively, which are the translates of the essential class $6x$.

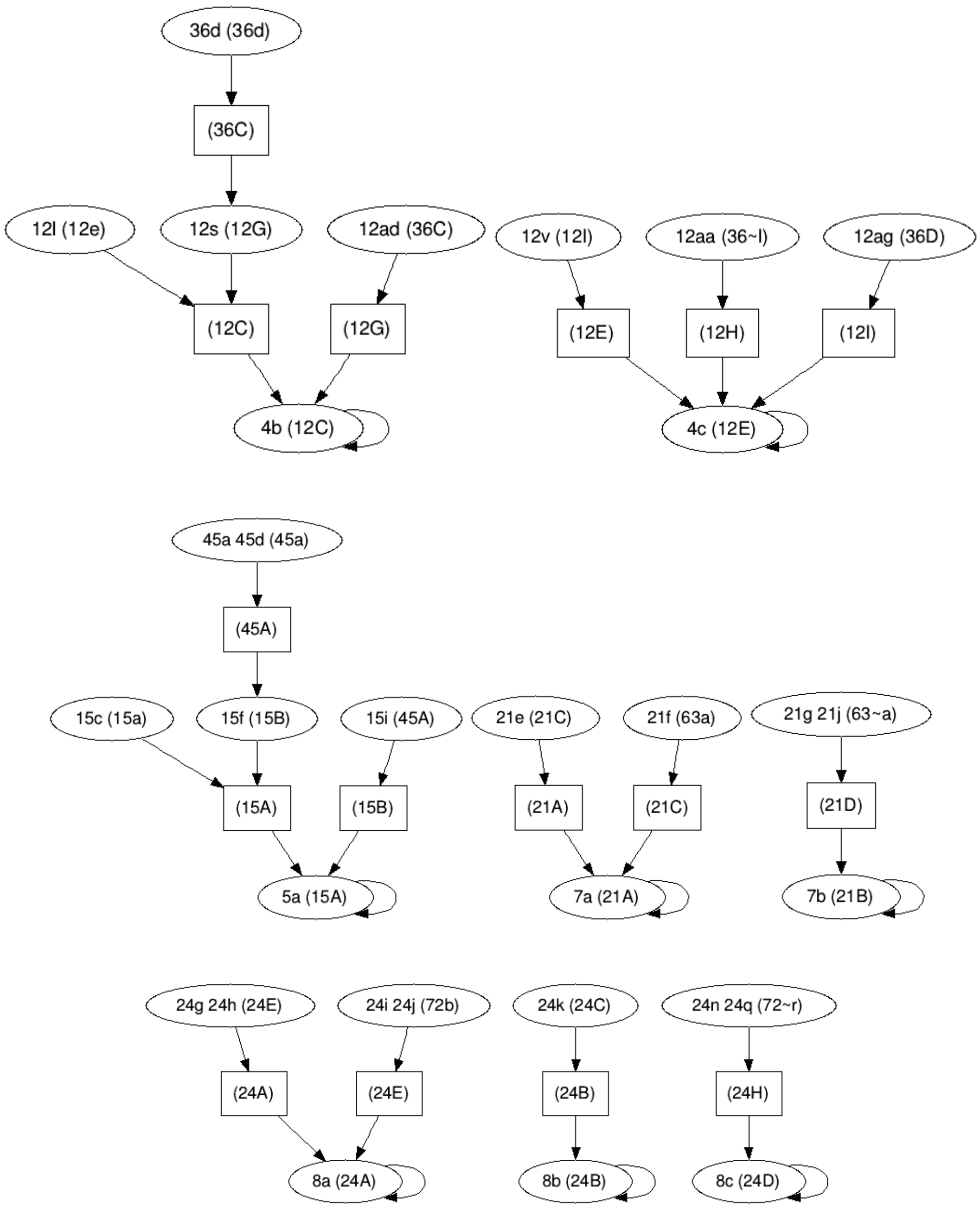
Below are the diagrams showing the power map structure of $3 \cdot F_{3+}$ and $3A$-replicability for all essential classes except $18m$ and $18p$. Every ellipse contains at least one class in $3 \cdot F_{3+}$ and the corresponding Hauptmodul between brackets. An arrow represents $\sqrt{3}$-replication so that a square box represents the $\sqrt{3}$-replicate of the McKay-Thompson series above it in the diagram. As we can see, every $\sqrt{3}$-replicate is again a Monstrous function.


27a 27b 27c (27b)
(27A)
9h (27A)
9a 9f (9B)
9g (9c)
9b (9b)
9e (9a)
(9B)
(9B)
(9A)
3g (9A)
3j (9B)
3c (3C)
3d (3B)
3k (9a)
(3B)
(3A)
(3C)
1a (3A)


18b (18h)
18f (18a)
18j (54A)
(18B)
(18A)
6e (6d)
6k (6D)
6r (18B)
6u (18A)
(6A)
(6D)
2a (6A)


18a (18e)
18e (18D)
18i (18d)
(18C)
(18D)
(18E)
6h (18~a)
6n (6F)
6o (6E)
6x (18E)
6aa (18C)
6ad (18D)
6ae (18e)
(6B)
(6C)
(6E)
(6F)
2b (6C)


36a 36b (36b)
36c (36B)
(36A)
(36B)
12g 12k (12D)
12h (12B)
12o (36A)
12r (36B)
12y 12z (36b)
(12A)
(12B)
(12D)
4a (12A)


36d (36d)
(36C)
12l (12e)
12s (12G)
12ad (36C)
(12C)
(12G)
4b (12C)
12v (12I)
12aa (36~I)
12ag (36D)
(12E)
(12H)
(12I)
4c (12E)
45a 45d (45a)
(45A)
15c (15a)
15f (15B)
15i (45A)
(15A)
(15B)
5a (15A)
21e (21C)
21f (63a)
(21A)
(21C)
7a (21A)
21g 21j (63~a)
(21D)
7b (21B)
24g 24h (24E)
24i 24j (72b)
(24A)
(24E)
8a (24A)
24k (24C)
(24B)
8b (24B)
24n 24q (72~r)
(24H)
8c (24D)

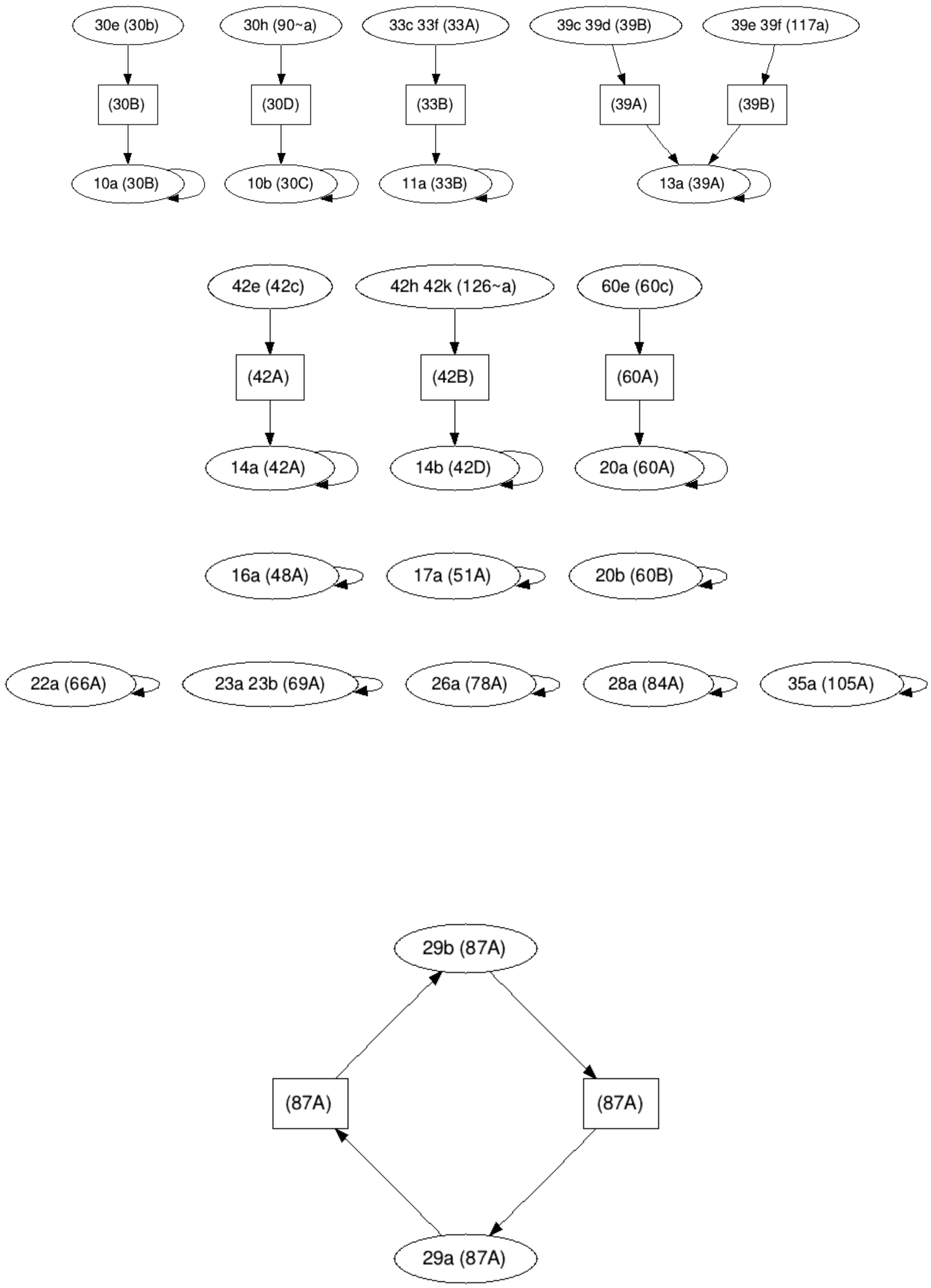


Figure 3.1: $3A$-replicability of essential classes in $3 \cdot F_{3+}$

3.4 Conclusion

In this chapter we applied the ideas from the previous chapter to an element of class $3A$ instead of class $2A$. We found that $3A$-replication respects the power map structure of $3 \cdot F_{3+}$ but only for those classes whose Hauptmodul is rational. In the previous chapter we saw that the $2A$-replication formulas for the McKay-Thompson series came from the denominator identity for the Baby Monster Lie algebra. We would like to build analogs of the vertex operator algebra W and the Baby Monster Lie algebra for $3 \cdot F_{3+}$. We believe that from the denominator identity for that Lie algebra it will be possible to deduce how $3A$-replication should work in the irrational cases. Work of Carnahan ([10]) might be relevant for this.

Two directions of future research could be, firstly, to find how $3A$-replication should work for the irrational functions and, secondly, how to extend these ideas to other groups that are not of the form $\Gamma_0(p)+$.

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