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SURVIVAL DISTRIBUTIONS WITH BATHTUB SHAPED HAZARD: A NEW DISTRIBUTION FAMILY

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Abstract

In this article we introduce an extension of Chen's (2000) family of distributions given by Lehman alternatives [see Gupta *et al.* (1998)] that is shown to present another alternative to the generalized Weibull and exponentiated Weibull families for modeling survival data. The extension proposed here can be seen as the extension to the Chen's distribution as the exponentiated Weibull is to the Weibull. A structural analysis of the density function in terms of tail classification and extremes is carried out similar to that of generalized Weibull family carried out in Mudholkar and Kollia (1994). The new model is also seen to fit well to the flood data used in fitting the exponentiated Weibull model in Mudholkar and Hutson (1996).

1 Introduction

In parametric modeling of a given set of random observations, Pearsonian family has played an important role, as it includes many common distributions. Some other generalized families that have been later introduced include Johnson S_B, or S_U family of curves [see Johnson *et al.* (1994) for a review] and Tukey Lambda family [see Friemer *et al.* (1988)]. These families serve as the basis of general modeling, however, for modeling survival data Weibull family of distributions [Weibull (1951)] is widely used. This family is a simple generalization of the exponential family and offers a simple alternative to modeling by gamma, lognormal and other commonly used survival distributions. The Weibull density, however, may not produce bathtub hazard rates that may be required for modeling in certain situations [see Rajarshi and Rajarshi (1988) and Nadarajah (2009)]. As a result, many generalizations have been proposed in literature; we refer the reader to the recent monograph by Murthy *et al.* (2003) for a comprehensive account of distributions and related issues connected with Weibull distribution.

In this paper our focus is on family of distributions given by Lehman alternatives (called exponentiated type family by Nadarajah and Kotz (2006)) considered by Gupta *et al.* (1998) where the generalization of a given cumulative distribution function (cdf) $F(t)$ is obtained by introduction of an additional parameter $\alpha > 0$, as given by

$$F_\alpha(t) = [F(t)]^\alpha. \tag{1.1}$$

This approach has been used in Gupta and Kundu (1999) in proposing a generalized exponential family by considering $F(x)$ to be an exponential distribution with a threshold parameter. This provides distributions with increasing or decreasing hazard rate depending on whether $\alpha < 1$ or $\alpha > 1$. On the other hand Mudholkar and Srivastava (1993) proposed an exponentiated Weibull family by considering $F(t)$ to be the cdf of a Weibull distribution [see Nadarajah *et al.* (2013) for an extensive review of the exponentiated Weibull distribution]. It was noted that the exponentiated Weibull family is richer than the generalized exponential as it may provide bath tub shaped hazard rates in addition to increasing and decreasing hazard rates.

It may be further noted that generalizations to a family of distributions may also be obtained by suitably generalizing the corresponding quantile function, e.g. see Friemer *et al.* (1988) for generalization of Tukey Lambda family and Mudholkar and Kollia (1994) for generalized Weibull family. A detailed structural analysis of generalized

Tukey Lambda family and that of generalized Weibull family and their closeness with the Pearsonian family have been systematically investigated in Friemer *et al.* (1988) and Mudholkar and Kollia (1994). It was observed in Mudholkar *et al.* (1996) that the generalized Weibull family exhibits a variety of hazard shapes, i.e. increasing, decreasing, bathtub and unimodal. This feature is also shared by the exponentiated Weibull family [see Mudholkar and Srivastava (1993)]. As such these families find effective applications in modelling survival data.

On the other hand, a simple two parameter distribution with bathtub shape or increasing hazard rate has been proposed by Chen (2000). Its distribution function is given by

$$F_C(t) = 1 - \exp\left[\lambda\left(1 - \exp\left(t^{\beta}\right)\right)\right], (t > 0). \tag{1.2}$$

Xie *et al.* (2002) extend this family by introduction of another parameter and named it the extended-Weibull distribution, that is given by the distribution function

$$F_{XTG}(t) = 1 - \exp\left\{\lambda\sigma\left[1 - \exp\left(\left(\frac{t}{\sigma}\right)^{\beta}\right)\right]\right\}, t \geq 0. \tag{1.3}$$

The term 'extended Weibull' is used for the above family of distributions as it resembles the Weibull family of distributions for large σ, since as $\sigma \to \infty, 1 - \exp[(t/\sigma)^{\beta}] \approx -(t/\sigma)^{\beta}$. It is to be noted that the special case $\sigma = 1$ gives the distribution by Chen (2000) introduced earlier. It also gives the distribution family proposed by Smith and Bain (1975) when $\lambda = 1/\sigma$ in order to model bathtub shaped failure rates given by

$$F_{SB}(t) = 1 - \exp\left\{1 - \exp\left(\left(\frac{t}{\sigma}\right)^{\beta}\right)\right\}, t \geq 0.. \tag{1.4}$$

This family as well as the Chen's family of distributions contain distributions with increasing and bathtub shape failure rate depending whether $\beta \geq 1$ or $\beta < 1$. This family has been extended by Xie *et al.* (2002) and further studied by Tang *et al.* (2003). It has been further extended by Pappas *et al.* (2012) recently, using the technique of Marshall and Olkin (1997). For other recent generalized families, the reader is referred to the recent articles by Bourguignon *et al.* (2014) [see also Zografos and Balakrishnan

(2009) and Gurvich *et al.* (1997)].

In this article we introduce another extension of the Chen's family (1.2) according to Lehman alternatives (1.1), called the extended Chen (EC) family with the distribution function given by

$$F_{EC}(t) = \left(1 - \exp\left[\lambda\left(1 - \exp\left(t^{\beta}\right)\right)\right]\right)^{\alpha}, \quad (1.5)$$

where $\alpha > 0, \beta > 0, \lambda > 0$ are the parameters of the distribution. This extension can be seen as the extension to the Chen's distribution as the exponentiated Weibull is to the Weibull.

A structural analysis of the corresponding density function in terms of the tail classification and extremes is carried out similar to that of generalized Weibull family as in Mudholkar and Kollia (1994). The new model is also seen to fit well to the standard flood data used in fitting the exponentiated Weibull model in Mudholkar and Hutson (1996).

Section 2 examines the density and tail shape classification depending on the parameters of the distribution along with an analysis of the corresponding hazard function. Section 3 presents an analysis of the corresponding hazard shapes and Section 4 provides an application of this distribution using flood data for the Floyd River at James, Iowa that has been used in Mudholkar and Hustson (1996). We use maximum likelihood method to estimate the parameters and give the confidence intervals for each parameter by using the bootstrap method and the likelihood ratio test to test some hypotheses about the distribution. The empirical TTT transform is used to justify the appropriateness of this distribution for the data used in the illustration.

2 Density Function and Tail Shape Classification

To simplify the discussions, we let $\lambda = 1$. The general case can be dealt similarly. The distribution function in this case becomes (where we have dropped the subscript 'EC')

$$F(t) = \left(1 - e^{1-e^{t^\beta}}\right)^\alpha, \qquad (t > 0, \alpha > 0, \beta > 0), \tag{2.1}$$

and the corresponding probability density function (pdf) is given by

$$f(t) = \alpha\beta \left(1 - e^{1-e^{t^\beta}}\right)^{\alpha-1} e^{1+t^\beta - e^{t^\beta}} t^{\beta-1}, \qquad (t > 0, \alpha > 0, \beta > 0). \tag{2.2}$$

The general nature of the density function is summarized in the following proposition, proof of which will be relegated to the Appendix A.1.

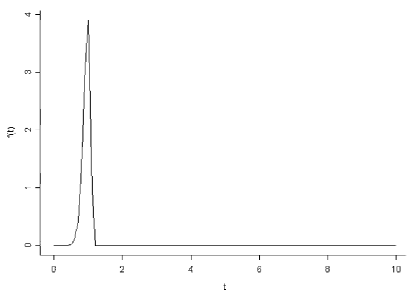
Proposition 2.1. *The density function corresponding to the distribution* (1.5)*, when* α *and* β *both are larger than 1, is unimodal, whereas, when both are smaller than 1, the density function is decreasing; in other cases, we may get unimodal or decreasing density function.*

Some graphs of the density function for various values of α and β are provided in Figures 2.1-2.6 for illustration.

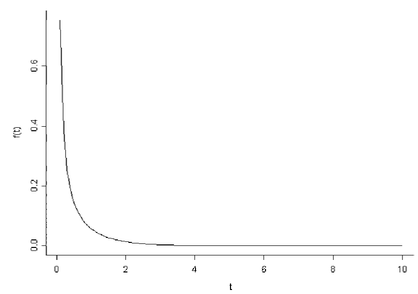
2.1 Parzen's Classification

The tail shapes may be classified [see Parzen (1979)] according to the limiting behavior of the extreme values as given in the following definition.

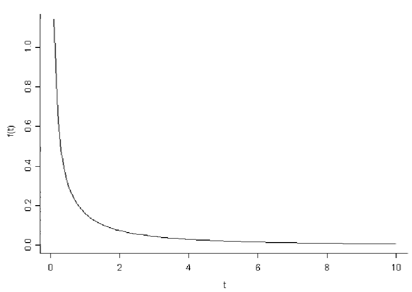
Definition 2.1. *Let* $X_{1:n}$ *and* $X_{n:n}$ *denote the minimum and maximum, respectively in a random sample of size* n*, from a population with d.f.* F*. If as* $n \to \infty$, $a_n X_{n:n} + b_n$ *converges in law to respectively to* $Y^{-1/\beta}, -Y^{1/\beta}$ *and* $-\log Y$*, where* $\beta > 0$*, and* Y *denotes the standard exponential random variable, i.e.,* $\Pr[Y \le y] = 1 - e^{-y}, y > 0$*, then the corresponding population distribution* F *is said to have long, short and medium right tail respectively. Similarly, if as* $n \to \infty$, $a_n X_{1:n} + b_n$ *converges in law to respectively to*



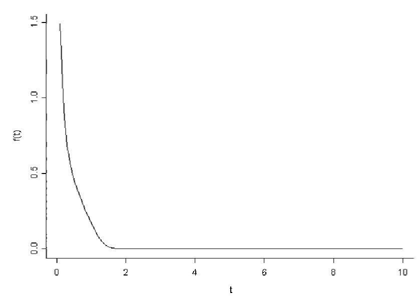
(a) $\alpha = 2, \beta = 5$



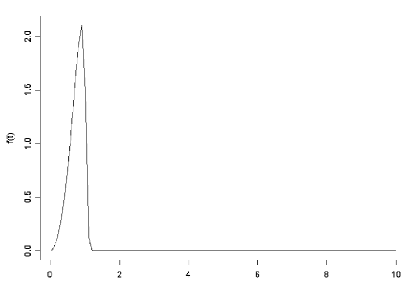
(b) $\alpha = 0.2, \beta = 0.5$



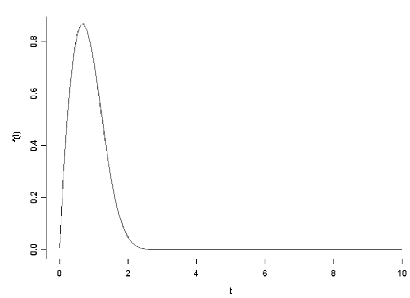
(c) $\alpha = 2, \beta = 0.2$



(d) $\alpha = 0.2, \beta = 1.5$



(e) $\alpha = 0.5, \beta = 6$



(f) $\alpha = 2, \beta = 0.9$

Figure 2.1: The density function curves

$-Y^{-1/\beta}, Y^{1/\beta}$ *and* $\log Y$, *where* $\beta > 0$, *then the corresponding population distribution* F *is said to have long, short and medium left tail, respectively.*

The tail classification may be achieved by the quantile function expansion as given in Friemer *et al.* (1989) for Tukey lambda family and generalized Weibull family. The following lemma is helpful in establishing this classification for the new family.

Lemma 2.1. *Let* $X_{1:n}$ *and* $X_{n:n}$ *be the minimum and maximum of a random sample of size* n, *respectively from the distribution* (2.1) *and let* $U_{1:n}$,*and* $U_{n:n}$ *denote those from the* $U(0,1)$ *distribution, then we have*

$$X_{1:n} \stackrel{\mathscr{L}}{=} (U_{1:n})^{\frac{1}{\alpha\beta}} + O_P(n^{-\frac{3}{\alpha\beta}}) \tag{2.3}$$

and

$$X_{n:n} \stackrel{\mathscr{L}}{=} \left(\log(1 - \log \frac{1 - U_{n:n}}{\alpha} + O_p(n^{-1}) \right)^{1/\beta}. \tag{2.4}$$

Proof: It is easily seen that the quantile function for the distribution in (2.1) is given by

$$Q(u) = \left(\log(1 - \log(1 - u^{1/\alpha})) \right)^{1/\beta}. \tag{2.5}$$

In order to study the probability law of the lower extreme, we expand $Q(u)$ around $u = 0$, and find that since, as $u \to 0$,

$$\log(1 - \log(1 - u^{\frac{1}{\alpha}})) = u^{\frac{1}{\alpha}} + O(u^{\frac{3}{\alpha}}). \tag{2.6}$$

Next use the fact that $X_{1:n} = Q(U_{1:n})$ and $nU_{1:n} \sim Y$ where Y has the exponential distribution with mean 1, to conclude the result in (2.3). In order to prove the result in (2.4), we obtain a similar expansion for $Q(u)$ around $u = 1$. Writing $1 - u^{1/\alpha}$ as

$$1 - u^{1/\alpha} = \frac{u - 1}{\alpha}(1 + O(u - 1)) \tag{2.7}$$

we have

$$\left(\log(1 - \log(1 - u^{1/\alpha})) \right)^{\frac{1}{\beta}} = \left(\log(1 - \log \frac{1 - u}{\alpha} + O(u - 1)) \right)^{\frac{1}{\beta}}. \tag{2.8}$$

Now we use the fact that $X_{n:n} = Q(U_{n:n})$ and $n(1-U_{1:n}) \sim Y$ where Y has the exponential distribution with mean 1, to conclude the result in (2.4). This proves the lemma.

□

The following theorem gives the Parzen's classification of the probability law studied in this article.

Theorem 2.1. *Let $X_{1:n}$ and $X_{n:n}$ be the minimum and maximum of a random sample of size n, respectively from the distribution* (2.1)*, and let Y denote the standard exponential random variable, i.e.* $\Pr[Y \leq y] = 1 - \mathrm{e}^{-y}, y \geq 0$. *Then as $n \to \infty$, we have*

$$n^{\frac{1}{\alpha\beta}} X_{1:n} \xrightarrow{\mathscr{L}} Y^{\frac{1}{\alpha\beta}} \tag{2.9}$$

$$\beta(\log(1+\log n))^{1-\frac{1}{\beta}}(1+\log n)X_{n:n} - \beta\log(1+\log n)(1+\log n) - \log\alpha \xrightarrow{L} -\log Y. \tag{2.10}$$

From these results we conclude as per Def. 2.1, that the left tails of the distribution are short and the right tails are medium.

Proof:

Eq. (2.9) follows by reexpressing (2.3) as

$$n^{\frac{1}{\alpha\beta}} X_{1:n} \stackrel{\mathscr{L}}{=} (nU_{1:n})^{\frac{1}{\alpha\beta}} + O_P(n^{-\frac{2}{\alpha\beta}}) \tag{2.11}$$

and the fact that $nU_{1:n} \xrightarrow{\mathscr{L}} Y$ as $n \to \infty$. To prove (2.4), we need a finer analysis. From Eq. (2.4), we note that as $n \to \infty$,

$$X_{n:n} \xrightarrow{\mathscr{L}} \left(\log(1+\log n - \log\frac{n(1-U_{n:n})}{\alpha})\right)^{\frac{1}{\beta}}. \tag{2.12}$$

Next to complete the proof, we use the well known result that if $Y_n \xrightarrow{\mathscr{L}} Y$ as $n \to \infty$, then $g_n(Y_n) \xrightarrow{\mathscr{L}} g(Y)$, provided $g_n(y) \to g(y)$ uniformly over all compact subsets. In the present case, we consider

$$g_n(y) = \frac{(\log(1+\log n - y))^{1/\beta} - (\log(1+\log n))^{1/\beta}}{\frac{1}{\beta}(\log(1+\log n))^{\frac{1}{\beta}-1}} \frac{1}{1+\log n}. \tag{2.13}$$

and $Y_n = \log \frac{n(1-U_{n:n}}{\alpha}$, and show that $g_n(y) \to -y$. We write

$$
\begin{aligned}
g_n(y) &= \frac{(\log(1+\log n - y))^{\frac{1}{\beta}} - (\log(1+\log n))^{\frac{1}{\beta}}}{\frac{1}{\beta}(\log(1+\log n))^{\frac{1}{\beta}-1}\frac{1}{1+\log n}} \\
&= \frac{(\log(1+\log n)+\log(1-\frac{y}{1+\log n}))^{\frac{1}{\beta}}}{\frac{1}{\beta}(\log(1+\log n))^{\frac{1}{\beta}-1}\frac{1}{1+\log n}} \\
&\quad - \frac{(\log(1+\log n))^{\frac{1}{\beta}}}{\frac{1}{\beta}(\log(1+\log n))^{\frac{1}{\beta}-1}\frac{1}{1+\log n}} \\
&= \frac{\left(1+\frac{\log(1-\frac{y}{1+\log n})}{\log(1+\log n)}\right)^{\frac{1}{\beta}} - 1}{\frac{1}{\beta}\frac{1}{\log(1+\log n)}\frac{1}{1+\log n}}
\end{aligned}
$$

For y in a compact set, there must exist a positive number M which makes $|y| < M$, we can choose n, such that $\frac{|y|}{1+\log n} < 1$, *i.e.* $n > e^{|y|-1}$. Choose $N = max(3, \lfloor e^{M-1} \rfloor)$, then for $n \geq N$,

$$
g_n(y) = \frac{1+\frac{1}{\beta}\frac{\log(1-\frac{y}{1+\log n})}{\log(1+\log n)} + o\left(\frac{\log(1-\frac{y}{1+\log n})}{\log(1+\log n)}\right) - 1}{\frac{1}{\beta}\frac{1}{\log(1+\log n)}\frac{1}{1+\log n}}. \tag{2.14}
$$

This shows that

$$
g_n(y) \approx \frac{\log(1-\frac{y}{1+\log n})}{\frac{1}{1+\log n}} \to -y \ (n \geq N) \text{ uniformly.} \tag{2.15}
$$

And since $Y_n = \log \frac{n(1-U_{n:n})}{\alpha}$ converges to $\log \frac{Y}{\alpha} = \log Y - \log \alpha$ in law, it follows from

Eq. (2.12) that,

$$\frac{(\log(1+\log n-\log\frac{n(1-U_{n:n})}{\alpha}))^{\frac{1}{\beta}}-(\log(1+\log n))^{\frac{1}{\beta}}}{\frac{1}{\beta}(\log(1+\log n))^{\frac{1}{\beta}-1}\frac{1}{1+\log n}}\xrightarrow{L}\log\alpha-\log Z$$

that in turn implies

$$\beta(\log(1+\log n))^{\frac{1}{\beta}-1}(1+\log n)Y_{n:n}-\beta\log(1+\log n)(1+\log n)-\log\alpha\xrightarrow{L}-\log Z. \tag{2.16}$$

This completes the proof of the theorem. □

2.2 Extreme Spacing Classification

For a random sample of size n from the population of a random variable X the right and left *Extreme spacings* (ES) are defined respectively as

$$S_{n:n} = X_{n:n}-X_{n-1:n} \tag{2.17}$$

$$S_{1:n} = X_{2:n}-X_{1:n}. \tag{2.18}$$

The tail classification introduced earlier in terms of extreme value behavior is somewhat crude as remarked in Schuster (1984), as many distributions such as normal and gamma distributions, with seemingly different tail behavior are classified as having the medium right tail. Schuster (1984) proposed the following definition of tail behavior based on the probability limit of $S_{n:n}$ as $n\to\infty$, to render further classification of medium right tails, which seems quite appealing.Here we focus on the right extreme spacings $S_{n:n}$ since they are useful in refinement of the right tail classification of a family of distributions. Similar definition holds for the left tail using $S_{1:n}$.

Definition 2.2. *If, as $n\to\infty$, $S_{n:n}$ converges in probability to 0, the right tail is ES short. If $S_{n:n}$ diverges in probability, then the right tail is ES long. It is ES medium, if $S_{n:n}$ remains bounded but non-zero in probability.*

Schuster (1984) proposed a relationship between outlier proneness and the ES classification. According to this relationship, a sample from a population with ES short tail "will rarely have outliers" in that tail, ES medium tail populations "will occasionally have moderate outliers" and those populations with ES long tail "will often have extreme outliers".

Friemer *et al.* (1989) show how the expansions of quantile functions may be used to obtain the convergence in law results for extreme spacings. We begin by noting a simple but useful observation in Friemer *et al.* (1989):

Proposition 2.2. *If $a_n X_{n:n} + b_n$ has a limiting distribution as $n \to \infty$, then the high probability magnitude of the length of the right tail of the distribution of X may be estimated by*

$$S_{n:n} = O_P\left(\frac{1}{a_n}\right). \tag{2.19}$$

Next, we obtain the limiting distributions of extreme spacings for the new family of densities based on the results of extreme value distribution presented in the previous sections. We state the following lemma from Friemer *et al.* (1989) needed for this purpose.

Lemma 2.2. [Friemer *et al.*, 1989] *Let $U_{1:n} \le U_{2:n} \le \ldots \le U_{n:n}$ denote the ordered values in a random sample from the uniform(0,1) distribution, then as $n \to \infty$, $(n(1-U_{n-1:n}), n(1-U_{n:n}))$ converges in law to (Z,Y), where (Z,Y) has the joint pdf*

$$f_{Z,Y}(z,y) = \begin{cases} \mathrm{e}^{-z} & \text{if} \quad 0 \le y \le z, \\ 0, & \text{otherwise} \end{cases} \tag{2.20}$$

The following theorem gives the limiting distribution of the extreme spacings corresponding to the distribution in Eq. (2.1).

Theorem 2.2. *For a random sample of size n from the family given in* (2.2) *and random variable (Z,Y) with joint p.d.f.*

$$f_{Z,Y}(z,y) = \begin{cases} e^{-z}, & \mathit{if}\, 0 \le y \le z, \\ 0, & \mathit{otherelse} \end{cases}$$

as $n \to \infty$

$$n^{\frac{1}{\alpha\beta}} S_{1:n} \xrightarrow{L} Z^{\frac{1}{\alpha\beta}} - Y^{\frac{1}{\alpha\beta}}$$

and

$$(\log(1+\log n))^{1-\frac{1}{\beta}}(1+\log n)S_{n:n} \xrightarrow{L} \frac{1}{\beta}(\log Z - \log Y)$$

Proof:

Since $S_{n:n} = Y_{n:n} - Y_{(n-1):n}$ and $S_{1:n} = Y_{2:n} - Y_{1:n}$, using expansion of $Q(u)$ as in Theorem 2.1, we get the following results:

$$n^{\frac{1}{\alpha\beta}} S_{1:n} \xrightarrow{L} Z^{\frac{1}{\alpha\beta}} - X^{\frac{1}{\alpha\beta}}, \tag{2.21}$$

and

$$(\log(1+\log n))^{1-\frac{1}{\beta}}(1+\log n)S_{n:n} \xrightarrow{L} \frac{1}{\beta}(\log Z - \log X). \tag{2.22}$$

□

Corrolary 2.1. *The left and right extreme spacings of a sample of size* n *from the distribution in Eq.* (2.2) *satisfy:*

$$S_{1:n} = O_p(n^{-\frac{1}{\alpha\beta}}), \tag{2.23}$$

and

$$S_{n:n} = O_p\left(\frac{(\log(1+\log n))^{\frac{1}{\beta}-1}}{1+\log n}\right). \tag{2.24}$$

Remark 2.1. *From the above Corollary, it can be seen that in Schuster's terminology, classically medium right tail of this distribution is always medium-short, and the convergence rate of the extreme spacings depends on the value of* β. *For example, the convergence is much faster for* $\beta > 1$ *than* $\beta < 1$.

3 The Hazard Function Shapes

The hazard function (also known as the failure rate, hazard rate, or force of mortality) $h(t)$ is the ratio of the density function $f(t)$ to the survival function $R(t) =$

$1 - F(t)$, given by

$$h(t) = \frac{f(t)}{R(t)}. \tag{3.1}$$

For the distribution, introduced in Eq. (2.1), the hazard function is given by:

$$h(t) = \frac{\alpha\beta(1 - e^{1-e^{t^\beta}})^{\alpha-1} e^{1+t^\beta - e^{t^\beta}} t^{\beta-1}}{1 - (1 - e^{1-e^{t^\beta}})^\alpha}. \tag{3.2}$$

The shape of the hazard function $h(t)$ depends on the values of α and β as depicted in the following proposition whose proof is relegated to Appendix A.2.

Proposition 3.1. *The shapes of the hazard function corresponding to the distribution given in Eq.* (2.1) *is given in the following table:*

Table 3.1: Four types of hazard shapes

α	β	failure behavior
1	1	constant
< 1	< 1	bathtub
> 1	> 1	increasing
< 1	> 1	increasing or bathtub
> 1	< 1	increasing or bathtub

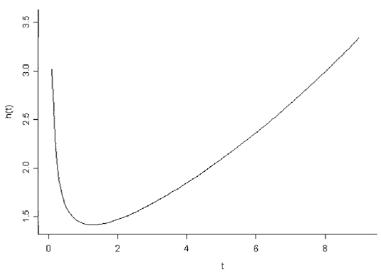
Figures 2(a-e) depict different shapes of the hazard function for the cases: i) $\alpha > 1,\ \beta > 1$, ii) $\alpha < 1,\ \beta < 1$, iii) $\alpha > 1,\ \beta < 1$ and $\alpha\beta < 1$, iv) $\alpha < 1,\ \beta > 1$ and $\alpha\beta < 1$, v) $\alpha < 1,\ \beta > 1$ and $\alpha\beta > 1$ and vi) $\alpha > 1,\ \beta < 1$ and $\alpha\beta > 1$. We note that when at least one of the α, β is less than 1 and the other one is larger than 1, the corresponding hazard function is increasing or bathtub shaped depending on whether $\alpha\beta > 1$ or not. Thus we conjecture that $h(t)$ is increasing if $\alpha\beta < 1$ and it is of bath tub shape if $\alpha\beta > 1$.

4 An Application

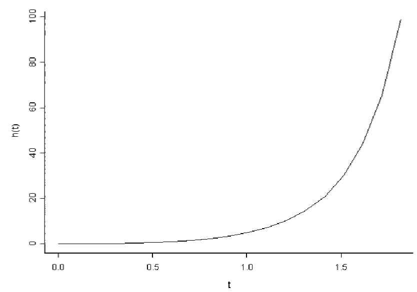
The flood rate of rivers have important economic, social, political and engineering implications. The modeling of flood data and analyses involving indications constitute an important application of the extreme value theory. Mudholkar and Hutson (1996) used the empirical TTT transform to demonstrate that exponential Weibull family provides a practical model for the analysis of the flood data. Here we use the similar method to examine the model introduced here for the flood data and compare it with the model used by Mudholkar and Hutson (1996).

Table 4.1: The Consecutive Annual Flood Discharge Rates of the Floyd River at James, Iowa

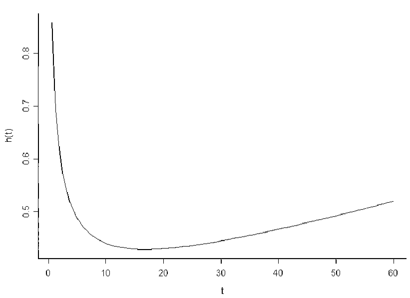
Year	Flood Discharge in(ft^3/s)				
1935-1944	1460	4050	3570	2060	1300
	1390	1720	6280	1360	7440
1945-1954	5320	1400	3240	2710	4520
	4840	8320	13900	71500	6250
1955-1964	2260	318	1330	970	1920
	15100	2870	20600	3810	726
1965-1973	7500	7170	2000	829	17300
	4740	13400	2940	5660	



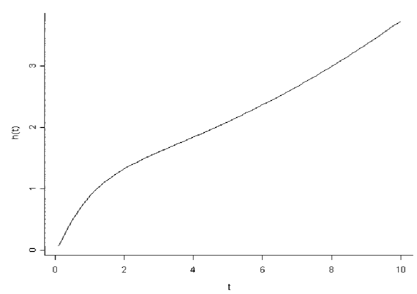
(a) $\alpha = 0.5, \beta = 0.5$



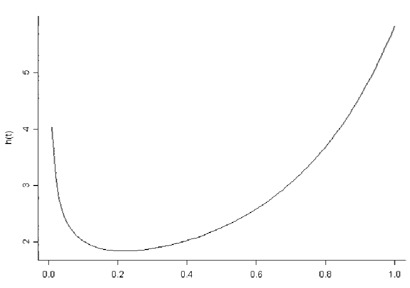
(b) $\alpha = 2, \beta = 2$



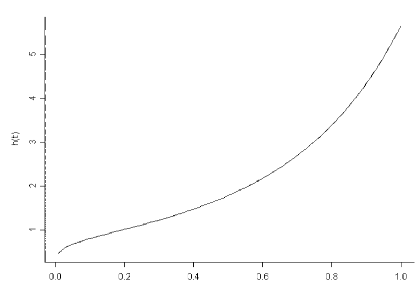
(c) $\alpha = 2, \beta = 0.3$



(d) $\alpha = 5, \beta = 0.5$



(e) $\alpha = 0.3, \beta = 2$



(f) $\alpha = 0.6, \beta = 2$

Figure 3.1: The hazard function curves

4.1 Models Considered and Parameter Estimates

The Floyd River flood rate data for the years 1935-1973 are given in Table 5.1. An exponentiated Weilbull model with the distribution function

$$F_{EW}(t) = [1 - \exp(-(t/\sigma)^{\beta})]^{\alpha}, t \geq 0$$

was used by Mudholkar and Hutson (1996) to demonstrate the application of their model and for checking the exponentiality of the data.

In order to check if simpler models may be adequate for the data we will consider three cases for the pdf corresponding to the distribution function (1.5):
i) $\lambda = 1$ ii) $\alpha = 1$ and iii) full model; these will be referred to as models EC(i), EC(ii) and EC(iii) respectively.
The standard maximum likelihood method is used estimating the parameters. The maximization of the likelihood is obtained using the `optim` routine of `R`-package (with the value of `reltol` set to 1e-20), as explicit solutions of the likelihood are not available. The likelihood function for the three cases are given below:

i) EC(i) model: In this case the probability density function is given by:

$$f(x) = \alpha\beta(1 - e^{1-e^{x^{\beta}}})^{\alpha-1} e^{1+x^{\beta}-e^{x^{\beta}}} x^{\beta-1}$$

and consequently the log-likelihood function is given by

$$\begin{aligned} \ell(\alpha,\beta) &= n\log\alpha + n\log\beta + (\alpha-1)\sum_{i=1}^{n}\log(1-e^{1-e^{x_i^{\beta}}}) + n + \sum_{i=1}^{n} x_i^{\beta} - \sum_{i=1}^{n} e^{x_i^{\beta}} \\ &\quad + (\beta-1)\sum_{i=1}^{n}\log x_i \end{aligned}$$

ii) EC(ii) model: The density in this case is given by

$$\begin{aligned} f(x) &= -e^{\lambda(1-e^{x^{\beta}})}(-\lambda)e^{x^{\beta}}\beta x^{\beta-1} \\ &= \lambda\beta e^{\lambda(1-e^{x^{\beta}})+x^{\beta}} x^{\beta-1} \end{aligned}$$

Table 4.2: Parameter Estimates, Likelihood and AIC for Different Models

Model	Parameter estimates	log-likelihood	AIC
EW	$\hat{\alpha}=0.2323,\ \hat{\theta}=77.9517,\ \hat{\sigma}=4.2423$	-376.3498	377.0355
$EC(i)$	$\hat{\alpha}=266.7735\ \hat{\beta}=0.08139$	-376.3688	376.7021
$EC(ii)$	$\hat{\beta}=0.1702,\ \hat{\lambda}=0.01138$	-387.7844	388.1177
$EC(iii)$	$\hat{\alpha}=387.3361, \hat{\beta}=0.07797, \hat{\lambda}=1.1326$	-376.3624	377.0481

and the likelihood function is consequently given by

$$\ell(\lambda,\beta)=n\log\lambda+n\log\beta+n\lambda-\lambda\sum_{i=1}^{n}e^{x_i^\beta}+\sum_{i=1}^{n}x_i^\beta+(\beta-1)\sum_{i=1}^{n}\log x_i.$$

iii) EC(iii) model: This is the case of unrestricted parameters and the full likelihood for the unrestricted parameters is given under the pdf

$$\begin{aligned} f(x) &= \alpha(1-e^{\lambda(1-e^{x^\beta})})^{\alpha-1}(-e^{\lambda(1-e^{x^\beta})})(-\lambda e^{x^\beta})\beta x^{\beta-1} \\ &= \alpha\lambda\beta(1-e^{\lambda(1-e^{x^\beta})})^{\alpha-1}e^{\lambda(1-e^{x^\beta})+x^\beta}x^{\beta-1} \end{aligned}$$

as

$$\begin{aligned} \ell(\alpha,\beta,\lambda) &= n\log\alpha+n\log\beta+n\log\lambda+(\alpha-1)\sum_{i=1}^{n}\log(1-e^{\lambda(1-e^{x_i^\beta})})+n\lambda \\ &\quad -\lambda\sum_{i=1}^{n}e^{x_i^\beta}+\sum_{i=1}^{n}x_i^\beta+(\beta-1)\sum_{i=1}^{n}\log x_i. \end{aligned}$$

The estimators under various models along with the log-likelihood are summarized in the table below.

The likelihood of the restricted Model (i) (with $\lambda=1$) compares closely to the full model however at the expense of an additional parameter, that is also close to the likelihood given by the Mudholkar-Hutson exponentiated Weibull model. Hence, we look at the Akaike information criterion (AIC) [see Burnham & Anderson (2002)] that penalizes for additional parameters given by

$$AIC = -\log(\text{likelihood}) + \frac{2k(k+1)}{n-k-1}. \tag{4.1}$$

The goodness of fit of a model based on the value of AIC is judged on the smaller values of AIC, i.e. smaller is the AIC, better is the considered model. Thus according to this criterion, the restricted Model 1 comes out to be best of the four models considered here; the three parameter new distribution provides almost the same AIC as the one for the exponentiated Weibull model. Thus we might be interested in testing the restrictions through a formal test of hypothesis. This can be done on a large sample basis through likelihood ratio test that considers the statistic

$$\Lambda = 2(\log L_1 - \log L_2), \tag{4.2}$$

where L_1 is the maximized likelihood under the full model and L_2 is that under the reduced model (H_0.) This statistic has a χ^2 distribution under H_0 with 1 degree of freedom and the null hypothesis will be rejected for larger values of the test statistic.

Thus for model (i) we are in a situation for testing $H_0 : \lambda = 1$ vs. $H_1 : \lambda \neq 1$. The value of the test statistic in this case is $\Lambda = 2(-376.362 + 376.369) = 0.014$ that is significantly lower than the right tail 5% value of χ^2 with 1 degree of freedom that equals 3.84. Hence model (i) is accepted in favor of $\lambda = 1$. On the other hand for Model (ii), we test $H_0 : \alpha = 1$ vs. $H_1 : \alpha \neq 1$. In this case the comparing the log-likelihood of model (ii) and model (iii) we get $\Lambda = 2(387.784 - 376.362) = 22.844$ that is significantly higher than 3.84 indicating that model (ii) should be rejected.

These conclusions are based on large sample theory but may also be validated using Bootstrap method [see Davison and Hinkley (1997)] that may be more appropriate for smaller samples. Below we provide 95% BCa (bias-corrected and adjusted) bootstrap confidence intervals (CI) [see Efron and Tibshirani (1985)] based on $B = 1000$ replications from the full model:

95% CI for α: (4.223883, 1414.707)
95% CI for λ: (0.04455415, 1.680493)
95% CI for β: (0.05785784, 0.1090082)

Based on the 95% CI for λ we accept the null hypothesis $H_0 : \lambda = 1$, where as the hypothesis $H_0 : \alpha = 1$ is rejected giving the same conclusions obtained using the large sample theory.

4.2 Model Suitability Based on Scaled TTT Transform

It has become a common practice to examine the TTT (total time on test) transform introduced by Barlow and Campo (1975) [see also Bergman and Klefsjö (1984, 1985)] in order to judge the shape of the hazard function and closeness of the data distribution with that of the model. Aarset (1987) proposed and illustrated the use of empirical TTT-tranform for identifying bathtub failure rates. The scaled TTT transform of a probability distribution with d.f. F(.) and quantile function Q(.) is:

$$\phi(u) = \frac{1}{\mu}\int_0^{Q(u)} (1 - F(t))dt. \tag{4.3}$$

where μ is mean of the distribution F. For the exponential distribution, that has a constant hazard function, $\phi(u) = u$. If $\phi(u)$ is convex then the hazard function $h(u)$ is decreasing, and $h(u)$ is increasing if $\phi(u)$ is concave. And If $\phi(u)$ is concave-convex then $h(u)$ is unimodal; and it is convex-concave if $h(u)$ is bathtub shaped. In practice, given a random sample $x_{(1)} \leq x_{(2)} \cdots \leq x_{(n)}$ from F, the TTT transform of the fitted model may be compared with the empirical TTT transform given as:

$$\phi_n(i/n) = \frac{\sum_{j=1}^{i} x_{(j)} + (n-i)x_{(i)}}{\sum_{j=1}^{n} x_{(j)}}. \tag{4.4}$$

For the three cases considered in this paper, the quantile functions are:

$$(i)\ Q(u) = (\log(1 - \log(1 - u^{\frac{1}{\alpha}})))^{\frac{1}{\beta}},$$

$$(ii)\ Q(u) = (\log(1 - \frac{\log(1-u)}{\lambda}))^{\frac{1}{\beta}}$$

and

$$(iii)\ Q(u) = (\log(1 - \frac{\log(1-u^{\frac{1}{\alpha}})}{\lambda}))^{\frac{1}{\beta}}.$$

respectively, and that for the exponentiated-Weibull is given by

$$Q(u) = \sigma[-\log(1-u^{1/\theta})]^{1/\alpha}, 0 \leq u \leq 1.$$

Figure 4.1 gives a graph of the empirical transform superimposed with those of the fitted distributions for the four models considered in this paper. Note that the circles represent the empirical transform.

We note that the model provided by the Chen's distribution ($\alpha = 1$) is not adequate for this data, however, the modification with two parameters ($\lambda = 1$) fits almost as well as the three parameter exponentiated Weibull model.

5 Conclusions

A new three-parameter lifetime distribution with bathtub shape or increasing failure rate function is introduced in this paper. We mainly studied the properties of the density function, tail shapes, hazard function and extremes and extreme spacings of this distribution in the similar method as the structural analysis of the Tukey lambda family in Friemer *et al.* (1988), of the Weibull family by Mudholkar and Kollia (1994) and Exponentiated-Weibull family by Mudholkar and Hutson (1996). The principal applications are in survival, reliability and the extreme-value analysis. For the analysis considered here, we consider $\lambda = 1$; for other values similar properties are postulated. It is shown here using a commonly used data set that the new distribution fits as well as the exponential Weibull distribution, that has been used earlier in the literature. We use this data set to demonstrate tests of hypotheses using resampling confidence intervals, for example the hypothesis $\lambda = 1$. Another use of this distribution is to test the composite goodness-of-fit hypothesis of the distribution given by Chen (2000) by testing $\alpha = 1$. Introducing an additional shape parameter α may provide a better model than Chen's model as demonstrated for the Floyd River flood data.

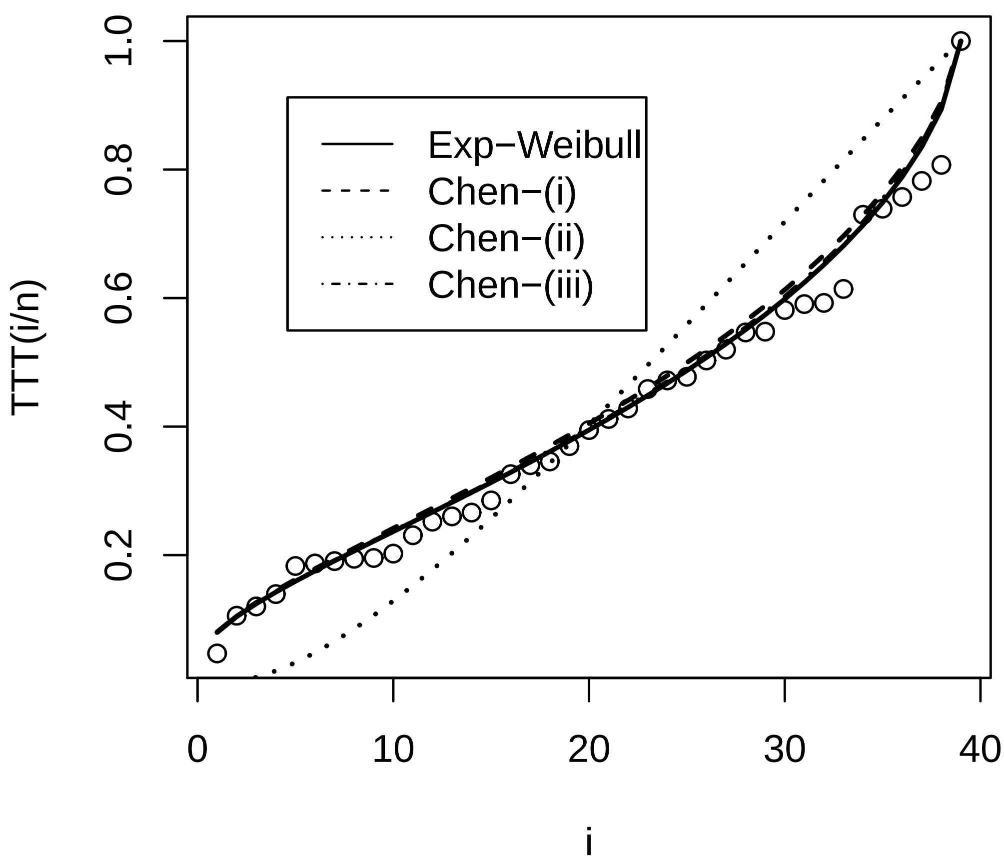


Figure 4.1: The Scaled TTT Transforms for the Floyd River Flood Data

This distribution, just like exponentiated-Weibull distribution, is very useful in the lifetime, reliability and extreme-value data analysis. Thus further research on inference problems for this model may be of interest.

6 Appendix

A.1. Proof of Proposition 2.1

Let

$$z \equiv z(t) = e^{t^{\beta}}.$$

Then $f(t) = g(z)$, where

$$g(z) = \alpha\beta(1-e^{1-z})^{\alpha-1}e^{1-z}z(\log z)^{\frac{\beta-1}{\beta}}. \tag{6.1}$$

It is seen that the derivative of $g(z)$ is given by :

$$g'(z) = \alpha\beta\{(1-e^{1-z})^{\alpha-2}(\log z)^{\frac{\beta-1}{\beta}-1}e^{1-z}[T_1(z)+T_2(z)+T_3(z)]\} \tag{6.2}$$

where

$$\begin{aligned} T_1(z) &= (\alpha-1)z\log z e^{1-z}, \\ T_2(z) &= \log z(1-e^{1-z})(1-z) \end{aligned}$$

and

$$T_3(z) = (1-e^{1-z})\frac{\beta-1}{\beta}.$$

Now we analyze the sign of each of the terms T_1, T_2 and T_3. At first, consider the case of $z \to \infty$ (that is $t \to \infty$). It can be easily seen that

$$\lim_{z\to\infty} T_1(z) = 0, \lim_{z\to\infty} T_2(z) = -\infty$$

and

$$\lim_{z\to\infty} T_3(z) = \frac{\beta-1}{\beta}.$$

Thus we conclude that for large z, $g'(z) < 0$ and hence as $t \to \infty$, $f(t)$ is decreasing. Therefore, $f(t)$ may be overall decreasing or unimodal. Now we consider four cases where we can explicitly discuss the nature of $f(t)$ for all $t > 0$.

Case I: $(\alpha < 1, \beta < 1)$
If $\alpha < 1$ and $\beta < 1$, then: Since,
$T_1(z) < 0$ for all $z > 1$ or $t > 0$,
$T_2(z) < 0$ for all $z > 1$ and
$T_3(z) < 0$, it follows from Eq.(3.4), that $g'(z) < 0$ for all $z > 1$. That means $g(z)$ is decreasing or $f(t)$ is strictly decreasing.

Case II: $(\alpha > 1, \beta > 1)$
For this case, we show that $f(t)$ is unimodal. Let:

$$\Psi(z) = (\alpha-1)z\log z e^{1-z} + \log z(1-e^{1-z})(1-z) + (1-e^{1-z})\frac{\beta-1}{\beta}$$

then

$$\Psi(z) \quad = \quad (\alpha-1)z\log z e^{1-z} + (1-e^{1-z})\psi(z)$$

where $\psi(z) = \log z - z\log z + \frac{\beta-1}{\beta}$. We find that,

$$\psi'(z) = \frac{1}{z} - \log z - z\frac{1}{z}$$
$$\psi''(z) = -\frac{1}{z^2} - \frac{1}{z}$$

It is obvious that $\psi''(z) < 0$ for $z > 1$. This implies that $\psi'(z)$ is decreasing function. Hence; $\psi'(z) < \psi'(1) = 0 \Rightarrow \psi(z)$ is a decreasing function. Since $\lim_{z\to\infty}\psi(z) = -\infty$, and $\psi(1) = \frac{\beta-1}{\beta} > 0$, there exists a z^* such that $\psi(z^*) = 0$ and, $0 < \psi(z) < \psi(1) = \frac{\beta-1}{\beta}$ for $1 < z < z^*$. This implies $\Psi(z) > 0$ for $1 < z < z^*$. Further since, $\lim_{z\to\infty} g'(z) = -\infty$, using the same argument, we find that there exists $z^{**} \geq z^*$, such that $\Psi(z^{**}) = 0$. This provides that $g(z)$ is unimodal or equivalently $f(t)$ is unimodal.

Case III: $(\alpha < 1, \beta > 1)$
Note that $g'(z)$ has the same sign as $\Psi(z)$. Since

$$\Psi(z) = (\alpha-1)z\log ze^{1-z} + (1-e^{1-z})\psi(z)$$

For $\alpha < 1$; $(\alpha-1)z\log ze^{1-z} < 0$ for all $z > 1$.
Also, $\psi(z)$ is decreasing function and $\psi(z) < \psi(1) = \frac{\beta-1}{\beta}$.
Let z^* be such that $\psi(z^*) = 0$; then
For $z > z^*$; $\Psi(z) < 0$, hence;
$g(z) \searrow$ for $z > z^*$.
For $z \leq z^*$,

$$\begin{aligned} \Psi(z) \geq 0 \quad &\Leftrightarrow \quad \alpha \geq 1 - \frac{(1-e^{1-z})\psi(z)}{z\log ze^{1-z}} \; \forall z^* \geq z \\ &\Leftrightarrow \quad \alpha \geq 1 - sup_{z\leq z^*} \frac{(1-e^{1-z})\psi(z)}{z\log ze^{1-z}} = 1 - u_{z^*}, \text{say.} \end{aligned}$$

If α satisfies the above condition, then $f(t)$ is unimodal, otherwise $f(t)$ is decreasing with t.

Case IV: $(\alpha > 1, \beta < 1)$
This case is very similar to the case III. Maybe $\Psi(z)$ is always non-positive, or at the beginning, it is non-negative and eventually becomes non-positive. That means $g(z)$ is decreasing or unimodal. It is equivalent to saying $f(t)$ may be decreasing or unimodal.

By the above analysis, it is clear that when α and β both are larger than 1, the density function is unimodal, whereas, when both are smaller than 1, the density function is decreasing; in other cases, we may get unimodal or decreasing density function as summarized in Table 1.

A.2. Proof of Proposition 2.2

As in the previous section, we analyze $h(t)$ in terms of $z = e^{t^\beta}$, and consider

$$h(t) = r(z) = \frac{\alpha\beta(1-e^{1-z})^{\alpha-1} z e^{1-z} (\log z)^{\frac{\beta-1}{\beta}}}{1-(1-e^{1-z})^{\alpha}} \tag{6.3}$$

Now

$$r'(z) = \alpha\beta \frac{\phi_1(z)(1-(1-e^{1-z})) - (1-e^{1-z})^{\alpha-1} e^{1-z} z (\log z)^{\frac{\beta-1}{\beta}} \phi_2(z)}{(1-(1-e^{1-z})^{\alpha})^2},$$

where

$$\phi_1(z) = \frac{d((1-e^{1-z})^{\alpha-1} e^{1-z} z (\log z)^{\frac{\beta-1}{\beta}})}{dz}$$

and

$$\phi_2(z) = \frac{d}{dz}(1-(1-e^{1-z})^{\alpha}).$$

It can be seen that

$$\begin{aligned} \phi_1(z) &= (1-e^{1-z})^{\alpha-2} (\log z)^{\frac{\beta-1}{\beta}-1} e^{1-z} z \log z ((\alpha-1)e^{1-z} - (1-e^{1-z})) \\ &\quad + (1-e^{1-z})^{\alpha-2} (\log z)^{\frac{\beta-1}{\beta}-1} e^{1-z} (1-e^{1-z}) (\log z + \frac{\beta-1}{\beta}) \quad (6.4) \end{aligned}$$

and

$$\phi_2(z) = -\alpha(1-e^{1-z})^{\alpha-1} e^{1-z}. \tag{6.5}$$

Then $r'(z)$ can be written as:

$$r'(z) = \phi_3(z)\phi_4(z)$$

where

$$\phi_3(z) = \alpha\beta \frac{(1-e^{1-z})^{\alpha-2}(\log z)^{\frac{\beta-1}{\beta}-1}e^{1-z}}{(1-(1-e^{1-z})^{\alpha})^2}$$

and

$$\begin{aligned}\phi_4(z) &= (z\log z((\alpha-1)e^{1-z}-(1-e^{1-z}))+(1-e^{1-z}(\log z+\frac{\beta-1}{\beta}))(1-(1-e^{1-z})^{\alpha}) \\ &\quad +(1-e^{1-z})^{\alpha}\alpha z e^{1-z}\log z.\end{aligned}$$

It is very easy to see that $\phi_3(z)$ is always greater than 0, so we need to consider $\phi_4(z)$ in detail. Write $\phi_4(z)$ as

$$\phi_4(z) = (1-e^{1-z})G_1(z)+G_2(z),$$

where

$$G_1(z) = (\log z+\frac{\beta-1}{\beta})(1-(1-e^{1-z})^{\alpha})$$

and

$$G_2(z) = z\log z(\alpha e^{1-z}-1+(1-e^{1-z})^{\alpha}).$$

We find that when $t\to\infty(z\to\infty)$:

$$\begin{aligned}\lim_{z\to\infty}G_1(z) &= \lim_{z\to\infty}\alpha z e^{1-z}(\log z)^2 \\ &= 0 \qquad (6.6)\end{aligned}$$

and

$$\begin{aligned}\lim_{z\to\infty}G_2(z) &= \lim_{z\to\infty}\frac{2z\log z}{\alpha(\alpha-1)e^{2z-2}} \\ &= 0. \qquad (6.7)\end{aligned}$$

Further

$$G_1(z) \overset{z\to\infty}{\approx} \log z(\alpha e^{1-z}-\frac{\alpha(\alpha-1)}{2}(e^{1-z})^2+o((e^{1-z})^2))$$

$$G_2(z) \overset{z\to\infty}{\approx} z\log z(\frac{\alpha(\alpha-1)}{2}(e^{1-z})^2+o((e^{1-z})^2))$$

and

$$\lim_{z\to\infty} ze^{1-z} = \lim_{z\to\infty}\frac{z}{e^{z-1}} = 0.$$

Thus $G_2(z)$ converges to zero at a faster rate than $G_1(z)$.
Suppose $k(\alpha) = \alpha e^{1-z} - 1 + (1-e^{1-z})^{\alpha}$ and $e^{1-z} = t,\qquad (0 < t < 1)$
so that

$$k(\alpha) = t\alpha - 1 + (1-t)^{\alpha},$$

$$k'(\alpha) = t + (1-t)^{\alpha}\log(1-t),$$

$$k''(\alpha) = (1-t)^{\alpha}\log(1-t)\log(1-t) = (1-t)^{\alpha}(\log(1-t))^2 > 0.$$

and for $\alpha = 0, k(0) = 0$, and for $\alpha = 1, k(1) = 0$. Let $k'(\alpha) = 0$, then $\alpha^* = \log_{1-t}^{-\frac{t}{\log(1-t)}}$, $\alpha = \alpha^*$ gives minima of $k(\alpha)$.
For further analysis of $k(\alpha)$ we need the following inequality:

$$1-t < -\frac{t}{\log(1-t)} < 1 \qquad (6.8)$$

The left hand side of the above inequality follows by considering the function $T(t) = (1-t)\log(1-t)+t$ and noting that

$$T(0) = 0$$

and

$$T'(t) = -\log(1-t) + \frac{1-t}{1-t}(-1) + 1 = -\log(1-t) > 0.$$

Thus we conclude that $T(t) \nearrow$, and $T(t) > T(0) = 0$, that is $-\frac{t}{\log(1-t)} > 1-t$ that proves the left hand side of (6.8). To prove the right hand side of (6.8) consider $g(t) = \log(1-t)+t$. We have

$$g(0) = 0,$$

and

$$g'(t) = \frac{-1}{1-t} + 1 < 0,$$

that implies that $g(t) \searrow$, and $g(t) < g(0) = 0$, that is $-\frac{t}{\log(1-t)} < 1$ and the inequality on the right hand side of (6.8) follows.

Now, we go back to consider the behavior of $k(\alpha)$. Since α^* is the minimum value of $k(\alpha)$

$$k'(\alpha) < 0, \qquad \text{for}(\alpha < \alpha^*)$$

and

$$k'(\alpha) > 0, \qquad \text{for}(\alpha > \alpha^*)$$

Also when $\alpha > 1$, $k'(\alpha) > 0$ and $k(1) = 0$; *i.e*, for $\alpha > 1$, $k(\alpha) \nearrow$, hence

$$k(\alpha) > 0, \qquad \text{for}(\alpha > 1)$$

and

$$k(\alpha) < 0, \qquad \text{for}(0 < \alpha < 1).$$

Thus we have four cases:

Case I: $(\alpha \geq 1,\ \beta > 1)$
When $\alpha \geq 1$, $\beta > 1$, $\phi_4(z) > 0$, so $r(z)$ is increasing, that is $h(t)$ is increasing.

Case II: $(\alpha > 1,\ \beta < 1)$
When $\alpha > 1$, $\beta < 1$, $\phi_4(z) > 0$, for all z or at the beginning, $\phi_4(z) < 0$, then it becomes positive. In this case, therefore $r(z)$ is increasing or bathtub, that is $h(t)$ is increasing or bathtub.

Case III: $(\alpha < 1,\ \beta > 1)$
When $\alpha < 1$, $\beta > 1$, $\phi_4(z) > 0$, for all z or at the beginning, $\phi_4(z) < 0$, then it becomes positive. Hence $r(z)$ is increasing or bathtub, that is $h(t)$ is increasing or bathtub.

Case IV: $(\alpha \leq 1,\ \beta < 1)$
When $\alpha \leq 1, \beta < 1$, at the beginning, if $z < e^{\frac{1-\beta}{\beta}}$ $\phi_4(z) < 0$, and then for $z > e^{\frac{1-\beta}{\beta}}$ $\phi_4(z) > 0$, hence $r(z)$ is bathtub, that is $h(t)$ is bathtub shaped.

References

[1] Aarset, M. V.(1987). How to identify bathtub hazard rate, *IEEE Trans. Reliability* **R-36**, 106-108.

[2] Barlow, Richard E. and Campo, R. (1975). Total time on test processes and applications to failure data analysis. In *Reliability and fault tree analysis* (Eds.: R. E. Barlow, J. B. Fussell and N. D. Singpurwalla), SIAM, Philadelphia, 451–481.

[3] Bergman, B. and Klefsjö, B. (1984). The total time on test concept and its use in reliability theory., *Operations Research* **32**, 596-606.

[4] Bergman, B. and Klefsjö, B. (1985). Burn-in models and TTT transforms. *Qual. and Reliab. Int.* **1,** 125-130.

[5] Bourguignon, M., Silva, R.B. and Cordeiro, G.M. (2014). The Weibull-G Family of Probability Distributions. *J. Data Sc.* **12**, 53–68.

[6] Burnham, K.P. and Anderson, D.R. (2002). *Model Selection and Multimodel Inference: A Practical Information-Theoretic Approach (2nd ed.)*, Springer-Verlag, Berlin.

[7] Chen, Z. (2000). A new two-parameter lifetime distribution with bathtub shape or increasing failure rate function. *Stat. & Prob. Lett.* **49,** 155-161.

[8] Davison, A.C. and Hinkley, D.V. (1997). *Bootstrap Methods and Their Application,* Cambridge University Press, London.

[9] Efron, B. and Tibshirani, R. (1985). Bootstrap Methods for Standard Errors, Confidence Intervals, and Other Measures fo Statistical Accuracy, *Statistical Science* **1,** 54-77.

[10] Friemer, M., Mudholkar, G.S., Kollia, G. and Lin, C.T. (1988). A study of the generalized Tukey lambda family, *Commun. Statist.-Theory Meth.* **17**(10), 3547-3567.

[11] Friemer, M., Mudholkar, G.S., Kollia, G. and Lin, C.T. (1989). Extremes, extreme spacings and outliers in the Tukey and Weibull families, *Commun. Statist.-Theory Meth.* **18**(11), 4261-4274.

[12] Gupta, R.D. and Kundu, D. (1999). Generalized exponential distributions. *Austr. NZ. Jour. Stat.* **41** 173–188.

[13] Gupta, R.C., Gupta, P.L. and Gupta, R.D. (1998). Modeling failure time data by Lehman alternatives. *Comm. Statist. Theory – Methods* **27**, 887-904.

[14] Gurvich, M. R., DiBenedetto, A.T. and Ranade, S.V. (1997). A new statisti- cal distribution for characterizing the random strength of brittle materials. *Journal of Materials Science* **32**, 2559–2564.

[15] Johnson, N.L., Kotz, S. and Balakrishnan, N. (1994). *Continuous Univariate Distributions*, Vol. 1 (2nd Ed.), Wiley, New York.

[16] Marshall, A.W. and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with applications to exponential and Weibull families. *Biometrika* **84** , 641–652.

[17] Mudholkar, G.S. and Hutson, A.D. (1996). The exponentiated weibull family: some properties and a flood data application, *Commun. Statist.-Theory Meth.* **25**, 3059-3083.

[18] Mudholkar, G.S., Kollia, G.D. (1994). Generalized Weibull family: a structural analysis. *Comm. Statist. Theory – Methods* **23**, 1149–1171.

[19] Mudholkar, G.S., Kollia, G.D., Lin, C.T. and Patel, K.R. (1991). A graphical procedure for comparing goodness-of-fit tests. *J. Roy. Statist. Soc. Ser. B* **53**, 221–232.

[20] Mudholkar, G.S. and Srivastava, D.K. (1993). Exponentiated Weibull family for analyzing bathtub failure-rate data, *IEEE Trans. Reliab.* **42**, 209-302.

[21] Mudholkar, G.S., Srivastava, D.K. and Kollia, G.D. (1996). A generalization of the Weibull distribution with application to the analysis of survival data. *J. Amer. Statist. Assoc.* **91**, 15751583.

[22] Murthy, D. N. P., Xie, M. and Jiang, R. (2003). *Weibull Models*, Hoboken, NJ, USA.

[23] Nadarajah, S. (2009). Bathtub-shaped failure rate functions. *Qual. Quant.* **43**, 855–863.

[24] Nadarajah, S.; Cordeiro, G.M. and Ortega, E.M.M. (2013). The exponentiated Weibull distribution: A survey. *Stat. Papers* **54**, 839-877.

[25] Nadarajah, S. and Kotz, S. (2006). The exponentiated type distributions. *Acta Appl. Math.* **92**, 97–111.

[26] Pappas, V., Adamaidis, K. and Loukas, S. (2012). A family of lifetime distributions. *Int. J. Qual., Stat. Rel.* **2012**, Article ID: 760687.

[27] Parzen, E. (1979). Nonparametric statistical data modelling. *Journal of the American Statistical Association* **74**, 105–131.

[28] Rajarshi, S. and Rajarshi, M.B. (1988). Bathtub distributions: A review. *Comm. Stat. – Theor. Methods* **17**, 2597–2621.

[29] Schuster, E.F. (1984). Classification of probability laws by tail behavior. *Jour. Amer. Statist. Assoc.* **79**, 936-939.

[30] Smith, R.M. and Bain, L.J.(1975). An exponential power life-testing distribution. *Comm. Statist.* **4**, 469-481.

[31] Tang, Y., Xie, M. and Goh, T.N. (2003). Statistical Analysis of a Weibull Extension Model. *Comm. Statist. Theory – Methods* **32**, 913–928.

[32] Weibull, W. (1951). A statistical distribution function of wide applicability. *J. Appl. Mech.-Trans. ASME* **18**, 293–297.

[33] Xie, M., Tang, Y. and Goh, T.N. (2002). A modified Weibull extension with bathtub failure rate function. *Reliab. Eng. System Saf.* **76**,279–285.

[34] Zografos, K. and Balakrishnan, N. (2009). On families of beta- and generalized gamma-generated distributions and associated inference. *Statistical Methodology* **6**, 344-362.

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