Matrix-based Ramanujan-Sums Transforms

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Abstract

In this paper, we study the Ramanujan Sums (RS) transform by means of matrix multiplication. The RS are orthogonal in nature and therefore offer excellent energy conservation capability. The 1D and 2D forward RS transforms are easy to calculate, but their inverse transforms are not defined in the literature for non-even function (mod M). We solved this problem by using matrix multiplication in this paper.

Keywords: Ramanujan Sums (RS); Fourier transform (FT); Gaussian white noise.

1. Introduction

The Ramanujan Sums (RS) were proposed by S. Ramanujan in 1918 [1], and were applied to time-frequency analysis, signal processing, moment invariants, and shape recognition recently ([2]-[9]). The RS are orthogonal in nature and therefore offer excellent energy conservation, similar to the Fourier transform (FT). The RS are operated on integers and hence can obtain a reduced quantization error implementation. Even though the RS transform has so many important properties, it does not have the inverse RS transform for non-even function (mod M) signals.

In this paper, we analyse the RS transform by means of matrix multiplication, which can invert the RS transform easily. We derive both the forward and inverse RS transforms for 1D signals and 2D images. A few examples are also tested and our method can recover the 1D signals and 2D images perfectly without any errors.

The organization of this paper is as follows. Section 2 presents a short review of the RS transform and proposes the matrix-based RS transforms for 1D signals and 2D images. The inverse RS transforms can recover the signals and images perfectly without any errors. Finally, Section 3 concludes the paper and proposes future research directions about the RS transform.

2. Matrix-based RS Transform

The RS transform has been used as means of representing arithmetical functions by an infinite series expansion. The basis of this transform is the building block of number-theoretic functions. The RS are sums of the n^{th} powers of q^{th} primitive roots of unity, defined as

$$c_q(n) = \sum_{p=1;\gcd(p,q)=1}^{q} \exp(2i\pi \frac{p}{q} n) \quad (1)$$

where gcd(p,q) =1 means that the greatest common divisor (GCD) is unity, i.e., p and q are co-prime. An alternate computation of RS can be given as

$$c_q(n) = \mu(\frac{q}{\gcd(q,n)}) \frac{\phi(q)}{\phi(\frac{q}{\gcd(q,n)})} \quad (2)$$

Let $q = \prod_i q_i^{\alpha_i}$ (q_i prime). Then, we have $\phi(q) = q\prod_i (1-\frac{1}{q_i})$. The Möbius function $\mu(n)$ is equal to 0 if n contains a square number; 1 if $n=1$; and $(-1)^k$ if n is a product of k distinct prime numbers. We tabulate $c_q(n)$ with q∈[1,15] in Table 1 in this paper.

The RS have the following multiplicative property:

$$c_{qq'}(n) = c_q(n)c_{q'}(n) \text{ if } \gcd(q,q')=1. \quad (3)$$

and the orthogonal property:

$$\sum_{n=1}^{qq'} c_q(n)c_{q'}(n) = 0 \text{ if } q \neq q'. \quad (4a)$$

$$\sum_{n=1}^{q} c_q^2(n) = q\phi(q) \text{ otherwise.} \quad (4b)$$

We can also derive $c_q(n)$ by using Euler's formula $e^{ix} = \cos x + i\sin x$ and basic trigonometric identities.

$$c_1(n) = 1\,.$$
$$c_2(n) = \cos n\pi\,.$$
$$c_3(n) = 2\cos\frac{2}{3}n\pi\,.$$
$$c_4(n) = 2\cos\frac{1}{2}n\pi\,.$$
$$c_5(n) = 2\cos\frac{2}{5}n\pi + 2\cos\frac{4}{5}n\pi\,.$$
$$c_6(n) = 2\cos\frac{1}{3}n\pi\,.$$
$$c_7(n) = 2\cos\frac{2}{7}n\pi + 2\cos\frac{4}{7}n\pi + 2\cos\frac{6}{7}n\pi\,.$$
$$c_8(n) = 2\cos\frac{1}{4}n\pi + 2\cos\frac{3}{4}n\pi\,.$$
$$c_9(n) = 2\cos\frac{2}{9}n\pi + 2\cos\frac{4}{9}n\pi + 2\cos\frac{8}{9}n\pi\,.$$
$$c_{10}(n) = 2\cos\frac{1}{5}n\pi + 2\cos\frac{3}{5}n\pi\,.$$

The 1D forward RS transform of a signal $x(m)$ is defined as

$$r(q) = \frac{1}{\phi(q)}\frac{1}{M}\sum_{m=1}^{M} x(m)c_q(m) \quad (5)$$

where M is the number of samples in the signal. However, there does not exist the inverse RS transform for the input signal in the literature. Haukkanen [10] claimed that every $x \in E_M$ can be written uniquely as

$$x(q) = M^{-1} \sum_{d|M} r(d) c_q(d) \tag{6}$$

where E_M is the set of all even functions *(mod M)*. A signal *x* is called even signal (mod M) if $x_M(n) = x_M(\gcd(n, M))$ for any *n*. It is easy to show that every even function *(mod M)* is a periodic function, but the converse does not hold. This means that for an ordinary input signal, its forward 1D RS transform exists, but the inverse transform cannot be calculated by using the above formula (6).

In this paper, we represent the 1D and 2D forward and inverse RS transforms by means of matrix multiplication. Let us define the matrix

$$A(q, j) = \frac{1}{\phi(q)M} c_q(\mathrm{mod}(j-1, q)+1) \tag{7}$$

where $q, j \in [1, M]$ and $\mathrm{mod}()$ means the modular operation. The input signal can be represented as $X = (x(1), x(2), ..., x(M))^T$, where the T means the transpose of the vector.

The forward 1D RS transform of a signal X can be realized as

$$Y = AX\ , \tag{8}$$

where $Y = (y(1), y(2), ..., y(M))^T$.

The inverse 1D RS transform can be obtained as

$$X = A^{-1}Y\ , \tag{9}$$

where A^{-1} means the inverse of matrix A. It has been proved in [11] that the determinant of the M×M matrix C, whose q, j entry is the Ramanujan sum $c_q(j)$, is $\det[c_q(j)] = M!$. Therefore,

$$\det[A(q, j)] = \frac{M!}{M^M \prod_{q=1}^{M} \phi(q)} \neq 0\ . \tag{10}$$

This means that the Matrix A(q,j) in always non-singular. Further reading about the determinant of C can be found in [12] and [13]. In order to be stable numerically, we can use QR decomposition to decompose matrix A, i.e., A=QR. Therefore,

$$X = R^{-1}Q^T Y\ . \tag{11}$$

We calculate the 1D forward and inverse RS transforms of three 1D signals that are used extensively in signal denoising literature. Fig. 1 shows the three input signals in the first row, their forward RS coefficients in the second row, their inverse RS transform signals in the third row, and the difference between the original signals and their reconstructed signals by using the method proposed in this paper in the last row. It can be seen that the error introduced in the transforms is nearly zero.

For 2D images, we can perform the forward RS transform as

$$Y(p, q) = \frac{1}{\phi(p)\phi(q)} \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} x(m, n) c_p(m) c_q(n) \tag{12}$$

This can be written in the matrix form

$$Y = AXA^T \tag{13}$$

where A is defined as in equation (7) and $X = (x(m, n))$ for $m \in [1, M]$ and $n \in [1, N]$.

The inverse 2D RS transform can be given as

$$X = A^{-1} Y (A^{-1})^T \tag{14}$$

As before, let A=QR. Then,

$$X = R^{-1} Q^T Y Q (R^{-1})^T\ . \tag{15}$$

We tested the Lena and Boat images of size 512×512 pixels for our 2D RS transform. Figs. 2 and 3 show the original Lena/Boat image, its RS coefficients, and the reconstructed image. For visual quality purpose, we only display a 50×50 region of the RS coefficients. It can be seen that the 2D RS transform compresses the energy of the image into a small number of RS coefficients. Our inverse 2D RS transform can recover the input image perfectly without errors.

The computational complexity of our matrix-based RS transforms is as follows. For 1D signal, both the forward and backward 1D RS transforms need $O(M^2)$ flops of operations, where M is the signal length. For 2D images, the forward 2D RS transform needs $O(M^2 N + MN^2)$ flops of operations. The inverse 2D RS transform also requires $O(M^2 N + MN^2)$ flops of operations.

3. Conclusions

In this paper, we have studied the 1D and 2D forward and inverse RS transforms by means of matrix multiplication. Our method can find the 1D and 2D inverse RS transforms for any kinds of signals and images. Currently, there is no existing inverse RS transform in the literature for non-even functions *(mod M)*. This paper fills in this gap by using the matrix notation.

Future research directions about the RS transform are given below. We would like to apply the 1D and 2D RS transforms to signal, image, or video compression. This is because the RS transform has very good property to compress the energy of the input signals, images, or videos into a few number of RS coefficients. We would also extend the RS transform to 3D data cube. This may have important applications in hyperspectral imagery analysis.

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Table 1. The RS basis $c_q(n)$ for q∈[1,15].

q															
1	1														
2	-1	1													
3	-1	-1	2												
4	0	-2	0	2											
5	-1	-1	-1	-1	4										
6	1	-1	-2	-1	1	2									
7	−1	−1	−1	−1	−1	−1	6								
8	0	0	0	−4	0	0	0	4							
9	0	0	−3	0	0	−3	0	0	6						
10	1	−1	1	−1	−4	−1	1	−1	1	4					
11	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	10				
12	0	2	0	−2	0	−4	0	−2	0	2	0	4			
13	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	12		
14	1	−1	1	−1	1	−1	−6	−1	1	−1	1	−1	1	6	
15	1	1	−2	1	−4	−2	1	1	−2	−4	1	−2	1	1	8

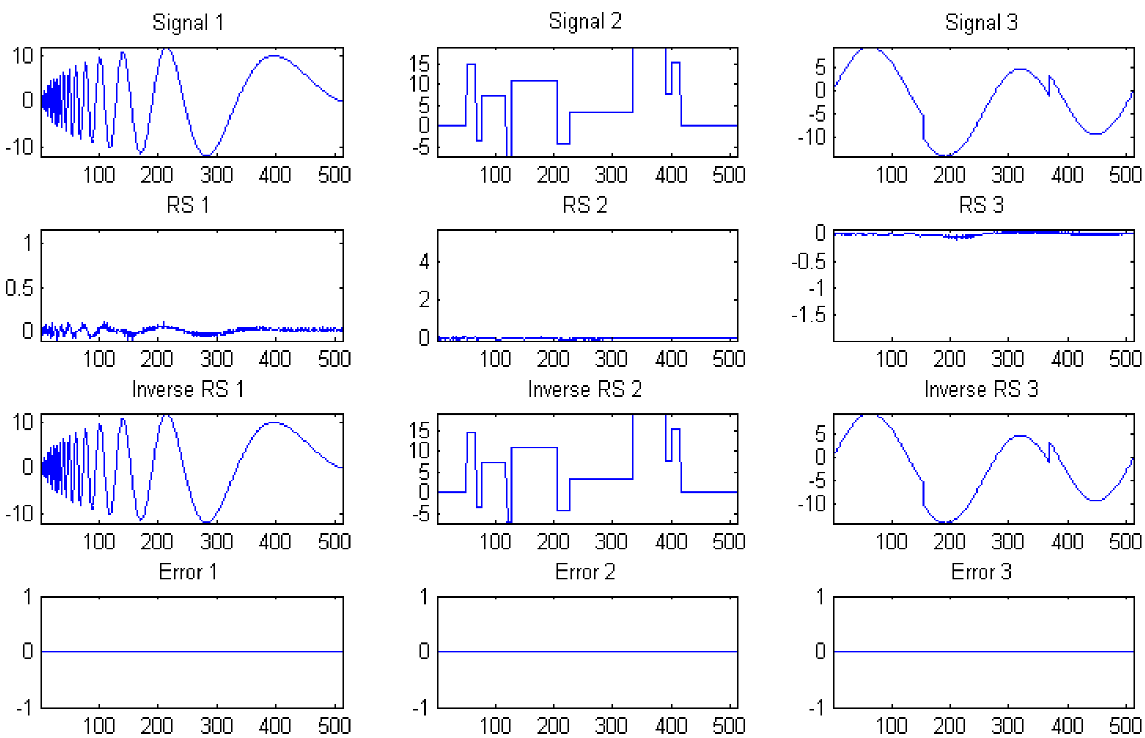

Figure 1. The forward and inverse RS transforms for three signals. It can be seen that the errors introduced is nearly zeros.

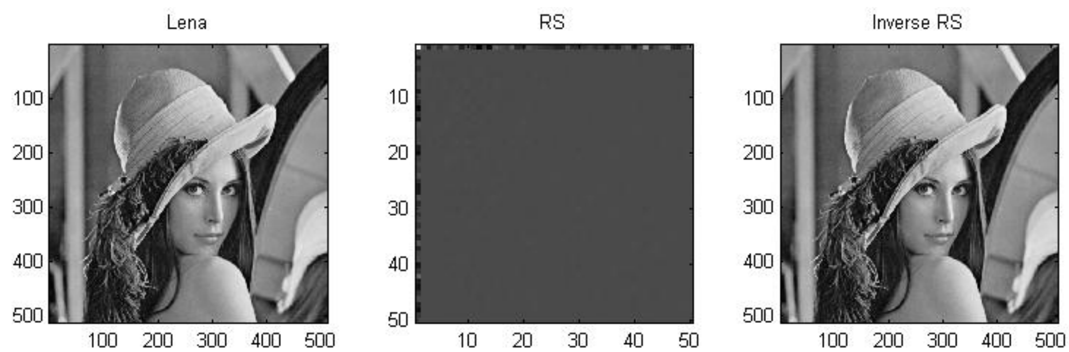

Figure 2. The forward and inverse 2D RS transforms for the Lena image.

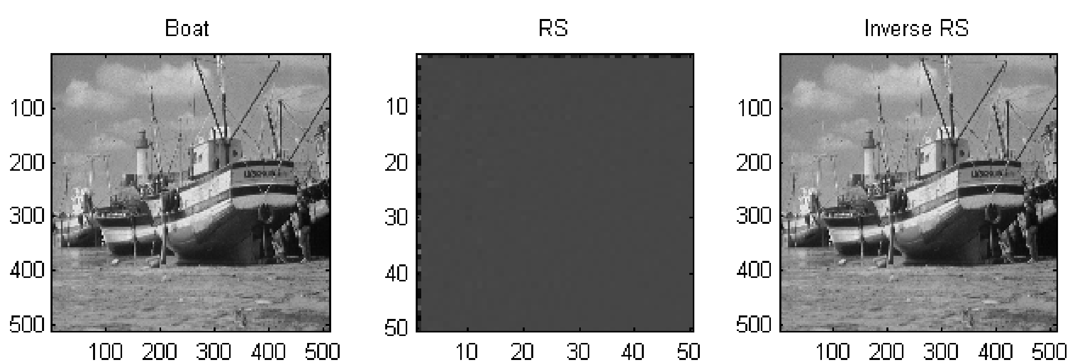

Figure 3. The forward and inverse 2D RS transforms for the Boat image.