Slope Conditions for Stability of ACIMs of Dynamical Systems

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ABSTRACT

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The family of $W-$shaped maps was introduced in 1982 by G. Keller. Based on the investigation of the properties of the maps, it was conjectured that instability of the absolutely continuous invariant measure (acim) can result only from the existence of small invariant neighbourhoods of the fixed critical point of the limiting map. We show that the conjecture is not true by constructing a family of $W-$shaped maps with acims supported on the entire interval, whose limiting dynamical behavior is described by a singular measure. We then generalize the above result by constructing families of $W-$shaped maps $\{W_a\}$ with a turning fixed point having slope s_1 on one side and $-s_2$ on the other. Each such W_a map has an acim μ_a. Depending on whether $\frac{1}{s_1}+\frac{1}{s_2}$ is larger, equal, or smaller than 1, we show that the limit of μ_a is a singular measure, a combination of singular and absolutely continuous measure or an acim, respectively. We also consider W-shaped maps satisfying $\frac{1}{s_1}+\frac{1}{s_2}=1$ and show that the eigenvalue 1 of the associated Perron-Frobenius operator is not stable, which in turn implies the instability of the isolated spectrum. Motivated by the above results,

we introduce the harmonic average of slopes condition, with which we obtain new explicit constants for the upper and lower bounds of the invariant probability density function associated with the map, as well as a bound for the speed of convergence to the density. Moreover, we prove stability results using Rychlik's Theorem together with the harmonic average of slopes condition for piecewise expanding maps.

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Chapter 1

Preliminaries

1.1 Absolutely continuous invariant measures and some functional spaces

Let X be a set equipped with a metric; X is usually assumed to be a compact metric space. Let a family $\mathfrak{B}$ of subsets of X be a $\sigma-$algebra.

Definition 1.1.1. A function $\mu : \mathfrak{B} \to \mathbf{R}^+$ is called a measure on $\mathfrak{B}$ if and only if:

1. $\mu(\emptyset) = 0$;

2. for any sequence $\{B_n\}$ of disjoint measurable sets, where $B_n \in \mathfrak{B}$, $n = 1, 2, 3, \ldots$,

$$\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n).$$

The triplet $(X, \mathfrak{B}, \mu)$ is called a measure space. We call it a normalized measure space or probability space if $\mu(X) = 1$.

Definition 1.1.2. A measure is called $\sigma-$finite if X is the countable union of measurable sets with finite measure.

Definition 1.1.3. Suppose there are two measures, μ and ν, on the same measure space $(X, \mathfrak{B})$. We say that μ is absolutely continuous with respect to ν if for any $B \in \mathfrak{B}$ satisfying $\nu(B) = 0$, it follows that $\mu(B) = 0$. We write $\mu << \nu$.

Definition 1.1.4. Let $1 \leq p < \infty$, and $(X, \mathfrak{B}, \mu)$ be a measure space. Consider the family of real valued measurable functions $f : X \to \mathbf{R}$ satisfying

$$\int_X \mid f(x) \mid^p \, d\mu < \infty \, .$$

This family of functions can be made into a normed space by taking quotient space with respect to the kernel of $\| \cdot \|_p$, where for each function f in this family,

$$\| f \|_p = \left(\int_X \mid f(x) \mid^p \, d\mu \right)^{\frac{1}{p}} \, .$$

This space is called the $L^p(X, \mathfrak{B}, \mu)$ space. It will be denoted by $L^p(\mu)$ when the underlying space is clearly known, and by L^p when both the space and the measure are known.

The Radon-Nikodym Theorem allows us to represent μ in terms of ν if $\mu << \nu$.

Theorem 1.1.1. *Let μ and ν be two normalized measures on $(X, \mathfrak{B})$. If $\mu << \nu$, then there exists a unique $f \in L^1(X, \mathfrak{B}, \nu)$ such that for every $A \in \mathfrak{B}$,*

$$\mu(A) = \int_A f \, d\,\nu.$$

The function f is called the Radon-Nikodym derivative and is denoted by $\frac{d\mu}{d\nu}$.

Let us introduce the measure-preserving transformation.

Definition 1.1.5. We say that the measurable transformation $\tau : X \to X$ preserves the measure μ or that μ is $\tau-$invariant if $\mu(\tau^{-1}(B)) = \mu(B)$ for all $B \in \mathfrak{B}$.

If a measure μ is invariant and also $\mu << \nu$, where ν is the underlying measure, then we call μ an absolutely continuous invariant measure (acim). Now, we can present the definition of a dynamical system.

Definition 1.1.6. Let $\tau : X \to X$ preserve the measure μ. We call the quadruple $(X, \mathfrak{B}, \mu, \tau)$ a dynamical system.

Let $\tau : X \to X$ be a measure-preserving transformation. Its nth iteration is denoted by τ^n: $\tau^n(x) = \tau \circ \cdots \circ \tau(x)$ n times. We are interested in properties of the orbit $\{\tau^n(x)\}_{n\geq 0}$. The Poincaré Recurrence Theorem states that a dynamical system will return to a state very close to the initial state after a sufficiently long time.

Theorem 1.1.2. *Let τ be a measure-preserving transformation on a normalized measure space $(X, \mathfrak{B}, \mu)$. Let $E \in \mathfrak{B}$ such that $\mu(E) > 0$. Then almost all points of E return infinitely often to E under iterations of τ, i.e.,*

$$\mu\left(\{x \in E \mid \textit{ There exists } N \textit{ such that } \tau^n(x) \notin E \textit{ for all } n > N\}\right) = 0.$$

Definition 1.1.7. Let $\tau : (X, \mathfrak{B}, \mu) \to (X, \mathfrak{B}, \mu)$ be a transformation preserving the measure μ; we call it

1. ergodic if for any $B \in \mathfrak{B}$ such that $\tau^{-1}(B) = B$, we have $\mu(B) = 0$ or $\mu(X \backslash B) = 0$;

2. mixing if for all A, $B \in \mathfrak{B}$, $\mu(\tau^{-n}(A) \cap B) \to \mu(A)\mu(B)$ as $n \to \infty$;

3. exact if for all $B \in \mathfrak{B}$, $\tau(B) \in \mathfrak{B}$, and moreover, if $\mu(B) > 0$, $\lim_{n\to\infty} \mu(\tau^n(A)) = 1$.

Ergodicity is a very useful concept since it reveals the property of indecomposability for measure-preserving transformations. There are some other properties equivalent to the ergodicity [Boyarsky and Góra, 1997]:

Theorem 1.1.3. *Let $\tau : (X, \mathfrak{B}, \mu) \to (X, \mathfrak{B}, \mu)$ be measure-preserving. Then the following statements are equivalent:*

1. *τ is ergodic.*

2. *If f is measurable and $(f \circ \tau)(x) = f(x)$ a.e., then f is constant a.e..*

3. *If $f \in L^2(\mu)$ and $(f \circ \tau)(x) = f(x)$ a.e., then f is constant a.e..*

The Birkhoff Ergodic Theorem [Birkhoff, 1931] is the fundamental theorem in ergodic theory.

Theorem 1.1.4. *Suppose $\tau : (X, \mathfrak{B}, \mu) \to (X, \mathfrak{B}, \mu)$ is measure-preserving, where $(X, \mathfrak{B}, \mu)$ is $\sigma-$finite, and $f \in L^1(\mu)$. Then there exists a function $f^* \in L^1(\mu)$ such that*

$$\frac{1}{n}\sum_{k=0}^{n-1} f(\tau^k(x)) \to f^*, \ \mu - a.e.$$

Furthermore,

$$f^* \circ \tau = f^* \ \mu - a.e.$$

and if $\mu(X) < \infty$, then

$$\int_X f^* \, d\mu = \int_X f \, d\mu$$

By Theorem 1.1.3, we obtain the following corollary.

Corollary 1.1.1. *If τ is ergodic, then f^* is constant $\mu-$a.e. and if $\mu(X) < \infty$, then*

$$f^* = \frac{1}{\mu(X)} \int_X f \, d\mu \ a.e.$$

Thus, if $\mu(X) = 1$ and τ is ergodic, for $E \in \mathfrak{B}$ we have

$$\frac{1}{n} \sum_{k=0}^{n-1} \chi_E(\tau^k(x)) \to \mu(E), \ \mu - a.e.,$$

and thus the orbit of almost every point of X occurs in the set E with asymptotic relative frequency $\mu(E)$.

Theorem 1.1.4 together with Corollary 1.1.1 shows that for an ergodic transformation $\tau : (X, \mathfrak{B}, \mu) \to (X, \mathfrak{B}, \mu)$, we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\tau^k(x)) = \frac{1}{\mu(X)} \int_X f \, d\mu \text{ a.e.},$$

i.e. the time average of $f \in L^1(\mu)$ equals its space average.

1.2 Functions of bounded variations and the Frobenius-Perron operator

We first introduce the total variation of $f : [a, b] \to \mathbf{R}$.

Definition 1.2.1. Let $f : [a, b] \to \mathbf{R}$ be a function. The number

$$\bigvee_{[a,b]} f = \sup_{\mathcal{P}} \left\{ \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \right\},$$

where the sup is taken over all partitions $\mathcal{P}$ of the interval $[a, b]$, is called the total variation or the variation of f on $[a, b]$.

We denote the space of functions of bounded variation by

$$BV([a,b]) = \left\{ f \in L^1([a,b]) | \inf_{f_1 = f\, a.e.} \bigvee_{[a,b]} f_1 < \infty \right\}.$$

For any $f \in BV([a,b])$, the norm on $BV([a,b])$ is defined as follows:

$$\| f \|_{BV} = \| f \|_1 + \inf_{f_1 = f\, a.e.} \bigvee_{[a,b]} f_1.$$

Let X be an interval, $I = [a,b]$. We call a transformation $\tau : I \to I$ nonsingular if $L(A) = 0$ implies $L(\tau^{-1}(A)) = 0$ whenever $A \in \mathfrak{B}$, and where L is Lebesgue measure. With the aid of Theorem 1.1.1, we now define the Frobenius-Perron operator:

Definition 1.2.2. Let $\tau : I \to I$ be a nonsingular transformation. For any $f \in L^1([a,b])$, $P_\tau f$ is the unique function in $L^1([a,b])$ such that

$$\int_A P_\tau f \, dL = \int_{\tau^{-1}(A)} f \, dL \,,$$

for any $A \in \mathfrak{B}$. The existence and uniqueness of $P_\tau f$ follows from Theorem 1.1.1.

For more details, we refer interested readers to the book [Boyarsky and Góra, 1997]. If $\tau \in \mathfrak{T}(I)$, the class of piecewise expanding transformations (see Definition 5.2.1), P_τ has the explicit representation:

$$P_\tau f = \sum_{i=1}^{q} \frac{f(\tau_i^{-1}(x))}{|\tau'(\tau_i^{-1}(x))|} \chi_{\tau([a_{i-1}, a_i])}(x) \,,$$

where $f \in L^1(I)$. It is well known that the fixed point (normalized one) of P_τ is the density of a $\tau-$invariant absolutely continuous measure. We call it the τ invariant probability density function (pdf).

To show the existence of the invariant pdf, the following lemma is important.

Lemma 1.2.1. *Let $\tau \in \mathcal{T}(I)$ and let $g(x) = \frac{1}{|\tau'(x)|}$, $\delta = \min_{1 \le i \le q} L(I_i)$. Then, for any $f \in BV(I)$,*

$$\bigvee_I (P_\tau f) \le A \bigvee_I f + B \int_I |f| \, dL \ ,$$

where $A = \frac{2}{\alpha} + \max_{1 \le i \le q} \bigvee_{I_i} g$, $B = \frac{2}{\alpha\delta} + \frac{1}{\delta} \max_{1 \le i \le q} \bigvee_{I_i} g$, and $\inf_{x \in I} |\tau'(x)| \ge \alpha > 1$.

Now, we introduce the well-known Lasota-Yorke Inequality which plays a crucial role in showing the existence of an acim.

Lemma 1.2.2. *Let $\tau \in \mathcal{T}(I)$. Then there exist constants $0 < r < 1$, $C > 0$, and $R > 0$ such that for any $f \in BV(I)$ and any $n \ge 1$,*

$$\| P_\tau^n f \|_{BV} \le C r^n \| f \|_{BV} + R \| f \|_1 \ .$$

Remark 1.2.1. Lemmas 1.2.1 and 1.2.2 show that the Frobenius-Perron operator P_τ can be viewed as an operator from $BV(I)$ into $BV(I)$, and that it is quasi-compact [Keller, 1982].

With the help of Lemmas 1.2.1 and 1.2.2, we have the following result for the existence of an acim for a piecewise expanding map.

Theorem 1.2.1. *Let $\tau \in \mathcal{T}(I)$. Then τ admits an acim whose density is of bounded variation.*

Remark 1.2.2. On the one hand, the constant R in Lemma 1.2.2 depends on the constant B in Lemma 1.2.1, and the latter again depends on the quantity δ which can be small if the partition of the map τ is very fine. On the other hand, in practice, to make the calculations easier, we consider the iterated system τ^n for some $n \ge 1$.

Moreover, we would like to study the stability of a system. In either of the cases, δ is small, and may even approach 0 as the perturbation goes to zero. See Fig. 4.1 in Chapter 4 for an example. This is why it is relatively easier to show the existence of acim than to show the stability of acim.

1.3 Analysis of dynamical systems: acims and their stability

It is well known that complicated behavior can be exhibited by deterministic dynamical systems. A popular example of deterministic chaotic behavior is the butterfly effect, which dates back to Lorenz's numerical model of a weather prediction [Lorenz, 1963]. Roughly speaking, chaos is a very spectacular long-term behavior of dynamical systems that are highly sensitive to initial conditions. For these kinds of dynamical systems, small differences in initial conditions can result in large differences to future outcomes. In reality, due to external noise or roundoff errors in computation, when we deal with a chaotic dynamical system, it is very hard or even impossible to predict its state past a certain short time range. Thus, we cannot study the limiting behavior following individual orbits which are generated by iterating the system from starting points.

Instead of viewing orbits in the phase space X, we apply the Frobenius-Perron operator P_τ associated with the transformation τ, which defines the evolution of the probability density function under τ. It is an operator acting on $L^1(X)$, the space of

Lebesgue integrable functions. The fixed point of P_τ, say $f(x)$ which is a pdf, describes how all the orbits will distribute in the future. This invariant density function f defines an invariant measure $\mu(B) = \int_B f dL$ for $B \in \mathfrak{B}$, where L is Lebesgue measure. Among invariant measures of a dynamical system, the acim (usually it is absolutely continuous with respect to Lebesgue measure) is of the greatest practical importance, as it is a "physical" measure and can be simulated by computer. With the help of the acim, we can study the ergodic properties of the dynamical system, for example, the Birkhoff Ergodic Theorem 1.1.4 and the Poincaré Recurrence Theorem 1.1.2.

In practice, there is a natural interest in the stability of properties of chaotic dynamical systems under small perturbations. If we consider a family of piecewise expanding maps $\tau_a : I \to I$, $a > 0$ with acims $\{\mu_a\}$, converging to a piecewise expanding map τ_0 with acim μ_0, then under general assumptions μ_a's converge to μ_0. One such assumption is that $\inf |\tau_a'| > 2$ for all $a > 0$ (see [Baladi and Smania, 2010], [Góra, 1979], [Góra and Boyarsky, 1989a] or [Keller and Liverani, 1999]). This is useful in the study of the metastable systems [Gonzaléz-Tokman et al., 2011], or to approximate the invariant densities [Góra and Boyarsky, 1989b].

[Keller, 1982] introduced the family of $\{W_a\}$ maps that are piecewise expanding, ergodic transformations with a "stochastic singularity", i.e. the μ_a's converge to a singular measure. This occurs because of the existence of diminishing invariant neighborhoods of the turning fixed point. The slopes of the W_a maps converge to 2 and -2 on the left and right hand sides of the turning fixed point, respectively. In Chapter 2 we will present more details.

Given two numbers, s_1 and s_2, greater than 1, we consider a W-shaped map with one turning fixed point having slope s_1 on one side and $-s_2$ on the other. A $W-$shaped map is a map with a graph in the shape of the letter W for which the middle vertex is a fixed point. More precisely, it is a map $\tau : I \to I$, piecewise monotonic on the partition $\{I_1, I_2, I_3, I_4\}$ of I, $I_i = [a_{i-1}, a_i], i = 1, 2, 3, 4$, such that $\tau(a_0) = \tau(a_4) = 1, \tau(a_1) = \tau(a_3) = 0$ and $\tau(a_2) = a_2$.

In Chapter 2 [Li et al., 2013], we consider the special case where $s_1 = s_2 = 2$. The perturbed maps W_a are piecewise expanding with slopes strictly greater than 2 in modulus and are exact with their acims supported on all of [0, 1]. The standard bounded variation method [Boyarsky and Góra, 1997] cannot be applied in this setting as the slopes of the maps in that family are not uniformly bounded away from 2. Other methods, for example, those studied in [Dellnitz et al., 2000], [Kowalski, 1979] and [Murray, 2005] cannot be applied either. Using the main result of [Góra, 2009], it can be shown that the μ_a's converge to $\frac{2}{3}\mu_0 + \frac{1}{3}\delta_{(\frac{1}{2})}$, where $\delta_{(\frac{1}{2})}$ is the Dirac measure at the point 1/2 and μ_0 is the acim of the W_0 map. Thus, the family of measures μ_a approach a combination of an absolutely continuous and a singular measure rather than the acim of the limit map. Similar instability was also shown in [Eslami and Misiurewicz, 2012] for a countable family of transitive Markov maps approaching Keller's W_0 map.

The result of Chapter 2 is generalized in Chapter 3, where a family of $W-$shaped maps is constructed, and various instabilities are presented. Depending on whether $\frac{1}{s_1} + \frac{1}{s_2}$ is larger, equal, or smaller than 1, we show that the limiting measure is a singular measure, a combination of singular and absolutely continuous measure or an absolutely continuous measure, respectively. The main result is Theorem 3.2.1. More importantly, this result inspired the introduction of the harmonic average of slopes condition, which motivated the result concerning a stronger Lasota-Yorke type inequality in [Eslami and Góra, 2012]. It is also applied in Chapter 5 and Chapter 6.

In Chapter 4, we study the instability of the $W-$shaped map by observing the unstable spectrum of its Frobenius-Perron operator. The second eigenvalues of the perturbed Frobenius-Perron operators converge to 1, which is the maximal eigenvalue of the Frobenius-Perron operator. We also discuss the relation between an unstable second eigenvalue and a metastable dynamical system.

In Chapter 5, whose main results are also presented in the joint work [Góra et al., 2012b], we apply the harmonic average of slopes condition to the transformation in the class $\mathcal{T}(I)$, the class of piecewise expanding transformations. Here we use the weak covering property, weak mixing and a generalized Lasota-Yorke type inequality [Eslami and Góra, 2012]. We weaken the slope 2 condition to ensure stability, obtain the explicit constants concerning the lower bound of the invariant pdf, and the explicit constants for the decay of correlations. We also extend our results to families of maps.

In Chapter 6, whose main results can also be found in the joint work [Góra et al., 2012a], we continue to use the harmonic average of slopes condition. Instead of the bounded variation technique (Lasota-Yorke type inequality), we introduce the summable oscillation condition, and use Rychlik's Theorem (see, e.g., [Boyarsky and Góra, 1997]) to show the existence of an acim and its stability.

Chapter 2

$W-$shaped Maps Having Singular Measure as a Limit of Acims

2.1 Introduction

Usually, the acim of a piecewise expanding map of an interval is stable under deterministic or even random perturbations. This means that if we consider a family of piecewise expanding maps τ_a, $a > 0$, with acims μ_a, converging to a piecewise expanding map τ_0 with acim μ_0, then under general assumptions μ_a's converge to μ_0. As we already discussed in Chapter 1, one such assumption is that for some positive ϵ, $|\tau_a'| > 2 + \epsilon$ for all $a \geq 0$.

[Keller, 1982] introduced a family of W-maps that are piecewise expanding and exhibit a wide variety of behaviour. This was done to understand whether in dimension one the expanding constant ensuring stability is really 2 rather than 1 as for zero-dimensional systems [Góra, 1979]. This regularity was later confirmed in [Góra and Boyarsky, 1989a] by showing that this constant for a piecewise expanding n-dimensional system with rectangular partition is $n+1$.

Key to the complexity of Keller's families is the fact that, as the parameter approaches 0, say, the behavior near a folded turning point plays a crucial role. This turning point has slope 2 on one side and -2 on the other. Thus, the entire family is uniformly piecewise expanding and each member has a unique acim. However, the stability of probability density functions that one might expect in families of uniformly piecewise expanding maps does not occur. Keller provided an example of a family for which the limit of acims is a singular measure. This occurred because of the existence of diminishing invariant neighbourhoods of the turning point. Keller conjectured that this is the only mechanism which can cause such limiting behaviour.

In this chapter we construct a family of simple W-maps which disproves Keller's conjecture. All our maps are piecewise expanding with slopes strictly greater than 2 in magnitude and are exact with their acims supported on all of $[0,1]$, but the limiting dynamical behaviour is captured by a singular measure.

Standard bounded variation methods [Boyarsky and Góra, 1997; Dellnitz et al., 2000; Kowalski, 1979; Murray, 2005] cannot be applied in this setting as the slopes of maps in our family are not uniformly bounded away from 2. In this chapter we shall utilize the main result of [Góra, 2009], which proves that the invariant probability density function (pdf) for any piecewise linear map which is eventually expanding has a convenient infinite series expansion. The estimates on the family of pdf's derived from this representation allows us to prove our main result, that the acims of the family of W-maps approach a combination of an absolutely continuous and a singular measure rather than the acim of the limit map.

In Section 2.2 we introduce our family and state the main theorem, which is proved in Section 2.4. In Section 2.5, we show computational results for some pdf's of the W_a maps when a is small.

The results obtained in this chapter (Sections , 2.4 and 2.5) were, after some modifications, published in the paper [Li et al., 2013].

2.2 Family of W_a maps and the main result

We consider the family $\{W_a : 0 \leq a\}$ of maps of $[0, 1]$ onto itself defined by

$$W_a(x) = \begin{cases} 1 - 4x \ , \quad \text{for} \ \ 0 \leq x < 1/4 \ ; \\ \\ (2 + a)(x - 1/4) \ , \quad \text{for} \ \ 1/4 \leq x < 1/2 \ ; \\ \\ 1/2 + a/4 - (2 + a)(x - 1/2) \ , \quad \text{for} \ \ 1/2 \leq x < 3/4 \ ; \\ \\ 4(x - 3/4) \ , \quad \text{for} \ \ 3/4 \leq x \leq 1 \ . \end{cases} \tag{2.1}$$

The map W_0 is Keller's W-map [Keller, 1982]. We consider only small $a > 0$ as we are interested in the limiting behaviour of the W_a's as $a \to 0$. Fig. 1 shows the graphs of W_a for $a = 0$ and $a > 0$. Every W_a is a piecewise linear, piecewise expanding map with minimal modulus of the slope equal to $2 + a$. Every W_a has a unique acim μ_a supported on $[0, 1]$ and is exact with respect to this measure. The transitivity of such maps is proven in [Eslami and Misiurewicz, 2012]. The uniqueness of the acim and exactness follow directly from the Li-Yorke paper [Li and Yorke, 1978].

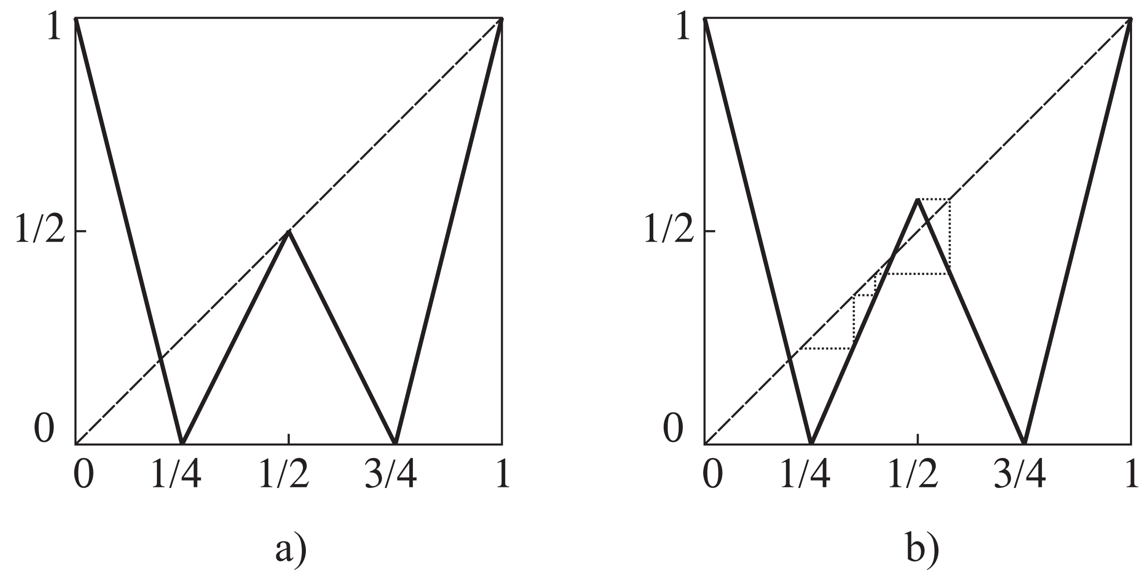

Figure 2.1: a) map W_0, b) map W_a, $a > 0$, with a few first points of the trajectory of 1/2.

Let h_a denote the normalized density of μ_a, $a \geq 0$. It is easy to check that for W_0, μ_0 has density

$$h_0 = \begin{cases} \frac{3}{2}\ , & \text{for}\ \ 0 \leq x < 1/2\ ; \\ \frac{1}{2}\ , & \text{for}\ \ 1/2 \leq x \leq 1\ . \end{cases} \tag{2.2}$$

Our goal is to prove the following theorem:

Theorem 2.2.1. *As $a \to 0$ the measures μ_a converge $*$-weakly to the measure*

$$\frac{2}{3}\mu_0 + \frac{1}{3}\delta_{(\frac{1}{2})}\ ,$$

where $\delta_{(\frac{1}{2})}$ is the Dirac measure at point 1/2.

The proof relies on the general formula for invariant densities of piecewise linear maps [Góra, 2009] and direct calculations. The calculations depend on the parameter a, but we suppress it whenever there is no confusion.

We spent some time on finding the proof to support this theorem. The first crucial step was finding a special sequence of a's, such that the $W-$shaped map is approximated by Markov ones, and then observing what happens as a approaches 0.

2.3 Markov subfamily, the heuristic idea for the proof of the general theorem

In this section we consider a Markov subfamily of the family $\{W_a\}_{a\geq 0}$. We consider a's such that for some finite $m \geq 2$, $W_a^m(1/2) = 1/4$. For these a's all calculations can be made in a finite form. Our method is different from the standard Markov map approach. Other Markov subfamilies can be considered in a similar way. A 3-parameter family of transitive Markov maps converging to the W-map was considered in [Eslami and Misiurewicz, 2012].

Let $W_{a,i}$ denote the i-th branch of the W_a map, $i = 1, 2, 3, 4$. Let $s_i = W_{a,i}^{-1}, i = 1, 2, 3, 4$; $I_0 = [0, \frac{1}{2} + \frac{a}{4}]$. The associated Frobenius-Perron operator of W_a is

$$P_a f = \frac{1}{4} f \circ s_1 + \frac{1}{2+a}(f \circ s_2)\chi_{I_0} + \frac{1}{2+a}(f \circ s_3)\chi_{I_0} + \frac{1}{4} f \circ s_4$$

Notice that $\chi_{I_0} \circ s_1 = 1, \chi_{I_0} \circ s_2 = \chi_{I_0}, \chi_{I_0} \circ s_3 = [(2+a)(\frac{1}{4} - \frac{a}{4}), \frac{1}{2} + \frac{a}{4}], \chi_{I_0} \circ s_4 = 0$; let $I_1 := [(2+a)(\frac{1}{4} - \frac{a}{4}), \frac{1}{2} + \frac{a}{4}]$ whose left end point is $W_a^2(\frac{1}{2})$, i.e. $W_a(\frac{1}{2} + \frac{a}{4})$.

Let a satisfy:

$$W_a^m(\frac{1}{2}) = \frac{1}{4}, \tag{2.3}$$

where $m \geq 2$ is the first time the trajectory of $\frac{1}{2}$ reaches $\frac{1}{4}$.

Let us take 1 as the initial function to do the iteration $P_a^n 1$ which will be denoted by $f_{n,m}$. Let

$$I_i = [W_a^i(\frac{1}{2}+\frac{a}{4}), \frac{1}{2}+\frac{a}{4}], i = 1, 2, \cdots, m.$$

After a certain number of iterations, using (2.3) we will get:

$$f_{n,m} = c_{n,0} + \alpha_{n,0}\chi_{I_0} + \alpha_{n,1}\chi_{I_1} + \alpha_{n,2}\chi_{I_2} + \cdots + \alpha_{n,m-1}\chi_{I_{m-1}} + \alpha_{n,m}\chi_{I_m},$$

where $c_{n,0}$ and $\alpha_{n,i}(i = 0, 1, \cdots, m)$ are constants. Now let us look at the $f_{n+1,m}$. By straightforward calculations, we obtain the following proposition.

Proposition 2.3.1. (1) *$c_{n,0} \circ s_1$ and $c_{n,0} \circ s_4$ are again constant functions, $c_{n,0} \circ s_2\chi_{I_0}$ and $c_{n,0} \circ s_3\chi_{I_0}$ are the characteristic function χ_{I_0};*

(2) *$\chi_{I_0} \circ s_1$ is a constant function, $\chi_{I_0} \circ s_2\chi_{I_0} = \chi_{I_0}$, $\chi_{I_0} \circ s_3\chi_{I_0} = \chi_{I_1}$, $\chi_{I_0} \circ s_4$ is 0;*

(3) *For $i = 1, 2, \cdots, m-1$, $\chi_{I_i} \circ s_1$ and $\chi_{I_i} \circ s_4$ are 0, $\chi_{I_i} \circ s_2\chi_{I_0} = \chi_{I_{i+1}}$, $\chi_{I_i} \circ s_3\chi_{I_0} = \chi_{I_1}$;*

(4) *$\chi_{I_m} \circ s_1$ and $\chi_{I_m} \circ s_4$ are 0, $\chi_{I_m} \circ s_2\chi_{I_0} = \chi_{I_0}$, $\chi_{I_m} \circ s_3\chi_{I_0} = \chi_{I_1}$.*

Thus, we have the following proposition.

Proposition 2.3.2. *For n sufficiently large, $f_{n,m}$ always has the form:*

$$f_{n,m} = c_{n,0} + \alpha_{n,0}\chi_{I_0} + \alpha_{n,1}\chi_{I_1} + \alpha_{n,2}\chi_{I_2} + \cdots + \alpha_{n,m-1}\chi_{I_{m-1}} + \alpha_{n,m}\chi_{I_m},$$

and

$$\begin{bmatrix} c_{n+1,0} \\ \alpha_{n+1,0} \\ \alpha_{n+1,1} \\ \vdots \\ \alpha_{n+1,m} \end{bmatrix} = A_m \begin{bmatrix} c_{n,0} \\ \alpha_{n,0} \\ \alpha_{n,1} \\ \vdots \\ \alpha_{n,m} \end{bmatrix}.$$

A_m is given by

$$A_m = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{2}{2+a} & \frac{1}{2+a} & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2+a} \\ 0 & \frac{1}{2+a} & \frac{1}{2+a} & \frac{1}{2+a} & \frac{1}{2+a} & \cdots & \frac{1}{2+a} & \frac{1}{2+a} \\ 0 & 0 & \frac{1}{2+a} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2+a} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & \frac{1}{2+a} & 0 \end{bmatrix}$$

where A_m is $(m+2) \times (m+2)$.

We also need the following proposition.

Proposition 2.3.3. *(2.3) is equivalent to:*

$$(2+a)^m - \sum_{i=0}^{m-1} (2+a)^i = \frac{1}{a} \ . \qquad (2.4)$$

Proof. Actually, equation (2.4) is:

$$\frac{a(2+a)^m+1}{1+a} = \frac{1}{a},$$

which is equivalent to

$$-\frac{1}{4}a\frac{a(2+a)^m+1}{1+a} + \frac{1}{2} = \frac{1}{4}.$$

By equation (2.3) and Lemma 2.4.3(I), we finish the proof. □

Using Proposition 2.3.3, we can find the fixed vector of A_m. If we denote it by $(c, \alpha_0, \alpha_1, \cdots, \alpha_m)^T$, then up to a multiplicative constant the invariant function of P_a is

$$g_m^* = c + \alpha_0\chi_{I_0} + \alpha_1\chi_{I_1} + \alpha_2\chi_{I_2} + \cdots + \alpha_{m-1}\chi_{I_{m-1}} + \alpha_m\chi_{I_m},$$

where

$$\begin{aligned}
c &= \frac{1}{2a} \\
\alpha_0 &= \frac{1}{a} \\
\alpha_1 &= (2+a)^{m-1} \\
\alpha_2 &= (2+a)^{m-2} \\
\cdots \\
\alpha_{m-2} &= (2+a)^2 \\
\alpha_{m-1} &= 2+a \\
\alpha_m &= 1.
\end{aligned}$$

Let us normalize g_m^*. First, we multiply g_m^* by a, and denote the new function by f_m^*:

$$f_m^* = C + \beta_0\chi_{I_0} + \beta_1\chi_{I_1} + \beta_2\chi_{I_2} + \cdots + \beta_{m-1}\chi_{I_{m-1}} + \beta_m\chi_{I_m},$$

where

$$\begin{aligned}
C &= \frac{1}{2} \\
\beta_0 &= 1 \\
\beta_1 &= a(2+a)^{m-1} \\
\beta_2 &= a(2+a)^{m-2} \\
&\cdots \\
\beta_{m-2} &= a(2+a)^2 \\
\beta_{m-1} &= a(2+a) \\
\beta_m &= a.
\end{aligned}$$

It follows from (2.4) that $(2+a)^m = \frac{1}{a^2}$, so

$$a(2+a)^{m-1} = \frac{1}{a(2+a)} \to \infty \text{ as } a \to 0.$$

The length of I_k is

$$|I_k| = \frac{1}{4}a((2+a)^k - \sum_{i=1}^{k-1}(2+a)^i), k = 1, 2, \cdots, m,$$

and

$$\begin{aligned}
\int_0^1 \beta_k \chi_{I_k}\, dL &= \int_0^1 a(2+a)^{m-k}\chi_{I_k}\, dL \\
&= \frac{1}{4}\frac{a^3(2+a)^m + a^2(2+a)^{m-k+1}}{1+a} \\
&= \frac{1}{4}\frac{a + \frac{1}{(2+a)^{k-1}}}{1+a)} := A_k, k = 1, 2, \cdots, m.
\end{aligned}$$

Thus,

$$\sum_{k=1}^{m}\int_0^1 \beta_k \chi_{I_k}\, dL = \sum_{k=1}^{m} A_k$$

$$
\begin{aligned}
&= \frac{1}{4}\frac{ma + \frac{1-\frac{1}{(2+a)^m}}{1-\frac{1}{2+a}}}{1+a} \\
&= \frac{1}{4}\frac{ma + \frac{1-a^2}{1-\frac{1}{2+a}}}{1+a}.
\end{aligned}
$$

Now, let us look at the term ma. By (2.4) we obtain:

$$
\begin{aligned}
\lim_{a\to 0} ma &= \lim_{a\to 0}\frac{-2a\ln a}{\ln(2+a)} \\
&= \lim_{a\to 0}\frac{2}{\ln(2+a)}(-a\ln a) \\
&= 0\,.
\end{aligned}
$$

This implies that $\lim\limits_{m\to\infty}\sum\limits_{k=1}^{m} A_k = \frac{1}{2}$.

On the other hand, let $m_1 = [m/2]$. We have

$$
\begin{aligned}
\lim_{m_1\to\infty}\sum_{k=1}^{m_1} A_k &= \lim_{m_1\to\infty}\frac{1}{4}\frac{m_1 a + \frac{1-\frac{1}{(2+a)^{m_1}}}{1-\frac{1}{2+a}}}{1+a} \\
&= \lim_{m_1\to\infty}\frac{1}{4}\frac{m_1 a + \frac{1-a}{1-\frac{1}{2+a}}}{1+a} \\
&= \frac{1}{2}.
\end{aligned}
$$

Moreover,

$$
|I_k| = \frac{1}{4}a((2+a)^k - \sum_{i=1}^{k-1}(2+a)^i) > \frac{a}{4}, k = 1, 2, \cdots, m\,.
$$

The left endpoint of I_k will be smaller than $\frac{1}{2}$ since the right endpoint of I_k is $\frac{1}{2}+\frac{a}{4}$, for $k = 1, 2, \cdots, m$. Notice that the length $|I_k|$ is increasing as k is increasing.

$$
\begin{aligned}
|I_{m_1}| &= \frac{1}{4}\frac{a^2(2+a)^{m_1} + a(2+a)}{1+a} \\
&= \frac{1}{4}\frac{a + a(2+a)}{1+a} \to 0, \text{as } a \to 0,
\end{aligned}
$$

so all the intervals $I_1, I_2, \cdots, I_{m_1}$ concentrate at $\frac{1}{2}$.

On the interval $[0, \frac{1}{2})$ and $(\frac{1}{2}, 1]$, we have

$$\int_0^{\frac{1}{2}} C + \beta_0 \chi_{I_0} \, dL = \frac{3}{4}$$

and

$$\int_{\frac{1}{2}}^1 C + \beta_0 \chi_{I_0} \, dL = \frac{1}{4} + \frac{a}{4} \to \frac{1}{4}, \text{as } a \to 0,$$

so the invariant measure is

$$\frac{3}{4} \cdot \frac{2}{3} \cdot 2L \mid_{[0,\frac{1}{2})} + \frac{1}{2} \cdot \frac{2}{3} \delta \mid_{\frac{1}{2}} + \frac{1}{4} \cdot \frac{2}{3} \cdot 2L \mid_{(\frac{1}{2},1]} = L \mid_{[0,\frac{1}{2})} + \frac{1}{3} \delta \mid_{\frac{1}{2}} + \frac{1}{3} \text{Ł} \mid_{(\frac{1}{2},1]},$$

where L and δ denote the Lebesgue measure and Dirac measure, respectively.

2.4 Proof of Theorem 2.2.1

This section contains the proof of Theorem 2.2.1, divided into a number of steps.

2.4.1 Formula for non-normalized invariant density of W_a

We adapt the general formulas of [Góra, 2009] to our case and obtain the following formula for f_a:

Lemma 2.4.1. *For small $a > 0$ there exists $A < -1$ such that*

$$f_a = 1 + 2A \left(\sum_{n=1}^{\infty} \frac{\chi^s(\beta(1/2, n), W_a^n(1/2))}{|\beta(1/2, n)|} \right). \tag{2.5}$$

is a W_a-invariant non-normalized density.

Here,

$$\chi^s(t, y) = \begin{cases} \chi_{[0,y]} & \text{for } \ t > 0 \ ; \\ \chi_{[y,1]} & \text{for } \ t < 0 \ , \end{cases}$$

and $\beta(1/2, n)$ is the cumulative slope along the n steps of the trajectory of $1/2$ defined by:

$$\beta(1/2, 1) = 2 + a, \text{ and}$$

$$\beta(1/2, n) = (2 + a) \cdot W_a'(W_a(1/2)) \cdot W_a'(W_a^2(1/2)) \cdots W_a'(W_a^{n-1}(1/2)), \text{ for } n \geq 2.$$

The detailed justification of formula (2.5) is in Subsection 2.4.2.

For small positive a, the first image of $1/2$ is $W_a(1/2) = 1/2 + a/4$ and the next image lands just below the fixed point slightly less than $1/2$. The following forward images of $1/2$ form a decreasing sequence until they go below $1/4$. Let k be the first iterate j when $W_a^j(1/2)$ is less than $1/4$. That is, $k = \min\{j \geq 1 : W_a^j(1/2) \leq 1/4\}$. Then, the consecutive cumulative slopes of $1/2$, namely $\beta(1/2, j), 1 \leq j \leq k$, are

$$(2 + a), -(2 + a)^2, -(2 + a)^3, \ldots, -(2 + a)^k \ ,$$

and

$$f_a = 1 + 2A \left(\frac{\chi_{[0, W_a(1/2)]}}{(2 + a)} + \sum_{j=2}^{k} \frac{\chi_{[W_a^j(1/2), 1]}}{(2 + a)^j} + \ldots \right). \tag{2.6}$$

2.4.2 Justification of the formula for f_a

Using the notation of [Góra, 2009], we have the following lemma:

Lemma 2.4.2. (a) *N=4, K=2, L=0;*

(b) $\alpha = (1, 1/2 + a/4, 1/2 + a/4, 1)$, $\beta = (-4, 2 + a, -(2 + a), 4)$, $\gamma = (0, 0, 0, 0)$*;*

(c) *The digits* $A = (a_1, a_2, a_3, a_4)$, *where* $a_1 = -1, a_2 = 1/2 + a/4, a_3 = -3/2 - 3a/4, a_4 = 3$*;*

(*d*) *There are two* c_i*'s, which are* $c_1 = (1/2, 2)$ *and* $c_2 = (1/2, 3)$*, and* $j(c_1) = 2$*,* $j(c_2) = 3$*. Then,* $W_u = \{c_1, c_2\}$*,*$W_l = \emptyset$*,* $U_l = \{c_2\}$*,*$U_r = \{c_1\}$*.*

(*e*) $\beta(c_1, 1) = 2 + a$ *since* $j(c_1) = 2$*, then* $\beta(c_1, 2) = -(2 + a)^2$ *and* $\beta(c_1, k) = -(2 + a)^k$ *up to* k *defined in Subsection 2.4.1,* $k = \min\{j \geq 1 : W_a^j(1/2) \leq 1/4\}$*;*

(*f*) $\beta(c_2, 1) = -(2 + a)$ *since* $j(c_2) = 3$*, then* $\beta(c_2, 2) = (2 + a)^2$ *and* $\beta(c_2, k) = (2 + a)^k$ *up to the same* k *in part* (e)*,* $W_a^n(c_1) = W_a^n(c_2)$ *for all* n*;*

(*g*) *Based on* (*f*)*, we have the following for the matrix* $S = S_{i,j}$*,*$i, j = 1, 2$*:*

For $c_1 \in U_r$

$$S_{1,1} = \sum_{n=1}^{\infty} \frac{\delta(\beta((c_1, n) > 0))\delta(W_a^n(c_1) > 1/2) + \delta(\beta((c_1, n) < 0))\delta(W_a^n(c_1) < 1/2)}{|\beta(c_1, n)|},$$

$$S_{1,2} = \sum_{n=1}^{\infty} \frac{\delta(\beta((c_1, n) > 0))\delta(W_a^n(c_1) > 1/2) + \delta(\beta((c_1, n) < 0))\delta(W_a^n(c_1) < 1/2)}{|\beta(c_1, n)|}.$$

For $c_2 \in U_l$

$$S_{2,1} = \sum_{n=1}^{\infty} \frac{\delta(\beta((c_2, n) < 0))\delta(W_a^n(c_2) > 1/2) + \delta(\beta((c_2, n) > 0))\delta(W_a^n(c_2) < 1/2)}{|\beta(c_2, n)|},$$

$$S_{2,2} = \sum_{n=1}^{\infty} \frac{\delta(\beta((c_2, n) < 0))\delta(W_a^n(c_2) > 1/2) + \delta(\beta((c_2, n) > 0))\delta(W_a^n(c_2) < 1/2)}{|\beta(c_2, n)|},$$

where $\delta(\text{“condition”})$ *is equal to* 1 *if the “condition” holds and to* 0 *if it does not.*

Remark 2.4.1. It follows from (e, f) of Lemma 2.4.2 that $S_{i,j}$ are equal for $i, j = 1, 2$. Let Id be the 2×2 identity matrix, $V = [1, 1]$. Then, for the solution, $D = [D_1, D_2]$, of the following system :

$$\left(-S^T + Id\right) D^T = V^T \ , \tag{2.7}$$

we have $D_1 = D_2$. Let us denote them by A.

Let I_1, I_2, I_3, I_4 be the partition of $I = [0, 1]$, where $I_1 = [0, 1/4), I_2 = (1/4, 1/2), I_3 = (1/2, 3/4)$ and $I_4 = (3/4, 1]$. Let $\beta_1 = -4, \beta_2 = 2 + a, \beta_3 = -(2 + a)$, and $\beta_4 = 4$.We define the following index:

$$j(x) = j \text{ for } x \in I_j, j = 1, 2, 3, 4,$$

and

$$j(c_1) = 2, j(c_2) = 3.$$

Already defined for Lemma 2.4.1 we have cumulative slopes for iterates of points:

$$\beta(x, 1) = \beta_{j(x)}, \quad \text{and } \beta(x, n) = \beta(x, n-1) \cdot \beta_{j(W_a^{n-1}(x))}, \qquad n \geq 2 \ ,$$

and

$$\chi^s(t, y) = \begin{cases} \chi_{[0,y]} & \text{for} \quad t > 0 \ ; \\ \\ \chi_{[y,1]} & \text{for} \quad t < 0 \ . \end{cases}$$

Using Theorem 2 in [Góra, 2009] directly, we obtain Lemma 2.4.2. Now, we can prove Lemma 2.4.1:

Proof. First, by part (g) of Lemma 2.4.2, since the first and fourth branches of W_a have slope of modulus $4 > 2 + a$,

$$S_{i,j} \leq \sum_{n=1}^{\infty} \frac{1}{(2+a)^n} = \frac{1}{1+a} < 1 \ .$$

On the other hand, for small a

$$S_{i,j} \geq \frac{1}{2+a} + \frac{1}{(2+a)^2} > 1/2.$$

Now, the solution of the system (2.7) will be $D_1 = D_2 = \frac{1}{1-2S_{1,1}} < -1$. By Theorem 2 in [Góra, 2009], it follows from (d, e, f) of Lemma 2.4.2 that:

$$\begin{aligned} f_a &= 1 + D_1 \sum_{n=1}^{\infty} \frac{\chi^s(\beta(c_1, n), W_a^n(c_1))}{|\beta(c_1, n)|} + D_2 \sum_{n=1}^{\infty} \frac{\chi^s(-\beta(c_2, n), W_a^n(c_2))}{|\beta(c_2, n)|} \\ &= 1 + A \sum_{n=1}^{\infty} \frac{\chi^s(\beta(c_1, n), W_a^n(1/2))}{|\beta(c_1, n)|} + A \sum_{n=1}^{\infty} \frac{\chi^s(-\beta(c_2, n), W_a^n(1/2))}{|\beta(c_2, n)|} \\ &= 1 + 2A \left(\sum_{n=1}^{\infty} \frac{\chi^s(\beta(1/2, n), W_a^n(1/2))}{|\beta(1/2, n)|} \right), \end{aligned}$$

which completes the proof. □

2.4.3 Estimates on f_a

Recall that $k = \min\{j \geq 1 : W_a^j(1/2) \leq 1/4\}$. Clearly, $k > 1$. Furthermore, we have the following lemma:

Lemma 2.4.3. *(I) for $2 \leq m \leq k$, $W_a^m(1/2) = -\frac{1}{4}a\frac{a(2+a)^{m-1}+1}{1+a} + \frac{1}{2}$;*

(II) $\lim_{a\to 0} ak = 0$;

(III) $\lim_{a\to 0} \frac{1}{a(2+a)^k} = 0$.

Moreover, if we let $k_1 = [\frac{2}{3}k]$ (integer part of $2k/3$), we have

(IV) $\lim_{a\to 0} \frac{1}{a(2+a)^{k_1}} = 0$;

(V) $\lim_{a\to 0} a^2(2+a)^{k_1} = 0$;

(VI) $\lim_{a\to 0} W_a^{k_1}(\frac{1}{2}) = \frac{1}{2}$.

Proof. Suppose (I) is true. By the definition of k, $0 \leq W_a^{k-1}(1/2) \leq 1/4$. That is,

$$0 \leq -\frac{1}{4}a\frac{a(2+a)^{k-1}+1}{1+a} + \frac{1}{2} \leq \frac{1}{4}. \tag{2.8}$$

The first inequality of (2.8) implies

$$a^2(2+a)^{k-2} \leq 1. \tag{2.9}$$

Thus

$$\begin{aligned}
ak &\leq a\frac{\ln(2+a)-2\ln a}{\ln(2+a)}+a=2a-\frac{2a\ln a}{\ln(2+a)},\\
a &\leq \frac{2+a}{(2+a)^{\frac{k}{2}}},\\
a^2(2+a)^{k_1} &\leq \frac{(2+a)^2}{(2+a)^{k-k_1}},
\end{aligned}$$

so we obtain (V), and since $\lim\limits_{a\to 0} a\ln a=0$, we obtain (II). and the second inequality of (2.8) implies

$$\frac{1}{a(2+a)^k}\leq\frac{a}{2+a}, \tag{2.10}$$

letting $a\to 0$, we obtain (III).

On the other hand, (2.10) implies that

$$\frac{1}{a(2+a)^{k_1}}\leq\frac{a(2+a)^{k-k_1}}{2+a}\leq\frac{\frac{2+a}{(2+a)^{\frac{k}{2}}}(2+a)^{k-k_1}}{2+a}=\frac{1}{(2+a)^{k_1-\frac{k}{2}}}.$$

By the definition of k_1, we obtain (IV). Finally (VI) follows from (I) and (V).

Now, let us prove (I). For $m=2$, it is easy to check that $W_a^2(1/2)=\frac{2-a-a^2}{4}$ which is the same as $-\frac{1}{4}a\frac{a(2+a)+1}{1+a}+\frac{1}{2}$. Suppose (I) holds for $m=i<k$, that is

$$W_a^i(1/2)=-\frac{1}{4}a\frac{a(2+a)^{i-1}+1}{1+a}+\frac{1}{2}.$$

Then, for $m=i+1$,

$$\begin{aligned}
W_a^{i+1}(1/2) &= (2+a)\Big(-\frac{1}{4}a\frac{a(2+a)^{i-1}+1}{1+a}+\frac{1}{2}-\frac{1}{4}\Big)\\
&= -\frac{1}{4}a\frac{a(2+a)^i+2+a}{1+a}+\frac{1}{2}+\frac{a}{4}\\
&= -\frac{1}{4}a\frac{a(2+a)^i+1}{1+a}+\frac{1}{2}\ .
\end{aligned}$$

This completes the proof. □

Let δ("condition") be equal to 1 if the "condition" holds and to 0 if it does not. Lemma 2.4.2 implies that

$$S_{1,1} = \sum_{n=1}^{\infty} \frac{\delta(\beta((1/2,n) > 0))\delta(W_a^n(1/2) > 1/2) + \delta(\beta((1/2,n) < 0))\delta(W_a^n(1/2) < 1/2)}{|\beta(1/2,n)|} .$$

Also, it was shown there that $A = \frac{1}{1-2S_{1,1}}$. Since

$$S_{1,1} \geq \sum_{n=1}^{k_1} \frac{1}{(2+a)^n} = \frac{\frac{1}{2+a} - \frac{1}{(2+a)^{k_1+1}}}{1 - \frac{1}{2+a}},$$

and

$$S_{1,1} \leq \sum_{n=1}^{\infty} \frac{1}{(2+a)^n} = \frac{1}{1+a},$$

we have

$$A_l = \frac{1+a}{a-1+\frac{2}{(2+a)^{k_1}}} \leq A \leq \frac{1+a}{a-1} = A_h \ . \tag{2.11}$$

Note that, for small a, both estimates A_l and A_h are smaller that -1.

Let us define,

$$g_l = \frac{\chi_{[0,W_a(1/2)]}}{(2+a)} + \sum_{j=2}^{k_1} \frac{\chi_{[W_a^j(1/2),1]}}{(2+a)^j} \ ,$$

and

$$g_h = g_l + \sum_{j=k_1+1}^{\infty} \frac{1}{(2+a)^j} = g_l + \frac{1}{(1+a)(2+a)^{k_1}} \ .$$

Let us further define $f_l = 1 + 2A_l g_h$ and $f_h = 1 + 2A_h g_l$. It follows from (2.6) and (2.11) that

$$f_l \leq f_a \leq f_h \ . \tag{2.12}$$

Let $\chi_1 = \chi_{[0,1/2+a/4]}$, $\chi_j = \chi_{[W_a^j(1/2),1/2+a/4]}, j = 2,3,\ldots,k_1$, $\chi_c = \chi_{(1/2+a/4,1]}$. Now we will represent the functions f_l and f_h as combinations of functions χ_j, $j = 1,\ldots,k_1$ and χ_c. After some calculations, we obtain:

$$f_l \quad = \quad \left(\frac{2}{2+a}A_l + 1\right)\chi_1 + 2A_l \sum_{n=2}^{k_1} \frac{\chi_n}{(2+a)^n} +$$

$$+\left(2A_l\frac{\frac{1}{2+a}-\frac{1}{(2+a)^{k_1}}}{1+a}+1\right)\chi_c+2A_l\frac{1}{(1+a)(2+a)^{k_1}};$$

$$f_h = \left[A_h\frac{2}{2+a}+1\right]\chi_1+2A_h\sum_{n=2}^{k_1}\frac{\chi_n}{(2+a)^n}+\left(2\frac{\frac{1}{2+a}-\frac{1}{(2+a)^{k_1}}}{a-1}+1\right)\chi_c.$$

Note that (2.11) implies that both A_l, A_h are smaller than $-(1+2a)$. Using this we can show that all the coefficients in the representation of f_l and f_h are negative for sufficiently small a.

2.4.4 Normalization

Let us define $J_1=[0,W_a^{k_1}(1/2)]$, $J_2=(W_a^{k_1}(1/2),1/2+a/4]$, $J_3=(1/2+a/4,1]$. We will calculate integrals of f_h over each of these intervals and use them to normalize f_h. We have

$$\begin{aligned} C_1=\int_{J_1}f_h\,dL &= \int_{J_1}\left[2\left(\frac{1+a}{a-1}\frac{1}{2+a}\right)+1\right]\chi_1\,dL \\ &= \left[2\left(\frac{1+a}{a-1}\frac{1}{2+a}\right)+1\right]W_a^{k_1}(\frac{1}{2})=\frac{a^2+3a}{(a-1)(2+a)}W_a^{k_1}(\frac{1}{2})\ . \end{aligned}$$

Using Lemma 2.4.3, we have $\lim_{a\to 0}\frac{C_1}{a}=-\frac{3}{4}$. In the same way we can see that for any $0<\alpha<1/2$, we obtain

$$\lim_{a\to 0}\frac{1}{a}\int_0^\alpha f_h\,dL=-\frac{3}{2}\alpha\ . \tag{2.13}$$

On the interval J_2, the integral of f_h is:

$$C_2=\int_{J_2}f_h\,dL = \int_{J_2}\left[2\left(\frac{1+a}{a-1}\frac{1}{2+a}\right)+1\right]\chi_1\,dL+2\frac{1+a}{a-1}\sum_{j=2}^{k_1}\int_{J_2}\frac{\chi_j}{(2+a)^j}\,dL$$

$$
\begin{aligned}
= & \frac{a^2+3a}{(a-1)(2+a)}\left(\frac{1}{2}+\frac{a}{4}-W_a^{k_1}(\frac{1}{2})\right) \\
& +2\frac{1+a}{a-1}\left[\frac{(k_1-1)a^2}{4(2+a)(1+a)}+\frac{a}{4(1+a)}\frac{1-\frac{1}{(2+a)^{k_1-1}}}{1+a}\right] .
\end{aligned}
$$

Using Lemma 2.4.3, we have

$$
\lim_{a\to 0}\frac{C_2}{a}=-\frac{1}{2}. \tag{2.14}
$$

On the interval J_3, the integral of f_h is:

$$
\begin{aligned}
C_3=\int_{J_3} f_h\, dL &= \int_{J_3}\left(2\frac{\frac{1}{2+a}-\frac{1}{(2+a)^{k_1}}}{a-1}+1\right)\chi_c\, dL \\
&= \left(2\frac{\frac{1}{2+a}-\frac{1}{(2+a)^{k_1}}}{a-1}+1\right)(\frac{1}{2}-\frac{a}{4}) .
\end{aligned}
$$

Using Lemma 2.4.3, we have

$$
\lim_{a\to 0}\frac{C_3}{a}=-\frac{1}{4} .
$$

In the same way we can see that for any $0<\alpha<1/2$, we obtain

$$
\lim_{a\to 0}\frac{1}{a}\int_{1/2+\alpha}^{1} f_h\, dL=-\frac{1}{2}\left(\frac{1}{2}-\alpha\right) . \tag{2.15}
$$

If we define $B=C_1+C_2+C_3$, then $\frac{f_h}{B}$ is a normalized density. We see that

$$
\lim_{a\to 0}\frac{B}{a}=-\frac{3}{2} .
$$

2.4.5 Conclusion of the proof

Now, we will use our foregoing calculations to show that the normalized measures $(f_h/B)\cdot L$ converge $*$-weakly to the measure $\frac{2}{3}\mu_0+\frac{1}{3}\delta_{(\frac{1}{2})}$, as $a\to 0$.

For any interval $[0, \alpha]$, $0 < \alpha < 1/2$ as $a \to 0$, formula (2.13) implies

$$\lim_{a \to 0} \int_0^\alpha \frac{f_h}{B}\, dL = \frac{-\frac{3}{2}\alpha}{-\frac{3}{2}} = \alpha \ . \tag{2.16}$$

For J_2, which converges to the point $1/2$, formula (2.14) implies

$$\lim_{a \to 0} \int_{J_2} \frac{f_h}{B}\, dL = \frac{-\frac{1}{2}}{-\frac{3}{2}} = \frac{1}{3} \ . \tag{2.17}$$

For any interval $[1/2 + \alpha, 1]$, $0 < \alpha < 1/2$, formula (2.15) implies

$$\lim_{a \to 0} \int_{1/2+\alpha}^1 \frac{f_h}{B}\, dL = \frac{-\frac{1}{2}\left(\frac{1}{2} - \alpha\right)}{-\frac{3}{2}} = \frac{1}{3}\left(\frac{1}{2} - \alpha\right) \ . \tag{2.18}$$

Formulas (2.16), (2.17) and (2.18) together show that measures $(f_h/B) \cdot L$ converge $*$-weakly to the sum of the measure with density $\chi_{[0,1/2]} + \frac{1}{3}\chi_{[1/2,1]}$ and $\frac{1}{3}$ of a unit point mass at $1/2$, i.e., to the measure $\frac{2}{3}\mu_0 + \frac{1}{3}\delta_{(\frac{1}{2})}$.

Now, we will show the same for the normalized measure defined using f_l. To this end, let us note that

$$\begin{aligned} f_h - f_l &= 2A_h g_l - 2A_l g_h = 2(A_h - A_l) g_l - 2A_l \frac{1}{(1+a)(2+a)^{k_1}} \\ &= 2\frac{1+a}{a-1} \frac{-2/(2+a)^{k_1}}{a - 1 + 2/(2+a)^{k_1}} g_l - 2A_l \frac{1}{(1+a)(2+a)^{k_1}} \ , \end{aligned}$$

where $|g_l| \leq 1$ and $\lim_{a \to 0} A_l = -1$. Using Lemma 2.4.3 once again, we can show that, for any subinterval $J \subset [0, 1]$, we have

$$\lim_{a \to 0} \frac{1}{a} \int_J (f_h - f_l)\, dL = 0 \ .$$

For $J = [0, 1]$ this means that the normalizations of f_l and f_h are asymptotically the same. Thus, the limit for a general J implies that the $*$-weak limit of normalized measures defined using f_l is the same as for those defined using f_h. Together with inequality (2.12) this proves Theorem 2.2.1.

2.5 Computational results

We present in Fig. 2 graphs of W_a normalized invariant densities for a): $a = 0.1$, b): $a = 0.05$ and c): $a = 0.01$. They were obtained using Maple 13. Note that the vertical scales of the graphs are very different.

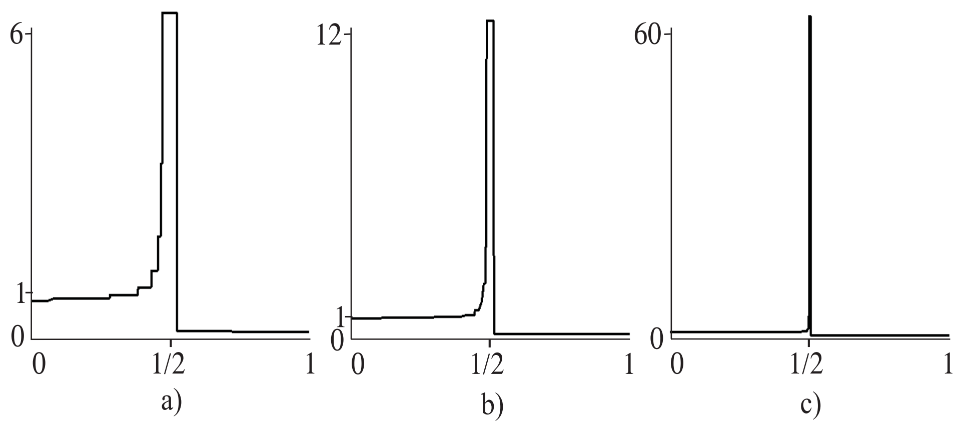

Figure 2.2: W_a-invariant pdf's for a): $a = 0.1$, b): $a = 0.05$ and c): $a = 0.01$.

Chapter 3

$W-$shaped Maps with Various Instabilities of Acims

3.1 Introduction

In this chapter, we construct a family of maps for which the instability of the acims has a global character, not a local one. In the more general case considered in this chapter, with s_1, s_2 not necessarily equal to 2, we will discuss the limits of the acims μ_a of the $\{W_a\}$ maps. We have three cases:

(I) If $\frac{1}{s_1} + \frac{1}{s_2} > 1$, then μ_a's converge $*$-weakly to $\delta_{(\frac{1}{2})}$.

(II) If $\frac{1}{s_1} + \frac{1}{s_2} = 1$, then μ_a's converge $*$-weakly to

$$\frac{(qs_1 + ps_2 - p - q)(s_2 + 2)}{(qs_1 + ps_2 - p - q)(s_2 + 2) + 2rs_1 s_2^2}\mu_0 + \frac{2rs_1 s_2^2}{(qs_1 + ps_2 - p - q)(s_2 + 2) + 2rs_1 s_2^2}\delta_{(\frac{1}{2})},$$

where p, q and r are parameters defining our family of maps.

(III) If $\frac{1}{s_1} + \frac{1}{s_2} < 1$, then μ_a's converge to μ_0.

Additionally, in Theorem 3.2.2, we prove that in case (III) the densities of the μ_a's are uniformly bounded. The first case of our result contains the example in which Keller [Keller, 1982] obtained the "stochastic singularity." In the second case, the limit measure is a combination of an absolutely continuous and a singular measure, and this combination is varying according to p, q and r for fixed s_1 and s_2. This is a generalization of the result of Chapter 2 [Li et al., 2013]. In the third case, we have a map with a stable acim.

At the end of this chapter, we use our main results to provide an interesting example. [Keller, 1978] and [Kowalski, 1979] proved that for a piecewise expanding map $\tau : I \to I$ with $\frac{1}{|\tau'(x)|}$ being a function of bounded variation, the density of the acim of τ has a uniform positive lower bound on its support. We construct a family of piecewise expanding, piecewise linear maps τ_n such that τ_n are exact on $[0, 1]$, τ_n converge to $\tau = W_0$ ($s_1 = s_2 = 2$), $|\tau_n'| > 2$ for all n but the densities of the acims μ_n's do not have a uniform positive lower bound.

In Section 3.2, we introduce our family of W_a maps and state the main result. In Section 3.3 we present the proofs. In Section 3.4, we show the example related to the results of [Keller, 1978] and [Kowalski, 1979].

The results obtained in this chapter (Sections 3.2, 3.3 and 3.4) were, after some modifications, published in the paper [Li, 2013].

3.2 Family of W_a maps and the main result

Let $s_1,\ s_2 > 1$ and $p,\ q,\ r > 0$. We consider the family $\{W_a : 0 \le a\}$ of maps of $[0,1]$ onto itself defined by

$$W_a(x) = \begin{cases} 1 - \frac{2(s_1+pa)}{s_1-1+pa-2ra}x\ , & \text{for } \ 0 \le x < \frac{1}{2} - \frac{\frac{1}{2}+ra}{s_1+pa}\ ; \\ (s_1+pa)(x-1/2)+1/2+ra\ , & \text{for } \ \frac{1}{2} - \frac{\frac{1}{2}+ra}{s_1+pa} \le x < 1/2\ ; \\ -(s_2+qa)(x-1/2)+1/2+ra\ , & \text{for } \ 1/2 \le x < \frac{1}{2} + \frac{\frac{1}{2}+ra}{s_2+qa}\ ; \\ 1 + \frac{2(s_2+qa)}{s_2-1+qa-2ra}(x-1)\ , & \text{for } \ \frac{1}{2} + \frac{\frac{1}{2}+ra}{s_2+qa} \le x \le 1\ . \end{cases} \tag{3.1}$$

For each choice of $s_1,\ s_2 > 1,\ p,\ q,\ r > 0$, we consider only $a > 0$ such that $0 \le W_a(x) \le 1$ for $x \in [0,1]$.

An example of a W_a map is shown in Fig.3.1. Fig.3.1(a) is the unperturbed W_0 map with turning fixed point at $1/2$ and $s_1 = 3/2$, $s_2 = 3$. Fig.3.1(b) is the perturbed map W_a, with $a = 0.05$, $r = 2$, $p = 3$, $q = 2$. The slope of the second branch is $s_1 + pa = 1.65$, the slope of the third branch is $s_2 + qa = 3.1$, and $W_{0.05}(1/2) = 1/2 + ra = 0.6$.

Every W_a has a unique acim μ_a since all the slopes are greater than 1 in modulus. We will show later that, for $\frac{1}{s_1} + \frac{1}{s_2} \le 1$, μ_a is supported on $[0,1]$ and for $\frac{1}{s_1} + \frac{1}{s_2} > 1$ it is supported on a subinterval around $1/2$. W_a is an exact map with the measure μ_a. Let h_a denote the normalized density of μ_a, $a \ge 0$. Since the W_0 map is a Markov

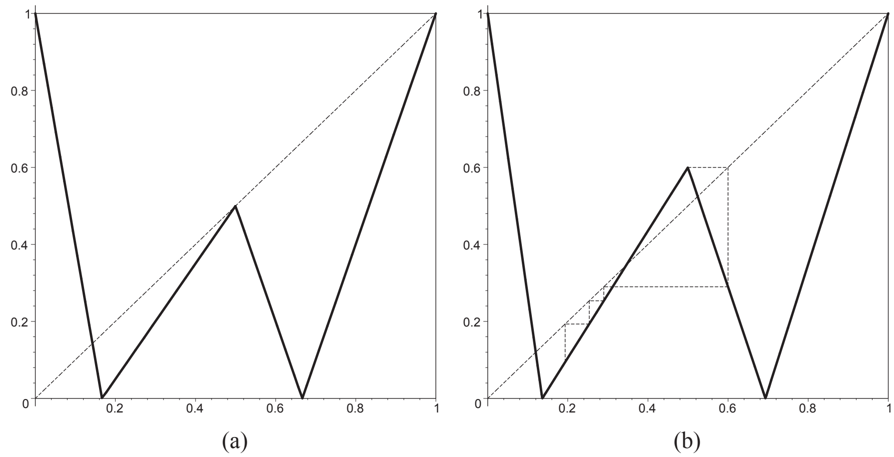

Figure 3.1: The W-shaped maps with $\frac{1}{s_1} + \frac{1}{s_2} = 1$: (a) W_0 with $s_1 = 3/2$ and $s_2 = 3$, (b) W_a with $s_1 = 3/2$, $s_2 = 3$; $a = 0.05$; $r = 2$, $p = 3$, $q = 2$; also several initial points of the trajectory of 1/2.

one, it is easy to check that

$$h_0 = \begin{cases} \frac{2s_1(s_2+1)}{2s_1s_2+s_1-s_2} \ , & \text{for } \ 0 \le x < 1/2 \ ; \\ \\ \frac{2s_2(s_1-1)}{2s_1s_2+s_1-s_2} \ , & \text{for } \ 1/2 \le x \le 1 \ . \end{cases} \tag{3.2}$$

Our main result is the following theorem

Theorem 3.2.1. *As $a \to 0$ the measures μ_a converge $*$-weakly to the measure*

(I) $\delta_{(\frac{1}{2})}$, *if* $\frac{1}{s_1} + \frac{1}{s_2} > 1$;

(II) $\frac{(qs_1+ps_2-p-q)(s_2+2)}{(qs_1+ps_2-p-q)(s_2+2)+2rs_1s_2^2}\mu_0 + \frac{2rs_1s_2^2}{(qs_1+ps_2-p-q)(s_2+2)+2rs_1s_2^2}\delta_{(\frac{1}{2})}$, *if* $\frac{1}{s_1} + \frac{1}{s_2} = 1$;

(III) μ_0, *if* $\frac{1}{s_1} + \frac{1}{s_2} < 1$,

where $\delta_{(\frac{1}{2})}$ *is the Dirac measure at point* $1/2$.

The proof relies on the general formula for invariant densities of piecewise linear maps [Góra, 2009] and direct calculations. Most objects and quantities we use depend on the parameter a. We suppress a from the notation to make it simpler.

In case (III), we actually prove a little more:

Theorem 3.2.2. *If* $\frac{1}{s_1} + \frac{1}{s_2} < 1$*, then the normalized invariant densities* $\{h_a\}$ *are uniformly bounded for given* p*,* q *and* r*. Consequently, we obtain Theorem 3.2.1(III).*

3.3 Proofs of Theorem 3.2.1 and Theorem 3.2.2

This section contains the proofs of Theorems 3.2.1 and 3.2.2, divided into a number of steps.

3.3.1 Assume $\frac{1}{s_1} + \frac{1}{s_2} > 1$

Let

$$x_l^* = \frac{s_1 - 1 + pa - 2ra}{2(s_1 - 1 + pa)}$$

and

$$x_r^* = \frac{s_2 s_1 - s_2 + (2rs_1 - q + ps_2 + qs_1)a + (2rp + pq)a^2}{2(s_1 - 1 + pa)(s_2 + qa)}.$$

x_l^* is the fixed point on the second branch of W_a, and x_r^* is the preimage of x_l^* under the third branch of W_a. Both x_r^* and x_l^* converge to $\frac{1}{2}$ as a approaches 0. For small a, we have

$$W_a(1/2) - x_r^* = \frac{ra\,[s_1 s_2 - s_1 - s_2 + a(qs_1 + ps_2 - p - q + pqa)]}{(s_1 - 1 + pa)(s_2 + qa)} < 0.$$

In this case, we have $W_a([x_l^*, x_r^*]) \subseteq [x_l^*, x_r^*]$. $W_a|_{[x_l^*, x_r^*]}$ is a skewed tent map with $W_a(1/2) > 1/2$; it is known that with acim μ_a, it is ergodic on $[W_a^2(1/2), W_a(1/2)]$. Since μ_a is concentrated on $[x_l^*, x_r^*]$, we conclude that μ_a converge $*$-weakly to $\delta_{(\frac{1}{2})}$. This proves Theorem 3.2.1(I).

Fig.3.2 shows an example with $a = 0.05, r = 2, p = 3, q = 2$; $s_1 = 4/3, s_2 = 5/2$.

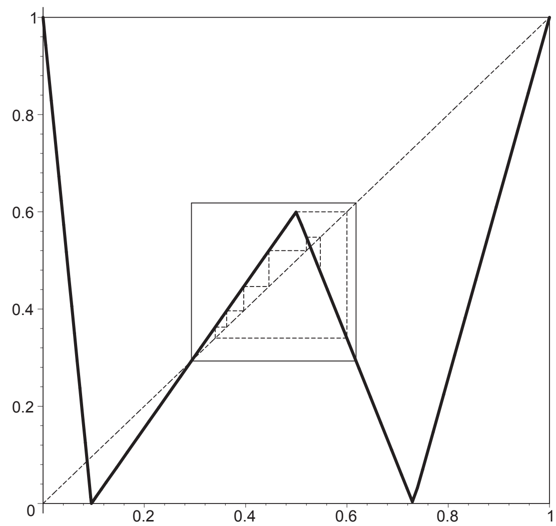

Figure 3.2: The W_a map with $\frac{1}{s_1} + \frac{1}{s_2} > 1$

3.3.2 Formula for the non-normalized invariant density of W_a if $\frac{1}{s_1} + \frac{1}{s_2} \leq 1$

An example of a map W_a is shown in Fig.3.1. We have the following proposition.

Proposition 3.3.1. *For $\frac{1}{s_1} + \frac{1}{s_2} \leq 1$, the map W_a has an acim μ_a supported on $[0,1]$ and the map W_a with respect to μ_a is exact.*

Proof. W_a is a piecewise expanding transformation. From the general theory (see for example [Boyarsky and Góra, 1997]), it follows that it is enough to show that the images $W_a^n(J)$ grow to cover all $[0,1]$ as $n \to \infty$, for any interval $J \subset [0,1]$. Since W_a is expanding, $W_a^n(J)$ grow until some image $W_a^{n_0}(J)$ contains an internal partition point. If this point is not $1/2$, then $W_a^{n_0+2}(J)$ contains the repelling fixed point 1. Then its images grow to cover all of $[0,1]$. If this point is $1/2$, we proceed as follows.

First, assume that $\frac{1}{s_1} + \frac{1}{s_2} < 1$. Consider a small neighborhood $J = (z_1, z_2)$ around 1/2 with length ℓ, then

$$\min_{z_2 - z_1 = \ell} \max \left\{ (\frac{1}{2} - z_1)(s_1 + pa), (z_2 - \frac{1}{2})(s_2 + qa) \right\} = \frac{1}{\frac{1}{s_1+pa} + \frac{1}{s_2+qa}} \ell > \ell.$$

Thus, the interval J will grow until its image covers two partition points of W_a. Then the second iteration afterward will cover $[0, 1]$. Therefore, W_a is exact with respect ot μ_a.

Assume $\frac{1}{s_1} + \frac{1}{s_2} = 1$. If $a \neq 0$, then $\frac{1}{\frac{1}{s_1+pa} + \frac{1}{s_2+qa}} > 1$, which implies W_a is exact with respect to μ_a. In the case $a = 0$, we first note that 1/2 is a turning fixed point. Take again a small interval $J = (z_1, z_2) \ni 1/2$. Its image is an interval $(z, 1/2)$. It will grow under iteration and its iterations still contain 1/2. It will grow until its image covers another partition point of W_a. Then, the second iteration afterward will covers all of $[0, 1]$. Thus, W_a is again exact with respect to μ_a. $\square$

We adapt the general formulas of [Góra, 2009] to our case and obtain the following lemma:

Lemma 3.3.1. *(I) N=4, K=2, L=0 ;*

(II) $\alpha = (1, 1/2 + ra, 1/2 + ra, 1)$, $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$, *where* $\beta_1 = -\frac{2(s_1+pa)}{s_1-1+pa-2ra}$, $\beta_2 = s_1 + pa$, $\beta_3 = -(s_2 + qa)$ *and* $\beta_4 = \frac{2(s_2+qa)}{s_2-1+qa-2ra}$, $\gamma = (0, 0, 0, 0)$ *;*

(III) The digits $A = (a_1, a_2, a_3, a_4)$, *where* $a_1 = -1$, $a_2 = \frac{s_1-1+pa-2ra}{2}$, $a_3 = -\frac{s_2+1+qa+2ra}{2}$, $a_4 = \frac{s_2+1+qa+2ra}{s_1-1+pa-2ra}$ *;*

(IV) There are two c_i*'s, which are* $c_1 = (1/2, 2)$ *and* $c_2 = (1/2, 3)$, *and* $j(c_1) = 2$, $j(c_2) = 3$. *Then,* $W_u = \{c_1, c_2\}$,$W_l = \emptyset$, $U_l = \{c_2\}$,$U_r = \{c_1\}$ *;*

(V) $\beta(c_1, 1) = s_1 + pa$ *since* $j(c_1) = 2$, *then* $\beta(c_1, 2) = -(s_1 + pa)(s_2 + qa)$ *and*

$\beta(c_1,k) = -(s_2+qa)(s_1+pa)^{k-1}$ up to some k which is the first moment j when the $W_a^j(1/2)$ is less than $\frac{1}{2} - \frac{1/2+ra}{s_1+pa}$, and is the same one defined in Lemma 3.3.4 ;

(VI) $\beta(c_2,1) = -(s_2+qa)$ since $j(c_2)=3$, then $\beta(c_2,2) = (s_2+qa)^2$ and $\beta(c_2,k) = (s_2+qa)^2(s_1+pa)^{k-2}$ up to the same k in part (e), $W_a^n(c_1) = W_a^n(c_2)$ for all n ;

(VII) Based on (VI), we have the following for the matrix $S = (S_{i,j})_{1\le i,j\le 2}$:

For $c_1 \in U_r$

$$S_{1,1} = \sum_{n=1}^{\infty} \frac{\delta(\beta(c_1,n)>0)\delta(W_a^n(c_1)>1/2) + \delta(\beta(c_1,n)<0)\delta(W_a^n(c_1)<1/2)}{|\beta(c_1,n)|},$$

$$S_{1,2} = \sum_{n=1}^{\infty} \frac{\delta(\beta(c_1,n)>0)\delta(W_a^n(c_1)>1/2) + \delta(\beta(c_1,n)<0)\delta(W_a^n(c_1)<1/2)}{|\beta(c_1,n)|}.$$

For $c_2 \in U_l$

$$S_{2,1} = \sum_{n=1}^{\infty} \frac{\delta(\beta(c_2,n)<0)\delta(W_a^n(c_2)>1/2) + \delta(\beta(c_2,n)>0)\delta(W_a^n(c_2)<1/2)}{|\beta(c_2,n)|},$$

$$S_{2,2} = \sum_{n=1}^{\infty} \frac{\delta(\beta(c_2,n)<0)\delta(W_a^n(c_2)>1/2) + \delta(\beta(c_2,n)>0)\delta(W_a^n(c_2)<1/2)}{|\beta(c_2,n)|}.$$

Remark 3.3.1. It follows from (V, VI) of Lemma 1 that

$$S_{1,1} = S_{1,2}\ ,\ S_{2,1} = S_{2,2} \text{ and } S_{1,1} = \frac{s_2+qa}{s_1+pa} S_{2,2}\ .$$

Let Id be the 2×2 identity matrix and let $V = [1,1]$. Then, for the solution, $D = [D_1, D_2]$, of the system :

$$\left(-S^T + Id\right) D^T = V^T, \tag{1}$$

we have $D_1 = D_2$. Let us denote them by Λ.

Let I_1, I_2, I_3, I_4 be the partition of $I=[0,1]$ into maximal intervals of monotonicity of W_a: $I_1 = [0, \frac{s_1-1+pa-2ra}{2(s_1+pa)})$, $I_2 = (\frac{s_1-1+pa-2ra}{2(s_1+pa)}, 1/2)$, $I_3 = (1/2, \frac{s_2+1+qa+2ra}{2(s_2+qa)})$ and $I_4 =$

$(\frac{s_2+1+qa+2ra}{2(s_2+qa)}, 1]$. We define the following index function:

$$j(x) = i \text{ for } x \in I_i, i = 1, 2, 3, 4,$$

and

$$j(c_1) = 2, j(c_2) = 3.$$

We define the cumulative slopes for iterates of points as follows:

$$\beta(x, 1) = \beta_{j(x)}, \quad \text{and } \beta(x, n) = \beta(x, n-1) \cdot \beta_{j(W_a^{n-1}(x))}, \qquad n \geq 2.$$

In particular, we have

$$\beta(1/2, n) = (s_1 + pa) \cdot W_a'(W_a(1/2)) \cdot W_a'(W_a^2(1/2)) \cdots W_a'(W_a^{n-1}(1/2)) \ ,$$

which is the cumulative slope along the n steps of trajectory of $1/2$. Recall that k is the first moment j when the $W_a^j(1/2)$ is less than $\frac{1}{2} - \frac{1/2+ra}{s_1+pa}$. Let $k_1 = [\frac{2}{3}k]$ (the integer part of $2k/3$). Note that $k_1 \to \infty$ as $a \to 0$. Let

$$\chi^s(t, x) = \begin{cases} \chi_{[0,x]} & \text{for } \ t > 0 \ ; \\ \\ \chi_{[x,1]} & \text{for } \ t < 0 \ . \end{cases}$$

Now, we can obtain the following formula for f_a:

Lemma 3.3.2. *Let*

$$f_a = 1 + (1 + \frac{s_1 + pa}{s_2 + qa})\Lambda \left(\sum_{n=1}^{\infty} \frac{\chi^s(\beta(1/2, n), W_a^n(1/2))}{|\beta(1/2, n)|} \right) .$$

Then f_a is W_a invariant non-normalized density. Furthermore, for small $a > 0$, we have:

(I) If $\frac{1}{s_1} + \frac{1}{s_2} = 1$, then $\Lambda < -1$;

(II) If $\frac{1}{s_1}+\frac{1}{s_2}<1$*, the sign of* Λ *depends on* s_1 *and* s_2*, can be either positive or negative depending on the sign of* $\vartheta = 1-\left(\frac{s_1+s_2}{s_1 s_2}+\frac{s_1+s_2}{s_2^2(s_1-1)}\right) = 1-\frac{s_1+s_2}{s_1 s_2}\left(1+\frac{s_1}{s_2(s_1-1)}\right)$*. The case when* $\vartheta = 0$ *is discussed at the end of Section 3.3.*

Proof. By the Theorem 2 in [Góra, 2009], it follows from (IV, V, VI) of Lemma 3.3.1 that:

$$\begin{aligned} f_a &= 1 + D_1 \sum_{n=1}^{\infty} \frac{\chi^s(\beta(c_1,n), W_a^n(c_1))}{|\beta(c_1,n)|} + D_2 \sum_{n=1}^{\infty} \frac{\chi^s(-\beta(c_2,n), W_a^n(c_2))}{|\beta(c_2,n)|} \\ &= 1 + \Lambda \sum_{n=1}^{\infty} \frac{\chi^s(\beta(c_1,n), W_a^n(1/2))}{|\beta(c_1,n)|} + \Lambda \sum_{n=1}^{\infty} \frac{\chi^s(-\beta(c_2,n), W_a^n(1/2))}{|\beta(c_2,n)|} \\ &= 1 + (1 + \frac{s_1+pa}{s_2+qa})\Lambda \left(\sum_{n=1}^{\infty} \frac{\chi^s(\beta(1/2,n), W_a^n(1/2))}{|\beta(1/2,n)|} \right). \end{aligned}$$

Let $s = \min\left\{\frac{2s_1}{s_1-1}, \frac{2s_2}{s_2-1}, s_1, s_2\right\}$. Note that $s > 1$. Since

$$\begin{aligned} S_{1,1} &\geq \frac{1}{s_1+pa} + \frac{1}{s_2+qa} \sum_{n=1}^{k_1-1} \frac{1}{(s_1+pa)^n} = \frac{1}{s_1+pa} + \frac{1}{s_2+qa} \frac{1-\frac{1}{(s_1+pa)^{k_1-1}}}{s_1+pa-1}, \\ S_{1,1} &\leq \frac{1}{s_1+pa} + \frac{1}{s_2+qa} \left(\sum_{n=1}^{k_1-1} \frac{1}{(s_1+pa)^n} + \frac{1}{(s_1+pa)^{k_1-1}} \sum_{n=1}^{\infty} \frac{1}{s^n} \right) \\ &= \frac{1}{s_1+pa} + \frac{1}{s_2+qa} \left(\frac{1-\frac{1}{(s_1+pa)^{k_1-1}}}{s_1+pa-1} + \frac{1}{(s_1+pa)^{k_1-1}} \frac{1}{s-1} \right), \end{aligned}$$

and $\Lambda = \frac{1}{1-\frac{s_1+s_2+pa+qa}{s_2+qa}S_{1,1}}$, we have

$$\Lambda_l = \frac{1}{1-(\kappa+\eta(1-\frac{1}{(s_1+pa)^{k_1-1}}))} \leq \Lambda \leq \frac{1}{1-(\kappa+\eta(1-\frac{1}{(s_1+pa)^{k_1-1}})+\omega)} = \Lambda_h, \tag{3.3}$$

where $\kappa = \frac{s_1+s_2+pa+qa}{(s_1+pa)(s_2+qa)}$, $\eta = \frac{s_1+s_2+pa+qa}{(s_2+qa)^2(s_1+pa-1)}$, $\omega = \frac{s_1+s_2+pa+qa}{(s_2+qa)^2}\frac{1}{(s_1+pa)^{k_1-1}}\frac{1}{s-1}$.

(I) Note that for small a both estimates Λ_l and Λ_h are smaller than -1 since both κ and η are smaller than 1 and close to 1. Furthermore, as a approaches 0, both κ and η approach 1, ω approaches 0.

(II) As a approaches 0, κ and η approach $\frac{s_1+s_2}{s_1 s_2}$ and $\frac{s_1+s_2}{s_2^2(s_1-1)}$, respectively. Again, note that for small a, estimates Λ_l and Λ_h can be either positive or negative, and they have the same sign. $\square$

For small positive a, the first image of $1/2$ is $W_a(1/2) = 1/2 + ra$ and the next one falls just below the fixed point x_l^* slightly less than $1/2$. The following images form a decreasing sequence until they go below $\frac{1}{2} - \frac{1/2+ra}{s_1+pa}$. Since k is the first iteration j when the $W_a^j(1/2)$ is less than $\frac{1}{2} - \frac{1/2+ra}{s_1+pa}$, the consecutive cumulative slopes of $1/2$ are

$$(s_1 + pa), -(s_1 + pa)(s_2 + qa), -(s_1 + pa)^2(s_2 + qa), \ldots, -(s_1 + pa)^{k-1}(s_2 + qa) \ ,$$

and

$$f_a = 1 + (1 + \frac{s_1 + pa}{s_2 + qa})\Lambda \left(\frac{\chi_{[0,W_a(1/2)]}}{(s_1 + pa)} + \sum_{j=2}^{k} \frac{\chi_{[W_a^j(1/2),1]}}{(s_1 + pa)^{j-1}(s_2 + qa)} + \ldots \right) . \quad (3.4)$$

3.3.3 Estimates, normalizations and integrals on f_a for $\frac{1}{s_1} + \frac{1}{s_2} \leq 1$

Remembering that $k = \min\{j \geq 1 : W_a^j(1/2) \leq \frac{1}{2} - \frac{1/2+ra}{s_1+pa}\}$ and $k_1 = [\frac{2}{3}k]$ (the integer part of $2k/3$), we will give the estimates on f_a.

Let us define

$$g_l = \frac{\chi_{[0,W_a(1/2)]}}{s_1 + pa} + \frac{1}{s_2 + qa} \sum_{j=2}^{k_1} \frac{\chi_{[W_a^j(1/2),1]}}{(s_1 + pa)^{j-1}} \ ,$$

and

$$g_h = g_l + \frac{1}{s_2 + qa} \sum_{j=1}^{\infty} \frac{1}{(s_1 + pa)^{k_1 - 1} s^j} = g_l + \frac{1}{(s_2 + qa)(s - 1)(s_1 + pa)^{k_1 - 1}} \ .$$

Also, let $\chi_1 = \chi_{[0,1/2+ra]}$, $\chi_j = \chi_{[W_a^j(1/2),1/2+ra]}, j = 2, 3, \ldots, k_1$, $\chi_c = \chi_{(1/2+ra,1]}$.

3.3.3.1 Estimates on f_a if $\frac{1}{s_1} + \frac{1}{s_2} = 1$

We have the following lemma:

Lemma 3.3.3. *For the family of W_a maps, if $\frac{1}{s_1} + \frac{1}{s_2} = 1$, we have*

(I) $W_a(1/2) = 1/2 + ra$, $W_a^2(1/2) = -ra(s_2 + qa) + 1/2 + ra$, and for $3 \leq m \leq k$, we have $W_a^m(1/2) = -a^2(s_1 + pa)^{m-2}\frac{r(qs_1+ps_2-p-q)+rpqa}{s_1+pa-1} + \frac{s_1-1+pa-2ra}{2(s_1+pa-1)}$;

(II) $\lim\limits_{a\to 0} ak = 0$;

(III) $\lim\limits_{a\to 0} \frac{1}{a(s_1+pa)^k} = 0$;

(IV) $\lim\limits_{a\to 0} \frac{1}{a(s_1+pa)^{k_1}} = 0$;

(V) $\lim\limits_{a\to 0} a^2(s_1 + pa)^{k_1} = 0$;

(VI) $\lim\limits_{a\to 0} W_a^{k_1}(\frac{1}{2}) = \frac{1}{2}$.

Proof. Suppose (I) is true. Let us first prove that (II) and (III) are true.

By the definition of k, we have:

$$0 \leq -a^2(s_1+pa)^{k-2}\frac{r(qs_1 + ps_2 - p - q) + rpqa}{s_1 + pa - 1} + \frac{s_1 - 1 + pa - 2ra}{2(s_1 + pa - 1)} \leq \frac{1}{2} - \frac{1/2 + ra}{s_1 + pa}. \tag{3.5}$$

The first inequality of (3.5) implies that $(s_1 + pa)^{k-2} \leq \frac{s_1-1+pa-2ra}{2a^2(r(qs_1+ps_2-p-q)+rpqa)}$, thus

$$ak \leq a\frac{\ln(s_1 - 1 + pa - 2ra) - \ln 2 - 2\ln a - \ln(r(qs_1 + ps_2 - p - q) + rpqa)}{\ln(s_1 + pa)} + 2a,$$

$$a \leq \frac{\sqrt{s_1 - 1 + pa - 2ra}(s_1 + pa)}{\sqrt{2(r(qs_1 + ps_2 - p - q) + rpqa)}(s_1 + pa)^{k/2}},$$

$$a^2(s_1 + pa)^{k_1} \leq \frac{(s_1 - 1 + pa - 2ra)(s_1 + pa)^2}{2(r(qs_1 + ps_2 - p - q) + rpqa)(s_1 + pa)^{k-k_1}},$$

so we obtain (V), and since $\lim\limits_{a\to 0} a\ln a = 0$, we obtain (II).

The second inequality of (3.5) implies

$$\frac{1}{a(s_1+pa)^{k-2}} \leq \frac{2a(r(qs_1+ps_2-p-q)+rpqa)(s_1+pa)}{s_1-1+pa-2ra}.$$

Therefore,

$$\frac{1}{a(s_1+pa)^{k}} \leq \frac{2a(r(qs_1+ps_2-p-q)+rpqa)}{(s_1-1+pa-2ra)(s_1+pa)}, \tag{3.6}$$

and as $a \to 0$, we obtain (III).

On the other hand, (3.6) implies

$$\begin{aligned}
\frac{1}{a(s_1+pa)^{k_1}} &\leq \frac{2a(r(qs_1+ps_2-p-q)+rpqa)(s_1+pa)^{k-k_1}}{(s_1+pa-2ra-1)(s_1+pa)} \\
&\leq \frac{\sqrt{2(r(qs_1+ps_2-p-q)+rpqa)}(s_1+pa)^{k-k_1}}{\sqrt{s_1+pa-2ra-1}(s_1+pa)^{k/2}} \\
&= \frac{\sqrt{2(r(qs_1+ps_2-p-q)+rpqa)}}{\sqrt{s_1+pa-2ra-1}(s_1+pa)^{k_1-k/2}}.
\end{aligned}$$

By the definition of k_1, we obtain (IV). (VI) follows from (V).

Now, let us prove (I).

The fixed point slightly less than 1/2 is $x_l^* = \frac{s_1-1+pa-2ra}{2(s_1-1+pa)}$, and

$$x_l^* - W_a^2(1/2) = \frac{ra^2(q(s_1-1)+p(s_2-1)+apq)}{s_1-1+pa} > 0,$$

which implies that $W_a^m(1/2)$ are all in the domain of the second branch of W_a for $3 \leq m \leq k$. For a linear map $T(x) = m_0x+b_0$, we have $T^n(x) = m_0^n x + \frac{m_0^n-1}{m_0-1}b_0$. This proves (I). □

Using (3.4) and (3.3) we see that for the functions $f_l = 1+(1+\frac{s_1+pa}{s_2+qa})\Lambda_l g_h$ and $f_h = 1+(1+\frac{s_1+pa}{s_2+qa})\Lambda_h g_l$, we have

$$f_l \leq f_a \leq f_h\ . \tag{3.7}$$

Now, we will represent functions f_l and f_c as combinations of functions $\chi_j,\ j = 1,\dots,k_1$ and χ_c. After some calculations, we obtain

$$
\begin{aligned}
f_l &= 1 + (1 + \frac{s_1+pa}{s_2+qa})\Lambda_l \Bigg(\frac{\chi_{[0,W_a(1/2)]}}{s_1+pa} + \frac{1}{s_2+qa}\sum_{j=2}^{k_1} \frac{\chi_{[W_a^j(1/2),1]}}{(s_1+pa)^{j-1}} \\
&\quad + \frac{1}{(s_2+qa)(s-1)(s_1+pa)^{k_1-1}} \Bigg) \\
&= \left(\frac{s_1+s_2+pa+qa}{(s_2+qa)(s_1+pa)}\Lambda_l + 1 \right)\chi_1 + \frac{s_1+s_2+pa+qa}{(s_2+qa)^2}\Lambda_l \sum_{j=2}^{k_1}\frac{\chi_j}{(s_1+pa)^{j-1}} \\
&\quad + \left(\frac{s_1+s_2+pa+qa}{(s_2+pa)^2}\Lambda_l \frac{1-\frac{1}{(s_1+pa)^{k_1-1}}}{s_1+pa-1} + 1 \right)\chi_c \\
&\quad + \frac{\frac{s_1+s_2+pa+qa}{s_2+qa}\Lambda_l}{(s_2+qa)(s-1)(s_1+pa)^{k_1-1}} \ ,
\end{aligned}
$$

$$
\begin{aligned}
f_h &= 1 + (1 + \frac{s_1+pa}{s_2+qa})\Lambda_h \left(\frac{\chi_{[0,W_a(1/2)]}}{s_1+pa} + \frac{1}{s_2+qa}\sum_{j=2}^{k_1} \frac{\chi_{[W_a^j(1/2),1]}}{(s_1+pa)^{j-1}} \right) \\
&= \left(\frac{s_1+s_2+pa+qa}{(s_2+qa)(s_1+pa)}\Lambda_h + 1 \right)\chi_1 + \frac{s_1+s_2+pa+qa}{(s_2+qa)^2}\Lambda_h \sum_{j=2}^{k_1}\frac{\chi_j}{(s_1+pa)^{j-1}} \\
&\quad + \left(\frac{s_1+s_2+pa+qa}{(s_2+qa)^2}\Lambda_h \frac{1-\frac{1}{(s_1+pa)^{k_1-1}}}{s_1+pa-1} + 1 \right)\chi_c \ .
\end{aligned}
$$

In the case we are considering, (3.3) implies that both Λ_l, Λ_h are smaller than -1. Using this, one can show that all the coefficients in the representation of f_l and f_h are negative for sufficiently small a. For example, let us consider the coefficient of χ_1 in f_h:

$$
\begin{aligned}
\frac{s_1+s_2+pa+qa}{(s_2+qa)(s_1+pa)}\Lambda_h + 1 &= \frac{\kappa}{1-(\kappa+\eta(1-\frac{1}{(s_1+pa)^{k_1-1}})+\omega)} + 1 \\
&= \frac{1-\eta+\frac{\eta}{(s_1+pa)^{k_1-1}}-\omega}{1-(\kappa+\eta(1-\frac{1}{(s_1+pa)^{k_1-1}})+\omega)} < 0 \ .
\end{aligned}
$$

3.3.3.2 Normalizations and integrals if $\frac{1}{s_1} + \frac{1}{s_2} = 1$

Let us define $J_1 = [0, W_a^{k_1}(1/2)]$, $J_2 = (W_a^{k_1}(1/2), 1/2 + ra]$, $J_3 = (1/2 + ra, 1]$. We will calculate integrals of f_h over each of these intervals J_1, J_2 and J_3, and use them to normalize f_h. We have

$$
\begin{aligned}
C_1 &= \int_{J_1} f_h \, dL = \int_{J_1} \left[\frac{s_1 + s_2 + pa + qa}{(s_2 + qa)(s_1 + pa)} \Lambda_h + 1 \right] \chi_1 \, dL \\
&= \left[\frac{s_1 + s_2 + pa + qa}{(s_2 + qa)(s_1 + pa)} \Lambda_h + 1 \right] W_a^{k_1}(\frac{1}{2}) \\
&= \left[\frac{\kappa}{1 - (\kappa + \eta(1 - \frac{1}{(s_1+pa)^{k_1-1}}) + \omega)} + 1 \right] W_a^{k_1}(\frac{1}{2}) \\
&= \left[\frac{a(2qs_1s_2 + ps_2^2 - 2qs_2 - p - q)}{(1 - (\kappa + \eta(1 - \frac{1}{(s_1+pa)^{k_1-1}}) + \omega))(s_2 + qa)^2(s_1 + pa - 1)} \right. \\
&\quad + \frac{a^2(2pqs_2 - q^2 + q^2 s_1) + pq^2a^3}{(1 - (\kappa + \eta(1 - \frac{1}{(s_1+pa)^{k_1-1}}) + \omega))(s_2 + qa)^2(s_1 + pa - 1)} \\
&\quad \left. + \frac{\frac{\eta}{(s_1+pa)^{k_1-1}} - \omega}{1 - (\kappa + \eta(1 - \frac{1}{(s_1+pa)^{k_1-1}}) + \omega)} \right] W_a^{k_1}(\frac{1}{2}) \, .
\end{aligned}
$$

Using Lemma 3.3.3, we obtain

$$
\lim_{a \to 0} \frac{C_1}{a} = -\frac{2qs_1s_2 + ps_2^2 - 2qs_2 - p - q}{2s_2^2(s_1 - 1)} = -\frac{2qs_1 + ps_2^2 - p - q}{2s_2s_1}.
$$

In the same way, we can see that for any $0 < \theta < 1/2$, we obtain

$$
\lim_{a \to 0} \frac{1}{a} \int_0^\theta f_h \, dL = -\frac{2qs_1 + ps_2^2 - p - q}{s_2s_1} \theta \, .
$$

On the interval J_2, the integral of f_h is:

$$
\begin{aligned}
C_2 = \int_{J_2} f_h \, dL &= \int_{J_2} \left[\frac{s_1 + s_2 + pa + qa}{(s_2 + qa)(s_1 + pa)} \Lambda_h + 1 \right] \chi_1 \, dL \\
&\quad + \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)^2} \Lambda_h \sum_{j=2}^{k_1} \int_{J_2} \frac{\chi_j}{(s_1 + a)^{j-1}} \, dL \\
&= \frac{1 - \eta + \frac{\eta}{(s_1+pa)^{k_1-1}} - \omega}{1 - (\kappa + \eta(1 - \frac{1}{(s_1+pa)^{k_1-1}}) + \omega)} \left(\frac{1}{2} + ra - W_a^{k_1}(\frac{1}{2}) \right)
\end{aligned}
$$

$$+\frac{s_1+s_2+pa+qa}{(s_2+qa)^2}\Lambda_h\left[\frac{ra(s_2+qa)}{s_1+pa}+\frac{ra(1-\frac{1}{(s_1+pa)^{k_1-2}})}{(s_1+pa-1)^2}\right.$$
$$\left.+\frac{a^2(k_1-2)}{s_1+pa}\frac{r(qs_1+ps_2-p-q)+rpqa}{s_1+pa-1}\right].$$

Using Lemma 3.3.3, we obtain

$$\lim_{a\to 0}\frac{C_2}{a}=-\frac{s_1+s_2}{s_2^2}\left[\frac{rs_2}{s_1}+\frac{r}{(s_1-1)^2}\right]=-rs_2.$$

On the interval J_3, the integral of f_h is:

$$\begin{aligned}
C_3=\int_{J_3}f_h\,dL &= \int_{J_3}\left(\frac{s_1+s_2+pa+qa}{(s_2+qa)^2}\Lambda_h\frac{1-\frac{1}{(s_1+pa)^{k_1-1}}}{s_1+pa-1}+1\right)\chi_c\,dL\\
&= \left[\frac{\left(1-\frac{1}{(s_1+pa)^{k_1-1}}\right)\eta}{1-(\kappa+\eta(1-\frac{1}{(s_1+pa)^{k_1-1}})+\omega)}+1\right](\frac{1}{2}-ra)\\
&= \frac{\frac{a(qs_1+ps_2-p-q)+pqa^2}{(s_1+pa)(s_2+qa)}-\omega}{1-(\kappa+\eta(1-\frac{1}{(s_1+pa)^{k_1-1}})+\omega)}(\frac{1}{2}-ra)\ .
\end{aligned}$$

Using Lemma 3.3.3, we obtain

$$\lim_{a\to 0}\frac{C_3}{a}=-\frac{qs_1+ps_2-p-q}{2s_1s_2}\ .$$

In the same way, we can see that for any $0<\theta<1/2$, we obtain

$$\lim_{a\to 0}\frac{1}{a}\int_{1/2+\theta}^{1}f_h\,dL=-\frac{qs_1+ps_2-p-q}{s_1s_2}\left(\frac{1}{2}-\theta\right)\ .$$

If we define $B=C_1+C_2+C_3$, then $\frac{f_h}{B}$ is a normalized density. We see that

$$\lim_{a\to 0}\frac{B}{a}=-\frac{(qs_1+ps_2-p-q)(s_2+2)+2rs_1s_2^2}{2s_1s_2}\ .$$

Our calculations show that the normalized measures $\{(f_h/B)\cdot L\}$ converge $*$-weakly to the measure

$$\frac{(qs_1+ps_2-p-q)(s_2+2)}{(qs_1+ps_2-p-q)(s_2+2)+2rs_1s_2^2}\mu_0+\frac{2rs_1s_2^2}{(qs_1+ps_2-p-q)(s_2+2)+2rs_1s_2^2}\delta_{(\frac{1}{2})}\ .$$

Now, we will show the same holds for the normalized measure defined by f_l. To this end, let us notice that

$$
\begin{aligned}
f_h - f_l &= (1+\frac{s_1+pa}{s_2+qa})\Lambda_h g_l - (1+\frac{s_1+pa}{s_2+qa})\Lambda_l g_h \\
&= (1+\frac{s_1+pa}{s_2+qa})(\Lambda_h - \Lambda_l) g_l - \Lambda_l \frac{1+\frac{s_1+pa}{s_2+qa}}{(s_2+qa)(s-1)(s_1+pa)^{k_1-1}} \\
&= \frac{\omega(1+\frac{s_1+pa}{s_2+qa})}{(1-(\kappa+\eta(1-\frac{1}{(s_1+pa)^{k_1-1}})+\omega))(1-\kappa-\eta(1-\frac{1}{(s_1+pa)^{k_1-1}}))} g_l \\
&\quad -\Lambda_l \frac{1+\frac{s_1+pa}{s_2+qa}}{(s_2+qa)(s-1)(s_1+pa)^{k_1-1}} \ ,
\end{aligned}
$$

where $|g_l| \leq \frac{2}{s_1}$ and $\lim\limits_{a\to 0} \Lambda_l = -1$. Using Lemma 3.3.3 once again, we can show that for any subinterval $J \subset [0,1]$, we have

$$
\lim_{a\to 0} \frac{1}{a} \int_J (f_h - f_l)\, dL = 0 \ .
$$

For $J = [0,1]$ this means that the normalizations of f_l and f_h are asymptotically the same. With this, the limit for a general J means in particular that the $*$-weak limit of normalized measures defined using f_l is the same as for those defined using f_h. In view of inequality (3.7), this proves Theorem 3.2.1(II).

3.3.3.3 Estimates on f_a if $\frac{1}{s_1} + \frac{1}{s_2} < 1$

We have the following lemma:

Lemma 3.3.4. *For the family of W_a maps, if $\frac{1}{s_1} + \frac{1}{s_2} < 1$, we have*

(I) $W_a(1/2) = 1/2 + ra$, $W_a^2(1/2) = -ra(s_2+qa) + 1/2 + ra$, and for $3 \leq m \leq k$, we have $W_a^m(1/2) = -a(s_1+pa)^{m-2}\frac{r[s_1s_2 - s_1 - s_2 + a(qs_1+ps_2-p-q+pqa)]}{s_1+pa-1} + \frac{s_1-1+pa-2ra}{2(s_1+pa-1)}$;

(II) $\lim\limits_{a\to 0} ak = 0$;

(*III*) $\lim_{a\to 0} a(s_1+pa)^{k_1} = 0$;

(*IV*) $\lim_{a\to 0} W_a^{k_1}(\frac{1}{2}) = \frac{1}{2}$.

Proof. Suppose (I) is true. Let us first prove that (II) and (III) are true.

By the definition of k, we have:

$$\begin{aligned} 0 \le &- a(s_1+pa)^{k-2}\frac{r\left[s_1s_2 - s_1 - s_2 + a(qs_1+ps_2-p-q+pqa)\right]}{s_1+pa-1} \\ &+ \frac{s_1-1+pa-2ra}{2(s_1+pa-1)}. \end{aligned} \tag{3.8}$$

The inequality (3.8) implies $a(s_1+pa)^{k-2} \le \frac{s_1-1+pa-2ra}{2r[s_1s_2-s_1-s_2+a(qs_1+ps_2-p-q+pqa)]}$, thus

$$\begin{aligned} ak \le \quad & a\frac{\ln(s_1-1+pa-2ra) - \ln 2 + 2\ln(s_1+pa) - \ln r - \ln a}{\ln(s_1+pa)} \\ & -a\frac{\ln(2r\left[s_1s_2-s_1-s_2+a(qs_1+ps_2-p-q+pqa)\right])}{\ln(s_1+pa)}, \\ a(s_1+pa)^{k_1} \le \quad & \frac{(s_1-1+pa-2ra)(s_1+pa)^2}{2r\left[s_1s_2-s_1-s_2+a(qs_1+ps_2-p-q+pqa)\right](s_1+pa)^{k-k_1}}, \end{aligned}$$

and since $\lim_{a\to 0} a\ln a = 0$, we obtain (II) and (III). (IV) follows from (III).

Now, let us prove (I).

The fixed point slightly less than 1/2 is $x_l^* = \frac{s_1-1+pa-2ra}{2(s_1-1+pa)}$, and

$$x_l^* - W_a^2(1/2) = \frac{ra\left[s_1s_2-s_1-s_2+a(qs_1+ps_2-p-q+pqa)\right]}{s_1-1+pa} > 0,$$

which implies that $W_a^m(1/2)$ are all in the domain of the second branch of W_a for $3 \le m \le k$. Now, (I) follows by the same reasoning as in Lemma 3.3.3. □

Lemma 3.3.5. *If the normalized densities $\{h_a\}_{a<a_0}$, for some $a_0>0$, are uniformly bounded, then $h_a \to h_0$ in L^1.*

Proof. The uniform boundedness implies $\{h_a\}_{a<a_0}$ is a weakly precompact set in L^1. Thus, any limit of $\{h_a\}_{a<a_0}$ is a invariant density by Proposition 11.3.1 [Boyarsky

and Góra, 1997]. At the same time, this limit is an L^1 function, thus defines an acim. Since the map W_0 is exact and has only one acim, we conclude that $h_a \to h_0$ in L^1. $\square$

Now, we will prove Theorem 3.2.2:

The main idea of the proof is the following: since non-normalized densities $\{f_a\}$ are uniformly bounded (formulas (3.9, 3.10, 3.11)), it is enough to show that $\{\int_0^1 f_a\, dL\}$ are uniformly separated from zero.

For small a, by Lemma 3.3.2, Λ (and then both Λ_l and Λ_h) can be either positive or negative. Thus, we can have the following cases.

Case (i): $\Lambda_l < 0$:

Comparing with (3.4) and (3.3), we see that for the functions $\widehat{f_l} = 1+(1+\frac{s_1+pa}{s_2+qa})\Lambda_l g_h$ and $\widehat{f_h} = 1+(1+\frac{s_1+pa}{s_2+qa})\Lambda_h g_l$, we have

$$\widehat{f_l} \le f_a \le \widehat{f_h} \ . \tag{3.9}$$

Note that $\widehat{f_l}$ and $\widehat{f_h}$ have the same form as f_l and f_h in Section 3.3.3.1, so their representations as combinations of functions χ_j, $j = 1, \ldots, k_1$ and χ_c are similar to that of f_l and f_h. At the same time, now we have $\frac{1}{s_1} + \frac{1}{s_2} < 1$, so the representation is as follows:

$$\begin{aligned}
\widehat{f_l} \;\; = \;\; & \left(\frac{s_1+s_2+pa+qa}{(s_2+qa)(s_1+pa)}\Lambda_l + 1\right)\chi_1 + \frac{s_1+s_2+pa+qa}{(s_2+qa)^2}\Lambda_l \sum_{j=2}^{k_1} \frac{\chi_j}{(s_1+pa)^{j-1}} \\
& + \left(\frac{s_1+s_2+pa+qa}{(s_2+pa)^2}\Lambda_l \frac{1-\frac{1}{(s_1+pa)^{k_1-1}}}{s_1+pa-1} + 1\right)\chi_c \\
& + \frac{\frac{s_1+s_2+pa+qa}{s_2+qa}\Lambda_l}{(s_2+qa)(s-1)(s_1+pa)^{k_1-1}} \ ,
\end{aligned}$$

$$\begin{aligned}\widehat{f}_h &= \left(\frac{s_1+s_2+pa+qa}{(s_2+qa)(s_1+pa)}\Lambda_h+1\right)\chi_1+\frac{s_1+s_2+pa+qa}{(s_2+qa)^2}\Lambda_h\sum_{j=2}^{k_1}\frac{\chi_j}{(s_1+pa)^{j-1}}\\ &+\left(\frac{s_1+s_2+pa+qa}{(s_2+qa)^2}\Lambda_h\frac{1-\frac{1}{(s_1+pa)^{k_1-1}}}{s_1+pa-1}+1\right)\chi_c\ .\end{aligned}$$

(3.3) implies that all the coefficients in the representation of $\widehat{f}_l$ and $\widehat{f}_h$ are negative for sufficiently small a.

We use the same notations J_1, J_2 and J_3 as in Section 3.3.3.2. First, we do the calculations assuming that $\vartheta = 1-\left(\frac{s_1+s_2}{s_1s_2}+\frac{s_1+s_2}{s_2^2(s_1-1)}\right)\neq 0$.

We will calculate the integrals of $\widehat{f}_h$ over each of J_1, J_2 and J_3, and use them to normalize $\widehat{f}_h$. We have

$$\begin{aligned}\widehat{C}_1 &= \int_{J_1}\widehat{f}_h\,dL=\int_{J_1}\left[\frac{s_1+s_2+pa+qa}{(s_2+qa)(s_1+pa)}\Lambda_h+1\right]\chi_1\,dL\\ &= \left[\frac{s_1+s_2+pa+qa}{(s_2+qa)(s_1+pa)}\Lambda_h+1\right]W_a^{k_1}(\frac{1}{2})\\ &= \left[\frac{\kappa}{1-(\kappa+\eta(1-\frac{1}{(s_1+pa)^{k_1-1}})+\omega)}+1\right]W_a^{k_1}(\frac{1}{2})\\ &= \Bigg[\frac{s_1s_2^2-s_1-s_2-s_2^2}{1-(\kappa+\eta(1-\frac{1}{(s_1+pa)^{k_1-1}})+\omega))(s_2+qa)^2(s_1+pa-1)}\\ &\quad+\frac{a(2qs_1s_2+ps_2^2-2qs_2-p-q)}{(1-(\kappa+\eta(1-\frac{1}{(s_1+pa)^{k_1-1}})+\omega))(s_2+qa)^2(s_1+pa-1)}\\ &\quad+\frac{a^2(2pqs_2-q^2+q^2s_1)+pq^2a^3}{(1-(\kappa+\eta(1-\frac{1}{(s_1+pa)^{k_1-1}})+\omega))(s_2+qa)^2(s_1+pa-1)}\\ &\quad+\frac{\frac{\eta}{(s_1+pa)^{k_1-1}}-\omega}{1-(\kappa+\eta(1-\frac{1}{(s_1+pa)^{k_1-1}})+\omega)}\Bigg]W_a^{k_1}(\frac{1}{2})\ .\end{aligned}$$

Using Lemma 3.3.4, we have

$$\lim_{a\to 0}\widehat{C}_1=\frac{1}{2}\frac{\frac{s_1s_2^2-s_1-s_2-s_2^2}{s_2^2(s_1-1)}}{1-\left(\frac{s_1+s_2}{s_1s_2}+\frac{s_1+s_2}{s_2^2(s_1-1)}\right)}=\frac{1}{2}\frac{1-\frac{s_1+s_2}{s_2^2(s_1-1)}}{1-\left(\frac{s_1+s_2}{s_1s_2}+\frac{s_1+s_2}{s_2^2(s_1-1)}\right)}.$$

On the interval J_2, the integral of $\widehat{f}_h$ is:

$$\widehat{C}_2=\int_{J_2}\widehat{f}_h\,dL = \int_{J_2}\left[\frac{s_1+s_2+pa+qa}{(s_2+qa)(s_1+pa)}\Lambda_h+1\right]\chi_1\,dL$$

$$
\begin{aligned}
&+\frac{s_1+s_2+pa+qa}{(s_2+qa)^2}\Lambda_h \sum_{j=2}^{k_1}\int_{J_2}\frac{\chi_j}{(s_1+pa)^{j-1}}\,dL \\
= &\ \frac{1-\eta(1-\frac{1}{(s_1+pa)^{k_1-1}})-\omega}{1-(\kappa+\eta(1-\frac{1}{(s_1+pa)^{k_1-1}})+\omega)}\left(\frac{1}{2}+ra-W_a^{k_1}(\frac{1}{2})\right) \\
&+\frac{s_1+s_2+pa+qa}{(s_2+qa)^2}\Lambda_h\left[\frac{ra(s_2+qa)}{s_1+pa}+\frac{ra(1-\frac{1}{(s_1+pa)^{k_1-2}})}{(s_1+pa-1)^2}\right. \\
&\left.+\frac{a(k_1-2)}{s_1+pa}\frac{r(s_1s_2-s_1-s_2+a(qs_1+ps_2-p-q+pqa))}{s_1+pa-1}\right].
\end{aligned}
$$

Using Lemma 3.3.4, we have $\lim_{a\to 0}\widehat{C}_2=0$.

On the interval J_3, the integral of $\widehat{f}_h$ is:

$$
\begin{aligned}
\widehat{C}_3=\int_{J_3}\widehat{f}_h\,dL &= \int_{J_3}\left(\frac{s_1+s_2+pa+qa}{(s_2+qa)^2}\Lambda_h\frac{1-\frac{1}{(s_1+pa)^{k_1-1}}}{s_1+pa-1}+1\right)\chi_c\,dL \\
&= \left[\frac{\eta\left(1-\frac{1}{(s_1+pa)^{k_1-1}}\right)}{1-(\kappa+\eta(1-\frac{1}{(s_1+pa)^{k_1-1}})+\omega)}+1\right](\frac{1}{2}-ra) \\
&= \frac{\frac{s_1s_2-s_1-s_2+a(qs_1+ps_2-p-q)+pqa^2}{(s_1+pa)(s_2+qa)}-\omega}{1-(\kappa+\eta(1-\frac{1}{(s_1+pa)^{k_1-1}})+\omega)}(\frac{1}{2}-ra)\ .
\end{aligned}
$$

Using Lemma 3.3.4 once again, we have

$$
\lim_{a\to 0}\widehat{C}_3=\frac{1}{2}\frac{1-\frac{s_1+s_2}{s_1s_2}}{1-\left(\frac{s_1+s_2}{s_1s_2}+\frac{s_1+s_2}{s_2^2(s_1-1)}\right)}\ .
$$

Note that if we define $\widehat{B}=\widehat{C}_1+\widehat{C}_2+\widehat{C}_3$, then

$$
\lim_{a\to 0}\widehat{B}=\frac{1}{2}\frac{2-\left(\frac{s_1+s_2}{s_1s_2}+\frac{s_1+s_2}{s_2^2(s_1-1)}\right)}{1-\left(\frac{s_1+s_2}{s_1s_2}+\frac{s_1+s_2}{s_2^2(s_1-1)}\right)}\ ,
$$

which is not 0. Since $\{\widehat{f}_h\}$ are uniformly bounded, we conclude that the normalized $\{\widehat{f}_h\}$ are also uniformly bounded.

Now, we will show that the normalized $\{\widehat{f}_l\}$ are also uniformly bounded. To this end, let us notice that

$$
\widehat{f}_h-\widehat{f}_l = (1+\frac{s_1+pa}{s_2+qa})\Lambda_h g_l-(1+\frac{s_1+pa}{s_2+qa})\Lambda_l g_h
$$

$$
\begin{aligned}
&= (1+\frac{s_1+pa}{s_2+qa})(\Lambda_h-\Lambda_l)g_l-\Lambda_l\frac{1+\frac{s_1+pa}{s_2+qa}}{(s_2+qa)(s-1)(s_1+pa)^{k_1-1}} \\
&= \frac{\omega(1+\frac{s_1+pa}{s_2+qa})}{(1-(\kappa+\eta(1-\frac{1}{(s_1+pa)^{k_1-1}})+\omega))(1-\kappa-\eta(1-\frac{1}{(s_1+pa)^{k_1-1}}))}g_l \\
&\quad -\Lambda_l\frac{1+\frac{s_1+pa}{s_2+qa}}{(s_2+qa)(s-1)(s_1+pa)^{k_1-1}} \ ,
\end{aligned}
$$

where $|g_l| \le \frac{1}{s_1}+\frac{1}{s_2(s_1-1)}$ and $\lim_{a\to 0}\Lambda_l = \frac{1}{1-\left(\frac{s_1+s_2}{s_1 s_2}+\frac{s_1+s_2}{s_2^2(s_1-1)}\right)}$. Thus, $\lim_{a\to 0}\widehat{f}_h-\widehat{f}_l = 0$. We conclude that the normalized $\{\widehat{f}_l\}$ are uniformly bounded since the normalized $\{\widehat{f}_h\}$ are uniformly bounded. Thus, after normalization, $\{f_a\}$ are also uniformly bounded.

Case (ii): $\Lambda_l > 0$:

This case implies that f_a given by (3.4) has the following properties:

$$f_a \ge 1 \ , \tag{3.10}$$

and all the coefficients of the characteristic functions appearing in (3.4) are positive. We note that Λ is always positive for small a. Thus,

$$f_a \le 1+(1+\frac{s_1+pa}{s_2+qa})\Lambda\sum_{n=1}^{\infty}\frac{1}{|\beta(1/2,n)|} \ , \tag{3.11}$$

which is finite since our maps $\{W_a\}$ are expanding. In view of (3.10), we conclude that the normalized $\{f_a\}$ are uniformly bounded.

If $\vartheta = 1-\left(\frac{s_1+s_2}{s_1 s_2}+\frac{s_1+s_2}{s_2^2(s_1-1)}\right) = 0$, then we have $\lim_{a\to 0}\frac{1}{\Lambda_l} = \lim_{a\to 0}\frac{1}{\Lambda_h} = 0$, Λ_l and Λ_h are still of the same sign. We can renormalize f_a. Let us take the $\widehat{f}_h$ as an example. Multiplying it by $\frac{1}{\Lambda_h}$, we obtain

$$
\begin{aligned}
\frac{1}{\Lambda_h}\widehat{f}_h &= \left(\frac{s_1+s_2+pa+qa}{(s_2+qa)(s_1+pa)}+\frac{1}{\Lambda_h}\right)\chi_1+\frac{s_1+s_2+pa+qa}{(s_2+qa)^2}\sum_{j=2}^{k_1}\frac{\chi_j}{(s_1+pa)^{j-1}} \\
&\quad +\left(\frac{s_1+s_2+pa+qa}{(s_2+qa)^2}\frac{1-\frac{1}{(s_1+pa)^{k_1-1}}}{s_1+pa-1}+\frac{1}{\Lambda_h}\right)\chi_c \ .
\end{aligned}
$$

Note that the coefficients of χ_1 and χ_c converge to $\frac{s_1+s_2}{s_1 s_2}$ and $\frac{s_1+s_2}{s_2^2(s_1-1)}$, respectively. Thus, $\{\int_0^1 \frac{1}{\Lambda_h}\widehat{f}_h\, dL\}$ are separated from 0. This implies $\{\frac{1}{\Lambda_h}\widehat{f}_h\}$ are uniformly bounded. A similar procedure can be applied to $\widehat{f}_l$. We conclude that $\{\frac{1}{\Lambda}f_a\}$ are uniformly bounded.

3.4 Example

One of the important properties of a piecewise expanding transformation of an interval is that its invariant density is bounded away from 0 on its support. The following result was proved, by [Keller, 1978] and by [Kowalski, 1979].

Theorem 3.4.1. *Let a transformation $\tau : I \to I$ be piecewise expanding with $\frac{1}{|\tau'(x)|}$ a function of bounded variation, and let f be a τ-invariant density which can be assumed to be lower semicontinuous. Then there exists a constant $c > 0$ such that $f|_{\text{supp } f} > c$.*

We provide an example showing that this result cannot be generalized to a family of expanding maps, even if they all have this property and converge to a limit map also with this property. Let $d(\cdot,\cdot)$ be the metric on the weak topology of measures.

Example 3.4.1. Let us fix

$$s_1 = s_2 = 2,\ p = q = 1.$$

For small $a > 0$, let $W_{a,r}$ denote the W_a maps with varying parameter r, and let $\mu_{a,r}$ denote the acim of $W_{a,r}$. We know that $\mu_{a,r}$ is supported on $[0,1]$ and $W_{a,r}$ with $\mu_{a,r}$ is exact. Using Theorem 3.2.1, we know that $\{\mu_{a,r}\}$ converge $*$-weakly to the measure

$$\mu_{0,r} = \frac{1}{1+2r}\mu_0 + \frac{2r}{1+2r}\delta_{(\frac{1}{2})}.$$

Let $r_n = n$, $n = 1, 2, 3, \cdots$. Also, let $\{a_n\}_1^\infty$ satisfy $r_n a_n < 1/2$ and be so small that

$$d(\mu_{a_n,r_n}, \mu_{0,r_n}) < \frac{1}{n} \ .$$

Now, for the family of maps $\tau_n = W_{a_n,r_n}$, $n = 1, 2, 3, \cdots$, τ_n converge to W_0 with $|\tau_n'(x)| > 2$, but the invariant densities μ_{a_n,r_n} converge to $\delta_{(\frac{1}{2})}$. This implies that the invariant densities $\{f_{a_n,r_n}\}$ corresponding to $\{\mu_{a_n,r_n}\}$ have no uniform positive lower bound.

Chapter 4

Instability of Isolated Spectrum for $W-$shaped Maps

4.1 Introduction

In previous chapters, we discussed one of the most important problems in the theory of dynamical systems: their stability and possible instability [Boyarsky and Góra, 1997; Keller, 1982; Keller and Liverani, 1999]. In particular, in the theory of piecewise expanding maps of an interval, it is interesting whether the given system has a stable acim, and more generally, if the isolated spectrum of the Perron-Frobenius operator is stable under small perturbations of the map.

In general, the setting of the stability problem we are interested in is as follows: let τ_0 be a piecewise expanding map of an interval with unique acim μ_0 and $\{\tau_a\}_{a>0}$ a family of its perturbations with acims μ_a, correspondingly. If maps τ_a converge to τ_0

(say, in Skorokhod metric), do their acims converge (say, in the $*$-weak topology) to μ_0? Or more generally, do the isolated spectra of P_{τ_a} converge to the isolated spectrum of P_{τ_0}, including multiplicities and eigenfunctions? P_τ is the Perron-Frobenius operator induced by τ on the space of functions of bounded variation and by isolated spectrum we mean the part of the spectrum which lies outside the essential spectral radius. Papers [Keller, 1982] and [Keller and Liverani, 1999] show that such stability takes place if the family $\{\tau_a\}_{a\geq 0}$ satisfies the Lasota-Yorke inequality ([Lasota and Yorke, 1973] or [Eslami and Góra, 2012] for strengthened form) with uniform constants. Usual conditions ensuring this are $|\tau_a'| > 2+\epsilon$ plus the minimal length of subintervals of defining partitions uniformly separated from 0.

One of the known sources of instability is the presence of a turning fixed or periodic point touching a map branch with slope 2 or smaller. In previous chapters, we studied the famous W-shaped map introduced in [Keller, 1982] (see previous chapters for detailed examples). Because of the turning fixed point we cannot use an iterate of the map to increase the minimal slope. It causes appearance of arbitrary short partition intervals in perturbed maps (see the discussion in Remark 1.2.2). This is depicted in Fig. 4.1.

The W_{s_1,s_2} map is a piecewise linear map of the interval $[0,1]$ onto itself with a graph in the shape of letter W. The original W-map of [Keller, 1982] is of $W_{2,2}$ type, and in Chapter 2 (see also [Eslami and Misiurewicz, 2012; Li et al., 2013]) we see that its acim is unstable under some family of localized perturbations. A more general situation was considered in Chapter 3 [Li, 2013]. The perturbations similar to that of Chapter 2 [Li et al., 2013] were considered. It was shown there that depending on

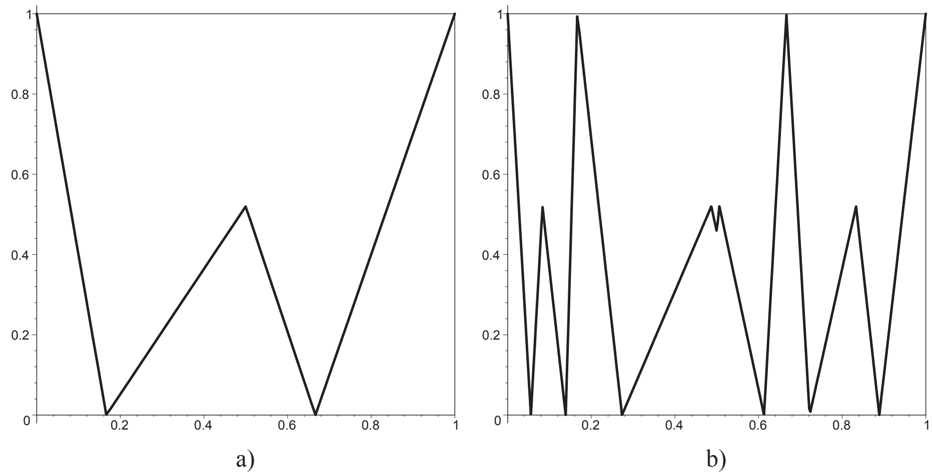

Figure 4.1: The W-shaped maps: a) W_a with $s_1 = 3/2$ and $s_2 = 3$; $a = 0.01$, b) W_a^2, the second iteration of W_a shown in a).

whether $\frac{1}{s_1} + \frac{1}{s_2}$ is larger, equal or smaller than 1 the limit of μ_a's is Dirac measure $\delta_{1/2}$, or a combination of $\delta_{1/2}$ and μ_0, or μ_0, correspondingly. This result suggested that condition $\frac{1}{s_1} + \frac{1}{s_2} < 1$ may actually imply stability which was later proved for a quite general setting in [Eslami and Góra, 2012].

In this chapter we consider map W_0 of type W_{s_1,s_2} with $\frac{1}{s_1} + \frac{1}{s_2} = 1$ and show that that eigenvalue 1 is not stable. We do this in a constructive way. For each perturbed map W_a we show the existence of the "second" eigenvalue λ_a,

$$\frac{1-2ra}{1+2ra} < \lambda_a < \frac{1}{1+2ra} \ ,$$

where r is a constant independent of a. Thus, as $a \to 0$ the eigenvalues $\lambda_a \to 1$ which shows instability of isolated spectrum of W_0. At the same time, the existence of second eigenvalues close to 1 causes the maps W_a behave in a metastable way. They have two almost invariant sets and the system spends long periods of consecutive iterations in each of them with infrequent jumps from one to the other.

For a fixed small $a_0 > 0$, the slopes of the middle branches of W_{a_0} satisfy $1/(s_1 + 2rs_1a_0) + 1/(s_2 + 2rs_2a_0) < 1$. By the stability result of [Eslami and Góra, 2012], there exists an $\epsilon > 0$ such that for all a satisfying $|a - a_0| < \epsilon$, the maps W_a have eigenvalues close to λ_{a_0}. Most of these maps are non-Markov. Thus, our Markov examples prove the existence of non-Markov examples with similar properties.

The results obtained in this chapter (Sections 4.2, 4.3 and 4.4) were, after some modifications, published in the paper [Li, 2013].

4.2 Markov W_a maps and their invariant densities

Let s_1, $s_2 > 1$ satisfy $\frac{1}{s_1} + \frac{1}{s_2} = 1$, and $r > 0$. Let us consider W-shaped map:

$$W_0(x) = \begin{cases} W_{0,1}(x) := 1 - 2s_2x, & 0 \le x < \frac{1}{2} - \frac{1}{2s_1}, \\ W_{0,2}(x) := s_1(x - \frac{1}{2} + \frac{1}{2s_1}), & \frac{1}{2} - \frac{1}{2s_1} \le x < \frac{1}{2}, \\ W_{0,3}(x) := s_2(\frac{1}{2} + \frac{1}{2s_2} - x), & \frac{1}{2} \le x < \frac{1}{2} + \frac{1}{2s_2}, \\ W_{0,4}(x) := 2s_1(x-1) + 1, & \frac{1}{2} + \frac{1}{2s_2} \le x < 1, \end{cases}$$

and its perturbations, W_a maps with parameter $a > 0$:

$$W_a(x) = \begin{cases} W_{a,1}(x) := 1 - 2s_2x, & 0 \le x < \frac{1}{2} - \frac{1}{2s_1}, \\ W_{a,2}(x) := (s_1 + 2rs_1a)(x - \frac{1}{2} + \frac{1}{2s_1}), & \frac{1}{2} - \frac{1}{2s_1} \le x < \frac{1}{2}, \\ W_{a,3}(x) := (s_2 + 2rs_2a)(\frac{1}{2} + \frac{1}{2s_2} - x), & \frac{1}{2} \le x < \frac{1}{2} + \frac{1}{2s_2}, \\ W_{a,4}(x) := 2s_1(x-1) + 1, & \frac{1}{2} + \frac{1}{2s_2} \le x < 1. \end{cases}$$

Let $\tau_i = W_{a,i}^{-1}$, $i = 1, 2, 3, 4$; $I_0 = [0, \frac{1}{2} + ra]$. The Frobenius-Perron operator (see Definition 1.2.2, or [Boyarsky and Góra, 1997] for more details) associated with W_a

is

$$P_a f = \frac{1}{2s_2} f\circ\tau_1 + \frac{1}{s_1+2rs_1a}(f\circ\tau_2)\chi_{I_0} \quad + \quad \frac{1}{s_2+2rs_2a}(f\circ\tau_3)\chi_{I_0}$$
$$+ \quad \frac{1}{2s_1} f\circ\tau_4$$

Note that

$$\chi_{I_0}\circ\tau_1 = 1 \quad , \quad \chi_{I_0}\circ\tau_2 = \chi_{I_0} \ ,$$
$$\chi_{I_0}\circ\tau_3 = \chi_{[W_a^2(1/2),\frac{1}{2}+ra]} \quad , \quad \chi_{I_0}\circ\tau_4 = 0 \ . \tag{4.1}$$

Let $I_1 = [W_a^2(1/2), \frac{1}{2}+ra]$ whose left end point is $W_a^2(\frac{1}{2}) = W_a(\frac{1}{2}+ra)$.

We will consider only parameters a such that W_a is a Markov map, i.e., some iterate of $1/2$ falls into an endpoint of the defining partition. Let a satisfy:

$$W_a^m(\frac{1}{2}+ra) = \frac{1}{2} - \frac{1}{2s_1}, \tag{4.2}$$

where $m \geq 1$ is the first time when the trajectory of $W_a(\frac{1}{2}) = \frac{1}{2}+ra$ reaches the partition point $\frac{1}{2}-\frac{1}{2s_1}$. Note that $\frac{1}{2}-\frac{1}{2s_1} = \frac{1}{2s_2}$. The point $W_a(\frac{1}{2}+ra)$ is just below the fixed point on the second branch of W_a and the consecutive images $W_a^i(\frac{1}{2}+ra)$ decrease until for $i=m$ the equality (4.2) is satisfied.

Let us take 1 as the initial function and iterate it using operator P_a. Let $P_a^n 1$ be denoted by $f_{n,m}$. Let

$$I_i = [W_a^i(\frac{1}{2}+ra), \frac{1}{2}+ra], i = 1, 2, \cdots, m.$$

Because of (4.2)and (4.1), after some number of iterations ($n \geq m+1$), we have:

$$f_{n,m} = c_{n,0} + \alpha_{n,0}\chi_{I_0} + \alpha_{n,1}\chi_{I_1} + \alpha_{n,2}\chi_{I_2} + \cdots + \alpha_{n,m-1}\chi_{I_{m-1}} + \alpha_{n,m}\chi_{I_m},$$

where $c_{n,0}$ and $\alpha_{n,i}$ ($i = 0, 1, \cdots, m$) are constants. Now, let us look at the $f_{n+1,m}$. We have the following proposition.

Proposition 4.2.1. *(I) $c_{n,0} \circ \tau_1$ and $c_{n,0} \circ \tau_4$ are again constant functions, $c_{n,0} \circ \tau_2 \chi_{I_0}$ and $c_{n,0} \circ \tau_3 \chi_{I_0}$ are the characteristic function χ_{I_0};*

(II) $\chi_{I_0} \circ \tau_1$ is constant function, $\chi_{I_0} \circ \tau_2 \chi_{I_0} = \chi_{I_0}$, $\chi_{I_0} \circ \tau_3 \chi_{I_0} = \chi_{I_1}$, $\chi_{I_0} \circ \tau_4$ is 0;

(III) For $i = 1, 2, \cdots, m-1$, $\chi_{I_i} \circ \tau_1$ and $\chi_{I_i} \circ \tau_4$ are 0, $\chi_{I_i} \circ \tau_2 \chi_{I_0} = \chi_{I_{i+1}}$, $\chi_{I_i} \circ \tau_3 \chi_{I_0} = \chi_{I_1}$;

(IV) $\chi_{I_m} \circ \tau_1$ and $\chi_{I_m} \circ \tau_4$ are 0, $\chi_{I_m} \circ \tau_2 \chi_{I_0} = \chi_{I_0}$, $\chi_{I_m} \circ \tau_3 \chi_{I_0} = \chi_{I_1}$.

Thus, we have the following proposition.

Proposition 4.2.2. *for n big enough, $f_{n,m}$ always has the form:*

$$f_{n,m} = c_{n,0} + \alpha_{n,0}\chi_{I_0} + \alpha_{n,1}\chi_{I_1} + \alpha_{n,2}\chi_{I_2} + \cdots + \alpha_{n,m-1}\chi_{I_{m-1}} + \alpha_{n,m}\chi_{I_m},$$

and

$$\begin{bmatrix} c_{n+1,0} \\ \alpha_{n+1,0} \\ \alpha_{n+1,1} \\ \vdots \\ \alpha_{n+1,m} \end{bmatrix} = A_m \begin{bmatrix} c_{n,0} \\ \alpha_{n,0} \\ \alpha_{n,1} \\ \vdots \\ \alpha_{n,m} \end{bmatrix},$$

where $(m+2) \times (m+2)$ matrix A_m is given by

$$A_m = \begin{bmatrix} \frac{1}{2s_1}+\frac{1}{2s_1} & \frac{1}{2s_2} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{s_1+2rs_1a}+\frac{1}{s_2+2rs_2a} & \frac{1}{s_1+2rs_1a} & 0 & 0 & 0 & \cdots & 0 & \frac{1}{s_1+2rs_1a} \\ 0 & \frac{1}{s_2+2rs_2a} & \frac{1}{s_2+2rs_2a} & \frac{1}{s_2+2rs_2a} & \frac{1}{s_2+2rs_2a} & \cdots & \frac{1}{s_2+2rs_2a} & \frac{1}{s_2+2rs_2a} \\ 0 & 0 & \frac{1}{s_1+2rs_1a} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{s_1+2rs_1a} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & \frac{1}{s_1+2rs_1a} & 0 \end{bmatrix}.$$

Since $\frac{1}{s_1}+\frac{1}{s_2}=1$, we can simplify the A_m to the form

$$A_m = \begin{bmatrix} \frac{1}{2} & \frac{1}{2s_2} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{1+2ra} & \frac{1}{s_1+2rs_1a} & 0 & 0 & 0 & \cdots & 0 & \frac{1}{s_1+2rs_1a} \\ 0 & \frac{1}{s_2+2rs_2a} & \frac{1}{s_2+2rs_2a} & \frac{1}{s_2+2rs_2a} & \frac{1}{s_2+2rs_2a} & \cdots & \frac{1}{s_2+2rs_2a} & \frac{1}{s_2+2rs_2a} \\ 0 & 0 & \frac{1}{s_1+2rs_1a} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{s_1+2rs_1a} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & \frac{1}{s_1+2rs_1a} & 0 \end{bmatrix}.$$

We also need the following proposition.

Proposition 4.2.3. *Equation (4.2) is equivalent to:*

$$(s_2+2rs_2a)(s_1+2rs_1a)^{m-1} - \sum_{i=0}^{m-1}(s_1+2rs_1a)^i = \frac{1}{2rs_1a}\ ,$$

or

$$(s_1+2rs_1a)^m = \frac{1}{4r^2s_2^2a^2}. \tag{4.3}$$

Proof. If

$$(s_2+2rs_2a)(s_1+2rs_1a)^{m-1} - \sum_{i=0}^{m-1}(s_1+2rs_1a)^i = \frac{1}{2rs_1a}\ ,$$

then

$$(s_1+2rs_1a)^{m-1}\left[(s_2+2rs_2a)(s_1+2rs_1a-1)-(s_1+2rs_1a)\right] = \frac{s_1-1}{2rs_1a} = \frac{1}{2rs_2a},$$

Thus, we obtain

$$(s_1 + 2rs_1a)^m = \frac{1}{4r^2s_2^2a^2}.$$

On the other hand, it is proven in [Li, 2013] that

$$\begin{aligned} &W_a^{m+1}(1/2) \\ &= -a^2(s_1+2rs_1a)^{m-1}\frac{r(2rs_1s_2+2rs_1s_2-2rs_1-2rs_2)+4r^3s_1s_2a}{s_1+2rs_1a-1} \\ &\qquad\qquad\qquad\qquad + \frac{s_1-1+2rs_1a-2ra}{2(s_1+2rs_1a-1)}. \end{aligned}$$

If equation (4.2) holds, then

$$a^2(s_1+2rs_1a)^{m-1}\frac{2r^2s_1s_2+4r^3s_1s_2a}{s_1+2rs_1a-1} = \frac{s_1-1}{2s_1(s_1+2rs_1a-1)},$$

hence,

$$(s_1+2rs_1a)^{m-1} = \frac{1}{a^2 4r^2 s_1 s_2^2(1+2ra)}$$

which is equivalent to equation (4.3). □

Using Proposition 4.2.2, we can find the fixed vector of A_m. Let us denote it by $(c, \alpha_0, \alpha_1, \cdots, \alpha_m)^T$. Then, the fixed function (not necessarily normalized) of P_a is:

$$g_m^* = c + \alpha_0\chi_{I_0} + \alpha_1\chi_{I_1} + \alpha_2\chi_{I_2} + \cdots + \alpha_{m-1}\chi_{I_{m-1}} + \alpha_m\chi_{I_m},$$

where

$$\begin{aligned} c &= \frac{1}{2rs_1s_2a} \\ \alpha_0 &= \frac{1}{2rs_1a} \\ \alpha_1 &= (s_1+2rs_1a)^{m-1} \\ \alpha_2 &= (s_1+2rs_1a)^{m-2} \end{aligned}$$

$$\cdots$$

$$\begin{aligned}
\alpha_{m-2} &= (s_1 + 2rs_1a)^2 \\
\alpha_{m-1} &= s_1 + 2rs_1a \\
\alpha_m &= 1.
\end{aligned}$$

Using equation (4.2) or (4.3), we can directly normalize g_m^* and find its limit, which will give us the convergent invariant density. The idea will be similar to what we did in Section 2.3. Alternatively, there is an easier way, it was proven in [Li, 2013] that after normalization the measures $g_m^* \cdot L$ converge to the measure

$$\frac{1}{2r(s_1+s_2)(s_2+2)+2rs_1s_2^2}\left(2r(s_1+s_2)(s_2+2)\mu_0 + 2rs_1s_2^2\delta_{1/2}\right) ,$$

where L is Lebesgue measure, μ_0 is the acim of W_0 and $\delta_{1/2}$ is Dirac measure at 1/2.

4.3 Second eigenvalues for Markov W_a maps

Now, instead for a fixed vector, we will look for an eigenvector corresponding to an eigenvalue $\lambda < 1$. Denote the eigenvector of A_m associated with λ by $(c, \alpha_0, \alpha_1, \cdots, \alpha_m)^T$. Then, the corresponding eigenfunction of P_a associated with λ is:

$$h_m = c + \alpha_0\chi_{I_0} + \alpha_1\chi_{I_1} + \alpha_2\chi_{I_2} + \cdots + \alpha_{m-1}\chi_{I_{m-1}} + \alpha_m\chi_{I_m} \ . \tag{4.4}$$

The equation

$$A_m h_m = \lambda h_m \ ,$$

is equivalent to the system

$$\lambda c = \frac{1}{2}c + \frac{1}{2s_2}\alpha_0$$

$$
\begin{aligned}
\lambda\alpha_0 &= \frac{1}{1+2ra}(c + \frac{1}{s_1}\alpha_0 + \frac{1}{s_1}\alpha_m) \\
\lambda\alpha_1 &= \frac{1}{s_2(1+2ra)}(\alpha_0 + \alpha_1 + \cdots + \alpha_m) \\
\lambda\alpha_2 &= \frac{1}{s_1(1+2ra)}\alpha_1 \\
\lambda\alpha_3 &= \frac{1}{s_1(1+2ra)}\alpha_2 \\
&\cdots \\
\lambda\alpha_{m-2} &= \frac{1}{s_1(1+2ra)}\alpha_{m-3} \\
\lambda\alpha_{m-1} &= \frac{1}{s_1(1+2ra)}\alpha_{m-2} \\
\lambda\alpha_m &= \frac{1}{s_1(1+2ra)}\alpha_{m-1}.
\end{aligned}
$$

We can solve this system starting from the last equation. Let $\alpha_m = 1$. Then,

$$
\begin{aligned}
\alpha_m &= 1 \\
\alpha_{m-1} &= \lambda s_1(1+2ra) \\
\alpha_{m-2} &= \lambda^2 s_1^2 (1+2ra)^2 \\
&\cdots \\
\alpha_2 &= \lambda^{m-2} s_1^{m-2}(1+2ra)^{m-2} \\
\alpha_1 &= \lambda^{m-1} s_1^{m-1}(1+2ra)^{m-1} \\
\alpha_0 &= \lambda s_2(1+2ra)\alpha_1 - (\alpha_1 + \alpha_2 + \cdots + \alpha_m) \\
&= \lambda^m s_1^{m-1} s_2 (1+2ra)^m - \frac{\lambda^m s_1^m (1+2ra)^m - 1}{\lambda s_1(1+2ra) - 1} \\
&= \frac{\lambda^m s_1^{m-1}(1+2ra)^m(\lambda s_1 s_2(1+2ra) - s_1 s_2) + 1}{\lambda s_1(1+2ra) - 1} \\
c &= \lambda(1+2ra)\alpha_0 - \frac{\alpha_0}{s_1} - \frac{1}{s_1} \\
c &= \frac{\alpha_0}{s_2(2\lambda - 1)}
\end{aligned}
\tag{4.5}
$$

We have two expressions for c. The system can be solved only if they are equal. Thus, we obtain equation

$$\lambda^m s_1^{m-2}(1+2ra)^m(\lambda s_1 s_2(1+2ra) - s_1 s_2)$$
$$= \frac{\lambda^m s_1^{m-1}(1+2ra)^m(\lambda s_1 s_2(1+2ra) - s_1 s_2) + 1}{s_2(2\lambda - 1)(\lambda s_1(1+2ra) - 1)} \ ,$$

or

$$\lambda^m s_1^{m-1} s_2 (1+2ra)^m (\lambda(1+2ra) - 1) \left[s_2(2\lambda - 1)(\lambda s_1(1+2ra) - 1) - s_1\right] = 1 \ .$$

We are going to prove that for small a this equation has a solution $\frac{1-2ra}{1+2ra} < \lambda < \frac{1}{1+2ra}$. Let us introduce an auxiliary function

$$\phi(\lambda) = \lambda^m s_1^{m-1} s_2 (1+2ra)^m (\lambda(1+2ra) - 1) \left[s_2(2\lambda - 1)(\lambda s_1(1+2ra) - 1) - s_1\right] \ .$$

Obviously $\phi(\frac{1}{1+2ra}) = 0$. We will show that $\phi(\frac{1-2ra}{1+2ra}) > 1$ if a is small enough. We have

$$\begin{aligned}
\phi\left(\frac{1-2ra}{1+2ra}\right) &= \left(\frac{1-2ra}{1+2ra}\right)^m s_1^{m-1} s_2 \left(1+2ra\right)^m \left(\left(\frac{1-2ra}{1+2ra}\right)(1+2ra) - 1\right) \cdot \\
&\quad \cdot \left[s_2 \left(2\left(\frac{1-2ra}{1+2ra}\right) - 1\right)\left(\left(\frac{1-2ra}{1+2ra}\right) s_1 \left(1+2ra\right) - 1\right) - s_1\right] \\
&= (1-2ra)^m s_1^{m-1} s_2 (-2ra) \frac{-2ra(s_2 + 5s_1 - 6s_1 s_2 ra)}{1+2ra} \\
&= (1-2ra)^m s_1^{m-1} s_2 4r^2 a^2 \frac{s_2 + 5s_1 - 6s_1 s_2 ra}{1+2ra}.
\end{aligned}$$

Using (4.3) we obtain

$$\begin{aligned}
\phi\left(\frac{1-2ra}{1+2ra}\right) &= \left(\frac{1-2ra}{1+2ra}\right)^m \frac{s_2 + 5s_1 - 6s_1 s_2 ra}{s_1 s_2 (1+2ra)} \\
&= \left(\frac{1-2ra}{1+2ra}\right)^m \frac{1 + \frac{4}{s_2} - 6ra}{1+2ra}.
\end{aligned}$$

Note that if $a < \frac{1}{2rs_2}$, then $\frac{1+\frac{4}{s_2}-6ra}{1+2ra} > 1$. Furthermore, $\lim\limits_{a\to 0} \frac{1+\frac{4}{s_2}-6ra}{1+2ra} = 1 + \frac{4}{s_2} > 1$.

Using (4.3) again, we can represent m as

$$m = \frac{-2\ln(2s_2ra)}{\ln(s_1 + 2s_1ra)} \ , \tag{4.6}$$

which gives

$$\begin{aligned}
\phi\left(\frac{1-2ra}{1+2ra}\right) &= \frac{1+\frac{4}{s_2}-6ra}{1+2ra}\left(\frac{1-2ra}{1+2ra}\right)^{\frac{-2\ln(2s_2ra)}{\ln(s_1+2s_1ra)}} \\
&= \frac{1+\frac{4}{s_2}-6ra}{1+2ra}\exp\left(-2\ln\left(\frac{1-2ra}{1+2ra}\right)\frac{\ln(2s_2ra)}{\ln(s_1+2s_1ra)}\right) \ .
\end{aligned}$$

Since $\lim\limits_{a\to 0}\ln\left(\frac{1-2ra}{1+2ra}\right)\ln(a) = 0$, the argument of exp converges to 0 as $a \to 0$. Thus

$$\lim_{a\to 0}\phi\left(\frac{1-2ra}{1+2ra}\right) = 1 + \frac{4}{s_2} \ .$$

This proves our claim for a small enough. We proved the following

Theorem 4.3.1. *Assume that a satisfies (4.3), for some integer m, i.e., W_a map is Markov with $W_a^{m+1}(1/2) = \frac{1}{2} - \frac{1}{2s_1}$. For a small enough, Perron-Frobenius operator P_a has an eigenvalue λ_a satisfying*

$$\frac{1-2ra}{1+2ra} < \lambda_a < \frac{1}{1+2ra} \ . \tag{4.7}$$

The corresponding eigenfunction is given by equations (4.4) and (4.5), up to a multiplicative constant.

Remark 4.3.1. Using tedious calculations we were able to show that ϕ'' is positive in a neighbourhood of 1. Since $\phi\left(\frac{1-2ra}{1+2ra}\right) > 1$, $\phi\left(\frac{1}{1+2ra}\right) = 0$ and $\phi(1) = 1$, for small a there is only one eigenvalue in the interval $\left(\frac{1-2ra}{1+2ra}, 1\right)$, i.e., λ_a we found in Theorem 4.3.1 is really the "second" eigenvalue.

4.4 Eigenfunction for $\lambda_a < 1$

In this section we take a closer look at the eigenfunction corresponding to the second eigenvalue λ_a found in Theorem 4.3.1. We omit the subscript "a" to simplify the notation. Let $(c, \alpha_0, \alpha_1, \cdots, \alpha_m)$ be the λ-eigenvector given by (4.5). We have

$$\alpha_j = \lambda^{m-j} s_1^{m-j} (1+2ra)^{m-j} > 0 \quad , \quad j = 1, \ldots, m \ .$$

Next,

$$\alpha_0 = \frac{\lambda^m s_1^{m-1}(1+2ra)^m(\lambda s_1 s_2(1+2ra) - s_1 s_2) + 1}{\lambda s_1(1+2ra) - 1} < 0 \ ,$$

since $\lambda(1+2ra) < 1$ but very close to 1 and using formula (4.6) we can show that $\lambda^m(1+2ra)^m$ approaches 1 as $m \to \infty$. As $\alpha_0 < 0$, we also have

$$c = \frac{\alpha_0}{s_2(2\lambda - 1)} < 0 \ .$$

The P_a eigenfunction h_m, defined in (4.4), is positive on some interval $G_m = [W_a^{m_1}(1/2), 1/2 + a/4]$ and negative outside this interval. Since, as a decreases, more and more of numbers $\alpha_m, \alpha_{m-1}, \alpha_{m-2} \ldots$ are necessary to balance $\alpha_0 + c$, we have $\lim_{a \to 0}(m - m_1) = +\infty$. This implies, that intervals G_m shrink to the point 1/2 as $a \to 0$.

Since $0 < \lambda < 1$ we have $\int_0^1 h_m d\mathrm{L} = 0$. Let $K_m = \int_0^1 |h_m| d\mathrm{L}$. The normalized signed measures $\frac{1}{K_m} h_m \cdot \mathrm{L}$ converge $*$-weakly to the measure

$$-\frac{1}{2}\mu_0 + \frac{1}{2}\delta_{(1/2)} \ ,$$

where μ_0 is W_0-invariant absolutely continuous measure and $\delta_{(1/2)}$ is Dirac measure at point 1/2.

As it is described in [Froyland and Stančević, 2010] the presence of the eigenvalue λ close to 1 makes the system behave in a metastable way. The sets $A^+ = \{t : h_m(t) \geq 0\}$ and $A^- = \{t : h_m(t) < 0\}$ are almost invariant with the escape rates bounded by $-\ln\lambda$ which is close to 0. This means that a typical trajectory stays for a long time in A^+, then jumps to A^-, stays there for a long time, then jumps to A^+, spend there long time, etc. Despite the small essential spectral radius (equal to $\max\{1/s_1, 1/s_2\}$), the system converges to equilibrium slowly at the rate given by $C\lambda^n$, for some constant C.

Fig. 4.2 shows graphs of normalized functions h_m produced using Maple 13. We used $s_1 = s_2 = 2$ and $r = 1/4$.

a) $m = 5$, $a = 0.14789903570478$, $\lambda = 0.8732372308$, $K_m = 3.819456626$;

b) $m = 7$, $a = 0.077390319202550$, $\lambda = 0.9365803433$, $K_m = 8.987509817$.

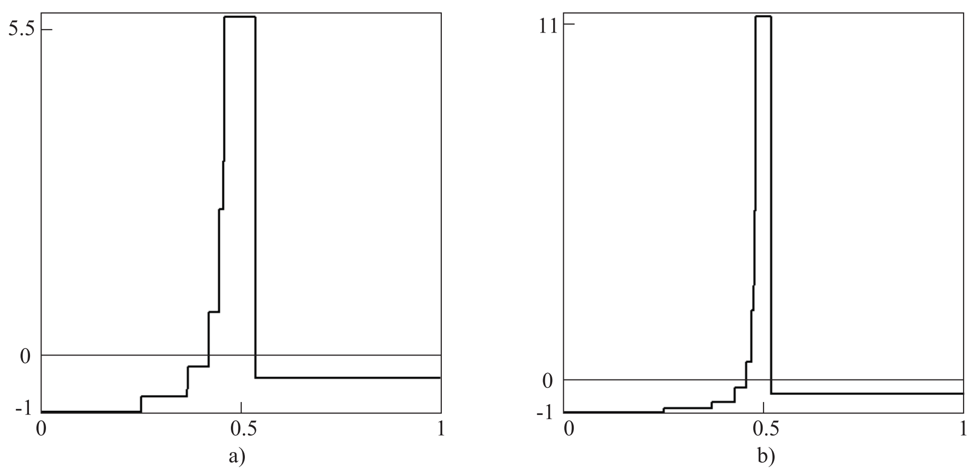

Figure 4.2: Normalized eigenfunctions h_m.

Note that the vertical scales of the pictures are very different.

4.5 Remarks

In [Li and Góra, 2012], for $s_1,\ s_2 > 1$ satisfying $\frac{1}{s_1} + \frac{1}{s_2} = 1$, $a > 0$, the case when $r = 1$ was considered for the following perturbing W_a maps:

$$W_a(x) = \begin{cases} W_{a,1}(x) := 1 - 2s_2 x, & 0 \le x < \frac{1}{2} - \frac{1}{2s_1}, \\ W_{a,2}(x) := (s_1 + 2s_1 a)(x - \frac{1}{2} + \frac{1}{2s_1}), & \frac{1}{2} - \frac{1}{2s_1} \le x < \frac{1}{2}, \\ W_{a,3}(x) := (s_2 + 2s_2 a)(\frac{1}{2} + \frac{1}{2s_2} - x), & \frac{1}{2} \le x < \frac{1}{2} + \frac{1}{2s_2}, \\ W_{a,4}(x) := 2s_1(x-1) + 1, & \frac{1}{2} + \frac{1}{2s_2} \le x < 1. \end{cases}$$

Now, $W_a(\frac{1}{2}) = \frac{1}{2} + a$ for each a. Using the idea above, it can be shown that the existence of the "second" eigenvalue λ_a and it satisfies,

$$\frac{1-2a}{1+2a} < \lambda_a < \frac{1}{1+2a} .$$

Figure 4.3 shows graphs of normalized functions h_m produced using Maple 13. We used $s_1 = s_2 = 2$.

a) $m = 5$, $a = 0.036974758926197$, $\lambda = 0.873237227279931$;

b) $m = 7$, $a = 0.019347579800639$, $\lambda = 0.936580332073165$.

Also note that the vertical scales of the pictures are very different. Thus, as $a \to 0$ the eigenvalues λ_a still converge to 1. In either case, the instability of isolated spectrum of W_0 is shown although the additional constant r slows the convergence.

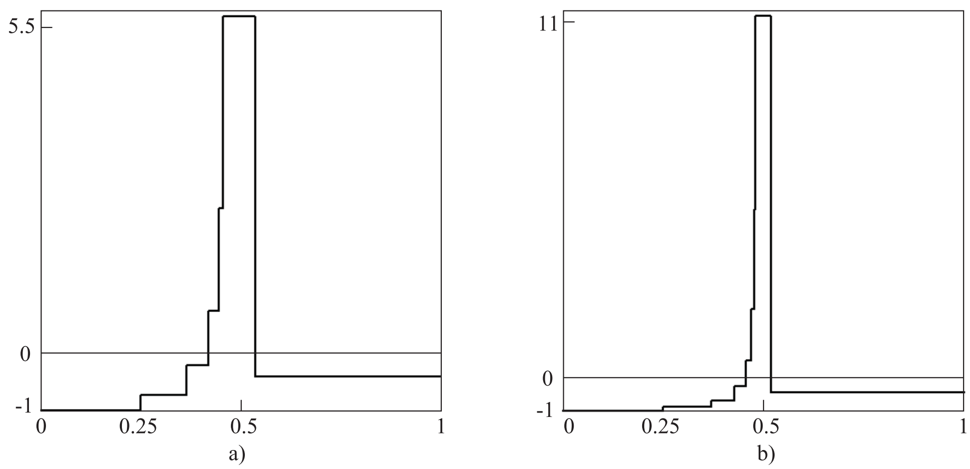

Figure 4.3: Normalized eigenfunctions h_m.

Chapter 5

Harmonic Averages and New Explicit Constants for Densities of Piecewise Expanding Maps of the Interval

5.1 Introduction

Let $I = [0, 1]$ and let $\mathcal{P}$ be a finite partition of I. Recall that $\mathcal{T}(I)$ denotes the class of piecewise expanding transformations on I with partition $\mathcal{P}$. We study statistical properties of the probability density function (pdf) associated with τ in $\mathcal{T}(I)$. To implement our approach we impose two conditions on τ : (1) weak covering, by which we mean there exists an integer K such that the union of forward images of every element of $\mathcal{P}$ equals I, and (2) harmonic average of slopes condition, which comes out motivated by the results in Chapter 3 and means that the harmonic average of the (inf of) slopes of every two adjacent intervals (except for the first and last interval) is strictly less than 2. We use these two conditions to derive explicit constants for the upper and lower bounds of the pdf as well as the constant that determines the speed of convergence to the pdf. Related results were obtained by [Liverani, 1995a] under the assumption that the magnitude of all slopes is strictly greater than 2. Without this condition many different behaviors for approximating maps can occur as shown in [Keller, 1982] for $W-$shaped maps. For example, the acims of approximating maps can converge to a singular, absolutely continuous or a mixed measure. We have studied some $W-$shaped maps in the previous chapters. An example of a $W-$shaped map is shown in Fig.5.3.2. $W-$shaped maps are continuous but our considerations do not depend on continuity and we do not assume it.

It is one of the objectives of this chapter to show how we can weaken the slope 2 condition with the help of the harmonic average of slopes condition and establish stability of acim for some $W-$shaped maps. For standard stability results we refer the reader to [Li, 2013].

In Section 5.2 we use the weak covering property and the harmonic average of slopes condition to derive an explicit bound on the number of iterations needed to obtain weak covering for any subinterval of a partition element. In Section 5.3 we use this result and a generalized Lasota-Yorke inequality to obtain explicit constants for the upper bound of the pdf and from this we derive an explicit upper bound for the pdf. We then show (Theorem 5.3.2) that we can extend our results to families of maps. We provide an example to show that the harmonic slope condition is essential. For a $W-$shaped map we calculate all the constants necessary to find the lower bound. In Section 5.4 we assume weak mixing and use our derived constants to find an explicit constant for the speed of convergence. Finding the rate of convergence of initial densities to the invariant pdf of the map is an important problem in many scientific fields. Our methods depend on using equipartitions rather than partitions of the inverse images of $\mathcal{P}$ and, as such, in most situations, results in sharper constants. We work out an example where the results of [Liverani, 1995a] do not apply.

The results obtained in this chapter (Sections 5.2, 5.3 and 5.4) were, after some modifications, published in the paper [Góra et al., 2012b].

5.2 Notation and preliminary results

Let $I = [0, 1]$ and let L be Lebesgue measure on I. We present the usual definition of a piecewise expanding map.

Definition 5.2.1. Suppose there exists a partition $\mathcal{P} = \{I_i := [a_{i-1}, a_i], i = 1, \ldots, q\}$ of I such that $\tau : I \to I$ satisfies the following conditions:

1. τ is monotonic on each interval I_i;
2. $\tau_i := \tau|_{I_i}$ is C^1 and $\lim_{x \to a_{i-1}^+} \tau'(x)$, $\lim_{x \to a_i^-} \tau'(x)$ exist (can be infinite);
3. $|\tau_i'(x)| \geq s_i > 1$ for any i and for all $x \in (a_{i-1}, a_i)$.

Then, we say $\tau \in \mathcal{T}(I)$, the class of piecewise expanding transformations.

We will also assume that τ is weakly covering, i.e.,

Definition 5.2.2. Map $\tau \in \mathfrak{T}(I)$ is called weakly covering if and only if there exists a $K \geq 1$ such that

$$\bigcup_{n=0}^{K} \tau^n(I_i) = [0,1] \ , \ i = 1, \ldots, q \ . \tag{5.1}$$

Let

$$s := \min_{1 \leq i \leq q} s_i > 1 \ . \tag{5.2}$$

Suppose $\tau \in \mathfrak{T}(I)$ satisfies the following condition.

$$s_H = \max_{i=1,\ldots,q-1} \left\{ \frac{1}{s_i} + \frac{1}{s_{i+1}} \right\} < 1 \ . \tag{5.3}$$

The number $H(a,b) = \frac{2}{\frac{1}{a}+\frac{1}{b}}$ is called the harmonic average of a and b. Condition $H(a,b) > 2$ is equivalent to condition $\frac{1}{a} + \frac{1}{b} < 1$. If τ satisfies $s_H < 1$ we say that τ satisfies the harmonic average of slopes condition.

Now, we prove a very simple minimax lemma with interesting consequences.

Lemma 5.2.1. *Let $z_1, z_2 > 1$ and $\alpha + \beta = c$, where $\alpha, \beta > 0$. Assume*

$$\frac{1}{z_1} + \frac{1}{z_2} < 1 \ .$$

Then,

$$\min_{\alpha,\beta} \max\{z_1\alpha, z_2\beta\} = \frac{1}{\frac{1}{z_1} + \frac{1}{z_2}} c > c \ .$$

Proof. We have

$$\min_{\alpha,\beta} \max\{z_1\alpha, z_2\beta\} = \min_{\alpha} \max\{z_1\alpha, z_2(c-\alpha)\} \ .$$

The line $f(\alpha) = z_1\alpha$ is increasing while the line $g(\alpha) = z_2(c-\alpha)$ is decreasing. The $\min_\alpha \max\{z_1\alpha, z_2(c-\alpha)\}$ occurs where the lines intersect, i.e., at

$$\alpha = \frac{z_2 c}{z_1+z_2}\ ,$$

which gives

$$\min_{\alpha,\beta} \max\{z_1\alpha, z_2\beta\} = \frac{z_1 z_2 c}{z_1+z_2} = \frac{1}{\frac{1}{z_1}+\frac{1}{z_2}} c > c\ .$$

□

Remark 5.2.1. If $\frac{1}{z_1}+\frac{1}{z_2} = 1$, then, $\min_{\alpha,\beta}\max\{z_1\alpha, z_2\beta\} = c$.

Lemma 5.2.1 implies the following

Proposition 5.2.1. *If $\tau \in \mathcal{T}(I)$ satisfies the harmonic average of slopes condition, then for any subinterval $J \subset I$ which does not contain two endpoints of partition $\mathcal{P}$ we have*

$$L(\tau(J)) \geq \frac{1}{s_H} L(J)\ . \tag{5.4}$$

Proof. Note that

$$s = \min_{1\leq i\leq q} s_i \geq \min_{1\leq i\leq q-1} \frac{1}{\frac{1}{s_i}+\frac{1}{s_{i+1}}} \geq \frac{1}{s_H}\ .$$

If J does not contain any endpoints of partition $\mathcal{P}$, then $J \subset I_i$, for some $1 \leq i \leq q$, and

$$L(\tau(J)) \geq s_i L(J) \geq \frac{1}{s_H} L(J)\ .$$

If J contains exactly one endpoint of partition $\mathcal{P}$, then let $L(J) = \alpha+\beta$, where α and β are the lengths of parts of J to the left and to the right of the partition point. By Lemma 5.2.1 we obtain $L(\tau(J)) \geq \frac{1}{s_H} L(J)$. □

Proposition 5.2.2. *If $\tau \in \mathcal{T}(I)$ satisfies the harmonic average of slopes condition, then for any subinterval $J \subset I$ there exists a positive integer $M(J)$ such that at least one connected component of $\tau^{M(J)}(J)$ contains two endpoints of partition $\mathcal{P}$ and, automatically, the interval between them. Moreover,*

$$0 \leq M(J) \leq \max\left\{ \left\lceil \frac{\ln \frac{L(J)}{\delta_{\max}}}{\ln(s_H)} \right\rceil, 0 \right\}, \tag{5.5}$$

where $\delta_{\max} = \max\{L(I_i \bigcup I_{i+1}) \mid i = 1, 2, \ldots, q-1\}$ and $\lceil t \rceil$ is the smallest integer equal or larger than t.

Proof. Let J be a subinterval of I. Then,

Case (i): If J contains two or more endpoints of $\mathcal{P}$, then $M(J) = 0$. In particular, this happens when $L(J) \geq \delta_{\max}$.

Case (ii): We assume $L(J) < \delta_{\max}$ and that J contains at most one endpoint of partition $\mathcal{P}$. First, let us assume that J contains exactly one endpoint of $\mathcal{P}$, and this endpoint divides J into two subintervals, $J_{0,1}$ and $J_{0,2}$. Lemma 5.2.1 implies

$$\max\{L(\tau(J_{0,1})), L(\tau(J_{0,2}))\} \geq \frac{1}{s_H} L(J).$$

We can assume $L(\tau(J_{0,1})) \geq \frac{1}{s_H} L(J)$. Notice that $\tau(J_{0,1})$ is also an interval since $\tau \in \mathcal{T}(I)$.

Second, if J contains no endpoint of $\mathcal{P}$, then $\tau(J)$ is again an interval, and $L(\tau(J)) \geq sL(J) \geq \frac{1}{s_H} L(J)$.

Thus, for an interval J that contains at most one endpoint of $\mathcal{P}$, we can find an interval in $\tau(J)$, denote it by J_1, such that $L(J_1) \geq \frac{1}{s_H} L(J)$. If J_1 contains two endpoints of $\mathcal{P}$, we stop the iteration. Otherwise, consider $\tau(J_1)$, we can again find one interval in $\tau(J_1)$, denote it by J_2, such that $L(J_2) \geq \frac{1}{s_H} L(J_1) \geq \frac{1}{s_H^2} L(J)$. Repeat this procedure, we can find an integer k such that $L(J_k) \geq \frac{1}{s_H^k} L(J) \geq \delta_{\max}$, which implies $\tau^k(J)$ contains at least two endpoints of $\mathcal{P}$. Therefore, we obtain

$$M(J) \geq \frac{\ln \frac{L(J)}{\delta_{\max}}}{\ln(s_H)}.$$

□

Corollary 5.2.1. *If $\tau \in \mathcal{T}(I)$ is weakly covering and satisfies the harmonic average of slopes condition, then for any subinterval $J \subset I$ we have*

$$\bigcup_{n=0}^{K} \tau^{M(J)+n}(J) = [0,1] \ , \tag{5.6}$$

where $M(J)$ is the number from Proposition 5.2.2.

Remark 5.2.2. Note that the weak covering property plus $s_H < 1$ does not imply topological exactness. The simplest example would be the map τ such that $\tau([0,1/2]) = [1/2,1]$ and $\tau([1/2,1]) = [0,1/2]$, and τ restricted to each of these intervals is a tent map (or other expanding map). An additional assumption is needed for topological exactness, see Theorem 5.2.1 and Corollary 5.2.2.

We define $\mathcal{P}^{(n)} = \{I_{i_0} \cap \tau^{-1}(I_{i_1}) \cap \tau^{-2}(I_{i_2}) \cap \cdots \cap \tau^{-(n-1)}(I_{i_{n-1}}) : 1 \leq i_0, i-1, i-2, \ldots, i_{n-1} \leq q\}$. Partition $\mathcal{P}^{(n)}$ is the partition of monotonicity of τ^n. Note that $\mathcal{P} = \mathcal{P}^{(1)}$.

Theorem 5.2.1. *Let $\tau \in \mathcal{T}(I)$ be piecewise C^{1+1} with $s_H < 1$ and satisfies* $\inf \phi \geq \beta$*, where ϕ is the τ-invariant density. If τ is weakly mixing with respect to Lebesgue measure, then there exists K_1 such that*

$$\tau^{K_1}(I_i) = [0,1] \ , \ i = 1, 2, \ldots, q \ .$$

Proof. We follow the proof of a similar theorem from [Liverani, 1995a]. For maps we consider weak mixing is equivalent to mixing and to exactness [Boyarsky and Góra, 1997]. Let $\chi = \chi_{I_i}/L(I_i)$ for some $1 \le i \le q$. Since τ is exact we have $P_\tau^n \chi \to \phi$ in L^1, as $n \to \infty$. Thus, for any n_1 (it will be fixed later) we can find an $N(n_1)$ such that for any $n \ge N(n_1)$ in every interval J of the partition $\mathcal{P}^{(n_1)}$ there is a point $x \in J$ with $P_\tau^n \chi(x) \ge \beta/2$. On the other hand, the Lasota-Yorke inequality implies that

$$\bigvee_{[0,1]} P_\tau^k \chi \le C \ ,$$

for all k and some constant C. Let $n \ge N(n_1)$ and

$$\mathcal{B} = \{J \in \mathcal{P}^{(n_1)} : \exists_{\ x \in J}\ P_\tau^n \chi(x) < \beta/4\} \ .$$

If $J \in \mathcal{B}$, then we have $\bigvee_J P_\tau^n \chi \ge \beta/4$ and

$$\bigvee_{[0,1]} P_\tau^n \chi \ge (\beta/4) \# \mathcal{B} \ .$$

Thus, $\#\mathcal{B} \le 4C/\beta = L_0$.

The Perron-Frobenius operator P_τ induced by τ, can be viewed as an operator on $BV(I)$, the space of functions of bounded variation on I (or more generally on $L^1(I)$). For $\tau \in \mathcal{T}(I)$, it has the following representation

$$P_\tau f = \sum_{i=1}^{q} \frac{f(\tau_i^{-1}(x))}{\left|\tau'(\tau_i^{-1}(x))\right|} \chi_{\tau[a_{i-1},a_i]}(x) \ .$$

For more detailed information about space $BV(I)$, operator P_τ and its properties we refer the reader to [Boyarsky and Góra, 1997]. Here, the most important will be the fact that an f is an invariant pdf (or a τ-invariant density) if and only if $P_\tau f = f$.

Using the representation of P_τ, we have the following inequality holding for all $x \in [0,1]$:

$$\beta \le \phi(x) = \sum_{y \in \tau^{-n}(x)} \frac{\phi(y)}{|(\tau^n)'(y)|} \le \sup(\phi)\#(\tau^{-n}(x))s^{-n} \ .$$

This shows that $\#(\tau^{-n}(x))$ goes to infinity as n goes to infinity, uniformly in x. In particular we can find an N_1 such that for all $x \in [0,1]$

$$\#(\tau^{-N_1}(x)) > L_0.$$

Let us fix $n_1 = N_1$ and $N_2 \ge N(N_1)$. Then,

$$P_\tau^{N_1+N_2}\chi(x) = \sum_{y \in \tau^{-N_1}(x)} \frac{P_\tau^{N_2}\chi(y)}{|(\tau^{N_1})'(y)|} \ge \frac{\beta}{4s^{N_1}} \ ,$$

since at least one preimage $y \in \tau^{-N_1}(x)$ belongs to an interval $J \notin \mathcal{B}$.

We have proved that $\tau^{N_1+N_2}(I_i) = [0,1]$. We choose K_1 as the maximum of constants $N_1 + N_2$ over all $i = 1, 2, \ldots, q$ to complete the proof. $\square$

The following result is an immediate consequence.

Corollary 5.2.2. *If $\tau \in \mathcal{T}(I)$ is weakly covering, weakly mixing and satisfies the harmonic average of slopes condition, then τ is topologically exact. For any subinterval $J \subset I$ we have*

$$\tau^{M(J)+K_1}(J) = [0,1] \ , \tag{5.7}$$

where $M(J)$ is the number from Proposition 5.2.2 and K_1 is the constant from Theorem 5.2.1.

5.3 Lower bound for the invariant density

From now on we assume that our $\tau \in \mathcal{T}(I)$ is piecewise C^{1+1}, i.e., each τ_i' satisfies Lipschitz condition with a constant M_i:

$$|\tau_i'(x) - \tau_i'(y)| \le M_i|x-y| \ , \text{ for all } x, y \in I_i \ , \ i = 1, 2, \dots, q \ .$$

This means τ is a piecewise expanding, piecewise C^{1+1} map of I. We introduce the following notation

$$M := \max_{1\le i\le q} M_i,$$

and

$$\delta_i^{\pm} := \delta_{\{\tau(a_i^{\pm})\notin\{0,1\}\}} = \begin{cases} 0 & \text{if } \tau(a_i^{\pm}) \in \{0,1\}, \\ 1 & \text{if } \tau(a_i^{\pm}) \notin \{0,1\}, \end{cases}$$

where $\tau(a_i^{\pm})$ means $\lim_{x\to a_i^{\pm}} \tau(a_i)$. For example, $\delta_i^{+} = 1$ means that the left endpoint of the $(i+1)$-st branch of τ is hanging (does not touch 0 or 1).

Also, let

$$\eta_i := \begin{cases} \max\left\{\frac{\delta_0^+}{s_1}, \frac{\delta_1^+}{s_2}\right\} & \text{if } i = 1, \\ \max\left\{\frac{\delta_{q-1}^-}{s_{q-1}}, \frac{\delta_q^-}{s_q}\right\} & \text{if } i = q, \\ \max\left\{\frac{\delta_{i-1}^-}{s_{i-1}}, \frac{\delta_i^+}{s_{i+1}}\right\} & \text{for } i = 2\dots q-1. \end{cases}$$

Now, we present the following proposition from [Eslami and Góra, 2012].

Proposition 5.3.1. *Let $\tau \in \mathcal{T}(I)$, and satisfy the above Lipschitz condition. Then, for every $f \in BV([0,1])$,*

$$\bigvee_I P_\tau f \le \eta \bigvee_I f + \gamma \int_I |f|\, dm, \tag{5.8}$$

where $\eta = \max_{1\le i\le q}\left\{\frac{1}{s_i}+\eta_i\right\}$, $\gamma = \left[\frac{M}{s^2}+\frac{2\max\limits_{1\le i\le q}\eta_i}{\min\limits_{1\le i\le q} L(I_i)}\right]$.

Note that we always have

$$\max_{1\le i\le q}\eta_i < \frac{1}{s}\ .$$

As proved in Theorem 3.2 of [Eslami and Góra, 2012], if $\tau(0),\tau(1)\in\{0,1\}$, then $\eta\le s_H<1$. If the condition $\tau(0),\tau(1)\in\{0,1\}$ is not satisfied one uses an extension method to arrive to the similar conclusion, as it is done in Theorem 3.3 of [Eslami and Góra, 2012]. For completeness, we describe the method. Let $I^\varepsilon=[0-\varepsilon,1+\varepsilon]$ for some fixed small positive ε. We define τ^ε on I^ε as follows

$$\tau^\varepsilon(x)=\begin{cases}-\varepsilon+\frac{1+\varepsilon}{\varepsilon}(x+\varepsilon) & ,\ x\in[-\varepsilon,0)\ ;\\ \tau(x) & ,\ x\in[0,1]\ ;\\ 0+\frac{1+\varepsilon}{\varepsilon}(x-1) & ,\ x\in(1,1+\varepsilon]\ .\end{cases}$$

See Fig. 5.1 for an illustration. The interval $[0,1]$ is the attractor of τ^ε. We choose ε so small that the constants s and s_H are the same for maps τ and τ^ε. We consider the subspace $BV^\varepsilon(I^\varepsilon)=\{f\in BV(I^\varepsilon): f(x)=0 \text{ outside } [0,1]\}$ of $BV(I^\varepsilon)$. It is easy to check that $P_{\tau^\varepsilon}(BV^\varepsilon(I^\varepsilon))\subset BV^\varepsilon(I^\varepsilon)$ and $(P_{\tau^\varepsilon}f)_{|[0,1]}=P_\tau(f_{|[0,1]})$ for $f\in BV^\varepsilon(I^\varepsilon)$. Now, we obtain inequality (5.8) for P_{τ^ε} on $BV(I^\varepsilon)$. In particular it holds for $f\in BV^\varepsilon(I^\varepsilon)$. The constants η_i are different but by the choice of ε we still have $\eta<s_H$ and $\max\limits_{1\le i\le q}\eta_i<\frac{1}{s}$. The additional partition subintervals $I_0=[-\varepsilon,0]$ and $I_{q+1}=[1,1+\varepsilon]$ do not show in the $\min\limits_{1\le i\le q}L(I_i)$ because the integrals $\int_{I_0}f dm$ and $\int_{I_{q+1}}f dm$ are 0 for $f\in BV^\varepsilon(I^\varepsilon)$. Thus, for $f\in BV^\varepsilon(I^\varepsilon)$, we obtain inequality

$$\bigvee_{I^\varepsilon}P_\tau f\le\eta\bigvee_{I^\varepsilon}f+\gamma\int_I|f|\,dm, \tag{5.9}$$

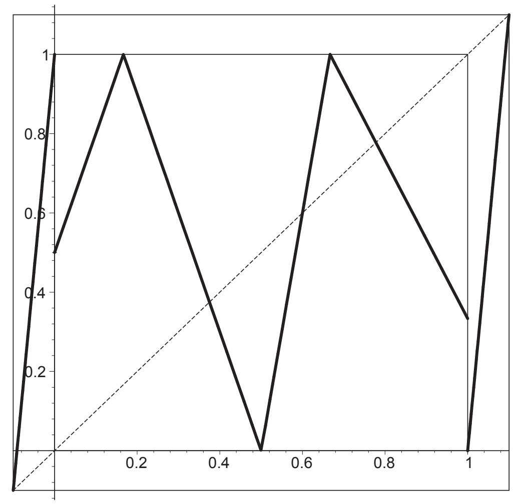

Figure 5.1: The extension of map τ to $[0-\varepsilon, 1+\varepsilon]$.

with $\eta \leq s_H < 1$ and $\gamma = \frac{M}{s^2} + \frac{2}{s \cdot \min\limits_{1 \leq i \leq q} L(I_i)}$.

It is well known (see [Boyarsky and Góra, 1997]) that (5.8) or (5.9) implies that τ admits an acim with a pdf of bounded variation. We denote this invariant density by ϕ. It follows from (5.8) or (5.9) that

$$\bigvee_I \phi \leq \frac{\gamma}{1-\eta}. \tag{5.10}$$

We now consider the uniform partition $\mathcal{P}^u$ of $[0,1]$ into $2([\frac{\gamma}{1-\eta}]+1)$ subintervals, where $[\frac{\gamma}{1-\eta}]$ is the integer part of $\frac{\gamma}{1-\eta}$. Thus, for each $J \in \mathcal{P}^u$, we have $L(J) < \frac{1-\eta}{2\gamma}$. Now, we prove the following lemma.

Lemma 5.3.1. *There exists $J_u \in \mathcal{P}^u$ such that*

$$\phi(x) \geq \frac{1}{2} \quad \textit{for all } x \in J_u\,.$$

Proof. Suppose the conclusion is not true. Then, for each $J \in \mathcal{P}^u$, there exists a point $x_J \in J$ such that

$$\phi(x_J) < \frac{1}{2}\,.$$

Using the inequality (5.10), we obtain

$$\begin{aligned} 1 &= \int_I \phi\, dm = \sum_{J \in \mathcal{P}^u} \int_J \phi\, dm \le \sum_{J \in \mathcal{P}^u} L(J) \left(\phi(x_J) + \bigvee_J \phi \right) \\ &< \sum_{J \in \mathcal{P}^u} \left(\frac{L(J)}{2} + \frac{1-\eta}{2\gamma} \bigvee_J \phi \right) = \frac{1}{2} + \frac{1-\eta}{2\gamma} \bigvee_I \phi \le \frac{1}{2} + \frac{1-\eta}{2\gamma} \frac{\gamma}{1-\eta} = 1\,. \end{aligned}$$

The contradiction completes the proof. □

Now, we can prove the existence of the lower bound for the invariant density of τ. This result for individual maps is not new, see [Keller, 1978], [Kowalski, 1979] or [Boyarsky and Góra, 1997]. What is new are the explicit constants we obtain, which allows us to prove the existence of the uniform lower bound for the invariant densities of a family of maps.

Theorem 5.3.1. *Let $\tau \in \mathcal{T}(I)$ be piecewise C^{1+1} and satisfy $s_H < 1$. Then there exists $\beta > 0$ such that* $\inf \phi \ge \beta$*, where ϕ is the τ-invariant density.*

Proof. Let $S_{\max}$ denote the biggest value of $|\tau'(x)|$ over I. Since ϕ is the invariant density, $P_\tau^n \phi = \phi$ for any natural number n. Lemma (5.3.1) implies that there exists interval $J_u \subseteq I$ with $L(J_u) = \frac{1}{2([\frac{\gamma}{1-\eta}]+1)}$ such that

$$\phi(y) \ge \frac{1}{2} \quad \text{for all } y \in J_u\,.$$

And, by Corollary 5.2.1, for each $x \in I$, we can find an integer $n_u \le M(J_u) + K$ and

$y_u \in J_u$ such that $\tau^{n_u}(y_u) = x$. Therefore,

$$\phi(x) = (P_\tau^{n_u}\phi)(x) = \sum_{y \in \tau^{-n_u}(x)} \frac{\phi(y)}{|(\tau^{n_u})'(y)|} \geq \frac{\phi(y_u)}{|(\tau^{n_u})'(y_u)|} \geq \frac{1}{2S_{\max}^{n_u}} .$$

Setting $\beta = (2S_{\max}^{n_u})^{-1}$ (or $\beta = \left(2S_{\max}^{M(J_u)+K}\right)^{-1}$ for an explicit formula) completes the proof. $\square$

The next theorem generalizes Theorem 5.3.1 to a family of maps uniformly satisfying the assumptions.

Theorem 5.3.2. *Let $\{\tau^{(r)}\} \subset \mathcal{T}(I)$ be a family of piecewise C^{1+1} maps. The defining partition for $\tau^{(r)}$ is $\mathcal{P}^{(r)} = \{I_1^{(r)}, \ldots, I_{q(r)}^{(r)}\}$. We assume we can find uniform constants $s_H < 1$, K, $\delta > 0$, $\delta_{\max}$, M, $s > 1$, $S_{\max}$ such that*

$$\begin{aligned}
s_H \geq s_H^{(r)} &= \max\{\min_{I_i^{(r)}} |(\tau^{(r)})'|^{-1} + \min_{I_{i+1}^{(r)}} |(\tau^{(r)})'|^{-1} : i = 1, 2, \ldots, q(r) - 1\} ; \\
K \geq K^{(r)}, &\ \textit{where } \cup_{n=0}^{K^{(r)}} (\tau^{(r)})^n (I_i^{(r)}) = [0,1] , \ i = 1, 2, \ldots, q(r) ; \\
\delta \leq \delta^{(r)} &= \min\{L(I_i^{(r)}) : i = 1, 2, \ldots, q(r)\} ; \\
\delta_{\max} \geq \delta_{\max}^{(r)} &= \max\{L(I_i^{(r)} \cup I_{i+1}^{(r)}) : i = 1, 2, \ldots, q(r) - 1\} ; \\
M \geq M^{(r)}, &\ \textit{the common Lipschitz constant for } (\tau_i^{(r)})' , \ i = 1, 2, \ldots, q(r) ; \\
s \leq s^{(r)} &= \min\left\{ \min_{I_i^{(r)}} |(\tau_i^{(r)})'| , \ i = 1, 2, \ldots, q(r) \right\} ; \\
S_{\max} \geq S_{\max}^{(r)} &= \max\left\{ \max_{I_i^{(r)}} |(\tau_i^{(r)})'| , \ i = 1, 2, \ldots, q(r) \right\} .
\end{aligned} \tag{5.11}$$

Let us define

$$\beta = \left(2S_{\max}^{\max\left\{ \left\lceil \frac{-\ln(2([\frac{\gamma}{1-s_H}]+1)) - \ln(\delta_{\max})}{\ln(s_H)} \right\rceil , 0 \right\} + K} \right)^{-1} , \tag{5.12}$$

where $\gamma = \frac{M}{s^2} + \frac{2}{s \cdot \delta}$. *Then, for all* r, $\inf \phi^{(r)} \geq \beta$, *where* $\phi^{(r)}$ *is the* $\tau^{(r)}$*-invariant density.*

Proof. This is just a combination of all previous results in this chapter. □

Below we refer to an example from [Li, 2013] (or Chapter 3) which shows that the condition $s_H < 1$ is necessary in Theorem 5.3.2. Another such example was constructed in [Eslami and Misiurewicz, 2012].

Example 5.3.1.

In [Li, 2013], a family $\{\tau^{(r)}\}$ of W-shaped maps was constructed which converged to the standard W-map τ_0 with a turning fixed point at 1/2 and slopes 2 to the left of 1/2 and -2 to the right of this point. The uniform constants K, $\delta > 0$, $\delta_{\max}$, M, $s > 1$, $S_{\max}$ can be found for this family. The constants $s_H^{(r)}$ converge to 1, as $\tau^{(r)} \to \tau_0$. Each $\tau^{(r)}$ is exact on the whole $[0, 1]$, but the acims of $\tau^{(r)}$ converge to Dirac measure $\delta_{(1/2)}$ as $\tau^{(r)} \to \tau_0$. Thus, the uniform positive lower bound cannot exist for the invariant densities of this family.

We now present here an example of a non-linear W-shaped map and calculate for it all the constants necessary to find the lower bound. The theoretical one turns out to be approximately 4×10^{-10}, while the computer simulation indicates that the actual lower bound for the invariant density is 0.54.

Example 5.3.2.

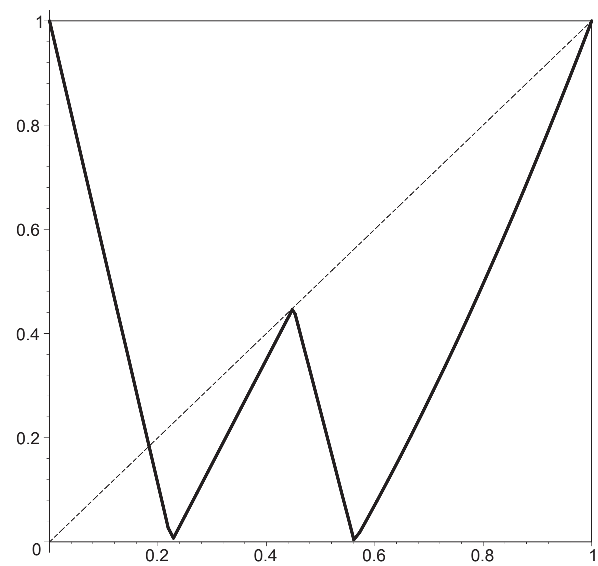

Figure 5.2: W-shaped map of Example 5.3.2

Let the map τ be defined as follows

$$\tau(x) = \begin{cases} \tau_1(x) := 1 - 40/9x, & 0 \leq x < 9/40, \\ \tau_2(x) := 2(x - 9/40), & 9/40 \leq x < 9/20, \\ \tau_3(x) := -4(x - 9/16), & 9/20 \leq x < 9/16, \\ \tau_4(x) := x^2 + 81/112x - 81/112, & 9/16 \leq x < 1. \end{cases}$$

The graph of τ is shown in Fig.5.2. We have

$$\tau_1'(x) = -40/9, \quad \tau_2'(x) = 2, \quad \tau_3'(x) = -4, \quad \tau_4'(x) = 2x + 81/112 \ ;$$

$$s_1 = 40/9, \quad s_2 = 2, \quad s_3 = 4, \quad s_4 = 207/112 \ ;$$

$$s = \min\{40/9,\ s_2 = 2,\ s_3 = 4,\ s_4 = 207/112\} = 207/112 \ ;$$

$$L(I_1) = L([0, 9/40)) = 9/40, \quad L(I_2) = L([9/40, 9/20)) = 9/40 \ ,$$

$$L(I_3) = L([9/20, 9/16)) = 9/80, \quad L(I_4) = L([9/16, 1]) = 7/16 \ ;$$

$$\delta = \min\{L(I_1),\ L(I_2),\ L(I_3),\ L(I_4)\} = 9/80 \ ;$$

$$\begin{aligned}
\delta_{\max} &= \max\{L(I_1) + L(I_2),\ L(I_2) + L(I_3),\ L(I_3) + L(I_4)\} \\
&= \max\{9/20,\ 27/80,\ 11/20\} = 11/20 \ ;
\end{aligned}$$

$$s_H = \max\{9/40 + 1/2,\ 1/2 + 1/4,\ 1/4 + 112/207\} = 655/828 \ ;$$

$$M_1 = 0, \quad M_2 = 0, \quad M_3 = 0, \quad M_4 = 2 \ ;$$

$$M = \max\{0,\ 0,\ 0,\ 2\} = 2 \ ;$$

$$\gamma = \frac{M}{s^2} + \frac{2}{s \cdot \delta} = 437248/42849 \ ;$$

$$\left\lceil \frac{\gamma}{1 - s_H} \right\rceil = \left\lceil \frac{1748992}{35811} \right\rceil = 48 \ ;$$

$$S_{\max} = 40/9 \ ;$$

$$L(J_u) = \frac{1}{98} \ ;$$

$$K = 2 \ .$$

The estimate for number N_u of iterations needed for any interval J_u to expand to the entire interval $[0, 1]$ comes from Corollary 5.2.1 and is

$$N_u \ \geq \ \max\left\{ \left\lceil \frac{-\ln(2([\frac{\gamma}{1-s_H}] + 1)) - \ln(\delta_{\max})}{\ln(s_H)} \right\rceil, 0 \right\} + K$$

$$= \left\lceil \frac{\ln(539/10)}{\ln(828/655)} \right\rceil + 1 + 2 = 20 \; .$$

which gives

$$\beta \geq \left(2S_{\max}^{N_u}\right)^{-1} = \left(2(40/9)^{20}\right)^{-1} \approx 5.53 \times 10^{-14} \; .$$

With the aid of a computer we found the actual value $N_u = 8$, which gives a much better, although still perhaps unsatisfactory estimate $\beta \geq 3.28 \times 10^{-6}$.

5.4 Explicit convergence constants

In this section we assume that $\tau \in \mathcal{T}(I)$ is weakly covering, weakly mixing and piecewise of class C^{1+1} with $s_H < 1$. In particular, this implies Theorem 5.2.1, Corollary 5.2.2 and Theorem 5.3.1. To obtain the exact convergence constants we follow the method of Liverani [Liverani, 1995a] with small improvements. For more information on Hilbert metrics and the use of cones in the theory of piecewise expanding maps we refer the reader to [Liverani, 1995b], [Baladi, 2000] or [Schmitt, 1986].

We consider the following cone:

$$C_\kappa = \left\{ g(x) \in BV(I) \mid g(x) \neq 0,\; g(x) \geq 0 \text{ for all } x \in [0,1];\; \bigvee_{[0,1]} g \leq \kappa \int_{[0,1]} g \; dm \right\}.$$

Let $\theta = \eta + \frac{\gamma}{\kappa}$.

Lemma 5.4.1. *If $\kappa > \frac{\gamma}{1-\eta}$, then $\theta < 1$ and $P_\tau C_\kappa \subset C_{\theta\kappa}$.*

Proof. First, $\theta = \eta + \frac{\gamma}{\kappa} < \eta + \gamma \frac{1-\eta}{\gamma} = 1$.

If $f \in C_\kappa$, using (5.8), we obtain

$$\bigvee_{[0,1]} P_\tau f \leq \eta \bigvee_{[0,1]} f + \gamma \int_{[0,1]} |f| dm \leq (\eta\kappa + \gamma) \int_{[0,1]} |f| dm = \kappa\theta \int_{[0,1]} |f| dm.$$

$\square$

Lemma 5.4.1 shows that the cone C_κ is invariant under the action of the operator P_τ. We now define the Hilbert metric $\Theta(f,g)$ on C_κ. For f, g in C_κ we define

$$\alpha(f,g) = \sup\{\lambda > 0 | \lambda f \leq g\} \ ,$$

$$\beta(f,g) = \inf\{\mu > 0 | g \leq \mu f\} \ ,$$

$$\Theta(f,g) = \ln\left[\frac{\beta(f,g)}{\alpha(f,g)}\right] \ ,$$

where we take $\alpha = 0$ or $\beta = \infty$ when the corresponding sets are empty.

We recall the following lemma from [Liverani, 1995a].

Lemma 5.4.2. *If Θ_κ is the Hilbert metric associated with the cone C_κ, then for each $\nu < 1$ and $g \in C_{\kappa\nu}$*

$$\Theta_\kappa(g,1) \leq \ln\left(\frac{\max\left\{(1+\nu)\int_{[0,1]} g\,dm, \sup_{x\in[0,1]} g(x)\right\}}{\min\left\{(1-\nu)\int_{[0,1]} g\,dm, \inf_{x\in[0,1]} g(x)\right\}}\right) .$$

A slight change of Lemma 5.3.1 leads to the following lemma.

Lemma 5.4.3. *Let $\mathcal{P}^u$ be the uniform partition of $[0,1]$ into $2([\frac{\gamma}{1-\eta}]+1)$ subintervals. For each $g \in C_\kappa$, there exists $J_{u^*} \in \mathcal{P}^u$ such that*

$$g(x) \geq \frac{1}{2}\int_{[0,1]} g\,dm \quad \textit{for all } x \in J_{u^*}\,.$$

Proof. Consider the normalized function, $\frac{g(x)}{\int_{[0,1]} g\,dm}$, which is a density function and also in C_κ. Lemma 5.3.1 implies that there exists $J_{u^*} \in \mathcal{P}^u$ such that

$$\frac{g(x)}{\int_{[0,1]} g\,dm} \geq \frac{1}{2} \quad \text{for all } x \in J_{u^*}\,.$$

This completes the proof. $\square$

Let numbers $M(J_{u^*})$ and K_1 be as in Proposition 5.2.2 and Theorem 5.2.1. Now, we now prove

Lemma 5.4.4. *For each* $\kappa > \frac{\gamma}{1-\eta}$, *there exists* $N_{u^*} \leq M(J_{u^*}) + K_1$ *and* $\Delta > 0$ *such that*

$$diam\left(P_\tau^{N_{u^*}}(C_\kappa)\right) \leq \Delta < \infty \ .$$

Proof. Let $g(x) \in C_\kappa$, Lemma 5.4.3 implies that there exists $J_{u^*} \in \mathcal{P}^u$ such that $\frac{g(x)}{\int_{[0,1]} g\,dm} \geq \frac{1}{2}$ for all $x \in J_{u^*}$. Corollary 5.2.2 implies that we can find an integer $N_{u^*} \leq M(J_{u^*}) + K_1$ and $y_{u^*} \in J_{u^*}$ such that $\tau^{N_{u^*}}(y_{u^*}) = x$. Therefore,

$$\begin{aligned}\left(P_\tau^{N_{u^*}} g\right)(x) &= \sum_{y \in \tau^{-N_{u^*}}(x)} \frac{g(y)}{|(\tau^{N_{u^*}})'(y)|} \geq \frac{g(y_{u^*})}{|(\tau^{N_{u^*}})'(y_{u^*})|} \\ &\geq \frac{\int_{[0,1]} g\,dm}{2S_{\max}^{N_{u^*}}} \geq \frac{\int_{[0,1]} g\,dm}{2S_{\max}^{M(J_{u^*})+K_1}} \ .\end{aligned}$$

Using Lemma 5.4.1, we obtain $P_\tau^{N_{u^*}} C_\kappa \subset C_{\theta_1 \kappa}$, where

$$\theta_1 = \eta^{N_{u^*}} + \frac{1-\eta^{N_{u^*}}}{1-\eta}\frac{\gamma}{\kappa} \ . \tag{5.13}$$

Let

$$\omega(g) = \frac{\inf\limits_{x\in[0,1]} \left(P_\tau^{N_{u^*}} g\right)(x)}{\int_{[0,1]} g\,dm} \ .$$

Then,

$$\frac{1}{2S_{\max}^{M(J_{u^*})+K_1}} \leq \omega(g) \leq 1 \ .$$

Note that

$$\bigvee_I P_\tau^{N_{u^*}} g \leq \eta^{N_{u^*}} \bigvee_I g + \frac{1-\eta^{N_{u^*}}}{1-\eta}\gamma \int_{[0,1]} g\,dm \ ,$$

which implies

$$\frac{\bigvee_I P_\tau^{N_{u^*}} g}{\int_{[0,1]} g\,dm} \leq \kappa\theta_1 \ .$$

Using Lemma 5.4.2, we obtain

$$\begin{aligned}
&\operatorname{diam}\left(P_\tau^{N_{u^*}}(C_\kappa)\right) \leq \\
\leq\ & \sup_{g\in P_\tau^{N_{u^*}}(C_\kappa)} 2\ln\left[\frac{\max\left\{(1+\theta_1)\int_{[0,1]} P_\tau^{N_{u^*}} g\, dm,\ \sup_{x\in[0,1]}\left(P_\tau^{N_{u^*}} g\right)(x)\right\}}{\min\left\{(1-\theta_1)\int_{[0,1]} P_\tau^{N_{u^*}} g\, dm,\ \inf_{x\in[0,1]}\left(P_\tau^{N_{u^*}} g\right)(x)\right\}}\right] \\
\leq\ & \sup_{g\in P_\tau^{N_{u^*}}(C_\kappa)} 2\ln\left[\frac{\max\left\{(1+\theta_1)\int_{[0,1]} g\, dm,\ \inf_{x\in[0,1]}\left(P_\tau^{N_{u^*}} g\right)(x)+\bigvee_I P_\tau^{N_{u^*}} g\right\}}{\min\left\{(1-\theta_1)\int_{[0,1]} g\, dm,\ \inf_{x\in[0,1]}\left(P_\tau^{N_{u^*}} g\right)(x)\right\}}\right] \\
\leq\ & 2\ln\left[\frac{\max\{1+\theta_1, 1+\kappa\theta_1\}}{\min\left\{1-\theta_1, \frac{1}{2S_{\max}^{M(J_{u^*})+K_1}}\right\}}\right] = \Delta\ .
\end{aligned}$$

□

Thus, exactly as in [Liverani, 1995a], we can obtain the following theorem on the decay of correlations.

Theorem 5.4.1. *Let $\tau \in \mathcal{T}(I)$ be weakly covering, weakly mixing and piecewise of class C^{1+1} with $s_H < 1$. Then, for each $f \in L^1([0,1])$ and a density $g \in BV([0,1])$,*

$$\left|\int_{[0,1]} g\cdot f\circ\tau^n\ dm - \int_{[0,1]} f d\mu\right| \leq K_n\Lambda^n ||f||_1 \left(1 + b\bigvee_{[0,1]} g\right)\ ,$$

where

$$\Lambda = \tanh\left(\frac{\Delta}{4}\right)^{\frac{1}{N_{u^*}}}\ ,$$

$$K_n = \left\{\exp\left[\Delta\Lambda^{n-N_{u^*}}\right]\right\}\Lambda^{-N_{u^*}}\Delta||\phi||_\infty\ ,$$

$$b = \left(\kappa - \frac{\gamma}{1-\eta}\right)^{-1}\ .$$

Note that $\phi \leq \bigvee_{[0,1]}\phi + \frac{||\phi||_1}{1-0} \leq \kappa+1$ and since $\Lambda < 1$, we have $\lim_{n\to\infty} K_n \leq \Lambda^{-N_{u^*}}\Delta(\kappa+1)$. Although we may not have an explicit formula for N_{u^*}, we can give the upper bound using Proposition 5.2.2.

In [Liverani, 1995a] the convergence constants are calculated for an example. We calculated them for the same example and obtained the same numbers. For maps with constant modulus of slope and without turning periodic points our method does not offer any advantages over the methods of [Liverani, 1995a] or [Keller, 1999]. Below, we continue Example 5.3.2 to which the methods of [Liverani, 1995a] and [Keller, 1999] do not apply.

Example 5.3.2. (continued) We use the directly calculated $N_{u^*} = 8$. We have $\frac{\gamma}{1-\eta} = \frac{\gamma}{1-s_H} = \frac{1748992}{35811}$. We choose $\kappa = \frac{1748995}{35811}$. By equation (5.13), we have $\theta_1 \sim 0.9999985478$ and

$$\Delta = 2\ln\left((1+\kappa\theta_1)2S_{\max}^{N_{u^*}}\right) \sim 33.07038934\ .$$

Then, $\Lambda \sim 0.9999999835$, $b = 11937$ and $K_n \leq\sim 1648\exp(33 \cdot 0.9999999835^{n-8})$.

Since all the constants in Theorem 5.4.1 are explicit we obtain a similar theorem for families.

Theorem 5.4.2. *Let a family $\{\tau^{(r)}\}$ satisfy the assumptions of Theorem 5.3.2. We assume that all maps $\tau^{(r)}$ are weakly mixing with uniform constant K_1 of Theorem 5.2.1. Then, Theorem 5.4.1 holds for family $\{\tau^{(r)}\}$ with uniform constants Λ, b and K_n.*

Chapter 6

Harmonic Average of Slopes and the Stability of Acim

6.1 Introduction

The main motivation for this chapter is to prove stability of the acims for some maps with fixed or periodic turning points, for example, the so called $W-$shaped maps introduced in previous chapters. The difficulty caused by periodic turning points was first noticed by [Keller, 1982]. We will study classes of maps more general than the $W-$shaped maps.

For almost forty years the Lasota-Yorke inequality [Boyarsky and Góra, 1997; Lasota and Yorke, 1973] has played a crucial role in establishing existence of acims and in studying properties of these measures. More precisely, in the setting where we have a single piecewise expanding map $\tau : I \to I$, the Lasota-Yorke method requires that we use an iterate τ^n for which we have $\inf |(\tau^n)'| > 2$. Then, the partition $\mathcal{P}^{(n)}$ of τ^n is used in an argument where the magnitude of the minimum length of $\mathcal{P}^{(n)}$ appears in the denominator of a term. This works if we are dealing with a single map or with a family of maps for which the n th iterate of all members of the family has slopes uniformly bounded away from 2 in modulus. Stability of acim in this situation was considered in [Keller, 1982; Keller and Liverani, 1999]. However, in some important situations this does not happen. Consider the example of $W-$shaped maps in previous chapters, where the limit map has $|\text{slope}| = 2$ at a turning fixed point 1/2. In this situation the standard Lasota-Yorke inequality cannot be applied to a family of approximating maps since taking an iteration of these maps creates partition elements which go to 0 length. The papers [Eslami and Misiurewicz, 2012; Li et al., 2013] show instability of acim for this map. In the paper [Li, 2013] (Chapter 3), stability of a more general W shaped map has been considered. The results of this paper inspired the introduction of the harmonic average of slopes condition.

Recently the Lasota-Yorke inequality has been strengthened [Eslami and Góra, 2012] by using the harmonic average of the slopes on each side of the partition points rather than the doubled reciprocal of the minimal slope. This allows us to show stability of the acim of the limit map for a larger class of maps. The smoothness assumption in [Eslami and Góra, 2012] is piecewise C^{1+1}.

In this chapter we generalize the use of the harmonic average of slopes condition to maps with much weaker smoothness properties, namely we assume only the summable oscillation condition for the reciprocal of the derivative. Unlike [Eslami and Góra, 2012], we do not use the bounded variation technique. Our main tool is Rychlik's

Theorem (see, e.g., [Boyarsky and Góra, 1997]). We show that the invariant densities of families of perturbed maps form a uniformly bounded set in L^∞ which implies that it is weakly compact in L^1. From this compactness property it follows that we have stability of the acim associated with the limit map.

In section 6.2 we will define the class of maps we will consider and introduce the harmonic average slope condition. Then, we will recall Rychlik's Theorem [Boyarsky and Góra, 1997, Theorem 6.2.1]. In section 6.3 we rewrite Rychlik's proof and show that the harmonic average condition is enough for the result to hold. In section 6.4 we prove the main result of this chapter, which establishes weak compactness in L^1 of the densities associated with the perturbing family of maps. This in turn proves stability of acim of the limit map. This result stays true in many situation not covered by previous works. An example is presented in section 6.5.

The results obtained in this chapter (Sections 6.2, 6.3, 6.4 and 6.5) were, after some modifications, published in the paper [Góra et al., 2012a].

6.2 Notation and preliminary results

Let $I = [0, 1]$ and let L be Lebesgue measure on I. In this chapter, we consider piecewise expanding map $\tau \in \mathcal{T}(I)$, see the Definition 5.2.1 for $\mathcal{T}(I)$. And, the condition 2 will be strengthened as:

$$2'.\tau_i := \tau|_{I_i} \text{ is } C^1 \text{ and } \lim_{x\to a_{i-1}^+} \tau'(x), \lim_{x\to a_i^-} \tau'(x) \text{ exist ; let } M = \max_{x\in I} |\tau'(x)|.$$

Let s and s_H be the same as defined in (5.2) and (5.3), respectively. The definition of the harmonic average of slopes condition is the same as defined in Chapter 5. Let

$$\delta := \min_{2 \le i \le q-1} L(I_i) \ . \tag{6.1}$$

Note, that to calculate the δ we do not use the first and the last subintervals of the partition.

Let

$$g_n = \frac{1}{|(\tau^n)'|},$$

wherever $(\tau^n)'$ is defined. Let $\mathcal{P}^{(n)} = \bigvee_{i=0}^{n-1} \tau^{-i}(\mathcal{P})$. Note that $\mathcal{P} = \mathcal{P}^{(1)}$. For any measurable subset A of $[a, b]$, let

$$\mathcal{P}(A) = \{J \in \mathcal{P} : \lambda(J \cap A) > 0\} \ .$$

Let $\gamma_n = \sum_{J \in \mathcal{P}^{(n)}} \ \sup_J \ g_n$.

For $J \in \mathcal{P}^{(n)}$, we define $\operatorname{osc}_J \frac{1}{|\tau'|} = \max_J \frac{1}{|\tau'|} - \min_J \frac{1}{|\tau'|}$ and

$$d_n = \max_{J \in \mathcal{P}^{(n)}} \operatorname{osc}_J \frac{1}{|\tau'|} \ .$$

Definition 6.2.1. We say that a map $\tau \in \mathcal{T}(I)$ satisfies the summable oscillation condition, or $\tau \in \mathcal{T}_\Sigma(I)$, if

$$\sum_{n \ge 1} d_n \le D < +\infty \ .$$

Note that usually the summable oscillation condition means a similar condition for $|\tau'|$ (e.g., [Góra, 1994]) and not $\frac{1}{|\tau'|}$ as here.

This condition is satisfied for example for the following maps:

(i) piecewise in $C^{1+\varepsilon}$, i.e., with bounded derivative satisfying a Hölder condition;

(ii) piecewise satisfying Collet's condition [Collet and Eckmann, 1985], i.e, the modulus of continuity of τ' satisfies

$$\omega(t) \leq \frac{K}{(1 + \log|t|)^{1+\gamma}} \ ,$$

as $t \to 0$, for some $K, \gamma > 0$ ($\gamma = 0$ is not enough);

(iii) satisfying Schmitt's condition [Góra, 1994; Schmitt, 1986], i.e, summable oscillation condition for $|\tau'|$.

6.3 Main result

We now recall Rychlik's Theorem. The proof can be found in [Rychlik, 1983] or [Boyarsky and Góra, 1997, Theorem 6.2.1].

Theorem 6.3.1. *Let τ be a piecewise monotonic transformation of an interval $[a, b]$ satisfying the following three conditions:*

1. *There exists $d > 0$ such that for any $n \geq 1$ and any $J \in \mathcal{P}^{(n)}$,*

$$\sup_J \ g_n \leq d \cdot \inf_J \ g_n;$$

2. *There exist $\varepsilon > 0$ and $r \in (0, 1)$ such that for any $n \geq 1$ and any $J \in \mathcal{P}^{(n)}$,*

$$L(\tau^n(J)) < \varepsilon \Rightarrow \sum_{J' \in \mathcal{P}(\tau^n(J))} \sup_{J'} g \leq r;$$

3. *$\gamma_1 = \sum_{J \in \mathcal{P}} \sup_J \ g < +\infty$.*

Then τ admits an acim. Moreover, if f is a τ-invariant density then

$$\|f\|_\infty \leq \gamma_1 \frac{d}{\varepsilon(1 - r)} \ . \tag{6.2}$$

Theorem 6.3.2. *If $\tau \in \mathcal{T}_\Sigma$ and satisfies the harmonic average of slopes condition $s_H < 1$, then it satisfies the assumptions of Rychlik's Theorem.*

Proof. Condition (1): Note that $\sup g \leq \frac{1}{s}$. Let $J \in \mathcal{P}^{(n)},\ x, y \in J$. We have

$$\frac{g_n(x)}{g_n(y)} = \frac{g(\tau^{n-1}(x))g(\tau^{n-2}(x))\dots g(\tau(x))g(x)}{g(\tau^{n-1}(y))g(\tau^{n-2}(y))\dots g(\tau(y))g(y)}.$$

For any $k = 0, \dots, n-1, \tau^k(x)$ and $\tau^k(y)$ belong to the same element J_k of $\mathcal{P}^{(n-k)}$. Using the inequality

$$\frac{a}{b} = 1 + \frac{a-b}{b} \leq \exp\left(|\frac{a-b}{b}|\right),$$

we get

$$\begin{aligned} \frac{g(\tau^k(x))}{g(\tau^k(y))} &\leq \exp\left(\frac{1}{g\left(\tau^k(x)\right)}|g(\tau^k(x)) - g(\tau^k(y))|\right) \\ &\leq \exp\left(M\, d_{n-k}\right), \end{aligned}$$

and thus,

$$\frac{g_n(x)}{g_n(y)} \leq \exp\left(M \sum_{k=0}^{n-1} d_{n-k}\right) \leq \exp\left(M \cdot D\right).$$

We have established condition (1) with

$$d = \exp\left(M \cdot D\right).$$

We now invoke the harmonic average of slopes condition to prove condition (2): let $\varepsilon = \frac{1}{2}\delta$ and $r = s_H < 1$. (It is important to note that we did not use the lengths of the first and the last interval of the partition to define δ.) It is enough to notice that, for any $J' \in \mathcal{P}^{(n)}$, $\tau^n(J')$ is an interval and if $L(\tau^n J') < \varepsilon$, then $\tau^n J'$ can intersect at most two intervals of $\mathcal{P}$. Thus, $\sum_{J \in \mathcal{P}(\tau^n J')} \sup\ g \leq s_H = r < 1$.

Condition (3) is satisfied by definition. This completes the proof. $\square$

Remark 6.3.1. Note that in the above proof, if we use the usual summable oscillation condition for $|\tau'|$ (e.g., [Góra, 1994]), we can alternatively obtain the following:

$$\begin{aligned}
\frac{g(\tau^k(x))}{g(\tau^k(y))} &= \frac{\frac{1}{g(\tau^k(y))}}{\frac{1}{g(\tau^k(x))}} \\
&\leq \exp\left(g\left(\tau^k(x)\right)\left|\frac{1}{g(\tau^k(x))} - \frac{1}{g(\tau^k(y))}\right|\right) \\
&\leq \exp\left(\frac{1}{s}\, d_{n-k}\right),
\end{aligned}$$

and thus,

$$\frac{g_n(x)}{g_n(y)} \leq \exp\left(\frac{1}{s}\sum_{k=0}^{n-1} d_{n-k}\right) \leq \exp\left(\frac{D}{s}\right).$$

Therefore, we can establish condition (1) in Theorem 6.3.1 with

$$d = \exp\left(\frac{D}{s}\right).$$

6.4 Stability of acim for families of maps

The main motivation for this investigation is to prove stability of the acim for maps with turning fixed or periodic points. The general setting is as follows. Let τ_0 be a map with an invariant density f_0 and and $\{\tau_\gamma\}_{\gamma>0}$ a family of maps with invariant densities f_γ such that τ_γ converge to τ_0 in some sense as γ converges to 0. Question: under what conditions does $f_\gamma \to f_0$ in some sense? Such problems were investigated in many articles but usually using bounded variation technique [Keller, 1982; Keller and Liverani, 1999].

Theorem 6.4.1. *Let the family $\{\tau_\gamma\}_{\gamma>0} \subset \mathcal{T}_\Sigma$ satisfies the assumptions of Rychlik's Theorem in a uniform way, i.e, with the same constants and $\tau_\gamma \to \tau_0$ almost uniformly* *as $\gamma \to 0$. If τ_0 has exactly one acim, then $f_\gamma \to f_0$ in L^1 as $\gamma \to 0$. In the general case every limit point of the family $\{f_\gamma\}$, as $\gamma \to 0$, is an invariant density of τ_0.*

Proof. The proof follows from Theorem 11.2.3 of [Boyarsky and Góra, 1997] which we recall below with appropriate changes. □

Theorem 6.4.2. *Let $\tau_\gamma \in \mathcal{T}$, $\gamma \geq 0$. Let the invariant densities of $\{f_\gamma\}_{\gamma \geq 0}$ be uniformly bounded in L^∞. If $\tau_\gamma \to \tau_0$ almost uniformly as $\gamma \to 0$, then any limit point of $\{f_\gamma\}_{\gamma > 0}$, as $\gamma \to 0$, is a τ_0-invariant density. If $\{\tau_0, f \cdot L\}$ is ergodic, then $f_\gamma \to f_0$ in L^1.*

We now describe two families of maps for which Theorem 6.4.1 applies.

Proposition 6.4.1. *Let $\tau_0 \in \mathcal{T}_\Sigma$ satisfy the harmonic average condition $s_H < 1$. Let τ_γ be defined on the same partition $\mathcal{P} = \{I_1, I_2, \ldots, I_q\}$ and $\tau_\gamma \to \tau_0$, as $\gamma \to 0$, in $C^1(\mathrm{int}(I_i))$ for all $i = 1, 2, \ldots, q$. We also assume that the summable oscillation condition is satisfied uniformly for $\{\tau_\gamma\}_{\gamma \geq 0}$. Then, the family $\{\tau_\gamma\}_{\gamma \geq 0}$ satisfies the assumptions of Theorem 6.4.1.*

Proposition 6.4.2. *Let $\tau_0 \in \mathcal{T}_\Sigma$ satisfy the harmonic average condition $s_H < 1$. Let each τ_γ be piecewise expanding on the partition $\mathcal{P}_\gamma = \{I_0^{(\gamma)}, I_2^{(\gamma)}, \ldots, I_{q+1}^{(\gamma)}\}$, $I_i^{(\gamma)} = [a_{i-1}^{(\gamma)}, a_i^{(\gamma)}]$, $i = 0, 1, 2, \ldots, q+1$. We allow the possibility that $I_0^{(\gamma)}$ or $I_{q+1}^{(\gamma)}$ or both of them are empty. We assume $a_i^{(\gamma)} \to a_i^{(0)}$ as $\gamma \to 0$, $i = 0, 1, 2, \ldots, q$. Then, automatically $a_{-1}^{(\gamma)} \to a_0^{(0)}$ and $a_{q+1}^{(\gamma)} \to a_q^{(0)}$ as $\gamma \to 0$. We also assume that the summable oscillation condition and harmonic average condition are satisfied uniformly for $\{\tau_\gamma\}_{\gamma \geq 0}$. If $\tau_\gamma \to \tau_0$ almost uniformly as $\gamma \to 0$, then the family $\{\tau_\gamma\}_{a \geq 0}$ satisfies the assumptions of Theorem 6.4.1.*

Results similar to those above were derived in [Góra and Boyarsky, 1989c] under additional much stronger conditions on the family of transformations. The two main stronger conditions assumed in [Góra and Boyarsky, 1989c] are

(i) There exists a constant $\delta > 0$ such that for any τ_γ in the family of maps there exists a finite partition $\mathcal{K}_\gamma$ such that for any $J \in \mathcal{K}_\gamma$, $\tau_{\gamma|_J}$ is one-to-one, $\tau_\gamma(J)$ is an interval, and

$$\min_{J \in K_\gamma} \operatorname{diam}(J) \quad > \quad \delta \; .$$

(ii) For any $m \geq 1$, there exists $\delta_m > 0$ such that if

$$\mathcal{K}_\gamma^{(m)} \quad = \quad \bigvee_{j=0}^{m-1} \tau_\gamma^{-j}(\mathcal{K}_\gamma)$$

then

$$\min_{J \in K_\gamma^{(m)}} \operatorname{diam}(J_m) \quad \geq \quad \delta_m > 0 \; .$$

From these conditions it follows that the family of densities is weakly compact in L^1.

6.5 Example

The results of this chapter allow us to answer a question posed in [Eslami and Misiurewicz, 2012], which is also studied in [Pendev, 2012].

Example 6.5.1.

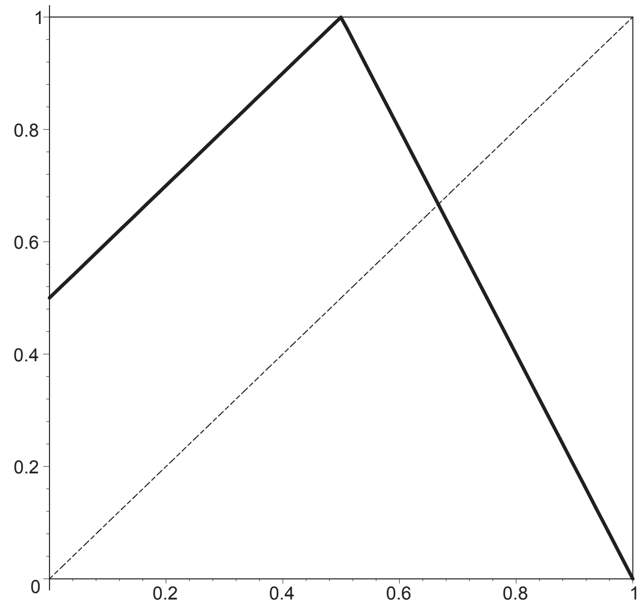

Figure 6.1: The graph of Example 6.5.1 for $\gamma = 0$.

Let τ_γ, $0 \le \gamma < \varepsilon_0 < 1/2$, be a map defined by

$$\tau_\gamma(t) = \begin{cases} \frac{1}{2} - \gamma + (1+2\gamma)t & , \quad 0 \le t < \frac{1}{2} \ ; \\ \\ 2 - 2t & , \quad \frac{1}{2} \le t \le 1 \ . \end{cases}$$

τ_0, which is shown in Fig. 6.1, is exact with invariant density $f_0 = \frac{2}{3}\chi_{[0,1/2]} + \frac{4}{3}\chi_{[1/2,1]}$. Is this acim stable under perturbation given by the family $\{\tau_\gamma\}_{\gamma>0}$? τ_0 has a turning point 1/2 which is periodic with period 3. Previously known methods did not give an answer to this question.

We will consider the family of third iterates $\{\tau_\gamma^3\}_{\gamma>0}$. τ_0^3 is shown in Fig. 6.2 (a) and a typical τ_γ^3 is shown in Fig. 6.2 (b). The slopes of τ_γ^3 are $s_1 = s_3 = s_7 = 2+8\gamma+8\gamma^2$, $s_2 = s_4 = s_6 = 4 + 8\gamma$, and $s_5 = 8$. Since τ_0 is exact, τ_0^3 is also exact with the same acim and stability of acim for τ_0^3 implies the same for τ_0. We can see that the family $\{\tau_\gamma^3\}_{\gamma>0}$ satisfies the conditions of Proposition 6.4.2. Thus, τ_0 has a stable acim.

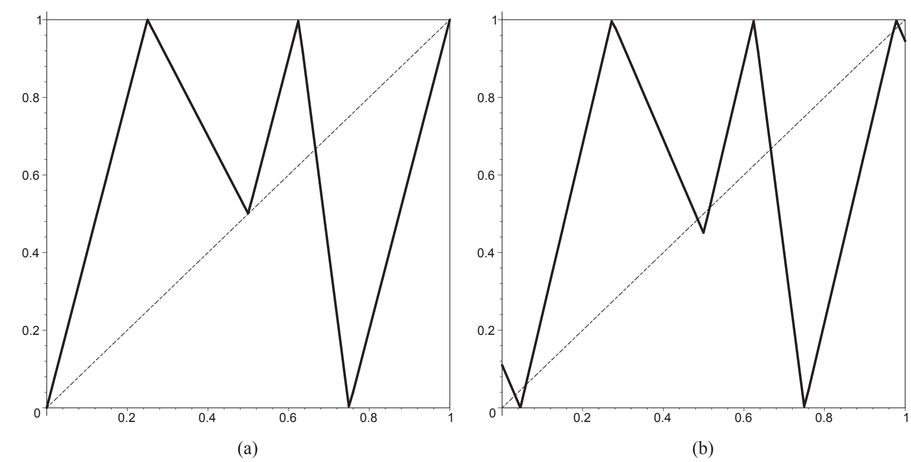

Figure 6.2: The 3rd iterates of maps of Example 6.5.1: (a) τ_0^3, (b) $\tau_{0.05}^3$.

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