EXPERIMENTING WITH DISCURSIVE AND NON-DISCURSIVE STYLES OF TEACHING ABSOLUTE VALUE INEQUALITIES TO MATURE STUDENTS

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ABSTRACT

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This research is a follow-up of Sierpinska, Bobos, and Pruncut's 2011 study, which experimented with three teaching approaches to teaching absolute value inequalities (AVI), visual, procedural, and theoretical, presented over an audio lecture with slides. The study demonstrated that participants treated with the visual approach were more likely to engage in theoretical thinking than those treated with the other two approaches. In the present experiment, two groups of participants enrolled in prerequisite mathematics courses at a large, urban North American University were taught AVI with the visual approach using two different teaching styles: discursive (permitting and actively encouraging teacher-student interactions during the lecture) and non-discursive (not allowing teacher-student interactions during the lecture). In Sierpinska et al.'s study, the non-discursive style was used in all three approaches (the lectures were recorded and the teacher was not present in person). In the present study, a live teacher was lecturing in both treatments. Another difference was that in Sierpinska et al.'s study, lectures were delivered individually to each participant, while in the present study, all participants in a group were treated simultaneously. Therefore, in the discursive approach, not only teacher-student but also student-student interactions during the lecture were possible.

The aim of this research was to explore the conjecture that the discursive approach is more likely to promote theoretical thinking in students. The group exposed to the discursive approach was, therefore, my experimental group and the other played the role of the control group. The conjecture was not confirmed, but the two approaches seem to have provoked different aspects of theoretical thinking. The experimental group was found to be more reflective, while the control group tended to be more systemic in their thinking. Some striking results, not predicted by Sierpinska et al.'s study, were also found with respect to reflective thinking, definitional thinking, proving behavior, and analytic thinking.

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Introduction

As mathematics learners, we have all most probably been in the situation of sitting submissively in a mathematics class, devotedly jotting down notes of examples and proofs, all the while thinking “this makes sense”. However, when confronted later with an exercise or proof to be written independently, most of us have also sometimes realized that we had not completely grasped all that was being taught as well as we had thought.

The question posed in this study, is whether expecting students to listen to a lecture in silence or, on the contrary, inviting their active participation in a mathematics lesson is more advantageous to a student’s learning. This issue has been the topic of an ongoing conversation between many mathematics educators and have long been an interest of mine. Therefore, the debate between discursive approaches and non-discursive approaches to teaching has been the inspiration for this experiment.

In the hopes of gaining an insight into the impact of these two teaching approaches on students’ learning, a follow-up experiment to Sierpinska, Bobos, and Pruncut (2011) was conducted. In the present experiment, two groups of students were exposed to different teaching styles; discursive and non-discursive. Discursive style refers to interactive lecturing where the instructor permits and actively encourages participation throughout the lecture by asking questions, soliciting participation from the students, tackling any difficulties or misunderstandings on the spot, and generally encouraging a classroom culture in which students may feel more comfortable asking their own questions, making interventions, and initiating discussions.

In the non-discursive approach, on the other hand, no student participation during the lecture was allowed. The participants had to listen to the lecture in silence. They could not interrupt the lecturer to clarify a point they did not understand. They were given the opportunity to ask their questions only after the lecture.

The aim of this experiment was to study the conjecture that the discursive approach to teaching is more likely to provoke theoretical thinking in students and produce better performance on solving AVI. The group exposed to the discursive approach was, thus, my experimental group and the other played the role of the control group.

The lectures presented to the participants in this experiment followed the visual approach lecture given to students in Sierpinska et al. (2011) but they took place in a classroom setting and not in the form of power point slides with audio-recording as was the case in the previous study. The lecture in Sierpinska et al. was similar in form to lectures used in online courses. Thus, in addition to comparing the discursive and non-discursive lecturing styles, the experiment allows for a comparison of two teaching settings, the classroom and the audio lecture with slides, for their possible impact on participants' performance and their engagement in theoretical thinking. Therefore, the results obtained in the present experiment may also contribute to the ongoing debate over online versus classroom settings for the teaching of mathematics courses.

By using the model for theoretical thinking in Sierpinska et al. (2011) as a lens to analyse participant solutions and oral contributions, I hope to obtain a deeper understanding about the level of student learning and thinking with respect to each teaching approach.

Thus, the questions addressed in this teaching experiment are:

1. Will the discursive or non-discursive approach lead to higher participant performance?

2. Which teaching approach will better promote participants' theoretical thinking?
3. How are participants' performance and theoretical thinking affected by a computer-based versus classroom setting?

One way to judge the effects of the discursive teaching style was to see whether this type of interacting with the participants would make them more mentally stimulated by the lecture: would participants in the experimental group be more likely to think theoretically about the material and ask more questions during the discussion period than those in the control group who would have listened to the lecture in silence?

Experimenting with the discursive versus non-discursive approaches to teaching is necessary because it is not obvious, *a priori*, which one has a better effect on students' performance and learning. That is, should we probe our students to make sure they are on track before moving on with the lecture and allow them to interrupt every time they have insecurity, or would it be better to let their thoughts percolate? Perhaps a question someone may have at one point during the lecture will be answered later on in the lecture, anyway. On the other hand, not having a point clarified on the spot might hold a student back from understanding the rest of the lecture. Some students may appreciate being able to pose questions immediately. Others may feel that their learning process is disturbed if the lecture is constantly interrupted by responding to some students' questions.

This is a small scale research, with only 6 participants in the experimental group and 5 in the control group. Therefore its results have to be interpreted with caution. At the most general level, we can say that the conjecture of the experiment was not entirely confirmed. The study yielded, however, some interesting results in relation to different aspects of theoretical thinking (reflective, systemic, and analytic) in participants subjected to different teaching styles. In particular, the experimental group was found to be more reflective. On the other hand, the control group tended to be more systemic in their thinking. These results suggest that the two approaches promoted different aspects of theoretical thinking. Some striking results were also found in terms of reflective thinking, definitional thinking, proving using the reasoning by cases technique, and analytic thinking when comparing the results of this experiment to those of Sierpinska et al. (2011).

Through this experiment, it is my hope to answer these fundamental questions and suggest some useful teaching strategies that will assist educators with one of the most discussed topics in mathematics education, the teaching approach.

Chapter 1. Literature Review

Before discussing the present experiment and its results, we will review research that was fundamental to the development of this experiment and to the analysis and discussion of its results. First, we will discuss documented student difficulties with the notion of absolute value and then look at a range of teaching approaches.

1.1 Student difficulties with absolute value

Many researchers have studied and discussed the difficulties faced by students when learning the definition of absolute value (AV) in order to find better ways to teach this concept. The difficulties explored by the researchers whose work will be discussed here are those dealing with difficulties in understanding the piecewise function definition of absolute value, the meaning of "$-x$", logical operators, and the notion of variable.

Chiarugi, Fracassina and Furinghetti (1990) discuss research by Marcovits, Eilon and Bruckheimer (1986) and by Tall and Vinner (1981). This research found that piecewise functions pose difficulties for students (p. 4). Brumfiel (1980) agrees by stating that many students "find it difficult to use more than one formula in the description of a function" (p. 24). In particular, he refers to the two-part analytic definition of absolute value.

Difficulties with the piece-wise function definition of AV have been found to lie in many factors. Students often identify a function with an algebraic expression, and therefore treat a piece-wise defined function not as one, but two (or more) functions. In addition to their difficulties with the notion of piecewise function in general, students have also been documented to struggle with the concept of "$-x$" which appears in the analytic definition of absolute value, as the function that multiplies by -1 (Brumfiel, 1980, p. 24; Chiarugi et al., 1990, p. 5). Chiarugi et al. explain that the concept image of a letter with no sign as a positive number “is so strong that the majority of students rejects the idea that $-a$ or $-x$ may be negative” (p. 5). In fact, their research showed that this misconception was held even by those students who correctly answered questions about the meaning of absolute value (Chiarugi et al., p. 5).

Another obstacle faced by many students is understanding absolute value as the operation of “ignoring the sign”. Duroux (1983) explains this common error by the way students are usually taught absolute value in elementary school where only absolute value of concrete numbers is tackled (e.g, $|-3| = 3$) (p. 51). In the context of school arithmetic, absolute value of a number might indeed appear as “ignoring the sign” of the number. This then becomes an obstacle to understanding the meaning of “$-x$” in the analytic definition of absolute value in algebra (Gagatsis and Thomaidis, 1994, p. 347).

Given the above mentioned obstacles, it is not surprising that, in Chiarugi et al.’s research (1990), students were found to score significantly higher on questions involving the geometrical representation of the notion of absolute value than on questions involving algebraic definitions (p. 5).

Logical operators have also been seen as a source of difficulty in research studies. In particular, Chiarugi et al. (1990) state that they ‘think the difficulty hidden in the standard definition of absolute value (and in the piecewise functions) is the presence of the logical operators “or” and “implies” (p. 4). Moreover, students tend to misunderstand the “or” as meaning “and”, and at times do not take into account the “if... then” statement (Chiarugi et al., p. 4).

Students in this same study were also found to have difficulties with reading algebraic sentences (Chiarugi et al., 1990, p. 4). This caused many to falsely claim that $|x| < 0$ if $x < 0$

when asked “for which values of $x \in \mathbb{R}$ is $|x| < 0$?”. The authors argue that these types of errors are caused by difficulties in identifying the domain and range of the absolute value function (Chiarugi et al., p. 4). That is, students may have the misconception that the “$x < 0$” portion of the function does not refer to the domain, but to the range of the function; thus, leading students to perceive the “$-x$” to be a piece of the function below the x-axis.

Another common difficulty for students discussed by Chiarugi et al. (1990) is the notion of variable (p. 2). The authors mention research by Olivier (1988), Rosnick (1981), and Wagner (1983) in which a common cause of difficulty is the use of letters instead of numbers (Chiarugi et al., p. 2). Further, the transition from the arithmetic to the algebraic domain can include difficulties with the “meaning of the operations and of the conventions in procedures which are different from those in the arithmetic” (Chiarugi et al., p. 2). The authors go on to mention that this commonly leads to errors in manipulations (p.2).

Gagtsis and Thomaidis (1994) and Duroux (1983) discuss epistemological, cognitive and didactic obstacles that can lead to errors with absolute value.

One obstacle discussed by Gagatsis and Thomaidis (1994) is the conception of number as absolute measure (p. 346). This obstacle, according to the researchers is what causes difficulty with the order of the real numbers (Gagatsis & Thomaidis, p. 347). This problem has already been noticed by Cauchy (1821/1968) in the introduction to his exposition of Mathematical Analysis in his “Cours d’Analyse de l’École Polytechnique”, as stressed by Duroux (1983), another mathematics educator who has done research on difficulties with the notion of absolute value in secondary school students. Cauchy did not take for granted that his readers will have the concept of number such as it is necessary for the development and understanding of Analysis. He distinguished between the absolute measure numbers (calling them “nombres”) used in

Arithmetic and numbers representing relative change (increase or decrease) (referring to them as "quantités") used in Analysis (p. 2).

One of the didactic obstacles mentioned by Gagatsis and Thomaidis (1994), is teaching students that the first thing to do when solving a problem with absolute value is to get rid of the absolute value bars (p. 347). It is documented that teaching students this leads to contradictions later on in their studies with absolute value. In particular, this error is frequently seen with students who remove the absolute value bars and solve as though they were never there. These students then come to an answer for x that sometimes does not hold depending on the initial conditions that were put on x by the absolute value sign (Gagatsis & Thomaidis, p. 348).

Researchers such as Brumfiel (1980), Wilhelmi, Godino, and Lacasta (2007) acknowledge that the various definitions of AV are not equivalent if we are considering them from an epistemic point of view. Moreover, these definitions "condition the operative and discursive practices in relation to the [absolute value notion]" (Wilhelmi et al., p. 78). In fact, these researchers view each definition of absolute value as only a partial meaning of the notion (Wilhelmi et al., p. 80). Wilhelmi et al. explain that these partial meanings *together* form the holistic meaning of the notion and that this holistic meaning comes from the "tensions, filiations and contradictions that are established between" the individual partial meanings (p. 80). Brumfiel (1980) does not adopt the terms *partial* and *holistic meanings*; however, he believes that the deliberate choice of a particular definition leads students' thoughts in specific directions (p. 26). Similarly, Wilhelmi et al. explain that partial meanings refer to a "coherent form for structuring the different contexts of use, the mathematical practices relating to them and the objects emerging from such practices" (p. 79).

Through their study, Wilhelmi et al. (2007) found that students who were introduced to the piece-wise function partial meaning were more likely to use the notion of absolute value symbolically and of systematizing the use of the arithmetical partial meaning (p. 86). However, students who had only been shown the arithmetical partial meaning associated absolute value with the algorithmic action of removing the minus sign from negative numbers (p. 86). These results demonstrate that although the two-part definitions are more challenging for students, they are a necessary part of the learning process and make a fundamental difference in the depth of student understanding.

The above discussions focused on the types of difficulties faced by students while learning absolute value definitions. It is, however, equally important to consider the ways in which students understand the concept and the process in which they learn if we wish to find more effective ways of teaching.

Vinner (1991) explains that 'defining a mathematical object involves "more than anything else the conflict between the structure of mathematics, as conceived by professional mathematicians, and the cognitive processes of concept acquisition"' (quoted in Wilhelmi et al., 2007, p.72). Wilhelmi et al. go on to discuss that in order to determine the didactic effectiveness of the particular definitions of AV, we must consider their epistemic, cognitive and instructional dimensions (Wilhelmi et al., p. 73).

Ozmantar and Monaghan (2006) study the different ways that the various definitions of AV are learned by students. They adopt an activity-theoretic model of abstraction that was proposed by Hershkowitz et al. in 2001, which places importance on theoretic thought in order to argue that an abstraction is a consolidated construction that can be used by students to create new constructions (Ozmantar & Monaghan, p. 94).

Research by Sierpinska et al. (2011) experimented with three approaches to teaching absolute value inequalities to mature students in order to identify the characteristics of an approach that will be more likely to promote theoretical thinking in students (p.1). The approaches were named "procedural", "theoretical", and "visual". The particular characteristics of these approaches are given in Sierpinska et al.'s article. In this experiment, only the visual approach was used, which offered students two methods of solving absolute value inequalities, graphical and algebraic.

Theoretical thinking, according to Sierpinska et al. (2011), can be described as thinking that is reflective, systemic, and analytic. A symptom of reflective thinking can be an investigative attitude when solving mathematical problems; for instance, reflecting back on one's solution to a problem and perhaps realizing that a less time-consuming solution method could have been used. Systemic thinking is demonstrated when a student's thinking is based on proofs and definitions, while analytic thinking is based on "sensitivities to formal symbolic notations and specialized terminology ... and to the structure and logic of mathematical language" (Sierpinska et al., p. 279). Moreover, "engaging with theoretical thinking may help avoid both 'conceptual' and 'procedural' errors in the sense of Porter and Masingila (2000)" (ibid.).

Further, Sierpinska et al. (2011) include the reasoning by cases (RBC) technique in all three lectures (theoretical, visual, and procedural) since it "has the potential of engaging [students] in theoretical thinking" (p. 7).

All the research reviewed up to this point focused only on student difficulties with the absolute value concept; none specifically mentioned difficulties with absolute value inequalities. Aside from Sierpinska et al. (2011), I am not aware of any other research studying these particular difficulties.

The study by Sierpinska et al. (2011) found that students who were taught absolute value inequalities (AVI) using a visual approach scored higher in terms of performance than the other two groups. In fact, these students were also more likely to be reflective, notice relationships between problems, solve more conceptually without the standard reasoning by cases technique, and more likely to check their answers.

As will be discussed later, the theoretical thinking model used by Sierpinska et al. (2011) is the theoretical framework of this research and is used as a lens to analyze the data collected. In the research, the visual approach to teaching AVI, identified as the most effective in promoting theoretical thinking in students, was used. But this research also experimented with a discursive teaching style in teaching AVI with the visual approach and therefore, in the remaining part of this chapter, a brief review of the literature on "discursive" teaching of mathematics will be offered.

1.2 Discursive teaching of mathematics

Approaches to the teaching of mathematics that have the potential to promote classroom discussion have been of interest to many researchers in mathematics education since the traditional ways of teaching such as the acquisition (sender-receiver) model and transmission model have been found ineffective by many mathematics educators and research studies (Van Oers, 2001, p. 65).

We will begin by covering teaching approaches that for the most part have been rejected by researchers. However, this is not to say that there are no educators who apply similar teaching approaches in today's classrooms. Then, we will explore the discursive approach to teaching and follow this discussion by different approaches studied by researchers who also advocate including dialogue in mathematics classrooms, but use this dialogue in a different way than is described by the discursive approach. That is, many researchers have conducted studies that test the effect of dialogue in classrooms or within small groups of students but the researchers have placed different roles on the students and teachers than those promoted by the discursive approach.

1.2.1 Rejected approaches to mathematics teaching

The acquisition model or sender-receiver model of communication, based primarily on the idea of language as a code, focused on the transmission of pieces of knowledge from one individual, in this case, the teacher, to another, the student (Sierpinska, 2005, p. 208-209). That is, the teacher (sender) would transmit his or her information in symbolic form to the receiver (student) who would interpret or decode this information (Sierpinska, p. 209). Van Oers (2001) explains that this approach leads to an authoritarian relationship between the teacher and his students (p. 62). Also, this classic Platonic model is based on the idea that it is not useful to let students discover mathematics on their own (Van Oers, p. 62). Sfard (2001) states, however, that the acquisition of knowledge can take place through the passive reception or active construction of the receivers, which can lead to a personalized version of the knowledge being transmitted (p. 22). That is, this type of communication can "'distill' cognitive activities from their context and thus tell us only a restricted part of the story of learning" (p. 22). At the same time, Sfard admits that we should not reject this theory completely, but only add to it the theory of other models that is lacking (p. 22). Nonetheless, public discourse in most schools today still follows this communication model (Van Oers, p. 62).

The transmission model holds knowledge as objective units of thought that can be transmitted from one person to another (Van Oers, 2001, p.64). This model has also been rejected by many due to its disappointing outcomes. Moreover, this view focuses on mathematical knowledge as dealing with "abstract structures that have to be applied to concrete situations and problems" (Van Oers, p. 62). Further, educators endorsing this approach place a great importance on exploration and communication in their classrooms, while having students deal with "abstract structures for the organization of practical situations or for the solution of quantitative and spatial problems" (Van Oers, p. 62-63).

Another approach to teaching discussed, but not applied by Van Oers (2001), is based on solving problems in the classroom that are based on realistic situations with the help of self-invented symbolic tools. Van Oers argues that this approach, however, lacks focus on mathematical proof, argumentation, and systemacy (p. 64). He also states that there is no guarantee that students will choose relevant propositions in their dialogues (Van Oers, p. 64). Moreover, "non-expert pupils lack the criteria to link their own actions to the meanings of the cultural (mathematical) practice" (Van Oers, p. 64).

1.2.2 The discursive approach to the teaching of mathematics

The Discursive Approach, although different in some respects to the views of Davydov and Vygotsky, maintains its roots in the works of these two researchers. Therefore, it is pertinent to discuss the perspectives endorsed by them before defining the Discursive Approach.

Vygotsky views culture and society as parts of our environment with which individuals interact, not as factors that uni-directionally influence development (Sierpinska, 2005, p. 209). While the environment acts on the person, the person simultaneously acts on the environment. Therefore, society emerges from the interactions between participants (Sierpinska, p. 209). So, students' and teachers' activities are both of equal importance in the teaching and learning process. Within this perspective, the teacher possesses a guiding role in the educational process by introducing his or her students to fundamental sociocultural practices of the mathematical culture, while also guiding and monitoring the students' individual activities (Van Oers, 2001, p. 67). In addition, mental functions originate in the "communication between individuals, their relationships between each other and in their relationships with the objects created by people" (Van Oers, p. 67). Thus, communication refers to individuals sharing communalities in meanings and constructively dealing with the meanings that individuals seem to have in common (Van Oers, p. 67).

Davydov, on the other hand, takes a position of dialectical materialism orthodoxy, similar to Marx, Engels, and Lenin (Sierpinska, 2005, p. 209). For him, thinking is a collective task and society is a 'super-entity that "confers [...] the historically developed forms of [...] activity" on the individual (Sierpinska, p. 209). Thus, Davydov's point of view is based on a *human society*, while Vygotsky's is based more on the idea of a *civil society*. That is, Davydov looked at society from afar so that its members are not distinguishable (society as "the masses"), while Vygotsky looked at society from its members' perspectives and focused on the mental functioning of these individuals (Sierpinska, p. 209).

The Discursive Approach to teaching mathematics is at times in favor of Davydov's point of view, and other times leans more toward the perspective of Vygotsky's civil society. The Discursive Approach (DA) to mathematics education, according to Sierpinska (2005), "promotes a certain *ideology* and encourages a particular *didactic action*, while developing a *theory* and conducting experimental research guided by this theory" (p. 206). The ideology for the learner of mathematics does not consider what the he is thinking to himself, and a learner's behavior is only "interesting insofar as it is interpreted by other members of the group as a sign; that is, as having some meaning for them" (Sierpinska, p. 207). That is, the learner is considered an apprentice of mathematical discourse (Sierpinska, p.206). The teacher creates conditions for the initiation of students into the mathematical culture and acts as a participant in the discourse, while simultaneously directing the discussion and helping to formulate students' ideas by revoicing what is being discoursed (Sierpinska, p. 207). Additionally, the classroom "is a community of mathematical discourse" (Sierpinska, p. 207). That is, students work on solving common problems, while agreeing or disagreeing on approaches and techniques with their peers and teacher (Sierpinska, p. 207). Sierpinska specifies that the particular dialogue in this classroom is a polylogue, a polyphonic discourse among all voices that helped to create the history of the community (Sierpinska, p. 207). Further, the curriculum should be relevant from the historical and cultural perspective in order to initiate students into the historically developed ways of doing and talking, while getting them involved in the mathematics (Sierpinska, p. 207). Hence, it is not enough to speak for conversation's sake. Moreover, instructors must use the meta-discursive rules of the mathematical "genre" in order for them to become second nature for students (Sierpinska, p. 207). DA considers discourse as any instance of communication; that is, diachronic or synchronic, with oneself or with others, mostly verbal, or using any symbolic system (p. 208). In addition, DA focuses on the participation in an activity of "sharing communalities and constructively dealing with the meanings people seem to have in common" (Sierpinska p. 208).

Van Oers (2001) also views teaching mathematics as a process of initiating students into the culture of the mathematical community (p. 59). The researcher refers to Buffee (1993) who speaks about reacculturation as "giving up, modifying, or renegotiating the language, values, knowledge, mores and so on that are constructed, established, and maintained by the community one is coming from, and becoming fluent instead in the language and so on of another community" (p.60).

1.2.3 Similar and not-so-similar discursive approaches

There are many other teaching approaches that are similar to DA, but which differ in terms of how concepts are discussed in the classroom and/or the role of the teacher in these discussions.

Sfard (2001) considers thinking as communication; in particular, she considers this as communication with oneself (p. 26). Thus, Sfard does not restrict communication to interactions mediated by language (p. 26). Moreover, any acts that influence the effectiveness of communication count as components of a discourse for Sfard; for instance, body movements, situational clues and the participants' histories (p. 28). In order to be a successful participant of a mathematical discourse, Sfard (2001) argues that a student should use two main factors of mathematical discourse: its reliance on symbolic artifacts and the meta-rules that regulate communication (p. 13).

Although Kieran (2001) takes on a DA to teaching mathematics, she mentions that thoughts are "potentially private" and can be hidden from others (p. 192). Kieran states that thoughts are sensitive to discursive content; however, she does not take the position that thoughts are a form of communication.

Lerman (2001) views interactions as discursive contributions that can lead students forward into their increasing participation in mathematical thinking and speaking, into what Vygotsky would call their Zones of Proximal Development (p. 89). The author also mentions other aspects of peer interactions that can be important to students; specifically, aspects of their peer interactions such as gender roles, ethnic stereotypes, body shape and size, abilities valued by peers, and their relationship to school life (Lerman, p. 99). These peer interactions lead to different choices of goals by students, different ways of behaving in the classroom, and to different things being learned by each student based on their goals and behavior (Lerman, p. 99). So, perhaps by promoting a discussion, we will be more likely to obtain students' attention and decrease the effects of these peer interactions; therefore, promoting a better grasp of what is being taught and discussed.

Although Forman and Ansell (2001) advocate the importance of promoting discussion during class time, they do not use DA. These researchers, in addition, try to move away from the traditional three-part initiation, response, evaluation/feedback sequence of class discussions. Rather, they refer to the National Council of Teachers of Mathematics (NCTM) which proposes that students "make sense of mathematics by explaining their invented strategies for solving problems and by listening to and reflecting on the strategies of others" (Forman & Ansell, p. 116). The teacher's role has thus become one of helping students structure their talk, revoicing, organizing turn-taking, asking students to reflect on and evaluate explanations provided, and orchestrating collective arguments (Forman & Ansell, p. 119).

The classroom discussion that is described by O'Connor (2001) is similar to that of Forman and Ansell (2001) in that it does not follow the discursive approach. However, O'Connor refers to his approach as a position-driven discussion which involves a teacher who leads his or her students in considering one main question which possesses a limited number of possible answers and to which the answer is usually unknown beforehand (p. 150). The aim of this type of discussion is for each student to take a position and defend this position with evidence (O'Connor, p. 150).

By applying a sociocultural perspective, Zack and Graves examine the degree of participation of three students in a mathematical discussion and the way in which their participation influenced interaction, their roles, and their identities as participants in a mathematical discourse (p. 229-230). These researchers propose a reconceptualization of Vygotsky's notion, Zone of Proximal Development. In particular, Zack and Graves wish to define a student's zone of proximal development as "an intellectual space, created in the moment as a result of the interaction of specific participants engaged with each other at that specific point in time" (p. 233). In this space, it is important for the teacher and students involved to allow room for students to grow, feel secure and able to explore (Zack & Graves, p. 234). Students in the classroom are expected to explain and justify their reasoning, listen to and try to understand others students' explanations, communicate respectfully, kindly, and in a way which expands possibilities for discussion and learning, and also credit other students whose ideas were helpful to them (Zack & Graves, p. 236). Further, the authors believe that instances of disagreement and misunderstanding can be triggers for "an intensive discussion leading to a more complete understanding among the participants" (Zack & Graves, p. 262). Zack and Graves specify that an expert teacher in such a setting should relate new material to what students already know, use multiple representations in explanations, show students how to coordinate and translate among alternative representations so they can see a concept in different ways, provide detailed justifications for each step in problem-solving, respond to indications of misunderstanding with elaborated explanations, encourage students to freely admit a lack of understanding, encourage students to express disagreement with others, and control the pace and content of their classroom activities (p. 263).

1.3 Conclusions

These various teaching approaches, including the discursive approach and its alternatives, have motivated me to test the discursive approach in a setting which mimics a mathematics classroom in order to find out whether this approach could be effective in promoting theoretical thinking in students. Although the last section describes approaches that are in some ways different from the Discursive Approach, they do provide us with some valuable tips on how we can use discursive approaches in our teaching. These will be discussed after the analysis of our study.

Chapter 2. Methodology and Theoretical Framework

This study aimed at exploring the conjecture that the discursive approach (DA) is more likely to provoke theoretical thinking in students of mathematics than an approach where lectures are separated from question periods. The impact of the approach on students' thinking was measured by certain numerical values assigned to the participants' solutions of exercises and their oral contributions.

2.1 Recruitment

All subjects were recruited from prerequisite mathematics courses at a large, urban North American University. By "prerequisite mathematics" course we mean college level mathematics courses offered by the university for candidates who have applied to a university academic program but lacked the mathematical prerequisites for admission. With the permission of their respective professors, participation invitations were distributed to all students before their class commenced (see Appendix C3). In all, 140 participation invitations were handed out in seven different classes. To avoid bias caused by previous knowledge of absolute value inequalities in students, courses not assuming this knowledge among their prerequisites were chosen.

Students were offered remuneration for their participation and were asked to send an email to a provided address if they were interested in participating. Thus, participation was completely voluntary and each time a student would send an email, his or her name would be placed alternately into one of the two groups, beginning with the experimental group. That is, the first participant was placed into the experimental group, the second into the control group, the third into the experimental group, and so on. This continued until all 14 spots were filled; there were 14 places available due to the funding available for remuneration. After all students were notified about the time and place, 11 of the initial 14 participants responded to confirm their attendance; 6 participants in the experimental group and 5 in the control group.

Students who expressed interest in participating were asked to sign consent forms (Appendix C1) and to fill out a short identification questionnaire (see Appendix C2).

2.2 The setting and structure of the classroom sessions

Each group attended a lecture that was conducted in a very similar manner to that of Sierpinska et al.'s (2011) visual approach (VA) audio lecture. The difference in the present study was that the lectures, experimental and control, were given in a classroom setting by an instructor. Certain measures were taken in order to avoid potential influencing factors within the classroom setting. In particular, the same classroom was used for both group lectures, the same instructor presented both lectures, and participants were spaced out as much as possible, without compromising a realistic classroom setting, in order to avoid as many distractions as possible.

Each participant was provided with a booklet containing all the slides from Sierpinska et al.'s (2011) VA slideshow (see Appendix C4) and was given a pen for taking down notes in his or her booklet.

Both lectures began with a review on graphing functions, then the concept of absolute value (AV) was introduced, and the lecture ended with a lesson on solving absolute value inequalities (AVI). The control group (CG) was instructed not to ask questions during the lecture and to mark the places where they had difficulty in the provided booklet so that they could ask their questions during the question period. On the other hand, the experimental group (EG) was encouraged, from the commencement of the lecture, to ask questions or make comments at any time during the lecture. Discussion was also promoted by probing the participants from time to time with questions. They too, were provided the same question period after the lecture; thus, they could ask any remaining questions before the exercise period.

After the lecture, each group underwent a collective question period in which participants were given the chance to ask any lingering questions they had before entering into the exercise period. During this question period, the instructor reinforced only the points already made during the lecture strictly using the visual approach to teaching and did not provide new examples. No new teaching approaches or examples were introduced during this time in order to keep the material learned amongst the groups as consistent as possible. Moreover, the instructor was to keep the lectures as close as possible to each other, content-wise, to avoid giving one group an advantage over the other.

After the question period, both groups were given identical exercises featuring one arithmetical exercise to calculate the numerical value of an AV expression and five AVI exercises. The exercises were the same as those given to the participants in Sierpinska et al.'s (2011) study. The participants were given the time they required to work on the exercises, thus, eliminating the potential pressure that can be placed on participants by a time constraint. This was considered vital to the results since any additional pressure on the participants had the potential to jeopardize their performance. In addition, the instructor was not permitted to answer any questions during this period in order to avoid distracting participants.

When all participants finished, the solutions to each exercise were discussed in the whole group. During this final discussion period, the exercises were worked through one by one. Participants were asked for their answers and then the solution to that exercise would be worked through either by the instructor or by the participants. In addition to being asked for answers, participants were asked how they arrived at their answers. They were questioned on what method or methods they used to solve, especially for the exercises involving AVI. After this discussion period, the instructor solicited participants' opinions about the experiment, in a form of group interview.

2.3 Exercises used to test participants' understanding and ways of thinking about absolute value inequalities

We list the exercises given to the participants here, because we will refer to them in describing the data analysis tools in section 2.5. The rationale for the choice of the exercises will be given in section 3.5.

1. Calculate: $||16-24|-|6-56||$

In exercises 2- 6, solve the given inequality

2. $|x-1|<|x+1|$
3. $|x+3|<-3|x-1|$
4. $|2x-1|<5$
5. $|2x-1|>5$
6. $|50x-1|<|x+100|$

2.4 Theoretical framework

The analysis of the data in this experiment was based on the Theoretical Thinking (TT) model in mathematics used by Sierpinska et al. (2011). The model allowed an analysis of the participants' solutions that went beyond calculating performance scores. This additional analysis was fundamental to the research since the number of correct answers obtained by a participant is often not enough to determine the extent to which material has been learned and the way in which a participant is thinking when solving exercises. It was used to evaluate the quality of participants' learning, written solutions, and oral contributions.

The model has already been briefly presented in Chapter 1 (section 1.1). TT is described as thinking that is reflective, systemic, and analytic. Reflective thinking is displayed through investigative behaviors such as seeking a simpler solution or reflecting back on a solution. Behaviors that demonstrate this type of thinking are the opposite of applying a solution method that was taught in class for each exercise without regard for its efficiency and forgetting about the solution once the exercise is solved. In reference to Belenky, Clinchy, Goldberger and Tarule (1997), Sierpinska et al. term positive reflective behaviors as *constructed behaviors* and negative reflective behaviors as *procedural behaviors*. Thus, a student who is being reflective is more likely to demonstrate constructed knowledge rather than procedural knowledge.

It is assumed that a student is demonstrating systemic thinking if “systems of concepts” are reflected in his or her oral or written reasoning (Sierpinska et al., 2011, p. 278). This thinking is characterized as “definitional”, i.e., referring to definitions rather than to intuitions or mental associations in reasoning; based on mathematical proofs; and hypothetical which includes “being aware of the conditional character of mathematical statements” (p. 279). A symptom of hypothetical thinking can be the use of conditional statements in the form of “if... then...” in discussing a mathematical problem. Thus, hypothetical thinking refers not only to being able to point out conditional statements in a piece of mathematical work, but also using this conditional reasoning in one’s own work.

The last component of the TT model, analytic thinking, refers to a student’s linguistic sensitivity and meta-linguistic sensitivity (Sierpinska et al., 2011, p. 279). Linguistic sensitivity refers to sensitivity to the formal symbolic notation and specialized terminology of mathematics. On the other hand, meta-linguistic sensitivity pertains to sensitivity to the structure and logic of the mathematical language.

2.5 Description of data analysis tools

The data, which in this case was the participants' written work and the transcripts of discussions in both groups, was analyzed first by evaluating participants' performance on the exercises. Performance was measured by the number of correct answers obtained by each participant.

Then, an analysis using the Theoretical Thinking (TT) model was conducted. As previously mentioned, this analysis was needed to better interpret the quality of the written solutions and the thoughts orally expressed by each participant during the question period and discussion period.

As mentioned in the previous section, the TT model takes into account three aspects of thinking: reflective, systemic, and analytic. Each student's written and oral productions were analyzed for "positive" and "negative" symptoms of each of these aspects of thinking. A positive symptom was a behavior suggesting that theoretical thinking is taking place; a negative symptom was a behavior suggesting a way of thinking that was contrary to the characteristics of theoretical thinking.

A student demonstrating a positive behavior in any of the three aspects of TT at least once was allotted 1 point. At the same time, a student reflecting a negative behavior at least once would lose one point; that is, the participant would be given a "-1". Each participant could only be awarded or deducted one point for each type of behavior. If the participant did not demonstrate a particular behavior, no points were given.

The following three sub-sections define the behaviors observed in Sierpinska et al.'s (2011) study. In addition to the behaviors from the 2011 study, some additions and changes that were found to be necessary for this experiment were made. These are also defined below.

2.5.1 Reflective thinking

Reflective thinking is divided into two categories, procedural and constructed behaviors, each consisting of 3 specific types of behaviors (Sierpinska et al., 2011, p. 292). Procedural behaviors (PB) are considered negative and constructed behaviors (CB) are counted as positive. Their descriptions, as formulated in Sierpinska et al. (p. 292), are given below.

Positive symptoms of reflective thinking (constructed behaviors):

CB1: Noticing that it is not necessary to solve exercises 2 and 3 in the same way and using a quicker method to solve 3

CB2: Using the relationship between exercises 4 and 5 to solve 5 in a quicker way or using elements in exercise 4 to solve exercise 5, without repeating steps

CB3: Including only the necessary elements used to solve an exercise within a written solution

Negative symptoms of reflective thinking (procedural behaviors):

PB1: Solving exercise 3 using the same approach as in exercise 2, without noticing that a quicker method could be used

PB2: Repeating steps already carried out in exercise 4 to solve exercise 5. For example, applying the definition of AV to $|2x - 1|$ in exercise 5 when it was already done in exercise 4

PB3: Including elements of the lecture in a solution, but not using them to find the answer; thus, using them in a "ceremonial" or "ritualistic" fashion

2.5.2 Systemic thinking

This aspect of theoretical thinking has been characterized as definitional, based on proofs, and hypothetical. Accordingly, Sierpinska et al. (2011) identified three categories of behaviors assumed to count as symptomatic of each of the particular characteristics of systemic thinking.

Two kinds of behaviors symptomatic of definitional thinking were distinguished in Sierpinska et al.'s research: DT-w (Definitional thinking, written) and DT-o (Definitional thinking, oral), depending on whether a behavior occurred in the participant's written work or in oral explanations. Both codes refer to behaviors in which a participant applies the formal definition of absolute value (AV) in at least one written solution (DT-w) or in his or her oral reasoning (DT-o) (Sierpinska et al., 2011, p. 293). However, the present study did not include individual participant interviews, as was done in Sierpinska et al., and the number of explanations expressed by participants during the discussion period is not equal among EG and CG. Therefore, only one category, DT, is included in this analysis. It is considered s positive symptom of theoretical thinking.

DT: A participant's written solution or oral explanation contains an explicit application of the formal definition of AV.

The following categories of behaviors are considered as positive symptoms of "proving". They were distinguished by the mathematical means used by a student to settle any uncertainty with respect to a particular solution (Sierpinska et al., 2011, p. 294):

P-numerical testing: Plugging numbers into the initial inequality of an exercise

P-graphing-physical: Drawing a graph as part of a written solution

P-graphing-mental: Saying that a mental visualization of a graph was important to the solution process of an AVI exercise

P-structural: Providing reasoning focused on the structural properties of the inequality and the functions involved in it

P-RBC: Using reasoning by cases (RBC) to solve an AVI

Two types of behaviors, one positive and one negative, were listed for hypothetical thinking in Sierpinska et al. (2011, p. 294). The third behavior in the list below, HT-initial-inequality-ignored (qualified as negative), was added in the present study since this behavior was observed in the solutions of two participants.

HT-interval-conditions-ignored (negative): The participant demonstrates the intention to use RBC in solving, but does not consider the interval conditions when analysing cases

HT-conditional-statements (positive): The participant uses conditional statements in discussing or writing his or her solutions

HT-initial-inequality-ignored (negative): The participant shows the intention to use RBC, but does not consider the initial inequality when analysing cases

2.5.3 Analytic thinking

Analyzing the solutions to AVI exercises makes it fairly difficult to identify positive symptoms of analytic thinking. Moreover, noticing a participant's sensitivity to formal mathematical notation is not obvious in this type of analysis because one cannot always be sure if the participant copied from the given notes or if the participant actually understood what he or she was writing and did so independently of the notes. Negative symptoms of analytical thinking were more obvious. Thus, four negative behaviors were listed by Sierpinska et al. (2011) as symptomatic of participants' weaknesses in analytical thinking (p. 299):

A-cases-listed: The participant does not combine the cases in at least one exercise and leaves the cases uncombined

A-combining-replaced: In using RBC, the cases are not combined and another method is used to replace this step

A-combining-by-conjunction: The participant combines the cases by conjunction in at least one exercise

A-combining-inconsistent: The participant makes inconsistent use of the logical connectives "and" and "or" within a single exercise

The next chapter is meant to give the reader a detailed description of both lectures and the question periods that followed before describing the six exercises and the participants' solutions.

Chapter 3. Description of the Experiment

In both groups, the teaching experiment started with an introduction where participants were given information on the sequence of activities in the session and the rules of behavior during the lecture (no questions allowed during the lecture in the control group (CG); questions allowed in the experimental group (EG)). The sequence of activities was as follows:

Introduction

Lecture

Question period

Exercise period

Discussion period

Group interview period

The lectures followed identical material and example exercises. The question period followed the lecture in both groups. For the CG, the question period was the first opportunity to ask questions about anything that wasn't clear when listening to the lecture. After the question period, participants in both groups received a set of six exercises to be solved individually. The exercises were the same in both groups. Their instructor was not to answer any questions during the exercise period. The exercise period was followed by a discussion period where participants could present and debate about their solutions to the exercises. In the interview period, participants were invited to share their opinions about the experiment.

In this chapter, more details will be given about how the lectures and question periods were conducted for both groups and the rationale for the more important didactic and pedagogical choices will be given.

3.1 Instructions for participants

The differences in the lectures rested solely in the interaction with the participants. The control group (CG) was instructed at the start of the lecture not to interrupt the instructor at any time for questions or to begin any sort of discussion. Further, participants were advised to mark any places in their booklets where they had questions so that they could ask their questions during the question period. On the other hand, participants in the experimental group (EG) were told that they were allowed to interrupt whenever they did not understand, had any questions, needed the instructor to slow down the pace, or wanted to make any comments on the instructor's explanation or on others' comments. They, too, were given the opportunity to ask questions during a question period following the lecture.

3.2 Instructor's strategy

In EG, the instructor did not leave the initiative of participating to the participants, but actively prompted them to participate many times throughout the lecture. This was done by stopping the lecture at certain points and asking the participants questions about solutions to the exercises being solved. Thus, questions posed to EG were meant to be answered by the participants. On the other hand, questions asked during the CG lecture were purely rhetorical and would be immediately answered by the instructor or the answer would be inferred by further lecturing. So, statements made during the CG lecture were very often reformulated as questions for EG.

The instructor did not always immediately respond to a participant's statement (or question or comment) but would first revoice (repeat louder) and/or rephrase it when necessary in order to clarify or confirm what was being said for the rest of the participants. The revoicing also served to help formulate participants' ideas or thoughts.

Since the participants had already completed the exercises by the discussion period, both discussions were conducted similarly by the instructor. That is, the instructor encouraged discussion and debate with respect to participants' answers with both groups, which is a characteristic of the discursive approach. This was done since the discussion would not be likely to affect the participants' written solutions (they were told not to scratch out any written work). Further, this discussion in both groups was considered as time to be used by the instructor to attempt to extract any information about participants' thinking and solving strategies during the exercise period. Thus, an interactive discussion was considered complimentary to the analysis of both groups' written solutions.

3.3 The lecture

Each lecture began with a brief review on graphing functions, an introduction to the notion of absolute value (AV), and then taught how to solve AV inequalities (AVI) graphically and algebraically. Although the same material was taught to both groups, the way in which this material was presented and taught to the participants was different.

3.3.1 A review on graphing functions

The lecture began with a brief review on graphing functions. This review was considered necessary since it is known that many students who are registered in prerequisite mathematics courses have not covered these topics in a long while. Thus, a review can refresh their memory to lessen the possibility of participants making errors during the exercise period because of a weak background in these skills. This portion of the lecture also serves as a "recall" for those who have seen the material more recently and reinforces it.

The first part of this review consists of plotting a graph of a linear function given its formula. The chosen function was $f(x) = 2x - 1$. A linear function was chosen because the lesson on solving absolute value inequalities was restricted to inequalities with absolute values of linear functions only. The function chosen was also a fairly simple example of a linear function (small integer coefficients) so as to obscure the demonstration of the procedure of graphing with irrelevant details. At the same time, the function was not completely trivial: a coefficient was placed by the x variable and a shift to the right was added to demonstrate how to calculate the value of a function that includes some additional challenges, This demonstration of the graphing procedure and then, in the next example, demonstration of reading the values of a function from its graph (see Figure 1), were intended to encourage participants to use the graphing technique of approaching absolute value inequalities whenever appropriate or helpful in getting a sense of what the solution might be.

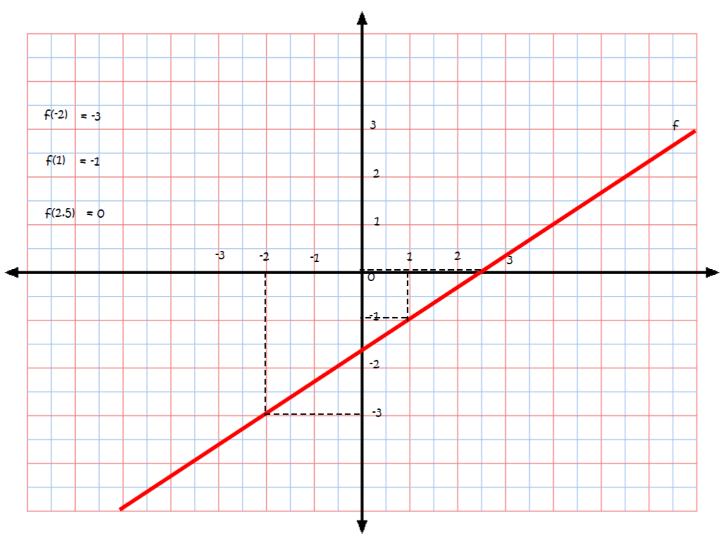

Figure 1. Reading the values of a function $g(x)$

A linear graph was drawn for the participants. The graph was called g and the teacher, planning it, had in mind the function $g(x) = \frac{2}{3}x - \frac{5}{3}$. However, this formula was not provided to the participants.

As mentioned, the instructor constantly probed participants for the answers to exercises being done during the EG lecture. For example, the instructor asks the participants what is the value of g at 2.5[1]. One of the participants, E2, starts off this discussion by offering the correct answer, “zero”, but cannot explain why. So, the instructor hints by reminding the participants of the previously asked question, finding the value of $g(-2)$: “You can think about, maybe, how we found the value of the function at minus two”. After this hint, E4 is able to explain why $g(2.5)$ equals zero: "We can try to draw a perpendicular line straight down to where it intersects the graph and we can’t do that (unintelligible) because it’s already there". Then, the instructor repeats and rephrases E4's explanation with the goal of ensuring that all participants have heard and understood the explanation: "What [E4] has said, is because two point five is a zero of the function, it is already intersecting the x-axis. So, all that’s left to do in this case is draw a line toward the y-axis, which happens to be along the x-axis, so we see that the value of the function is zero, as [E2] said. So, g of two point five is equal to zero. Great". The instructor did this not only to rephrase what E4 said more clearly to the other participants and avoid confusion, but at the same time, to reinforce what E4 appeared to be thinking. Further, the explanation was altered to match how E4 explained his thinking process in a more precise mathematical language. It has to be stressed, however, that the answer and explanation for this exercise were given by the participants and only rephrased by the instructor. Thus, the solution process and explanation

[1] Experimental group transcript, lines 12-18.

were similar to that of the CG lecture; however, the participants in EG were actively involved in the solution process, not passive recipients of an answer given by the teacher.

3.3.2 The notion of absolute value

The second part of this lecture was an introduction to absolute value. It began with an anecdote about three characters, Jane and Joe, measuring the circumference of a coin with a string, and Tom who knows the true value of the circumference. Jane's measurement gives 55 mm, Joe's – 58 mm, and Tom says that since the true length is 56 mm, Jane's error is $56 - 55 = 1$, and Joe's error is $56 - 58 = -2$. Tom concludes that since $1 > -2$, then Jane's error is greater than Joe's. This story aims to introduce the concept of absolute in a way that demonstrates the need for taking the absolute value of a number. It starts by distinguishing the magnitude of the difference between two numbers from the result of subtraction of one number from another. The idea of magnitude of the difference between numbers is then carried to that of the magnitude of a number, given the name of "absolute value", and formalized in the form of the piece-wise functional definition of absolute value. This introduction takes a historical approach to teaching the notion of AV so that participants can be taken through a similar thought process of those who first found the need for taking the AV of a number.

In the first phase of this introduction, after the measurement situation described above was told to the participants, the instructor asked, in both EG and CG, "Do you agree with Tom?" Due to the instructions given to CG, these participants knew that they were not expected to respond and that this was only a thought provoking question which would be answered in one way or another by their instructor. At least, we can assume that they hoped the question would be answered. On the other hand, EG immediately began to offer responses to this question[2]. E2 was

[2] Experimental group transcript, lines 35-41.

the first to respond, “no”. In order to gain more perspective on E2’s reasoning, the instructor asks: “No? And why not?” E2 replies: “Because fifty-six is an exact value and the other two are.... But, how to measure whether the mistake is bigger or not it depends on the difference that is the absolute value”. She later states that “this is the true value”. Due to this interaction, the instructor spontaneously confirmed the answer to the initial question (“do you agree with Tom?”), something that was not done during the CG lecture. Thus, EG obtained an explicit confirmation to the question because of the spontaneous and discursive nature of the lecture.

After this anecdote, the link to the magnitude of the difference between two numbers is made stronger through a visual representation of the number line and the explanation of the inequality $|2 - 1| < |2 - 4|$ (see Figure 2).

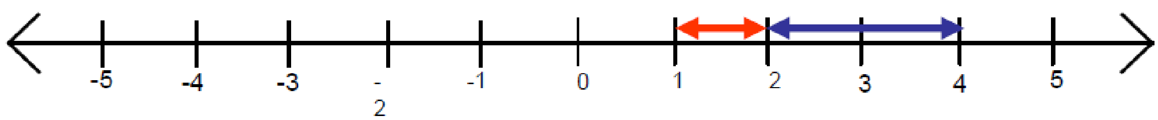

Figure 2. Visual representation of the magnitude of the difference between two numbers

This was done so that participants could answer, or be convinced of the answer, to the question “do you agree with Tom?" The two cases, answer or be convinced of the answer, are purposely stated here because of how this question was approached differently in the two lectures. Hopefully, comparing two absolute values of smaller numbers would provide a simpler example to those participants who still were not sure about the answer to this question. During this phase of the EG lecture, the class was asked for the magnitude of the difference between 2 and 4. Consequently, a student difficulty was encountered when E6 answered “negative two”. The instructor’s reaction to this response was whether everyone agreed with E6's answer[3]. Only

[3] Experimental group transcript, lines 43-49.

one participant shyly said, “I’m not exactly sure that that is minus two. I would say positive two” The instructor confirmed it and explained, “We said that the magnitude of the difference between two and one is one. Now, two minus four … We’re talking about the distance between the two numbers, right? And the distance between the two numbers is the magnitude between the two numbers. So, the distance between two and four is two”. Seeing that there is a problem, the instructor decided to change the order of the instruction by immediately introducing another numerical example[4], “Let’s now consider what the absolute value of a number means. Say we have the absolute value of three. Does anybody know what this would mean in terms of the definition of absolute value?” This example was supposed to follow the function definition of AV, as it did during the CG lecture, but was purposely asked at this point since the instructor saw the need to settle any difficulties before introducing the formal definition. Moreover, this was seen as necessary since certain elements of the function definition can pose many difficulties for students, such as the meaning of “$-x$” and the logical operators (Brumfiel, 1980; Chiarugi et al., 1990, p. 4-5). Therefore, students should understand the simple numerical example before moving to the generalization of this notion. The example was quickly revisited with this group after the definition of AV was given to reinforce this concept. On the other hand, since CG was not permitted to speak, the instructor could not identify any possible difficulties and simply stuck to the planned order of the lecture.

This rearrangement seems to have benefitted the EG participants since it led to another participant interruption by E4: “But, what if it was negative three?” The instructor answers, “if we had the absolute value of negative three, it is still the difference between zero and the number. And the distance between zero and minus three is still three, right?” Due to this explanation, E4 was able to overcome his difficulty and responds, “So, the absolute value

[4] Experimental group transcript, lines 51-62.

symbol means that only positives can come out". This question demonstrates that although the instructor may have thought that the notion was well understood, there were still insecurities held by at least one of the students.

The piece-wise function definition of AV was introduced next; first, in words and then as a function. For negative values of x, the AV of x was said, in words, as equal to the "opposite of x" (not as "negative x", as it is customary to say in mathematics classes) so as to remove the support for the misconception that "$-x$" always represents a negative number. Another visual representation of the number line was given to demonstrate where a number is sent when taking the absolute value (see Figure 3). That is, the absolute value of -2 is 2. Thus, -2 is sent to 2 and an arrow points from -2 to 2. This visual representation was shown to participants to provide a more concrete example of how we operate on x when we take its absolute value.

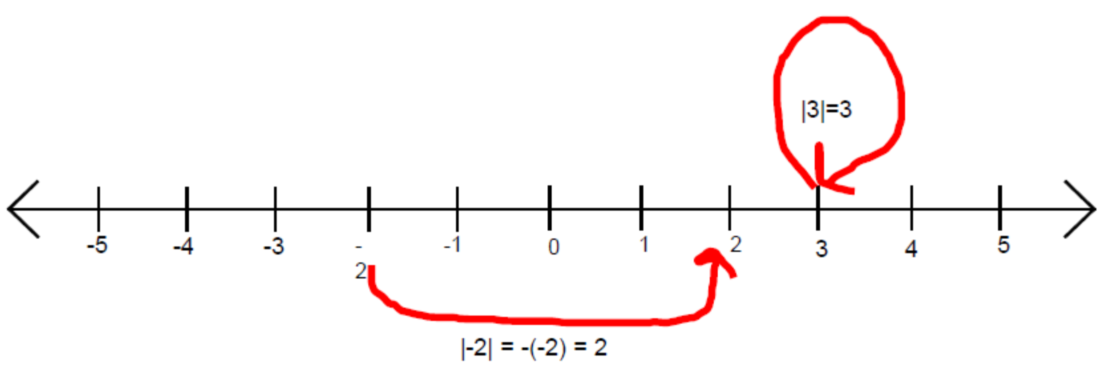

Figure 3. Diagram accompanying the discussion of the piece-wise function definition of AV

Using the function definition explicitly captures the idea that there are two possibilities for the AV of a number x if x is a variable, which could greatly aid participants later in understanding why we have two cases attached to each side of an AVI. Second, using the function definition also aims to help students recognize that they could picture each side of the inequality as a different absolute value function.

3.3.3 Graphing four absolute value functions

The last exercise before introducing AVI was to plot four different absolute value functions; $f(x) = |x|$, $g(x) = |x - 1|$, $h(x) = |x + 2|$, and $k(x) = |2 * x - 1|$. The purpose of this exercise was to review the graphing of AV functions, beginning with the simplest, $f(x) = |x|$. The next two, g and h, were added to show participants how to graph absolute value functions that have been shifted to the right or left. The last function, k, was added in order to highlight the effect of the x-coefficient on the slope of the function. Of course, all four of these functions were drawn on the same Cartesian plane to demonstrate this. The multiplication sign in in the formula of $k(x)$ was represented by an asterisk to highlight the operations involved. This way, an error could more likely be blamed on misunderstandings with the concepts involved with AVI and not on small calculation errors.

While graphing $h(x)$ with EG, the instructor stated: "When x equals zero, we see that zero plus two is two. So, here is our second point and what about when x equals one?" However, the instructor pointed at negative one while asking this and E4 quickly interrupted, "minus one?" to correct the instructor's wording. This interaction demonstrates the importance of permitting student involvement in a lecture and promoting this kind of classroom culture. Moreover, if students were not comfortable speaking up during the lecture, some may have been very confused by the instructor's explanation.

3.3.4 Solving $|x - 1| < |x + 2|$

In the final part of the lecture, the inequality $|x - 1| < |x + 2|$ was solved in two ways, graphically and using the technique of reasoning by cases (RBC). This particular inequality was chosen because it was sufficiently simple (monic linear polynomials with small integers as constants) to allow for a graphical solution and sufficiently complex (absolute value functions on both sides of the inequality) to make reference to the formal definition useful in solving the inequality algebraically. The two functions g and h in the previous graphing exercise were identical with the two functions appearing on both sides of this inequality for a purpose; their choice in the graphing exercise was meant to make the graphical solution of the inequality easier to obtain, since the graphs of both sides were already prepared.

The inequality was first solved using the graphical approach. The absolute value function on left hand side was labeled as $f(x)$ and the right hand side, $g(x)$. The solution was then shown by explaining how to find the point of intersection of the two functions, $x = -1/2$, and then where $f(x)$ is less than $g(x)$.

The rationale for solving the inequality again but using algebra and logic this time was explained by saying, "What happens when we cannot see straight from the graph where the values of the two functions are equal or when one is greater than or less than the other? Well, we are going to use some algebra and some reasoning". The inequality was solved by writing the four possible cases:

$x \geq 1$ and $x \geq -2$ and $x - 1 < x + 2$

$x \geq 1$ and $x < -2$ and $x - 1 < -x - 2$

$x < 1$ and $x \geq -2$ and $-x + 1 < x + 2$

$x < 1$ and $x < -2$ and $-x + 1 < -x - 2$

The cases were then analyzed, and the combination of the conclusions from them was supported by a visual representation of the union of intervals on the number line.

It was stressed that both methods, graphical and algebraic, produce the same solution, thus showing that there is more than one way of solving an AVI, and one can choose the most appropriate one for a particular problem.

One important difference between the lectures in the two groups during this particular phase is that CG was not informed that the cases were derived from applying the AV definition to each side of the inequality. EG, on the other hand, was asked if they knew how the instructor got to these four statements. After E2 responded, “from the definition of absolute value”, the instructor explained the connection between what was written and the piece-wise function definition of AV that was given earlier: “Exactly, so if you recall, we had written the definition of absolute value. We had said that it is x for x greater than or equal to zero and it’s the opposite of x for x less than zero. So, that’s where we’re getting this from. Pretend that this is x [pointing to the function inside the AV bars], it’s equal to x for x greater than or equal to zero and likewise for when x is below zero”. The point here is not that the discussion *allowed* the instructor to inform the participants about this, since this could have been pointed out to CG through teaching only as well, but that promoting interaction with students allowed the teacher to be more spontaneous and ask questions about what was being taught. Perhaps if the teacher would not have asked this question, a participant might have interrupted to ask where the statements came from. Moreover, this point was only brought up to CG when a participant asked about it in the question period, as we will discuss later. All that was mentioned during the lecture by the instructor was: “We have two cases attached to each side of our inequality and now this will give us four cases to consider”.

Another key difference between the lectures here is that the majority of this exercise was solved by the participants during the EG lecture through the instructor's constant probing. For instance, EG was asked for what the instructor should be writing in terms of the cases attached to each side of the inequality and for the conclusions to each of the four cases. For instance, the instructor asked the participants to define $f(x)$ in terms of the function definition: “How about you guys help me writing it for *f* of x? So, the absolute value of x plus two is equal to?" Participant E4 answers by stating, "x plus two", and the instructor continues leading the students in the right direction by probing again: "Yes, and when does it equal x plus two?" In addition, when a contradiction arose, the participants were asked to justify why: "Can anybody else tell me why this [$x \geq 1$ and $x < -2$] is a contradiction?" E5 responded to this question: "Because x can't be equal or greater than one and at the same time be smaller than minus two". Again, we can see that the reasoning behind almost each step was explained by the participants, which helped the instructor determine whether the participants were ready to move on with the exercise.

Another interesting interaction during the EG lecture occurred while the instructor and the participants were considering the first case; $x \geq 1$, and $x \geq -2$, and $x - 1 < x + 2$. The instructor paused and asked, "Let's look at our inequality here, so, can we say anything about this?" E4 answered, "No matter what value you put in for x, when you take one away from it, it's going to be less than if you would take the same value of x and add two to it". The instructor replied, "Great. So, that is a good way to reason and if you wanted to check that, all you had to do was... We had x minus one less than x plus two. If we were to subtract x from each side we would just get that minus one is less than two, which is true". This answer made this interaction an important one since E4's response was not one that the instructor expected: it was a non-algebraic reasoning based on number sense. Offering also an algebraic reasoning allowed the participants of this group to hear two interpretations or reasons why this inequality is true for all x.

3.4 Question period

The question period was intended to give the participants an opportunity to ask any lingering questions they might have had before moving on to the exercise period. The length of this period for the control group was approximately 7 minutes, while the time taken by the experimental group was 11 minutes. Although this does not seem a big difference, the time actually occupied by asking questions in CG was approximately 3 minutes and 15 seconds, while the time occupied by asking questions during the experimental group's session was approximately 10 minutes. Furthermore, CG participants asked only 2 questions, while EG participants asked 7 questions.

In addition, participants in CG appeared not very comfortable when asking their questions. They stayed silent for a long time after being invited to ask questions: "I hope you have some questions that you marked and prepared for me." The instructor ended up asking her own prepared question: "How about I start? Is everybody alright with this? How we got from this here, to this conclusion. That x is greater than minus one half. Everything is okay with that? Yes?" After yet another long pause she then needed to probe participants who seemed confused before any questions were asked by students. In particular, she asked C2: "[C2], you seem like you're pondering a little". He points to his confusion about the cases obtained from the definition, which were not explained to have been derived from the definition to this group, as mentioned before: "I wasn't quite sure how you got the..." The instructor explains: "I got that from the definition of absolute value. I will write it out and we will look at this together. We defined the absolute value of a number x as the following function: x is equal to, the absolute value of x is equal to x for x positive or zero and then we said that it's equal to minus x or the opposite of x for x negative. So, when we look at our inequality, we first start with our function

f and we define it. We say, well, what does x minus one equal to if x minus one were positive or equal to zero? So, if x minus one was positive or equal to zero it fell in this category so the absolute value of x minus one equals to x minus one. Similarly, if it were negative, it would equal to minus x plus one. Is that alright now?" C2, still confused, asks about how $x - 1$ was obtained in the algebraic solution of the AVI: "I was more wondering about the x minus one is greater than or equal to zero. I didn't quite understand where that came from." Here, we can see the confusion that this participant goes through in order to understand the two cases obtained from the function definition of AV. After answering C2's question, the instructor still needed to encourage questions by stating: "Come on guys, I like questions. We like to investigate things together like this". Only one more question was asked by C3: "Do we have another way to solve, like the square root?" Desiring the participants to prioritize the presented solving strategies, the instructor responded, "We want to stay away from that."[5] After answering this question, the teacher waited for a long while as the participants flipped through their papers, but no further questions were asked. Notice, as well, that only one question, the one posed by C3, was voluntarily asked.

On the other hand, participants in EG began asking questions almost immediately and no extra encouragement on the part of the lecturer was required. Further, once the questioning began, the questioning was constant. That is, there were no gaps in time such that the participants were not speaking as they were in the CG question period. The first question was asked by E6, who requested further explanation with respect to the last two contradictions found: "Can you just (explain) that?" The instructor explained the second case in this exercise, "Definitely. Here we see that x is greater than or equal to one and it's stating that it's also less than minus two. So,

[5] The student seemed to be alluding to the square root definition of absolute value $|x| = \sqrt{x^2}$.

let's look at the number line, it will be a lot clearer this way. So, here is minus two and here is one. The first condition says x is greater than or equal to one. The second condition says x is less than minus two. Now, can x be here and here at the same time? No. See? That's where the contradiction arises. You can't have x be greater than or equal to one and x be less than minus two at the same time. So, because we had a contradiction there we disregarded case two." After another elaborate explanation for the fourth case of the exercise, the participant asked yet another question about the second case, which demonstrated a deep misunderstanding, "When you say it is contradictory, shouldn't it be if one of them contradicts against if x is greater than one and also greater than zero? But, imagine if x is minus one. It cannot be bigger than one". The instructor responded, "Well, you see... We are not saying that our value of x has to satisfy both of these at the same time. We are saying that if x is greater than or equal to one, it is definitely going to be greater than or equal to minus two. Okay?" Finally, E6 reached an understanding of this not-so-easy to grasp reasoning.

The third and fourth questions asked by EG were by E1: "At the end are you going to have to do a union of the two no matter what the values are?" After the instructor answered, "Yes", he asked another, very good question: "Every time you're going to end up with two things, you're going to add them up to one?"

E4 asked the fifth question, which pertained to the exercise period, "Will we be allowed to look this over while the…?" Here, we can see that the participant was concerned about whether he would be allowed to use his notes during the exercise period.

The instructor asked the sixth question, referring back to the presented example's four cases:

$x \geq 1$ and $x \geq -2$ and $x - 1 < x + 2$

$x \geq 1$ and $x < -2$ and $x - 1 < -x - 2$

$x < 1$ and $x \geq -2$ and $-x + 1 < x + 2$

$x < 1$ and $x < -2$ and $-x + 1 < -x - 2$

She asked, “We saw [cases] two and four were impossible. Do you think they will always be impossible? I want to refer you back to page eight. So, let’s try to find those cases. Try to find where we’ve got the first function above zero and we’ve got the second function below zero. You can also look for the fourth case. You can look for where they are both below zero”. E5 attempted to answer, “I guess they’re never below zero. Does that answer your question?” This response allowed the instructor to realize that her question may not have been phrased clearly and she answered: “Well, maybe that’s my fault because I didn’t word [my question] properly. I didn’t mean that the function is below zero because they are never below there in this case. What I mean is the slope of the function is negative”. The instructor goes on to explain more clearly, but notices E5 is still confused and says, “I see you are pondering away”. E5 responds, “Well, because I don’t understand what slope has to do with it”. After more discussion, the question was not answered by the participants and the instructor responded to other questions from the participants. However, we will later see that the question was revisited by another participant.

The problem with the instructor’s wording, here, was in the use of the expression “the function … is below zero”. This slip of the tongue may have led participants into thinking that the instructor was referring to the function $|x - 1|$, or to the function $|x + 2|$, while, she was actually referring to the functions $x - 1$ or $x + 2$. Further, using the word “slope” did not address the main source of the ambiguity that the instructor wished to reflect about AVI exercises. Considering these imprecisions in the instructor's choice of words it is, therefore, quite understandable why the participants were confused.

E6 was next to ask, "What would happen if the coefficient is before the absolute value? Do you consider that? Or cancel out the coefficient or just consider that?" Here, E6 is wondering how adding a coefficient other than 1 before the AV bars will affect the solution process.

After this question by E6, the discussion moved back to answering the instructor's previous question. E1 asked, "And number two and number four are always not going to line up because?" The instructor interpreted the question as referring to the second and fourth case as leading to a contradiction, and responded, "It depends what the function is, right? When there is a coefficient in front, it is going to change the shape of the function, the direction of the function. But, in this case, where they were both above the x axis and crossing, then yes, but not always." This revisiting of the instructor's question by a participant is important since it demonstrates the participants' interest and deep thought about the material. At the same time, the wording "cases 2 and 4" may have had a literal significance to the participants and was not clarified by the instructor. That is, the instructor seems to have been unaware that some participants may have gotten the impression the second and the fourth cases would always be rejected, no matter how the conditions would be ordered. By not explaining that the consistency of the cases depends on the interval conditions within each case, this could have caused yet another source of misunderstanding for the participants.

In this chapter, many examples of participant interactions during the EG lecture were discussed and as expected, many more questions were asked by EG during the question period. However, did asking more questions benefit this group in the exercise period? This is what we will look at next.

3.5 Description of the exercises

Participants were allowed the time they needed to complete the 6 exercises since our interest was not in how long it took them to get to their answers, but only in how they arrived at their answers, what approach was taken, and whether this answer was correct. The exercises used in the present study were exactly the same as those used in Sierpinska et al. (2011). The rationale behind the exercises was given in this paper. Here, we only summarize the main points from the article.

Overall, participants' reflective thinking was probed mainly in exercises 3-6 since these exercises were unlike the example solved in the lecture and therefore required an adaptation of the given procedures; analytic thinking – mainly in exercise 6 since this exercise could not be solved graphically; and systemic thinking in exercises 2-6, since all exercises except the first one provided students with opportunities to use the definition of absolute value (definitional thinking), to check their solutions (proving) and to take into account the conditions under which the inequality takes one form or another (hypothetical thinking).

The first exercise asked participants to calculate $\left||16-24|-|7-56|\right|$ in order to check if they had at least an understanding of the "arithmetic notion" of absolute value, for which seeing negative numbers as compound objects made of a number, in the sense of absolute measure, and a sign (plus or minus) is sufficient.

In exercises 2 to 6, participants were expected to solve absolute value inequalities. Hopefully, using the methods that were taught during the lecture or when appropriate, a creative, but still correct method.

Exercise 2 started the participants off gradually with the inequality, $|x-1|<|x+1|$. This exercise served to let the participants start off slow, with an exercise that closely resembled something which had been covered during the lecture. In fact, they had already seen the instructor graph $g(x) = |x - 1|$, the function that is on the left hand side of this AVI.

Exercise 3 challenged them with $|x + 3| < -3|x - 1|$. The multiplication in front of the absolute value bars on the right hand side of the inequality was placed in this exercise to investigate how participants would cope with a problem substantially different from the example given in the lecture. Would they use the RBC technique (which would suggest unreflective thinking), or, the graphical method before concluding that the inequality has no solution? If only a graphical method is used, and a correct conclusion is derived it may be due to some developed intuition that this is a simpler and quicker method in this case. If participants use RBC, would they still understand how to apply the definition of absolute value and also when, or if, they would carry out the multiplication by -3. Solving the inequality requires not a straightforward application of a given procedure but an adaptation of a learned procedure to a more complex situation, since the only example that the participants saw in the lecture was a multiplication of the x inside the absolute value bars, $k(x) = |2x - 1|$. Participants might also first solve the problem using the graphical method and then solve the problem again using RBC. This could happen not only for the purpose of verifying if the graphical solution led them to a correct answer, but because, through my own experience and reading of research, I have found that many students consider a graphical approach less mathematical or a less convincing solution. Research conducted by Dreyfus and Eisenberg suggests that teachers tend to convey that a visual approach is inferior to one that is analytical, whether deliberately or not (Harel, Selden, & Selden, p. 149, 2006).

Exercises 4 and 5 were very similar: Exercise 4. $|2x - 1| < 5$ and Exercise 5. $|2x - 1| > 5$. The absolute value function in these exercises had been graphed in the lecture; therefore participants were not expected to have problems with just the graphing, which could distract them from the main point of these exercises, which was to probe the participants' ability to apply the knowledge presented in the lecture in a creative and constructive way. Both inequalities were different from the example in the lecture by the fact that the absolute value appeared on one side of the inequality only, and the coefficient by x within the absolute value brackets was not 1. Moreover, Exercise 5 contained the "greater than" inequality, which was not included in any lecture examples, besides the example in the review session which graphically compared the values of two linear functions. Therefore, again, the procedures for solving the inequalities learned in the lecture could not be applied directly. The number 5 on the right-hand side of the inequalities could be interpreted as a constant function. This way of thinking about it could facilitate using the graphical method, and also adapting the RBC technique to this particular situation. It is known, however, that students often do not see constants as functions, and this could create an obstacle to applying the knowledge presented in the lecture. The point of using the same absolute value function in the two exercises and thus making the solution to exercise 5 complementary to the solution to exercise 4 was to check if the participants would notice it and display reflective thinking or blindly repeat the same procedure in exercise 5 as they did in exercise 4.

Exercise 6, $|50x - 1| < |x + 100|$, was included to check how participants would adapt to inequalities with larger numbers than those shown in the lecture examples. Because of the larger numbers, solving the inequality with the graphical method is not easy and the result may not be reliable. The RBC technique would be more advisable here. If a participant used the graphical method in exercises 2-5, he or she would have to use a more analytic method in exercise 6, giving us the occasion to probe analytical thinking in the participants.

The next chapter presents the participants' overall performance and discusses their solutions to the above-described exercises in detail.

Chapter 4. Descriptions of Participants' Solutions

After the exercise period, participants were asked for their answers as a group during the discussion period. Their overall performance, theoretical thinking and a detailed overview of their solutions to each question will now be given. Before doing this, however, it is necessary to provide an account on the participants' backgrounds.

4.1 Participants' backgrounds

Participants in this experiment were all enrolled in prerequisite mathematics courses in a large, urban North American University. These courses ranged from Pre-Calculus for Commerce to Calculus 2 and Linear Algebra for Science. The participants were considered qualified at the time of recruitment since they would not have covered AVI before the experiment. In general, however, AVI's are not usually presented using the RBC method or the graphical approach in these courses. This can be assumed from their course material. Thus, even if a participant was taking a course for the second time, it would be unlikely that they would have seen these approaches in the past.

Further, the participants' ages ranged from under 21 years old to over 30 since students registered in these courses come from various backgrounds. For instance, a student can enrol into one of these courses right after finishing college or he or she could be a mature student returning to university. In general, 80% of CG and 50% of EG were 25 years old or younger. Since there were no participants from the age range 26-29, more participants over 30 years of age were in EG.

4.2 Overall performance amongst groups

In this section, we will review participant performance for each group. For now, the percentage of correct answers will be noted, and later, a deeper look into the quality of the responses will be discussed.

Students were awarded a maximum of one point per correctly answered question. Half of a point was given for an almost perfect solution that contained minor errors or if it was evident that the question was copied wrongly, but the participant solved the considered question correctly. Otherwise, no point was given. *Minor errors* in this analysis refer to small computation errors or sign errors. A computational error is considered as, for instance, a participant writing "$2 + 3 = 6$". This is an error that most likely occurs when one is writing too quickly and although this is a careless mistake, most would agree that it is quite obvious that a student at this level knows simple addition and similar manipulations. Moreover, if a participant displayed the ability to make simple calculations before a mistake of this kind, the benefit of the doubt was given to him or her. Similarly, sign errors were considered as minor errors if they could be categorized in the same manner. For instance, a student who forgot to include a minus sign in an equation, when that sign was on the previous line, most probably made a careless error similar to the student who wrote "$2 + 3 = 6$". That is, a student who writes on one line, "$3 + (x + 2) = 4 - 2x + x$" and then on the next writes, "$3 - (x + 2) = 4 - x$" is given the benefit of the doubt to have been skilled enough to know that the addition sign on the left hand side of the equation did not suddenly change to a subtraction sign.

It is important to note that a participant's answer was evaluated not only based on the written answer, but also on what he or she said during the discussion period. Of course, judgment was also important in determining whether or not a participant was giving a thoughtful answer or whether this answer was simply given because it was a previous participant's answer.

A group's score on a particular exercise was calculated as the percent of the sum of points obtained by each member of the group out of the maximum number of points for that group. There were six participants in EG and five in CG. Thus, the maximum that EG, as a whole, can obtain on an exercise is 6. Similarly, the maximum that the CG can obtain is 5. For example, if the EG containing six participants scored 83.3% on a particular exercise, this implies that the sum of the group's "marks" was 5 points out of the maximum of 6 points the group could score on that exercise. It also means that 5 members of this group obtained correct answers and one obtained an incorrect answer.

The coding E1-E6 and C1-C5 represents the six EG participants and the five CG participants, respectively. Table 1 displays the performance results by individual participant, by individual exercise, and by group. The totals per exercise are given in percents since the number of participants in each group is unequal. This allows us to compare the scores. The last row of Table 1 describes the difference in EG and CG averages. Each percent is either attached to EG or CG depending on which group had a higher average.

Both groups obtained 100% on Exercise 1, which suggests that the participants all possessed at least a solid arithmetic background.

In Exercise 2, the control group fared better than the experimental group. Specifically, the EG obtained 75% (4 students out of 6 gave correct answers), while the CG obtained 80% (4 students out of 5 gave correct answers).

On the other hand, the EG achieved far better scores than the control group on Exercise 3. Their score was 66.7%, while the CG only obtained 40%.

The results were reversed for Exercise 4 as the EG was again given 66.7%, but the CG, 80%.

Performance was similar on Exercise 5. The EG obtained 58.3% ($3\frac{1}{2}$ correct answers out of 6), while the CG achieved 80% (4 correct answers out of 5).

The performance of EG and CG was the same (50%) on exercise 6.

Four out of the all the participants obtained perfect scores; E2, E3, C4, and C5. That is, 33.3% of the EG and 40% of the CG participants obtained a perfect score. All three of these participants solved the AVI exercises using methods covered in the lectures (graphical approach and RBC). E2 used the graphical approach to answer exercises 2, 3, and 5 and used RBC to solve 4 and 6. C4 behaved similarly to E2, using the graphical approach for exercises 2, 4 and 5 and RBC for 3 and 6. On the other hand, E3 and C5 applied the RBC method for every AVI exercise.

In total, the group average on all six exercises was approximately 68.1% for EG and 71.7% for CG. In this sense, we can say that CG performed slightly better than EG. The question is if this measure implies that lecturing without interruptions by teacher-student interactions is superior to lecturing in the discursive style.

We surmise that such conclusion would not be justified even if the sample was statistically significant. Aside from the fact that the difference in performance between the groups is minimal, the key difference between the two groups appears to lie less in the percentage of correct answers and more within the ways of thinking and mathematical behaviors displayed by the participants with respect to different exercises. These differences are already hinted at in the significantly higher performance on the non-trivial exercise 3 (the AVI with a contradiction) of EG over CG. To identify those more subtle differences, we will look at participants' responses through the lens of the model of theoretical thinking.

Student	Exercise 1	Exercise 2	Exercise 3	Exercise 4	Exercise 5	Exercise 6	Average (%)
E1	1	1	0.5	0.5	1	1	83.3
E2	1	1	1	1	1	1	100
E3	1	1	1	1	1	1	100
E4	1	1	1	0.5	0.5	0	66.7
E5	1	0.5	0.5	1	0	0	41.7
E6	1	0	0	0	0	0	16.7
EG score (% of 6)	**100**	**75**	**66.7**	**66.7**	**58.3**	**50**	**68.1**
C1	1	1	0	1	1	0	66.7
C2	1	0	0	1	1	0.5	58.3
C3	1	1	0	0	0	0	33.3
C4	1	1	1	1	1	1	100
C5	1	1	1	1	1	1	100
CG score (% of 5)	**100**	**80**	**40**	**80**	**80**	**50**	**71.7**
EG % − CG %	**0**	**-5**	**26.7**	**-13.3**	**-21.7**	**0**	**-3.6**

Table 1. Participants' performance on exercises

4.3 General assessment of participants' theoretical thinking

This section presents the results obtained after using the TT framework to analyze participants' responses. As already detailed in section 2.5, each student's written and oral productions were analyzed for "positive" and "negative" symptoms of each of the three aspects of theoretical thinking: reflective, systemic (definitional, proving, hypothetical) and analytical thinking. Positive and negative symptoms of these aspects of thinking were sought in students' written and oral productions. A student demonstrating a positive behavior in any of the three aspects of TT at least once is given 1 point. A student reflecting a negative behavior at least once is given “-1”. If the participant did not demonstrate a particular behavior, 0 points were given.

4.3.1 Reflective thinking behaviors observed in EG and CG

Table 2 shows the results for each reflective thinking behavior identified in section 2.5.1 by participant and by group. The reflective thinking (RT) totals for each group reflect the balance between positive and negative reflective behaviors in the groups. The balances are 0 for both groups. This could be interpreted as saying that in each group, participants were as likely as not to display positive reflective behaviors at least once during the experiment. The balances do not allow us, however, to distinguish between the two groups. To identify the differences in RT, the groups are compared with respect to positive RT behaviors, negative RT behaviors, and across individual behaviors.

Overall, EG fared better on RT. That is, EG participants scored lower in all procedural behaviors (negative) and higher in all constructed behaviors (positive). Specifically, 50% of EG participants showed at least one procedural behavior, while approximately 67% of CG participants did so. On the other hand, 50% of the EG participants displayed at least one constructed behavior; meanwhile, only 33% of CG did so. This gives EG an advantage of approximately 17% over CG for both procedural behaviors and constructed behaviors. Furthermore, two EG participants (33.3%) showed no signs of procedural behaviors, while 100% of CG participants displayed at least one procedural behavior. In addition, 66.7% of EG demonstrated at least one PB1 and PB2, and an impressive 16.7% showed PB3. Meanwhile, 100% of CG participants displayed PB1, 80% showed PB2 and 20% demonstrated PB3. Also, 33.3% of EG showed all three constructed behaviors, but no one in the CG showed all three of these positive reflective behaviors.

Student	PB1	PB2	PB3	PB Total	CB1	CB2	CB3	CB Total	RT Total
E1	-1	-1	0	**-2**	0	0	1	**1**	**-1**
E2	0	0	0	**0**	1	1	1	**3**	**3**
E3	-1	-1	0	**-2**	0	0	1	**1**	**-1**
E4	0	0	0	**0**	1	1	1	**3**	**3**
E5	-1	-1	0	**-2**	0	0	1	**1**	**-1**
E6	-1	-1	-1	**-3**	0	0	0	**0**	**-3**
EG score (% of 6)	66.7	66.7	16.7	**50**	33.3	33.3	83.3	**50**	**0**
C1	-1	-1	0	**-2**	0	0	1	**1**	**-1**
C2	-1	-1	0	**-2**	0	0	1	**1**	**-1**
C3	-1	-1	-1	**-3**	0	0	0	**0**	**-3**
C4	-1	0	0	**-1**	0	1	1	**2**	**1**
C5	-1	-1	0	**-2**	0	0	1	**1**	**-1**
CG score (% of 5)	100	80	20	**66.7**	0	20	80	**33.3**	**-5**
EG % - CG %	-33.3	-13.3	-3.3	**-16.7**	33.3	13.3	3.3	**16.7**	**5**

Table 2. Percentage of procedural and constructed behaviors amongst groups

4.3.2 Systemic thinking behaviors observed in EG and CG

The scores for systemic thinking are divided into the three categories of behaviors: definitional thinking, proving, and hypothetical thinking. The results are listed in Table 3.

All participants in both groups displayed at least one definitional behavior. So, all eleven participants understood the need for using the AV definition to solve AVI. If we look at the way the definition was applied in all cases, we see that 67% of EG participants (E1-E3, and E5) always applied the definition correctly, while 60% (C1, C4, and C5) of CG did so correctly.

CG showed slightly more proving behaviors overall. In section 2.5.2, we listed the proving behaviors identified among the participants of both the Sierpinska et al. (2011) study and the present study: numerical testing (P-num-test); drawing a graph (P-graph-phys); mentally visualizing a graph (P-graph-mental); reasoning based on structural properties of an AVI (P-struc), and RBC (P-RBC). Therefore, altogether, the EG participants could potentially display $5 \times 6 = 30$ identified proving behaviors, and CG could display $5 \times 5 = 25$ identified proving behaviors. In reality, the EG group displayed only 30% of the potential 30 proving behaviors while the CG group displayed 36% of the potential 25 proving behaviors.

Only one participant out of both groups, E4, demonstrated P-numerical-testing. Thus, 16.7% of the EG displayed this behavior, while none of the participants in CG showed this. Although E4 demonstrated a will to think creatively, he only accomplished to find half of the correct interval, as will be discussed in section 4.4.4.

33.3% of EG participants displayed at least one P-graphical-physical behavior, but, 40% of CG did this. From another perspective, however, all EG participants (E2 and E4) who included drawings of graphs in their written solutions gave correct representations of the graphs under consideration, but only half of CG participants (C4) did so.

At the same time, no participants in the EG demonstrated P-graphical-mental, while 20% the CG did. On the other hand, the CG participant (C2) who represents this 20% did not use his mental representations to solve the exercises correctly. In fact, he claimed to have known that two of his written solutions using RBC were incorrect because he could visualize the desired function’s graphical representations, but did not attempt to solve using this knowledge; yet again, another participant who seems to have been lacking confidence in the graphical approach. C2 explains, “I didn’t plot the graphs. I just know what they look like. I knew the answer. I just tried it this way [RBC] and messed up somewhere, that's it"[6]. The second instance of his insecurity in using his visual representation as a method to prove can be found on line 257 of the CG transcript.

At the same time, 16.7% of the EG participants showed P-structural and 20% of CG students showed this behavior. The participants (E2 and C4) who reasoned using the structure of the AVI and the functions involved did so correctly.

Lastly, 83% of the EG used the RBC method, thus, showing P-RBC and 100% of the CG showed this behavior. In order to consider those who applied this method correctly, the written solutions were considered only up to the setting up of the steps, since many small errors could have been made in combining the interval conditions afterwards. In other words, being able to use the AV functions, interval conditions, and the initial inequality correctly in order to set up the four cases was all that was considered in order to determine which participants used the RBC method correctly. However, it is important to make clear that this was not the method involved to determine whether participants would obtain a point for a correct solution. Now, considering only those who applied the RBC method correctly, 60% of both CG and EG participants who used RBC set their cases up correctly.

Concerning the third aspect of systemic thinking, hypothetical thinking, in section 2.5.2 two negative and one positive behaviors were identified: not taking into account the interval conditions when analyzing cases in RBC (negative; HT-int-c-n-t); ignoring the initial inequality when analyzing cases in RBC (negative; HT-init-inq-ign); making conditional statements in reasoning (positive; HT-cond-stm). Therefore, in EG, there could potentially be $2 \times 6 = 12$ positive and $1 \times 6 = 6$ negative hypothetical thinking behaviors; in CG there could potentially be $2 \times 5 = 10$ positive and $1 \times 5 = 5$ negative hypothetical thinking behaviors. In reality, the

[6] Control group transcript, line 99.

EG group exhibited 25% of the potential negative hypothetical thinking behaviors. No positive hypothetical thinking behaviors (conditional statements) were observed in EG. CG demonstrated only 10% of the potential negative hypothetical thinking behaviors, but as many as 40% of the CG members were found to be using conditional statements in their written solutions. Two EG participants (33.3%) displayed HT-interval-conditions-not-taken, while no CG participants showed this flaw in hypothetical thinking. On the other hand, 16.7% of the EG and 20% of the CG demonstrated HT-initial-inequality-ignored.

Student	**DT**	**P-num-test**	**P-g-phys**	**P-g-mental**	**P-struc**	**P-RBC**	**HT-int-c-n-t**	**HT-init-inq-ign**	**HT-cond-stm**	**ST Total**
E1	1	0	0	0	0	1	0	0	0	**2**
E2	1	0	1	0	1	1	0	0	0	**4**
E3	1	0	0	0	0	1	0	0	0	**2**
E4	1	1	1	0	0	0	0	0	0	**4**
E5	1	0	0	0	0	1	-1	0	0	**1**
E6	1	0	0	0	0	1	-1	-1	0	**0**
EG avg (% of 6)	**100**	**16.7**	**33.3**	**0**	**16.7**	**83.3**	**33.3**	**16.7**	**0**	**-**
C1	1	0	0	0	0	1	0	0	1	**3**
C2	1	0	0	1	0	1	0	0	0	**3**
C3	1	0	1	0	0	1	0	-1	0	**2**
C4	1	0	1	0	1	1	0	0	0	**4**
C5	1	0	0	0	0	1	0	0	1	**3**
CG score (% of 5)	**100**	**0**	**40**	**20**	**20**	**100**	**0**	**20**	**40**	**-**

Table 3. Percent of systemic thinking behaviors amongst groups

4.3.3 Analytic thinking behaviors in EG and CG

In section 2.5.3, four negative symptoms of analytical thinking were identified: not combining cases at all when using RBC (A-cases-listed); replacing the combining of cases by something else (A-combining-replaced); combining the cases using conjunction (A-combining-conjunction), and combining some cases by "or" and other by "and" in a single exercise (A-combining-inconsistent). In that section, we also explained why only negative symptoms were identified.

In each EG and CG, a negative symptom of analytic thinking appeared only once. It appeared in the work of participant E6 in EG and in that of the student C3 in CG. In both cases it was the first of the above-mentioned symptoms. The participants only listed the cases but did not combine them in any way.

Although E2 demonstrated A-combining-by-conjunction in one of his written solutions, he was not penalized for this since he quickly realized his carelessness during the discussion and corrected himself. That is, this participant used the conjunction to combine the two interval conditions for his final answer in exercise 5. However, during the discussion period he stated, "I did write 'and' when it should be 'or' though"[7].

4.4 Detailed description of participants' solutions

In this section, we will discuss the participants' solutions to particular exercises for each group.

[7] Experimental group transcript, line 444.

4.4.1 Exercise 1

Both groups obtained 100% on the first exercise. However, one participant, E4, had not noticed the external absolute value bars and gave -41 as a written answer. Since he promptly corrected himself during the discussion period, he was given credit for this oral answer. In general, answers to Exercise 1 proved that the participants had at least an arithmetic understanding of absolute value.

4.4.2 Exercise 2

Both groups performed fairly well on exercise 2. Four out of five participants in the control group (CG) obtained the correct answer (80%), while four out of six in the experimental group (EG) obtained the correct answer and one got half of a point (75%).

Two of the four CG participants used RBC, while the other two used the graphical approach. The four EG participants answered using the same two methods and in exactly the same proportions as CG.

However, one participant (C2) from the four correct in CG provided an incorrect response during the discussion period. Although the participant solved correctly using the graphical method in his written solution, he then started to solve algebraically and did not finish. He then appeared to lose confidence in his graphical answer and provided an incorrect answer to the group. One possible reason for his incorrect answer during the discussion period could be that he became confused while solving algebraically and, consequently, doubted his graphical solution. Since C2 was the first asked to share his answer with the group, this doubt could not have been spurred by another participant's answer. At first, he provides the answer "x bigger than one", which could have been a slip of the tongue for the correct answer, x greater than zero, since he did have this written down. However, when asked to come to the blackboard to solve the problem, he hesitated and said, “I think I got it wrong”. Therefore, it is more likely that he lacked confidence in his graphical solution. There are a few possibilities for C2 not trusting his graphical solution. One could be that he perceived his graphical solution as insufficient. This negative perception of his graphical solution is exactly what was discussed in the research of Dreyfus and Eisenburg discussed by Harel et al. (2006) and why their research is so interesting for this experiment. Another reason for this lack of confidence could be that he did not think this type of answer would be accepted by the instructor. Nonetheless, he was given a point for having the correct solution in his notes.

One of the participants who obtained a full grade for this exercise, C5, made a small error in her written solution. However, she promptly corrected herself during the discussion period, as E4 did for exercise 1. The following exchange displays this realization:

> C5: I think my answer is wrong. There is no solution.
>
> INSTRUCTOR: Okay. [C5] which one was your answer?
>
> C5: My answer is the third one.
>
> INSTRUCTOR: This one here? So, why do you now think that there is no solution?
>
> C5: I got this because I also did the 4 steps and there is just one step that has the possible solution. But I just say x, like my answer is x between one and three and another condition should be satisfied, which is x smaller than minus three. So, it should satisfy both, so there is no solution.
>
> INSTRUCTOR: Okay, so you didn't notice the contradiction?
>
> C5: Yes.

On the other hand, one participant, E5, made a minor error in isolating x in the expression “$-x+1<x+1$.” When dividing by -2, she appears to have forgotten to reverse the inequality “<” to be “>”. This caused her to incorrectly conclude that the conjunction of the conditions in case 3 ($x<1 \wedge x \geq -1 \wedge -x+1<x+1$) gives $-1 \leq x<0$ rather than $0<x<1$. This was considered a minor error since the participant correctly reversed the inequality when dividing by a negative number in case 2 of the same exercise. E5 was, therefore, given half of a point for exercise 2.

Both participants (C2 and E6) who obtained incorrect answers for this exercise attempted to use the RBC method. C2 came to correct conclusions for the first two and last cases, but ignored one interval condition in the third case and, therefore, came to an incorrect conclusion for case 3. Although C2's answer for the exercise is incorrect, it correctly follows from the final conclusions of his cases. The second participant who obtained an incorrect answer (E6) left out the interval conditions when applying the AV definition to both AV functions involved in this exercise, thus writing:

$$|x-1| = x-1$$

$$|x-1| = -(x+1)$$

As can be seen, ignoring the interval condition was not the only error made; E6 also incorrectly writes "-(x+1)" on the second line instead of "-x+1" or "-(x-1)". After disregarding the interval conditions when applying the definition of AV, E6 proceeds to consider only the interval conditions when looking at the four cases. In other words, the participant left out the initial inequality in the four cases and considered only the interval conditions, giving the four cases:

1. $x \geq 1$ and $x \geq 1$
2. $x \geq 1$ and $x < -1$
3. $x < 1$ and $x \geq 1$
4. $x < 1$ and $x < -1$

Once again, we can see that many other errors were made besides those mentioned.

This participant made the similar errors in exercise 3. Therefore, he will not be discussed again until exercise 4.

4.4.3 Exercise 3

Two out of five CG participants obtained the correct answer for exercise 3 , while three out of six in the EG obtained the correct answer and two participants were given a half point . Both participants (C4 and C5) who obtained a full grade in CG used RBC. On the other hand, two out of the three (E2 and E4) from EG used the graphical approach, while the two who got half of a point (E1 and E5) used RBC. E1 applied the RBC method as well and was given half a point for an almost correct answer due to a small error in combining the interval conditions in the fourth case. This was considered a minor error since he was able to combine correctly in previous cases. Although her written solution contained similar errors to those made in exercise 2, E5 obtained half of a point because when solving on the board, she did so correctly and with minimal input from the instructor[8].

All three CG (C1-C3) and both EG participants who were incorrect for exercise 3 used RBC. C1's errors laid in the combining of the interval conditions in two cases. For example, in case 1 of this exercise he claimed that the combination of $x \geq -3$ and $x \geq 1$ and $x < 0$ is $-3 \leq x < 0$, rather than concluding that the conditions are contradictory. C2 began by incorrectly copying the exercise and made several errors afterwards. C3 also made several errors within his written solution. A few of them were: incorrectly applying the AV definition to the two considered functions, incorrectly manipulating the interval conditions, and not taking the initial inequality into consideration when looking at cases. E5 and E6 made similar mistakes to those made in exercise 2.

[8] Experimental Group transcript, lines 342-385.

4.4.4 Exercise 4

Four out of five CG participants attained correct answers, while three out of six in the EG obtained the correct answer and two students were given half of a point. Three of the CG participants (C1, C2, and C5) used RBC and only one used the graphical approach (C4). All three EG participants (E2, E3, and E5) who were allotted a full point used RBC, as did one participant who received a half point (E1). The second participant, E4, who got a half point, used numerical testing. E1 was given half of a point once again for a similar error to the one he made in exercise 3. He was invited to the board to solve this exercise and did so successfully, but with the help of the instructor. Thus, his grade was not increased to a full point. E4 was only awarded half point since his method only considered one portion of the answer to this exercise. In particular, his answer was "$(-\infty, 3)$", but the full answer should have been $(-2, 3)$.

The only participant who was incorrect in CG (C3) used RBC, while the only participant who received “0” for EG (E6) used an incorrect method. C3 made similar errors to those he made in exercise 3. Therefore, he will not be discussed again in this section. E6’s error, however, was interesting since he abandoned the RBC method suddenly and decided to equate $2x - 1$ and 5 instead; consequently, he came to $x = 3$, which he then used to derive a final answer of $x < 3$.

4.4.5 Exercise 5

Again, four out of five CG participants had correct answers, while three out of six in the EG obtained the correct answer and one participant was given a half point (58.3%). The same CG participants were correct for this exercise as for exercise 4. As is probably expected, C1, C2, and C5 used RBC and only C4 used the graphical approach. It is interesting to note that the three using RBC did not use elements in their solutions from exercise 4 in order to facilitate their solving this exercise. Only C4, who used the graphical approach, used the graph she drew in exercise 4 to answer exercise 5. Out of the three correct EG participants, two used RBC (E1 and E3), while one (E2) used the graphical approach. Both E1 and E3 did not solve this exercise using information from their solution to exercise 4. In contrast, E2 used the graph from exercise 4 to quickly solve this exercise. The same participant (E4) who used numerical testing in the previous exercise was given a half point since his answer followed from that in exercise 4. Moreover, although his answer was still incorrect, this participant acknowledged the relationship between exercises 4 and 5 and used this awareness to quickly solve exercise 5.

The three participants with incorrect answers for this exercise are: C3, E5, and E6. As mentioned, C3 and E6's errors will not be discussed here since they are analogous to those made in exercise 4. In fact, E6 wrote the identical expressions for both exercises and came to the exact same answer. What is interesting to notice is the error made by E5 since her answer for exercise 4 was correct. Moreover, although she obtained an interval answer for the previous exercise, she wrote that there is no solution for the complementary AVI. This might suggest a misconception about the disjunction ("or") operator (identifying it with conjunction) or the union of sets (identified with intersection).

4.4.6 Exercise 6

Only two CG students obtained full grades and one was given a half point for the last exercise; also, half of the EG obtained correct answers for the last exercise. All five participants (C4, C5, E1-E3) with correct answers used the RBC method, which was expected given the numbers involved in this exercise. C2, who was given half of a point, noticed that he had an incorrect answer because he could visualize the graphs of the two AV functions involved. However, this participant did not provide the correct answer upon acknowledging his mistake.

Therefore, he was not given a full point. All participants who made errors used the RBC method, except for one (E6) who used equations, as he did in the previous two exercises.

Up to this point, we have discussed the results and described the participants' solutions in depth. In the next chapter, the results will be discussed in more detail and then a discussion of the results will be in order.

Chapter 5. Live teacher versus pre-recorded audio slide presentation

The results presented in Chapter 4 will now be compared and contrasted with those of Sierpinska et al. (2011) whose lectures were presented to each individual participant through PowerPoint slides and audio recordings on individual computers. It is of interest to compare these results since the debate over computer versus classroom settings for the teaching mathematics courses is becoming more and more prominent among prerequisite mathematics courses in North American universities.

These results are comparable since the participants were recruited from the same types of prerequisite courses as Sierpinska et al. (2011) and also because the same visual approach (VA) lecture conducted in Sierpinska et al.'s experiment was presented. This lecture was selected among the three used in their experiment since it produced the highest student success compared to the procedural (PA) and theoretical approach (TA) lectures, both in terms of performance and theoretical thinking. Each group in Sierpinska et al. consisted of six participants, allowing for a reasonable and fair comparison to VA.

In the discussion below, we will label the group treated with the VA lecture in Sierpinska et al. (2011) simply as "VA", and the experimental and control groups in the present research – which were both treated with a visual approach as well but with a live teacher – as "EG" and "CG" as before.

5.1 Performance comparison of the VA, EG and CG groups

In terms of the participants' overall performance on the exercises, Sierpinska et al.'s (2011) VA group obtained an average of 72.2%, which surpasses CG by 0.5% and EG by 4.1%. Interestingly, EG and CG both surpassed the group averages of the PA and TA groups in

Sierpinska et al. Although VA performed slightly better than the EG and CG, the differences between these averages are not very substantial. However, all three groups (VA, EG, and CG) performed better overall than PA and TA. Thus, the results of the present study are consistent with the results of Sierpinska et al. across both settings (computer and classroom).

Comparing the averages for individual exercises amongst EG, CG, and VA, the results become more interesting. In particular, both EG and CG obtained 100% on exercise 1, while VA obtained 66.7%. Thus, both groups in classroom settings performed better on exercise 1. With an average of 83.3%, VA performed better than CG (80%) and EG (75%) on exercise 2. Here, we can also see that both groups which did not have interaction during the instruction showed higher success on this exercise. On the other hand, both VA and EG obtained 66.7% on exercise 3, while CG obtained only 40%. Thus, VA and EG maintained a higher standard than CG when faced with an unfamiliar and non-trivial problem. The results reversed once again for exercises 4 and 5, with VA at 83.3% for both, CG at 80% for both, and EG at 66.7% for exercise 4 and only 58.3% for exercise 5. This further demonstrates a higher performance among participants who had no interruptions during their lectures. All three groups had the same average on exercise 6, 50%, which suggests that regardless of the setting or the number of interruptions during a lecture, a problem of this kind produces a 50% success rate on average.

5.4.2 Reflective thinking comparison of VA, EG and CG groups

EG demonstrated less negative reflective thinking behaviors and more positive behaviors than CG. However, VA displayed procedural behaviors less often than both EG and CG and more constructed behaviors than both groups. That is, 27.8% of the total potential procedural behaviors were demonstrated by VA, as opposed to 50% for the EG and 66.7% for the CG. In addition, 61.1% of the total potential constructed behaviors were identified in VA, in contrast to 50% for the EG and 33.3% for the CG.

With respect to the individual procedural behaviors, VA displayed PB1 (solving exercise 3 using the same technique as in exercise 2) 16.7% less often than EG and 50% less often than CG. The differences for PB2 (repeating steps in exercise 4 to solve exercise 5) were the most surprising since VA showed this behavior 50% less than EG and 63.3% less often than CG. VA's behavior was similar to EG and CG for the last procedural behavior, however, with occurrences of PB3 (using elements of the lecture in a "ritualistic" manner), 16.7% of the time.

Again, VA's demonstration of constructed behaviors surpassed EG and CG for CB1 (using a quicker method than that used in exercise 2 to solve 3) and CB2 (using the relationship between exercises 4 and 5 to solve 5 quicker), but was similar with respect to CB3 (including only necessary elements in a written solution). In particular, 50% of VA participants showed CB1, while 33.3% of EG and 0% of CG did so. At the same time, 50% of VA displayed CB2, but only 33.3% of EG and 20% of CG did so. Further, 83.3% of VA and EG demonstrated CB3, while 80% of CG did so.

5.4.3 Systemic thinking comparison among the VA, EG and CG groups

Since the TT model used by Sierpinska et al. (2011) had two categories of definitional thinking (DT-w and DT-o), as was explained earlier, those results were added to match the one category, DT, in the present study. After these calculations were made, 16.7% of VA demonstrated definitional thinking, but 100% of both EG and CG showed this type of thinking.

Many more participants in VA relied on numerical testing than EG and CG. Specifically, 50% of the VA participants showed this behavior, while only 16.7% of EG and 0% of CG did so.

This can either suggest that participants in VA were more open to finding different solving strategies, or had less of an understanding of the presented approaches.

The results for P-graphical-physical were fairly close among the three groups, with 33.3% of both VA and EG and 40% of CG displaying this behavior in their written solutions. On the other hand, VA and CG were more likely to visualize graphs of AV functions, with 16.7% of VA, 20% of CG and 0% of EG demonstrating P-graphical-mental.

EG was found to be less likely than both VA and CG to solve exercises using the structure of the functions or the AVI since 16.7% of this group demonstrated P-structural, while 20% of VA and CG did so.

The most surprising results with respect to the proving behaviors were those observed for P-RBC since only 16.7% of VA were noted as using RBC, but 83.3% of EG and 100% of CG used this method at least once in their written solutions. This result is extremely curious since all three groups were presented the same lecture and will be discussed in more detail later.

With regards to hypothetical thinking, both VA and CG did not demonstrate the negative hypothetical thinking behavior referred to as HT-interval-conditions-not-taken, while two participants in EG (33.3%) displayed this negative symptom of hypothetical thinking. In addition, neither VA, nor EG displayed the positive hypothetical thinking behavior, HT-conditional-statements; however, 40% of the CG did. Since the TT model did not include the hypothetical thinking behavior, HT-initial-inequality-ignored, this will not be discussed in this section.

5.4.4 Analytic thinking comparison among the VA, EG and CG groups

Although VA has proved to have some advantage over EG and CG in many of the categories of behaviors up to this point, this group displayed more instances of lacking the necessary analytical thinking to properly solve AVI's than EG and CG combined. Moreover, VA displayed 2 negative symptoms of analytic thinking, while in EG and CG, a negative symptom of analytic thinking appeared only once in each group; hence, slightly less than VA.

This chapter has listed many thought-provoking results. In the next chapter, these results and the research questions will be discussed.

Chapter 6. Discussion

In this chapter, we will discuss the participants' questions during the question period, their performance on the exercises, and their theoretical thinking.

6.1 Discussion of participants' questions during the question period

Beginning this discussion with the first possible comparison of the two groups, their interaction with the instructor during the question period, we are already able to notice a difference between the two approaches to teaching. There appears to be a difference in the scope and amount of questions asked by EG and CG.

In particular, CG's questions tended to be general, non-specific questions about the material. For example, C3 asked "Do we have another way to solve like the square root?[9]" Here, we can see that this participant is asking a question that can either be considered as trying to find a way to avoid the methods shown in the lecture, or as trying to be reflective about the two solutions given in the lecture for the AVI. Since C3 solved exercise 2 using the graphical approach correctly, but then lost faith in it and used RBC for the rest of the exercises, this suggests that he was insecure with the methods shown and was curious to know whether he could use a technique that he was more comfortable with. Further, C3 did not trust his graphical solution, as was already discussed, and decided to use RBC all the way through the rest of the exercises even though he could not complete one exercise using this technique. Further, C3 did not continue his solutions past his incorrect setup of the four cases. In addition, he drew a graph in exercise 3 (incorrectly), and did not use it in his solution. Thus, again, reflecting this participant's perception that graphical solutions are insufficient or unreliable.

[9] Control group transcript, line 59.

On the other hand, participants in the EG tended to ask more focused or specific questions. For instance, E6 asked "what would happen if the coefficient is before the absolute value?[10]" The first point about this question is that it is very focused on one detail or possibility that could arise in the future. Thus, the participant is demonstrating that he is not only concerned with understanding the presented AVI example, but also with being able to solve other types of exercises. It can be argued that E6 had looked over the list of exercises in advance and was preparing himself for the exercise period. However, many specific questions were asked by EG participants. Therefore, it is reasonable to assume he had not seen exercise 3 in advance. The other point to be made about E6 is that he was the lowest scoring participant out of both EG and CG. So, the fact that these types of very detailed questions were posed by all levels of participants in EG suggests that the discursive approach did aid students in reflecting about the material in a more focused manner; thus, being able to ask relevant questions that would benefit them in the exercise period.

As we can see, the questions asked by participants in EG were very detailed and pointed to specific difficulties they had with the solution to the presented AVI or curiosities about the possibilities for future exercises. On the other hand, CG participants were more likely to ask more general questions having to do with different solution methods, or even the origin of the four cases (C2)[11]. These different questions suggest that not permitting students to ask spontaneous questions during the lecture may have left some CG participants with a rather superficial understanding of the presented concepts and solution methods. So, rather than being able to focus on understanding the solution process, they were possibly left stuck on particular uncertainties that should have been cleared up on the spot. Nevertheless, we could see from the

[10] Experimental group transcript, line 174.
[11] Control group transcript, lines 50-57.

results that these “uncertainties” did not dramatically affect their performance ability. However, this may have still affected certain aspects of their TT, which some may argue is even more pertinent.

This difference in the amount of questions posed also suggests that there was not as much deep thought being provoked among CG as in EG. For example, the last question by a CG participant had to do with another solving process, suggesting that some participants may have been insecure with the solving methods given. This leads us to an obvious question: could this have been avoided and could their understanding have been enhanced if the CG participants would have been permitted to ask questions as soon as they experienced a misunderstanding?

Recall the confusion displayed by E6 with respect to the contradictions in the lecture example as he asked for a re-explanation. This confusion was not displayed by any of the CG participants. This raises the question of whether all participants in CG understood why the cases were being discarded or whether they just did not want to ask for further explanation.

Furthermore, these participants were part of a classroom setting that could cause students to feel as though their questions are not important to the teacher or the teacher is not interested in their understanding, but only in finishing the lecture. On the other hand, the EG participants were part of a classroom culture where discussion, comments, and questions were constantly welcomed and encouraged. Therefore, the EG participants were possibly not only more comfortable asking questions because of the teaching approach applied during their lecture, but were also more likely to feel their understanding was important to the instructor. In other words, the discursive approach may have been more effective in reflecting the instructor’s interest in the participants’ understanding, while by using the non-discursive approach may have given off the impression that the instructor was indifferent to their learning and success.

In contrast, one may argue that it appears as though the EG participants demonstrated more difficulties due to the larger amount of questions asked by them. However, this is taken as a positive result of the EG lecture since the amount of questions EG asked also reflects that they were aware of what they did not understand. Furthermore, the EG participants were able to deal with their insecurities by asking the right questions.

6.2 Discussion of participants' performance

Although the difference in performance scores between EG and CG is not large enough to make any categorical claims, the performance of EG on AVI exercises was less scattered than that of CG. We will say that EG demonstrated a more "consistent" performance than CG. Moreover, EG maintained their average for individual AVI exercises between 50% and 75%, while CG's average ranged from 40% to 80%. Specifically, participants in CG performed better on the "simpler" problems (average of 80% on exercises 2, 4, and 5), but their grades dropped significantly when faced with the more unexpected problems in exercises 3 and 6. In exercise 3, where the absolute value function on the right hand side of the AVI was multiplied by -3, making the inequality inconsistent, the CG average dropped to 50% (30% decrease). In exercise 6, where large numbers were used in the place of the usual smaller numbers like 1, 2, and 3, the average dropped to 40% on exercise 3 (40% decrease). On the other hand, EG's average of 66.7% was maintained for exercises 2 through 4 and only dropped to 58.3% on exercise 5 (8.4% decrease) and went down to 50% on exercise 6 (16.7% decrease). This suggests that although CG had a slightly higher overall average, they were not prepared to tackle unexpected problems as well as EG. It also seems as though EG was provoked to think in different ways than CG, since the groups excelled on different exercises.

The multiplication of an absolute value function by a negative number (-3) in exercise 3 appeared to be a huge obstacle to some participants when looking over their written solutions. One EG participant could not manipulate the -3 correctly, and two CG participants had problems with this. In EG, E6 simply ignored the negative factor and tried to solve the AVI without its disturbance. In CG, C2 and C3 did take -3 into account but made mistakes when manipulating with this number. One explanation for the fact that more students had problems with the negative factor in CG than in EG could be found in the questions which were asked by each group during the question period. As was discussed in the last subsection, E6 asked a question referring directly to the case where there is a number outside of the AV bars. Since this specific question was asked, it may have helped students in EG to be prepared for AVIs such as the one in exercise 3.

The next question that needs to be answered is: what could have benefitted CG in order to outshine EG on exercises 2, 4, and 5? One possibility is that the constant interruptions of participants' questioning or comments may have been a source of distraction for particular students. Further, some EG participants may have been in critical points in their learning process when an intervention could have been made, causing a "gap" or void in their understanding.

Advocates of the discursive approach sometimes give the impression that letting students ask questions during the lecture is enough to promote better learning. This idea was proven to not be true during the experimental group's question period. Judging from the resulting interactions to the instructor's question "Do you think [cases two and four] will always be impossible?", it seems as though the effectiveness of the discursive approach depends on the quality of the instructor's responses to students' questions. Referring, again, to the imprecision in the instructor's use of "the function … is below zero" and "slope", it can be argued that the instructor may not have noticed what she said caused confusion, may not have remembered her slip of the tongue, or may not have interpreted the participant's question correctly. Therefore, an instructor's explanation of a concept may be more complicated than the concept itself and result only in obscuring the concept. This can occur to any instructor because students' questions are often awkwardly formulated and it is not always easy to understand what they mean. Researchers have the time to look over the transcripts, think about a student's questions, make informed conjectures about its intended meaning, and then dwell on the interaction to figure out what the most appropriate response should have been. But, it is very different when one is in the middle of the action, in class, thinking about what he or she will do next in the lecture, and suddenly a student comes up with a badly formulated question, interrupting one's train of thought. Teachers require many years of experience, presence of mind, and knowledge of students' language in order to understand all students' questions on the spot. Thus, the discursive style might produce more confusion on the spot for both the students and teacher.

All in all, it is clear that these participants may have been swayed to think differently with respect to each exercise. Now, we will discuss these differences in terms of the theoretical thinking model and attempt to uncover whether the discursive approach was in fact more advantageous to the learning of participants. We will also comment on the comparison to the VA group in order to discuss which teaching setting may be more beneficial to student learning.

6.3 Discussion of participants' theoretical thinking

Although EG was found to show much more reflective thinking than CG, participant E6 demonstrated a great lack of reflective thinking in exercises 4 and 5. As was previously discussed, E6 wrote identical equations, $2x - 1 = 5$, and obtained the exact same answers for each of these exercises, $x < 3$. It appeared as though E6 abandoned the RBC technique when he experienced difficulties in applying it. His converting to using equations and ignoring the AV bars could have extended from two possible sources: his past algebra courses and learning to solve for x in linear equations, or the lecture phase where the intersection point of two functions graphs was sought. Additionally, this participant acted as if believing that the first thing to do when dealing with absolute value equation or inequality is to remove the AV bars, a didactic obstacle highlighted by Gagatsis and Thomaidis (1994).

Recall that EG not only outshone CG by demonstrating less procedural behaviors and more constructed behaviors, but 33.3% of EG did not demonstrate any procedural behaviors and all of CG demonstrated at least one. Further, 33.3% of EG displayed all three constructed behaviors, but none of the CG participants did this. Thus, there appears to be a connection between the teaching approach and reflective thinking. Some may argue that it is not reasonable to claim that EG's significantly higher scores for reflective thinking are due to the discursive approach to teaching since Sierpinska et al.'s (2011) VA group achieved higher scores than EG in this category of thinking. Moreover, VA did not interact with an instructor as EG did, so, we cannot conjecture that the discursive approach promoted reflective thinking. However, VA participants were given one-on-one time with an instructor after the audio-recording in order to ask any questions they had before their exercise period. Perhaps, these individual interviews (or question periods) gave VA an advantage over CG because they could ask any questions they had without the pressures that could be added by being in a collective question period, where some students may feel less comfortable asking their questions. EG also had a collective question period, however, the culture of this group had already been transformed so that the EG participants were more at ease posing questions and being open about their insecurities, as we have already discussed.

Looking back at the results for definitional thinking, 60% of CG and 67% of EG applied the definition of AV correctly in all cases. This suggests that the discursive approach could have led to a better understanding of the definition of AV, and therefore, a more successful application of it in the exercises. However, it is also important to acknowledge that all 11 participants in the present experiment reflected definitional thinking at least once in their written solutions. In contrast, only one member of the VA group displayed definitional thinking. With such a large difference in proportions, it is very likely that the classroom setting, regardless of the teaching style, is much more effective than the pre-recorded audio lecture for promoting definitional thinking. Referring back to the lectures, both EG and CG were presented the definition of AV by a live person, which could have more forcefully impressed the definition upon a student's mind than a definition that is only written and read over a recorded lecture. Further, the instructor of the EG and CG lectures repeated the definition multiple times. When introducing the definition for the first time she states, "we're going to define the absolute value of a number x as being x for positive x and as being equal to the opposite of x when x is negative, or negative x [said orally]. Let's define the absolute value of a number x by the following function [now, writing on the board]: the absolute value of x is equal to x for positive x or zero and it's equal to negative x or the opposite of x for negative x. Like I said before, we're going to assume that the absolute value of zero is zero. So, we can see that the absolute value of a number is thus never negative." Later the instructor recaps, "The absolute value of a positive number is equal to itself and the absolute value of a negative number is equal to its opposite." However, in VA, the definition was read only once and the formulation was very close to the written formula. Further, a live instructor can potentially display more emotions and be more animated when speaking directly to students than one who is reading for an audio-lecture. This may also influence students' mental stimulation and learning.

The discursive approach could also be responsible for promoting the correct use of a physical graph in participants' solutions. Moreover, all EG participants used this proving method correctly, but only half of the control group participants used a graph to correctly solve an inequality.

It is also interesting that all CG participants used RBC at least once, but one EG participant, namely E4, did not use this method at all. In particular, E4 did not solve exercise 6, which leaves us wondering what approach he might have taken. On the one hand, one may argue that E4 was a strong reflective thinker, which is definitely suggested by his score, and he would have probably been creative enough to use another way of reasoning to solve the last exercise due to his explicit avoidance of this method. On the other hand, one could argue that E4 did not understand this technique and avoided RBC. However, as was already mentioned, E4 admitted to avoiding the use of RBC because it seemed very long and tedious. Therefore, it seems reasonable to assume that this participant' reason for avoiding this particular technique was not due to confusion.

Further, EG participants had a higher success rate with RBC than CG, even though they were found to apply the RBC technique less often. Going back to the graphical approach, it was noted that all solutions using this method led to correct answers. However, it is important to remember that one participant, C3, had no confidence in his graphical solution and switched to RBC. It is difficult to say whether he did so because he thought this solution method would be unacceptable, or whether he was unsure about his own accuracy in using this method. It is possible that C3 was not sure how accurate his graphical representations were and needed to use another method to verify his answer. What is significant here is that EG participants were not only more successful in using RBC, but also appeared more confident in the graphical method since none of the participants in EG doubted the approach. In fact, E4 admitted that he felt more desire to use graphs to solve than RBC, as explained earlier.

The comparison of VA to both CG and EG for the proving behavior using RBC makes this discussion even more interesting. That is, 10 out of the 11 participants in the present experiment applied the RBC technique, but only one member of the VA group did so. Although we have made previous comments about the positive aspects of students diverging from this approach for particular exercises, there also appears to be something significant occurring when most participants diverge from the presented method and this cannot simply be ignored. It can be, therefore, argued that there is a strong association between classroom settings and proving behaviors.

On the other hand, it can also be said that EG and CG were more reliant on the RBC technique and did not reflect on different solving strategies that could have been more time efficient. It is important to keep this possibility in mind since VA still maintained their performance average and, in fact, did slightly better performance-wise. To further this point, the VA group was found to do numerical testing much more often than EG and CG combined, suggesting that this group was more reflective and creatively exploring other solving strategies. Nevertheless, VA was still using a method that is not commonly encouraged by mathematics educators and was still avoiding the two possible methods taught during the given lecture.

All participants in EG and CG used proving methods presented during the lectures, except two participants, E6 and E4. E6 incorrectly equated the two functions given inside the AV bars. This procedure could have been translated from the portion of the lecture on calculating when two linear functions have the same value. E4 used numerical testing in order to avoid the tedious steps involved in the RBC technique. As well, it is interesting to note that E4 used the graphical approach, but thought that it was not a presented method[12]. This shows that EG participants may have had too many distractions and may have forgotten some fundamental aspects of the presented lecture.

The difference in the types of hypothetical thinking behaviors demonstrated by both groups is even more curious. We have already noted that 33.3% of EG demonstrated a lack in analytical thinking at least once and did not demonstrate any instances of conditional thinking, while 20% of CG displayed a lack in hypothetical thinking and 40% showed at least one instance of using conditional statements. Thus, CG scored higher in both the positive and negative aspects of hypothetical thinking. These differences among groups could be explained either by the EG's lecture interruptions or could be due to participants' backgrounds in mathematics. That is, we can make conjectures about the lecturing styles, but we still do not know the participants' backgrounds and previous experiences in mathematics. Thus, their sensitivity to mathematical notation may have been poor before this experiment was conducted and is not a product of this experiment. However, this is the case for all aspects of TT.

Another result that strengthens the possibility of the classroom setting being more advantageous to students' learning comes from comparing the episodes observed of weaknesses in analytic thinking. Moreover, it was mentioned that VA was found to have demonstrated more weaknesses in this thinking than EG and CG combined –16.7% versus 9.8%, or 5% and 4.8%. Since we could not obtain students' grades in mathematics, and all three groups' performances

[12] Experimental group transcript, line 209.

were fairly close in average, this suggests that the setting in which the participants were presented the lectures had an impact on their analytic thinking.

6.4 Discussion of the results in the light of the research questions

Although these results did not provide sufficient information to confirm the main conjecture of this experiment, many interesting points were uncovered that all added up to help us answer our research questions. In all, performances were similar in terms of average scores, but the experimental group's performance was more consistent. Further, the experimental group clearly outshone the control group in terms of reflective thinking, but the control group demonstrated more systemic thinking. The experimental group correctly applied the graphical and RBC techniques more often than the control group. Further, both demonstrated identical weaknesses and similar frequency in the weaknesses in analytic thinking. Therefore, it is difficult to place one approach above the other since each appears to have its advantages in promoting students' theoretical thinking. However, the experimental group does appear to have the slight advantage since the two proving techniques presented in the lecture (graphical and RBC) were applied correctly more frequently by the experimental group.

The obstacles discussed in the literature review of this paper were of analytic nature and focused on the notion of absolute value, thus, none specifically mentioned difficulties with absolute value inequalities, apart from Sierpinska et al. (2011). Since this research was closely related to that of Sierpinska et al., most of the difficulties related to the notion of absolute value were not identified. Difficulties with the disjunction and conjunction, conditional statements, reading algebraic sentences, and removing the absolute value bars were confirmed and discussed in the descriptions of the participants' solutions.

In terms of the setting of the lesson, it is again difficult to claim the superiority of the live teacher in the classroom setting over individual learning from pre-recorded lectures with slides. That is, the VA group slightly outperformed EG and CG in the number of correct answers and reflective thinking, but both groups in the classroom settings were much more likely to apply the RBC technique and display definitional thinking. Furthermore, the experimental and control groups were much less likely to demonstrate weaknesses in analytic thinking.

The discussion of the participants' questions, performance, and theoretical thinking has had one purpose, making progress in the teaching of mathematics. Therefore, some recommendations for teaching will be offered in the next chapter, along with suggestions for future research of this nature and other possible avenues of research.

Chapter 7. Concluding Remarks

The aim of this research was to explore the conjecture that the discursive approach is more likely to promote theoretical thinking in students. Although this was not confirmed, the experimental group was found to be more reflective and the control group tended to be more systemic. These results suggest that the two teaching approaches promote different aspects of theoretical thinking. Moreover, the experimental group's performance scores were more consistent than those of the control group. In addition, the discursive approach appears to have encouraged participants not only to be more inquisitive about the material, but also to be more focused in their questions.

7.1 Some recommendations for teaching

Using elements of the discursive approach seems to have provoked participants to ask more specific questions and think in a much more focused manner about the material presented. However, promoting discussion may have distracted participants at inconvenient points during the lecture, which could have been critical to their understanding. Therefore, it may be more accommodating to balance both approaches throughout one lecturing period; for instance, allowing students to ask questions at designated times throughout the lecture. That is, not allowing interruptions during explanations and allocating specific periods in between explanations for students to ask their questions may help to avoid students forgetting their specific questions, as CG appears to have done, without placing students who might be at a critical point in their learning process at a disadvantage.

One regret about the instructor's approach to the discussion period is the way it was treated as more of a "correction" period, although the intentions were to keep the discursive approach throughout. Moreover, it was very difficult for the instructor to stray away from old or preconceived habits, thus, the questions were not dwelled on enough with each student and discussion and debate were not encouraged as much as was intended. Therefore, some suggestions for future teaching would be not only to ask students if they agree with each other's answers, but to be conscious of not hinting at the correct answer until the students have debated their techniques and decided which answer is correct. This should aid in turning this time into more of a discussion than was achieved here, while simultaneously keeping students more involved and mentally stimulated. The point is also to have students be more critical and reflective of their own and others' solutions.

Another suggestion for educators who wish to use the discursive approach would be to let more students explain their solutions on the blackboard and to request those who used different solution techniques. If teachers could find the time for this, it could not only keep the students more mentally involved, but also open them to the possibility of different solution strategies besides their own. Also, it was seen through this experiment that encouraging students to come to the blackboard and explain their solutions also made participants, such as E5, notice their own errors; thus, giving them a chance to independently uncover where they went wrong, as opposed to having students rely on their teacher for evaluation. This also could create an environment in which students are less dependent on their teachers.

7.2 Suggestions for future studies of this nature

One drawback of this experiment was the timeframe, since it consisted of only one lecture with each group. In order to establish a classroom culture in which most students would feel comfortable asking questions and initiating discussions, many more lectures with the same students would probably be required. Moreover, it was very difficult to have the groups begin discussions during the discussion period and even more difficult to get them to debate their answers. However, as we have been able to see, this did not hold back most of the EG participants from asking questions.

As discussed, some suggested changes in the instructor's approach to the EG discussion would be to let participants solve on the blackboard more often and to request participants who used different solution techniques. Another suggestion would be to let participants discuss and debate on which solution is correct. We have already mentioned, however, that this can be difficult to achieve in such a short timeframe. Thus, testing these approaches in a real mathematics classroom, keeping the instructor and course constant, could eliminate the timeframe limitation and provide more meaningful and interesting results. That is, this has the potential to produce more dramatic differences between the two groups since the participants will become more assimilated to the respective classroom cultures over time.

The reason for suggesting the above changes to the instructor's strategy is to better match the guidelines of the discursive approach. In teaching, however, it is always a challenge to react differently than we are used to. Thus, it would also be useful for future researchers to think about ways in which they would react or respond to particular situations to encourage discussion beforehand. This might increase the likelihood of spontaneously promoting discussion among participants.

Further, instructors should dwell more on participants' statements, difficulties, and solution strategies in order to gain more perspective on how the participants are reasoning. This has the potential of contributing to the TT analysis, as Sierpinska et al.'s (2011) individual interviews did. Thus, rather than conducting a collective interview, perhaps individual interviews would also be beneficial to the analysis.

Some other additions to this research would be to consider participants' previous grades in mathematics and whether they recall learning to solve AVI's. This was done in Sierpinska et al. (2011), but not in the present experiment. Considering these factors may have aided the interpretation and discussion of the results.

7.3 Possible avenues for future research

Not only were participants' responses to the exercises important for this experiment, but the depth of the questions asked and interventions made were also noted. What would be interesting to look into further is the quality of these questions asked and interventions made by participants and the instructor. Moreover, this analysis could help in understanding and measuring the level of stimulation of each group.

Of course, a model would need to be defined for this type of analysis. For instance, answering what can be considered a "good question" or "good intervention". This question was partially answered, although a bit informally, with the following categories. A good student's question or intervention could be one that:

i. the teacher is not expecting
ii. shows that the student is thinking about possible questions that he/she has not seen examples of (different situations that could arise)
iii. points to an aspect of the lecture that was not obvious
iv. reveals a deep misunderstanding (this makes the teacher think about where this misunderstanding could have originated; was it due to the way something was stated in the lecture?)

v. refers to something the teacher was prepared to ask the students about or hoping that students would catch onto (demonstrates that the student's thinking is taking the path that the teacher desires).

The categories listed here are only those which were observed in this experiment and are not implied to be the only types of "good questions" or interventions. Moreover, human behavior is clearly unpredictable and it is obvious that any type of question can arise, which is what makes this type of investigation very interesting, but at the same time very difficult. Even more important, these eight types of questions are treated as separate and are in no way a list of criteria to be satisfied for one good question/intervention. For example, a good question could be one that was not expected by the instructor or one that reveals a student's deep misunderstanding.

Another possible avenue could be comparing multiple teaching settings using the same visual approach lecture and slides. Of course, one would have to hold the instructor and the teaching approach constant. A setting that would be interesting to add would be the live webinar in which students can virtually raise their hands to ask questions and professors can pose questions since this type of course if becoming more and more popular among universities.

Alternatively, the discursive and non-discursive approaches still did reflect advantages of both the control group and the experimental group. Therefore, it would also be interesting to experiment with each setting (including the live webinar) and vary the teaching approach. In addition, one could still hold the teaching approach constant, but instead use that which was suggested in section 7.1. That is, not allowing participation during the lecture, but stopping for discussion and questions at the end of each lecture phase.

Another emergent result not having to do with the teaching approach or setting was that participants who used the graphical approach, as opposed to RBC to solve exercises 4 and 5, were more likely to use the relationship between these exercises to solve exercise 5 quicker. Furthermore, no written solutions using the graphical approach were incorrect, while there were obviously mistakes in many solutions using RBC. Specifically, if we look solely at the written solutions using RBC, then we observe a 65.8% success rate among EG and a 56.8% success rate among CG. Therefore, it is easy to see that RBC led to a lower success rate for both groups. A thought that arises here is whether the graphical approach led to more solving success or whether it was used by the stronger participants, therefore, leading to correct answers. Thus, it would be interesting to research the following conjecture: Is using the graphical approach to solve AVI more likely to promote reflective thinking in students than the RBC technique?

7.4 Closing remarks

Through this research, many interesting and curious results were found, some in favor of the discursive approach and some in favor of the non-discursive approach. At the same time, the experimental group's performance was found to be more consistent across exercises than the control group. In addition, the study suggests that the classroom setting can be beneficial to the development of students' definitional and analytic thinking, but perhaps detrimental to their reflective thinking capacity.

It is true that as mathematics educators, we wish that our students could efficiently use the theory given to them, a few examples presented, and their past knowledge to correctly solve any exercises they are given. However, with the structure of today's education, it is becoming more and more common to hear about students who "cram" for examinations and then immediately forget what they have "learned". Thus, it is clear that finding a way to effectively introduce new material could be highly beneficial to our students. Moreover, this could begin with the encouragement of discussion and the setting in which this is done. It is my hope that through this research, we have learned a bit more on the effect of teaching approaches and settings. Further, it is my hope that research of this nature is extended to longer timeframes in order to add significance to these results since a less controlled setting would be needed in order to create a classroom where a culture is slowly established.

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Appendix A

This appendix outlines the analyses of each participant's work. The titles C1 to C5 refer to students in the control group, while the titles E1 to E6 refer to students in the experimental group. Each individual obtained a two different total scores; one for the theoretical thinking (TT) analysis and another for the number of total correct exercises. These scores appear at the end of each analysis. The following are some other necessary abbreviations with their respective meanings:

A-cases-list	Analytical thinking behavior; A-cases-listed
A-comb-conj	Analytical thinking behavior; A-combining-by-conjunction
A-comb-incons	Analytical thinking behavior; A-combining-inconsistent
A-comb-repl	Analytical thinking behavior; A-combining-replaced
AV	Absolute value
AVI	Absolute Value Inequality
CB1	Reflective thinking behavior; Constructed behavior 1
CB2	Reflective thinking behavior; Constructed behavior 2
CB3	Reflective thinking behavior; Constructed behavior 3
DT	Systemic thinking behavior; Definitional thinking
HT-int-cond-n-t	Hypothetical thinking behavior; HT-interval-conditions-not-taken
HT-cond-stm	Hypothetical thinking behavior; HT-conditional-statements
HT-init-inq-ign	Hypothetical thinking behavior; HT-initial-inequality-ignored
PB1	Reflective thinking behavior; Procedural behavior 1
PB1	Reflective thinking behavior; Procedural behavior 2
PB1	Reflective thinking behavior; Procedural behavior 3
P-g-phy	Proving behavior; P-graphing-physical
P-g-mental	Proving behavior; P-graphing-mental
P-num-test	Proving behavior; P-numerical-testing
P-RBC	Proving behavior; P-RBC
P-struct	Proving behavior; P-structural
RBC	Reasoning by cases

C.1. Analysis of participant: C1

PB1 (-1)	The participant uses the algebraic approach to solve in exercise 3, as he does in exercise 2.
PB2 (-1)	The definition of AV is applied to the function $\lvert 2x - 1 \rvert$ a second time in exercise 5 and the same procedure is used to solve the AVI. This displays that this participant did not use the connection between exercises 4 and 5 to solve in a quicker, more efficient way.
CB3 (+1)	The participant only includes elements in his solution that are necessary in his solution process.
DT (+1)	The participant applies the definition of AV in many of his solutions. This displays definitional thinking.
P-RBC (+1)	The participant uses reasoning by cases in many of his solutions.
HT-cond-stm (+1)	The participant demonstrates conditional thinking when making statements such as "if $x \geq$ ½" and later stating "$x < 3$" (See *Figure A1*).

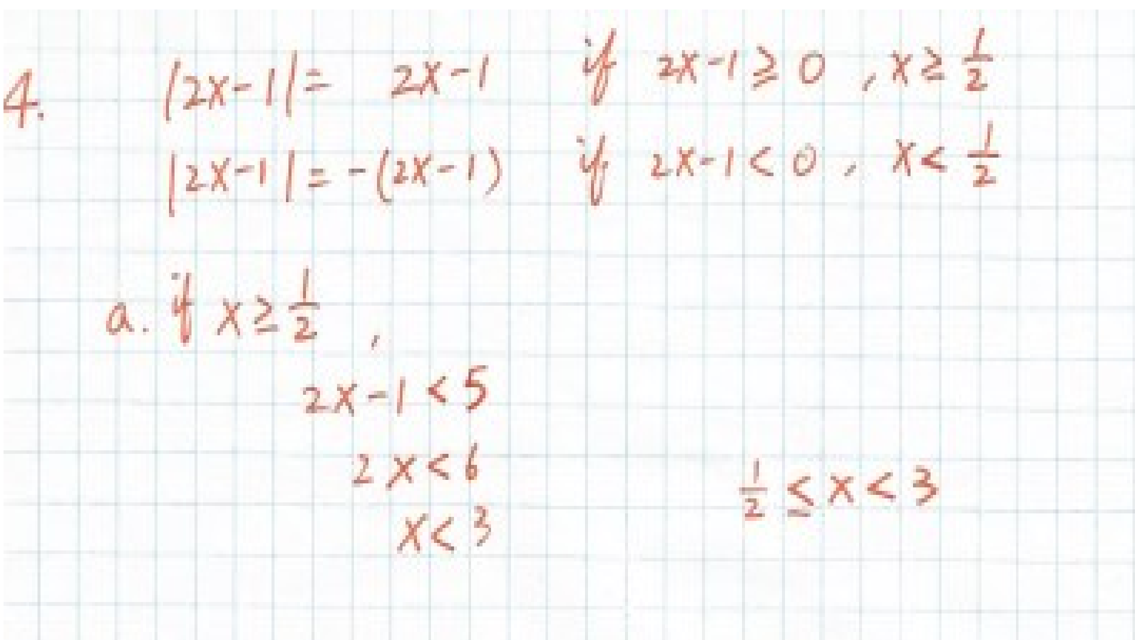

***Figure A1*. C1 – Hypothetical Thinking**

Total Score for TT Analysis (C1): 2

Total Correct Exercises (out of 6): 4

C.2. Analysis of participant: C2

PB1 (-1)	Student uses the algebraic approach to solve exercise 3 in the same way he does in exercise 2. That is, he does not acknowledge a more efficient way of solving.
PB2 (-1)	The definition of AV is applied to the function $\|2x - 1\|$ a second time in exercise 5 and the participant uses an almost identical procedure to solve this AVI.
PB3 (-1)	The participant includes information that is pertinent, but does not use it in finding the solutions.
DT (+1)	The participant applies the definition of AV in every exercise.
P-g-mental (+1)	The participant stated during the discussion “I knew I was wrong just because I know what the absolute value of x minus one and the absolute value of x plus one look like. So, I could visually tell where the answer was. I just got it wrong” and later explains, “I didn’t plot the graphs. I just know what they look like. I knew the answer, I just tried it this way (steps) and messed up somewhere, that’s it” (lines 97-99).
P-RBC (+1)	Although sometimes not used correctly, this participant attempts to use the reasoning by cases method.

Total Score for TT Analysis (C2): 2

Total Correct Exercises (Out Of 6): 3.5

C.3. Analysis of participant: C3

PB1 (-1)	The participant uses the algebraic approach to solve exercise 3 and uses both the algebraic and graphical approach in exercise 2. While the participant arrives at the solution using the graphical method in exercise 2, he does not use this method in exercise 3. Thus, a longer method was used in solving exercise 3.
PB2 (-1)	This behavior is displayed since the participant uses an almost identical solution method in exercise 5 as that of exercise 4. The usefulness of his written work in number 4 was not, therefore, recognised as being valuable to the next exercise.
PB3 (-1)	Although the graphical method was used to find the solution in exercise 2, the participant does not use the graph drawn in exercise 3 toward his solution. It appears to have been drawn and then ignored. Also, RBC is attempted in each

exercise, but the participant never goes further than organising the four cases and then rejecting the cases he thinks possess some sort of contradiction. Thus, the steps seem to also play a ritualistic role because they are never used to find a solution.

In addition, the participant does not seem to understand how to recognise when a case should be disregarded. This can be seen clearly in exercise 3, where the participant crosses out cases 2 and 4 in a ceremonial fashion. Moreover, all cases in this exercise should possess some contradiction since there is no solution to this AVI.

DT (+1) — The participant attempts to apply the definition of AV in all exercises. Although some elements are lacking, definitional thinking still remains evident.

P-g-phy (+1) — As mentioned above, the graphical method was used to come to a solution in exercise 2.

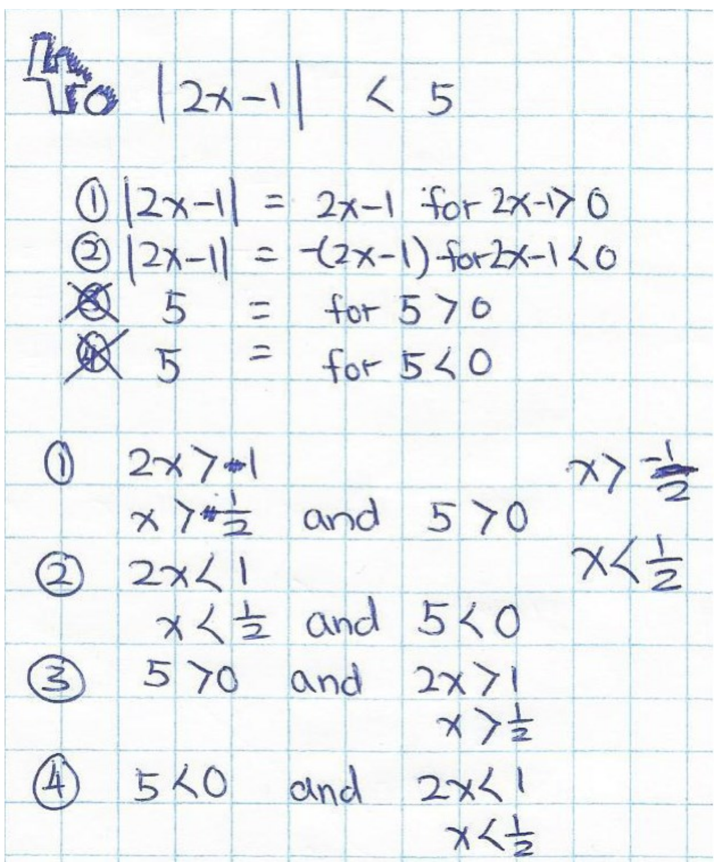

***Figure A2*. C3 – HT-initial-inequality-not-taken**

HT-init-ineq (-1)	This participant disregards the initial inequality in the four cases laid out in all exercises from 2 to 6 (See *Figure A2*).
P-RBC (+1)	The participant tries to use reasoning by cases in all exercises, although he is unsuccessful in arriving at any solutions. For instance, this is the sole method used in exercise 4. Therefore, it was used to attempt finding an answer, but the participant did not complete the exercise. Nonetheless, his attempt reflects this type of systemic thinking.
A-cases-list (-1)	This participant shows an inability to combine the conditions he lists in the four cases. This is displayed, for example, in exercise 2 where the participant leaves case 1 as "$x > 1$ and $x > -1$". That is, the participant does not acknowledge that we do not need to consider $x > -1$.

Total Score for TT Analysis (C3): -2

Total Correct Exercises (Out Of 6): 2

C.4. Analysis of participant: C4

PB1 (-1)	Although this participant solves exercise 2 using the graphical method, RBC is used in exercise 3, rather than a quicker method.
CB2 (+1)	The participant uses the graph drawn in exercise 4 in order to quickly find the solution to exercise 5.
CB3 (+1)	The participant only includes information that is pertinent to finding the solution in all exercises.
DT (+1)	The participant applies the definition of AV in at least one exercise; for example, exercise 3.
P-g-phy (+1)	Exercises 2, 4, and 5 are solved using the graphical representations of the AV functions under consideration.
P-struct (+1)	This participant is given the benefit of the doubt for solving number 5 as the opposite exercise of number 4. This behaviour demonstrates that the participant acknowledged the structure of the inequalities in exercises 4 and 5 (See *Figure A3*).
P-RBC (+1)	At least one exercise is solved using reasoning by cases; for instance, exercise 3.

Total Score or TT Analysis (C4): 5

Total Correct Exercises (Out Of 6): 6

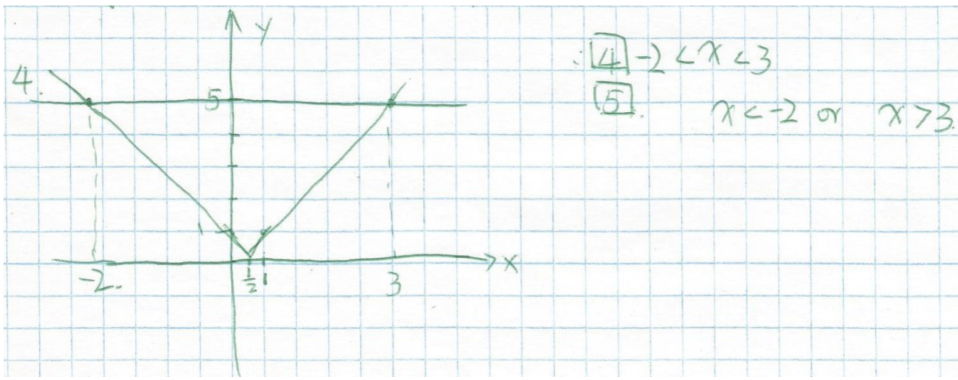

***Figure A3*. C4's demonstration of P-structural**

C.5. Analysis of participant: C5

PB1 (-1) The algebraic method is used to solve exercise 3, as it was in exercise 2. Therefore, this participant did not realize there was a quicker way to solve exercise 3.

PB2 (-1) The algebraic method is used in exercise 5, exactly as it was applied in exercise 4. That is, the participant applied the definition of absolute value a second time to the function $|2x - 1|$ and then used reasoning by cases.

CB3 (+1) Only necessary elements are included to arrive at the solutions for all exercises.

DT (+1) The definition of AV was applied to the AV functions in all exercises from 2 to 6; that is, in all exercises on AVI's.

P-RBC (+1) This participant demonstrates systemic thinking by using RBC in exercises 2 to 6.

HT-cond-stm (+1) As seen in the image below, the participant uses hypothetical thinking when making statements such as "if $1 < 2$" and then later stating that "$x \geq 1$". By writing "if $1 < 2$" the participant refers to the part of the domain that she will consider in the first case. See *Figure A4* below.

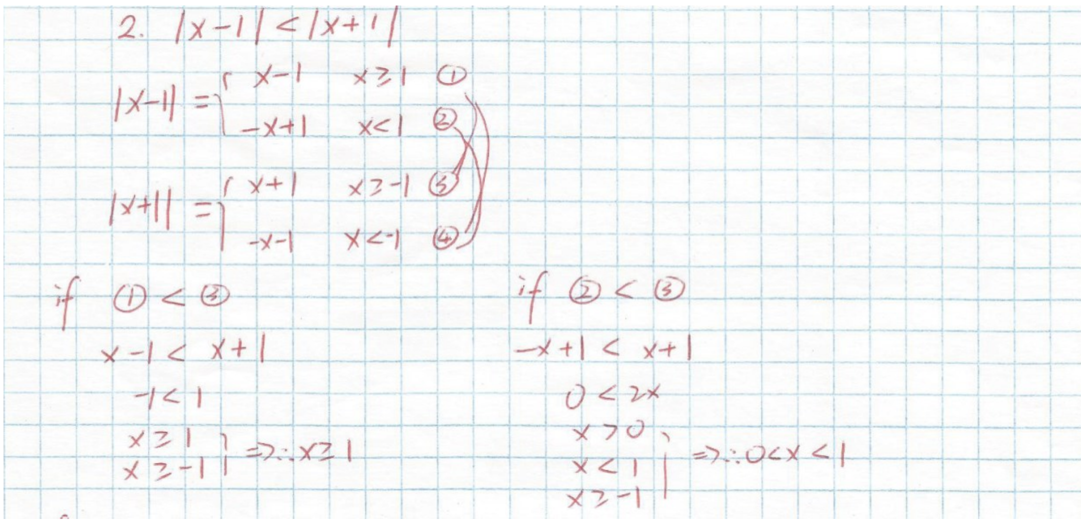

Figure A4. **C5 – Hypothetical Thinking**

Total Score for TT Analysis (C5): 2

Total Correct Exercises (Out Of 6): 6

E.1. Analysis of participant: E1

PB1 (-1)	The participant uses the algebraic approach in exercise 2 and then again, in exercise 3.
PB2 (-1)	Exercise 5 is solved using the algebraic approach, just as it was used in exercise 4. The connection between the two exercises was not used to solve exercise 5 quicker.
CB3 (+1)	The participant only includes relevant information to his solutions.
DT (+1)	The participant applies the definition of AV in every AVI exercise solved.
P-RBC (+1)	Reasoning by cases is used to solve each of the AVI exercises.

Total Score for TT Analysis (E1): 1

Total Correct Exercises: 5

E.2. Analysis of participant: E2

CB1 (+1)	Exercise 3 is solved using the graphical method and logical reasoning about the structure of the respective AV functions. See *Figure A5*.

CB2 (+1)	Exercise 5 is solved using the graphical method, while exercise 4 is done using the algebraic method. Thus, no steps from exercise 4 are repeated in exercise 5.
CB3 (+1)	No superfluous information is included in this participant's solutions.
DT (+1)	Although the participant did not explicitly write out the formula of one of the AV functions, it still remains clear that the definition of AV was applied since reasoning by cases was used. That is, each case is clearly written out using the necessary interval conditions on x, attained from the domain of both of the AV functions. This can be seen in *Figure A6*.
P-g-phy (+1)	Exercises 2, 3, and 5 are solved using the graphical approach.
P-struct (+1)	As mentioned in the justification of CB1, exercise 3 is solved using some logical reasoning about the structure of the functions involved.
P-RBC (+1)	Reasoning by cases is used to solve exercises 4 and 6.

Total Score for TT Analysis (E2): **7**

Total Correct Exercises (Out Of 6): **6**

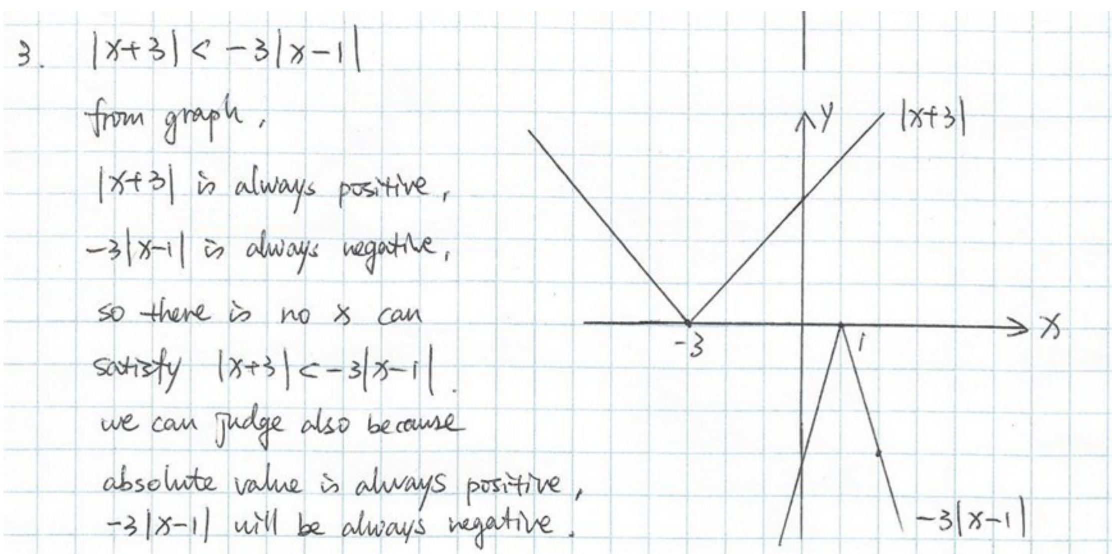

Figure A5. **E2 – Reflective Behavior, CB1**

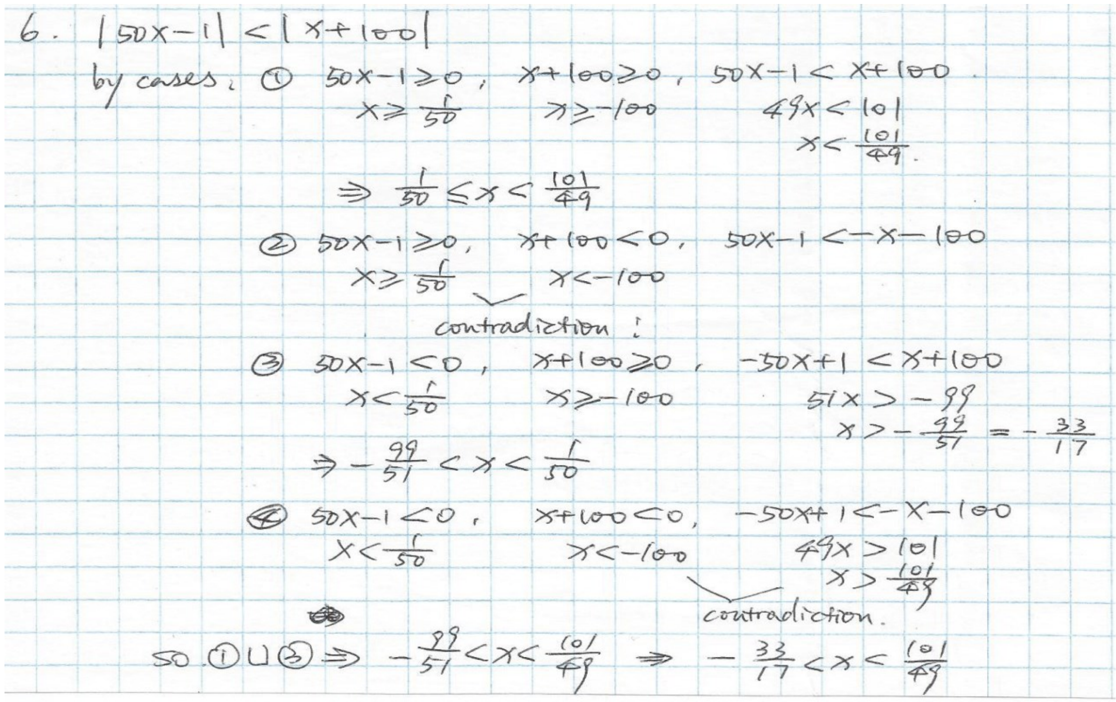

Figure A6. **E2 – Definitinal Thinking**

E.3. Analysis of participant: E3

PB1 (-1)	Reasoning by cases is used in exercise 3, as it was in exercise 2.
PB2 (-1)	The definition of AV is applied to the function $\|2x - 1\|$ a second time in exercise 5. Thus, the participant repeated actions in exercise 5, which had already been carried out in the previous exercise.
CB3 (+1)	This participant did not include superfluous information in her solutions.
DT (+1)	The definition of AV is explicitly applied to the AV functions involved in all the AVI exercises.
P-RBC (+1)	Reasoning by cases was used to solve all five of the AVI exercises.

Total Score for TT Analysis (E3): 1

Total Correct Exercises (Out Of 6): 6

E.4. Analysis of participant: E4

CB1(+1)	Although this participant spent most of his time on exercise 3, he made clear that the reason for this was to avoid the strenuous and lengthy work involved in the RBC method. When asked what method he used, he stated, "I just graphed it" and later explained, "it's not even that I thought it was especially quicker. I just found the other way had too many steps" (lines 211-213). Since the participant acknowledged that a faster way was possible, he was allotted one point.
CB2 (+1)	Exercise 5 is solved by taking the opposite of the interval that was found in exercise 4, although the answers are incorrect. That is, the interval $(-\infty, 3)$ was given as an answer to exercise 4 and the answer to exercise 5 was $(3, \infty)$.
CB3 (+1)	Although one may get the impression that E4 includes much information that is not used toward his answer for exercise 3, this participant spends a great deal of time thinking about this particular exercise. Moreover, everything that he writes has a purpose and it is clear that he is not mimicking elements from the lecture for the sake of writing something that resembles a solution.
DT (+1)	The definition of AV is applied in exercises 2, 3, and 6.
P-num-test (+1)	Numerical testing is used to solve exercise 4, although the answer is incorrect.
P-g-phy (+1)	Exercises 2 and 3 are solved using the graphical method.

Total Score for TT Analysis (E4): **6**

Total Correct Exercises (Out Of 6): **4**

E.5. Analysis of participant: E5

PB1 (-1)	The algebraic method is used to solve in exercise 3. So, a quicker method was not found to solve this exercise.
PB2 (-1)	An identical procedure is used in solving exercises 4 and 5. Thus, the connection between 4 and 5 did not help this participant to solve exercise 5 faster. These solutions can be seen in *Figure A7*.

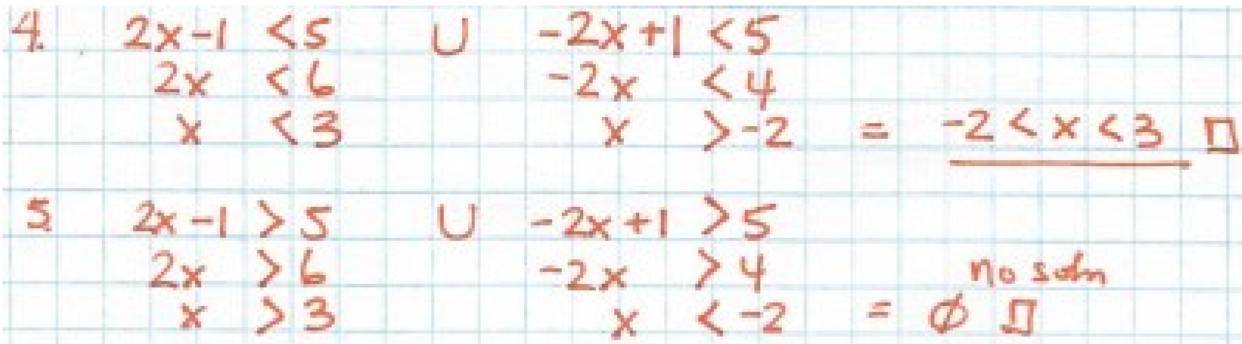

***Figure A7*. E5 – Procedural Behavior, PB2**

CB3 (+1)	All information included in E5's solutions is used to get to the final answer. Although exercise 6 was not completed, the written content was not considered as ceremonial since she never attempted to find an answer and later explained that this was because the procedure (algebraic) was taking her too long and she became too tired to proceed. She states: "I wanted to do it algebraically, but I was just too tired to go through the whole steps again" (line 453).
DT (+1)	The definition of AV is applied in exercises 2 and 3.
P-RBC (+1)	Reasoning by cases is used in all exercises from 2 to 6 (even though number 6 is not completed).
HT-int-c-n-t (+1)	Student did not consider the interval conditions when using RBC. Instead, she considered the individual pieces of the inequality in the place of the interval conditions. This can be seen more clearly in *Figure A8.*

Total Score for TT Analysis (E5): **1**

Total Correct Exercises (Out Of 6): **3**

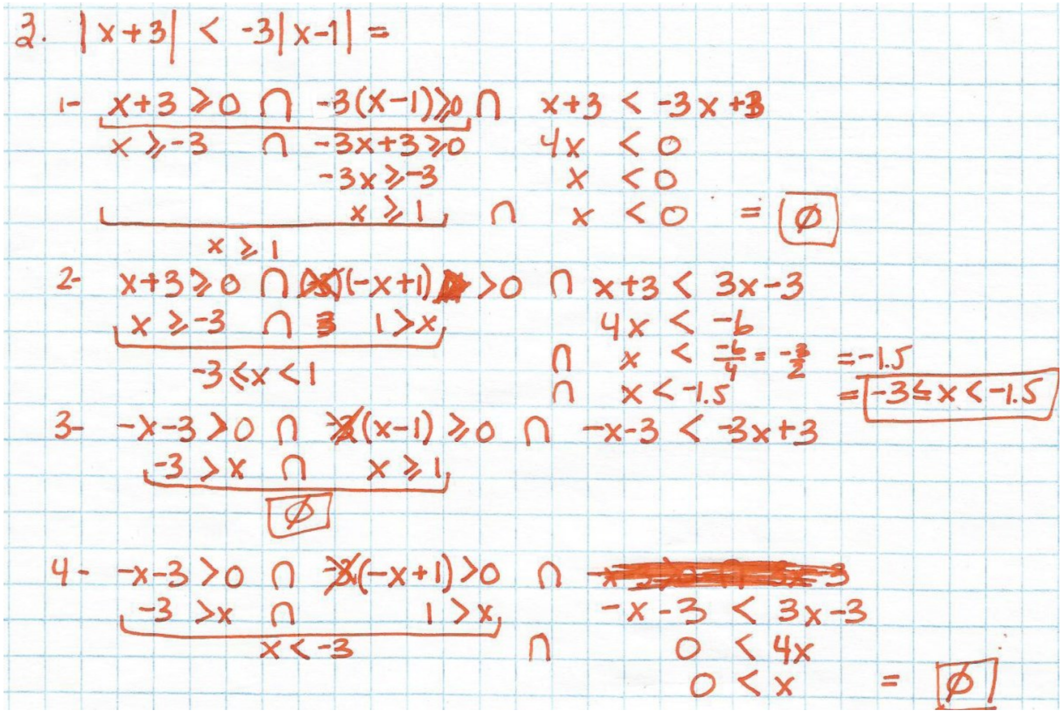

Figure A8. **E5- HT-interval-conditions-not-taken**

E.6. Analysis of participant: E6

PB1 (-1)	The participant attempts to solve exercise 3 in the same way he solves exercise 2. Thus, no new solving strategy was attempted.
PB2 (-1)	Solved exercises 4 and 5 in the exact same way. That is, the participant solves the equality $2x-1=5$ and then gives the same condition on x, $x<3$, as the answer for both exercises. See *Figure A9*.
PB3 (-1)	In exercise 3, the participant solves the equality $x+3=x+1$ and obtains the result $x=-1.50$. This is considered ceremonial because the result is not used toward finding the solution to the exercise.
DT (+1)	Student applies the definition of AV implicitly in exercises 2 and 3, although not rigorously. That is, the interval conditions are omitted. This fact is taken into account when his hypothetical thinking is evaluated.
P-RBC (+1)	This participant attempts to use the RBC method to solve exercises 2 and 3. Although he is unsuccessful, this attempt still demonstrates a way of thinking that is systemic.

HT-int-c-n-t (-1) As can be seen in *Figure A10*, the participant does not take interval conditions into consideration when defining the AV functions.

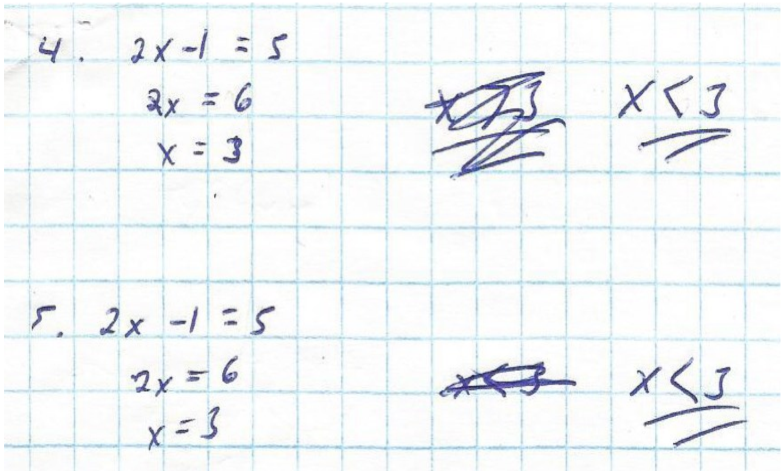

***Figure A9*. E6 – Procedural Behavior, PB2**

HT-init-inq-ign (-1) In *Figure A10* we can also see that this participant disregards the initial inequality in the four cases laid out.

A-cases-list (-1) As we can see in *Figure A10*, this participant is unable to combine the conditions on x. For instance, he checks off the first case in order to show that this can be used toward his solution. However, the fact that one of the conditions, $x \geq -3$, is unneeded is never mentioned. Further, no other "checks" were made to indicate any sort of thinking process, nor was it made explicit that any cases were disregarded due to contradictions.

Total Score for TT Analysis (E6): -4

Total Correct Exercises (Out Of 6): 1

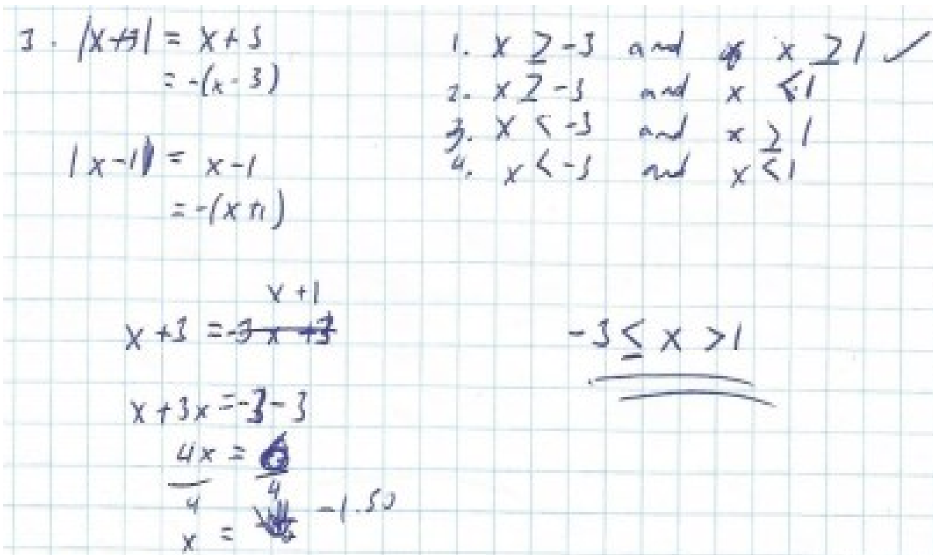

***Figure A10*. E6 – Lack of hypothetical thinking and demonstration of A-cases-listed**

Appendix B

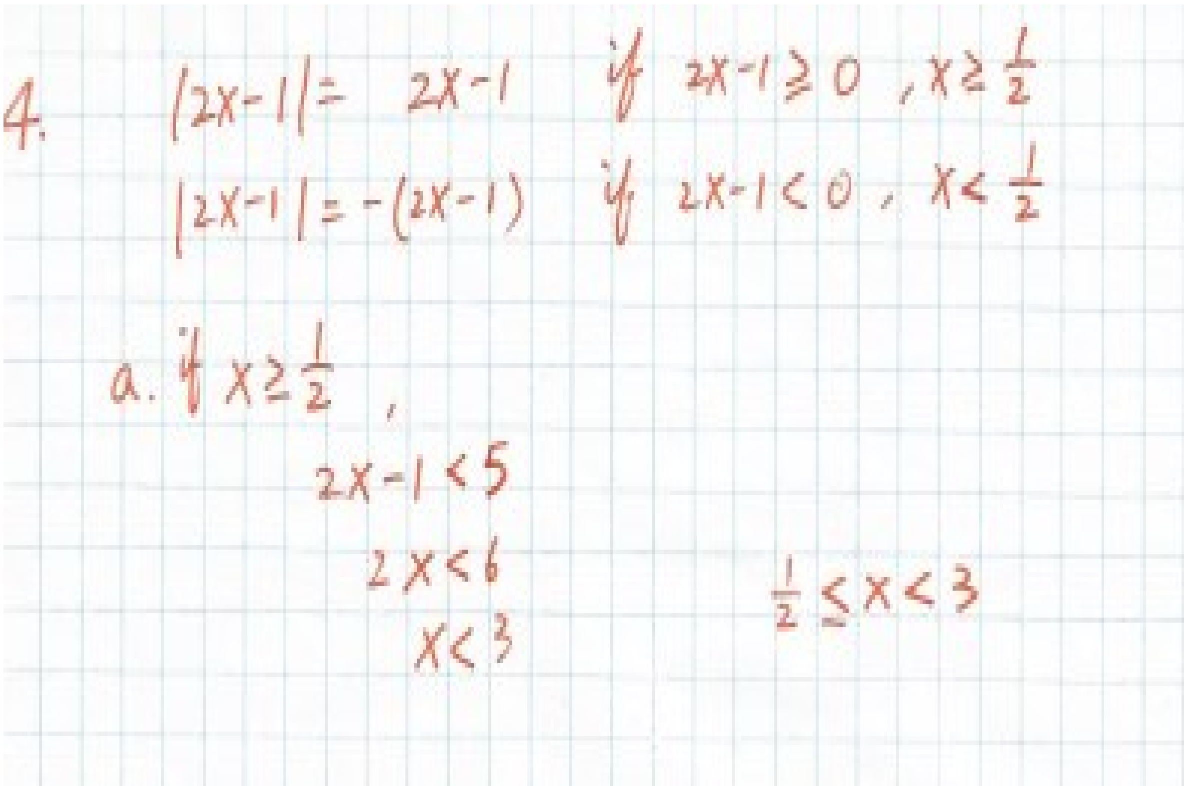

Figure A11. C1 – Hypothetical Thinking

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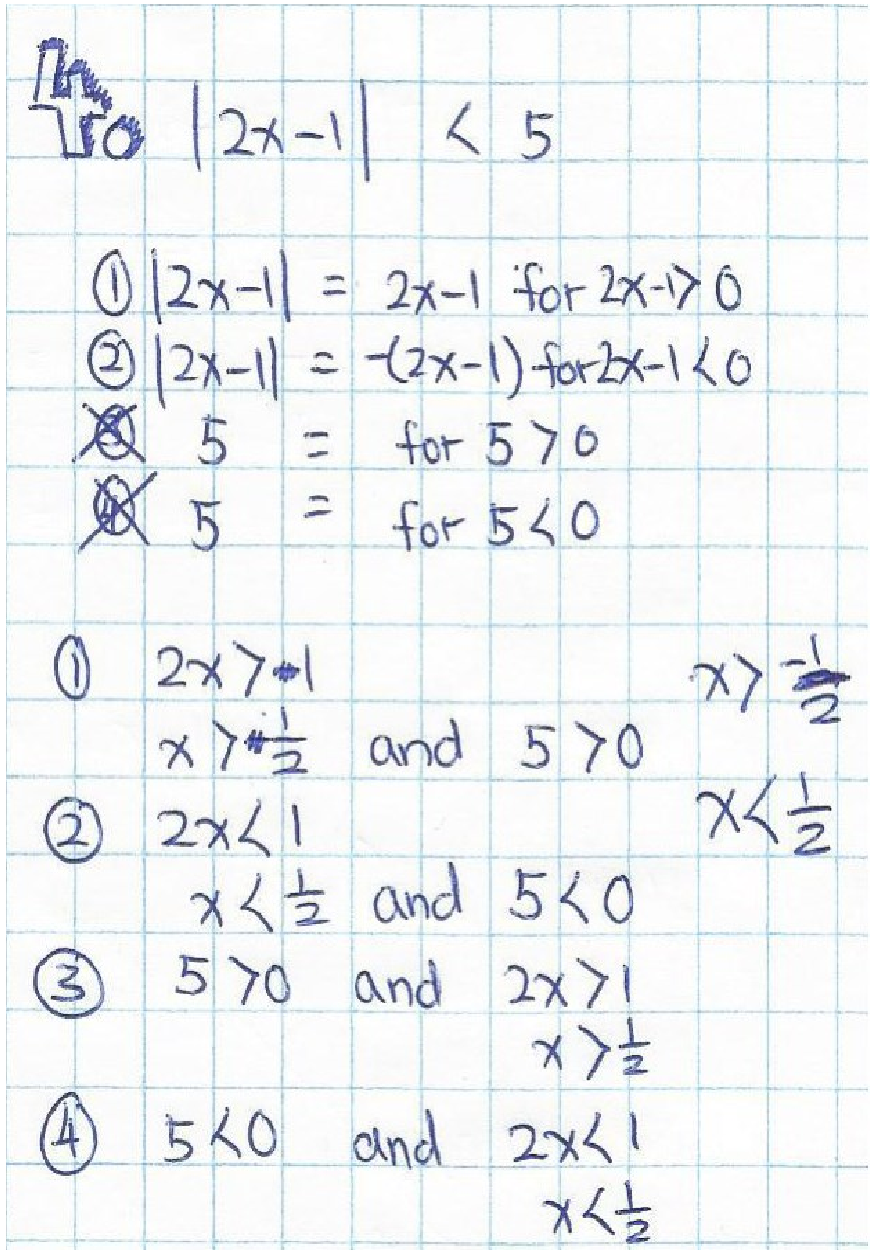


***Figure A12*. C3 – HT-initial-inequality-ignored**

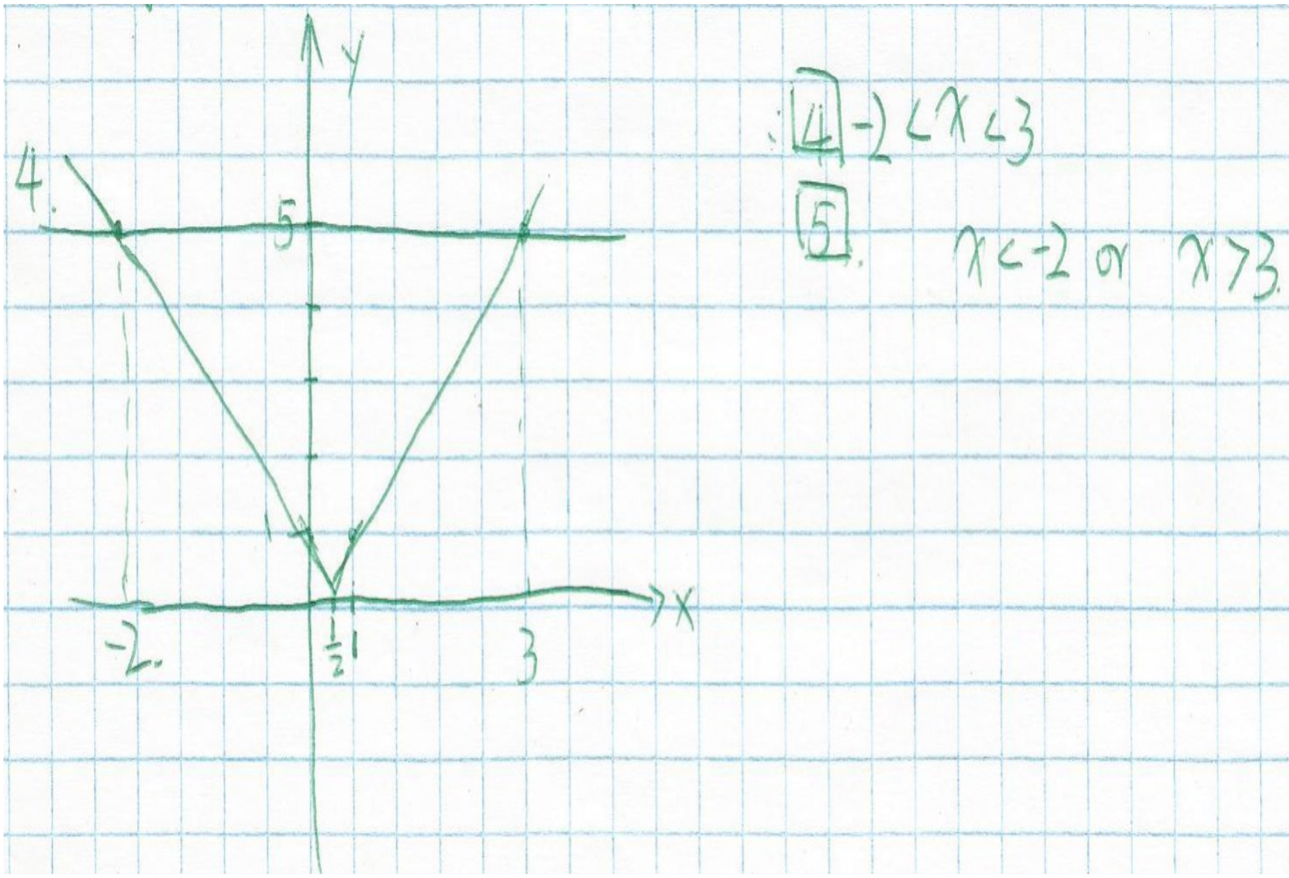


Figure A13. **C4's demonstration of P-structural**

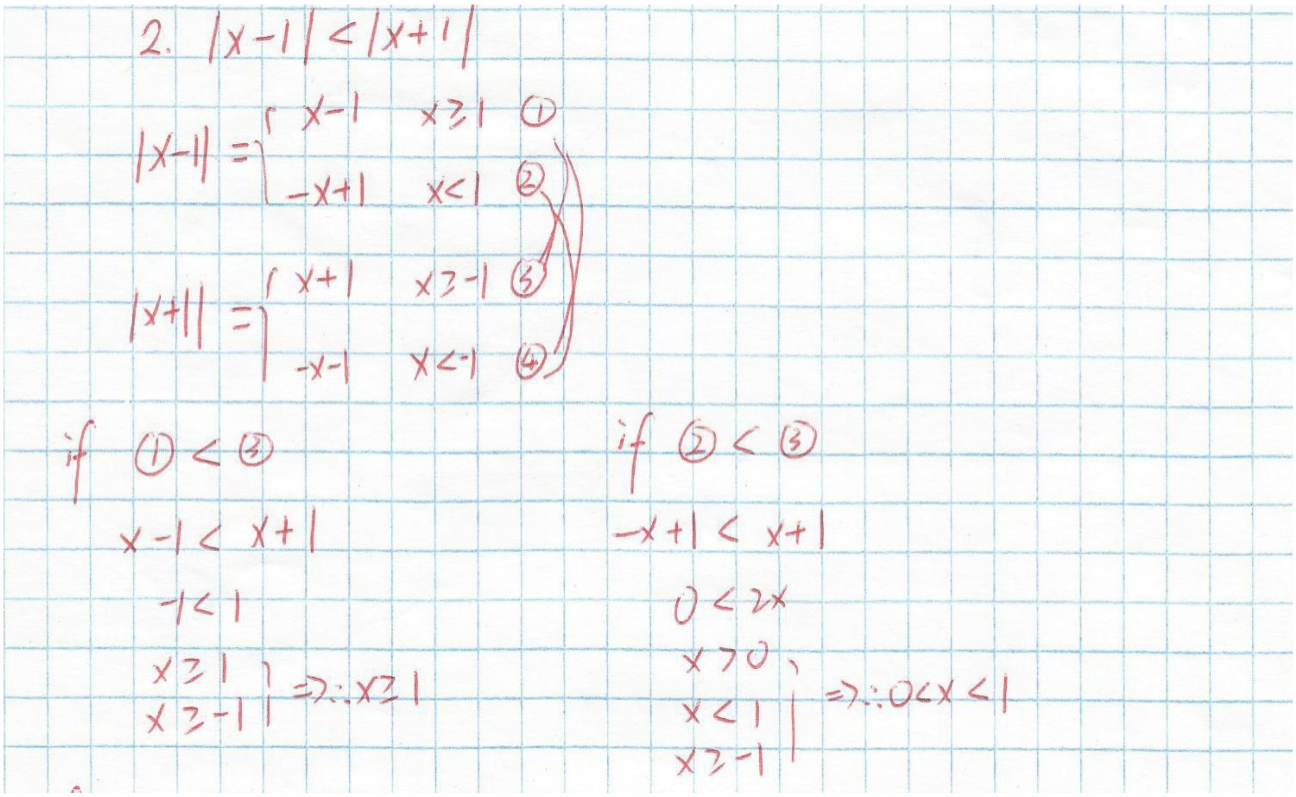


Figure A14. C5 – Hypothetical Thinking

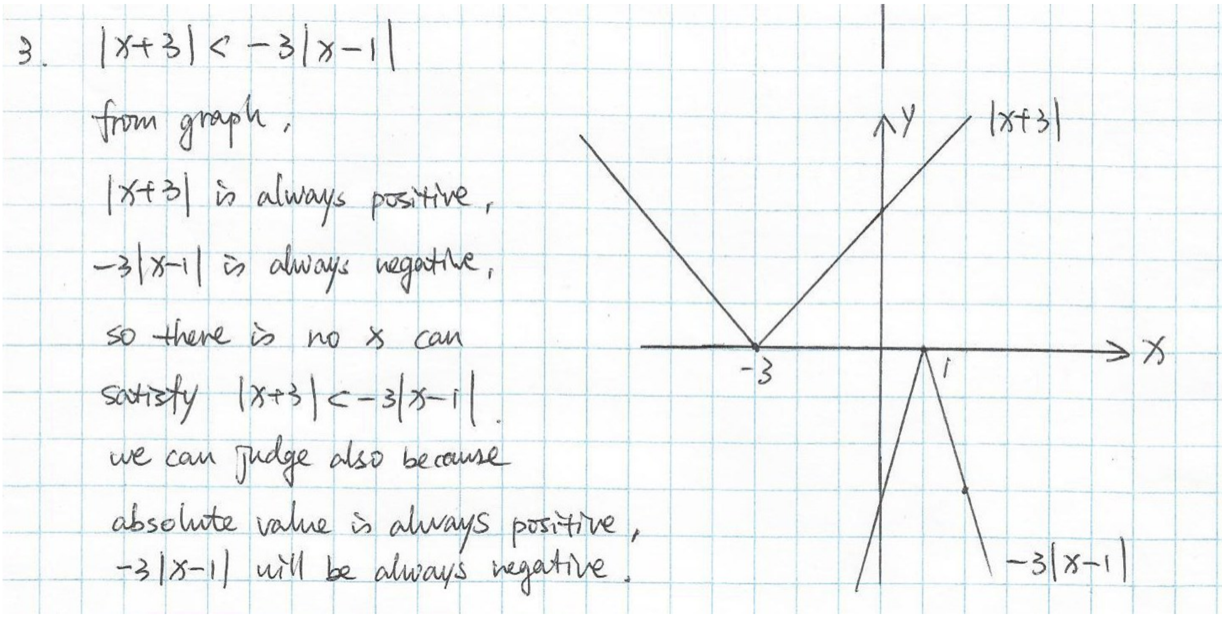
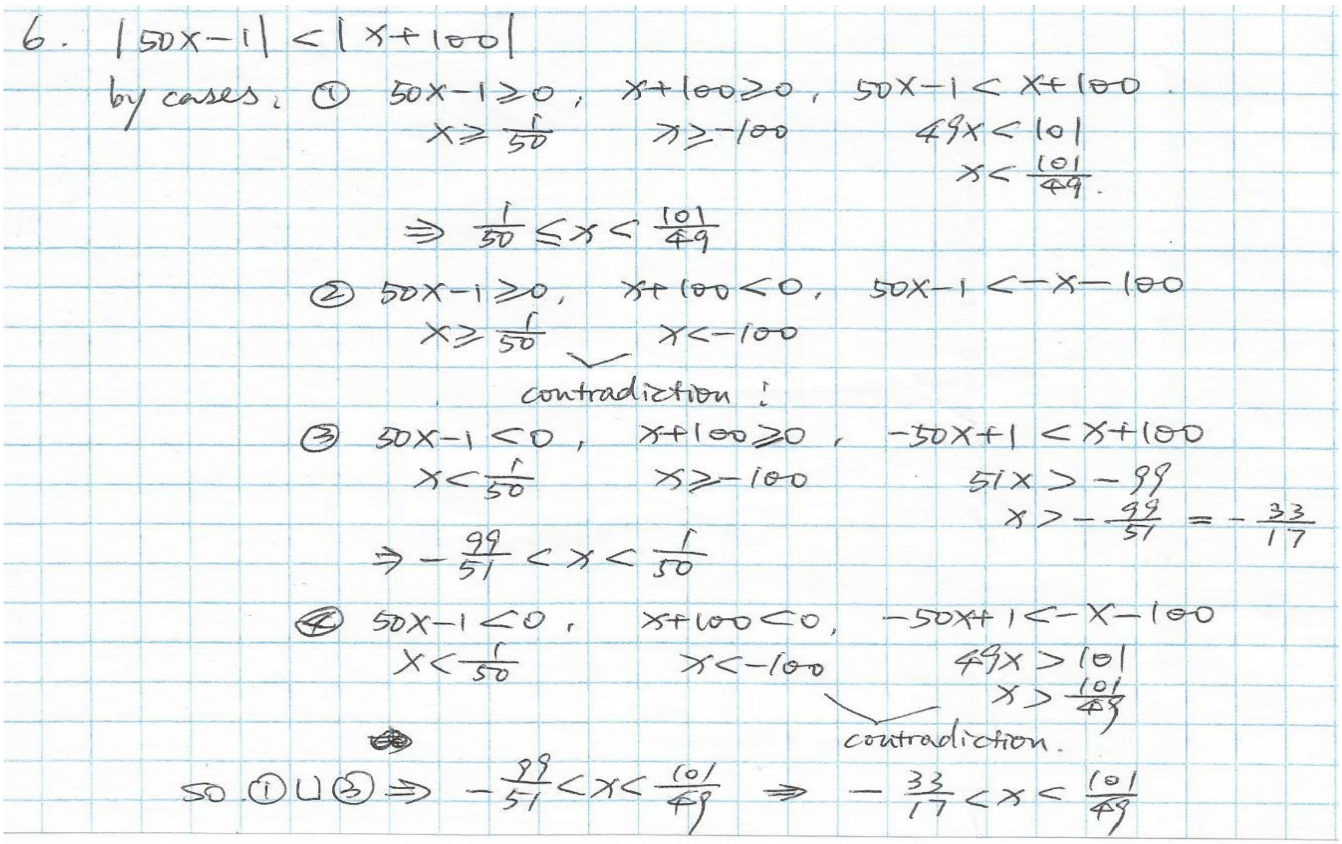


Figure A15. E2 – Reflective Behavior, CB1



6. $|50x-1| < |x+100|$

by cases:

① $50x-1 \geq 0$, $x+100 \geq 0$, $50x-1 < x+100$

$x \geq \frac{1}{50}$ $\quad x \geq -100$ $\quad 49x < 101$

$x < \frac{101}{49}$.

$\Rightarrow \frac{1}{50} \leq x < \frac{101}{49}$

② $50x-1 \geq 0$, $x+100 < 0$, $50x-1 < -x-100$

$x \geq \frac{1}{50}$ $\quad x < -100$

contradiction!

③ $50x-1 < 0$, $x+100 \geq 0$, $-50x+1 < x+100$

$x < \frac{1}{50}$ $\quad x \geq -100$ $\quad 51x > -99$

$x > -\frac{99}{51} = -\frac{33}{17}$

$\Rightarrow -\frac{99}{51} < x < \frac{1}{50}$

④ $50x-1 < 0$, $x+100 < 0$, $-50x+1 < -x-100$

$x < \frac{1}{50}$ $\quad x < -100$ $\quad 49x > 101$

$x > \frac{101}{49}$

contradiction.

so ① ∪ ③ $\Rightarrow -\frac{99}{51} < x < \frac{101}{49} \Rightarrow -\frac{33}{17} < x < \frac{101}{49}$

***Figure A16*. E2 – Definitinal Thinking**

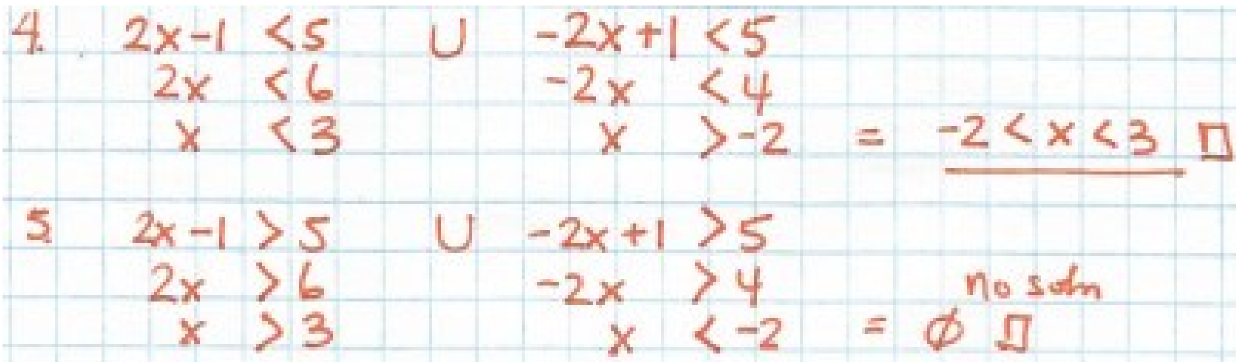


Figure A17. E5 – Procedural Behavior, PB2

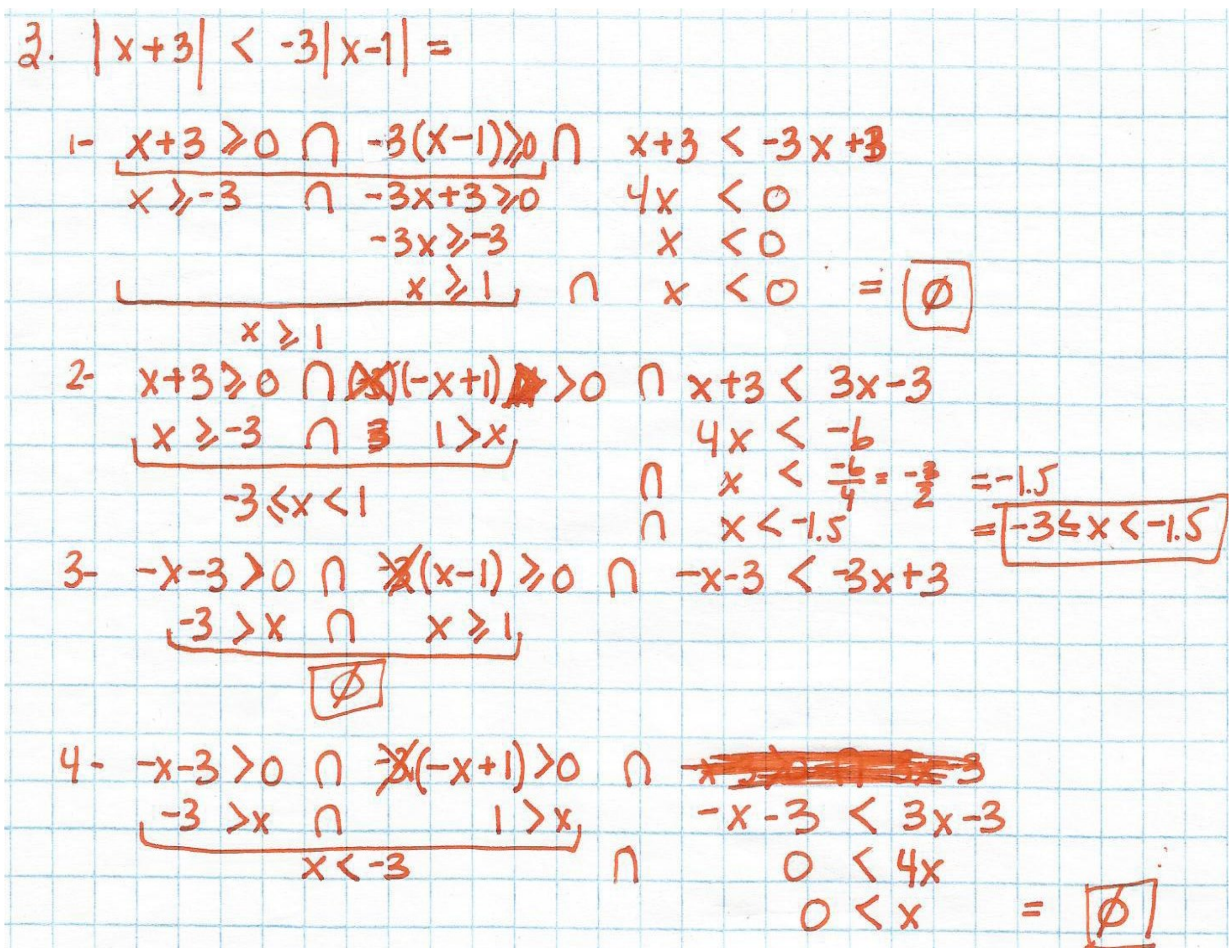


***Figure A18*. E5- HT-interval-conditions-not-taken**

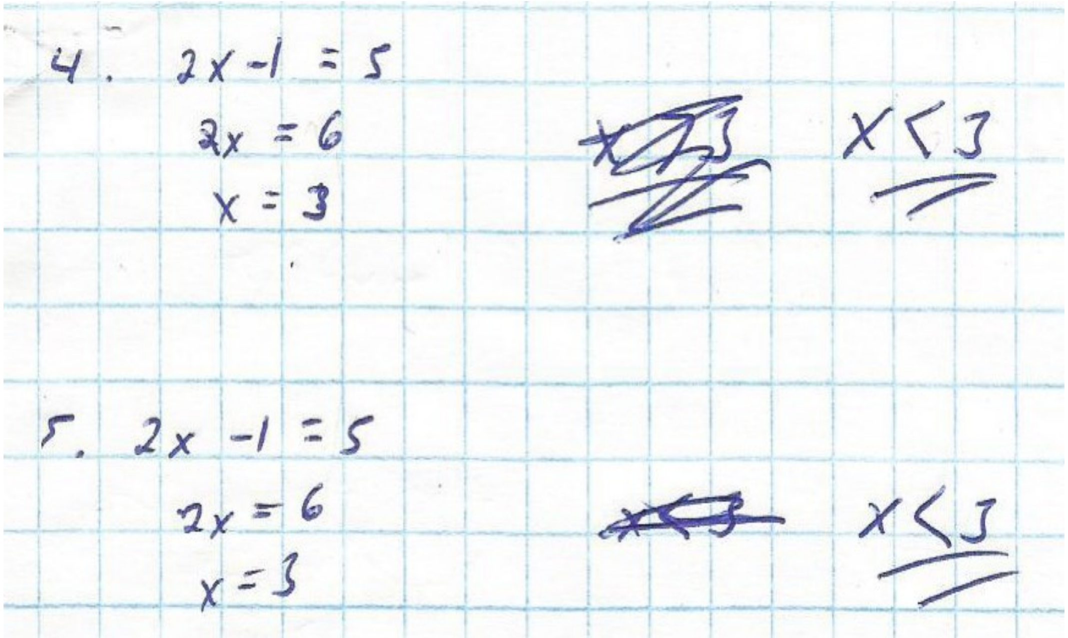


Figure A19. E6 – Procedural Behavior, PB2

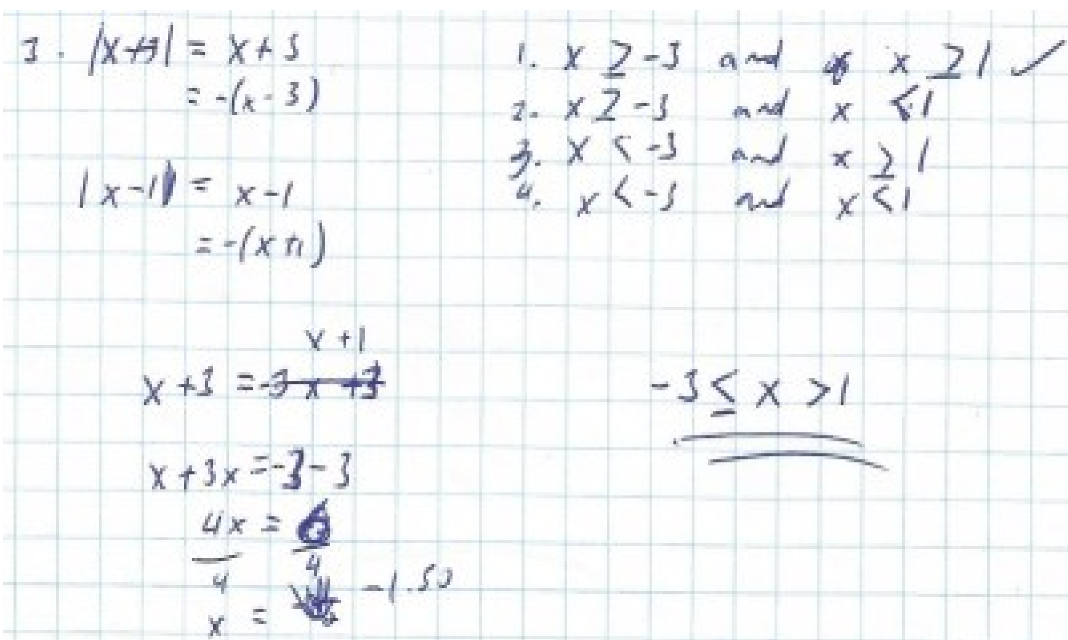


***Figure A20*. E6 – Lack of hypothetical thinking and demonstration of A-cases-listed**

Appendix C

C1. Identification questionnaire

Concordia University
Department of Mathematics and Statistics
Research Project: The Effectiveness of a Visual Approach to Teaching Mathematics
Researcher: Maria Tutino
Supervisor: Dr. Anna Sierpinska

Identification Questionnaire

Your first name or nickname: ____________________

Your age:

- Less than 21
- 21 to 25
- 26 to 30
- Over 30

Which of the mathematics courses listed are you currently enrolled in:

- MATH 200
- MATH 202
- MATH 203
- MATH 204
- MATH 205
- MATH 206
- MATH 208
- MATH 209

Why are you taking the course(s)?

- The course is a prerequisite for the course: ____________________________
- I need the course for admission into: _________________________________
- I'll need the math in my future study
- I'll need the math in my future profession
- Other: ____

How would you qualify your experience of taking the math course(s)?

Enjoyable:	very	somewhat	not at all
Stressful:	very	somewhat	not at all

Other: ____

If you agree to be contacted again, please provide your email address: ______________________

Thank you for answering this questionnaire!

Maria Tutino

C2. Consent to participate

CONSENT TO PARTICIPATE IN A STUDY OF THE EFFECTIVENESS OF A VISUAL APPROACH TO TEACHING MATHEMATICS

This is to state that I agree to participate in a research experiment conducted by Maria Tutino, tel. (514) 651- 5702, maria.tutino@mail.mcgill.ca under the supervision of Dr. Anna Sierpinska of the Department of Mathematics and Statistics of Concordia University.

A. PURPOSE

I have been informed that the purpose of the research is to test the effectiveness of a visual approach to the teaching of mathematics.

B. PROCEDURES

I have been informed that I will participate in the study in the role of a student. The participation will consist in spending approximately 2.5 hours with a researcher on the following activities:

1. Filling out a short questionnaire about myself and the math courses I am taking (10 min)
2. Taking part in a mathematical lecture presented by the researcher (20 min)
3. Communicating my possible difficulties of understanding the lecture in an interview following the lecture; the interviewer will help me to overcome these difficulties (20 min)
4. Working on a few exercises based on the lecture; I will have a copy of the slides used in the lecture to consult while solving the exercises (40 min)
5. Explaining how I solved the exercises to the interviewer (30 min)
6. Commenting on the teaching experiment as a whole; I will be asked to say what I liked and did not liked about it, and to share any suggestions to improve it (15-30 min).

C. RISKS AND BENEFITS

I am aware that I may experience some discomfort in the session due to frustration with the lecture or the problems to solve. I am also aware that my frustration and my reasons for it could be a means for the researcher to measure the effectiveness of the approach. Understanding how I felt during the experiment and what I found difficult will contribute to improving the approach, in general, and devising better ways to teach mathematics.

D. CONDITIONS OF PARTICIPATION

- I understand that I am free to withdraw my consent and discontinue my participation at any time without negative consequences
- I understand that my participation in this study is confidential (i.e., the researcher will know, but will not disclose my identity)
- I understand that my conversations may be recorded for the purpose of research analysis
- I understand that the data from this study may be published, but my identity will not be disclosed in them

I HAVE CAREFULLY STUDIED THE ABOVE AND UNDERSTAND THIS AGREEMENT. I FREELY CONSENT AND VOLUNTARILY AGREE TO PARTICIPATE IN THIS STUDY.

NAME (please print): ________________________

SIGNATURE: __________________________

If at any time you have questions about your rights as a research participant, please contact Adela Reid, Research Ethics and Compliance Officer, Concordia University, at (514) 848-2424, ext. 7481 or by email at areid@alcor.concordia.ca.

Thank you for your participation.

C3. Invitation to participate

Invitation to Participate

in a mathematics teaching experiment

Students enrolled in at least one of the MATH 200-209 courses this term are invited to participate in a teaching experiment conducted by Maria Tutino under the supervision of Dr. Anna Sierpinska of Concordia University's Mathematics and Statistics Department.

The purpose of this experiment is to study students' responses to a visual approach to the teaching of mathematics in the MATH 200-209 courses. The goal is to study the efficacy of this approach.

You are invited to participate in this teaching experiment in the role of a student. The participation will consist in spending about 2.5 hours with an interviewer (the researcher) on the following activities:

1. Filling out a short questionnaire about yourself and the math courses you are taking (10 min)
2. Taking part in a mathematical lecture presented by the researcher (20 min)
3. Communicating your possible difficulties of understanding the lecture in an interview following the lecture; the interviewer will help you to overcome these difficulties (20 min)
4. Working on a few exercises based on the lecture; You will have a copy of the slides used in the lecture to consult while solving the exercises (40 min)
5. Explaining how you solved the exercises to the interviewer (30 min)
6. Commenting on the teaching experiment as a whole; you will be asked to say what you liked and did not liked about it, and to share any suggestions to improve it (15-30 min).

Your reactions and responses will be kept strictly confidential and treated as anonymous in any publications that may result from this study. Your decision to participate or not will have no adverse effects on your grades in the courses you are taking. However, you might find the experience useful for your personal development, or understanding of mathematics, and even enjoyable.

There will be a $30 remuneration for your participation.

If you are interested in participating in this experiment, please send an email to maria.tutino@mail.mcgill.ca.

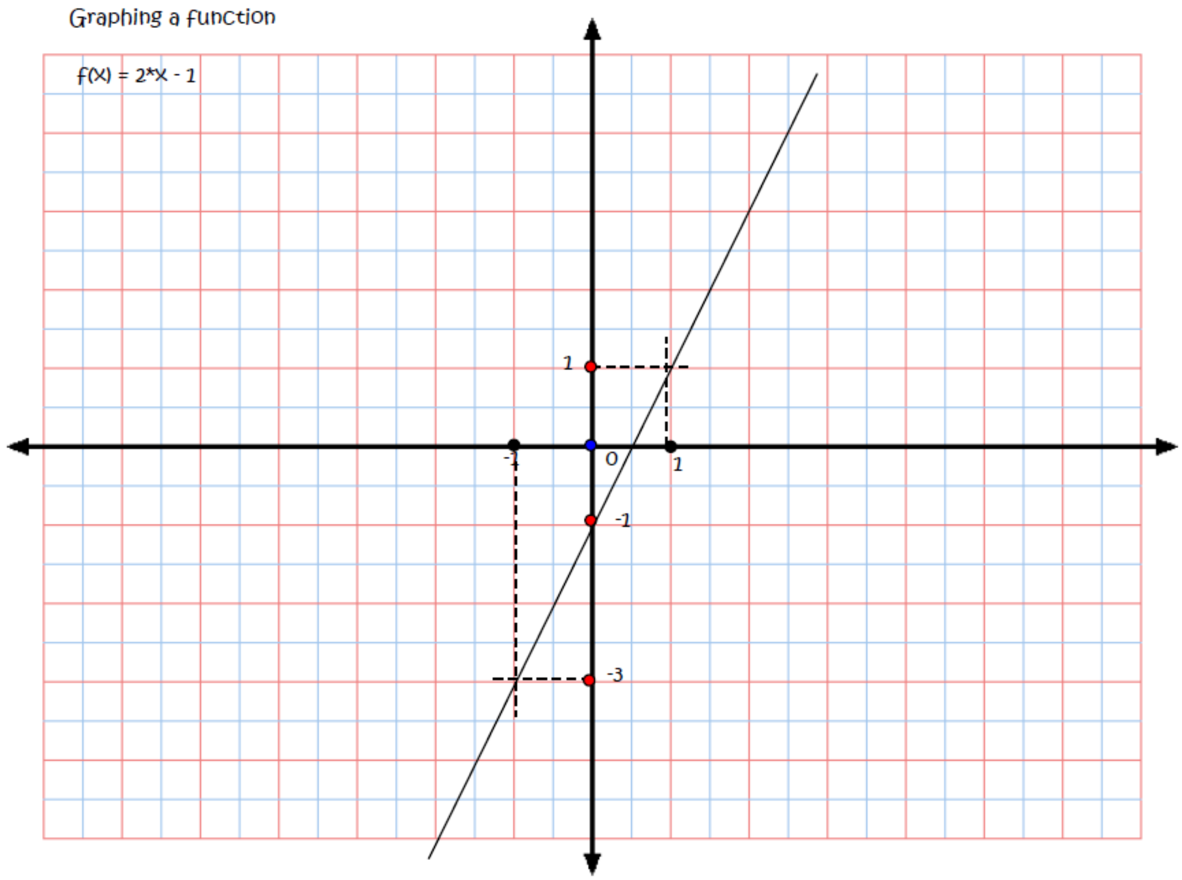
A convenient time will then be arranged for the experiment with you.

"Just a darn minute! — Yesterday you said that X equals two!"

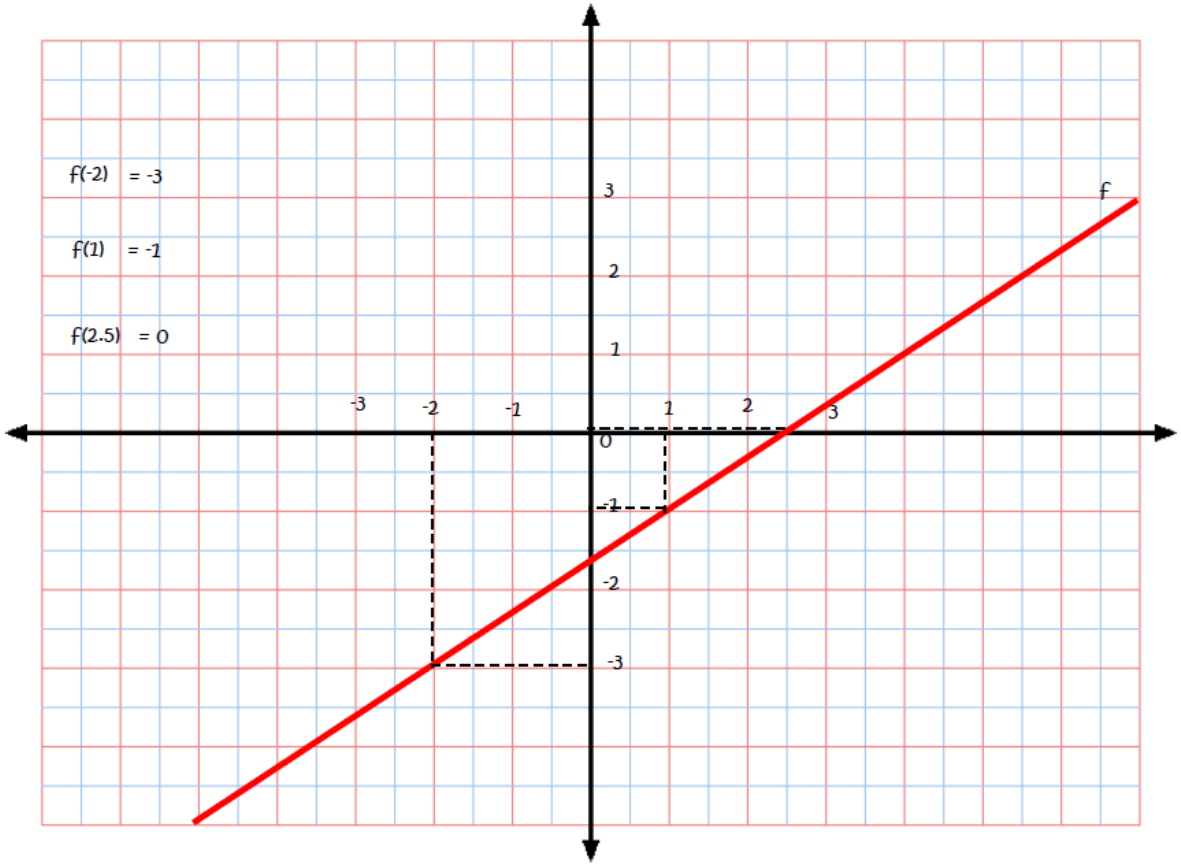
C4. Slides distributed to participants, from Sierpinska et al. (2011)

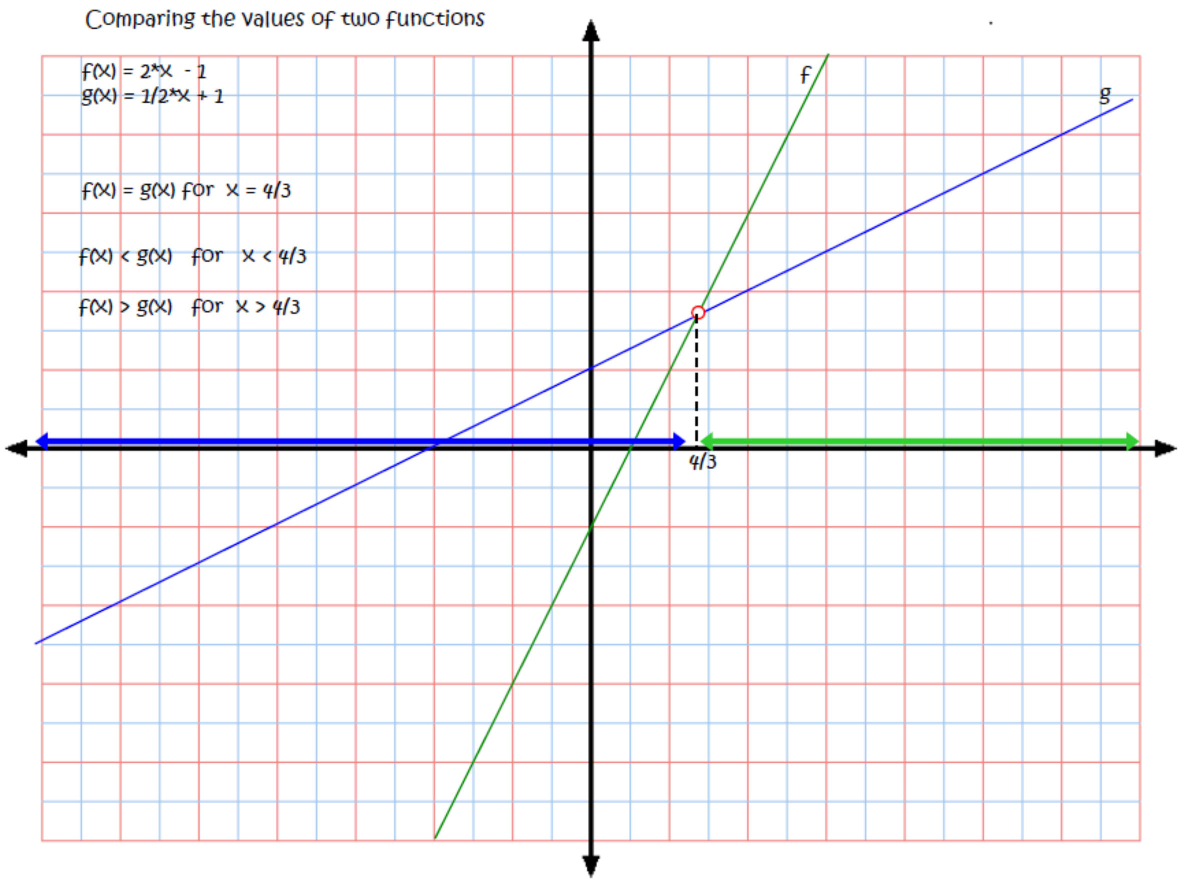
A visual approach to solving inequalities with absolute values

- We start by a brief review of graphing functions
- Next, we introduce the concept of absolute value
- And, finally, we learn how to solve inequalities with absolute value


Graphing a function
f(X) = 2*X - 1
1
-1
0
1
-1
-3

Exercise: To find the value of a function from its graph

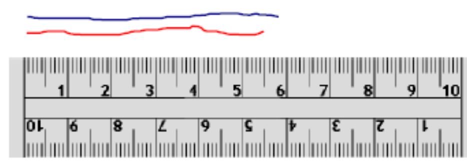



Comparing the values of two functions
f(x) = 2*x - 1
g(x) = 1/2*x + 1
f(x) = g(x) for x = 4/3
f(x) < g(x) for x < 4/3
f(x) > g(x) for x > 4/3
f
g
4/3

11 A LESSON ON ABSOLUTE VALUE

Jane and Joe are measuring the circumference of a dime with a string.

Jane' s result is: 55 mm
Joe's result is: 58 mm



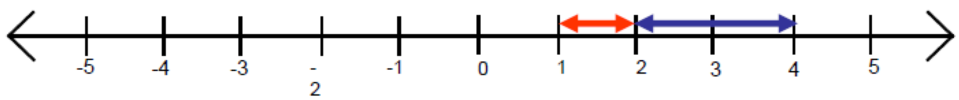
Tom knows the true length of the circumference: 56 mm.
He calculates the difference between the true length and the measurements:

56 - 55 = 1 56 - 58 = -2

He says: Since 1 > -2 then Jane made a bigger mistake than Joe. Do you agree with Tom?

Sometimes we are not interested in knowing whether a measurement was less than or greater than the true value
but only in the MAGNITUDE of the difference.

We call this magnitude THE ABSOLUTE VALUE of the number obtained as the difference.



$$|2 - 1| < |2 - 4|$$

The absolute value of 2 - 1 is equal to 1.

$$|2 - 1| = |1| = 1$$

The absolute value of 2 - 4 is equal to 2.

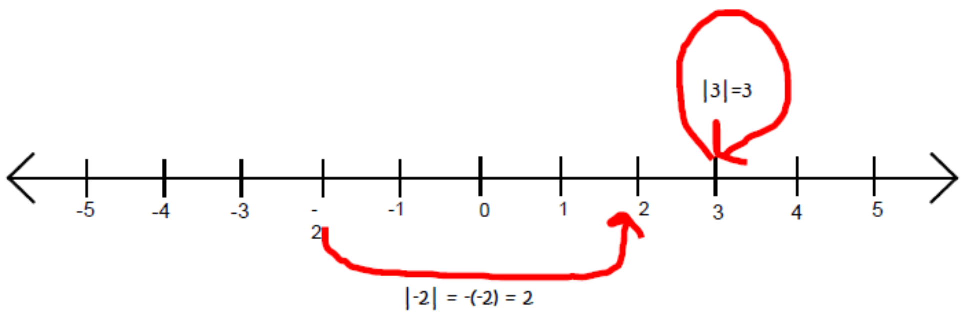
$|2 - 4| = |-2| = 2 =$ the opposite of $-2 = -(-2)$

In general, we define the absolute value of a number x as being equal to x for positive numbers x, and equal to the opposite of x for negative numbers x.
As for the number zero, we decide that $|0| = 0$. The absolute value of a number is thus never negative.

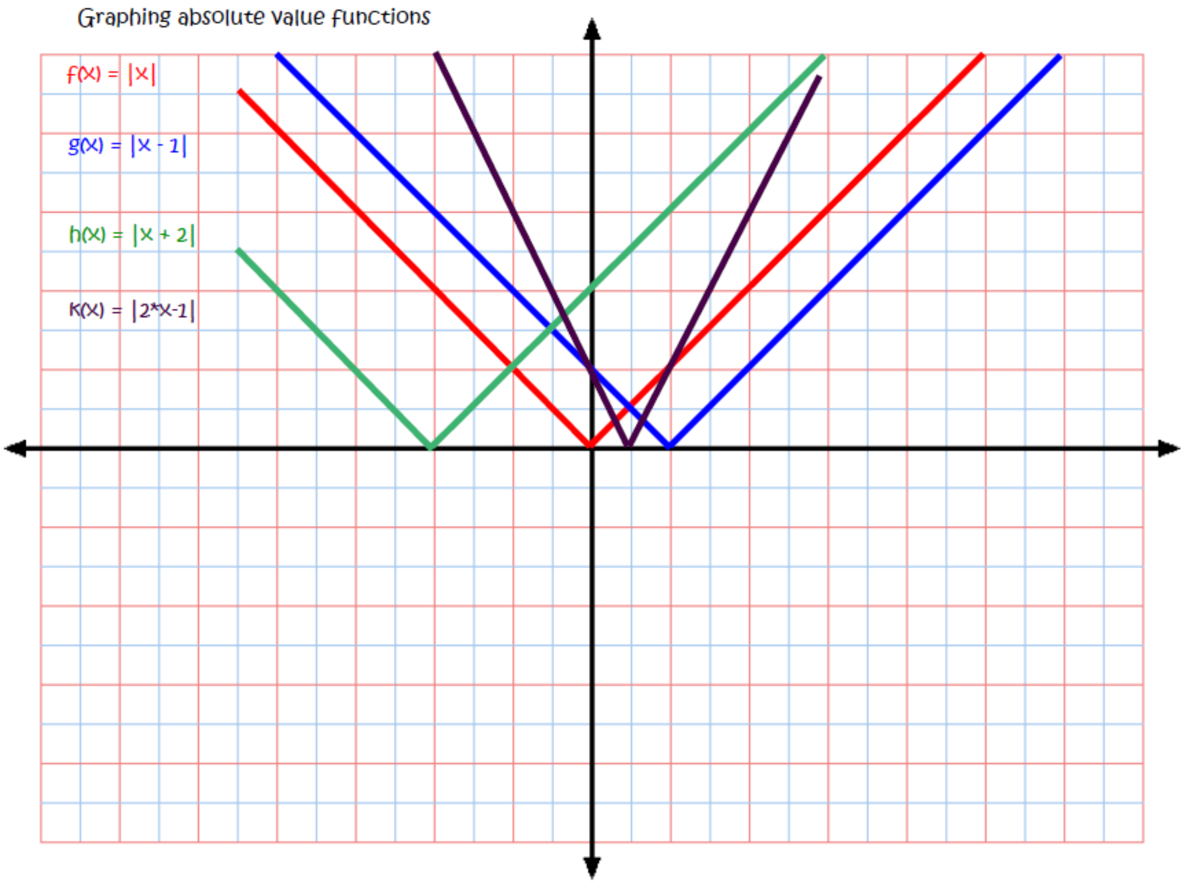
We can think of the absolute value of a number as a function, defined as follows:

$$|x| = \begin{cases} x & \text{if } x \text{ positive or } 0 \\ -x & \text{if } x \text{ is negative} \end{cases}$$

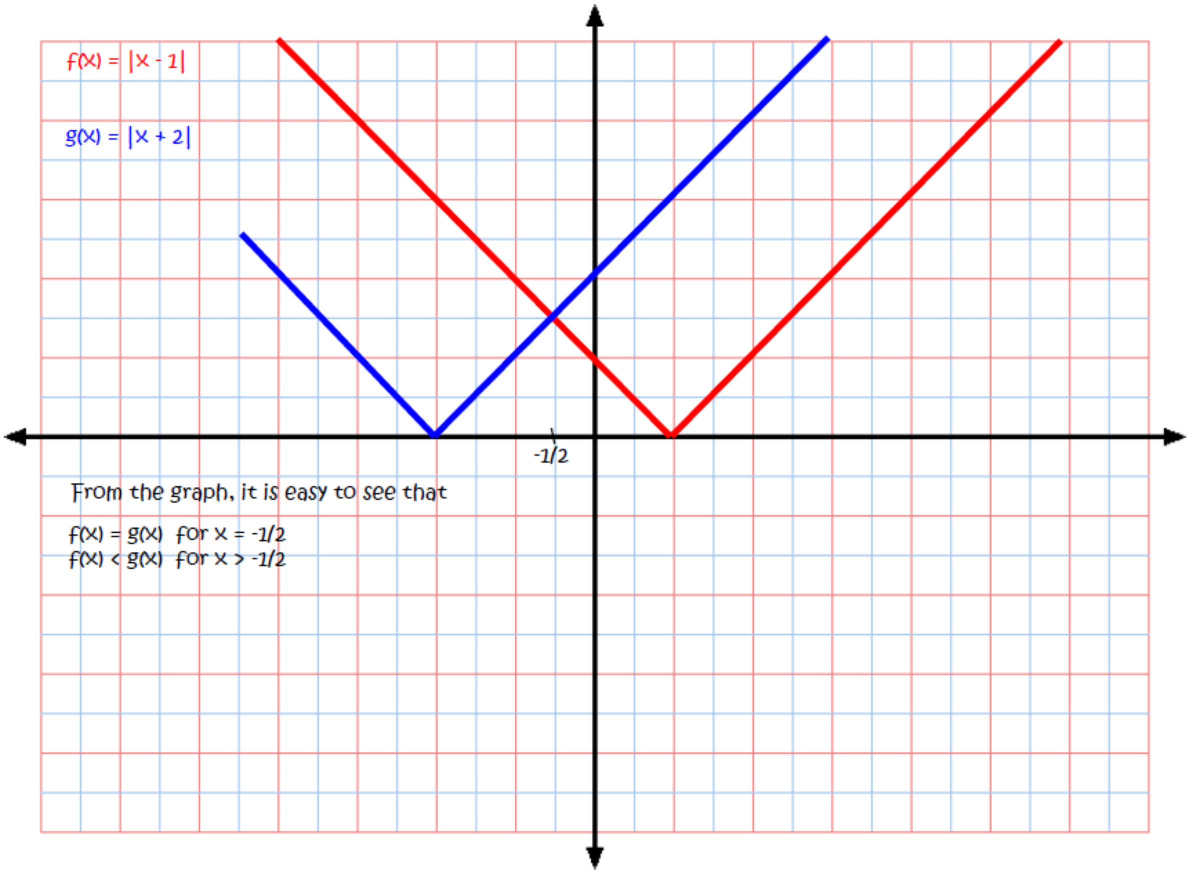
Here is a visual representation of this definition, using the number line:



The absolute value of a positive number is equal to itself.
The absolute value of a negative number is equal to its opposite.


Graphing absolute value functions
f(x) = |x|
g(x) = |x - 1|
h(x) = |x + 2|
k(x) = |2*x-1|

Exercise: Find all values of x for which |x - 1| < |x + 2|



Suppose it is not so easy to see from the graph for what values of x one function is equal to another
or where one function is less than the other one.
What does one do in such cases?

We can use reasoning and algebra.

Let's look again at the inequality $|x - 1| < |x + 2|$.

$|x - 1| = x - 1$ for $x - 1 > 0$ or $x - 1 = 0$
$|x - 1| = -(x - 1)$ for $x - 1 < 0$

$|x + 2| = x + 2$ for $x + 2 > 0$ or $x + 2 = 0$
$|x + 2| = -(x + 2)$ for $x + 2 < 0$

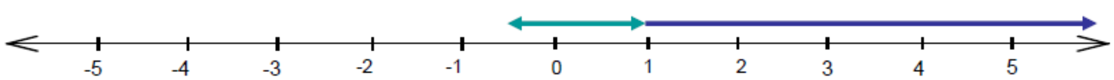
So we have 4 cases to consider:

$x \geqslant 1$ and $x \geqslant -2$ and $x - 1 < x + 2$ ← $x \geqslant 1$

$x \geqslant 1$ and $x < -2$ and $x - 1 < -x - 2$ ← This case is impossible

$x < 1$ and $x \geqslant -2$ and $-x + 1 < x + 2$ ← $-2 \leqslant x < 1$ and $-1/2 < x$ → $-1/2 < x$

$x < 1$ and $x < -2$ and $-x + 1 < -x - 2$ ← This case is also impossible



Exercises

1. Calculate: $||16-24| - |7-56||$

In each of the following exercises, find the values of the number x for which the given inequality is true

2. $|x - 1| < |x + 1|$

3. $|x + 3| < -3|x - 1|$

4. $|2x - 1| < 5$

5. $|2x - 1| > 5$

6. $|50x - 1| < |x + 100|$

Thank you for your attention!