Some Results in Extremal Combinatorics

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Abstract

Some Results in Extremal Combinatorics

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Extremal Combinatorics is one of the central and heavily contributed areas in discrete mathematics, and has seen an outstanding growth during the last few decades. In general, it deals with problems regarding determination and/or estimation of the maximum or the minimum size of a combinatorial structure that satisfies a certain combinatorial property. Problems in Extremal Combinatorics are often related to theoretical computer science, number theory, geometry, and information theory. In this thesis, we work on some well-known problems (and on their variants) in Extremal Combinatorics concerning the set of integers as the combinatorial structure.

The *van der Waerden number* $w(k; t_0, t_1, \ldots, t_{k-1})$ is the smallest positive integer n such that every k-colouring of $1, 2, \ldots, n$ contains a monochromatic arithmetic progression of length t_j for some colour $j \in \{0, 1, \ldots, k-1\}$. We have determined five new exact values with $k = 2$ and conjectured several van der Waerden numbers of the form $w(2; s, t)$, based on which we have formulated a polynomial upper-bound-conjecture of $w(2; s, t)$ with fixed s. We have provided an efficient SAT encoding for van der Waerden numbers with $k \geqslant 3$ and computed three new van der Waerden numbers using that encoding. We have also devised an efficient problem-specific backtracking algorithm and computed twenty-five new van der Waerden numbers with $k \geqslant 3$ using that algorithm.

We have proven some counting properties of arithmetic progressions and some unimodality properties of sequences regarding arithmetic progressions. We have generalized Szekeres' conjecture on the size of the largest sub-sequence of $1, 2, \ldots, n$ without an arithmetic progression of length k for specific k and n; and provided a construction for the lower bound corresponding to the generalized conjecture.

A Strict Schur number $S(h, k)$ is the smallest positive integer n such that every 2-colouring of $1, 2, \ldots, n$ has either a blue solution to $x_1 + x_2 + \cdots + x_{h-1} = x_h$ where $x_1 < x_2 < \cdots < x_h$, or a red solution to $x_1 + x_2 + \cdots + x_{k-1} = x_k$ where $x_1 < x_2 < \cdots < x_k$. We have proven the exact formula for $S(3, k)$.

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I dedicate this thesis to my wife Andalib

for her unconditional love, constant support, and immense patience.

Contents

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List of Symbols

$\mathbb{N}$	$\{0,1,2,\ldots\}$, the set of natural numbers
$\mathbb{Z}$	$\{\ldots,-2,-1,0,1,2,\ldots\}$, the set of integers
$\mathbb{Z}^+$	$\{1,2,3,\ldots\}$, the set of positive integers
$\mathbb{Z}_p$	$\{0,1,2,\ldots,p-1\}$, the set of integers with arithmetics taken modulo p
t-AP	arithmetic progression $a, a+d, \ldots, a+(t-1)d$ for $a \in \mathbb{Z}$ and $d \in \mathbb{Z}^+$
$w(k;t_0,t_1,\ldots,t_{k-1})$	van der Waerden number
$w(k,t)$	van der Waerden number with $t_0 = t_1 = \cdots = t_{k-1} = t$
$ap(t,n)$	The set of t-APs in $1,2,\ldots,n$
$c(t,n)$	The number of t-APs in $1,2,\ldots,n$
$ap(t,n;x)$	The set of t-APs in $1,2,\ldots,n$ that contain x
$c(t,n;x)$	The number of t-APs in $1,2,\ldots,n$ that contain x
$c_i(t,n;x)$	The number of t-APs in $1,2,\ldots,n$ that contain x as the i-th element
$r(t,n)$	Length of the longest t-AP free sub-sequence in $1,2,\ldots,n$
$b(t,n)$	Number of t-AP free sub-sequence of length $r(t,n)$ in $1,2,\ldots,n$
$S(h,k)$	2-colour Strict Schur number
L_m	System of inequalities given by $x_1 + x_2 + \cdots + x_{m-1} = x_m$ and $x_1 < x_2 < \cdots < x_m$
r_i	i-th red number
b_i	i-th blue number
p_k	Sum of the first k red numbers in $1,2,\ldots,n$ corresponding to $S(3,k)$
N_k	$\begin{cases} 3k^2/2 - 7k/2 + 3 & \text{if } k \equiv 0,1 \pmod 4, \\ 3k^2/2 - 7k/2 + 4 & \text{if } k \equiv 2,3 \pmod 4. \end{cases}$

Chapter 1

Introduction

Extremal Combinatorics is one of the central and heavily contributed areas in discrete mathematics, and has seen an outstanding growth during the last few decades. In general, it deals with problems regarding the determination and/or estimation of the maximum or the minimum size of a combinatorial structure that satisfies a certain combinatorial property. Problems in Extremal Combinatorics are often related to theoretical computer science, number theory, geometry, and information theory. In this thesis, we work on some well-known problems (and on their variants) in extremal combinatorics concerning the set of integers as the combinatorial structure.

1.1 Some basic definitions and notations

An *arithmetic progression* of length t, or a t-AP is a sequence of integers of the form $a, a+d, \ldots, a+(t-1)d$, where $a \in \mathbb{Z}$ and $d \in \mathbb{Z}^+$. For example, $3, 7, 11, 15$ is a 4-AP with $a = 3$ and $d = 4$. Let $ap(t, n)$ denote the set of t-APs in $1, 2, \ldots, n$.

A *group* G is a set together with a binary operation $+$ (addition) with the following properties:

(1) G is closed under $+$,

(2) $+$ is associative,

(3) there exists an indentity element in G, and an inverse for each $a \in G$.

For A, B subsets of an additive group $\mathbb{Z}$, the *sumset* $A + B$ is defined as $\{a + b : a \in A, b \in B\}$.

For a single-element set $\{b\}$, the set $\{b\} + A$ is called a *translate* of A. The set $\{ka : a \in A\}$ is called a *dilate* of A. It can be observed that a t-AP remains a t-AP after translations and dilations.

A subset A of an additive group is called *sum-free* if $A \cap (A + A) = \emptyset$. For example, the set $\{1, 3, 5\}$ is sum-free, but the set $\{1, 2\}$ is not.

Given positive integers k, $t_0, t_1, \ldots, t_{k-1}$, the *van der Waerden number* $w(k; t_0, t_1, \ldots, t_{k-1})$ is the smallest positive integer n such that every k-colouring of $1, 2, \ldots, n$ contains a monochromatic arithmetic progression of length t_j for some colour $j \in \{0, 1, \ldots, k-1\}$. If $t_0 = t_1 = \cdots = t_{k-1} = t$, then the number is denoted by $w(k, t)$ and is generally regarded as a *Classical* van der Waerden number.

Given positive integer k, the Schur number $S(k)$ is the smallest positive integer n such that every k-colouring of $1, 2, \ldots, n$ contains a monochromatic solution to $x + y = z$.

1.2 Some Classical problems in Extremal Combinatorics

Now, we list a few classical questions in Extremal Combinatorics. Details including references or results on the problems relevant to this thesis will be discussed in Chapter 2.

- ⋄ *Dense progression-free integer sets*: given a large positive integer N, what is the largest size of a subset in the interval $[1, N]$ free of t-APs?
- ⋄ *Large sum-free subsets of integer sets*: what is the size of the largest sum-free subset of a given set?
- ⋄ What is the order of growth of the van der Waerden numbers $w(k, t)$?
- ⋄ *Arithmetic progressions in sets with diverging reciprocals*: Suppose that $A \subset \mathbb{N}$ has the property that the sum of the reciprocals of the elements of A diverges. Does A contain an arbitrarily long arithmetic progression?

1.3 Outline of the thesis and our contributions

In Chapter 2, we discuss some results on sum-free integer sets, dense progression-free integer sets, and van der Waerden numbers. In each of those cases, small proofs are included as part of the brief survey.

In Chapter 3, we have reported five previously unknown van der Waerden numbers of the form $w(2; t_0, t_1)$, that have been computed using a distributed application of an efficient implementation of the DPLL algorithm [22, 5]. We have used local-search based algorithms to establish lower bounds of several other two-colour van der Waerden numbers. Based on the experimental results, we have formulated polynomial upper-bound conjectures for two-colour van der Waerden numbers. We have used a new encoding as well as a problem-specific backtrack algorithm to compute a total of twenty-eight previously unknown van der Waerden numbers of the form $w(k; t_0, t_1, \ldots, t_{k-1})$ with $k \geqslant 3$. We have also listed the new van der Waerden numbers as they appear in the On-line Encyclopædia of Integer Sequences (OEIS) [86].

In Chapter 4, we have proven some counting properties of arithmetic progressions and some unimodality properties of the sequence $c(k, n; i)$ for $i = 1, 2, \ldots, n$, where $c(k, n; i)$ is the number of k-APs in $1, 2, \ldots, n$, each containing i as an element. Based on experimental data, we have generalized Szekeres' conjecture on the size of the largest k-AP free sub-sequence of $1, 2, \ldots, n$ for specific k and n, and provided a construction for the lower bound corresponding to the generalized conjecture.

Finally, in Chapter 5, we have proven the exact formula for $S(3, k)$, which is the smallest positive integer n such that every (blue,red)-colouring of $1, 2, \ldots, n$ either contains a blue solution to $x_1 + x_2 = x_3$ where $x_1 < x_2 < x_3$, or a red solution to $x_1 + x_2 + \cdots, x_{k-1} = x_k$ where $x_1 < x_2 < \cdots < x_{k-1} < x_k$.

Chapter 2

Background

2.1 Sum-free integer sets

Schur's theorem on sum-free sets is considered as the earliest result in Ramsey theory.

2.1.1 Schur's theorem

Theorem 2.1.1 (Schur [72], 1916). *Given a positive integer k, there exists a positive integer n such that every k-colouring of $1, 2, \ldots, n$ contains a monochromatic solution to $x + y = z$.*

To prove Theorem 2.1.1, we need a special case of the following result from Ramsey theory.

Theorem 2.1.2 (Ramsey [66], 1930). *Given positive integers k and t, there exists a positive integer n such that every k-colouring of the edges of a complete graph K_n contains a monochromatic complete subgraph K_t.*

The *Ramsey number* $R(k, t)$ is the smallest positive integer satisfying Ramsey's theorem. The only known exact values of $R(k, t)$ are $R(2, 3) = 6$, $R(2, 4) = 18$, and $R(3, 3) = 17$, all three of which are discovered by Greenwood and Gleason [41] in 1955. Given positive integer k and positive integers $t_0, t_1, \ldots, t_{k-1}$, the generalized Ramsey number $R(k; t_0, t_1, \ldots, t_{k-1})$ is the smallest integer n such that every k-colouring of the edges of K_n contains a monochromatic complete subgraph K_j with t_j vertices for some $j \in \{0, 1, \ldots, k-1\}$. In this notation, $R(2, 3) = R(2; 3, 3)$. Some known values are

$R(2;3,4) = 9$, $R(2;3,5) = 14$, $R(2;3,6) = 18$, $R(2;3,7) = 23$, $R(2;3,8) = 28$, $R(2;3,9) = 36$, and $R(2;4,5) = 25$. For a survey on the values and bounds of Ramsey numbers, see Radziszowski [65].

Lemma 2.1.3. *Given positive integers k, there exists a positive integer n such that every k-colouring of the edges of a complete graph K_n contains a monochromatic triangle. (This is a special case of Theorem 2.1.2 with $t = 3$)*

Proof. We do induction on k. For $k = 2$, it can be easily verified that for $n_2 \geqslant 6$, there is a monochromatic triangle in every 2-colouring of the complete graph with n_2 vertices. Suppose there exists a positive integer n_{k-1} such that every $(k-1)$-colouring of the edges of the complete graph with n_{k-1} vertices contains a monochromatic triangle. Consider a k-colouring of the complete graph with $n_k = k(n_{k-1}-1)+2$ vertices and choose an arbitrary vertex v. Divide the other $k(n_{k-1}-1)+1$ vertices connected to v, according to the colour of the edges that connect them to v. One of these k colour-classes, say colour-class C_j for some colour j, will contain n_{k-1} or more vertices. Then colour j cannot be used between the vertices in C_j, or else we have a triangle. So the set C_j is internally coloured with $k-1$ colours and must contain a monochromatic triangle. □

Proof of Theorem 2.1.1. Consider the edges of the complete graph K_n with vertices $1, 2, \ldots, n$ be coloured with colours $1, 2, \ldots, k$. Define a colouring such that edge (i, j) belongs to the colour-class $|i-j|$. Choose $n = R(k, 3)$, which implies K_n contains a monochromatic triangle. If $i < j < k$ be the vertices of this triangle, listed in increasing order, then writing $x = j - i$, $y = k - j$, and $z = k - i$, we get a monochromatic solution to $x + y = z$. □

The smallest positive integer for a given positive integer k, satisfying Schur's theorem is known as the *Schur number* $S(k)$. Only four exact values of Schur numbers are $S(1) = 2$, $S(2) = 5$, $S(3) = 14$, and $S(4) = 45$. The first three values are easy to verify and $S(4)$ is given by Baumert [11]. Exoo [30] gave the lower bound of $S(5) \geqslant 161$. The lower limits $S(6) \geqslant 537$ and $S(7) \geqslant 1681$ are due to Fredricksen and Sweet [33].

Theorem 2.1.1 has been generalized by Rado [64] and Sanders [70], though it is named after Folkman by Graham, Rothschild and Spencer [39].

Theorem 2.1.4 (Folkman's theorem)**.** *Given positive integers k and t, there exists a positive integer n such that every k-colouring of $1, 2, \ldots, n$ contains a monochromatic solution to $x_1+x_2+\cdots+x_{t-1} = x_t$.*

2.2 Dense progression-free integer sets

A subsequence of $1, 2, \ldots, N$ is called *k-AP free* if it does not contain any k-term arithmetic progression. Let $r(k, N)$ denote the length of the longest k-AP free subsequences in $1, 2, \ldots, N$. The study of the function $r(3, N)$ was initiated by Erdős and Turán [29]. They determined the values of $r(3, N)$ for $N \leqslant 23$ and $N = 41$. They proved that for $N \geqslant 8$

$$r(3, 2N) \leqslant N$$

and they conjectured that

$$\lim_{n\to\infty} r(3, N)/N = 0.$$

This conjecture was proved in 1975 by Szemerédi [80]. Erdős and Turán also conjectured that $r(3, N) < N^{1-c}$, which was shown to be false by Salem and Spencer [69], who proved

$$r(3, N) > N^{1-c/\log\log N}.$$

This result was further improved by Behrend [13] to

$$r(3, N) > N^{1-c/\sqrt{\log N}}.$$

Recently, Elkin [25] has further improved this lower bound by a factor of $\Theta(\sqrt{\log N})$. The first non-trivial upper bound was due to Roth [67] who proved that

$$r(3, N) < CN/\log\log N.$$

This result has been improved by Bourgain [16] to

$$r(3, N) < cN\sqrt{\log\log N/\log N}.$$

Sharma [74] showed that Erdős and Turán gave the wrong value of $r(3, 20)$ and determined the values of $r(3, N)$ for $n \leqslant 27$ and $41 \leqslant N \leqslant 43$. Recently, Dybizbański [24] has computed the exact values of $r(3, N)$ for all $N \leqslant 123$ and proved that $r(3, 3N) \leqslant N$ for $N \geqslant 16$.

In Chapter 4, we generalize a conjecture of Szekeres on $r(k, N)$ based on experiemntal data.

2.2.1 Behrend's construction of 3-AP free subsets

In this section, we describe Behrend's construction for $r(3, N)$.

Theorem 2.2.1 (Behrend [13], 1946)**.** *There is a set A in $1, 2, \ldots, N$ which is 3-AP free and satisfies*

$$|A| \gg N \exp(-c\sqrt{\log N}),$$

where c is an absolute positive constant.

Proof. Consider the point $x = (x_1, x_2, \ldots, x_n) \in \{1, 2, \ldots, M\}^n$. There are M^n such points and for each of the M^n points, we have

$$r^2 = x_1^2 + x_2^2 + \cdots + x_n^2,$$

which is an integer in the interval $[n, nM^2]$. By pigeonhole principle, there must exist a sphere $S_n(M)$ with radius r containing at least

$$|S_n(M)| \geqslant \left\lceil \frac{M^n}{nM^2 - n + 1} \right\rceil \geqslant \frac{M^n}{n(M^2-1)} > \frac{M^{n-2}}{n}$$

points. Let $P : \mathbb{Z}^n \to \mathbb{Z}$ for mapping $S_n(M)$ to integers be defined by

$$P(x) = \frac{1}{2M} \sum_{i=1}^{n} x_i (2M)^i,$$

which is integer-valued and for each $x \in \{1, 2, \ldots, M\}^n$

$$1 \leqslant P(x) \leqslant (2M)^n.$$

It can also be observed that $P(x)$ is linear. Take $x, y \in \mathbb{Z}^n$ and $a, b \in \mathbb{Z}$. We have

$$\begin{aligned} P(ax+by) &= \frac{1}{2M}\sum_{i=1}^{n}(ax_i+by_i)(2M)^i \\ &= a\left(\frac{1}{2M}\sum_{i=1}^{n}x_i(2M)^i\right)+b\left(\frac{1}{2M}\sum_{i=1}^{n}y_i(2M)^i\right) \\ &= aP(x)+bP(y). \end{aligned}$$

Now, we show that P is one-to-one in the domain $\{1, 2, \ldots, M\}^n$. Let $P(x) = P(y)$ for $x, y \in \{1, 2, \ldots, M\}^n$. By the linearity of P, we have $P(x)-P(y) = P(x-y)$. By the assumption that $P(x) = P(y)$, we have $P(x-y) = 0$. It remains to show that for $w \in \{-2M, -2M+1, \ldots, 2M-1, 2M\}^n$, $P(w) = 0$ if and only if $w = 0$. If $w = 0$, then $P(w) = 0$ by definition. Let $P(w) = 0$ and $w \neq 0$, then there is a least coordinate j such that $w_j \neq 0$. Then we have

$$P(w) = \frac{1}{2M}\sum_{i=1}^{n} w_i(2M)^i = \frac{1}{2M}\sum_{i=j}^{n} w_i(2M)^i = 0.$$

which implies

$$|w_j| = \sum_{i=j+1}^{n} w_i(2M)^{i-j} = 2M\sum_{i=0}^{n-(j+1)} w_{i+(j+1)}(2M)^i = 2M \cdot k,$$

where k is an integer. Since we are assuming that $1 < |w_j| < 2M$, we need $0 < k < 1$ which is impossible. Hence $w = 0$. Therefore, $P(x-y) = 0$ implies $x - y = 0$, that is, $x = y$.

Take $n = \left\lceil \sqrt{\log N} \right\rceil$ and $M = \left\lfloor N^{1/n}/2 \right\rfloor$. Define $A = \{P(x) : x \in S_n(M)\}$. Since P is integer-valued and $1 \leqslant P(x) \leqslant (2M)^n$ for each $x \in \{1, 2, \ldots, M\}^n$, we have $A \subseteq [1, (2M)^n] \subseteq \{1, 2, \ldots, N\}$. Since P is one-to-one, we have $|A| = |S_n(M)|$. Finally, we show that A does not contain a 3-AP. We can observe that for $x, y, z \in \{1, 2, \ldots, M\}^n$, if $P(z) - P(y) = P(y) - P(x)$, then $z - y = y - x$. By assumption, $P(z) - 2P(y) - P(x) = 0$, and by linearity of P, we have $P(z - 2y + x) = 0$. Since $(z - 2y + x) \in \{-2M, -2M+1, \ldots, 2M-1, 2M\}^n$, we have $z - 2y + x = 0$, that is, $z - y = y - x$.

So any non-trivial 3-AP in A corresponds to a non-trivial 3-AP in $S_n(M)$, which is impossible since a line can intersect at most two points in an Euclidean sphere.

Therefore,

$$\begin{aligned}
|A| &= |S_n(M)| \geqslant \frac{M^{n-2}}{n} = \left\lfloor N^{1/n}/2 \right\rfloor^{n-2}/n \geqslant \left\lfloor N^{1/n}/e \right\rfloor^{n-2}/n = N \cdot exp(2-n) \cdot N^{-2/n} \cdot 1/n \\
&= N \cdot exp(2 - \lceil \sqrt{\log N} \rceil) \cdot N^{(-2/\lceil \sqrt{\log N} \rceil)} \cdot 1/\lceil \sqrt{\log N} \rceil \\
&\geqslant N \cdot exp(2 - (\sqrt{\log N} - 1)) \cdot exp(-2 \log N/\sqrt{\log N}) \cdot 1/(\sqrt{\log N} + 1) \\
&\geqslant N \cdot exp(2 - (\sqrt{\log N} - 1)) \cdot exp(-2 \log N/\sqrt{\log N}) \cdot exp(-1 - \sqrt{\log N}) \\
&= N \cdot exp(-4\sqrt{\log N}).
\end{aligned}$$

□

2.3 Van der Waerden numbers

The following theorem, widely known as one of the "Three Pearls of Number Theory", provides a basic result on arithmetic progressions. Since this thesis presents results on the values and bounds of van der Waerden numbers (Chapter 3), we provide a brief history of results starting from van der Waerden's theorem in 1927.

Theorem 2.3.1 (Van der Waerden [83], 1927)**.** *Given positive integers k and t, there is an integer n such that every k-colouring of the set $\{1, 2, \ldots, n\}$ has a monochromatic t-AP.*

The smallest integer $w(k, t)$ satisfying van der Waerden's theorem is known as the *van der Waerden number*. Given positive integers k, t, and n, a *good k-colouring* of $1, 2, \ldots, n$ contains no monochromatic t-AP. We call such a good k-colouring a *certificate* of the lower bound $w(k, t) > n$. We write a certificate as a sequence $c_1 c_2 \ldots c_n$ where each c_i in $\{0, 1, \ldots, k-1\}$ represents the colour assigned to the integer i. For example, 00110011 or $0^2 1^2 0^2 1^2$ for short, is a certificate of $w(2, 3) > 8$. Every 2-colouring of $1, 2, \ldots, n$ with $n \geqslant 9$ leaves a monochromatic 3-AP, and so $w(2, 3) = 9$.

2.3.1 Lower bounds of van der Waerden numbers

In this section, we discuss known results on the lower bounds. For each result, necessary background materials are also provided to make the literature-review self-contained. Much effort has been made to obtain theoretical lower bounds of $w(k,t)$, and more specifically of $w(2,t)$. We present several probabilistic lower bounds that ensure the existence of a good-colouring of $1, 2, \ldots, n$ (which can be derandomized to construct certificates of lower bound), and an algebraic lower bound that explicitly constructs a good-colouring of $1, 2, \ldots, n$.

2.3.1.1 Probabilistic lower bounds

The earliest lower bound, which is due to Erdős and Rado, is probabilistic and requires the following lemma:

Observation 2.3.2. *The number of t-APs in $ap(t,n)$ containing integer x on the i-th position is*

$$c_i(t,n;x) = \begin{cases} \left\lfloor (n-x)/(t-1) \right\rfloor & \text{if } i = 1, \\ \left\lfloor (x-1)/(t-1) \right\rfloor & \text{if } i = t, \\ \min\left\{ \left\lfloor (n-x)/(t-i) \right\rfloor, \left\lfloor (x-1)/(i-1) \right\rfloor \right\} & \text{otherwise.} \end{cases}$$

Lemma 2.3.3. *Let the total number of t-APs in $ap(t,n)$ be denoted by $c(t,n)$. Then $c(t,n) < n^2/t$.*

Proof. By definition of $c(t,n)$ and $c_1(t,n;x)$, we have,

$$c(t,n) = \sum_{x=1}^{n-t+1} c_1(t,n;x) \leqslant \sum_{x=1}^{n-t+1} \left(\frac{n-x}{t-1}\right) < \sum_{x=1}^{n-1} \left(\frac{n-x}{t-1}\right) = \frac{n(n-1)}{2(t-1)} < \frac{n^2}{t}.$$

where the leftmost inequality is due to Observation 2.3.2. □

Theorem 2.3.4 (Erdős and Rado [27], 1952)**.** *Given positive integers k and t,*

$$w(k,t) > \sqrt{t}k^{(t-1)/2}.$$

Proof. Let $w(k,t) = n$. Then every k-colouring of $1, 2, \ldots, n$ contains a monochromatic t-AP, that is, $P(\text{A monochromatic } t\text{-AP exists}) = 1$. Randomly k-colour $1, 2, \ldots, n$, where each i picks a colour with probability $1/k$. Let A_S be the event that a given t-AP, say S, is monochromatic. So $P(A_S) = k\,(1/k^t) = k^{1-t}$ and therefore using Lemma 2.3.3,

$$1 = P(\text{A monochromatic } t\text{-AP exists}) \leqslant \sum_{S \in ap(t,n)} P(A_S) < (n^2/t) \cdot k^{1-t},$$

which gives $n > \sqrt{t}k^{(t-1)/2}$, and hence the desired bound. □

A constructive proof of $w(2,t) > \sqrt{t}2^{(t-1)/2}$. An algorithm (Erdős and Selfridge [28]) for constructing a certificate to prove the lower bound $w(2,t) > \sqrt{t}2^{(t-1)/2}$ is as follows: Let $n \leqslant \sqrt{t}2^{(t-1)/2}$, that is, $(n^2/t)2^{1-t} \leqslant 1$, and there exists a good 2-colouring of $1, 2, \ldots, n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f(x_1, x_2, \ldots, x_n) = \sum_{s \in ap(t,n)} \left[\prod_{i \in s} x_i + \prod_{i \in s} (1 - x_i) \right],$$

where in a 2-colouring of $1, 2, \ldots, n$, we have $x_i \in \{0, 1\}$ if integer i has been assigned a colour, and $x_i = 1/2$ otherwise. Then $f(x_1, x_2, \ldots, x_n)$ represents the number of monochromatic t-APs in a 2-colouring of $1, 2, \ldots, n$ using colours 0 and 1. So, a 2-colouring is not good if and only if $f(x_1, x_2, \ldots, x_n) \geqslant 1$. We have to colour $1, 2, \ldots, n$ so that $f(x_1, x_2, \ldots, x_n) < 1$. Initially, when no integer $i \in \{1, 2, \ldots, n\}$ is assigned a colour, we have

$$f(1/2, 1/2, \ldots, 1/2) \quad = \sum_{s \in ap(t,n)} \left[(1/2^t) + (1/2^t) \right] < (n^2/t)2^{1-t} \leqslant 1,$$

Suppose we have already coloured integers $1, 2, \ldots, i-1$ with colours $c_1, c_2, \ldots, c_{i-1}$, respectively, such that $c_j \in \{0, 1\}$ for $1 \leqslant j \leqslant i-1$. Inductively, $f(c_1, c_2, \ldots, c_{i-1}, 1/2, \ldots, 1/2) < 1$.

We observe that

$$\begin{aligned} \frac{1}{2}\left[f(c_1, c_2, \ldots, c_{i-1}, 0, 1/2, \ldots, 1/2)\right. &+ \left. f(c_1, c_2, \ldots, c_{i-1}, 1, 1/2, \ldots, 1/2)\right] \\ &= f(c_1, c_2, \ldots, c_{i-1}, 1/2, \ldots, 1/2). \end{aligned}$$

So we set $c_i = 0$ if $f(c_1, c_2, \ldots, c_{i-1}, 0, 1/2, \ldots, 1/2) \leqslant f(c_1, c_2, \ldots, c_{i-1}, 1, 1/2, \ldots, 1/2)$ and $c_i = 1$ otherwise. The final vector $(x_1, x_2, \ldots, x_n)$ will be a good 2-colouring, that is, a certificate of the lower bound $w(2,t) > \sqrt{t}2^{(t-1)/2}$. $\square$

The Lovász Local Lemma [26] gives the following lower bound:

Theorem 2.3.5 (Lovász [26], 1973)**.** *$w(k,t) \geqslant k^t/ekt$.*

In particular, Theorem 2.3.5 gives $w(2,t) \geqslant 2^t/(2et)$. The following theorem improves this bound, which is the best known bound, again using Lovász Local Lemma.

Theorem 2.3.6 (Szabó [78], 1990)**.** *$w(2,t) \geqslant 2^t/t^\epsilon$ for $\epsilon > 0$ and for sufficiently large t.*

2.3.1.2 Berlekamp's construction

Berlekamp provided the only known general constructive lower bound using field theory. A *field* is a set F with two binary operations $+$ (addition) and $\cdot$ (multiplication) with the following properties:

(1) F is closed under $\cdot$ and $+$,

(2) $\cdot$ and $+$ are associative and commutative,

(3) $\cdot$ is distributive over $+$,

(4) there exist additive and multiplicative identity, and an additive inverse for each $a \in F$; and

(5) there exists a multiplicative inverse for each $a \in F \setminus \{0\}$.

A *Galois field* is a field with exactly p^t elements, denoted by $GF(p^t)$, where p is prime and t is a positive integer. For each $q = p^t$, there exists a unique (up to isomorphism) field of order q. Integers $0, 1, 2, \ldots, p-1$ form a Galois field $GF(p)$, with arithmetics taken modulo p, namely $\mathbb{Z}_p$. $GF(p^t)$ can be seen as a vector-space of dimension t over $\mathbb{Z}_p$, that is, there is a *basis* $\{b_1, b_2, \ldots, b_t\}$ such that every element $e \in GF(p^t)$ can be written uniquely as $e = e_1b_1 + \cdots + e_tb_t$ where $e_1, \ldots, e_t \in \mathbb{Z}_p$.

A Galois field $GF(p^t)$ may be represented by the set of all polynomials of degree at most $t-1$, with coefficients in $\mathbb{Z}_p$, that is,

$$GF(p^t) = \left\{a_{t-1}x^{t-1} + a_{t-2}x^{t-2} + \cdots + a_1x + a_0 | a_i \in \mathbb{Z}_p\right\}.$$

Note that $(1, x, x^2, \ldots, x^{t-1})$ is a basis for $GF(p^t)$ over $\mathbb{Z}_p$. A non-zero element $\alpha \in GF(q)$ is called a *primitive element* if $\alpha, \alpha^2, \ldots, \alpha^{q-1}$ are precisely the non-zero elements of $GF(q)$. It can be shown that every $GF(p^t)$ has a primitive element. So $GF(p^t)$ can be written as $\left\{0, \alpha, \alpha^2, \ldots, \alpha^{p^t-2}, 1\right\}$. Let

$ip(x)$ be a t-degree irreducible polynomial with coefficients in $\mathbb{Z}_p$. The set $GF(p^t)$ can be constructed with addition over Z_p and multiplication modulo $ip(x)$.

For example, the elements of $GF(2^3)$ modulo x^3+x+1 (which is irreducible over $\mathbb{Z}_2$) are

$$\left\{0, x^1, x^2, x^3, x^4, x^5, x^6, 1\right\} = \left\{0, x, x^2, x+1, x^2+x, x^2+x+1, x^2+1, 1\right\}.$$

Theorem 2.3.7 (Berlekamp [14], 1968)**.** *If t is a prime, then $w(2, t+1) > t(2^t-1)$.*

Proof. Let α be a primitive element of $GF(2^t)$, that is, $GF(2^t) - \{0\} = \left\{1, \alpha, \alpha^2, \ldots, \alpha^{2^t-2}\right\}$. Let a basis of $GF(2^t)$ over $\mathbb{Z}_2$ be $(b_1, b_2, \ldots, b_t)$. Now, consider

$$S = \left\{1, \alpha, \alpha^2, \ldots, \alpha^{2^t-2}, \alpha^{2^t-1}, \ldots, \alpha^{t(2^t-1)-1}\right\}$$

(which contains repeated elements), and express the elements in terms of the basis and $a_{i,j} \in \mathbb{Z}_2$ such that

$$\alpha^j = \sum_{i=1}^{t} a_{i,j} b_i.$$

Let the colour classes be S_0 and S_1. For $0 \leqslant j < t(2^t-1)$, set $j \in S_\epsilon$ if and only if $a_{1,j} = \epsilon$. Also define set of non-zero field elements T_ϵ such that $\alpha^j \in T_\epsilon$ for each $j \in S_\epsilon$. We need to show that this is a good 2-colouring. If not, then there is a monochromatic $(t+1)$-AP, say $\{c, c+d, \ldots, c+td\} \subset S_\epsilon$. Here, $c+td < t(2^t-1)$ and $d < (2^t-1)$, that is, $\alpha^d \neq 1$. Since $\alpha^d \in GF(2^t)$ and $\alpha^d \notin \{0,1\}$, the minimal polynomial over $\mathbb{Z}_2$ of which α^d is a root has degree t.

Now, we have the following cases:

1. $\epsilon = 0$: We have $\{c, c+d, \ldots, c+td\} \subset S_0$ and $\left\{\alpha^c, \alpha^{c+d}, \ldots, \alpha^{c+td}\right\} \subset T_0$. Since T_0 is a $(t-1)$-dimensional subspace spanned by $(b_2, b_3, \ldots, b_t)$ over $\mathbb{Z}_2$, any t distinct elements in T_0 are linearly dependent, that is, there exists $\beta_0, \beta_1, \ldots, \beta_{t-1} \in \mathbb{Z}_2$, not all zero, such that

$$\sum_{i=0}^{t-1} \alpha^{c+di}\beta_i = 0 \text{ implying } \sum_{i=0}^{t-1} \beta_i (\alpha^d)^i = 0,$$

that is, α^d is a root of the polynomial of degree at most $t-1$ with coefficients in $\mathbb{Z}_2$, which is not possible.

2. $\epsilon = 1$: $\{c, c+d, \ldots, c+td\} \subset S_1$ and $\{\alpha^c, \alpha^{c+d}, \ldots, \alpha^{c+td}\} \subset T_1$. All the $t+1$ elements, when expressed in the basis $(b_1, b_2, \ldots, b_t)$, have coefficient 1 for b_1.

   Now each of the t elements in $\left\{\alpha^c(\alpha^d - 1), \ldots, \alpha^c\left((\alpha^d)^t - 1\right)\right\}$, when expressed in the basis, has a coefficient zero for b_1, that is, the t elements are in $(t-1)$-dimensional space and are linearly dependent. So there exists $\beta_0, \beta_1, \ldots, \beta_{t-1} \in \mathbb{Z}_2$, not all zero, such that

$$\sum_{i=0}^{t-1} \alpha^c \left[(\alpha^d)^i - 1\right] \beta_i = 0 \text{ implying } \sum_{i=0}^{t-1} \beta_i \left[(\alpha^d)^i - 1\right] = 0,$$

   that is, α^d is a root of the polynomial of degree at most $t-1$ with coefficients in $\mathbb{Z}_2$, which is not possible.

□

The above construction will work for any choice of basis. Let $k = 2$, $t = 3$, and $n = 21$. Consider the polynomial $a(x) = x^3 + x + 1$, which is irreducible over $GF(2)$ and let α be a root of $a(x)$. Take the basis $b_1 = 1$, $b_2 = \alpha$, and $b_3 = \alpha^2$. For each $j = 0, 1, \ldots, 20$ find $(a_{1,j}, a_{2,j}, a_{3,j}) \in \{0,1\}^3$ such that $a_{1,j}b_1 + a_{2,j}b_2 + a_{3,j}b_3 = \alpha^j$. We get $a_{i,j}$ for $1 \leqslant i \leqslant 3$ and $0 \leqslant j \leqslant 20$:

1001011 1001011 1001011

0101110 0101110 0101110

0010111 0010111 0010111

Putting $j \in S_\epsilon$ for $\epsilon \in \{0,1\}$ if and only if $0 \leqslant j \leqslant n-1$ and $a_{1,j} = \epsilon$, we get

$$S_1 = \{0, 3, 5, 6, 7, 10, 12, 13, 14, 17, 19, 20\}; S_0 = \{1, 2, 4, 8, 9, 11, 15, 16, 18\}.$$

The above bound can be extended to include t additional consecutive integers with a choice of a specific basis as follows:

$$b_1 = 1,\ b_2 = 1 + \alpha,\ \ \ldots, b_{(t-1)/2} = 1 + \alpha^{(t-1)/2};$$

$$b_{(t+3)/2} = 1 + \alpha^{-1},\ b_{(t+5)/2} = 1 + \alpha^{-2},\ \ \ldots, b_t = 1 + \alpha^{-(t-1)/2}.$$

Theorem 2.3.8 (Berlekamp [14], 1968)**.** *If t is a prime, then $w(2, t+1) > t \cdot 2^t$.*

Proof. Use the construction in Theorem 2.3.7 with the above basis, to partition $\{0, 1, \dots, n-1\}$ into disjoint sets S_0 and S_1 where no block contains a $(t+1)$-AP and with the property that

$$\{0, 1, \dots, (t-1)/2\} \cup \{n-1, n-2, \dots, n-(t-1)/2\} \subset S_1.$$

Set $S_0^+ = S_0 \cup A \cup B$, where $A = \{-(t-1)/2, \dots, -2, -1\}$ and $B = \{n, n+1, \dots, n+(t-1)/2\}$. It remains to show that S_0^+ contains no $(t+1)$-AP. We have the following cases:

1. $S_0^+ = S_0 \cup \{x\} \cup \{y\}$ where $x \in A$ and $y \in B$: not possible as $y - x$ is not a multiple of t for any choice of x and y.

2. $S_0^+ = S_0 \cup T$ where $T \subseteq A$ (or $T \subseteq B$) and $|T| \geqslant 2$: If $T \subseteq A$, then a there is no $(t+1)$-AP in S_0^+ because $\{0, 1, \dots, (t-1)/2\} \cap S_0 = \emptyset$. Similarly, if $T \subseteq B$, then there is no $(t+1)$-AP in S_0^+ because $\{n-1, n-2, \dots, n-(t-1)/2\} \cap S_0 = \emptyset$.

3. $S_0^+ = S_0 \cup \{x\}$ where $x \in A$ (or $x \in B$) and S_0 has a t-AP: By construction a t-AP in S_0 has common-difference at least $2^t - 1$. A $(t+1)$-AP will have a span at least $t(2^t - 1)$ which contradicts the maximum span of S_0^+.

Therefore, in the partition $\{-(t-1)/2, -(t-1)/2+1, \dots, n+(t-1)/2\} = S_0^+ \cup S_1$, no block contains a $(t+1)$-AP. The partition can be translated to a partition of integers $\{1, 2, \dots, t2^t\}$ by adding $(t+1)/2$ to each element of S_0^+ and S_1. □

Let $k = 2$, $t = 3$, and $n = 21$. Consider the polynomial $a(x) = x^3 + x + 1$, which is irreducible over $GF(2)$ and let α be a root of $a(x)$. Take the basis $b_1 = 1$, $b_2 = 1 + \alpha = \alpha^3$, and $b_3 = 1 + \alpha^{-1} = \alpha^2$. For each $j = 0, 1, \dots, 20$ find $(a_{1,j}, a_{2,j}, a_{3,j}) \in \{0,1\}^3$ such that $a_{1,j}b_1 + a_{2,j}b_2 + a_{3,j}b_3 = \alpha^j$. We get $a_{i,j}$ for $1 \leqslant i \leqslant 3$ and $0 \leqslant j \leqslant 20$:

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0101110 0101110 0101110

0010111 0010111 0010111

Putting $j \in S_\epsilon$ for $\epsilon \in \{0,1\}$ if and only if $0 \leqslant j \leqslant n-1$ and $a_{1,j} = \epsilon$, we get

$$S_1 = \{0,1,4,6,7,8,11,13,14,15,18,20\}\,;\, S_0 = \{2,3,5,9,10,12,16,17,19\}\,;\, S_0^+ = S_0 \cup \{-1,21,22\}\,.$$

2.3.1.3 Folkman's construction

We discuss a constructive method by Folkman for computing lower bounds of $w(k,t)$, which is not general, but gives the best known lower bounds of certain van der Waerden numbers.

Find a prime p of the form $k\ell+1$ for some integer ℓ. Find a primitive element ρ of $GF(p)$. Then ρ^t for $t = 1,2,\ldots,(p-1)$ cover all non-zero values modulo p.

Partition $\{1,2,\ldots,p\}$ for $i = 0,1,\ldots,k-1$, as follows

$$\begin{aligned} C_i &= \left\{\left(\rho^{i+qk} \pmod p\right) + 1 : q = 0,1,\ldots,\ell-1\right\}, \text{and} \\ C_0 &= C_0 \cup \{1\}\,. \end{aligned}$$

The partition gives a potential certificate for $w(k,t) > p$, which has to be validated. For example, $w(2,4) > 11$ can be shown by constructing the partition $\{1,2,4,5,6,10\} \cup \{3,7,8,9,11\}$ using the primitive root modulo 11, $\rho = 2$. The corresponding certificate is 00100011101. A certificate for $w(k,t) > n$ is *cyclic* if it is still a certificate under the transformation $j = j + m \pmod n$ for each m with $j \in \{1,2,\ldots,n\}$. Rabung [62] showed that if a certificate of length p is cyclic, then repeating the certificate $(t-1)$ times and adding one additional number to C_{k-1}, we can construct a certificate of length $(t-1)p+1$ to show that $w(k,t) > (t-1)p+1$. For example, 00100011101 is cyclic and hence we get a new certificate

0010001110100100011101001000111011

giving $w(2,4) > 34$, which is the best possible bound as we know that $w(2,4) = 35$ [18].

2.3.2 Upper bounds of van der Waerden numbers

All known upper bounds of $w(k,t)$ are enormous. In this section, we provide an overview on the upper bounds in chronological order.

2.3.2.1 Upper bound from van der Waerden's proof

The original combinatorial proof of van der Waerden bounded $w(2,t)$ from above by $Ack(t)$, an Ackermann function in t. Given a natural number n, $f_n : \mathbb{N} \to \mathbb{N}$ is recursively defined by

$$f_n(t) = \begin{cases} 2 \cdot t & \text{if } n = 1, \\ \underbrace{f_k(f_k(\cdots f_k}_{t}(1)\cdots)) & \text{if } n = k+1 > 1. \end{cases}$$

Thus $f_2(t) = 2^t$, and $f_3(t) = 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$. The right hand side of the latter is a stack of 2's of height t, which is known as the tower function $tower(t)$. The tower function can be defined as

$$tower(t) = \begin{cases} 2 & \text{if } t = 1, \\ 2^{tower(t-1)} & \text{if } t > 1. \end{cases}$$

Now the Ackermann function $Ack(t)$ is defined as $f_t(t)$. To observe how enormously this function grows, we can check $f_4(4)$, which is

$$f_3(f_3(f_3(f_3(1)))) = f_3(f_3(f_3(tower(1)))) = f_3(f_3(tower(2))) = f_3(tower(4)) = f_3(65536),$$

or a tower of 65536 2's. Since it was known that $w(2,4) = 35$, there were significant reasons to believe that improved upper bounds were possible. For many years, Graham's question "$w(2,t) \leqslant tower(t)$?" was open. The function $Ack(t)$ grows much faster than primitive recursive functions[1]. The following section briefly describes how a primitive recursive upper bound for $w(k,t)$ is obtained (though it does not settle Graham's question).

[1] A primitive recursive function can be computed using only for-loops, which have a fixed iteration limit.

2.3.2.2 Improved upper bound from Hales-Jewett's theorem

Hales and Jewett proved the following theorem, which derives van der Waerden's theorem as a corollary. Let A be an alphabet with t symbols namely $\{0, 1, \ldots, t-1\}$. Let $\star \notin A$ be a new symbol. Define *roots* as points in $\{A \cup \{\star\}\}^m$ with at least one $\star$ as a coordinate. For a root τ, and a symbol $a \in A$, we write $\tau(a) \in A^m$ for the point obtained replacing each $\star$ in τ by a. A *combinatorial line* rooted at τ is the set of t points $L_\tau = \{\tau(0), \tau(1), \ldots, \tau(t-1)\}$. For example, if $\tau = (0, 1, \star, 2, \star, 1)$ is a root in $A = \{0, 1, 2\}$ (with $t = 3$), then L_τ is the set $\{(0, 1, 0, 2, 0, 1), (0, 1, 1, 2, 1, 1), (0, 1, 2, 2, 2, 1)\}$.

Theorem 2.3.9 (Hales-Jewett [44], 1963)**.** *Given $A = \{0, 1, \ldots, t-1\}$, there is a dimension $m \in \mathbb{N}$ such that for every k-colouring of the m-dimensional cube A^m, there exists a monochromatic combinatorial line.*

Let the least m satisfying Hales-Jewett theorem be denoted by $HJ(k, t)$. Clearly, $HJ(k, 1) = 1$ for all k.

Lemma 2.3.10. *$HJ(k, 2) = k$.*

Proof. Here, we have $A = \{0, 1\}$. Take $m = k$. It can be observed that two of the $k+1$ points formed by a (possibly void) sequence of ones followed by a (possibly void) sequence of zeros (for example, for $k = 4$, the points are $(0, 0, 0, 0)$, $(0, 0, 0, 1)$, $(0, 0, 1, 1)$, $(0, 1, 1, 1)$, and $(1, 1, 1, 1)$) must have the same colour and any two of these points form a monochromatic combinatorial line. Therefore, $HJ(k, 2) \leqslant k$.

To show that $HJ(k, 2) \geqslant k$, we construct a k-colouring of $\{0, 1\}^{k-1}$ without a monochromatic combinatorial line. In the set of 2^{k-1} binary strings of length $k-1$, assign a different colour to each element of the subset of strings containing an equal number of ones. Since any such subset does not contain a combinatorial line, there is no monochromatic combinatorial line in $\{0, 1\}^{k-1}$. (For example, if $k = 4$, then $\{0, 1\}^3$ can be be partitioned into the following colour classes: $\{(0, 0, 0)\}$, $\{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}$, $\{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$, and $(1, 1, 1)$.) $\square$

The original proof (1963) of Hales-Jewett theorem used double induction, which was avoided in an alternate combinatorial proof given by Shelah in 1986.

Proof of van der Waerden's theorem using Hales-Jewett Theorem. Set $n = m(t-1)+1$ where $m = HJ(k,t)$. Consider k-colouring $\{1, 2, \ldots, n\}$. Set $A = \{0, 1, \ldots, t-1\}$ and for $x = (x_1, x_2, \ldots, x_m) \in A^m$, define the mapping $f : A^m \to \{1, \ldots, n\}$ as $f(x) = x_1 + x_2 + \ldots + x_m + 1$. Therefore, f induces a colouring of A^m. Every combinatorial line $L_\tau = \{\tau(0), \tau(1), \ldots, \tau(t-1)\}$ is mapped to an arithmetic progression of length t with $a = f(\tau(0)) + 1$, and common difference d being the number of $\star$'s in τ. □

So, we have $w(k,t) \leqslant (t-1)HJ(k,t) + 1$. Since $w(2,3) = 9$, we have $HJ(2,3) \geqslant 4$.

2.3.2.3 Erdős-Turán conjecture, Szemerédi's theorem, and the consequences

With the aim of strengthening the upper bound for van der Waerden numbers, Erdős and Turán realized that one should be able to find t-APs in any sufficiently dense set of integers, and came up with the following conjecture:

Conjecture 2.3.11 (Erdős and Turán [29], 1936)**.** *Let $\delta > 0$ and t be a positive integer. There is a number $N = N_0(\delta, t)$ such that any set $A \subset \{1, 2, \ldots, N\}$ with $|A| \geqslant \delta N$ contains a non-trivial t-AP.*

Very first result on the above conjecture was given by Roth (using Fourier analysis), showing its validity for 3-APs.

Theorem 2.3.12 (Roth [67], 1953)**.** *There is a positive constant C such that if $A \subset \{1, 2, \ldots, N\}$ with $|A| \geqslant CN/\log\log N$, then A has a non-trivial 3-AP. In other words, for given $0 < \delta \leqslant 1$,*

$$N_0(\delta, 3) \leqslant \exp\left(\exp(C/\delta)\right).$$

Szemerédi [79] showed in 1969 that the Erdős-Turán conjecture holds for $t = 4$ and settled the conjecture in 1975 for arithmetic progressions of arbitrary length. It can be observed that Szemerédi's theorem strengthens van der Waerden's theorem and that $w(k,t)$ can be chosen to be $N_0(1/k, t)$.

Szemerédi's combinatorial proof [80] used van der Waerden's theorem and hence provided no improvement in the upper-bound of $w(k,t)$. Furstenberg [34] gave another proof of Szemerédi's theorem in 1977 using ergodic theory, which again gave no improved bound, but the techniques became useful in dealing with other previously inaccessible problems in the area. Gowers made a major breakthrough by giving another proof of Szemerédi's theorem (using 'quadratic theory of Fourier analysis') by generalizing Roth's argument, and providing a significant improvement in the upper bound of van der Waerden numbers.

Theorem 2.3.13 (Gowers [38], 2001). *There exists a positive constant c_t such that any subset A of $\{1, 2, \ldots, N\}$ with $|A| \gg N/(\log\log N)^{c_t}$ contains a non-trivial t-AP. More specifically, given $0 < \delta \leqslant 1/2$, a positive integer t, $N \geqslant 2^{2^{(\delta^{-1})^{2^{2^{(t+9)}}}}}$, and a subset $A \subseteq \{1, 2, \ldots, N\}$ of size at least δN; the set A contains a t-AP.*

Gowers' result gives the current best upper bound $w(k,t) \leqslant 2^{2^{k^{2^{2^{(t+9)}}}}}$ for van der Waerden numbers. The ideas used by Gowers have been proved useful in many marvelous results, most notably the work of Green and Tao on the existence of arithmetic progressions on primes.

For several hundred years, mathematicians have investigated patterns in primes, one of the simplest of which is that the set of primes contain arbitrarily long arithmetic progressions. The earliest non-trivial result in this direction came from van der Corput:

Theorem 2.3.14 (van der Corput [20], 1939). *There are infinitely many arithmetic progressions consisting of three primes.*

There are numerous related conjectures about the existence of arithmetic progressions, the most famous of which is the following:

Conjecture 2.3.15 (Erdős, 1973). *If*

$$\sum_{n \in A} \frac{1}{n} = \infty,$$

then A contains arbitrarily long arithmetic progressions.

A corollary of the above would be that primes contain arbitrarily long arithmetic progressions since $\sum_p 1/p$ diverges. After more than four decades of Corput's result, the following partial result for 4-AP was shown by Heath-Brown:

Theorem 2.3.16 (Heath-Brown [45], 1981). *There are infinitely many arithmetic progressions consisting of three primes and a number with only two prime-factors, counted with multiplicity.*

In a slightly different direction, Balog [9, 10] shows that for any positive integer k, there exists infinitely many k-tuples of distinct primes $p_1 < p_2 < \cdots < p_k$ such that $(p_i + p_j)/2$ is a prime for all $i, j \in \{1, 2, \ldots, k\}$. Theorem 2.3.14 is a special case ($k = 2$) of Balog's result. Finally, Green and Tao settles the problem which remained open for centuries.

Theorem 2.3.17 (Green and Tao [40], 2008)**.** *The set of primes in $\{1, 2, \ldots, N\}$ contains a t-AP if N is large enough.*

According to the prime number theorem, the number of primes less than or equal to x is approximately $x/\log x$, which implies that primes have density zero, and hence Szemerédi's theorem cannot be applied directly to prove the Green-Tao Theorem.

An outline of the proof of Green-Tao Theorem is as follows:

- There is a pseudorandom set $X \subseteq \{1, 2, \ldots, N\}$ such that primes have positive density in X.
- If A is a subset of a pseudorandom set X such that $|A| \geqslant \delta|X|$ where $\delta > 0$, then there is a set $M \subseteq \{1, 2, \ldots, N\}$ indistinguishable from A, such that $|M| \geqslant \delta N$.
- By Szemerédi's theorem, M must contain a t-AP and so does A.

2.3.3 Computational aspects of van der Waerden numbers

Due to the huge gap between the lower and upper bounds, computing exact values of $w(k, t)$ remains extremely difficult. Only seven values, namely, $w(2, 3) = 9$ [18], $w(2, 4) = 35$ [18], $w(2, 5) = 178$ [77], $w(2, 6) = 1132$ [56][2], $w(3, 3) = 27$ [18], $w(3, 4) = 293$ [57], and $w(4, 3) = 76$ [12] are known, all of which were computed using computers (though the first one can be computed using pen and paper). All but $w(2, 6)$ and $w(3, 4)$ were computed using primitive computer search algorithms.

Dransfield et al. [23] introduced advanced techniques like encoding the van der Waerden problem as an instance of the Satisfiability Problem (or SAT for short), and using SAT-solvers to compute certificates for the lower bounds of $w(k, t)$. Herwig et al. [46] improved the lower bounds of $w(k, t)$ by considering a certain symmetry in certificates and using SAT-solvers for computation. Table 2.1 shows the current state of the known values and bounds of $w(k, t)$. These van der Waerden numbers are known as *diagonal* or *classical* van der Waerden numbers in the literature.

[2]The claim remains to be verified by others.

Table 2.1: Diagonal van der Waerden numbers

$k\backslash t$	3	4	5	6	7	8	9
2	9	35	178	1132	> 3703	> 11495	> 41265
3	27	293	> 2173	> 11191	> 48811	> 238400	
4	76	> 1048	> 17705	> 91331	> 420217		
5	> 170	> 2254	> 98740	> 540025			
6	> 223	> 9778	> 98748	> 816981			

2.3.3.1 Using SAT solvers to determine van der Waerden numbers

We formulate an instance F of the satisfiability problem (described in the following paragraph) with n variables for the van der Waerden number $w(k; t_0, \ldots, t_{k-1})$ such that F is satisfiable if and only if $n < w(k; t_0, \ldots, t_{k-1})$.

To describe the satisfiability problem, we require a few other definitions. A *truth assignment* is a mapping f that assigns each variable in $\{x_1, x_2, \ldots, x_n\}$ a value in $\{0, 1\}$. The complement $\bar{x}_i$ of each variable x_i is defined by $f(\bar{x}_i) = 1 - f(x_i)$ for all truth assignments f. Both x_i and $\bar{x}_i$ are called *literals*. A *clause* is a set of (distinct) literals and a *formula* is a family of (not necessarily distinct) clauses. A truth assignment *satisfies a clause* if it maps at least one of its literals to 1. The assignment *satisfies a formula* if and only if it satisfies each of its clauses. A formula is called *satisfiable* if it is satisfied by at least one truth assignment; otherwise, it is called *unsatisfiable*. The problem of recognizing satisfiable formulas is known as the *satisfiability problem*, or SAT for short. These definitions are taken from Chvátal and Reed [19].

To check the satisfiability of the generated instance, we need to use an algorithm that either solves the instance providing a satisfying assignment, or says that the formula is unsatisfiable. We have, at our disposal, two kinds of algorithms: complete and incomplete. A complete algorithm like DPLL (see [22, 21]; Algorithm 1 is a slightly modified version of the one given in [2]) finds a satisfying assignment if one exists; otherwise, correctly says that no satisfying assignment exists and the formula is unsatisfiable. SAT solving has progressed much beyond this simple algorithm (see the handbook [15] for general information), however on this special problem class this basic algorithm together with a basic heuristic is very competitive.

Algorithm 1 DPLL algorithm

function DPLL(F)
    while TRUE **do**
        if $\{u\} \in F$ and $\{\bar{u}\} \in F$ **then return** UNSATISFIABLE
        else if there is a clause $\{v\}$ **then** $F = F|v$
        else break
    end while
    if $F = \emptyset$ **then return** SATISFIABLE
    Choose an unassigned literal u using a branching rule
    if DPLL($F|u$) = SATISFIABLE **then return** SATISFIABLE
    if DPLL($F|\bar{u}$) = SATISFIABLE **then return** SATISFIABLE
    return UNSATISFIABLE
end function

Given a formula F and a literal u in F, we let $F|u$ denote the *residual formula* arising from F when u is set to true: explicitly, this formula is obtained from F by (i) removing all the clauses that contain u, (ii) deleting $\bar{u}$ from all the clauses that contain $\bar{u}$, (iii) removing both u and $\bar{u}$ from the list of literals. Each recursive call of DPLL may involve a choice of a literal u. Algorithms for making these choices are referred to as *branching rules*. It is customary to represent each call of DPLL(F) by a node of a binary tree. By branching on a literal u, we mean calling DPLL($F|u$). If this call leads to a contradiction, then we call DPLL($F|\bar{u}$). Every node that is not a leaf has at least one child and may not have both children. We refer to this tree as the *DPLL-tree* of F. For an efficient implementation of the DPLL Algorithm, see [5].

Local-search based incomplete algorithms (see Ubcsat-suite [82]) are generally faster (as they try to find a satisfying assignment as fast as possible when we can control the heuristic, number of iterations, runs, and several other parameters) than a DPLL-like algorithm, but may fail to deliver a satisfying assignment when there exists one. A good partition is a proof of a lower bound for a certain van der Waerden number irrespective of its means of achievement. In addition, certain symmetry in the certificates allows us to obtain them even faster. Incomplete algorithms are handy for obtaining good partitions and improving lower bounds of van der Waerden numbers. When they fail to improve the lower bound any further, we may turn to a complete algorithm.

2.3.3.2 Parallelization and distribution of SAT problems

For computationally hard problems like determining van der Waerden numbers, a single processor, even when run for a long time, is by far not adequate. Hence some form of parallelisation or distribution of the work is needed. Four levels of parallelisation have been considered for SAT solving (in a variety of schemes):

Processor-level parallelisation works only for very special algorithms, and can only achieve some relatively small speed-up. See [47] for an example which exploits parallel bit-operations. It seems to play no role for the problems we are considering.

Currently a standard single computer may contain up to, say, 16 relatively independent processing units, working on shared memory. So threads (or processes) can run in parallel, using one (or more) of the following general forms of collaboration:

(a) Partitioning the work via partitioning the instance (see below); [84, 55] are "classical" examples.

(b) Using the same algorithm running in various nodes on the same problem, exploiting randomization and/or sharing of learned results; see [51, 43] for recent examples.

(c) Using some portfolio approach, running different algorithms on the same problem, exploiting that various algorithms can behave very differently and unpredictably; see [42] for the first example.

Often these approaches are combined in various ways; see [71, 37, 52, 53] for recent examples. Approaches (b) and (c) do not seem to be of much use for the well-specified problem domain of hard instances from Ramsey theory. Only (a) is relevant, but in a more extreme form (see below).

On a cluster of computers, hundreds of computers may be considered with restricted communication (though typically still non-trivial). In this case, partitioning the work via partitioning the instance becomes more dominant. For hard problems, this form of computation is a common approach.

Internet computation with completely independent computers and only very basic communication between the centre and the machines, can be used to solve problems with massive search space.

In principle, the number of computers is unbounded. There is no practical example for a SAT computation at this level.

We remark that the classical area of "high performance computing" seems to be of no relevance for SAT solving, since the basic SAT algorithms like unit-clause propagation are (unlike the typical forms of numerical computation) inherently sequential.

Chapter 3

On the Computation of van der Waerden Numbers

As discussed in Chapter 2, there is a huge gap between the lower and upper bounds of van der Waerden numbers. The lower bound is exponential and the upper bound is enormous, yet the known exact values are not even close to the theoretical upper bound. Having another data point, that is, computing a new van der Waerden number, which would be useful to better understand the actual pattern of growth of the van der Waerden numbers, is extremely difficult due to its immense search space.

In this chapter, we report exact values of five previously unknown van der Waerden numbers of the form $w(2; t_0, t_1)$, some lower bounds (which we conjecture to be exact), polynomial upper-bound conjecture for $w(2; s, t)$, an efficient SAT-encoding, and a problem-specific backtracking algorithm for computing certain van der Waerden numbers. Most of the results described in this chapter can be found in Ahmed [1, 2, 3, 4] and Ahmed, Kullmann, and Snevily [8]. We have added and extended respective entries in the Online Encyclopædia of Integer Sequences (OEIS) using the results reported in this chapter.

As explored in Dransfield et al.[23], Herwig et al. [46], Kouril [56], and Ahmed [1, 2], we can generate an instance F of the satisfiability problem (for definition, see any of the above references) corresponding to $w(k; t_0, t_1, \ldots, t_{k-1})$ and integer n, such that F is satisfiable if and only if $n < w(k; t_0, t_1, \ldots, t_{k-1})$. In particular, an instance corresponding to $w(2; t_0, t_1)$ with n variables can be generated with the following clauses:

(a) $\{x_a, x_{a+d}, \cdots, x_{a+d(t_0-1)}\}$ with $a \geqslant 1, d \geqslant 1, a + d(t_0 - 1) \leqslant n$ and

(b) $\{\bar{x}_a, \bar{x}_{a+d}, \cdots, \bar{x}_{a+d(t_1-1)}\}$ with $a \geqslant 1, d \geqslant 1, a + d(t_1 - 1) \leqslant n$,

where $x_i = \varepsilon$ encodes the assignment of colour $\varepsilon \in \{0, 1\}$ to integer i (if x_i is not assigned a colour but the formula is satisfied, then i can be arbitrarily assigned). Clauses (a) prohibit the existence of a monochromatic t_0-AP of colour 0 and clauses (b) prohibit the existence of a monochromatic t_1-AP of colour 1.

3.1 New exact values of some $w(2; t_0, t_1)$

Discovering a new van der Waerden number has always been a challenge as it requires exploring the search space completely, which has a size exponential in the number of variables in the corresponding satisfiability instance. To prove that an instance with n variables is unsatisfiable, the DPLL algorithm implicitly enumerates all the 2^n solutions, that is, systematically evaluates all possible cases without explicitly evaluating all of them. To finish the search-space corresponding to the current lower bound of a certain van der Waerden number would require years of CPU-time, perhaps trillions of years for the bigger lower bounds.

Dividing the computation of a search into parts that have no inter-process communication among themselves is straightforward. DPLL has this desirable property like many backtrack algorithms. We may pick a level, say ℓ, of the DPLL-tree and distribute the subtrees rooted at that level among the processors. The appropriate value of ℓ may depend on many factors like the number of computers available, and should be decided on a case-by-case basis. Distributing the tasks evenly will not guarantee the reduction of computation time by a factor close to the number of CPUs, as some of the subtrees may be considerably larger than others, or that a computer involved may be relatively slow. Advanced splitting techniques for problem-specific DPLL-tree would help to predict which branch would require to be splitted recursively.

The size of the DPLL-tree greatly varies with the choice of the branching rule (or heuristic). It is hard to find a branching rule that works well on every SAT instance. Given a formula F, define $w(u) = \sum_k 2^{-k} d_k(u)$, where $d_k(u)$ denotes the number of clauses of length k containing u. First we find a variable x that maximizes $w(x) + w(\bar{x})$, and then we choose x if $w(x) \geqslant w(\bar{x})$, and $\bar{x}$ otherwise. This branching rule is known as Two-sided Jeroslaw-Wang (2sJW), (by Hooker and Vinay [50]) in the literature. Our DPLL implementation reads branching suggestions, if there are any, up to a prescribed level, and then explores the search space corresponding to the residual formula after those branches. It also periodically stores the current state of the search as a sequence of pairs (u, t), where $u \in \{x_1, \bar{x}_1, x_2, \bar{x}_2, \ldots, x_n, \bar{x}_n\}$ and $t \in \{0, 1\}$. A pair $(u, 0)$ indicates that $\bar{u}$ has to be explored, and $(u, 1)$ indicates that $\bar{u}$ has been explored already. A generator version of the solver generates the branching suggestions up to a prescribed level, and then the solver runs on different suggestions corresponding to the respective branches of the DPLL-tree. This involves a slight modification in Algorithm 1, which can easily be done.

We extend the following table of known values of $w(2; s, t)$:

s/t	3	4	5	6	7	8	9	10	11	12	13	14	15	16
3	9	18	22	32	46	58	77	97	114	135	160	186	218	238
4		35	55	73	109	146								
5			178	206										
6				1132										

Table 3.1: Known values of $w(2; s, t)$ (excluding the numbers discovered by us)

In Sections 3.1.1 through 3.1.5, we report some new exact values of van der Waerden numbers. In each case, we provide a certificate of the lower bound corresponding to $w(2; s, t) - 1$. To compute the exactness of each of the values $w(2; 3, 17)$, $w(2; 3, 18)$, $w(2; 3, 19)$,$w(2; 4, 9)$, and $w(2; 5, 7)$, we have used a distributed application of our SAT-solver.

3.1.1 $w(2; 3, 17) = 279$:

We have computed the exact value of $w(2; 3, 17)$ (Ahmed [2]), using 2.2 GHz 64 bit AMD Opteron processors (64 of them) from `cirrus` cluster at Concordia taking a total of 301 days of CPU-time (5 days of run-time).

$$1^{4}01^{6}01^{5}01^{3}01^{10}01^{6}01^{5}01^{8}0^{2}1^{10}0101^{6}01^{7}0101^{8}01^{12}0101^{16}01^{9}0^{2}1^{2}0^{2}1^{9}01^{16}$$

$$0101^{12}01^{8}0101^{7}01^{6}0101^{10}0^{2}1^{8}01^{5}01^{6}01^{10}01^{3}01^{5}01^{6}01^{4}$$

3.1.2 $w(2;3,18) = 312$:

We have computed the exact value of $w(2;3,18)$ (Ahmed [2]), using 2.2 GHz 64 bit AMD Opteron processors (80 of them) from `cirrus` cluster at Concordia taking a total of 13.6 years of CPU-time (70 days of run-time).

$$1^{6}01^{7}0^{2}1^{3}01^{17}0^{2}1^{3}0^{2}101^{10}01^{16}01^{7}01^{9}0^{2}10^{2}1^{4}01^{9}01^{7}01^{4}01^{14}01^{10}01^{11}01^{10}01^{14}01^{4}01^{7}$$

$$01^{9}01^{4}0^{2}10^{2}1^{9}01^{7}01^{16}01^{10}010^{2}1^{3}0^{2}1^{17}01^{3}0^{2}1^{7}01^{6}$$

Observe that both the certificates corresponding to $w(2;3,17)$ and $w(2;3,18)$ are palindromic (reads the same backward or forward).

3.1.3 $w(2;3,19) = 349$:

We have computed the exact value of $w(2;3,19)$ (Ahmed, Kullmann, and Snevily [8]), using 2.2 GHz 64 bit AMD Opteron processors (200 of them) from `cirrus` cluster at Concordia taking a total of 196 years of CPU-time (300 days of run-time).

$$1^{4}01^{6}01^{18}01^{3}01^{4}01^{5}01^{4}01^{11}01^{9}01^{3}01^{6}01^{7}01^{5}01^{14}01^{16}0101^{2}0^{2}1^{2}01^{15}01^{4}01^{12}0$$

$$1^{15}01^{2}01^{5}01^{7}01^{10}01^{13}01^{2}01^{15}01^{12}01^{4}01^{15}01^{2}0^{2}1^{2}0101^{9}01^{6}01^{14}01^{5}01^{14}01^{2}.$$

3.1.4 $w(2;4,9) = 309$:

We have computed the exact value of $w(2;4,9)$ (Ahmed [3]), using 2.2 GHz 64 bit AMD Opteron processors (200 of them) from `cirrus` cluster at Concordia for part of the computation, and 2.8 GHz Intel Xeon E5462 processors (256 of them) at Université de Sherbrooke (under Quebec High Performance Computing Network, RQCHP) for the remaining part, taking a total of 176 years of CPU-time (330 days of run-time).

$$1^8 01^3 0101^2 0^3 1^2 0^2 101^3 01010^2 1^5 01^2 01^3 01^8 0^2 1^8 01^3 01^2 01^5 0^2 10101^3 010^2 1^2 0^3 1^2 0101^7 0101^2 0^3 1^2 0^2 101^3 01010^2 1^5 01^2 01^3 01^8$$

$$0^2 1^8 01^3 01^2 01^5 0^2 10101^3 010^2 1^2 0^3 1^2 0101^7 0101^2 0^3 1^2 0^2 101^3 01010^2 1^5 01^2 01^3 01^8 0^2 1^8 01^3 01^2 01^5 0^2 10101^3 010^2 1^2 0^3 1^2 0101^3 01^8$$

Observe that the certificate corresponding to $w(2;4,9)$ is palindromic.

3.1.5 $w(2;5,7) = 260$:

We have computed the exact value of $w(2;5,7)$ (Ahmed [4]), using 2.2 GHz 64 bit AMD Opteron processors (200 of them) from `cirrus` cluster at Concordia and 2.8 GHz Intel Xeon E5462 processors (256 of them) at Université de Sherbrooke (under Quebec High Performance Computing Network, RQCHP), taking a total of 266 years of CPU-time (220 days of run-time).

$$1^6 01^4 01^5 0^4 1^4 0^4 10^3 1^3 01^4 010^2 1^5 0^2 101^4 01^3 0^3 10^4 101^2 0^2 10^2 1^2 010^4 10^3 1^3 01^4 010^2 1^5 0^2 101^4 01^3 0^3 10^4 101^2$$

$$0^2 10^2 1^2 010^4 10^3 1^3 01^4 010^2 1^5 0^2 101^4 01^3 0^3 10^4 101^2 0^2 10^2 1^2 01010^2 10^3 1^3 01^6 0^2 1^2 01^2 0^2 101^4 01^4 01^2 0^3 1^2 01^2 0101^4 01^3$$

3.2 All known values of $w(2;s,t)$

We have extended the following table of known values of $w(2;s,t)$:

We have extended the following entries in the OEIS based on the above results:

1. A171081: $w(2;3,t)$ for $t \geqslant 3$.

$$9, 18, 22, 32, 46, 58, 77, 97, 114, 135, 160, 186, 218, 238, \mathbf{279}, \mathbf{312}, \mathbf{349}.$$

2. A171082: $w(2;4,t)$ for $t \geqslant 4$.

$$35, 55, 73, 109, 146, \mathbf{309}.$$

3. A217037: $w(2;5,t)$ for $t \geqslant 5$.

$$178, 206, \mathbf{260}.$$

t/s	3	4	5	6
3	9			
4	18	35		
5	22	55	178	
6	32	73	206	1132
7	46	109	**260**	
8	58	146		
9	77	**309**		
10	97			
11	114			
12	135			
13	160			
14	186			
15	218			
16	238			
17	**279**			
18	**312**			
19	**349**			

Table 3.2: Known values of $w(2;s,t)$ (the bold ones are discovered by us)

3.3 Conjectured values of some $w(2;t_0,t_1)$

In this section, we provide conjectured values of $w(2;3,t)$ for $t = 20, 21, \ldots, 30$. We have used the Ubcsat suite [82] of local-search based satisfiability algorithms for generating the corresponding certificates. Since local search based algorithms are incomplete (they may fail to deliver a satisfying assignment, and hence a good partition when there exists one), it remains to prove exactness of these numbers using a complete satisfiability solver or some complete colouring algorithm.

$w(2;3,20) \geqslant 389$**:**

$$1^{19}01^{11}01^{4}01^{7}0101^{4}01^{13}01^{9}01^{4}0101^{14}0^{2}1^{3}0^{2}101^{9}01^{18}01^{9}01^{4}0101^{5}01^{3}01^{10}01^{16}01^{8}01^{16}01^{10}01^{3}01^{5}0101^{4}01^{9}$$

$$01^{18}01^{9}010^{2}1^{3}0^{2}1^{14}0101^{4}01^{9}01^{13}01^{4}0101^{7}01^{4}01^{11}01^{19}$$

$w(2;3,21) \geqslant 416$**:**

$$1^{8}01^{17}0101^{6}0101^{9}01^{12}0^{2}1^{19}01^{18}01^{3}0101^{2}01^{12}0^{2}1^{5}0^{2}101^{10}01^{18}01^{3}010^{2}1^{8}01^{12}01^{20}01^{18}01^{3}0101^{7}01^{8}01^{12}0^{2}$$

$$101^{17}01^{18}01^{3}010^{2}1^{8}01^{5}01^{6}0^{2}1^{12}01^{6}01^{15}0101^{4}0101^{11}01^{20}$$

$w(2;3,22) \geqslant 464$:

$$1^{2}0^{2}1^{17}0^{2}1^{9}01^{12}0101^{12}01^{15}01^{2}01^{10}01^{4}01^{7}01^{5}01^{12}01^{9}0101^{3}01^{8}01^{3}0101^{9}01^{12}01^{5}01^{7}01^{4}01^{10}01^{2}01^{15}01^{12}$$

$$0101^{12}01^{9}0^{2}1^{17}0^{2}1^{9}01^{12}0101^{12}01^{15}01^{2}01^{10}01^{4}01^{7}01^{5}01^{12}01^{9}0101^{3}01^{8}01^{5}01^{9}01^{12}01^{5}01^{7}01^{11}$$

$w(2;3,23) \geqslant 516$:

$$1^{4}0^{2}1^{2}01^{17}01^{4}0101^{15}01^{16}01^{4}01^{5}01^{20}0101^{2}01^{8}0^{2}101^{4}01^{15}01^{2}01^{4}01^{16}01^{9}01^{10}0101^{9}01^{10}01^{7}01^{17}01^{6}0101^{19}01^{16}0$$

$$1^{21}0^{2}1^{19}01^{6}01^{2}01^{12}010^{2}1^{4}0101^{20}01^{13}0^{2}1^{11}01^{9}0^{2}1^{6}01^{4}01^{13}0101^{3}01^{8}01^{9}01^{20}01^{5}01^{18}01^{3}01$$

$w(2;3,24) \geqslant 593$:

$$1^{21}01^{18}01^{16}01^{4}01^{7}01^{6}0101^{14}01^{3}0^{2}1^{8}01^{7}01^{3}01^{2}01^{20}0^{2}1^{3}01^{7}01^{15}01^{7}01^{3}0^{2}1^{20}01^{2}01^{3}01^{7}01^{9}01^{18}0101^{6}01^{21}01^{7}0$$

$$1^{10}01^{7}01^{21}01^{6}0101^{18}01^{9}01^{7}01^{3}01^{2}01^{20}0^{2}1^{3}01^{7}01^{15}01^{7}01^{3}0^{2}1^{20}01^{2}01^{3}01^{7}01^{8}0^{2}1^{3}01^{14}0101^{6}01^{7}01^{4}01^{16}01^{18}01^{21}$$

$w(2;3,25) \geqslant 656$:

$$1^{16}01^{2}01^{19}01^{8}01^{7}01^{19}01^{8}01^{4}0101^{5}01^{17}01^{10}01^{21}0^{2}1^{2}0101^{10}01^{7}01^{12}01^{4}01^{3}01^{6}01^{7}01^{11}01^{3}01^{13}01^{4}01^{17}0101^{3}$$

$$01^{6}0^{2}1^{6}01^{17}01^{8}01^{7}01^{13}01^{14}01^{2}01^{4}01^{13}0^{2}1^{8}01^{7}01^{19}01^{8}01^{4}0101^{7}01^{15}01^{10}01^{9}01^{11}0^{2}1^{2}0101^{10}01^{5}01^{14}0$$

$$1^{4}01^{3}01^{6}01^{7}01^{11}01^{3}01^{13}01^{4}01^{17}0101^{3}01^{6}0^{2}1^{24}01^{8}01^{9}$$

$w(2;3,26) \geqslant 727$:

$$1^{10}01^{23}01^{10}0101^{20}01^{4}01^{11}01^{6}01^{11}0^{2}1^{2}01^{5}01^{6}01^{5}01^{23}01^{4}0101^{7}01^{17}01^{16}0101^{11}0^{2}1^{2}0101^{3}01^{4}01^{2}01^{18}$$

$$01^{3}01^{5}01^{14}01^{12}01^{16}01^{4}01^{19}01^{8}010^{2}1^{4}01^{13}01^{14}0101^{20}01^{4}01^{18}01^{11}0^{2}1^{2}01^{5}01^{6}01^{5}01^{23}01^{4}0101^{7}01^{15}0101^{16}$$

$$0101^{11}0^{2}1^{2}0101^{3}01^{4}01^{2}01^{18}01^{9}01^{14}01^{12}01^{16}01^{4}01^{19}01^{8}01^{2}01^{18}01^{3}01^{25}$$

$w(2;3,27) \geqslant 770$:

$$1^{24}01^{3}01^{5}01^{18}01^{17}0101^{2}01^{21}01^{3}01^{7}01^{2}01^{20}0^{2}1^{12}01^{15}01^{10}01^{11}01^{3}01^{9}01^{6}01^{13}01^{22}01^{3}0^{2}$$

$$1^{8}01^{5}01^{20}01^{6}0101^{16}01^{7}01^{3}01^{5}010^{2}101^{21}01^{2}0^{2}1^{14}01^{9}01^{17}0101^{2}01^{3}01^{17}01^{3}01^{5}0101^{2}01^{20}0^{2}1^{12}01^{15}01^{22}$$

$$01^{3}01^{9}01^{6}01^{13}01^{22}01^{3}0^{2}1^{8}01^{5}01^{20}01^{8}01^{16}01^{7}01^{3}01^{5}010^{2}101^{24}01^{6}01^{8}01^{20}01^{15}01^{16}01^{5}$$

$w(2;3,28) \geqslant 827$:

$$1^{27}01^{10}01^{22}0101^{16}01^{13}01^{16}01^{20}01^{4}01^{16}0101^{11}0^{2}1^{2}0101^{3}01^{6}0^{2}1^{18}01^{3}01^{5}01^{14}01^{12}01^{21}01^{14}0$$

$$1^{13}010^{2}1^{4}01^{23}01^{4}0101^{25}01^{18}01^{11}01^{3}0101^{3}01^{4}01^{7}01^{17}01^{10}0101^{7}01^{17}01^{11}01^{4}01^{13}0^{2}1^{2}0101^{3}01^{7}$$

$$01^{18}01^{3}01^{5}01^{14}01^{12}01^{16}01^{4}01^{14}01^{13}010^{2}1^{18}01^{9}01^{4}0101^{25}01^{18}01^{11}01^{9}01^{4}0101^{23}01^{10}01^{9}01^{12}01^{16}01^{10}$$

$w(2;3,29) \geqslant 868$**:**

$$1^{15}01^{21}01^{18}01^{17}01^{18}01^{21}01^{2}01^{7}01^{25}0101^{12}01^{17}01^{6}01^{7}0^{2}1^{2}01^{17}01^{5}0^{2}101^{10}01^{6}01^{13}0101^{2}01^{17}01^{16}01^{7}01^{4}$$

$$01^{20}0101^{2}01^{5}01^{11}01^{25}01^{10}01^{25}01^{2}0^{2}1^{16}01^{3}01^{10}01^{4}01^{20}0101^{2}01^{6}01^{20}01^{14}0^{2}1^{2}01^{17}01^{5}0^{2}101^{10}01^{6}01^{13}0101^{2}$$

$$01^{9}01^{7}01^{9}01^{6}01^{7}01^{4}01^{20}0101^{2}01^{17}01^{25}01^{2}01^{7}01^{25}01^{3}01^{16}01^{6}01^{12}01^{25}01^{6}01^{27}01^{8}$$

$w(2;3,30) \geqslant 903$**:**

$$1^{22}01^{16}01^{22}01^{26}01^{7}0101^{8}0101^{22}01^{12}01^{16}01^{7}01^{6}01^{9}01^{11}01^{6}0^{2}1^{15}0^{2}1^{13}01^{7}01^{8}0101^{23}01^{6}01^{28}01^{7}01^{3}01^{4}01^{22}$$

$$01^{8}0101^{2}0^{2}1^{11}0101^{22}01^{11}01^{14}01^{3}01^{19}01^{4}01^{16}0^{2}1^{11}0101^{5}01^{28}01^{6}0^{2}101^{10}01^{2}01^{14}01^{7}01^{10}01^{23}01^{6}01^{28}$$

$$01^{7}01^{5}01^{2}0^{2}1^{18}01^{2}01^{8}0101^{3}01^{11}0101^{16}01^{25}0101^{4}01^{23}01^{16}01^{4}0^{2}1^{13}01^{12}01^{3}01^{27}01^{10}01^{14}$$

$w(2;5,8) \geqslant 331$**:**

$$1^{2}0^{4}1^{2}01^{2}0^{3}1^{6}0^{2}1^{4}01^{4}01^{2}01010101^{4}01010^{4}1^{3}01^{4}010^{2}1^{2}01^{2}01^{4}01^{4}01^{3}01^{3}0^{3}1^{4}0^{2}10^{2}101^{3}010101^{4}01^{4}$$

$$01^{4}0^{3}1^{5}010^{2}10^{3}1^{2}0^{4}1^{3}010^{3}101^{7}01^{2}01^{2}0^{2}101^{3}0^{4}10^{4}10^{2}101^{3}010^{3}1^{4}01^{7}010^{3}101^{3}0^{4}10^{2}101^{2}0^{3}1^{6}0^{2}$$

$$1^{4}01^{4}01^{2}01^{2}01^{2}01^{4}01^{3}0^{3}10^{2}101^{4}0^{2}101^{2}01^{6}0^{2}1^{4}01^{3}01^{3}0^{3}1^{4}0^{3}1^{4}0^{3}1^{2}01^{2}$$

$w(2;5,9) \geqslant 473$**:**

$$1^{3}01^{3}0^{2}1^{3}01^{3}0101^{4}01^{2}0^{2}1^{5}01^{4}01010101^{2}01^{3}0^{2}1^{2}0^{2}101^{3}0101^{4}0101^{4}0^{2}10101^{2}01^{8}0^{2}10^{2}10^{3}1^{3}0101^{3}01010^{2}$$

$$1^{5}01^{2}01^{3}01^{8}0^{2}1^{3}01^{4}01^{3}01^{2}01^{5}0^{2}10101^{3}010^{2}1^{2}0^{3}1^{2}0101^{3}01^{3}0101^{2}0^{3}1^{2}0^{2}101^{3}01010^{2}1^{5}01^{2}01^{3}01^{8}$$

$$0^{2}1^{8}01^{3}01^{2}01^{5}0^{2}10101^{3}010^{2}1^{2}0^{3}1^{2}0101^{3}01^{3}0101^{2}0^{3}1^{2}0^{2}101^{3}01010^{2}1^{5}01^{2}01^{3}01^{4}01^{3}0^{2}1^{8}01^{3}01^{2}$$

$$01^{5}0^{2}10101^{3}0101^{3}0^{3}10^{2}10^{2}1^{8}01^{2}01010^{2}1^{4}0101^{4}0101^{3}010^{2}1^{2}0^{2}1^{3}01^{2}01010101^{4}01^{5}0^{2}1^{2}01^{4}0101^{3}01^{3}0^{2}1^{3}01^{3}$$

3.4 Upper bound conjectures for $w(2;s,t)$ when s is fixed

We observe that for $t = 24, 25, \ldots, 30$ we have $w(2;3,t) > t^2$, which refutes the assumption that $w(2;3,t) \leqslant t^2$, as suggested in Brown, Landman, and Robertson [17], based on the exact values for $5 \leqslant t \leqslant 16$ known by then. But, still a polynomial upper bound of $w(2;s,t)$ seems possible when s is fixed.

Conjecture 3.4.1 (Ahmed, Kullmann, and Snevily [8]). *For $t \geqslant 3$ and fixed $s \geqslant 3$, there exists $c \geqslant 1$ such that*

$$w(2; s, t) \leqslant ct^{s-1}.$$

Define

$$d_{max}(s) = \max\left\{w(2; s, j) - w(2; s, j-1) : j \geqslant s+1\right\},$$

a lower bound of which can be obtained from all known values and bounds of $w(2; s, t)$.

Example 3.4.2. *Based on the known values as given in Section 3.1 and lower bounds as given in Section 3.3 for $w(2; 3, t)$, we obtain the following recursion*

$$w(2; 3, t) \leqslant w(2; 3, t-1) + d_{max}(3) \cdot (t-1)$$

for $t \geqslant 3$ with $w(2; 3, 3) = 9$. From our data, considering the largest gap between consecutive values or bounds, we observe $d_{max}(3) \geqslant (593 - 516)/23 \approx 3.35$. Solving the recurrence, we get the following specific conjecture with $s = 3$:

$$w(2; 3, t) \leqslant w(2; 3, t-1) + d_{max}(3) \cdot (t-1) < 1.675t^2.$$

Similarly, for known values corresponding to $s = 4$ and $s = 5$, we conjecture $w(2; 4, t) < 2t^3$ and $w(2; 5, t) < t^4$, respectively.

3.5 On $w(k; t_0, t_1, \ldots, t_{k-1})$ for $k \geqslant 3$

In this section, we discuss an efficient SAT encoding of van der Waerden numbers $w(k; t_0, t_1, \ldots, t_{k-1})$ with $k \geqslant 3$. Using that encoding, we have computed three new numbers. Then we discuss an idea to reduce the backtrack search space of some specific van der Waerden numbers considering symmetry, and compute several new values using that idea. The results reported in this section are published in Ahmed [3, 4].

3.5.1 An efficient encoding

For any partition $P_0 \cup P_1 \cup \cdots \cup P_{k-1}$ of the set $\{1, 2, \ldots, n\}$, we want to prohibit the existence of arithmetic progressions of length t_j in block P_j $(0 \leqslant j \leqslant k-1)$. Here, we present a simple but useful idea of binary-encoding of the blocks of partition for SAT-encoding of van der Waerden numbers. The idea of using binary variables to represent the bits of a binary representation of a nonnegative integer is not new. This formulation is well-known in integer linear programming (see Garfinkel and Nemhauser [35]), but the idea was never used in the present context until we used it in 2009. Recently, Kullmann [58] has used a more generalized idea of binary encoding to compute Green-Tao numbers.

Instead of taking nk variables $x_{i,j}$ with $1 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant k-1$ (as described in [1]), we take nr variables $x_{p,q}$ with $1 \leqslant p \leqslant n$ and $0 \leqslant q \leqslant r-1$, where $r = \lceil \log_2(k) \rceil$.

To prove that an instance with v variables is unsatisfiable; the DPLL algorithm implicitly enumerates all the 2^v possible cases, that is, systematically evaluates all possible cases without explicitly evaluating all of them. As we have fewer variables in the proposed encoding, we have less number of cases to enumerate.

Let a block P_j for $0 \leqslant j \leqslant k-1$ be represented in binary such that $j = \sum_{q=0}^{r-1} b_{j,q} 2^q$, where $b_{j,q}$ is the q-th bit in the r-bit binary representation of j.

An integer i in $\{1, 2, \ldots, n\}$ belongs to a block P_j if and only if j equals $\sum_{q=0}^{r-1} x_{i,q} 2^q$, that is, $x_{i,q}$ has the same truth-value as $b_{j,q}$. To prohibit the existence of arithmetic progressions $a, a+d, \ldots, a+d(t_j-1)$, with $a \geqslant 1, d \geqslant 1, a+d(t_j-1) \leqslant n$ in block P_j, we add the following clauses:

$$\left\{u_{a,r-1}, \ldots, u_{a,0}, u_{a+d,r-1}, \ldots, u_{a+d,0}, \ldots, u_{a+d(t_j-1),r-1} \cdots, u_{a+d(t_j-1),0}\right\},$$

where literal $u_{i,q}$ (for $i \in \{a, a+d, \ldots, a+d(t_j-1)\}$) is defined as follows:

$$u_{i,q} = \begin{cases} \bar{x}_{i,q} & \text{if } b_{j,q} = 1, \\ x_{i,q} & \text{otherwise.} \end{cases}$$

The above clauses are added for each j in $\{0, 1, \ldots, k-1\}$. The clauses are long, but can be handled efficiently as described in Section 3 of [2].

To ensure that an integer i is not placed in a block P_j with $k \leqslant j \leqslant 2^r - 1$ (since there is no such block in the partition of $\{1, 2, \ldots, n\}$), we add the following clauses:

$$\{v_{i,r-1}, v_{i,r-2}, \ldots, v_{i,1}, v_{i,0}\} \quad (i = 1, 2, \ldots, n),$$

where literal $v_{i,q}$ is defined as follows:

$$v_{i,q} = \begin{cases} \bar{x}_{i,q} & \text{if } b_{j,q} = 1, \\ x_{i,q} & \text{otherwise.} \end{cases}$$

The double-subscript variables $x_{p,q}$ with $1 \leqslant p \leqslant n$ and $0 \leqslant q \leqslant r-1$ can be converted to single-subcript variables $y_{(p-1)r+q+1}$. In the following example, we write j and $-j$ to mean the literals y_j and $\bar{y}_j$ respectively. An instance corresponding to $w(3; 2, 3, 3)$ for $n = 5$ can be constructed with 10 variables $1, 2, \ldots, 10$ and the following 23 clauses:

(*i*) `{1,2,3,4}, {1,2,5,6}, {1,2,7,8}, {1,2,9,10}, {3,4,5,6}, {3,4,7,8}, {3,4,9,10}, {5,6,7,8}, {5,6,9,10}, {7,8,9,10},`

(*ii*) `{1,-2,3,-4,5,-6}, {1,-2,5,-6,9,-10}, {3,-4,5,-6,7,-8}, {5,-6,7,-8,9,-10},`

(*iii*) `{-1,2,-3,4,-5,6}, {-1,2,-5,6,-9,10}, {-3,4,-5,6,-7,8}, {-5,6,-7,8,-9,10},`

(*iv*) `{-1,-2}, {-3,-4}, {-5,-6}, {-7,-8}, {-9,-10}.`

Clauses (*i*), (*ii*), and (*iii*) prohibit the existence of arithmetic progressions of lengths 2 (in block P_0), 3 (in block P_1), and 3 (in block P_2) respectively. Clauses (*iv*) prohibit the placement of any integer in block P_3.

3.5.2 A new number: $w(3; 3, 3, 6) = 107$

It took 992 days of CPU-time (roughly 17 days of run-time) using 2.2 GHz AMD Opteron 64 bit processors (64 of them) from the `cirrus` cluster at Concordia to prove that the instance corresponding to 107 is unsatisfiable, that is, there is no good partition of the set $\{1, 2, \ldots, 107\}$. A good partition of the set $\{1, 2, \ldots, 106\}$ corresponding to $w(3; 3, 3, 6)$ is as follows:

```
22122112 22212222 00220021 22102021 22200220 22220202 11221222
22110122 02022002 20222122 12202210 11022220 22022221 22.
```

3.5.3 Some new lower bounds of $w(3; t_0, t_1, t_2)$

In this section, we provide the following new lower bounds of $w(3; t_0, t_1, t_2)$, which were published in Ahmed [3].

$$w(3; 2, 4, 8) > 155, \quad w(3; 3, 3, 7) > 149, \quad w(3; 3, 3, 8) > 185,$$
$$w(3; 3, 3, 9) > 221, \quad w(3; 3, 3, 10) > 265, \quad w(3; 3, 4, 5) > 163, \quad \text{and}$$
$$w(3; 3, 5, 5) > 243.$$

Recently, Kouril [57] proved that $w(3; 2, 4, 8) = 157$.

3.5.4 An observation

We observe that the more blocks in the partition, the better the performance of the encoding. Here, we present the following two previously unknown van der Waerden numbers

$$\begin{aligned} w(7; 2, 2, 2, 2, 2, 3, 5) &= 55 \text{ (taking 21 days) and} \\ w(8; 2, 2, 2, 2, 2, 2, 3, 4) &= 40 \text{ (taking 16 days).} \end{aligned}$$

For both these numbers, the encoding in [1, 23] takes more than a couple of months to prove the corresponding instances unsatisfiable.

3.5.5 Backtracking considering symmetry

In this section, we propose a problem-specific backtracking algorithm for computing van der Waerden numbers $w(k; t_0, t_1, \ldots, t_{k-1})$ with $t_0 = t_1 = \cdots = t_{j-1} = 2$, where $k \geqslant j + 2$ and $t_i \geqslant 3$ for $i \geqslant j$. We report some previously unknown numbers using this method.

3.5.5.1 On $w(k; 2, 2, \ldots, 2, t_j, t_{j+1} \ldots, t_{k-1})$

Suppose in $w(k; t_0, \ldots, t_{j-1}, t_j, \ldots, t_{k-1})$ where $k - j \geqslant 2$, we have $t_0 = t_1 = \cdots = t_{j-1} = 2$, and $t_i \geqslant 3$ for $i = j, j + 1, \ldots, k - 1$. Any certificate of a lower bound of this van der Waerden number will contain each of $0, 1, \ldots, j - 1$ exactly once. Hence the certificate will still remain valid after any in-place permutation of $0, 1, \ldots, j-1$. For example, 898998879898031546989829988989 is a certificate of lower bound of

$$w(10; 2, 2, 2, 2, 2, 2, 2, 2, 3, 3) > 30,$$

which uses 10 colours. Keeping 8 and 9 in place, there are 8! certificates that prove the same lower bound.

In such a case, any certificate containing k colours can be transformed into an equivalent certificate replacing each of $0, 1, \ldots, j-1$ with a symbol x, and keeping the remaining $k-j$ colours. When we extend a certificate, we prohibit any t_i-term arithmetic progressions for $i = j, j+1, \ldots, k-1$ and check that the number of x does not exceed j. This observation greatly reduces the search space (the backtrack search-tree becomes $(k-j+1)$-ary instead of k-ary) of a trivial backtrack algorithm and makes way for computing new van der Waerden numbers.

From the above discussion, an equivalent certificate in our example is

$$8989988x9898xxxxxx9898x9988989,$$

which uses only two colours and a symbol x. For computational convenience, we can write this certificate as

$$121221102121000000212102211212,$$

with symbol x being replaced by integer colour 0 and colour c being replaced by integer colour $c-j+1$.

3.5.5.2 On $w(k; 2, 2, \ldots, 2, t, t \ldots, t)$ with $t \geqslant 3$

Let $t_0 = t_1 = \cdots = t_{j-1} = 2$ and $t_i = t \geqslant 3$ for $i = j, j+1, \ldots, k-1$. We can further minimize the backtrack search-space by extending only one certificate from the set of isomorphic certificates under symmetry. Consider the forty-eight certificates of the lower bound $w(3; 3, 3, 3) > 26$ with the colours named 1, 2, and 3.

1:	11221123233131121223133232	2:	11223113132233223131132211
3:	11232113132233223131123211	4:	11323112123322332121132311
5:	11331132322121131332122323	6:	11332112123322332121123311
7:	12112123322332121123311313	8:	12122113223231133113232231
9:	12122113223231133113232232	10:	12122321131332322121331133
11:	12332321122112323321133131	12:	13113132233223131132211212
13:	13133112332321122112323321	14:	13133112332321122112323323
15:	13133231121223233131221122	16:	13223231133113232231122121
17:	21211223113132233223131131	18:	21211223113132233223131132
19:	21211312232331311212332233	20:	21221213311331212213322323
21:	21331312211221313312233232	22:	22112213133232212113233131
23:	22113223231133113232231122	24:	22131223231133113232213122
25:	22313221213311331212231322	26:	22331221213311331212213322
27:	22332231311212232331211313	28:	23113132233223131132211212
29:	23223231133113232231122121	30:	23233132212113133232112211
31:	23233221331312211221313312	32:	23233221331312211221313313
33:	31221213311331212213322323	34:	31311213323221211313223322
35:	31311332112123322332121121	36:	31311332112123322332121123
37:	31331312211221313312233232	38:	32112123322332121123311313
39:	32322123313112122323113311	40:	32322331221213311331212212
41:	32322331221213311331212213	42:	32332321122112323321133131
43:	33112332321122112323321133	44:	33113312122323313112322121
45:	33121332321122112323312133	46:	33212331312211221313321233
47:	33221331312211221313312233	48:	33223321211313323221311212

Table 3.3: All certificates of $w(3;3,3,3) > 26$

Let a permutation π of $1, 2, \ldots, k$ be a sequence $\pi(1), \pi(2), \ldots, \pi(k)$. Let $S(k)$ denote the set of all permutations of $1, 2, \ldots, k$. We write the permutations in $S(k)$ in parenthesized notation with respect to the indices $1, 2, \ldots, k$. For example,

$$S(3) = \{(1)(2)(3), (1)(2,3), (1,2)(3), (1,2,3), (1,3,2), (1,3)(2)\} .$$

Let a certificate of the lower bound $w(k; t, t, \ldots, t) > n$ be denoted by $C = c_1 c_2 \cdots c_n$. Let $T_\pi(C)$ and $T_{S(k)}(C)$ be defined by $\pi(c_1)\pi(c_2)\ldots\pi(c_n)$ and $\{T_\pi(C) : \pi \in S(k)\}$, respectively.

For example, $T_{S(3)}(11221123233131121223133232)$ equals the set with the following elements

$$\begin{array}{ll} 11221123233131121223133232, & 11331132322121131332122323, \\ 22112213133232212113233131, & 22332231311212232331211313, \\ 33113312122323313112322121, & 33223321211313323221311212. \end{array}$$

Similarly, all 48 certificates can be generated from the following 8 certificates:

1:	11221123233131121223133232	2:	11223113132233223131132211
3:	11232113132233223131123211	7:	12112123322332121123311313
8:	12122113223231133113232231	9:	12122113223231133113232232
10:	12122321131332322121331133	11:	12332321122112323321133131

Table 3.4: Representative certificates of $w(3; 3, 3, 3) > 26$

Thus instead of generating and extending all certificates, we can consider only one from the 3! equivalent certificates. To do so, we can observe that, in a certificate $c_1 c_2 \cdots c_n$ of $w(k; t, t, \ldots, t) > n$, if c_i is greater than c_ℓ for $1 \leqslant \ell \leqslant i - 1$, then we can ignore branching on $c_i + 1, c_i + 2, \ldots, k$ at position i.

3.5.5.3 The algorithm

We have the following algorithm for computing $w(k; t_0, t_1, \ldots, t_{k-1})$, where $t_0 = t_1 = \cdots = t_{j-1} = 2$ and $k \geqslant j + 2$ that combines the ideas in Sections 3.5.5.1 and 3.5.5.2.

Algorithm 2 RECURSIVE ALGORITHM RUN(k, j, $index$, x)

```
function Run(k, j, index, x)
    if zeroCount > j then return end if
    if index > 0 and x > 0 then
        if the indices of t_{x+j-1} x's in c_1 c_2 ··· c_{index} form an AP then
            return
        end if
    end if
    if index > max then max = index end if
    for i = 0 to k - j do
        if i = 0 then zeroCount = zeroCount + 1 end if
        c_{index+1} = i
        Run(k, j, index + 1, i)
        if i = 0 then zeroCount = zeroCount - 1 end if
        if i > 0 and t_j = t_{j+1} = ··· = t_{k-1} = t then
            if index ⩽ j + (i - 1)(t - 1) + 1 then
                if c_{index+1} > c_ℓ for 1 ⩽ ℓ ⩽ index then
                    break
                end if
            end if
        end if
    end for
end function
```

We can observe that function RUN in Algorithm 2 returns with

$$max + 1 = w(k; 2, 2, \ldots, 2, t_j, t_{j+1}, \ldots, t_{k-1})$$

when called as RUN(k,j,0,0) with $zeroCount$ and max initialized to zero.

3.5.5.4 Experiments with some known values

In Table 3.5, we report test-results of Algorithm 2 with parameters corresponding to some known van der Waerden numbers. We do so to verify the correctness of the algorithm. We consider only numbers that are relevant to the algorithm and take less than half an hour of run-time.

	$(t_j, t_{j+1}, \ldots, t_{k-1})$	$max + 1$	$time(s)$
RUN(2, 0, 0, 0)	(3,3)	$9 = w(2; 3, 3)$	0.00
RUN(2, 0, 0, 0)	(4,4)	$35 = w(2; 4, 4)$	0.00
RUN(3, 1, 0, 0)	(3,3)	$14 = w(3; 2, 3, 3)$	0.00
RUN(3, 1, 0, 0)	(4,4)	$40 = w(3; 2, 4, 4)$	0.38
RUN(3, 0, 0, 0)	(3,3,3)	$27 = w(3; 3, 3, 3)$	0.12
RUN(4, 2, 0, 0)	(3,3)	$17 = w(4; 2, 2, 3, 3)$	0.00
RUN(4, 2, 0, 0)	(3,4)	$25 = w(4; 2, 2, 3, 4)$	0.07
RUN(4, 2, 0, 0)	(3,5)	$43 = w(4; 2, 2, 3, 5)$	2.20
RUN(4, 2, 0, 0)	(3,6)	$48 = w(4; 2, 2, 3, 6)$	42.93
RUN(4, 2, 0, 0)	(4,4)	$53 = w(4; 2, 2, 4, 4)$	10.25
RUN(4, 1, 0, 0)	(3,3,3)	$40 = w(4; 2, 3, 3, 3)$	4.97
RUN(5, 3, 0, 0)	(3,3)	$20 = w(5; 2, 2, 2, 3, 3)$	0.00
RUN(5, 3, 0, 0)	(3,4)	$29 = w(5; 2, 2, 2, 3, 4)$	0.84
RUN(5, 3, 0, 0)	(3,5)	$44 = w(5; 2, 2, 2, 3, 5)$	38.11
RUN(5, 3, 0, 0)	(4,4)	$54 = w(5; 2, 2, 2, 4, 4)$	208.74
RUN(5, 2, 0, 0)	(3,3,3)	$41 = w(5; 2, 2, 3, 3, 3)$	102.71
RUN(6, 4, 0, 0)	(3,3)	$21 = w(6; 2, 2, 2, 2, 3, 3)$	0.05
RUN(6, 4, 0, 0)	(3,4)	$33 = w(6; 2, 2, 2, 2, 3, 4)$	7.66
RUN(6, 4, 0, 0)	(3,5)	$50 = w(6; 2, 2, 2, 2, 3, 5)$	522.64
RUN(6, 3, 0, 0)	(3,3,3)	$42 = w(6; 2, 2, 2, 3, 3, 3)$	1615.73
RUN(7, 5, 0, 0)	(3,3)	$24 = w(7; 2, 2, 2, 2, 2, 3, 3)$	0.31
RUN(7, 5, 0, 0)	(3,4)	$36 = w(7; 2, 2, 2, 2, 2, 3, 4)$	59.64
RUN(8, 6, 0, 0)	(3,3)	$25 = w(8; 2, 2, 2, 2, 2, 2, 3, 3)$	1.38
RUN(8, 6, 0, 0)	(3,4)	$40 = w(8; 2, 2, 2, 2, 2, 2, 3, 4)$	434.12
RUN(9, 7, 0, 0)	(3,3)	$28 = w(9; 2, 2, 2, 2, 2, 2, 2, 3, 3)$	5.58

Table 3.5: Experiment on some known values

3.5.5.5 New values of $w(k; t_0, t_1, \ldots, t_{k-1})$

We have computed the following new values of $w(k; t_0, t_1, \ldots, t_{k-1})$ using Algorithm 2.

$w(k; t_0, t_1, \ldots, t_{k-1})$	
$w(7; 2, 2, 2, 2, 2, 3, 6)$	$= 65$
$w(7; 2, 2, 2, 2, 2, 4, 4)$	$= 66$
$w(7; 2, 2, 2, 2, 3, 3, 3)$	$= 45$
$w(8; 2, 2, 2, 2, 2, 2, 3, 5)$	$= 61$
$w(8; 2, 2, 2, 2, 2, 2, 3, 6)$	$= 71$
$w(8; 2, 2, 2, 2, 2, 2, 4, 4)$	$= 67$
$w(8; 2, 2, 2, 2, 2, 3, 3, 3)$	$= 49$
$w(9; 2, 2, 2, 2, 2, 2, 2, 3, 4)$	$= 42$
$w(9; 2, 2, 2, 2, 2, 2, 2, 3, 5)$	$= 65$
$w(9; 2, 2, 2, 2, 2, 2, 3, 3, 3)$	$= 52$
$w(10; 2, 2, 2, 2, 2, 2, 2, 2, 3, 3)$	$= 31$
$w(10; 2, 2, 2, 2, 2, 2, 2, 2, 3, 4)$	$= 45$
$w(11; 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3)$	$= 33$
$w(11; 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 4)$	$= 48$
$w(12; 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3)$	$= 35$
$w(12; 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 4)$	$= 52$
$w(13; 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3)$	$= 37$
$w(13; 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 4)$	$= 55$
$w(14; 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3)$	$= 39$
$w(15; 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3)$	$= 42$
$w(16; 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3)$	$= 44$
$w(17; 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3)$	$= 46$
$w(18; 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3)$	$= 48$
$w(19; 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3)$	$= 50$

$w(20; 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3)$	$= 51$

Table 3.6: New values of $w(k; t_0, t_1, \ldots, t_{k-1})$

Based on the results in Table 3.6, we have added and extended (as shown in bold fonts) the following entries in the OEIS:

1. A217005: $w(j+2; t_0, t_1, \ldots, t_{j-1}, 3, 3)$ for $j \geqslant 0$ with $t_i = 2$, $0 \leqslant i \leqslant j-1$.

$$9, 14, 17, 20, 21, 24, 25, 28, \mathbf{31}, \mathbf{33}, \mathbf{35}, \mathbf{37}, \mathbf{39}, \mathbf{42}, \mathbf{44}, \mathbf{46}, \mathbf{48}, \mathbf{50}, \mathbf{51}.$$

2. A217058: $w(j+2; t_0, t_1, \ldots, t_{j-1}, 3, 4)$ for $j \geqslant 0$ with $t_i = 2$, $0 \leqslant i \leqslant j-1$.

$$18, 21, 25, 29, 33, 36, 40, \mathbf{42}, \mathbf{45}, \mathbf{48}, \mathbf{52}, \mathbf{55}.$$

3. A217059: $w(j+2; t_0, t_1, \ldots, t_{j-1}, 3, 5)$ for $j \geqslant 0$ with $t_i = 2$, $0 \leqslant i \leqslant j-1$.

$$22, 32, 43, 44, 50, 55, \mathbf{61}, \mathbf{65}.$$

4. A217060: $w(j+2; t_0, t_1, \ldots, t_{j-1}, 3, 6)$ for $j \geqslant 0$ with $t_i = 2$, $0 \leqslant i \leqslant j-1$.

$$32, 40, 48, 56, 60, \mathbf{65}, \mathbf{71}.$$

5. A217007: $w(j+2; t_0, t_1, \ldots, t_{j-1}, 4, 4)$ for $j \geqslant 0$ with $t_i = 2$, $0 \leqslant i \leqslant j-1$.

$$35, 40, 53, 54, 56, \mathbf{66}, \mathbf{67}.$$

6. A217008: $w(j+3; t_0, t_1, \ldots, t_{j-1}, 3, 3, 3)$ for $j \geqslant 0$ with $t_i = 2$, $0 \leqslant i \leqslant j-1$.

$$27, 40, 41, 42, \mathbf{45}, \mathbf{49}, \mathbf{52}.$$

Chapter 4

Some Properties of Arithmetic Progressions

In this chapter, we prove some basic counting lemmas on arithmetic progressions, and generalize a conjecture of Szekeres based on experimental data. The content of this chapter is based on joint work with Hunter Snevily and Janusz Dybizbański [6].

4.1 Notation

Recall the definitions: Let $c(k,n)$ denote the size of the set $ap(k,n)$; let $ap(k,n;x)$ denote the set of k-APs each containing x and $c(k,n;x)$ denote the size of $ap(k,n;x)$; let $c_i(k,n;x)$ be the number of k-APs in $ap(k,n;x)$ each of which contains x as the i-th element. Clearly, $c(k,n;x) = \sum_{j=1}^{k} c_j(k,n;x)$. Also let $c_{max}(k,n)$ denote the maximum of $c(k,n;x)$ over $x = 1,2,\ldots,n$.

Example 4.1.1. *Consider $k = 4$ and $n = 17$. Then we have*

j/x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	5	5	4	4	4	3	3	3	2	2	2	1	1	1	0	0	0
2	0	1	2	3	4	5	5	4	4	3	3	2	2	1	1	0	0
3	0	0	1	1	2	2	3	3	4	4	5	5	4	3	2	1	0
4	0	0	0	1	1	1	2	2	2	3	3	3	4	4	4	5	5
$c(4,17;x)$	5	6	7	9	11	11	13	12	12	12	13	11	11	9	7	6	5

Here $c(4,17;x) = c_1(4,17;x) + c_2(4,17;x) + c_3(4,17;x) + c_4(4,17;x)$*. For example,* $c(4,17;13) = 1+2+4+4$ *due to the following 4-APs in* $ap(4,17)$ *that contain* 13*, namely* $ap(4,17;13)$*:*

$$\{13,14,15,16\}\,,\quad \{12,13,14,15\}\,,\quad \{11,13,15,17\}\,,\quad \{5,9,13,17\}\,,$$
$$\{7,10,13,16\}\,,\quad \{9,11,13,15\}\,,\quad \{11,12,13,14\}\,,\quad \{1,5,9,13\}\,,$$
$$\{4,7,10,13\}\,,\quad \{7,9,11,13\}\,,\quad \{10,11,12,13\}\,.$$

Observation 4.1.2. *For* $x = 1,2,\ldots,\lfloor n/2\rfloor$, $c(k,n;x) = c(k,n;n-x+1)$*.*

Proof. From Observation 2.3.2, $c_1(n,k;x) = c_k(n,k;n-x+1)$.

For other values of j, if $\lfloor (n-x)/(k-j)\rfloor \leqslant \lfloor (x-1)/(j-1)\rfloor$, then taking $x' = n-x+1$ and $j' = k-j+1$,

$$c_j(k,n;x) = \Big\lfloor \frac{(n-x)}{(k-j)} \Big\rfloor = \Big\lfloor \frac{(n-x+1)-1}{(k-j+1)-1} \Big\rfloor = \Big\lfloor \frac{(x'-1)}{(j'-1)} \Big\rfloor = c_{j'}(k,n;x').$$

The last equality follows from the fact that

$$\Big\lfloor \frac{(x'-1)}{(j'-1)} \Big\rfloor \leqslant \Big\lfloor \frac{(x-1)}{(j-1)} \Big\rfloor = \Big\lfloor \frac{(n-x')}{(k-j')} \Big\rfloor.$$

Similarly, if $\lfloor (n-x)/(k-j)\rfloor > \lfloor (x-1)/(j-1)\rfloor$, then

$$c_j(k,n;x) = c_{k-j+1}(k,n;n-x+1).$$

Therefore, $c(k,n;i)$ with $i = 1,2,\ldots,n$ is symmetric. ∎

Lemma 4.1.3. *Given positive integers* k *and* n*, let* $2 \leqslant j \leqslant k-1$, $n = m(k-1)+r$ *with* $0 \leqslant r \leqslant k-2$*. Then* $c_j(k,n;x)$ *equals*

(a) $\lfloor (x-1)/(j-1) \rfloor$ if $x = 1, 2, \ldots, m(j-1)$,

(b) $\lfloor (n-x)/(k-j) \rfloor$ if $x = n - m(k-j) + 1, n - m(k-j) + 2, \ldots, n$,

(c) m otherwise, that is, for $x = m(j-1) + 1, \ldots, n - m(k-j)$.

Proof. (a) Take $y \leqslant m(j-1)$ and assume $\lfloor (n-y)/(k-j) \rfloor < \lfloor (y-1)/(j-1) \rfloor$. Now we have, $\lfloor (y-1)/(j-1) \rfloor \leqslant m-1$, and

$$\begin{aligned} y \leqslant m(j-1) \quad &\Rightarrow \quad n - y \geqslant n - m(j-1) = m(k-j) + r, \\ &\Rightarrow \quad \left\lfloor \frac{n-y}{k-j} \right\rfloor \geqslant m, \end{aligned}$$

which is a contradiction.

(b) Take $y \geqslant n - m(k-j) + 1$ and assume $\lfloor (n-y)/(k-j) \rfloor > \lfloor (y-1)/(j-1) \rfloor$. Similar reasoning as (a) leads to a contradiction.

(c) Here, $n - m(k-j)$ can be written as $m(j-1) + r$ and $m(j-1) + 1$ can be written as $n - m(k-j) - (r-1)$. There are exactly r elements in $m(j-1) + 1, \ldots, n - m(k-j)$ and for any x in this range,

$$c_j(k, n, x) = \lfloor (x-1)/(j-1) \rfloor = \lfloor (n-x)/(k-j) \rfloor = m.$$

□

Lemma 4.1.4. *Given positive integers k and n, let $n = m(k-1) + r$ with $0 \leqslant r \leqslant k-2$. Denote a sequence $a, a, \ldots, a$ with a repeated b times as a^b and consider a^0 to be an empty sequence. Then for $1 \leqslant j \leqslant k$, the sequence $c_j(k, n; x)$ with $1 \leqslant x \leqslant n$ has the form*

$$0^{j-1} 1^{j-1} \cdots (m-1)^{j-1} m^r (m-1)^{k-j} (m-2)^{k-j} \cdots 0^{k-j}.$$

Proof. From Observation 2.3.2,

$$c_1(k, n; x) = \left\lfloor \frac{n-x}{k-1} \right\rfloor = \left\lfloor m + \frac{r-x}{k-1} \right\rfloor = m + \left\lfloor \frac{r-x}{k-1} \right\rfloor.$$

Now, we have

$$c_1(k,n;x) = \begin{cases} m & \text{for } x = 1,2,\ldots,r, \\ (m-1) & \text{for } x = r+1, r+2, \ldots, r+(k-1), \\ (m-2) & \text{for } x = r+(k-1)+1, r+(k-1)+2, \ldots, r+2(k-1), \\ & \vdots, \\ 1 & \text{for } x = r+(m-2)(k-1)+1, \ldots, n-(k-1), \\ 0 & \text{for } x = r+(m-1)(k-1)+1, \ldots, n. \end{cases}$$

Hence, the sequence $c_1(k,n;x)$ with $x = 1,2,\ldots,n$ is

$$m^r(m-1)^{k-1}(m-2)^{k-1}\cdots 1^{k-1}0^{k-1}.$$

Similarly, the sequence $c_k(k,n;x)$ with $x = 1,2,\ldots,n$ is

$$0^{k-1}1^{k-1}\cdots(m-2)^{k-1}(m-1)^{k-1}m^r.$$

For $2 \leqslant j \leqslant k-1$, we have (by Lemma 4.1.3):

x	$c_j(k,n;x)$
$1,2,\ldots,j-1$	$\lfloor (x-1)/(j-1) \rfloor = 0$
$(j-1)+1, (j-1)+2, \ldots, 2(j-1)$	$\lfloor (x-1)/(j-1) \rfloor = 1$
$\vdots$	$\vdots$
$(m-2)(j-1)+1, (m-2)(j-1)+2, \ldots, (m-1)(j-1)$	$\lfloor (x-1)/(j-1) \rfloor = m-2$
$(m-1)(j-1)+1, (m-1)(j-1)+2, \ldots, m(j-1)$	$\lfloor (x-1)/(j-1) \rfloor = m-1$
$m(j-1)+1, m(j-1)+2, \ldots, n-m(k-j)$	m
$n-m(k-j)+1, \ldots, n-(m-1)(k-j)$	$\lfloor (n-x)/(k-j) \rfloor = m-1$
$n-(m-1)(k-j)+1, \ldots, n-(m-2)(k-j)$	$\lfloor (n-x)/(k-j) \rfloor = m-2$
$\vdots$	$\vdots$
$n-2(k-j)+1, n-2(k-j)+2, \ldots, n-(k-j)$	$\lfloor (n-x)/(k-j) \rfloor = 1$
$n-(k-j)+1, n-(k-j)+2, \ldots, n$	$\lfloor (n-x)/(k-j) \rfloor = 0$

Hence, we get the sequence $c_j(k,n;x)$ for $x = 1, 2, \ldots, n$ as

$$0^{j-1}1^{j-1}\cdots(m-1)^{j-1}m^r(m-1)^{k-j}(m-2)^{k-j}\cdots 0^{k-j}.$$

□

Corollary 4.1.5. *Given positive integers k and n, let $m = \lfloor n/(k-1) \rfloor$ and $n = m(k-1)+r$. Then*

$$c(k,n) = \binom{m}{2}(k-1) + mr.$$

Proof. From the definition of $c_1(k,n;x)$ for $1 \leqslant x \leqslant n$, we have (by Lemma 4.1.4),

$$\begin{aligned} c(k,n) &= \sum_{x=1}^{n-k+1} c_1(k,n;x), \\ &= mr + [(m-1) + (m-2) + \cdots + 2 + 1 + 0](k-1), \\ &= \binom{m}{2}(k-1) + mr. \end{aligned}$$

□

Corollary 4.1.6. *Given positive integers k and n, let $n = m(k-1) + r$ with $r < k-1$. Then for $1 \leqslant j \leqslant k$ and $x = 1, 2, \ldots, n$,*

$$c_j(k,n;x) \leqslant m.$$

Proof. Follows from Lemmas 4.1.3 and 4.1.4. □

Corollary 4.1.7. *Given positive integers k and n, let $n = m(k-1) + r$ with $r < k-1$. Then*

$$c_{max}(k,n) \leqslant m(k-1).$$

Proof. For any $x \in \{1, 2, \ldots, n\}$, we have the following values of $c_1(k,n;x) + c_k(k,n;x)$ using Lemma 4.1.4.

x	$c_1(k,n;x)+c_k(k,n;x)$
$1,2,\ldots,r$	$m+0=m$
$r+0(k-1)+1, r+0(k-1)+2,\ldots,k-1$	$(m-1)+0=m-1$
$1+(k-1), 2+(k-1),\ldots,r+1(k-1)$	$(m-1)+1=m$
$r+1(k-1)+1, r+1(k-1)+2,\ldots,2(k-1)$	$(m-2)+1=m-1$
$1+2(k-1), 2+2(k-1),\ldots,r+2(k-1)$	$(m-2)+2=m$
$\vdots$	$\vdots$
$r+(m-2)(k-1)+1, r+(m-2)(k-1)+2,\ldots,(m-1)(k-1)$	$1+(m-2)=m-1$
$1+(m-1)(k-1), 2+(m-1)(k-1),\ldots,r+(m-1)(k-1)$	$1+(m-1)=m$
$r+(m-1)(k-1)+1, r+(m-1)(k-1)+2,\ldots,m(k-1)$	$0+(m-1)=m-1$
$1+m(k-1), 2+m(k-1),\ldots,r+m(k-1),$	$0+m=m$

Hence, using Corollary 4.1.6, we have

$$\begin{aligned} c(k,n;x) &= c_1(k,n;x)+c_k(k,n;x)+\sum_{j=2}^{k-1} c_j(k,n;x) \\ &\leqslant m+m(k-2)=m(k-1). \end{aligned}$$

Therefore, $c_{max}(k,n) \leqslant m(k-1)$. □

It can be observed that the upper bound in Corollary 4.1.7 is the best possible for $c_{max}(k,n)$.

Recall from Chapter 2 that $r(k,n)$ denotes the length of the longest k-AP free subsequences in $1,2,\cdots,n$. The following theorem gives an upper bound of $r(k,n)$, which is very close to actual values (as suggested by experimental results).

Theorem 4.1.8. *Given positive integers k and n, let $m=\lfloor n/(k-1)\rfloor$ and $n=m(k-1)+r$ where $r<k-1$. Then*

$$r(k,n) \leqslant n-\lfloor m/2\rfloor.$$

Proof. Using Corollaries 4.1.6 and 4.1.7, we have

$$\begin{aligned} r(k,n) &\leqslant n - \left\lceil \frac{c(k,n)}{c_{max}(k,n)} \right\rceil \\ &\leqslant n - \left\lceil \frac{m(m-1)(k-1)/2 + mr}{m(k-1)} \right\rceil \\ &= n - \left\lceil \frac{m-1}{2} + \frac{r}{k-1} \right\rceil = n - f(m,k,r), \text{(say)} \end{aligned}$$

It can be observed that

$$f(m,k,r) = \begin{cases} y+1 & \text{if } m = 2y+1, \\ y & \text{if } m = 2y \text{ and } 2r \leqslant k-1, \\ y+1 & \text{if } m = 2y \text{ and } 2r > k-1. \end{cases}$$

Hence,

$$r(k,n) \leqslant n - \lfloor m/2 \rfloor.$$

□

4.2 Unimodality Lemmas

A sequence is called *unimodal* if it is first increasing and then decreasing.

Lemma 4.2.1. *Given positive integers k and n, take $2 \leqslant j \leqslant k-1$. Then the sequence $c_j(k,n;x)$ for $x = 1, 2, \ldots, n$ is unimodal.*

Proof. Follows directly from Lemma 4.1.4. □

Lemma 4.2.2. *The sequence $c(3,n;i)$ with $i = 1, 2, \ldots, n$ is unimodal.*

Proof. From Observation 2.3.2,

$$c_j(3,n;x) = \begin{cases} \lfloor (n-x)/2 \rfloor & \text{if } j=1 \\ x-1 & \text{if } j=2 \text{ and } x \leqslant \lfloor n/2 \rfloor \\ n-x & \text{if } j=2 \text{ and } x > \lfloor n/2 \rfloor \\ \lfloor (x-1)/2 \rfloor & \text{if } j=3 \end{cases}$$

By Observation 4.1.2, $c(3,n;i)$ equals $c(3,n;n-i+1)$ for $i = 1,2,\ldots,\lfloor n/2 \rfloor$.

Now, we consider the following two cases:

1. $(n = 2m)$. For $i = 1,2,\ldots,m-1$, we have

$$c_2(3,n;i+1) = c_2(3,n;i) + 1,$$

and for $i = 1,2,\ldots,m$,

$$c_1(3,n;i) + c_3(3,n;i) = \left\lfloor \frac{2m-i}{2} \right\rfloor + \left\lfloor \frac{i-1}{2} \right\rfloor = \left\lfloor m - \frac{i}{2} \right\rfloor + \left\lfloor \frac{i}{2} - \frac{1}{2} \right\rfloor.$$

If $i = 2j$ $(j \geqslant 1)$, then

$$c_1(3,n;i) + c_3(3,n;i) = (m-j) + \left\lfloor j - \frac{1}{2} \right\rfloor = (m-j) + (j-1) = m-1.$$

If $i = 2j+1$ $(j \geqslant 0)$, then

$$\begin{aligned} c_1(3,n;i) + c_3(3,n;i) &= \left\lfloor m - j - \frac{1}{2} \right\rfloor + \left\lfloor \frac{(2j+1)-1}{2} \right\rfloor \\ &= (m-j-1) + j = m-1. \end{aligned}$$

Therefore, for $i = 1,2,\ldots,m-1$,

$$c(3,n;i+1) = (m-1) + c_2(3,n;i) + 1 = c(3,n;i) + 1.$$

2. $(n = 2m+1)$. For $i = 1, 2, \ldots, m$, we have

$$c_2(3, n; i+1) = c_2(3, n; i) + 1,$$

and

$$c_1(3, n; i) + c_3(3, n; i) = \left\lfloor \frac{2m+1-i}{2} \right\rfloor + \left\lfloor \frac{i-1}{2} \right\rfloor = \left\lfloor m + \frac{1}{2} - \frac{i}{2} \right\rfloor + \left\lfloor \frac{i}{2} - \frac{1}{2} \right\rfloor.$$

If $i = 2j$ $(j \geqslant 1)$, then

$$c_1(3, n; i) + c_3(3, n; i) = \left\lfloor m + \frac{1}{2} - j \right\rfloor + \left\lfloor j - \frac{1}{2} \right\rfloor = (m-j) + (j-1) = m-1.$$

If $i = 2j+1$ $(j \geqslant 0)$, then

$$\begin{aligned} c_1(3, n; i) + c_3(3, n; i) &= \left\lfloor m + \frac{1}{2} - j - \frac{1}{2} \right\rfloor + \left\lfloor \frac{(2j+1)-1}{2} \right\rfloor \\ &= (m-j) + j = m. \end{aligned}$$

Therefore, for $i = 1, 2, \ldots, m$,

- If i is odd, then

$$\begin{aligned} c(3, n; i+1) &= c_1(3, n; i+1) + c_2(3, n; i+1) + c_3(3, n; i+1) \\ &= c_2(3, n; i) + 1 + (m-1) = c_2(3, n; i) + m \\ &= c_2(3, n; i) + c_1(3, n; i) + c_3(3, n; i) = c(3, n; i). \end{aligned}$$

- If i is even, then

$$\begin{aligned} c(3, n; i+1) &= c_1(3, n; i+1) + c_2(3, n; i+1) + c_3(3, n; i+1) \\ &= c_2(3, n; i) + 1 + m = c_2(3, n; i) + (m-1) + 2 \\ &= c_2(3, n; i) + c_1(3, n; i) + c_3(3, n; i) + 2 = c(3, n; i) + 2. \end{aligned}$$

Hence, $c(3, n; i)$ with $i = 1, 2, \ldots, n$ is unimodal. $\square$

Lemma 4.2.3. *For $k \geqslant 4$, there are infinitely many n such that the sequence $c(k, n; i)$ with $i = 1, 2, \ldots, n$ is unimodal.*

Proof. We show that $c(k, n; i)$ for $1 \leqslant i \leqslant n$ with $n = lcm\{1, 2, \ldots, k-1\} \cdot m$ (where $m \geqslant 1$) is unimodal. Since n is even, $n/2$ is an integer. By Observation 4.1.2, the sequence $c(k, n; i)$ with $1 \leqslant i \leqslant n$ is symmetric. So assume $i \leqslant n/2$. Let $lcm\{1, 2, \ldots, k-1\}$ be equal to $h_r \cdot r$ with $2 \leqslant r \leqslant k-1$, and $i \equiv s \pmod{k-1}$ with $t = \lfloor i/(k-1) \rfloor$. Now,

$$\begin{aligned} c_1(k, n; i) &= \left\lfloor \frac{(n-i)}{(k-1)} \right\rfloor = \left\lfloor mh_{k-1} - \frac{i}{(k-1)} \right\rfloor = \left\lfloor mh_{k-1} - t - \frac{s}{k-1} \right\rfloor \\ c_1(k, n; i+1) &= \left\lfloor \frac{(n-i-1)}{(k-1)} \right\rfloor = \left\lfloor mh_{k-1} - \frac{i+1}{(k-1)} \right\rfloor = \left\lfloor mh_{k-1} - t - \frac{s+1}{k-1} \right\rfloor \end{aligned}$$

Therefore,

$$c_1(k, n; i+1) = \begin{cases} c_1(k, n; i) - 1 & \text{if } s = 0, \\ c_1(k, n; i) & \text{otherwise.} \end{cases}$$

Similarly,

$$c_k(k, n; i+1) = \begin{cases} c_k(k, n; i) + 1 & \text{if } s = 0, \\ c_k(k, n; i) & \text{otherwise.} \end{cases}$$

Hence, $c_1(k, n; i) + c_k(k, n; i)$ remains constant for $1 \leqslant i \leqslant n$.

Again, for $1 \leqslant i \leqslant n/2$, and we have

$$i - 1 \leqslant n - i.$$

For $2 \leqslant j \leqslant k-1$, we want to show

$$c_j(k, n; i+1) + c_{k-j+1}(k, n; i+1) \geqslant c_j(k, n; i) + c_{k-j+1}(k, n; i).$$

Assume $j > \lfloor k/2 \rfloor$. This implies $k - j \leqslant j - 1$. Since $i - 1 \leqslant n - i$, we have

$$\Big\lfloor \frac{(i-1)}{(j-1)} \Big\rfloor \leqslant \Big\lfloor \frac{(n-i)}{(k-j)} \Big\rfloor.$$

So for $1 \leqslant i \leqslant n/2 - 1$, and considering $i \equiv s \pmod{j-1}$ and $t = \lfloor i/(j-1) \rfloor$, we have,

$$\begin{aligned} c_j(k,n;i+1) &= \Big\lfloor \frac{i}{j-1} \Big\rfloor = t \\ c_j(k,n;i) &= \Big\lfloor \frac{i-1}{j-1} \Big\rfloor = \Big\lfloor t + \frac{s-1}{j-1} \Big\rfloor = \begin{cases} t-1 & \text{if } s = 0, \\ t & \text{if } s \geqslant 1. \end{cases} \end{aligned}$$

Take $j' = k - j + 1$, and then $j' - 1 \leqslant k - j'$. If $\lfloor (i-1)/(j'-1) \rfloor \leqslant \lfloor (n-i)/(k-j') \rfloor$, then

$$0 \leqslant c_{j'}(k,n;i+1) - c_{j'}(k,n;i) \leqslant 1,$$

else

$$\begin{aligned} c_{j'}(k,n;i) &= \Big\lfloor \frac{n-i}{k-j'} \Big\rfloor = \Big\lfloor \frac{n-i}{j-1} \Big\rfloor = \Big\lfloor mh_{j-1} - t - \frac{s}{j-1} \Big\rfloor \\ &= \begin{cases} (mh_{j-1} - t) & \text{if } s = 0, \\ (mh_{j-1} - t - 1) & \text{otherwise.} \end{cases} \\ c_{j'}(k,n;i+1) &= \Big\lfloor \frac{n-i-1}{k-j'} \Big\rfloor = \Big\lfloor mh_{j-1} - t - \frac{s+1}{j-1} \Big\rfloor = (mh_{j-1} - t - 1) \end{aligned}$$

Therefore,

$$c_j(k,n;i+1) + c_{j'}(k,n;i+1) \geqslant c_j(k,n;i) + c_{j'}(k,n;i).$$

So the sequence $c(k,n;x)$ with $1 \leqslant x \leqslant n/2$ is non-decreasing and hence the sequence $c(k,n;x)$ with $1 \leqslant x \leqslant n$ is unimodal for infinitely many n. $\square$

4.3 Uniqueness conjectures

Let $b(k,n)$ denote the number of k-AP free subsequences of length $r(k,n)$ in $1,2,\cdots,n$. Szekeres' conjectures exact value of $r(k,n)$ for certain values of n.

4.3.1 Szekeres' conjecture

Erdős and Turán [29] noted that there is no 3-term arithmetic progression in the sequence of all numbers $n, 0 \leqslant n \leqslant \frac{1}{2}(3^t - 1)$, which do not contain the digit 2 in the ternary scale. Hence for every $t \geqslant 1$,

$$r\left(3, (3^t+1)/2\right) \geqslant 2^t$$

as we obtain the 3-AP-free sequence of length 2^t in $\left\langle (3^t+1)/2 \right\rangle$ by adding 1 to each of those numbers that does not contain digit 2 in the ternary scale. Szekeres conjectured that for every $t \geqslant 1$,

$$r\left(3, (3^t+1)/2\right) = 2^t,$$

and more generally, for any t and any prime p,

$$r\left(p, \frac{(p-2)p^t+1}{p-1}\right) = (p-1)^t.$$

4.3.2 Generalization of Szekeres' conjecture

Define

$$J(k,L) = \left\{(n,m) : n \leqslant L, r(k,n) = m \text{ and } b(k,n) = 1\right\}.$$

We have the following experimental data, based on which we formulate Conjectures 4.3.1 and

4.3.2:

$$\begin{aligned}
J(3,123) &= \{(2,2),(5,4),(14,8),(30,12),(41,16),(74,22),(84,24),(104,28),\\
&\quad (114,30),(122,32)\},\\
J(5,105) &= \{(2,2),(3,3),(4,4),(9,8),(14,12),(19,16),(44,32),(69,48),(94,64)\},\\
J(7,139) &= \{(2,2),(3,3),(4,4),(5,5),(6,6),(13,12),(20,18),(27,24),(34,30),\\
&\quad (41,36),(90,72),(139,108)\},\\
J(11,117) &= \{(2,2),(3,3),(4,4),(5,5),(6,6),(7,7),(8,8),(9,9),(10,10),(21,20),\\
&\quad (32,30),(43,40),(54,50),(65,60),(76,70),(87,80),(98,90),(109,100)\},\\
J(13,161) &= \{(2,2),(3,3),(4,4),(5,5),(6,6),(7,7),(8,8),(9,9),(10,10),(11,11),\\
&\quad (12,12),(25,24),(38,36),(51,48),(64,60),(77,72),(90,84),(103,96),\\
&\quad (116,108),(129,120),(142,132),(155,144)\}.
\end{aligned}$$

Conjecture 4.3.1 (The Uniqueness Conjecture)**.** *Consider a prime $p > 3$ and an integer $t \geqslant 1$. Then for $1 \leqslant i \leqslant p-1$,*

$$r\left(p, \frac{(ip-i-1)p^t+1}{(p-1)}\right) = i \cdot (p-1)^t,$$

and $b(p,x) = 1$, where $1 \leqslant x \leqslant p-2$ or else

$$x = \frac{(ip-i-1)p^t+1}{(p-1)}.$$

It can be observed that Szekeres' conjecture is a special case of Conjecture 4.3.1 with $i = 1$.

Conjecture 4.3.2 (Strong Uniqueness Conjecture)**.** *Consider a prime $p > 3$ and an integer $t \geqslant 1$. Then $b(p,x) = 1$ if and only if $1 \leqslant x \leqslant p-2$ or else*

$$x = \frac{(ip-i-1)p^t+1}{(p-1)}$$

with $1 \leqslant i \leqslant p-1$.

4.3.3 Construction for the lower-bound of Conjecture 4.3.1

For a prime $p > 3$ and $1 \leqslant i \leqslant p-1$, take

$$n = \frac{(ip-i-1)p^t+1}{p-1} = ip^t - p^{t-1} - p^{t-2} - \cdots - p - 1.$$

We can construct a p-AP free subset of $\{1, 2, \ldots, n\}$ of size $i \cdot (p-1)^t$ as follows:

$$\begin{aligned}
T_0 &= \{1, 2, \ldots, n\}, \\
T_1 &= T_0 - \{j : j \equiv 0 \pmod{p}\} = T_0 - S_0, \\
T_2 &= T_1 - \left\{j_1 p^2 - p + j_2 : 1 \leqslant j_1 \leqslant \lfloor n/p^2 \rfloor, 1 \leqslant j_2 \leqslant p-1\right\} = T_1 - S_1, \\
T_3 &= T_2 - \left\{j_1 p^3 - p^2 + j_2 p - j_3 : 1 \leqslant j_1 \leqslant \lfloor n/p^3 \rfloor, 1 \leqslant j_2, j_3 \leqslant p-1\right\} = T_2 - S_2, \\
T_4 &= T_3 - \left\{j_1 p^4 - p^3 + j_2 p^2 - j_3 p + j_4 : 1 \leqslant j_1 \leqslant \lfloor n/p^4 \rfloor, 1 \leqslant j_2, j_3, j_4 \leqslant p-1\right\} = T_3 - S_3, \\
&\vdots, \\
T_t &= T_{t-1} - \left\{j_1 p^t - p^{t-1} + \sum_{\ell=2}^{t} p^{t-\ell} j_\ell (-1)^\ell : 1 \leqslant j_1 \leqslant \lfloor n/p^t \rfloor, 1 \leqslant j_2, j_3, \ldots, j_t \leqslant p-1\right\} \\
&= T_{t-1} - S_{t-1}.
\end{aligned}$$

It can be observed that

$$\begin{aligned}
|S_0| &= \lfloor n/p \rfloor = ip^{t-1} - p^{t-2} - \cdots - p - 2, \\
|S_1| &= (p-1)\lfloor n/p^2 \rfloor = (p-1)\left(ip^{t-2} - p^{t-3} - \cdots - p - 2\right), \\
|S_2| &= (p-1)^2 \lfloor n/p^3 \rfloor = (p-1)^2\left(ip^{t-3} - p^{t-4} - \cdots - p - 2\right), \\
&\vdots, \\
|S_{t-1}| &= (p-1)^{t-1}\lfloor n/p^t \rfloor = (p-1)^{t-1}(i-1).
\end{aligned}$$

Lemma 4.3.3. *The S_ℓ's for $0 \leqslant \ell \leqslant t-1$ are disjoint.*

Proof. Each element in S_0 is divisible by p, and no element in any other S_ℓ is divisible by p. So S_0

is disjoint from every other S_ℓ. For $2 \leqslant \ell \leqslant t-1$,

$$S_\ell = \left\{ (xp + (-1)^{\ell-1} j_{\ell+1}) \in T_0 : x \in S_{\ell-1}, 1 \leqslant j_{\ell+1} \leqslant p-1 \right\},$$

and hence $S_\ell \cap S_u = \emptyset$ for $1 \leqslant u \leqslant \ell - 1$. □

Lemma 4.3.4. *For a prime $p > 3$ and $1 \leqslant i \leqslant p-1$,*

$$|T_t| = i \cdot (p-1)^t.$$

Proof. We can write the summation $\sum_{j=0}^{t-1} |S_j|$ as follows:

$$\begin{aligned}
\sum_{j=0}^{t-1} |S_j| &= ip^{t-1} \left(\sum_{a=0}^{t-1} \binom{a}{0} \right) + \cdots + (-1)^{\ell-1} ip^{t-\ell} \left(\sum_{a=\ell-1}^{t-1} \binom{a}{\ell-1} \right) + \cdots - i - \sum_{\ell=0}^{t-1} p^\ell, \\
&= ip^{t-1} \binom{t}{1} + \cdots + (-1)^{\ell-1} ip^{t-\ell} \binom{t}{\ell} + \cdots - i - \sum_{\ell=0}^{t-1} p^\ell, \\
&= i \sum_{\ell=1}^{t} \binom{t}{\ell} p^{t-\ell} (-1)^{\ell-1} - \sum_{\ell=0}^{t-1} p^\ell.
\end{aligned}$$

The fact

$$\sum_{a=\ell-1}^{t-1} \binom{a}{\ell-1} = \binom{t}{\ell}$$

can be easily proven using induction on t and using the fact that

$$\binom{t}{\ell} + \binom{t}{\ell-1} = \binom{t+1}{\ell}.$$

Now, we have

$$\begin{aligned}
|T_t| &= n - \sum_{j=0}^{t-1} |S_j|, \\
&= ip^t - \sum_{\ell=0}^{t-1} p^\ell - \left(i \sum_{\ell=1}^{t} \binom{t}{\ell} p^{t-\ell} (-1)^{\ell-1} - \sum_{\ell=0}^{t-1} p^\ell \right), \\
&= ip^t - i \sum_{\ell=1}^{t} \binom{t}{\ell} p^{t-\ell} (-1)^{\ell-1} = i \cdot (p-1)^t.
\end{aligned}$$

□

Lemma 4.3.5. *Given a prime $p > 3$, $n = ip^t - \sum_{\ell=0}^{t-1} p^\ell$ with $1 \leqslant i \leqslant p-1$, and the set $T = \{1, 2, \ldots, n\}$; the set T_1 contains no p-AP with*

$$d \in \{1 \leqslant d_1 \leqslant \lfloor (n-1)/(p-1) \rfloor : d_1 \not\equiv 0 \pmod p\}.$$

Proof. Assume T_1 contains a p-AP $a, a+d, \ldots, a+(p-1)d$. Here $a \not\equiv 0 \pmod p$. Suppose $a \equiv j \pmod p$ for some $1 \leqslant j \leqslant p-1$. Then $a + d(p-z) \equiv 0 \pmod p$ for some $1 \leqslant z \leqslant p-1$ when $dz \equiv j \pmod p$.

For each $d \in \{1 \leqslant d_1 \leqslant \lfloor (n-1)/(p-1) \rfloor : d_1 \not\equiv 0 \pmod p\}$,

$$\bigcup_{z=1}^{p-1} \{dz \pmod p\} = \{1, 2, \ldots, p-1\}$$

and so there exists $1 \leqslant z \leqslant p-1$ for any $1 \leqslant j \leqslant p-1$ such that $dz \equiv j \pmod p$. But, this is a contradiction since there is no number in T_1 which is divisible by p. Hence, T_1 contains no p-AP with $d \in \{1 \leqslant d_1 \leqslant \lfloor (n-1)/(p-1) \rfloor : d_1 \not\equiv 0 \pmod p\}$. □

Lemma 4.3.6. *The set T_t is p-AP free.*

Proof. By contruction, T_t contains no p-AP with $d \in \{1, p, p^2, \ldots, p^t\}$. By Lemma 4.3.5, T_t does not contain a p-AP with any other d. Hence, T_t is p-AP free. □

4.3.4 A construction algorithm for $r(k, n)$

In this section, we propose a greedy algorithm for construction of k-AP free subsequence of $1, 2, \ldots, n$. We call this algorithm *Bi-symmetric Greedy Algorithm* (BGA) as it builds a fully symmetric subsequence that is k-AP free.

1. Take $T = \{1, n\}$.

2. Choose the smallest $j \in \{1, 2, \ldots, n\} - T$ such that $T \cup \{j, n-j+1\}$ is k-AP free. Set $T = T \cup \{j, n-j+1\}$.

3. Repeat step 2 until no such j can be found.

4. Output T.

Clearly,

$$r(k,n) \geqslant |BGA(k,n)|.$$

From experimental data, we have the following observation:

Observation 4.3.7. *Consider a prime $p > 3$. Then $|BGA(p,x)| = x$ if $1 \leqslant x \leqslant p-2$, or else for $1 \leqslant i \leqslant p-1$ and $t \geqslant 1$,*

$$\left|BGA\left(p, \frac{(ip-i-1)p^t+1}{(p-1)}\right)\right| = i \cdot (p-1)^t.$$

Chapter 5

Strict Schur Numbers

In this chapter, we describe a variant of Schur numbers, namely, Strict Schur numbers. The content of this chapter is based on the joint work with Michael G. Eldredge, Jonathan J. Marler, and Hunter Snevily [7].

Let $\mathbb{N}$ denote the set of positive integers and $[a, b] = \{n \in \mathbb{N} : a \leqslant n \leqslant b\}$. A mapping $\chi : [a, b] \to [1, t]$ is called a t-colouring of $[a, b]$. Let L_m denote the system of inequalities given by

$$x_1 + x_2 + \ldots, x_{m-1} = x_m$$

$$x_1 < x_2 < \ldots < x_m.$$

A solution $n_1, n_2, \ldots, n_m$ to L_m is *monochromatic* if $\chi(n_i) = \chi(n_j)$ for $1 \leqslant i, j \leqslant m$. Henceforth, we assume a *two-colouring* ($t = 2$) of the interval and denote each colour by *red* and *blue*. Furthermore, a monochromatic solution to L_m such that $\chi(n_1) = \chi(n_2) = \ldots = \chi(n_m) = red$ will be called a "*red* solution", and likewise for a "*blue* solution". Lastly, we define $S(h, k)$ to be the least positive integer such that every colouring of the interval $[1, S(h, k)]$ by the colours *red* and *blue* contains either a *red* solution to L_k or a *blue* solution to L_h.

In the following proofs, we show that $S(3,3) = 9$, $S(3,4) = 16$, and for all $k \geqslant 5$,

$$S(3,k) = \begin{cases} 3k^2/2 - 7k/2 + 3 & \text{if } k \equiv 0, 1 \pmod 4, \\ 3k^2/2 - 7k/2 + 4 & \text{if } k \equiv 2, 3 \pmod 4. \end{cases}$$

For simplicity, we write

$$N_k = \begin{cases} 3k^2/2 - 7k/2 + 3 & \text{if } k \equiv 0, 1 \pmod 4, \\ 3k^2/2 - 7k/2 + 4 & \text{if } k \equiv 2, 3 \pmod 4. \end{cases}$$

5.1 The Lower Bound

Lemma 5.1.1 (Lower Bound)**.** *For $k \geqslant 3$,*

$$S(3,k) \geqslant N_k = \begin{cases} 3k^2/2 - 7k/2 + 3 & \textit{if } k \equiv 0, 1 \pmod 4, \\ 3k^2/2 - 7k/2 + 4 & \textit{if } k \equiv 2, 3 \pmod 4. \end{cases}$$

Proof. Consider a colouring of $\chi : [1, N_k - 1] \to \{blue, red\}$ defined as follows. For $n \in [1, N_k - 1]$, let

$$\chi(n) = \begin{cases} blue & \text{if } n \equiv 1 \pmod 2 \text{ and } n \leqslant k(k-1)/2, \\ blue & \text{if } n \equiv 0 \pmod 2 \text{ and } n \geqslant k(k-1), \\ red & \text{otherwise.} \end{cases}$$

We claim this colouring has no *blue* solution to L_3 and no *red* solution to L_k.

Suppose $n_1 + n_2 = n_3$, where $n_1 < n_2 < n_3$, is a *blue* solution to L_3 on the interval $[1, N_k - 1]$. Suppose $n_2 \equiv 1 \pmod 2$. Then $n_1 < n_2 \leqslant k(k-1)/2$, which implies $n_1 \equiv 1 \pmod 2$ and $n_3 < k(k-1)$. Since $n_3 = n_1 + n_2 \equiv (1+1) \pmod 2 \equiv 0 \pmod 2$, we must have $n_3 \geqslant k(k-1)$, which is a contradiction. Therefore, $n_2 \equiv 0 \pmod 2$.

Hence, $n_3 > n_2 \geqslant k(k-1)$, which implies $n_3 \equiv 0 \pmod 2$, which then implies $n_1 \equiv 0 \pmod 2$. Therefore, $n_2 > n_1 \geqslant k(k-1)$, which implies $n_3 = n_1 + n_2 \geqslant k(k-1) + [k(k-1)+2] > N_k - 1$, another contradiction implying no such *blue* solution to L_3 exists.

Next, suppose $n_1 + n_2 + \cdots + n_{k-1} = n_k$, where $n_1 < n_2 < \cdots < n_k$, is a *red* solution to

L_k on the interval $[1, N_k - 1]$. Let q denote the minimal sum of $k-2$ *red* numbers. Clearly, $q = \sum_{i=1}^{k-2} 2i = k^2 - 3k + 2$.

If $n_{k-1} \leqslant k(k-1)/2$, then $n_i \equiv 0 \pmod 2$ for $i \in [1, k-1]$. This implies $n_k \equiv 0 \pmod 2$, which then implies $n_k < k(k-1)$, but this is a contradiction since $k(k-1) > n_k \geqslant q + 2(k-1) = k(k-1)$. Therefore, $n_{k-1} > k(k-1)/2$.

If $k \equiv 0, 1 \pmod 4$, then $n_k > q + k(k-1)/2 = 3k^2/2 - 7k/2 + 2 = N_k - 1$, a contradiction that implies $k \equiv 2, 3 \pmod 4$.

If $n_{k-1} = k(k-1)/2 + 1$, then $n_i \equiv 0 \pmod 2$ for all $i \in [1, k-1]$, which implies $n_k \equiv 0 \pmod 2$. Since n_k is *red*, $n_k < k(k-1)$, which is a contradiction since $n_k \geqslant q + k(k-1)/2 + 1$. Therefore, $n_{k-1} > k(k-1)/2 + 1$, which implies $n_k \geqslant q + k(k-1)/2 + 2 = 3k^2/2 - 7k/2 + 4 = N_k$. □

5.2 The Upper Bound

Throughout this section, let p_k denote the sum of the first k *red* numbers and let r_i and b_i denote the i^{th} *red* and *blue* numbers, respectively. Then, $r_i < r_j$ and $b_i < b_j$, for all $i < j$.

Lemma 5.2.1. *For $n \geqslant 3$, if at least $n+1$ numbers in the interval $[1, 2n]$ are coloured blue, then the only colouring that avoids a blue solution to L_3 is given by*

$$\chi(x) = \begin{cases} \textit{red} & \textit{if } x \in [1, n-1], \\ \textit{blue} & \textit{if } x \in [n, 2n]. \end{cases}$$

Proof. Since the case $n = 3$ is trivial, assume $n > 3$.

Case 1: $\chi(2n) = \textit{red}$ (By induction). For some $n > 3$, assume the claim holds for $n-1$. To avoid a *blue* solution on the interval $[1, 2(n-1)]$, we must have $\chi(x) = \textit{blue}$ for all $x \in [(n-1), 2(n-1)]$ and $\chi(x) = \textit{red}$ for all $x \in [1, n-2]$. Since we need another *blue* in the interval $[1, 2n]$, the number $(2n-1)$ must be *blue*, but then $(n-1) + n = (2n-1)$ is a *blue* solution to L_3.

Case 2: $\chi(2n) = \textit{blue}$. By the Pigeonhole principle, $\chi(n) = \textit{blue}$, since otherwise one of the pairs $\{x, 2n-x\}$ with $1 \leqslant x < n$ would be all *blue* giving us the *blue* solution $(x) + (2n-x) = 2n$. Now suppose $\chi(1) = \textit{blue}$, which implies the pair $\{n-1, n+1\}$ is all *red*. Hence, some other pair $\{x, 2n-x\}$ with $1 \leqslant x < n-1$ is all *blue* and we get a contradiction. Therefore, $\chi(1) = \textit{red}$; hence

$\chi(2n-1) = blue$, otherwise some other pair $\{x, 2n-x\}$ with $2 \leqslant x \leqslant n-2$ would be all *blue*, giving us a contradiction as before. But then we must have $\chi(n-1) = red$ since $(n-1)+n = (2n-1)$; hence $\chi(n+1) = blue$. But then we must have $\chi(n-2) = red$ since $(n-2)+(n+1) = (2n-1)$; hence $\chi(n+2) = blue$. Continuing in this manner, we get the desired colouring. □

Corollary 5.2.2. *To avoid a blue solution to* L_3, $r_i \leqslant 2i+1$ *for all* i.

Proof. The claim is easily proven for r_1 and r_2. Suppose $r_i > 2i+1$ for some $i \geqslant 3$. This would imply at least $i+1$ numbers are coloured *blue* in the interval $[1, 2i]$. Applying Lemma 5.2.1 gives us that $\chi(i) = \chi(i+1) = blue$. Since there are $2i+1-(i-1) = i+2$ *blue* integers in $[1, 2i+1]$, $\chi(2i+1) = blue$ as well, but this yields the *blue* solution $i + (i+1) = 2i+1$. □

Corollary 5.2.3. *Let* $i > b_1$ *and* $i \geqslant 3$. *To avoid a blue solution to* L_3, $r_i \leqslant 2i$.

Proof. Suppose $r_i > 2i$. Then Corollary 5.2.2 implies $r_i = 2i+1$. Therefore, the interval $[1, 2i+1]$ must contain exactly $i+1$ blue numbers. Since $i \geqslant 3$, Lemma 5.2.1 implies that the interval $[1, i-1]$ is all *red*, but this contradicts the hypothesis $b_1 < i$. □

Lemma 5.2.4 (P Lemma). *(i) If* $b_1 = 1$, *then* $p_k \leqslant k^2 + k - 12 + r_1 + r_2 + r_3$,

(ii) If $1 < b_1 \leqslant k$, *then* $p_k \leqslant k^2 + k + 1 - b_1(b_1-1)/2$,

(iii) If $b_1 > k$, *then* $p_k = k(k+1)/2$.

Proof. (*i*) If $b_1 = 1$, then by using Corollary 5.2.3, we have

$$\begin{aligned} p_k = r_1 + r_2 + r_3 + \sum_{i=4}^{k} r_i &\leqslant r_1 + r_2 + r_3 + \sum_{i=4}^{k} 2i \\ &= k^2 + k - 12 + r_1 + r_2 + r_3. \end{aligned}$$

(*ii*) If $1 < b_1 \leqslant k$, then by using Corollaries 5.2.2 and 5.2.3, we have

$$\begin{aligned} p_k = \sum_{i=1}^{b_1-1} r_i + r_{b_1} + \sum_{i=b_1+1}^{k} r_i &\leqslant \sum_{i=1}^{b_1-1} i + (2b_1+1) + \sum_{i=b_1+1}^{k} r_i \\ &\leqslant \sum_{i=1}^{b_1-1} i + (2b_1+1) + \sum_{i=b_1+1}^{k} 2i \\ &= k^2 + k + 1 - b_1(b_1-1)/2 \end{aligned}$$

(*iii*) If $b_1 > k$, then $p_k = \sum_{i=1}^{k} r_i = \sum_{i=1}^{k} i = k(k+1)/2$.

□

Given a valid colouring, the upper bound of p_k can be improved by modifying Lemma 5.2.4. For example, for $k \geqslant 6$, if $r_1 = 1$, $b_1 = 2$, $r_2 = 3$, and $r_3 = 4$, then

$$p_k \leqslant k^2 + k + 1 - b_1(b_1-1)/2 - (5 - r_2) - (6 - r_3) = k^2 + k - 4.$$

Fact 5.2.5. *If $k \geqslant 6$, then $k^2 + k - 5 \leqslant N_k$.*

Corollary 5.2.6. *If $p_k - r_j \leqslant N_k$ for some $j \in [1, k]$, then to avoid a red solution to L_k, $\chi(p_k - r_i) =$ blue for all $i \in [j, k]$.*

Proof. If $\chi(p_k - r_i) = red$ for some $i \in [j, k]$, then we get the *red* solution to L_k

$$r_1 + r_2 + \ldots + r_{i-1} + r_{i+1} + r_{i+2} + \ldots + r_k = p_k - r_i,$$

where $p_k - r_i \leqslant p_k - r_j \leqslant N_k$ (by hypothesis).

Hence, $\chi(p_k - r_i) = blue$ for all $i \in [j, k]$. □

Note that Corollary 5.2.6 shows that p_k exists, that is, that are at least k numbers coloured *red*.

Corollary 5.2.7. *If $k \geqslant 6$ and $b_1 > 1$, then $p_k - r_i \leqslant N_k$ for all r_i.*

Proof. We have $p_k - r_i \leqslant p_k - 1$. In view of Fact 5.2.5, if $p_k \leqslant k^2 + k - 4$, then $p_k - r_i \leqslant N_k$ for all r_i. If $b_1 = 2$, then modifying Lemma 5.2.4, we get the following cases:

#	1	2	3	4	5	6	7	8	9	10	11	12	n s.t. $p_k \leqslant n$
1.	r_1	b_1	b_2	b_3	r_2	r_3	r_4	b_4	b_5	r_5	r_6	r_7	k^2+k-4
2.	r_1	b_1	b_2	b_3	r_2	r_3	r_4	b_4	r_5	r_6			k^2+k-4
3.	r_1	b_1	b_2	b_3	r_2	r_3	r_4	r_5	b_4	b_5	r_6		k^2+k-4
4.	r_1	b_1	b_2	b_3	r_2	r_3	r_4	r_5	b_4	r_6			k^2+k-5
5.	r_1	b_1	b_2	b_3	r_2	r_3	r_4	r_5	r_6				k^2+k-6
6.	r_1	b_1	b_2	r_2	r_3	b_3	b_4	r_4	r_5	r_6			k^2+k-5
7.	r_1	b_1	b_2	r_2	r_3	b_3	r_4	r_5					k^2+k-5
8.	r_1	b_1	b_2	r_2	r_3	r_4							k^2+k-4
9.	r_1	b_1	r_2	b_2	b_3	r_3	r_4	b_4	r_5				k^2+k-4
10.	r_1	b_1	r_2	b_2	b_3	r_3	r_4	r_5					k^2+k-5
11.	r_1	b_1	r_2	b_2	r_3	r_4							k^2+k-5
12.	r_1	b_1	r_2	r_3									k^2+k-4

Similarly, if $b_1 = 3$ then modifying Lemma 5.2.4, we get the following cases:

#	1	2	3	4	5	6	7	8	9	10	n s.t. $p_k \leqslant n$
1.	r_1	r_2	b_1	b_2	b_3	b_4	r_3	r_4	r_5	r_6	k^2+k-5
2.	r_1	r_2	b_1	b_2	b_3	r_3	r_4				k^2+k-4
3.	r_1	r_2	b_1	b_2	r_3						k^2+k-4
4.	r_1	r_2	b_1	r_3							k^2+k-5

For $4 \leqslant b_1 \leqslant k$, we have, by using Lemma 5.2.4,

$$p_k \leqslant k^2 + k - 5.$$

For $b_1 > k$, $p_k = k(k+1)/2$ by Lemma 5.2.4(iii).

For $k \geqslant 6$ and $b_1 > 1$, we have $p_k \leqslant k^2 + k - 4$, and hence by using Fact 5.2.5,

$$p_k - r_i \leqslant p_k - 1 \leqslant (k^2 + k - 4) - 1 = k^2 + k - 5 \leqslant N_k$$

for all r_i with $i \in [1, k]$. □

Remark 5.2.8. *Combining Corollaries 5.2.6 and 5.2.7, we see that, for* $k \geqslant 6$ *and* $b_1 > 1$, $\chi(p_k - r_j) =$ *blue for all* $j \in [1, k]$.

Lemma 5.2.9 (Upper Bound)**.** *For* $k \geqslant 6$,

$$S(3,k) \leqslant N_k = \begin{cases} 3k^2/2 - 7k/2 + 3 & \text{if } k \equiv 0,1 \mod 4, \\ 3k^2/2 - 7k/2 + 4 & \text{if } k \equiv 2,3 \mod 4. \end{cases}$$

Proof. Suppose to the contrary that N_k is not an upper bound for $k \geqslant 6$. This occurs if and only if there exists a colouring of $[1, N_k]$ without a *blue* solution to L_3 and a *red* solution to L_k. Consider the following two cases:

(1) $\chi(1) = blue$ (with $k \geqslant 6$). Suppose $\chi(2) = blue$. Then $r_1 = 3$ and $r_2 \leqslant 5$ (by Corollary 5.2.2) to avoid *blue* solutions $1 + 2 = 3$ and $1 + 4 = 5$, respectively. Therefore, by using Lemma 5.2.4 and Corollary 5.2.3, we have

$$p_k \leqslant k^2 + k - 4 + r_3 \leqslant k^2 + k + 2,$$

which implies (by Fact 5.2.5) $p_k - r_i \leqslant k^2 + k - 5 \leqslant N_k$ for $r_i \geqslant 7$.

If $\chi(x) = blue$ for some $x \in [6, 7]$, then this implies $\chi(x+1) = \chi(x+2) = red$ to avoid the *blue* solutions $1 + x = x + 1$ and $2 + x = x + 2$. Corollary 5.2.6 implies $\chi(p_k - x - 1) = \chi(p_k - x - 2) = blue$, and then $1 + (p_k - x - 2) = p_k - x - 1$ is a *blue* solution since $p_k - x - 2 \geqslant k(k+1)/2 - 9 \geqslant 12 > 1$. Therefore $\chi(6) = \chi(7) = red$, in which case Corollary 5.2.6 gives $\chi(p_k - 6) = \chi(p_k - 7) = blue$. Thus $1 + (p_k - 7) = p_k - 6$ is a *blue* solution in view of $p_k - 8 \geqslant k(k+1)/2 - 8 \geqslant 13 > 1$. So we conclude that $\chi(2) = red$.

If $b_2 \geqslant 5$, then $r_2 = 3$, $r_3 = 4$, and by using Lemma 5.2.4 and Fact 5.2.5, we have

$$p_k - r_1 \leqslant k^2 + k - 12 + r_2 + r_3 = k^2 + k - 5 \leqslant N_k,$$

which leads to a contradiction since Corollary 5.2.6 gives us the *blue* solution $1 + (p_k - 4) = p_k - 3$.

If $b_2 = 4$, then $r_2 = 3$ and $r_3 = 5$, and by Lemma 5.2.4 and Fact 5.2.5, we have

$$p_k - r_2 \leqslant k^2 + k - 12 + r_1 + r_3 = k^2 + k - 5 \leqslant N_k.$$

By Corollary 5.2.6, $\chi(p_k - 3) = \chi(p_k - 5) = \mathit{blue}$. To avoid a *blue* solution $1 + (p_k - 6) = p_k - 5$, we need $\chi(p_k - 6) = \mathit{red}$, but by Corollary 5.2.6, this implies $\chi(6) = \mathit{blue}$. In that case, to avoid the *blue* solution $1 + 6 = 7$, we need $\chi(7) = \mathit{red}$, but that yields the *blue* solution $4 + (p_k - 7) = p_k - 3$.

Now, suppose $b_3 > r_3$.

If $b_2 = 3$, then $r_2 = 4$, $r_3 = 5$ (since $b_3 > r_3$), and by using Lemma 5.2.4 and Fact 5.2.5, we have

$$p_k - r_2 \leqslant k^2 + k - 5 \leqslant N_k,$$

but that yields the *blue* solution $1 + (p_k - 5) = p_k - 4$ (by Corollary 5.2.6).

Therefore, $b_3 < r_3$, which implies the interval $[3, 5]$ has two *blue* numbers. Since these *blue* numbers cannot be adjacent, the only valid colouring is $\chi(3) = \chi(5) = \mathit{blue}$ and $\chi(4) = \chi(6) = \mathit{red}$. With this colouring, Corollary 5.2.2, Lemma 5.2.4, and Fact 5.2.5 conclude that $\chi(p_k - r_i) = \mathit{blue}$, for $i \in [3, k]$.

To avoid the *blue* solution $1 + (p_k - r_{i+1}) = p_k - r_i$, we must have $r_{i+1} > r_i + 1$, that is, $\chi(r_i + 1) = \mathit{blue}$, for all $i \in [3, k-1]$. Thus $\chi(7) = \mathit{blue}$. Also, to avoid the *blue* solution $1 + b_i = b_{i+1}$, we must have $\chi(b_i + 1) = \mathit{red}$, for all $i > 1$. Thus $\chi(8) = \mathit{red}$, which implies $\chi(9) = \mathit{blue}$, which then implies $\chi(10) = \mathit{red}$, and continuing in this manner, we get for all $x \in [1, 2k]$,

$$\chi(x) = \begin{cases} \mathit{red} & \text{if } x \equiv 0 \mod 2, \\ \mathit{blue} & \text{if } x \equiv 1 \mod 2. \end{cases}$$

Furthermore, for all $x \in [1, 2k-3]$ with $\chi(x) = \mathit{blue}$, we must have $\chi(x + (2k-1)) = \mathit{red}$, otherwise we get the *blue* solution $x + (2k-1) = x + 2k - 1$. This implies $\chi(y) = \mathit{red}$ for all $y \in [2k, 4k-4]$ with $y \equiv 0 \pmod 2$.

Using the block of even *red* numbers, we can extend the *blue* interval. Clearly, the sum of any $k-1$ *red* numbers which is less than N_k must be *blue*. The maximal sum of $k-1$ *red* numbers from the block is $\sum_{i=0}^{k-2}((4k-4)-2i) = 3k^2-5k+2$, which is clearly greater than N_k. Furthermore, the minimal sum of $k-1$ *red* numbers from the block is $\sum_{i=1}^{k-1} 2i = k^2-k$. Since we can always replace a *red* number in the minimal sum by an adjacent even number which is also *red*, and the maximal sum is greater than N_k, we get that all even numbers greater than or equal to k^2-k must be *blue*. This yields the extended colouring

$$\chi(x) = \begin{cases} red & \text{if } x \equiv 0 \pmod 2 \text{ and } x \in [2, 4k-4], \\ blue & \text{if } x \equiv 1 \pmod 2 \text{ and } x \in [1, 2k-1], \\ blue & \text{if } x \equiv 0 \pmod 2 \text{ and } x \in [k^2-k, N_k]. \end{cases}$$

It can easily be shown that $N_k \equiv 1 \pmod 2$, implying $\chi(N_k-1) = blue$. Since $\chi(1) = \chi(3) = blue$, we must have $\chi(N_k) = \chi(N_k-2) = \chi(N_k-4) = red$. Let q be the sum of first $k-2$ *red* numbers. Then $q = k^2-3k+2$. To avoid a *red* solution to L_k, we must have

$$\chi(N_k-q) = \chi(N_k-2-q) = \chi(N_k-4-q) = blue,$$

since $N_k-4-q > r_{k-2} = 2(k-2)$.

If $k \equiv 0, 1 \pmod 4$, then we get the *blue* solution

$$(N_k-q)+(N_k-2-q) = 2(3k^2/2-7k/2+3)-2(k^2-3k+2)-2 = k^2-k.$$

Likewise, if $k \equiv 2, 3 \pmod 4$, then we get the *blue* solution

$$(N_k-q)+(N_k-4-q) = 2(3k^2/2-7k/2+4)-2(k^2-3k+2)-4 = k^2-k.$$

Therefore, $\chi(1) \neq blue$.

(2) $\chi(1) = red$ (with $k \geqslant 6$). Since $k \geqslant 6$ and $b_1 > 1$, $\chi(p_k - r_i) = blue$ for all $i \in [1, k]$. Let a be the minimum *red* number such that $\chi(a-1) = blue$.

If $a < r_k$, then $\chi(p_k - a) = blue$, which gives us a potential *blue* solution $(a-1)+(p_k-a) =$

$p_k - 1$. In order for it to be a valid solution, we must have $a - 1 \neq p_k - a$. However, since $a = r_i$ for some $i \in [1, k]$, this has already been proven in Corollary 5.2.6 ($p_k - r_i > r_k$ for all $i \in [1, k]$).

If $a = r_k$, then $\chi(p_k - r_k) = \chi(\sum_{i=1}^{k-1} i) = blue$ to avoid a *red* solution to L_k. But $k + (k+1) = 2k+1$ is a *blue* solution since $2k+1 < \sum_{i=1}^{k-1} i$ for $k \geqslant 6$. Therefore, $a > r_k$, that is, the first k numbers must be *red* giving us $b_1 \geqslant k+1$.

Suppose $b_1 \leqslant 3k/2$. To avoid the *blue* solution, $b_1 + (b_1 + 1) = 2b_1 + 1$, either $b_1 + 1$ or $2b_1 + 1$ must be *red*, which implies $a \leqslant 2b_1 + 1 \leqslant 3k + 1$. Now consider,

$$\begin{aligned}(p_k - r_k) - 1 + a &= \sum_{i=1}^{k-1} i + (a-1) = k(k-1)/2 + (a-1) \\ &\leqslant k(k-1)/2 + (3k+1) - 1 = (k^2 + 5k)/2 \leqslant N_k.\end{aligned}$$

Since $\chi(a) = red$, to avoid the *red* solution $a + \sum_{i=2}^{k-1} i = (p_k - r_k) - 1 + a$, we have $\chi(p_k - r_k + a - 1) = blue$, which yields the potential *blue* solution

$$(p_k - r_k) + (a-1) = (p_k - r_k) - 1 + a.$$

To be a valid solution, we must have $a - 1 \neq p_k - r_k$. If $a - 1 = p_k - r_k$, then $p_k - r_k + 1 = a$, which implies $\chi(p_k - r_k + 1) = red$. However, this is a contradiction since $\chi(p_k - r_{k-1}) = blue$ and $r_{k-1} = r_k - 1$.

Therefore, $b_1 > 3k/2$, which implies $\chi(x) = red$ for all $x \in [1, 3k/2]$. Using this *red* interval, we can create another *blue* interval. The minimum sum of $k-1$ *red* numbers in this interval is $k(k-1)/2$, and the maximal sum is $k^2 - 1$. Since every integer in this new interval can be represented by a sum of $k-1$ *red* numbers, the interval $[k(k-1)/2, k^2 - 1]$ must be coloured *blue* to avoid a *red* solution. Since $k^2 - k + 1$ is in the *blue* interval, we have the *blue* solution $k(k-1)/2 + [k(k-1)/2 + 1] = k^2 - k + 1$.

Hence, for $k \geqslant 6$, every colouring of $[1, N_k]$ has a *blue* solution to L_3 or a *red* solution to L_k. □

5.3 The Cases $3 \leqslant k \leqslant 5$

In this section, we formally prove the exact values of $S(3,3)$ and $S(3,4)$, and provide the computer proof for the exact values of $S(3,5)$.

5.3.1 $S(3,3) = 9$

Lemma 5.3.1. $S(3,3) = 9$.

Proof. Let $\chi(1) = \chi(2) = \chi(4) = \chi(8) = red$ and $\chi(3) = \chi(5) = \chi(6) = \chi(7) = blue$. This colouring has no *red* or *blue* solution to L_3. Therefore, $S(3,3) > 8$.

Suppose to the contrary that $S(3,3) > 9$. Without loss of generality, let *blue* be the colour used 5 or more times from 1 to 9. If $\chi(9) = red$, then Lemma 5.2.1 gives us the *red* solution $1 + 2 = 3$. Therefore, $\chi(9) = blue$. We are left with two cases.

Case 1: $\chi(8) = blue$. To avoid the *blue* solution $1 + 8 = 9$, we have $\chi(1) = red$. If $\chi(5) = blue$, then we must have $\chi(3) = red$ (to avoid the *blue* solution $3 + 5 = 8$) and $\chi(4) = red$ (to avoid the *blue* solution $4 + 5 = 9$). But then we have the *red* solution $1 + 3 = 4$. Therefore, $\chi(5) = red$. To avoid the *red* solutions $1 + 4 = 5$ and $1 + 5 = 6$, we must have $\chi(4) = \chi(6) = blue$. Then $\chi(2) = red$ (to avoid the *blue* solution $2+4 = 6$), which implies $\chi(3) = blue$ (to avoid the *red* solution $1+2 = 3$), which gives the *blue* solution $3 + 6 = 9$.

Case 2: $\chi(8) = red$. If $\chi(7) = red$, then Lemma 5.2.1 gives us the *red* solution $1 + 7 = 8$. Therefore, $\chi(7) = blue$, which leads to a contradiction after a chain of implications:

$\chi(2) = red$ (to avoid the *blue* solution $2 + 7 = 9$),

$\chi(6) = blue$ (to avoid the *red* solution $2 + 6 = 8$),

$\chi(1) = red$ (to avoid the *blue* solution $1 + 6 = 7$),

$\chi(3) = blue$ (to avoid the *red* solution $1 + 2 = 3$),

and hence the *blue* solution $3 + 6 = 9$. □

5.3.2 $S(3,4) = 16$

Lemma 5.3.2. $S(3,4) = 16$.

Proof. For all $x \in [1, 15]$, let $x \in [6, 12]$ be *blue* and x be *red* otherwise. This colouring has no *blue* solution to L_3 and no *red* solution to L_4. Therefore, $S(3,4) > 15$.

Suppose to the contrary that $S(3,4) > 16$. Then suppose $\chi(1) = blue$. Corollary 5.2.3 implies $r_i \leqslant 2i$, for all $i \geqslant 3$.

Since $r_1 + r_2 + r_4 \leqslant 3 + 5 + 8 = 16$, we have $\chi(r_1 + r_2 + r_3) = \chi(r_1 + r_2 + r_4) = blue$. If $r_4 > r_3 + 2$, we get the *blue* solution $1 + (r_3 + 1) = r_3 + 2$, and if $r_4 = r_3 + 1$, we get the *blue* solution $1 + (r_1 + r_2 + r_3) = r_1 + r_2 + r_4$. Hence, $r_4 = r_3 + 2$. To avoid the *blue* solution $2 + (r_1 + r_2 + r_3) = r_1 + r_2 + r_4$, we must have $\chi(2) = red$, that is, $r_1 = 2$, which implies $r_1 + r_3 + r_4 \leqslant 2 + 6 + 8 = 16$. Thus $\chi(r_1 + r_3 + r_4) = blue$.

Applying the same reasoning to r_3 as we did to r_4, we get that $r_3 = r_2 + 2$. Then to avoid the *blue* solution $4 + (r_1 + r_2 + r_3) = r_1 + r_3 + r_4$, we must have $\chi(4) = red$. If $\chi(3) = red$, then $r_2 = 3$, and so $r_3 = 4$. But $r_3 \neq r_2 + 1$. Therefore, $\chi(3) = blue$, which implies $r_2 = 4$, and so $r_3 = 6$ and $r_4 = 8$. Then we must have $\chi(5) = \chi(7) = \chi(12) = blue$, but then we get the *blue* solution $5 + 7 = 12$. Therefore, $\chi(1) = red$.

Corollary 5.2.2 gives us that $r_2 = 2, 3, 4$, or 5. We handle these four cases separately.

Case 1: $r_2 = 5$. This implies $\chi(2) = \chi(3) = \chi(4) = blue$. Therefore, $\chi(6) = red$ and $\chi(7) = red$ to avoid the *blue* solutions $2+4 = 6$ and $3+4 = 7$, respectively. Hence $\chi(12) = blue$ and $\chi(14) = blue$ to avoid the *red* solutions $1 + 5 + 6 = 12$ and $1 + 6 + 7 = 14$, respectively. But, then we get the *blue* solution $2 + 12 = 14$.

Case 2: $r_2 = 4$. This implies $\chi(2) = \chi(3) = blue$. Therefore, $\chi(5) = red$ (to avoid the *blue* solution $2+3 = 5$), which implies $\chi(10) = blue$ (to avoid the *red* solution $1+4+5 = 10$). Therefore, $\chi(7) = red$ and $\chi(12) = red$ to avoid the *blue* solutions $3 + 7 = 10$ and $2 + 10 = 12$, respectively. But then we get the *red* solution $1 + 4 + 7 = 12$.

Case 3: $r_2 = 3$. This implies $\chi(2) = blue$. If $r_4 = 9$, then Lemma 5.2.1 implies $\chi(2) = red$. Thus $r_4 \leqslant 8$.

If $r_3 \geqslant 6$, then $\chi(4) = \chi(5) = blue$, which implies $r_3 = 6$ (to avoid the *blue* solution $2 + 4 = 6$). Therefore, $\chi(7) = red$ (to avoid the *blue* solution $2 + 5 = 7$), which implies $\chi(14) = blue$ (to avoid the *red* solution $1 + 6 + 7 = 14$) and $\chi(16) = blue$ (to avoid the *red* solution $3 + 6 + 7 = 16$). But then we get the *blue* solution $2 + 14 = 16$. Therefore, $r_3 \leqslant 5$.

This implies $3 + r_3 + r_4 \leqslant 16$, which gives us that $\chi(1 + r_3 + r_4) = \chi(3 + r_3 + r_4) = blue$, but then we get the *blue* solution $2 + (1 + r_3 + r_4) = 3 + r_3 + r_4$.

Case 4: $r_2 = 2$. Suppose $\chi(7) = red$. Then $\chi(4) = blue$ (to avoid the *red* solution $1 + 2 + 4 = 7$) and $\chi(10) = blue$ (to avoid the *red* solution $1+2+7 = 10$). This implies $\chi(6) = red$ and $\chi(14) = red$ to avoid the *blue* solutions $4+6 = 10$ and $4+10 = 14$, respectively. But then we get the *red* solution $1 + 6 + 7 = 14$. Therefore, $\chi(7) = blue$.

Suppose $\chi(3) = blue$. Then $\chi(4) = red$ and $\chi(10) = red$ to avoid the *blue* solutions $3 + 4 = 7$ and $3 + 7 = 10$, respectively. This implies $\chi(13) = blue$ and $\chi(16) = blue$ to avoid the *red* solutions $1+2+10 = 13$ and $2+4+10 = 16$, respectively. Therefore, $\chi(3) = red$, which leads to a contradiction after a chain of implications:

$\chi(6) = blue$ (to avoid the *red* solution $1 + 2 + 3 = 6$),

$\chi(13) = red$ (to avoid the *blue* solution $6 + 7 = 13$),

$\chi(9) = blue$ (to avoid the *red* solution $1 + 3 + 9 = 13$),

$\chi(16) = red$ (to avoid the *blue* solution $7 + 9 = 16$),

and hence the *red* solution $1 + 2 + 13 = 16$. □

5.3.3 $S(3, 5) = 23$ (Computer assisted proof)

Let us write a colouring of $[1, n]$ as a bit-string of length n where the i-th bit is zero if $\chi(i) = blue$, and one if $\chi(i) = red$.

By Lemma 5.1.1, the lower bound is $S(3, 5) > 22$. We consider all of the ten colourings of $[1, 22]$ (obtained by computer search) without a *blue* solution to L_3 and a *red* solution to L_5.

1. For each of the following four colourings

   0010110111111111111110,

   0010110111111011111110,

   0010110111111011011110, and

   0010110111101111011110,

   if $\chi(23) = blue$, then we have a *blue* solution $1 + 22 = 23$ to L_3; and

   if $\chi(23) = red$, then we have a *red* solution $3 + 5 + 6 + 9 = 23$ to L_5.

2. For each of the following four colourings

   0010111011111111111101,

   0010111011111110111101,

   0010111011111101111101, and

   0010111011111011111101,

   if $\chi(23) = blue$, then we have a *blue* solution $2 + 21 = 23$ to L_3; and

   if $\chi(23) = red$, then we have a *red* solution $3 + 5 + 6 + 9 = 23$ to L_5.

3. For each of the following two colourings

   0101010101111111101010, and

   0101010101111111111010,

   if $\chi(23) = blue$, then we have a *blue* solution $1 + 22 = 23$ to L_3; and

   if $\chi(23) = red$, then we have a *red* solution $2 + 4 + 6 + 11 = 23$ to L_5.

Therefore, $S(3, 5) = 23$.

Chapter 6

Conclusion

6.1 Summary of contributions

- Determination of five van der Waerden numbers of the form $w(2; t_0, t_1)$,
- Computed conjectures of several van der Waerden numbers of the form $w(2; t_0, t_1)$,
- Upper bound conjectures for $w(2; s, t)$ when s is fixed,
- An efficient encoding for $w(k; t_0, t_1, \ldots, t_{k-1})$ and determination of three numbers using that encoding,
- A problem-specific efficient backtracking algorithm and determination of twenty-five values of $w(k; t_0, t_1, \ldots, t_{k-1})$ using that algorithm,
- Some counting properties of arithmetic progressions,
- Some unimodality properties of sequences regarding arithmetic progressions,
- Generalization of Szekeres' conjecture on the size of the largest p-AP free sub-sequence of $1, 2, \ldots, n$, with a construction of the lower bound, and
- Characterization of Strict Schur numbers.

6.2 Ongoing work

6.2.1 Snevily numbers

Define

$$\begin{aligned} \nu(X,d) &= |\{(x,y) : x,y \in X, y > x, y-x = d\}|, \\ (a_1, a_2, \ldots, a_{t-1}; d) &= \text{a collection } X \text{ s.t. } \nu(X, d \cdot i) \geqslant a_i \\ &\quad\ \text{for } 1 \leqslant i \leqslant t-1. \end{aligned}$$

The t-AP $\{x, x+d, \ldots, x+(t-1)d\}$ (say T) has $\nu(T, d \cdot i) = t - i$ for $1 \leqslant i \leqslant t-1$. On the other hand, a set $(t-1, t-2, \ldots, 1; d)$ (say Y) has $\nu(Y, d \cdot i) \geqslant t - i$ for $1 \leqslant i \leqslant t-1$, but not necessarily contains a t-AP.

We define a weaker version of van der Waerden numbers based on the concept of distance pairs, and call them *Snevily numbers*. A Snevily number $ww(k,t)$ is the smallest integer n such that every k-colouring of $1, 2, \ldots, n$ contains a monochromatic set $(t-1, t-2, \ldots, 1; d)$ for some $d > 0$. Here, $(t-1, t-2, \ldots, 1; d)$ is a *t-AP distance-set* of size at least t. The existence of Snevily numbers is guaranteed by van der Waerden's theorem. The following Lemma is trivially true:

Lemma 6.2.1. $ww(k,t) \leqslant w(k,t)$.

Note that a certificate of lower bound is not required to avoid a monochromatic arithmetic progression. For example, while looking for a certificate of lower bound of $ww(2,4)$, if the set $X = \{1, 2, 3, 5, 9, 10\}$ (which does not contain a 4-AP) is monochromatic, then the colouring is "bad" as $\nu(X,1) = 3$, $\nu(X,2) = 2$, and $\nu(X,3) = 1$.

We are working on characterizing Snevily numbers.

6.2.2 A constructive proof of $w(2,t) > 2^t$

Recall that $w(2,t)$ is the smallest positive integer n such that every 2-colouring of $1, 2, \ldots, n$ contains a monochromatic t-AP. The known computed values and bounds are $w(2,3) = 9$, $w(2,4) = 35$, $w(2,5) = 178$, $w(2,6) = 1132$, $w(2,7) > 3703$, $w(2,8) > 11495$, and $w(2,9) > 41265$. These values comply with Erdős conjecture that $w(2, t) > 2^t$, but as of yet, no constructive proof for all values of t is known. We are working towards such an algorithm.

6.2.3 Engineering fast computation of van der Waerden numbers

The search space corresponding to the theoretical upper bounds of van der Waerden numbers is still beyond the reach of our computers. Since the exact values appear to be far, far less than those upper bounds, there is scope for algorithmic and data-structural developments (to compute unknown numbers), which would make an indirect contribution to computer science and engineering.

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