Cooperative Control of a Network of Multi-Vehicle Unmanned Systems

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ABSTRACT

Cooperative Control of a Network of Multi-Vehicle Unmanned Systems

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Development of unmanned systems network is currently among one of the most important areas of activity and research with implications in variety of disciplines, such as communications, controls, and multi-vehicle systems. The main motivation for this interest can be traced back to practical applications wherein direct human involvement may not be possible due to environmental hazards or the extraordinary complexity of the tasks. This thesis seeks to develop, design, and analyze techniques and solutions that would ensure and guarantee the fundamental stringent requirements that are envisaged for these dynamical networks.

In this thesis, the problem of team cooperation is solved by using synthesis-based approaches. The consensus problem is defined and solved for a team of agents having a general linear dynamical model. Stability of the team is guaranteed by using modified consensus algorithms that are achieved by minimizing a set of individual cost functions. An alternative approach for obtaining an optimal consensus algorithm is obtained by invoking a state decomposition methodology and by transforming the consensus seeking problem into a stabilization problem.

In another methodology, the game theory approach is used to formulate the consensus seeking problem in a "more" cooperative framework. For this purpose, a team cost function is defined and a min-max problem is solved to obtain a cooperative optimal solution. It is shown that the results obtained

yield lower cost values when compared to those obtained by using the optimal control technique. In game theory and optimal control approaches that are developed based on state decomposition, linear matrix inequalities are used to impose simultaneously the decentralized nature of the problem as well as the consensus constraints on the designed controllers. Moreover, performance and stability properties of the designed cooperative team is analyzed in presence of actuator anomalies corresponding to three types of faults. Steady state behavior of the team members are analyzed under faulty scenarios. Adaptability of the team members to the above unanticipated circumstances is demonstrated and verified. Finally, the assumption of having a fixed and undirected network topology is relaxed to address and solve a more realistic and practical situation. It is shown that the stability and consensus achievement of the network with a switching structure and leader assignment can still be achieved. Moreover, by introducing additional criteria, the desirable performance specifications of the team can still be ensured and guaranteed.

To my parents,

for their love and support all the way since the very first days of my life and

to my husband, Amin,

for his inspiring attitude towards learning and

to the memory of my grandfather who always wished to see my achievements.

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List of Symbols and Abbreviations

ARE Algebraic Riccati Equation

BDF Backward Differentiation Formula

DRE Differential Riccati Equation

HJB Hamilton-Jacobi-Bellman

ISR Intelligence, Surveillance, and Reconnaissance

LIP Lock-In-Place

LL Leaderless

LMI Linear Matrix Inequality

LOE Loss of Effectiveness

LQR Linear Quadratic Regulator

MIMO Multi-Input Multi-Output

MLF Modified Leader-Follower

NBS Nash Bargaining Solution

ND Negative Definite

NSD Negative Semi-Definite

PDE Partial Differential Equation

PD Positive Definite

PSD Positive Semi-Definite

SN Sensor Networks

UAVs Unmanned Aerial Vehicles

UGVs Unmanned Ground Vehicles

UMSN Unmanned Systems Networks

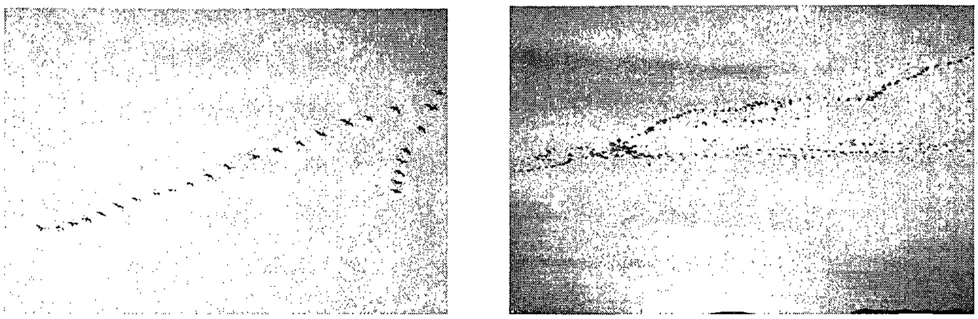
UUVs Unmanned Underwater Vehicles

Chapter 1

Introduction

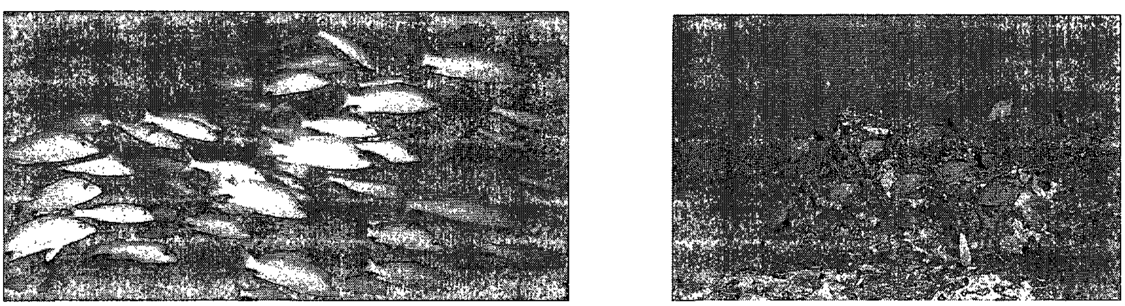
1.1 Motivation

Recently, there has been a growing interest towards development of Sensor Networks (SN), and in general a network of unmanned autonomous systems that can operate without an extensive involvement of humans. Consideration of problems in these networks is currently one of the strategic areas of research. The motivations for this focus may be traced to applications where direct human intervention is not possible due to either environmental hazards, extraordinary complexity of the tasks, or other restrictions. On the other hand, observations made based on natural behavior of animals operating as a team have inspired scientists in different disciplines to investigate the possibilities of networking a group of systems to accomplish a given set of tasks without requiring an explicit supervisor. Some examples of such natural behaviors can be found in the migration of birds, motion of fish searching for food, and team work of other animals which have a group living style, see Figures 1.1-1.2. In all these examples, the animals work together in a team with an intergroup co-operation and with no supervision from outside the team in order to perform complex tasks. Furthermore, advances in wireless communication networks have made it feasible to connect a number of systems distributed over a large geographic area.



(a) (b)

Figure 1.1: Flocks of birds (figures borrowed from Google images).



(a) (b)

Figure 1.2: Swarms of fish (figures borrowed from Google images).

These advances and observations have led the scientists to the area of designing Unmanned Systems Networks (UMSN). The advantages of UMSN are enormous and applications in various fields of research are being developed. Some of the advantages for deploying autonomous network of unmanned systems are enhanced group robustness to individual faults, increased and improved instrument sensing and resolution, reduced cost of operation, and adaptive reconfigurability capabilities which have been discussed in [4]. A team of agents can cooperate to accomplish a complicated task which is impossible to be done by a single unit.

These networks may be potentially made up of a large number of dynamical systems (agents), such as Unmanned Aerial Vehicles (UAVs), Unmanned Ground Vehicles (UGVs), or Unmanned Underwater Vehicles (UUVs). Some examples are networks of satellites, submarines, or mobile robots. Any of these systems usually has a number of sensors, actuators and decision makers and so the network of all these systems is a network of thousands of sensors and actuators or as is known in the literature, a sensor network.

Based on the above discussion and due to the multi-disciplinary nature of the problem, design of a network of sensors, actuators, and decision makers is currently one of the important trends of research in various disciplines, such as communications, control theory and mechanics. In order to fully take advantage of these large-scale networks, several prerequisites have to be satisfied. Some of these prerequisites are development of reliable communication, optimal power consumption management, security, optimal cooperation, and team collaboration that are discussed in [5]. Team cooperation and coordination for accomplishing predefined goals and requirements are one of the main prerequisites for these networked agents that are intended to be deployed in challenging missions. Although, a large body of work has been devoted to design requirements for UMSN, unsolved problems still exist. Some of these challenges are: (a) lack of complete information for all the agents in the network, (b) cohesion requirement and connectivity of the team in presence of uncertainties and partial information, (c) tackling inaccurate information due to the scale of the network, (d) presence of adversarial and environmental uncertainties, (e) robustness issues, and (f) fault diagnosis and recovery, to name a few. Other issues are to address dynamic nature of the team, and to use more complicated agents dynamical models rather than point-mass models. Hence, it is imperative that one designs reliable and high performance networks that can ensure and accommodate these requirements.

1.2 Applications

Wireless UMSN provide significant capabilities, and hence has received extensive attention in the past several years and numerous applications in various fields of research are being considered and developed. Some applications that necessitate development of these systems are in space explorations; satellite deployment for distributed deep space observations; automated factories; maneuvers of a group of UAVs; utilization of network of UGVs, e.g. mobile robotics, and UUVs for search and rescue; and teams of robots deployed in a hazardous environment where human involvement is dangerous. In [5], more applications are mentioned such as home and building automation, intelligent transportation systems, health monitoring and assisting, and commercial applications. There are also military applications in Intelligence, Surveillance, and Reconnaissance (ISR) missions in the presence of environmental disturbances, vehicle failures, and in battlefields subject to unanticipated uncertainties and adversarial actions [6], see Figures 1.3-1.5.

1.3 Literature review

Cooperation in a network of unmanned systems, known as formation, network agreement, collective behavior, flocking, consensus, or swarming in different contexts, has received extensive attention in the past several years. Several approaches to this problem have been investigated within different frameworks and by considering different architectures [7–17]. Moreover, the problem of

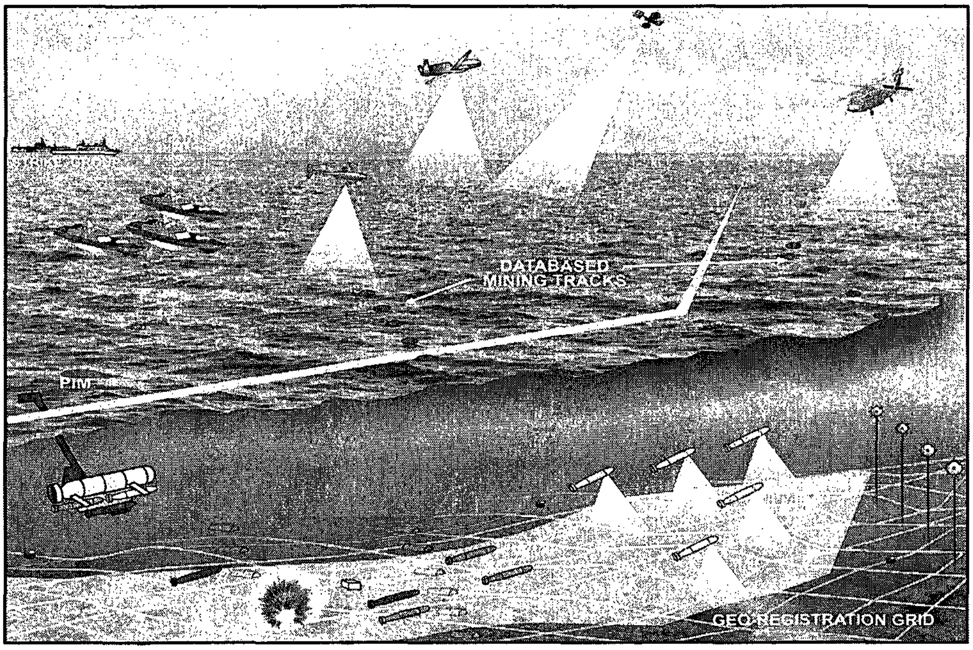


Figure 1.3: A network of unmanned vehicles (figure borrowed from [1])

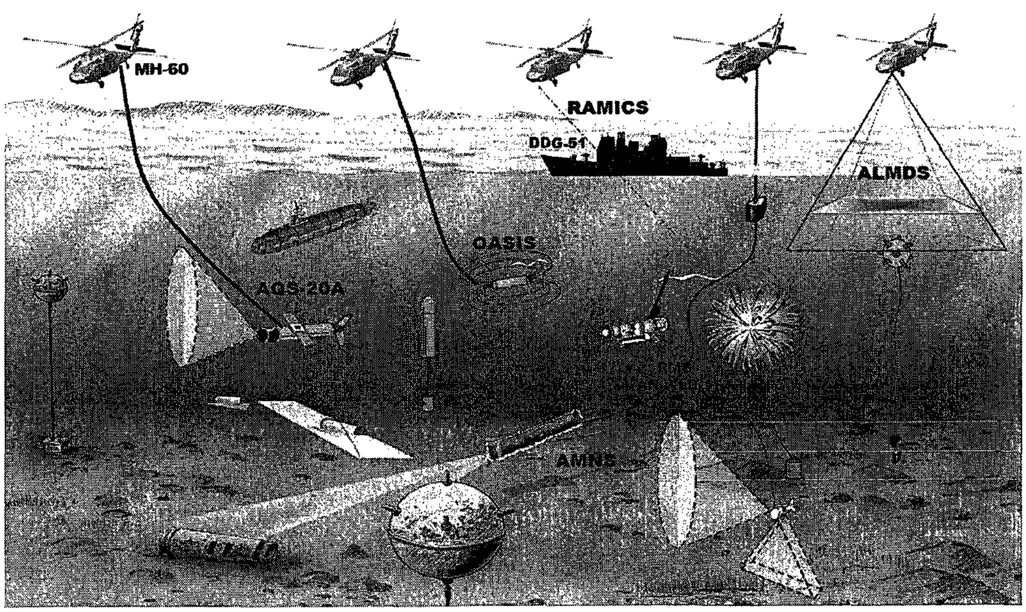
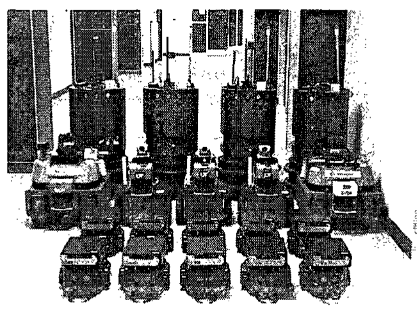
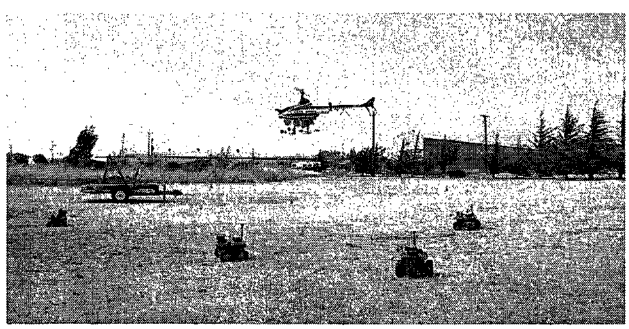


Figure 1.4: A network of unmanned vehicles (figure borrowed from Google images archive)



(a)



(b)

Figure 1.5: a) A team of mobile robots and b) a team of UGVs and UAVs (figures borrowed from DI Lab Robots archive and [2])

cooperation in a network has been considered at different levels. At the high-level, one can refer to task assignment, timing and scheduling, navigation and path planning, reconnaissance and map building [18–20] to name a few (see Figure 1.6 at the end of this chapter). In the mid-level, cooperative rendezvous, formation keeping, application of consensus algorithms, collective motion, and formal methods based on flocking/swarming ideas can be mentioned [4,12,13, 17,21,22].

Since the present research is aimed at cooperation in the mid-level, most of the literature reviewed in this section are on the low-level cooperation, i.e. formation keeping, application of consensus algorithms, and formal methods based on flocking/swarming ideas. The formation control problems can be distinguished from the consensus seeking and flocking/swarming-based approaches based on the degree of autonomy as well as the degree of distribution of the proposed algorithm. Generally, these two problems are characterized as follows:

- **Formation control:** Based on the definition presented in the survey paper done by Scharf, *et al.*, [23], a formation control law couples the dynamics of each member of a group of vehicles through a common control law. Basically, the formation control should have two properties: 1) at least one member of the group must track a predefined state relative to another member and 2) the corresponding control law should be dependant on the state of this member [23]. Some of the characteristics of this type of problem are as follows:

  1. The formation control is usually solved based on a centralized approach to the problem [24], although in some cases decentralized solutions are suggested, e.g. [25]. However, the assumption of modular architecture is common in formation problem [4,26,27].

2. The number of vehicles is limited (5-6 vehicles) [28].

3. Usually conventional control methods are used, e.g. adaptive and nonlinear control [24].

4. Information structure is not highlighted and not embedded in the design.

5. Autonomy is up to the level where a tracking path is given, a leader guides the group, or a coordination vector is provided [24].

6. For different structures, e.g. virtual structure [24], and Multi input-Multi output [29] the problem of formation has been addressed for complicated dynamics such as aircraft, UAVs, and spacecraft.

7. Various work has covered different aspects of control in this area such as formation keeping, tracking, formation stability (e.g. input-output stability), and fuel optimality [4, 30], as well as high-level tasks, e.g. initialization and reconfiguration [31, 32].

8. The present challenges are on experimental and practical issues such as design of high-resolution space instruments, fuel optimality and decrease of computational time. Also, the estimation problem in formation keeping [33–35] and design of decentralized and distributed algorithms are some current trends of research.

- **Flocking/consensus/agreement:** The problem is to have network agreement on a scalar state or a vector of states or on a function of states, while other behaviors (e.g. formation) are guaranteed. Some of the characteristics of these types of problems are as follows:

  1. They are based on a distributed approach and a large number of agents can be addressed [36].

2. Information is considered as one part of modelling where there are influences of communication topology on the stability and other dynamical properties [37].

3. The mathematical tool for problem formulation is graph theory for a majority of the work except for few references, e.g. [9, 38–43].

4. Very simple agents' dynamical model (first or second order integrator models) is assumed except in few works, e.g. [37, 38, 40, 41, 44].

5. Usually there is an analysis of the response and not a synthesis, except in few work, e.g. [38–41].

6. There are not much experimental considerations in the literature.

7. Different aspects of the problem that have already been considered are as follows:

–Basic properties such as convergence [10], finding equilibrium state [45], and in [46] controllability definition are provided.

–Behaviors such as formation keeping, collision avoidance, obstacle avoidance, as well as generating feasible planar trajectories to get the maximum coverage of the region, and finding minimal information structure needed for stability of a swarm are addressed in [10, 42, 47, 48].

–In [49] the main purpose is to find the consensus algorithm weighting such that the fastest convergence speed is achieved.

8. The present challenges and open problems are:

–To define the basic properties, e.g. group stability, controllability, and observability conditions.

–To extend the existing methods to the conditions where more complicated agents' dynamical models are considered.

–To add conditions such as uncertainty in the mission plan (leader command) for the followers, model uncertainties, disturbances, communication/sensor noise, appearance of fault in the leader and the followers.

–To put constraints on the input signal or other practical constraints.

–To assume the time-varying neighboring set, switching structure, and dynamic network topology.

–To formulate the estimation problem in the network framework.

–To assume stochastic frameworks with probabilistic information links.

–To consider delayed information exchange between the leader and the followers or among the followers.

In the following subsections, I will present a detailed literature review on different issues that arise in the cooperative control. However, since the topic of this research is specifically on consensus seeking algorithms, the majority of the reviewed and referred works are related to this issue.

1.3.1 Formation control

Based on its definition, in formation control the main focus is that the team achieves a predefined and given geometry and shape. This shape (formation) should be preserved during the mission and so the team of agents should act as a rigid body. Based on this property, a predefined trajectory is usually provided for the team motion e.g. a leader command, or a trajectory for the virtual structure mass center. The team should track this trajectory while its members keep their relative positions and preserve the required shape, i.e. the stability of the formation should be maintained. Other requirements can be added to these objectives, as well. A main prerequisite to keep formation is guidance and control of vehicles, while they perform tasks such as initialization, contraction, and expansion.

As discussed in [23] different architectures can be considered for the formation of a team of agents, namely: i) considering the formation as a single Multi-Input Multi-Output (MIMO) system [29], ii) leader-follower [50], iii) virtual structure [4] (or virtual leader [51]), in which the entire formation is considered as a virtual structure, iv) cyclic with non-hierarchical control architecture [52], and v) behavioral [53].

In the MIMO architecture, the entire dynamics of the system is considered as one MIMO model. Hence, in this architecture any of the conventional control strategies, e.g. optimal, nonlinear, or robust control strategies can be applied to the system.

The leader-follower architecture has been used very often in the literature [6, 45, 50, 54–57]. In this approach a hierarchical control architecture is considered with one or more of the agents as the leader(s), and other agents as the followers. The followers should track the position and orientation of the leader(s). This structure can also be constructed in a tree form, in which an agent is the leader of some other agents who are the leaders of some other ones and so on. The advantages of this approach are that it has an easy and understandable behavior, the formation is preserved even if the leader is perturbed, and that group behavior can be inspected by defining the behavior of the leader. However, lack of the explicit feedback to the formation, from the followers to the leader, is a disadvantage of this structure. Also, the failure of the leader implies the failure of the formation as mentioned in [4, 7].

In the virtual structure approach, the entire formation is treated as a unit. In this approach, three steps are considered for control design: i) to define the desired dynamics of the virtual structure, ii) to transform the states of the virtual structure into the states of individual agents, and iii) to design control laws for each agent, correspondingly. In [4, 7], the advantages of this method are considered as its simplicity in defining the coordinated behavior of the group, keeping the formation during different maneuvers, and existence of feedback from the agents to the virtual structure. The weakness of this structure is in its limitation in applications to time-varying or frequently reconfigurable formations. In [4, 7, 27] the idea of adding a feedback from the vehicles to the coordination unit is presented. Authors in [7] proposed virtual structure as a solution to the problem of multiple-spacecraft formation. In their approach, a feedback from a vehicle to the virtual structure is considered for keeping the vehicles in formation and improve the robustness of formation in case of disturbances or when the virtual structure moves very fast and some of the slow members may lose the group.

In the cyclic architecture, the agents are connected to each other in a cyclic form rather than a hierarchical architecture [52]. The disadvantage of this method is that the stability analysis of the proposed controller is not straight forward due to the dependency of individual controller on others' in a cyclic form.

In the behavioral approach several commands are combined to reach different and probably competing goals or several behaviors, e.g. collision avoidance, obstacle avoidance, and formation keeping for agents. The control law for each agent is a weighted average of the control for each behavior. Since competing behaviors are averaged, occasionally strange and unpredicted behaviors may occur. Despite the advantages of simple derivation of control strategies, explicit feedback to the formation in this approach, and capability of decentralized implementation, there are some weaknesses as well. For some examples, group behavior cannot be explicitly defined and mathematical analysis, e.g. stability, is difficult to be accomplished as is mentioned in [4].

Although the above structures were originally introduced for formation keeping, some of them can be used for flocking and consensus-based algorithms as well. Due to the supervised nature of structures such as leader-follower and virtual structure, they are not commonly used for flocking and consensus seeking, which are autonomous in their very nature (there is no predefined path, trajectory for the leader or virtual coordinate to be tracked). However, these structures can also be redefined for utilization in flocking/swarming or consensus seeking problems, see e.g. [45, 55].

One of the main challenges that arises in the development of a formation keeping strategy in a team of unmanned systems is the lack of complete information and the presence of uncertainties, faults and unpredictable events in the team. This necessitates the design and application of adaptive methods for formation control in some applications [6, 16, 24, 50, 58]. In [6], leader commands are unknown for the follower vehicle in a leader-follower architecture and so an adaptive controller is used for formation keeping. Missing leader commands may occur during a mission when the leader spots an approaching threat, and quickly reacts to avoid it. In this case there is no adequate time for the leader to send its new commands to the follower vehicles and hence the leader commands need to be assumed to be unknown in the control design. Also, in [50] the authors considered uncertainties in the vehicle dynamics, in which vortex forces are considered as unknown functions. In a formation mode, each vehicle experiences an upwash field generated by the other vehicles and so the aircraft motion is affected by the vortex of the adjacent vehicles. These effects are usually unknown and depend on the area, gross mass, span and dynamic pressure as well as velocity and position of the vehicle.

The ideas in [6] and [50] are integrated in [16] by using the framework that is introduced in [6] for design of an adaptive controller in order to compensate for time-varying unknown leader commands and vortex forces. The idea in [6] was extended to the case where the vortex forces are presented in the dynamical model of the system and both the vortex forces of the follower and leader commands are treated as time-varying unknown parameters. The control objective is to design the follower control input such that the relative distances between the followers and the leader are maintained close to their desired values in the presence of these uncertainties. For the case that vortex forces are considered in the velocity dynamics, adaptive updating laws are introduced for two cases of time-varying and constant forces. In this case the stability and tracking of relative distances are guaranteed. If in addition, the forces are applied in the heading angle dynamics, stability and tracking of relative distances is guaranteed for the case of constant vortex forces. The proposed algorithm is applied to formation control of UAVs.

Some of the work performed in coordination in a group of vehicles assume graph theory as the mathematical framework for modelling a distributed network of agents [8, 37, 47, 59]. In some approaches the analysis is also done in this framework, using the graph properties. The main difference between the previously reviewed work on formation and the above references is that in the latter, the final goal is to achieve a stabilized formation in an autonomous manner. No path or trajectory is provided for the group motion and the formation is stabilized based on the inter-group exchange of information and the desired shape provided *a priori*. In some other work, the graph theory tool is used only for modelling and other tools are used for mathematical analysis.

In [47], a local distributed bounded control input is designed for formation stabilization in a group of multiple autonomous vehicles. Point-mass dynamical models are assumed for agents and the information flow graph can be either directed or undirected. The concept is to use some potential functions which are constructed using desired properties, such as collision-free and stable formation. Similarly, results for a global stabilization and tracking are presented in [8] for a group of agents with linear dynamics. The problem is first solved for two agents and then extended to the general case using "dynamic node augmentation". The collision avoidance is also guaranteed and shown. In this approach the problems of stabilization and tracking are decoupled into three sub-problems, namely control of shape dynamics, rotational dynamics, and translational dynamics. In [59], the same problem is considered where different maneuvers such as split, rejoin and reconfiguration are assumed for the group. In [37], graph theory is used to model the communication network and to find the relation between the topology and the stability of the formation. Based on the graph Laplacian matrix properties, a Nyquist criterion is obtained to show the formation stabilization for groups of agents with linear dynamics. The formation stability is divided into two parts: stabilization of the information flow and stabilization of the individual vehicles. The leader-follower architecture can be addressed in this framework.

In [9], the formation and alignment goals are translated into an error framework. A decentralized robust controller is then designed for the error dynamics based on an overlapping design methodology. The assumed structure is leader-follower and a constant velocity command and formation structure is provided for the entire team. As for the formation of satellites, in [24] an adaptive approach is proposed to achieve formation of three satellites which in turn are used as a free-flying interferometer. The goal is that the formation follows a desired attitude trajectory. In [60], formation flight control of UAVs is addressed.

There are other important issues that are discussed in the formation of multi-agent systems. In the work of Smith, *et. al* [33], a parallel estimation structure based on the error covariance is suggested for control of a formation, however the solution is not necessarily an optimal one. In their approach, each vehicle with a linear dynamical model estimates the state vector of the entire group based on the noisy output it receives. Then in design of a controller, each vehicle uses its own estimation. The same authors have investigated results on adding communication in order to remove disagreement dynamics in [34, 61]. For this purpose, they have assumed that some of the nodes (at least $N-1$ receivers) send their estimation result to the rest of the group. In [35, 62], a similar idea is used for applications in spacecraft formation. In [63], convex optimization is used to develop a framework for distributed estimation or equivalently for data fusion. For solving the corresponding optimization problem, sub-gradient method and dual decomposition are utilized.

1.3.2 Flocking/swarming-based approaches

As presented in [64], the definition of flocking in a group of agents is:

"A group of mobile agents is said to (asymptotically) flock, when all agents attain the same velocity vectors and furthermore distances between the agents are asymptotically stabilized to constant values."

In [10], a dynamic graph theoretic framework is presented for formalizing the problem of flocking in the presence of some obstacles which are assumed to be in convex and compact sets and their boundaries are closed differentiable Jordan curves. An energy function is constructed for the team of agents in which different tasks of flocking are considered. Dissipation of this energy function through the protocols that are inspired by the Reynolds rules [65], namely I) alignment, II) flock centering, and III) collision avoidance, results in achieving all the predefined goals of the network. The main contribution of [10] is to derive and analyze an advanced form of Reynolds rules, specifically the last two rules. The authors have considered the point-mass dynamics for the agents and a flocking protocol is defined for the interactions among the agents which results in reduction of the constructed potential function. This protocol results in alignment, i.e. convergence of each agent to the weighted average position of its neighbors, and obstacle avoidance in the network of agents. Two tasks of split/rejoin and squeezing were presented in the provided simulation results. Similarly, in [36], a particle-based approach to flocking is considered for two scenarios, i.e. in the presence of multiple obstacles and in a free space. The suggested algorithms address the three rules of Reynolds as mentioned above.

In [22, 66, 67] a stable flocking motion law is introduced for a group of mobile agents with a connected graph. This law guarantees a collision free and cohesive motion and an alignment in the headings of the agents. The authors assumed a 2-component control law in which the first part is produced by using a potential function that regulates the relative distances among the agents as well as avoiding the collision and the second part that regulates the velocities. The potential function can define the final shape of the formation. In the case of a switching topology [22, 67], control laws may be switching and so Fillippov and non-smooth system frameworks are used for stability analysis. In this case, neighboring is based on the distance between the agents and so it is dynamic. In [68, 69], the flocking problem is solved by decomposing the entire team dynamics into the dynamics of the group formation and dynamics of the motion of center-of-mass. Each of these dynamics are analyzed and stabilized separately such that both formation keeping and velocity regulation to a constant value are guaranteed.

1.3.3 Consensus algorithms

Investigation of effects of information structure on a control decision was initiated by the early work on the theory of teams and was first introduced in [70] and later followed by [71–75]. These works can be considered as the first results in which the control in a team is discussed where only part of the information is available to the team members. However, since these early work were performed and specially in recent years, a large body of research has been conducted in which the effects of information on the control design problems are discussed, see for example [10, 11, 13, 15, 37, 45, 76–78]. In most of these works, each team member has access to limited information from other agents, or to information of its neighbors. The final state of the team is decided by the team members.

Consensus algorithms are one of the tools that are used for analysis of distributed systems where the network information structure has a vital effect on the control design but only part of the information is available to each member. As mentioned in [13], consensus problems deal with the agreement of a group of agents upon specific "quantities of interest". In this configuration the agents try to decide and agree among themselves upon what the final state should be. The state where all the "quantities of interest" are the same is called the consensus state. In the present research, we are specifically interested in problems in which the "quantities of interest" are related to the motion of the agents, e.g. their velocities or positions. However, as indicated in [79], several applications for consensus algorithms exist such as applications in decentralized computation, clock synchronization, sensor networks, and in coordinated control of multi-agent networks.

In [13], linear and nonlinear consensus protocols are applied to directed and undirected networks with fixed and switching topologies. A disagreement function was introduced as a Lyapunov function to provide a tool for convergence analysis of an agreement protocol in a switching network topology. The authors have shown that the maximum time-delay that can be tolerated by a network of integrators applying a linear consensus protocol is inversely proportional to the largest eigenvalue of the Laplacian of the information flow graph or the maximum degree of the nodes of the network. Similar results are obtained in [11] where convergence analysis are developed by using Nyquist plots for linear protocols. For nonlinear protocols, the notion of action graphs and disagreement cost are introduced and the problem was solved in a distributed manner.

In [45], the coordination problem is discussed for a team of agents using "nearest neighbor rule" for both leaderless and leader-follower configurations. The main focus of this work is on heading angle alignment in undirected graphs where the agents have simple integrator dynamics and the agents have the same speed but have different headings. In the leader-follower case, the leader can affect the followers whenever it is in their neighboring set. However, there is no feedback from the followers to the leader. It is shown that the connectivity of the graph on average (connection of union of graphs) is sufficient for convergence of the heading angles of the agents. The neighboring set assignment is switching and so the team structure is dynamic. In [80], asynchronous protocols for consensus seeking are introduced. Some updating rules for the control input of agents with discrete-time dynamical equations are suggested so that the consensus state would take a desirable predefined value.

In [38], passivity is used as a tool to achieve network agreement (or consensus) for a class of agents with dynamics which can satisfy the passivity conditions. The group main goal is to reach at a predefined common velocity (or any other interpretation of the derivative of a state), while the relative positions (the difference between a common state in the group) converge to a desired compact set. Based on this method a Lyapunov function can be constructed for stability analysis in a distributed communication network with bidirectional links. The designed controller is a filter which has a nonlinear function of the relative states as its input and is designed based on passivity properties. The relation between the topology and the stability of the formation is provided. In [81], a wider class of systems, i.e. nonlinear dissipative systems are considered and synchronization in a strongly connected network of agents with this dynamical property is discussed.

In [46, 82, 83], an interpretation of controllability is defined and proved for a first order integrator model. The main purpose in these work is to find the effects of external decisions on the agreement dynamics, in particular the conditions where some of the nodes do not follow the agreement protocols (decisions). In other words, there are some nodes that follow the agreement protocol while others have external inputs. The authors try to answer the question whether these "anchored" nodes are able to guide the rest of the group to the desired point. Similar to the ideas presented in [46] for a fixed network topology, the controllability conditions for a network with fixed and switching topologies are discussed in [84]. The authors have considered a leader-follower structure with one-way links from the leader to the followers. They have shown that controllability of the team is highly dependant on how the followers are connected to the leader.

The goal in [42] is to find the minimum communication that is required for guaranteeing the stability of a swarm of vehicles. The approach is to first define a centralized cost function for the group and then divide it into some individual costs. A vector of parameters is introduced for quantization of the information which should be exchanged among the agents in order to achieve a stable formation and the optimization is accomplished with respect to these parameters. Similar approach is pursued more recently in [43].

In [49], a fixed and a given network structure is assumed and the question addressed is how to find the weights of the interconnection links such that the convergence to consensus value is achieved at the fastest rate. To solve the problem a set of criteria is introduced to be minimized. The resulting optimization problem is non-convex which is then converted into a convex one. In [85], an estimate of the convergence rate of consensus seeking is obtained. The communication links are assumed to be time-varying. In [86], a lower bound on convergence rate of some of the consensus algorithms is provided. Towards this end, two approaches based on the properties of stochastic matrices and the concept of random walks are used. In [87], it is shown that connectivity of a network with a fixed number of links can be significantly increased by selecting the inter-agent information flow links properly. This in turn can result in a an ultra fast consensus seeking procedure. The idea is best applicable to small-world networks where any two nodes can be connected using a few links though the total size of the network can be large.

In some references, the communication delay is considered in modelling of a network of agents with point-mass model. For an example, one can refer to [13] in which directed and undirected networks with fixed and switching topologies are considered. It is assumed that the delayed information from other agents is compared with the delayed value of the agent's own dynamics at each time step. On the other hand in [88] the delayed information of the neighbors are compared with the current value of the agents' state. In this work, uniformly delayed communication links are analyzed for consensus algorithms. In [89], the agreement protocol is analyzed when there are non-uniform time delays in the links amongst the agents. Linear protocols are used for fixed networks with and without communication time-delays and communication channels that have filtering effects. Similarly, nonlinear protocols are applied to dynamic networks to achieve consensus. In all these cases, the effects of time delay are analyzed for the agreement protocol only and the analysis is performed in the frequency domain. In [90], the authors have considered time-varying time delays in communication links and presented conditions for consensus achievement in a network of nonlinear, locally passive systems. Considering time delays in other coordination problems such as formation control and target tracking is still an open area of research.

The problem of team cooperation, and specifically consensus seeking with switching topology, has received a wide attention in recent years and has been discussed in the literature from different perspectives [91]- [101]. The work performed in [91] can be considered as one of the pioneer work in which algorithms for distributed computation in a network with a time-varying network structure are analyzed. Specifically, in [92] for a discrete-time model of processors and a given number of tasks, convergence of a consensus algorithm in a time-varying structure is discussed given that some restrictions are imposed on the frequency of availability of the inter-agent communication links. One of the underlying assumptions in many of the related work on switching networks is that the graph describing the information exchange structure is a balanced graph. The authors in [101] considered balanced information graphs and shown the stability under switching time-delayed communication links. The analysis is performed by introducing a Lyapunov functional and then showing the feasibility of a set of linear matrix inequalities. In [15, 95], switching control laws are designed for a network of agents with undirected and connected underlying graphs whereas in [99], consensus in a directed, jointly connected and balanced network is discussed. The necessary conditions for achieving consensus in a network are discussed in [97]. The concept of "pre-leader-follower" is introduced as a new approach to achieve consensus in a network of discrete-time systems. The basic properties of stochastic matrices are used to guarantee consensus achievement in a network with switching topology and time-delayed communication links.

In [94], higher order consensus algorithms are discussed. The author's approach to handle the switching network structure with a spanning tree is to find an appropriate dwell time with their own provided definition. It is shown that the final consensus value depends on the information exchange structure as well as the controller weights. In [102] analysis is performed for a time-varying network of agents with discrete-time models. In this work, milder assumptions on connectivity of the agents over time are imposed when compared to [45] and necessary and sufficient results for consensus achieving are presented. The work in [22, 66, 67] are extensions of the approach in [45] for second order dynamics for fixed and dynamic topologies in undirected and connected graphs. In [96], the authors used a similar approach to the one used in [22, 66, 67] to analyze consensus achievement in a team with fixed and switching topologies. They divided the control law into several parts and used non-smooth analysis framework to address the problem. In [13], consensus achievement for a connected graph subject to certain switching in the network structure is addressed. The underlying assumption there is that the graph under consideration is a balanced graph.

In [77,99], consensus in directed, jointly connected and balanced network is discussed. The authors in [77] have considered information consensus in multi-agent networks with dynamically changing interaction topologies in the presence of limited information. They have shown that consensus can be achieved asymptotically under these conditions if the union of the directed interaction graphs have a spanning tree frequently enough for agents with discrete and continuous time dynamics. This condition is weaker than the assumption of connectedness that is made in [13] and [45], and implies that one half of the information exchange links that are required in [45] can be removed without affecting the convergence result. However, the final achieved equilibrium points will depend on the property of the directed graph, e.g. its connectedness. This work is an extension of [45] to digraphs case with more flexible weight selections in information update schemes. Some simulation examples of this work are presented in [103].

One of the recent research topics in consensus seeking is analysis of behavior of the consensus algorithms in presence of measurement noise or even design of consensus algorithms which can compensate for the lack of measurements or inaccuracy of them. In [104], the performance of first and second order consensus algorithms is discussed in the presence of measurement noise. A relation between the measurement error and the consensus error is derived. In [105], the measurement noise is considered for a leader-follower structure. Having used the stochastic analysis and by assuming time-varying weights the authors could guarantee a mean square consensus achievement in the presence of measurement noise. Similar idea of using time-varying weights is used in [106] to guarantee consensus on a Gaussian random variable. The authors in [107], have linked the consensus problem into a multi-inventory system control problem, where bounded disturbances affect the first order agents' dynamics.

One of the approaches to solve the consensus problem persuaded in the present research is based on the decentralized optimal control theory. Therefore, at this point I provide a brief literature review on the optimal approaches to consensus seeking problem. Among the first work where optimal control was discussed in a team of agents I can refer to [71]. There, the Linear Quadratic Regulator (LQR) problem was solved by "using" a team of decision makers and not "in" a team of decision makers. In other words, each decision maker is responsible for design of an optimal control at one (or some) time instant where the other decision makers should decide what the best (optimal) actions for the next time instants are to minimize a common cost function. Therefore, although the problem is dynamic in the sense that at the outset an optimal controller is designed to minimize a cost function with a given dynamical constraint, none of the decision makers has an individual dynamics. This implies that each decision maker can be interpreted as the state of a discrete-time system at one time only and not as an independent dynamical system.

In contrary, in optimal approach to the consensus problem which is introduced in this thesis, each decision maker has its own continuous-time dynamics, and its own cost function which is coupled to the state and control (action) of the other members. Therefore, each decision maker should decide for a set of actions and not only for a single one-time action. In other words, in the present work the purpose is that N decision makers design N control actions for all the time period whereas in [71], N decision makers design <u>one</u> control action each for only <u>one</u> time instant. Moreover, decentralization in the context of [71] implies that a single control action (decision) should be designed by the involvement of several decision makers, whereas in the framework presented in this thesis the decentralization refers to design of several controllers where the goal of controllers are coupled to each other.

In more recent literature, an optimal approach to team cooperation problem is considered in [29, 40, 41, 108, 109] for formation keeping and in [39, 98, 110, 111] for consensus seeking. The approach in [41] is based on individual agent cost optimization for achieving team goals under the assumption that the states of the other team members are constant. The concepts of Nash equilibrium, penalty function as well as Pareto optimality are used for design of optimal controllers. In [109], for a formation keeping problem the effect of the amount of information on the value of the cost is investigated. The authors have shown that the centralized architecture will result in the lowest cost value whereas the decentralized solution will increase the cost value. In order to solve an optimal consensus problem, the authors in [98] have assumed an individual agent cost for each team member. In evaluating the minimum value of each individual cost, the states of the other agents are assumed to be constant. For a switching network structure the dwell time that provides stability of the network subject to the switching structure is found. In [110], an H_2 optimal semistable methodology for stabilization of linear discrete-time systems is proposed. The authors then proposed a consensus algorithm and have shown that this protocol is a semistable controller which can solve the consensus seeking problem. The authors in [111] have shown that a specific type of graphs, i.e. de Bruijn's graph, is optimal for consensus seeking problem and with respect to a given cost function.

In all the above referenced work the optimal control problem is based on the individual cost definition for team members. However, to the best of my knowledge, a single team cost function formulation has been proposed in only a few work [40, 108]. In [108], optimal control strategy is applied for formation keeping and a single team cost function is utilized. The authors in [40] assumed a distributed optimization technique for formation control in a leader-follower structure. The design is based on dual decomposition of local and global constraints. However, in this approach, the velocity and position commands are assumed to be available to the *entire* team. In [39], the dynamics of the entire network are decomposed into two components, namely one in the consensus space and the other in its orthogonal subspace. A set of Linear Matrix Inequalities (LMIs) are then used to guarantee the stability and consensus achievement using an H_2 design strategy.

1.4 General problem statement and research objectives

In [112], coordination in a network of unmanned vehicles is concerned with either a cooperative motion such as formation control problem, network agreement, consensus, flocking or swarming behavior, or other high-level cooperative control problems such as task assignment, timing, search, and path-planning. Moreover, the reader is referred to [113] for more information on different direction of research in coordination in networked control systems. For implementing any of the above problems several levels of abstraction should be considered, namely: high-level, mid-level, and low-level (agent level). For a complete description of these three levels and different tasks which can be defined, refer to the block diagrams in Figures 1.6 and 1.7 at the end of this chapter. The "big picture" problem can be reformulated as four main subproblems:

1. Control design: stabilization, controllability, observability, robustness;
2. Network structure design: minimization of communication required to achieve a predefined goal by finding some conditions on the structure or on different strategies for information exchange;
3. Decentralized/distributed estimation: stability and convergence analysis; and

4. Low-level communication problems: congestion, routing, resource allocation.

The main challenges in solving these problems are:

- Real-time sensor processing and decision making: real-time implementation;
- Communication constraints: delayed and missed information due to bandwidth limitations, time-varying communication structures;
- Computational constraints;
- Design of a distributed collaborative team consisting of agents with more complex dynamics than point-mass models, and incomplete access to the information of the entire group;
- Distributed estimation in a network; and
- Analysis of team cooperative behavior in the presence of faults and anomalies in the team members.

Though a large body of work has been produced to accomplish the requirements of coordination in the UMSN and to address the above problems, there are still unsolved problems in this area. Some of these problems are listed below:

1. Complete information is not provided for all the agents. In a leader-follower structure and in some circumstances, leader command is not known to the followers. Moreover, in any structure it may happen that an agent is just aware of its neighbors' states. The team goals should be accomplished even if the full information of the team is not available to the entire group,

e.g. cohesion and connectivity of the group should be satisfied in the presence of uncertainties and partial information.

2. Most of the results appeared in the literature are for point-mass model of agents, see [8,45,47,59]. How to extend these results to more complicated agents' dynamics?

3. Robustness issues and considerations for different network structures should be investigated. What happens if the leader fails in a leader-follower structure? How can it be recovered? How about fault diagnosis and recovery in the formation keeping or consensus seeking problems?

4. Dynamic network topologies with time-varying structures and changing number of the nodes present in the network should be considered.

5. The behavior of the team in the presence of agents' faults and malfunctions should be predicted and analyzed.

The main goal in this research is to design a team of multi-agents so that it can accomplish several goals and missions. This team can perform different maneuvers within several scenarios that are required in different applications. The main focus is on design of a protocol for a distributed network of agents so that they can work cooperatively, e.g. to achieve a cohesive motion. There are some constraints on the availability of the information as well as the available communication links during the mission. Therefore, the network topology may be time-varying. To make the problem more practical it is assumed that for specific reasons an anomaly may happen in team members which may deteriorate the performance of the team.

In this research I use a new formulation for the problem to address these important issues. The problem of design of cooperating members with distributed information network, called team collaboration in control engineering and economics, is of considerable importance and plays a key role in understanding these complex systems. Hence, the main focus of this work is to introduce a unified approach for solving the problem of team cooperation in a framework which is broad enough to be able to address the problem subject to partial team information for a team of agents with more flexible dynamical characteristics and network topology. In my work the agents are mainly mobile robots which have to work together to keep their team cohesion. The main challenge of the present problem is that the autonomy of the team should be satisfied while there is no supervisor or command manager, and moreover even the information of the team may not be available to the entire group. However, the given relative specifications (e.g. regulation of relative distance or velocity) imposed on the vehicles, introduce challenging problems for research. Hence, the nature of the problem is such that the decision makers should be designed based on a (semi)decentralized manner, while on the other hand due to the autonomy and cooperative nature, some of them may not have access to the exact set-point or tracking path for their team decision. The proposed team cooperative strategy will accomplish agreement or command tracking goals using optimal control, game-theory, and LMIs by introducing interaction terms as a means to overcome relative specifications and dependencies of individual goals on other agents' outputs or states.

1.5 Main challenges and thesis contribution

The main challenges of this research are listed below:

1. **Synthesis of the controller** in a systematic way (and not just analysis) in order to optimize some performance index or achieve some goals. In most of the existing literature on coordination problems a controller is suggested and then the closed-loop system is analyzed to determine what

properties are satisfied by the corresponding controller. However, very few work has appeared in the literature in which at the outset the design of a control satisfying a set of predefined specifications and requirements is addressed. In fact, many of the earlier work in the literature have focused on analysis only, e.g. [11, 13, 22, 36, 45, 77].

2. Tackling the **coupling** due to the relative dependence of the agents' goals on others' specifications. Even when the commands are available for the entire team both the **relative performance** as well as the individual behavior are important. This dependency and relative performance requirement results in a highly coupled (centralized) solution. The techniques such as introducing coordinating vector in virtual structure or leader in leader-follower structure, dual decomposition, and error dynamic framework are used to decompose this relative dependency and provide a reference point.

3. Accommodating the **non-availability of the command or required information** for the entire team which requires a decentralized solution for the proposed problem. The challenge is how to formulate the distributed nature of the problem in the classical frameworks such as optimal control, model predictive control, or game theory.

4. Extending the existing results for a point-mass model to **general linear dynamical model** of agents. The results obtained in most of the existing literature on cooperative motion are for point-mass dynamical model of agents, whereas in practice dynamics of agents may be governed by more complicated linear models or even nonlinear models.

In order to overcome the existing challenges, this research has made the following contributions to the literature. The corresponding full explanations of each contribution are provided in the following chapters.

- To provide a **synthesis-based** methodology for consensus seeking that satisfy the predefined objectives and constraints for coordination in a network of agents in an aggregated and systematic way rather than analysis of a proposed algorithm. The approaches pursued in this thesis are based on the utilization of formal control design techniques like game theory and optimal control theory. Hence, controllers are formally designed to address the output consensus over a common value for a team of agents.

- To **derive a stable average consensus algorithm** or a modified version of it formally as the solution to the optimization problem, i.e. a performance index is introduced that is minimized by the consensus protocol through the proposed methodology.

- To compensate for the existing coupling among the team members due to the relative dependency by introduction and incorporation of the notion of **interaction terms** while respecting partial information availability. This novel modelling approach provides a framework in which local and global control requirements may be partially decoupled and the interconnections among agents can be described. This is an advantage of the proposed methodology when compared to the synthesis methods such as the ones introduced in [9,38,39].

- To utilize a **modified structure** with corrective feedback which accommodates partial availability of commands. Due to the special characteristics of the Modified Leader-Follower (MLF) architecture employed here, where the leader control input may also be affected by the followers through a corrective feedback from followers to the leader, this structure provides an embedded robustness capability for the team subject to the agents' faults as well as adaptability of the leader and followers to the uncertainties and unanticipated situations.

- To generalize the proposed concepts and methodologies for solving the consensus problem to agents with a general **linear dynamical model** properties rather than a point-mass model.

- To emphasize on the **information structure** and using its properties.

- To use proper **indices to measure the performance of the team** in achieving the predefined goals and to compare the performance of different methods.

- To design a **semi-decentralized optimal algorithm** (controller) for solving the consensus problem using two methods:

  - Finding a solution to Hamilton-Jacobi-Bellman (HJB) equations, using properties of the network underlying graph, and incorporating the notion of interaction terms; and

  - Finding a solution by decomposing the state vector into two components in consensus subspace and its orthogonal subspace and using the LMI formulation.

- To utilize **cooperative game theory** technique to formulate the consensus problem and design the consensus algorithm in a formal way to address the output consensus over a common value in a cooperative manner

  - To find a unique Pareto-efficient solution while distributed nature of the problem is respected by using the LMI formulation and Nash-bargaining concept.

- To analyze and predict the performance of the previously designed consensus algorithms as well as team behavior in the presence of leader and followers **actuator anomalies and faults**
  - Stability analysis of the steady state error in the case of fault occurrence for three types of actuator faults, i.e. float fault, Loss of Effectiveness (LOE), and Lock-In-Place (LIP);
  - Evaluation of the final value to which each agent converges subject to actuator faults; and
  - Adaptability analysis of team members to the changes that occur as a result of fault happening.
- To guarantee team stability and design a switching control strategy for a team with a switching topology and time-varying leader assignment for directed, and **unbalanced** graphs. Introducing a criterion, i.e. performance-control effort tradeoff to guarantee optimal performance of consensus algorithms.

The outline of the remaining parts of this thesis is as follows: in Chapter 2, the required background for the following chapters is provided. Chapter 3 is devoted to the semi-decentralized optimal control design based on the solution of HJB equations. In Chapter 4, two practical and crucial scenarios are discussed for the strategy that is proposed in Chapter 3. In the first scenario, failure analysis is performed when some of the agents in the team are subject to an actuator fault. The second scenario discusses the switching control design and stability analysis for a dynamic network of agents where both the network structure and the leader assignment are switching. In Chapter 5, consensus seeking problem is solved by using a game-theoretic framework and an optimal control approach by utilizing an LMI formulation. In the optimal control solution the ideas of state decomposition technique and LMI formulation are used. Finally, the thesis is concluded in Chapter 6 and some directions for future work are provided.

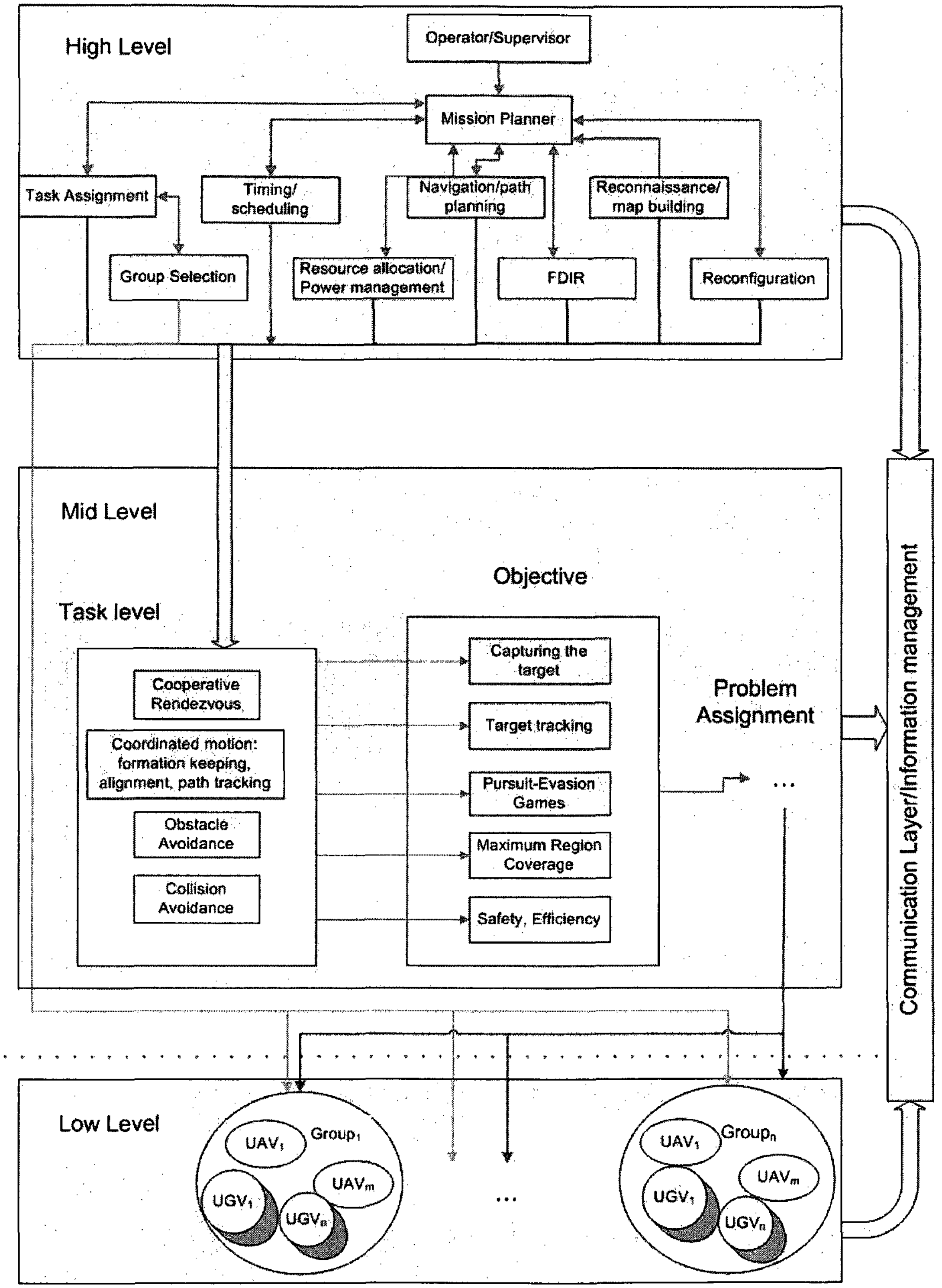


Figure 1.6: Hierarchical decomposition of a cooperative team design

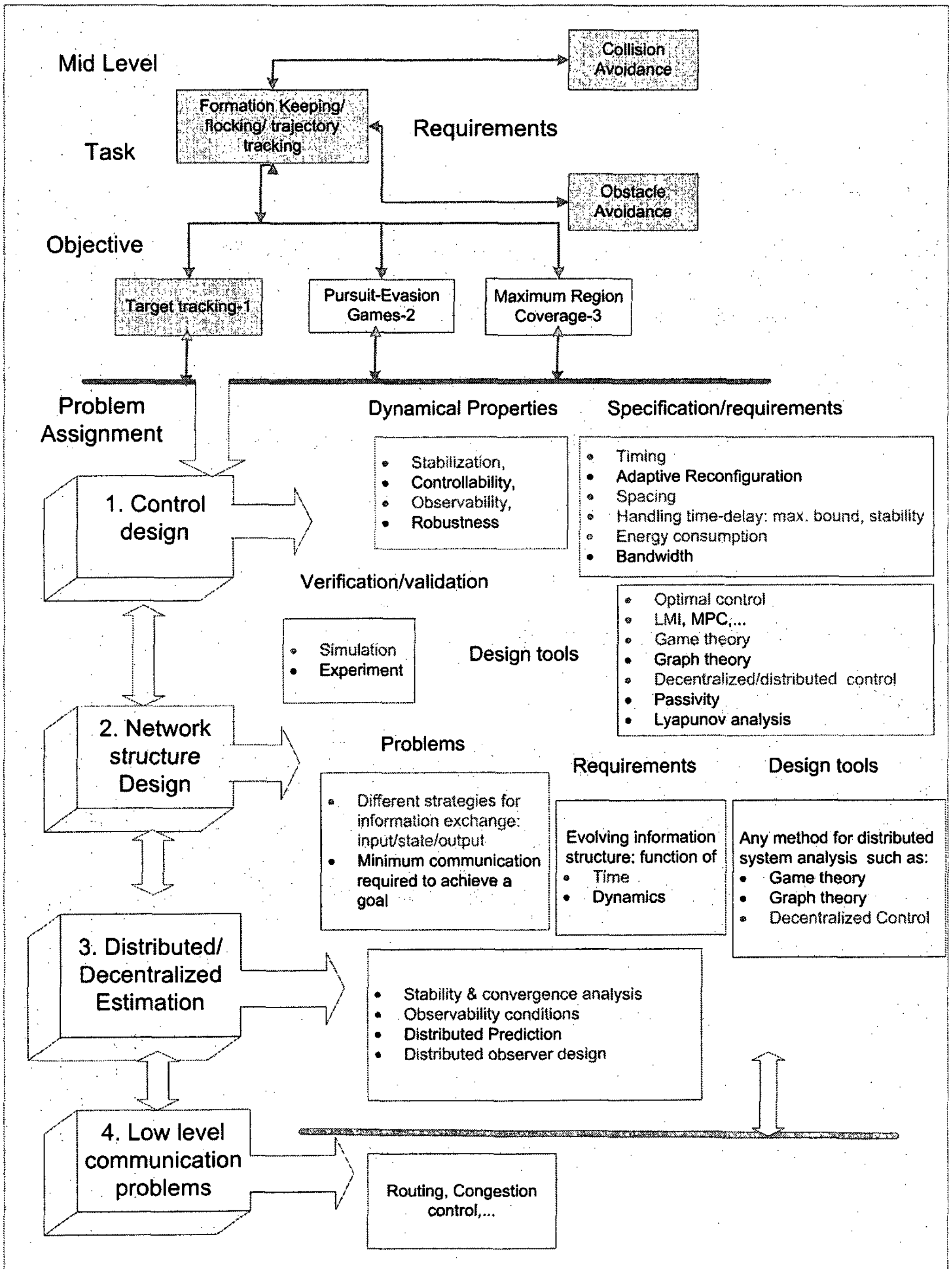


Figure 1.7: Problem assignment levels in a team design

Chapter 2

Background

2.1 Multi-agent teams

In general, an agent refers to a dynamical system. However, in the context of this thesis the term "agent" is interchangeable with "vehicle", where a vehicle may be a mobile robot or any other ground vehicle.

Assume a set of agents $\Omega = \{i = 1, \ldots, N\}$, where N is the number of agents. Each member of the team which is denoted by i is placed at a vertex of the network information graph. In general the dynamical representation of each agent is governed by

$$\dot{X}^i = A^i X^i + B^i u^i,\ Y^i = c^i X^i,\ i = 1, \ldots, N \tag{2.1}$$

where $X^i \in R^q$, $u^i \in R^m$ and $Y^i \in R^n$ are the state, input, and output vectors of agent i. The corresponding vectors of the entire team are designated by X, U, and Y, respectively, which are the concatenation of all the state, input, and output vectors of the agents and are given by:

$$X_{Nq\times 1} = [(X^1)^T \ \dots \ (X^N)^T]^T, \ U_{Nm\times 1} = [(u^1)^T \ \dots \ (u^N)^T]^T, \qquad (2.2)$$
$$Y_{Nn\times 1} = [(Y^1)^T \ \dots \ (Y^N)^T]^T.$$

2.2 Information structure and neighboring sets

In order to ensure cooperation and coordination among the members of a team, each member has to be aware of the status of other members (i.e. the output or the state vector), and therefore the members have to communicate with each other. For a given agent i in a network with an undirected graph, the set of agents from which it can receive information is called a neighboring set N^i, that is

$$\forall\ i = 1, \dots, N, \quad N^i = \{j = 1, \dots, N | (i, j) \in E\} \qquad (2.3)$$

where E is the edge set that corresponds to the underlying graph of the network. In other words, i and j have either a communication link to transfer their status (output vector) to each other or any other way to obtain information such as measurements of other agents states. Furthermore, all the members have two-way links with the agents that are connected to them. Based on this formulation the number of neighbors of the agent i is $|N^i|$ (the cardinality of the set N^i).

Leaderless (LL) structure: Assume an information exchange structure as shown in Figure 2.1(a). In the LL configuration no desired command is specified and a consensus state is decided and agreed upon by all agents. In other words, in this structure no external command is provided to the members of the team, and the goal is to make the agents' output, e.g. velocity, converge to a common value which is decided upon by the team members, i.e. $\forall i, j\ Y^i \rightarrow Y^j$. This state is referred to as the consensus state.

Modified Leader-Follower (MLF) structure: Coordination problems may be treated in a different architecture in which the desired value to which agents should converge is provided for the team via a commander which can be either a leader as in the leader-follower structure [50, 56, 57] or a coordination vector as in the virtual structure configuration [7]. In the standard leader-follower configuration, the leader can affect the followers whenever it is in their neighboring set but there is **no** feedback from the followers to the leader. In [4,7,27] the idea of adding a feedback from the virtual structures to the coordination unit is introduced. Other than leader-follower and virtual structure there can be an alternative architecture in which the command information is available to part of the team, e.g. to the leader, and the rest of the team should receive this information via links that exist among the neighbors. The work in [45] can be considered as an example of this structure for a team of agents with single integrator dynamical model where the leader can affect the followers but there is no feedback from the followers to the leader. Using the nearest neighbor rule, "each agent's heading is updated using the average of its own heading plus the headings of its neighbors" [45].

In this thesis an MLF structure is introduced and utilized. The difference between the MLF architecture that is introduced here and the above mentioned architectures is that here only the leader is aware of the command, and the rest of the team is connected to each other or to the leader through a predefined topology. The command can be a set point reference or a time-varying signal specified for the output or a trajectory to be followed by the agents. Here, the leader control input can also be affected by the followers through a corrective feedback from the followers. To be more specific, assume

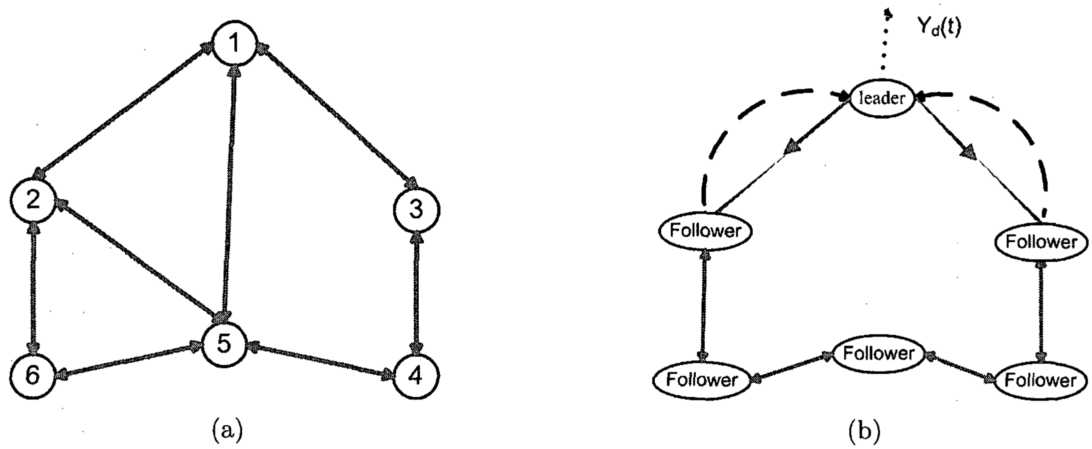


Figure 2.1: Information structure in a) an LL structure and b) an MLF structure.

an information exchange structure as shown in Figure 2.1(b). An external command is provided to one of the members designated as the leader, and the goal is to make the agents' output, e.g. velocity, converge to the external command, i.e. $\forall i,\ Y^i \to Y^d$. Other agents should follow the leader by communicating through their links with each other or with the leader. Furthermore, the leader has two-way links with the followers that are connected to it, i.e. it receives feedback from *some* of the followers. This implies that although the leader may follow a given trajectory without any feedback from the followers, it receives the status of the followers, which may contribute to improving the robustness of the team cohesion, as shown formally in Chapter 4.

<u>Ring Topology:</u> In this topology as can be seen in Figure 2.2, each agent is connected to its two adjacent neighbors, i.e.

$$\forall i = 2, \ldots, N-1,\ N^i = \{i-1, i+1\},\ N^N = \{N-1, 1\},\ N^1 = \{2, N\} \tag{2.4}$$

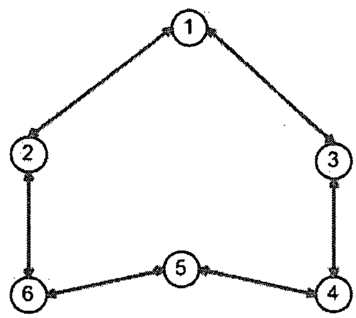


Figure 2.2: Information structure in a ring topology.

and the corresponding Laplacian matrix, as will be defined in Section 2.5, is given as

$$L = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ -1 & 0 & \dots & 0 & -1 & 2 \end{bmatrix} \tag{2.5}$$

2.3 Model of interaction among the team members

In decentralized control, the multi-level control concept is used as a means to compensate for the coupling dynamics in an interconnection of several subsystems. In the simplest case, the control u^i is partitioned into two parts, namely local and global components, i.e. $u^i = u^i_g + u^i_l$. The approach pursued here is clearly different from what is used in standard decentralized control, even though the general idea seems to be similar. Specifically, first an interconnection (interaction) term is introduced in the agent's dynamics in order to represent the connectivity of the members as required by the team goal. The interaction terms play the role of a global controller in the decentralized approach, so that they can eliminate the effects of relative dependency in design of individual controllers.

To formalize this idea, assume that the dynamical model of each agent is given by (2.1). This model defines the dynamics of an isolated agent of the team, however team agents do indeed have certain interactions among neighboring agents through existing information flows. This interaction may be considered for each agent through its input channel and available information from outputs of other agents in its neighboring set. These concepts may be represented by the following dynamical model

$$\dot{X}^i = A^i X^i + B^i u_l^i(X^i) + B^i u_g^i(Y^j), \;\; Y^i = c^i X^i \tag{2.6}$$

In other words, the actual control input $u^i(X^i, Y^j), i = 1, \ldots, N$ is decomposed into two components,

$$u^i(X^i, Y^j) = u_l^i(X^i) + u_g^i(Y^j), \quad j \in N^i$$

in which $u_g^i(Y^j)$ defines the dependence of the control input of agent i on its neighbors' information explicitly and $u_l^i(X^i)$ defines the dependence of the control input of agent i on its local information. The controllers u_l^i have the same role as local controllers in the decentralized approach. For simplicity, I assume that $u_g^i(Y^j) = \sum_{j \in N^i} \mathcal{F}^{ij} Y^j, \; i = 1, \ldots, N$, where $\mathcal{F}^{ij}$ is the interaction matrix to ensure compatibility of agent's input and output channels dimensions. The above formulation illustrates why the method that will be proposed in the following chapters will be called semi-decentralized. Whereas, u_l^i is only a function of X^i, u_g^i incorporates the effects of neighbors' information (output) into the control law so that the control input is not fully decentralized in the conventional sense.

By incorporating the interaction terms in the agents dynamics, the dynamical representation of the entire network can be written as follows:

$$\dot{X} = AX + BU, \quad Y = CX \tag{2.7}$$

in which X and Y are the entire team state and output vectors defined previously. Vector U is defined as in (2.2) by replacing u^i with u^i_l for all i. Matrices A, B and C are defined as follows:

$$A = \begin{bmatrix} A^1, \; 0, \ldots, B^1\mathcal{F}^{1j}c^j, \ldots, 0 \\ \vdots \\ 0, \ldots, B^N\mathcal{F}^{Nj}c^j, \ldots, 0, A^N \end{bmatrix}, \; B = Diag\{B^1, \; \ldots, \; B^N\}, \tag{2.8}$$
$$C = Diag\{c^1, \ldots, c^N\}$$

The terms $B^i\mathcal{F}^{ij}C^j$ represent the interactions that exist among the agents.

Remark 2.1. The phrase "interaction terms" is used in decentralized control of large scale systems quite often and refers to part of the dynamics which describes the **existing** coupling terms among the subsystems. However, the "interaction terms" in the context of this thesis refers to the **externally added** couplings that are added in the input channels and as part of the control law. The proposed interactions guarantee that the team goals and specifications are satisfied as opposed to compensating for the dynamical couplings among the agents.

2.4 Dynamical model of an agent

In this section I will discuss and introduce two linear dynamical representations for the agents. These models will be used in the future discussions. It is worth noting that many of the mobile vehicles have nonlinear dynamical equations due to their existing non-holonomic constraints [114, 115]. However, it will be shown in the following sections that a common nonlinear model of a mobile robot can be transformed into a linear one by using a nonlinear feedback of the states. Therefore, the assumption of linear dynamical model for mobile robots is reasonable even if the vehicles have non-holonomic constraints.

2.4.1 Mobile robot dynamical model: double integrator dynamical model

In the present work, the agents are mainly assumed to be mobile robots. There are different models for mobile robots in the literature. In [114], a 7th order model is considered for a mobile robot with non-slipping and pure-rolling motion. The robots are considered to have 2 steering wheels and a free one with non-holonomic constraints. Based on some simplifications that are discussed in [114] and due to planar motion these equations can be simplified into the following 5th order model:

$$\begin{cases} \dot{x}^i = \eta_1^i \sin\theta^i \\ \dot{y}^i = \eta_1^i \cos\theta^i \\ \dot{\theta}^i = \eta_2^i \\ \dot{\eta}_1^i = a_1^i = \frac{1}{M^i}\mathbb{F}^i \\ \dot{\eta}_2^i = a_2^i = \frac{1}{\mathbb{J}^i}\tau^i \end{cases} \tag{2.9}$$

in which x^i and y^i are the position components, η_1^i and η_2^i are the linear and the angular velocities, θ^i is the heading angle, a_1^i and a_2^i are the linear and angular accelerations, $\mathbb{F}^i$ and τ^i are the force and torque inputs, and M^i and $\mathbb{J}^i$ are the mass and moment of inertia of each robot, respectively.

The above model is in general a nonlinear model, however as the focus of this work is on linear models, in the following I will show how to linearize this model. For some definitions of output this model can be transformed into a linear model. Three cases are assumed for linearizing the model based on different output definitions:

Case I: Output of each agent is one of the three last states or a combination of them. For example assume that $Y^i = [\theta^i \quad \eta_1^i]^T$. Based on this definition of output vector, the first two equations may be eliminated as they do not have any effect on the input-output channel. Hence, a reduced order model may be considered which is given by:

$$\begin{cases} \dot{\theta}^i = \eta_2^i \\ \dot{\eta_1}^i = a_1^i = \frac{1}{M^i}\mathbb{F}^i \\ \dot{\eta_2}^i = a_2^i = \frac{1}{J^i}\tau^i \end{cases} \tag{2.10}$$

The new state vector may now be considered as $X^i = [\theta^i,\ \eta_1^i,\ \eta_2^i]^T$.

Case II.a: Output of each agent consists of at least one of the first two states, e.g.:

$$Y^i = [x^i \quad y^i]^T \tag{2.11}$$

In this case the first two equations cannot be separated anymore. Hence, one has to go through feedback linearization procedure. Based on the definition of the relative degree presented in [116], the relative degree of the system in (2.9) with respect to the output in (2.11) is $\{2, 2\}$. However, the system is not feedback linearizable. Hence, we need to modify the input-output definition. One common solution is to consider a dynamic input definition. Towards this end, define $z^i = [\dot{a}_1^i \quad a_2^i]^T$ as the new input vector. Hence, equation (2.9) can be written as:

$$
\begin{cases}
\dot{x}^i = \eta_1^i \sin\theta^i \\
\dot{y}^i = \eta_1^i \cos\theta^i \\
\dot{\theta}^i = \eta_2^i \\
\dot{\eta}_1^i = a_1^i = \frac{1}{M^i}\mathbb{F}^i \\
\dot{a}_1^i = z_1^i \\
\dot{\eta}_2^i = z_2^i = \frac{1}{J^i}\tau^i
\end{cases}
\tag{2.12}
$$

This system is feedback-linearizable with relative degree $\{3, 3\}$. By utilizing state transformation

$$
X = \begin{bmatrix}
x^i \\
y^i \\
\eta_1^i \sin\theta^i \\
\eta_1^i \cos\theta^i \\
\eta_1^i\eta_2^i \cos\theta^i + a_1^i \sin\theta^i \\
-\eta_1^i\eta_2^i \sin\theta^i + a_1^i \cos\theta^i
\end{bmatrix}
\tag{2.13}
$$

the system will be transformed into the linear model (2.1) with the following parameters:

$$
A^i = \begin{bmatrix}
0_{2\times2} & I_{2\times2} & 0_{2\times2} \\
0_{2\times2} & 0_{2\times2} & I_{2\times2} \\
0_{2\times2} & 0_{2\times2} & 0_{2\times2}
\end{bmatrix}, B^i = \begin{bmatrix}
0_{2\times2} \\
0_{2\times2} \\
I_{2\times2}
\end{bmatrix}, c^i = \begin{bmatrix} I_{2\times2} & 0_{2\times2} & 0_{2\times2} \end{bmatrix}
\tag{2.14}
$$

and

$$
u^i = \begin{bmatrix}
\sin\theta^i & \eta_1^i \cos\theta^i \\
\cos\theta^i & -\eta_1^i \sin\theta^i
\end{bmatrix} z^i + \eta_2^i \begin{bmatrix}
2a_1^i \cos\theta^i - \eta_1^i\eta_2^i \sin\theta^i \\
-2a_1^i \sin\theta^i - \eta_1^i\eta_2^i \cos\theta^i
\end{bmatrix}
\tag{2.15}
$$

The coefficient matrix of z^i is nonsingular for $\eta_1^i \neq 0$ and so the control law for u^i can be transformed to a value for z^i and then to the v^i (defined below). The model presented in (2.1) with parameters as in (2.14) is a chain of integrators. It is simple to verify that whereas the first two states in the new coordinate are the position states, the third and fourth states are linear velocity components and the last two states are acceleration components. If we denote the position, the velocity and the acceleration vectors of each agent by r^i, v^i and a^i respectively, it can be seen that:

$$\begin{cases} \dot{r}^i = v^i \\ \dot{v}^i = a^i \\ \dot{a}^i = u^i \end{cases} \tag{2.16}$$

Case II.b: The definition of the output is similar to the previous case, but here the first four states are assumed as the state vector and the input vector is $z^i = [a_1^i \ \ \eta_2^i]^T$. This can be done as the dynamics of η_2^i does not have any effect on the input-output channel. In this case the relative degree of the system is $\{2, 2\}$. By introducing the following state transformation:

$$X = \begin{bmatrix} x^i \\ y^i \\ \eta_1^i \sin\theta^i \\ \eta_1^i \cos\theta^i \end{bmatrix} \tag{2.17}$$

the system will be transformed into the linear model (2.1) with the following parameters:

$$A^i = \begin{bmatrix} 0_{2\times2} & I_{2\times2} \\ 0_{2\times2} & 0_{2\times2} \end{bmatrix}, B^i = \begin{bmatrix} 0_{2\times2} \\ I_{2\times2} \end{bmatrix}, c^i = \begin{bmatrix} I_{2\times2} & 0_{2\times2} \end{bmatrix} \tag{2.18}$$

and $u^i = \begin{bmatrix} \sin\theta^i & \eta_1^i\cos\theta^i \\ \cos\theta^i & -\eta_1^i\sin\theta^i \end{bmatrix} z^i$. The coefficient matrix of z^i is nonsingular for $\eta_1^i \neq 0$ and so the control law for u^i can be transformed to a value for z^i and then a_1^i and η_2^i can be found. The model presented in (2.1) with parameters as in (2.18) is again a chain of integrators. It is simple to verify that while the first two states in the new coordinate are the position states, the last two states are velocity components. If one denotes the position and the linear velocity of each agent by r^i and v^i respectively, then:

$$\begin{cases} \dot{r}^i = v^i \\ \dot{v}^i = u^i, \ \ i = 1, \ldots, N \end{cases} \tag{2.19}$$

To summarize the above discussion, I assume that the vehicles are mobile robots that are represented by double integrator dynamical models as in (2.19), and that $Y^i = v^i, i = 1, \ldots, N$. Using the formulation proposed in Section 2.3, the dynamical model of each agent with interaction terms is as follows

$$\begin{cases} \dot{r}^i = v^i \\ \dot{v}^i = u_l^i + u_g^i, \ \ u_g^i = \sum_{j\in N^i} \mathcal{F}^{ij} v^j \\ Y^i = v^i, i = 1, \ldots, N \end{cases} \tag{2.20}$$

2.4.2 Linear dynamical model

As discussed in previous subsection, in general the dynamical equation of most ground vehicles can be transformed into a generalized form of a set of integrators (canonical form) and due to the focus of this work on applications of teams of ground vehicles, assuming this kind of dynamical equation for the agents is an admissible assumption. In this work I assume that the vehicles' dynamics consist of position and velocity, i.e. $X^i = [X_1^i \ \ X_2^i]^T = [(r^i)^T \ \ (v^i)^T]^T$.

Therefore, in the presence of interaction terms the following agents' dynamical equations is considered

$$\begin{cases} \dot{r}^i = \bar{A}^i v^i, \\ \dot{v}^i = A^i v^i + B^i(u_l^i + u_g^i), \;\; u_g^i = \sum_{j \in N^i} \mathcal{F}^{ij} Y^j, \\ Y^i = v^i \in R^n, \;\; i = 1, \ldots, N \end{cases} \tag{2.21}$$

where B^i is a non-singular matrix, A^i and $\bar{A}^i$ are $n \times n$ matrices, and r^i, v^i denote the ith agent position and velocity vectors, respectively.

Remark 2.2. Dynamical model of each agent consists of position and velocity states. However, since the main objective in this thesis is to have a common output, namely velocity, for analysis I will only consider the velocity dynamics to describe the dynamical behavior of the agents. It should be noted that for the purpose of simulations naturally the position dynamics is also included.

2.5 Terminologies and definitions

- **Laplacian matrix [37]:** This matrix is used to describe the graph $\mathcal{G}$ associated with information exchanges in a network of agents and is defined as $L = [l_{ij}]_{N \times N}$, where

$$l_{ij} = \begin{cases} d(i) & i = j \\ -1 & (i,j) \in E \;\text{ and }\; i \neq j \\ 0 & \text{otherwise} \end{cases} \tag{2.22}$$

where E is the edge set of the graph $\mathcal{G}$, $d(i)$ is equal to the cardinality of the set N^i [117], $|N^i|$, and is called the degree of vertex i. For an undirected graph, the degree of a vertex is the number of edges incident to that vertex (total number of links connected to that vertex). For directed graphs, instead of the degree either the in-degree or the out-degree might be used (the total number of the links entering or leaving a node).

- **Normalized Laplacian matrix:** The normalized Laplacian matrix $\hat{L}$ is defined similar to the Laplacian matrix of a graph, where $\hat{L} = [\hat{l}_{ij}]_{N\times N}$ and

$$\hat{l}_{ij} = \begin{cases} 1 & i = j \\ -1/d(i) & (i,j) \in E \ \text{ and } \ i \neq j \\ 0 & \text{otherwise} \end{cases} \tag{2.23}$$

- **Adjacency matrix [118]:** The adjacency matrix of a graph, denoted by $\mathbf{A}$ is a square matrix of size N, defined by $\mathbf{A}(i,j) = 1$ if $(i,j) \in E$ and $i \neq j$, and is zero otherwise.

- **Normalized adjacency matrix:** The normalized adjacency matrix of a graph, denoted by $\hat{\mathbf{A}}$ is a square matrix of size N, defined by $\hat{\mathbf{A}}(i,j) = \frac{1}{d(i)}$ if $(i,j) \in E$ and $i \neq j$, and is zero otherwise.

- **Connected graph [64]:** An undirected graph consisting of a vertex set, V, and an edge set, E, is connected if there is a path between any two vertices and the path lies in the edge set.

- **Balanced graph:** If the Laplacian matrix of a graph, L, has the property that $\mathbf{1}^T L = 0$, then the graph is called a balanced graph. For balanced connected graphs one has the property that $L + L^T$ can be considered as the Laplacian matrix of an undirected and connected graph [13].

- **Tree [118]:** A connected graph with no cycles (acyclic) is called a tree.

- **Spanning Tree [118]:** A spanning tree of a connected undirected graph is a connected subgraph of the original graph with the same vertex set as the original graph and no cycles.

- **Forest [118]:** A forest is a disjoint union of trees.

- **Path [119]:** In a tree, every two vertices are connected by a unique path. The length of this path is the distance between the two vertices.

The followings are some other useful definitions from the linear algebra literature:

- **Spectrum of a matrix [120]:** The set of eigenvalues of a matrix is called its spectrum.

- **Inertia of a Hermitian matrix [120]:** is the number of positive, negative, and zero eigenvalues of a matrix.

- **Spectral Radius [120]:** The spectral radius of a matrix A, i.e. $\rho(A)$, is the largest of the absolute value of the eigenvalues (or magnitude of complex eigenvalues) of that matrix, that is

$$\rho(A) = \max_i |\lambda_i| \tag{2.24}$$

where λ_i is an eigenvalue of matrix A.

Definition 2.1. Schur complement [121]: Assume that matrix M is given by $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where D is invertible. Then the Schur complement of the block D of matrix M is $A - BD^{-1}C$. This conversion can be used to transform bilinear (nonlinear) matrix inequalities into LMIs, i.e.

$$\begin{pmatrix} A & B \\ B^T & D \end{pmatrix} \geq 0,\ D > 0 \Leftrightarrow A - BD^{-1}B^T \geq 0 \tag{2.25}$$

The following is the well-known Perron-Frobenius theorem for nonnegative matrices. This theorem will be used in Chapter 4.

Theorem 2.1. Perron-Frobenius Theorem [120]

Let $M \in M_n$ and suppose that M is irreducible and nonnegative. Then

1) $\rho(M) > 0$

2) $\rho(M)$ is an eigenvalue of M

3) There is a positive vector x such that $Mx = \rho(M)x$

4) $\rho(M)$ is an algebraically simple eigenvalue of M,

where M_n is the set of matrices of order n and $\rho(M)$ is the spectral radius of matrix M [120].

Definition 2.2. In a network consisting of nodes denoted by i and with a group system dynamic as $\dot{x} = f(x)$, $x = [x_1^T, \ldots, x_i^T, \ldots, x_N^T]^T$, a protocol asymptotically solves the χ-consensus problem if and only if there exists an asymptotically stable equilibrium x^* satisfying $x_i^* = \chi(x(0))$ for all nodes i [13], where $x(0)$ is the initial value of the state vector x. If $\chi(x) = Ave(x) = \frac{1}{N}\sum_{i=1}^{N} x_i$ the protocol is called average-consensus. Similarly, other χ-consensus protocols such as max-consensus and min-consensus can be defined.

2.6 Actuator fault types

In [122], the following three types of actuator faults that are of interest in this thesis are introduced, namely

i) **Lock-In-Place (LIP):** "In this case the actuator "freezes" at a certain value and does not respond to subsequent commands."

ii) **Float:** "The float fault occurs when the actuator output "floats" with zero value and does not contribute to the control authority."

iii) **Loss of Effectiveness (LOE):** "The loss of effectiveness (LOE) fault is characterized by lowering the actuator gain with respect to its nominal value."

These faults may be formally represented as follows

$$u_f^i = \begin{cases} \alpha u^i & \alpha = 1, \ \forall t \geq 0 & \text{No Failure} \\ \alpha u^i & 0 < \epsilon < \alpha < 1, \ \forall t \geq t_f & \text{Loss of Effectiveness (LOE)} \\ 0 & \alpha = 1, \ \forall t \geq t_f & \text{Float} \\ u^i(t_f) & \alpha = 1, \ \forall t \geq t_f & \text{Lock-In-Place (LIP)} \end{cases} \tag{2.26}$$

where u_f^i corresponds to the actual input that is produced by the actuator, u^i is the input commanded by the controller, t_f denotes the time when a fault is injected, and α represents the effectiveness coefficient of the actuator and is defined to be $\alpha \in [\epsilon, 1], \ \epsilon > 0$.

2.7 Hamilton-Jacobi-Bellman (HJB) equations

Minimization of a general nonlinear cost function, either unconstrained or subject to certain constraints, may be solved by using the HJB equations. In this thesis the unconstrained minimization problem is considered. The HJB equations in the general form for a finite horizon scenario are described below [123, 124]. Assume that the model of a dynamical system is given by

$$\dot{X}^i = f^i(t, X^i, u^i) \tag{2.27}$$

and the cost function to be minimized is given as

$$J^i = \int_0^T g^i(t, X^i, u^i)dt + h^i(X^i(T)) \tag{2.28}$$

The solution to the following optimization problem

$$\begin{aligned} \min_{u^i \in \mathcal{U}^i} \ J^i &= \int_0^T g^i(t, X^i, u^i)dt + h^i(X^i(T)) \\ \text{s.t.} \ \ \dot{X}^i &= f^i(t, X^i, u^i) \end{aligned}$$

is obtained if the following HJB equations have a solution:

$$\begin{aligned} & -\frac{\partial V^i}{\partial t}(t, X^i) = \min_{u^i \in \mathcal{U}^i} \Lambda^i(t, X^i, u^i), \\ & \Lambda^i(t, X^i, u^i) = g^i(t, X^i, u^i) + \frac{\partial V^i}{\partial X^i}(t, X^i) f^i(t, X^i, u^i), \\ & u^{i*}(t, X^i) = \arg\min_{u^i \in \mathcal{U}^i} \{\Lambda^i(t, X^i, u^i)\}, \ \ V^i(T, X^i) = h^i(X^i(T)) \end{aligned} \tag{2.29}$$

where V^i is a value function to be chosen such that the above Partial Differential Equation (PDE) is satisfied and $\mathcal{U}^i$ is the set of all strategies for player i. It is an indicator of the minimum value of the cost function (2.28) at any time t, i.e.:

$$V^i(t) = \int_t^T g^i(t, X^i, u^i)dt + h^i(X^i(T))$$

The term $h^i(X^i(T))$ is known as the "cost to go" and is the final value of the V^i, i.e. $V^i(T)$. An important issue to consider is the existence of a solution corresponding to the above equations and how the value function V^i is to be selected to satisfy the PDE given in (2.29).

2.8 LMI formulation of Linear Quadratic Regulator (LQR) problem

Consider the following cost function

$$d = \int_0^\infty \{X^T Q X + U^T R U\} dt \tag{2.30}$$

where Q is a Positive Semi-Definite (PSD) matrix, R is a Positive Definite (PD) matrix, and the corresponding linear dynamical system is as follows

$$\dot{X} = AX + BU \tag{2.31}$$

The problem of minimizing the cost function (2.30) subject to the dynamical constraint (2.31) is an LQR problem. The solution to this minimization problem can be found if the following Algebraic Riccati Equation (ARE) has a solution for P

$$PA + A^T P - PBR^{-1}B^T P + Q = 0 \tag{2.32}$$

where $U = -R^{-1}B^T PX$.

In [121]- [126] it is shown that the LQR problem can be formulated as a minimization or a maximization problem that is constrained to a set of matrix inequalities. In other words, instead of solving the ARE (2.32), a set of matrix inequalities can be solved. Using this formulation, the controller $U = KX$ that minimizes the cost function (2.30) subject to (2.31) is achieved by solving for and determining the appropriate matrix P:

$$\begin{aligned} &\min X(0)^T P X(0) \quad s.t. \\ &PA + A^T P - PBR^{-1}B^T P + Q \leq 0, \\ &P \geq 0 \end{aligned} \tag{2.33}$$

where $X(0)$ is the initial value of the state vector. Equivalently the following formulation in terms of P and K can be written as:

$$\begin{aligned} &\min X(0)^T P X(0) \quad s.t. \\ &P(A + BK) + (A + BK)^T P + Q + K^T R K \leq 0, \\ &P \geq 0 \end{aligned} \tag{2.34}$$

where $K = -R^{-1}B^T P$ yields the optimal solution. In [127] it is shown that if instead of the cost function (2.30) its expected value is considered and certain assumptions on the initial conditions of the system are imposed, the above minimization problem reduces to:

$$\begin{aligned} &\min trace(P) \quad \text{s.t.} \\ &P(A + BK) + (A + BK)^T P + Q + K^T R K \leq 0, \\ &P \geq 0 \end{aligned} \tag{2.35}$$

which can be transformed into an LMI optimization problem by introducing new variables $\bar{X} = P^{-1}$, and $\bar{Y} = KP^{-1}$ [127].

The following is another formulation that can be used for this purpose using a semi-definite programming framework [126], namely

$$\begin{aligned}
&\max trace(P) \quad \text{s.t.} \\
&PA + A^T P - PBR^{-1}B^T P + Q \geq 0, \qquad (2.36)\\
&P \geq 0
\end{aligned}$$

where the optimal control law is then selected as $U = -R^{-1}B^T PX$. By using the Schur complement decomposition and given that $R > 0$, this formulation can be translated into an LMI maximization problem.

2.9 Cooperative game theory

In this section, I will provide a general description of the "cooperative game theory" and in Chapter 5, I will modify the formulation introduced here to make it compatible with my specific problem, i.e. consensus seeking problem.

Assume a team of N players with the following dynamical model

$$\dot{x} = Ax + \sum_{i=1}^{N} B^i u^i \qquad (2.37)$$

where x is the state vector of the entire team, u^i is the individual control input, and the matrix A has an arbitrary structure. Each player wants to optimize its own cost

$$J^i = \int_0^T (x^T Q^i x + (u^i)^T R^i u^i) dt \qquad (2.38)$$

in which Q^i and R^i are symmetric matrices and R^i is a PD matrix.

If the players decide to minimize their cost in a non-cooperative manner, a strategy (control input u^i) chosen by the ith player can increase the cost of other players through dynamics of the system that couples the different players together. However, if the players decide to cooperate, the individual costs may be minimized. In other words, if each agent is aware of the others' decisions the agents can reduce their team cost by selecting a suitable cooperative strategy. Hence, in a cooperative strategy depending on which agent requires more resource the resulting minima can be different. The cooperation ensures that the total cost of the team is less than any other non-cooperative optimal solution obtained.

In a cooperative approach it is intuitively assumed that if a set of strategies for the team results in a lower cost for all the members, all the players will switch to that set. Hence, by excluding this situation the set of desired solutions consists of those strategies that if the team strategy changes to another one at least one of the players ends up with a higher cost. In other words, there is no alternative strategy that improves all the members' cost simultaneously. This property can be formally defined by the set of Pareto-efficient solutions as follows.

Pareto efficient strategies [124]: A set of strategies $U^* = [u^{1*}, \ldots, u^{N*}]$ is Pareto-efficient if the set of inequalities

$$J^i(U) \leqslant J^i(U^*), \quad i = 1, \ldots, N$$

with at least one strict inequality does not have a solution for U. The point $J^* = [J^1(U^*), \ldots, J^N(U^*)]$ is called a Pareto solution. This solution is never fully dominated by any other solution.

Now consider the following optimization problem and assume that $Q^i \geq 0$, specifically

$$\min_{u^i \in \mathcal{U}^i} J^i = \int_0^T (x^T Q^i x + (u^i)^T R^i u^i) dt$$

$$\text{s.t.} \quad \dot{x} = Ax + \sum_{i=1}^{N} B^i u^i$$

where $\mathcal{U}^i$ is the set of all strategies for player i. The above is a convex optimization problem. It can be shown that the following set of strategies results in a set of Pareto efficient solutions for this problem. In other words, the solution to the following minimization problem cannot be dominated by any other solution

$$U^*(\alpha) = \arg\min_{U \in \mathcal{U}} \sum_{i=1}^{N} \alpha^i J^i(U) \tag{2.39}$$

where $\alpha \in \mathcal{A}$, $\mathcal{A} = \{\alpha = (\alpha^1, \ldots, \alpha^N) | \alpha^i \geq 0 \text{ and } \sum_{i=1}^{N} \alpha^i = 1\}$, and $\mathcal{U}$ is the set of all strategies for all players. The cost values correspond to the optimal strategies U^* will be $J^1(U^*(\alpha)), \ldots, \ J^N(U^*(\alpha))$. It is worth noting that although this minimization is over the set of strategies $\mathcal{U}$, the controller parameters (matrices) are in fact being optimized. In other words, the control strategies $\mathcal{U}$ are assumed to be in the form of state feedback and the coefficient matrices are obtained through the above optimization problem.

The strategies obtained from the above minimization as well as the optimal cost values, here referred to as the "solutions", are functions of the parameter α. Therefore, the Pareto-efficient solution is in general not unique and the set of these solutions, i.e. Pareto frontier, is denoted by $\mathcal{P}$ which is an edge in the space of possible solutions (cost values), i.e. Ξ. It can be shown that in both infinite horizon and finite horizon cases, the Pareto frontier will be a smooth function of α [124]. Due to the non-uniqueness of Pareto solutions the next step is to decide how to choose one solution among the set of Pareto solutions (or to choose an α from the set of α's). This solution should be selected according to a certain criterion as the final strategy for the team cooperation problem. For this purpose, we need to solve the bargaining problem as defined below.

Bargaining problem [124]: In this problem two or more players have to agree on the choice of some strategies from a set of solutions while they may have conflicting interests over this set. However, the players understand that better outcomes may be achieved through cooperation when compared to the non-cooperative outcome (called threat-point). Some of the well-known axiomatic approaches to this problem are Nash bargaining, Kalai-Smorodinsky, and Egalitarian.

Applying any of the above mentioned methods to the Pareto efficient solutions will yield a unique cooperative solution. Due to the interesting properties of the Nash Bargaining Solution (NBS) such as symmetry and Pareto optimality [124], I invoke this method for obtaining a unique solution among the set of Pareto-efficient solutions obtained above.

Nash Bargaining Solution (NBS) [124]: In this method a point in Ξ, denoted by Ξ^N, is selected such that the product of the individual costs from d is maximal ($d = [d^i]^T$ is the threat-point or the non-cooperative outcome of agents), namely

$$\Xi^N(\Xi, d) = \arg\max_{J\in\Xi} \prod_{i=1}^{N} (d^i - J^i), \quad J = [J^i]^T \in \Xi \text{ with } J \preceq d \tag{2.40}$$

in which d^i's (the threat point) are the cost values calculated by using the non-cooperative solution that is obtained by minimizing the cost in (2.38) individually and constrained to (2.37). It can be shown that the NBS is on the Pareto frontier and therefore the above maximization problem is equivalent to the following problem

$$\alpha^N = \arg\max_{\alpha} \prod_{i=1}^{N} (d^i - J^i(\alpha, U^*)), \quad J \in \mathcal{P} \text{ with } J \preceq d \tag{2.41}$$

in which $J = [J^i]^T$, and where J^i's are calculated by using the set of strategies given in (2.39). By solving the maximization problem (2.41), a unique value for the coefficient α can be found.

Remark 2.3. Theorem 6.10 in [124] can be used to determine the relationship that exists between the coefficients α^i, $i = 1, \ldots, N$ and the achievable improvements in the individual costs due to cooperation in the team. According to this theorem the following relationship holds between the value of the cost functions at the NBS, $(J^{1*}(\alpha^*, U^*), \ldots, J^{N*}(\alpha^*, U^*))$, the threat-point d, and the optimal weight $\alpha^* = (\alpha^{1*}, \ldots, \alpha^{N*})$, that is

$$\begin{aligned} &\alpha^{1*}(d^1 - J^{1*}(\alpha^*, U^*)) = \ldots = \alpha^{N*}(d^N - J^{N*}(\alpha^*, U^*)) \quad \text{or} \\ &\alpha^{j*} = \frac{\prod_{i \neq j} (d^i - J^{i*}(\alpha^*, U^*))}{\sum_{i=1}^{N} \prod_{k \neq i} (d^k - J^{k*}(\alpha^*, U^*))} \end{aligned} \tag{2.42}$$

The expression in (2.42) describes the kind of cooperation that exists among the players. It shows that if during the team cooperation, i.e. minimization of the team cost, a player has improved its cost more, it will receive a lower weight in the minimization scheme (Pareto solution) whereas the one who has not gained a great improvement as a result of participation in the team cooperation receives a greater weight. Therefore, all the players benefit from the cooperation in almost a similar manner, and hence have the incentive to participate in the team cooperation.

Remark 2.4. Selection of the NBS is motivated by the fact that this solution enjoys several appealing properties (axioms). As pointed out in [124], in this method each agent does not need to have information about the utility value or the threat point of other agents. In other words, no "interpersonal comparison" of utility functions is required. Moreover, this solution satisfies four axioms namely, Pareto optimality, symmetry, independence of irrelevant alternatives, and affine transformation invariance, which are all defined in [124].

2.10 Problem statement: consensus in a team of multi-agents

The main goal here is to make the agents' output, or state vector, converge to a common value, which is either determined by the team members or enforced from outside through a supervisor. The output can be velocity, position, or any other state on which the team should have a consensus. This means that $\forall i,j\ X^i \longrightarrow X^j$. In other words, we desire that the team reaches to a consensus in the subspace spanned by the vector $\mathbf{1}$, that is:

$$X_{ss} = [(X^1)^T_{ss}\ \dots\ (X^N)^T_{ss}]^T = [1\ 1\ \dots 1]^T \otimes \xi = \mathbf{1} \otimes \xi$$

where ξ is the final state vector to which the states of all the agents converge and X_{ss} stands for the steady state vector of the entire team.

Figures 2.3 and 2.4 show two examples of a consensus on velocity in a team of aerial vehicles.

Definition 2.3. (consensus to $\mathcal{S}$) [39]: Let S be an orthonormal matrix in $R^{Nn \times m}$. The system (2.21) or in (2.20) achieves consensus to the subspace $\mathcal{S} = span\{S\}$ if $\mathcal{S}$ is a minimal set such that for any initial condition, the state $X(t)$ converges to a point in $\mathcal{S}$.

In this thesis it is assumed that the desired consensus subspace $\mathcal{S}$ is spanned by the unity vector, i.e. $S = \mathbf{1}$.

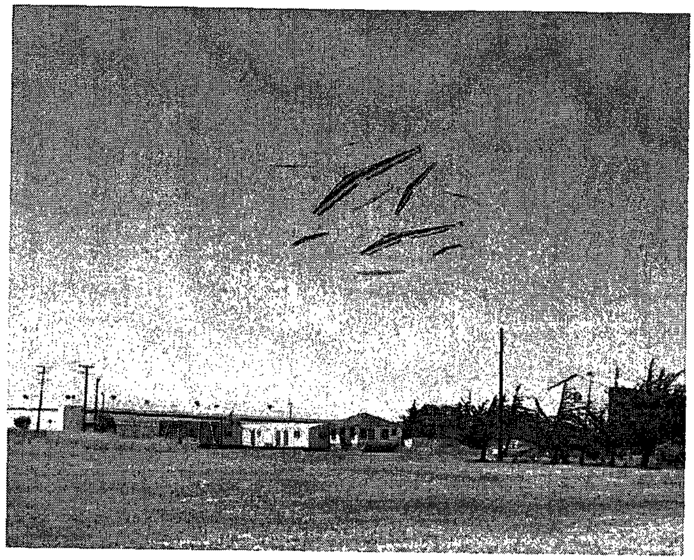


Figure 2.3: Consensus in a team of aerial vehicles (figure borrowed from [2]).

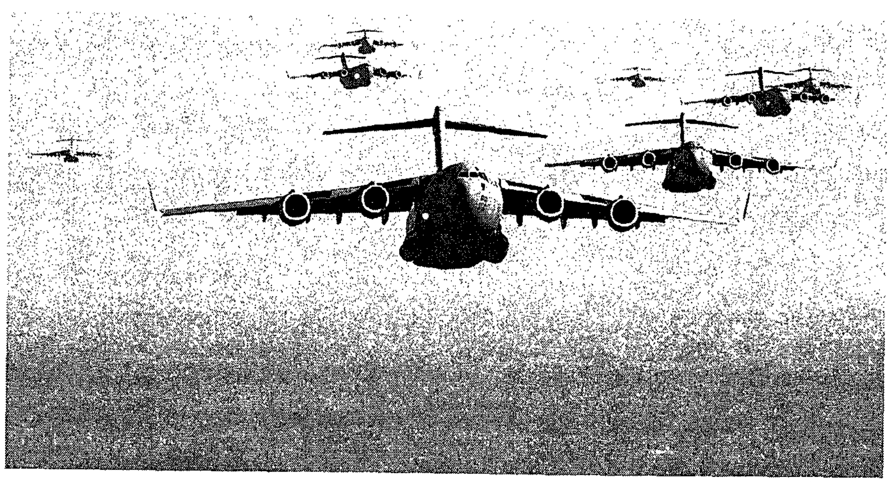


Figure 2.4: Consensus in a team of aircraft (figure borrowed from [3]).

Chapter 3

Semi-decentralized optimal control for team cooperation seeking

The objectives of this thesis are the development and design of controllers for a team of agents that accomplish consensus for agents' output in both LL and MLF architectures. The main goal of the team is to make the agents' output converge over time to a common value, which is either determined by the team members or enforced through a supervisor. Towards this end, the main feature of this research is to introduce synthesis-based protocols that have the advantage of simultaneously addressing several specifications and constraints while guaranteeing optimality of the solution for a team of agents with general linear dynamical models. Modelling of the agents' relative specifications as interaction terms provides a deeper insight into design of "local" and "global" controllers that will address the corresponding "local" and "global" specifications. Therefore, interactions among agents due to information flows are represented through the control channels in characterization of the dynamical model of each agent as discussed in Chapter 2.

Towards this end, in this chapter a semi-decentralized optimal control strategy is designed based on minimization of individual cost functions defined for the team members using local information. Furthermore, it is shown that for a general linear dynamical model of agents by appropriate selection of the cost functions gains through a set of LMIs, the team is guaranteed to remain stable and the controller yields results that satisfy the predefined team goal, i.e. cohesive motion of the team. It is shown that minimization of the proposed individual cost function in the infinite horizon case, results in a control law which is the well-known "average consensus protocol" [13] or its modified version for both the LL and the MLF structures. This implies that by using the present framework and methodology the average consensus protocol is derived in a formal way. In other words, we have introduced a performance index that is minimized by the consensus protocol through the proposed methodology.

Remark 3.1. In the context of the present work, the terms centralized and decentralized can be a source of confusion. Here, a decentralized strategy refers to the case in which the neighbors' information is available for each agent in that set and no exchange of information exists with the agents which are outside the neighboring set. The centralized strategy refers to the availability of the entire team information for each agent.

A summary of the following materials is published in [128]- [132].

3.1 Semi-decentralized optimal control design

3.1.1 Definition of cost functions

To achieve output consensus, the difference among the output of the agents in the team should converge to zero. Towards this end, and due to the connectivity assumption, it suffices to have consensus for the agents in a neighboring set. Hence, the difference between the output of each agent and its neighbors is used as a performance index for each agent. For the LL structure and the agents dynamical equation as given in (2.1) the cost functions for all the agents are defined as shown below

$$d^i = \int_0^T \{\sum_{j\in N^i} [(Y^i - Y^j)^T Q^{ij} (Y^i - Y^j)] + (u^i)^T R^i u^i\} dt$$

$$+ Y^i(T)^T E^i Y^i(T) + (F^i)^T Y^i(T) + G^i \tag{3.1}$$

If the MLF structure is considered, superscript 1 is used to denote the leader, while superscripts $i = 2, \ldots, N$ are used to correspond to the variables of the followers. The leader's cost is selected according to

$$\begin{aligned} d^1 = \int_0^T & \{[(Y^1 - Y^d)^T \Gamma (Y^1 - Y^d)] + \sum_{j\in N^1} [(Y^1 - Y^j)^T Q^{1j} (Y^1 - Y^j)] \\ & + (u^1)^T R^1 u^1\} dt + Y^1(T)^T E^1 Y^1(T) + (F^1)^T Y^1(T) + G^1 \end{aligned} \tag{3.2}$$

The cost functions for the followers are defined as in (3.1). In the above definitions, Q^{ij}, Γ, E^i and R^i are symmetric and PD matrices, F^i is a vector with a proper dimension, and G^i is a scalar. The variables Y^i, u^i are defined in (2.1), T denotes the time horizon over which the cost optimization is performed and Y^d is the desired output. The term $Y^i(T)^T E^i Y^i(T) + F^i Y^i(T) + G^i$ is the "cost to go". This final value can be either zero or non-zero and its structure is similar to that of the value function. For example, if there is no linear term in the value function, the corresponding linear term in the final value should also be zero. It should be noted that the first term in (3.2) is used to ensure that the leader follows its own command, i.e. ($Y^1 \rightarrow Y^d$). The second term incorporates the effects of the difference between the output of the leader and the output of its neighbors.

If certain necessary conditions are satisfied, minimization of the above cost functions is guaranteed. This is further discussed in Subsections 3.1.2 and 3.2.1. If these cost functions are minimized then the consensus will be achieved, i.e. all agents in a neighboring set will reach to the same output vector in steady state. Due to the connectivity of the information graph, any neighboring set has at least one common member with one of the other neighboring sets. Now assume that the member i has a final output vector Y^{i*}. Any other member k is either inside or outside its neighboring set. In the former case, $Y^{k*} \rightarrow Y^{i*}$, in which Y^{k*} is the final output of the kth member. In the latter case, due to the connectivity of the information graph, there is a path between these two members. Hence, the final output of member i is passed on to one of its neighbors which in turn passes it to another member in its own neighboring set until it reaches member k. Therefore, the two members will have the same final output. Hence, for the case of the LL structure all the members decide on a consensus value Y^c, namely: $\forall i, j,\ Y^{i*} \rightarrow Y^{j*} \rightarrow Y^c$. The same discussion applies to the MLF structure.

Remark 3.2. The cost functions introduced in (3.1) and (3.2) are quadratic functions, however as we will see later due to the dependency of each individual cost function on the outputs of other agents, and due to partial availability of information for individual agents, this quadratic optimal control problem cannot be solved by using conventional LQR methods. Specifically, application of these methods to the current problem requires full access to information of the team for each agent which is not available in the present case. Therefore, in order to find the minimum value of the above cost function we have to use the general approach, i.e. to solve the HJB equations [123, 124].

To this end, let us formulate the consensus problem and determine what problems may arise in obtaining the solution to the HJB equations subject to the availability of partial information. This will provide a clear motivation for utilizing the previously introduced interaction terms that will overcome these difficulties. We can choose any of the two previously defined structures, i.e. LL or MLF. I select LL structure for this purpose and later on the consensus problem is solved for both the LL and MLF structures as shown in Subsections 3.2.1 and 3.2.2.

3.1.2 The HJB equations for the consensus problem

Assume a team with an LL structure where the agent's dynamical equations are governed by (2.1) and the agent's cost function is as in (3.1). Based on the discussion in Chapter 2, the HJB equations corresponding to our specific problem will be as follows [123]

$$\begin{aligned}
&-\frac{\partial V^i}{\partial t}(t, X^i) = \min_{u \in \mathcal{U}} \Lambda^i(t, X^i, u^i), \\
&\Lambda^i = (X^i)^T Q^i X^i + k_1^i(t, \hat{X}_i^j) X^i + (u^i)^T R^i u^i + k_2^i(t, \hat{X}_i^j) \\
&+ \frac{\partial V^i}{\partial X^i}(t, X^i)(A^i X^i + B^i u^i) = -\frac{\partial V^i}{\partial t}, \\
&V^i(T) = X^i(T)^T \bar{E}^i X^i(T) + (\bar{F}^i)^T X^i(T) + G^i
\end{aligned} \tag{3.3}$$

in which $Q^i = (c^i)^T \sum_{j \in N^i} Q^{ij} c^i$, $\bar{E}^i = (c^i)^T E^i c^i$, $\bar{F}^i = (c^i)^T F^i$, $k_1^i(t, \hat{X}_i^j) = -2 \sum_{j \in N^i} (c^j X^j(t))^T Q^{ij} c^i$, $k_2^i(t, \hat{X}_i^j) = \sum_{j \in N^i} (X^j(t))^T (c^j)^T Q^{ij} c^j X^j(t)$, $\hat{X}_i^j$ denotes the set of state vectors of all agents $j \in N^i$, and V^i is a value function to be chosen such that the PDE (3.3) is satisfied.

Remark 3.3. Note that equation (3.3) is a function of not only X^i but also X^j, $j \in N^i$. Hence, normally the dynamics of the jth agent should also be considered as a constraint in the HJB equations. However, in that case the solution would be *centralized* in which the control input of each agent depends on the output of *all* the other agents and not necessarily the ones which are in its neighboring set. That is the reason for using only the dynamic constraint of each individual agent in its HJB equation and the other agents' states are assumed as time-varying functions.

In view of the above discussion the minimization of d^i is performed with respect to only $u^i(t, X^i)$ and therefore $u^{i*}(t, X^i) = \arg\{\min_{u^i \in \mathcal{U}^i} \Lambda^i(t, X^i, u^i)\}$, $i = 1, \ldots, N$. In this case optimality refers to finding the minimum value of the cost function (3.1) with the dynamical constraint given by (2.1) while the other agents' control (or states) are treated as being frozen. Therefore, in the context of this chapter optimality refers to an agent-by-agent optimality rather than the global optimality. Moreover, optimality refers to the optimal performance that is achievable within the specific semi-decentralized structure that is introduced in this work. In other words, the objective is not to show that the semi-decentralized approach is optimal as compared to the centralized case, which indeed it is not. The goal is to obtain an optimal solution for the semi-decentralized approach and this is shown to be achieved if a solution to the proposed optimization problem is obtained.

Discussion on the existence of a solution: In the above minimization problem, a choice for V^i may be specified as

$$V^i = \frac{1}{2}(X^i)^T K^i(t) X^i + (g^i(t))^T X^i + \gamma^i \tag{3.4}$$

in which K^i, g^i and γ^i, $i = 1, \ldots, N$ are time-varying parameters to be selected. The reason for this selection of V^i is that the optimal control law is of tracking type and moreover the terms correspond to $V^i(T)$ given in (3.3) have similar structure as V^i defined in (3.13). Then, the HJB equations in (3.3) can be solved if K^i and the vector γ^i can be calculated through the following differential equations

$$-\dot{K}^i = 2Q^i - \tfrac{1}{2}K^iB^i(R^i)^{-1}(B^i)^TK^i + K^iA^i + (A^i)^TK^i, \;\; K^i(T) = 2\bar{E}^i$$
$$-\dot{g}^i = -\tfrac{1}{2}K^i(t)B^i(R^i)^{-1}(B^i)^Tg^i + (A^i)^Tg^i + k_1^i(t, \hat{X}_i^j)^T, \;\; g^i(T) = \bar{F}^i$$

The main issue here is the existence of a solution for these equations. The first equation is a Differential Riccati Equation (DRE) and can be solved if certain conditions are provided by matrices A^i, B^i, c^i. Unfortunately, the term $k_1^i(t, \hat{X}_i^j)^T$ in the second equation is a function of the output of other agents. Although, this term is available for agent i at any time instant due to being in the neighboring set of agent i, the agent i cannot have its value in advance which is needed due to the nature of the equations (a two-point boundary value problem), and hence the second equation does not have a solution since $Y^j, j \in N^i$ are not known for the entire interval $[0, T]$. Hence, in this form there is no solution for g^i. In order to remedy the presence of this unwanted term, the idea of incorporating interaction terms introduced in Chapter 2 is used. These interaction terms compensate the effects of Y^j in the cost function of the ith agent. Moreover, this adopted concept allows one to design a *semi-decentralized* controller based on the available information for each agent. This idea is applied in the following.

Solution of the HJB equations in consensus problem: The HJB equations given in (3.3) have to be modified to properly address the case when the interaction terms are included in the model as given in (2.6). Moreover, the cost functions (3.1) and (3.2) are modified by replacing u^i, u^1 with u_l^i, u_l^1, respectively to have

$$d^i = \int_0^T \{\sum_{j \in N^i} [(Y^i - Y^j)^T Q^{ij} (Y^i - Y^j)] + (u_l^i)^T R^i u_l^i\} dt$$

$$+ Y^i(T)^T E^i Y^i(T) + (F^i)^T Y^i(T) + G^i \tag{3.5}$$

$$\begin{aligned} d^1 = \int_0^T & \{[(Y^1 - Y^d)^T \Gamma (Y^1 - Y^d)] + \sum_{j \in N^1} [(Y^1 - Y^j)^T Q^{1j} (Y^1 - Y^j)] \\ & + (u_l^1)^T R^1 u_l^1\} dt + Y^1(T)^T E^1 Y^1(T) + (F^1)^T Y^1(T) + G^1 \end{aligned} \tag{3.6}$$

Then, the resulting HJB equations will modify to

$$\begin{aligned} \Lambda^i &= (X^i)^T Q^i X^i + k_1^i(t, \hat{X}_i^j) X^i + k_2^i(t, \hat{X}_i^j) + (u_l^i)^T R^i u_l^i \\ &+ \frac{\partial V^i}{\partial X^i}(t, X^i)(A^i X^i + B^i u_l^i(X^i) + B^i u_g^i(Y^j)) = -\frac{\partial V^i}{\partial t} \end{aligned} \tag{3.7}$$

The solution to the HJB equation in (3.7) yields the following differential equations

$$\begin{cases} -\dot{K}^i = 2Q^i - \frac{1}{2} K^i B^i (R^i)^{-1} (B^i)^T K^i + K^i A^i + (A^i)^T K^i, \ K^i(T) = 2\bar{E}^i \\ -\dot{g}^i = [(A^i)^T - \frac{1}{2} K^i(t) B^i (R^i)^{-1} (B^i)^T] g^i + k_1^i(t, \hat{X}_i^j)^T + K^i(t) B^i u_g^i(Y^j), \\ g^i(T) = \bar{F}^i \end{cases} \tag{3.8}$$

The "undesirable" terms, $k_1^i(t, \hat{X}_i^j)$ and $K^i(t) B^i u_g^i(Y^j)$, in the second equation in (3.8) may be cancelled out by a suitable selection and design of the interaction terms. Specifically, one solution to consider is to assume that

$$\begin{aligned} & k_1^i(t, \hat{X}_i^j)^T + K^i(t) B^i u_g^i(Y^j) = 0 \ \Rightarrow \\ & -2 \sum_{j \in N^i} (c^i)^T Q^{ij} c^j X^j(t) + K^i(t) B^i \sum_{j \in N^i} \mathcal{F}^{ij} Y^j = 0 \end{aligned} \tag{3.9}$$

This is reduced to the following equation if $K^i(t)B^i$ is a non-singular matrix,

$$\mathcal{F}^{ij} = 2(K^i(t)B^i)^{-1}(c^i)^T Q^{ij} \tag{3.10}$$

Consequently, one can guarantee that the differential equations in (3.8) have a solution, and therefore a control law may be found to minimize the cost function (3.5). Moreover, using the properties of Riccati equations and assuming that the pair (A^i, B^i) is reachable and (A^i, Ω^i) is observable, where $Q^{ij} = (\Omega^i)^T\Omega^i$, by putting $\dot{K}^i = 0$ in (3.8), we conclude that there exists a PD solution K^i for the Riccati equation, i.e. the first equation in (3.8) [133]. Therefore, a square and non-singular B^i suffices for the existence of a solution to the second differential equation. In applications that are related to the ground vehicles, such as in mobile robots, this assumption is reasonable.

3.2 Agents with double integrator dynamical model

3.2.1 Consensus problem in a Leaderless (LL) multi-vehicle team

Consider a team of agents with a structure similar to Figure 2.1(a), the agents dynamics as given by (2.20), and the cost function as specified in (3.5) by replacing Y^i with v^i. Matrix Q^{ij} should be selected such that the pair $(0, \sqrt{Q^{ij}})$ would be observable. The corresponding HJB equation is then similar to (3.3) by assuming that $X^i = [r^i \; v^i]^T$ and Λ^i is defined as follows

$$\Lambda^i(t, X^i, u^i) = \sum_{j\in N^i} (v^i - v^j)^T Q^{ij}(v^i - v^j) + (u^i_l)^T R^i u^i_l$$

$$+\frac{\partial V^i}{\partial v^i}(t,X^i)(u_l^i+\sum_{j\in N^i}\mathcal{F}^{ij}v^j)+\frac{\partial V^i}{\partial r^i}(t,X^i)v^i \tag{3.11}$$

in which V^i at any time t has the following value:

$$\begin{aligned} V^i(t) &= \min_{u_l^i}\int_t^T [\sum_{j\in N^i}[(v^i-v^j)^T Q^{ij}(v^i-v^j)]+(u_l^i)^T R^i u_l^i]dt \\ &+ v^i(T)^T E^i v^i(T)+(F^i)^T v^i(T)+G^i \end{aligned} \tag{3.12}$$

and a choice of V^i may be specified as

$$V^i=\frac{1}{2}(v^i)^T K^i(t)v^i+\gamma^i(t) \tag{3.13}$$

in which K^i,γ^i $i=1,\ldots,N$ are time-varying parameters to be defined.

Existence of a solution: To guarantee existence of a solution to the above minimization problem, controllability of the open-loop system should be verified [133], i.e. the pair (A^i,B^i) in (2.6) should be controllable. Based on the representation of the system given in (2.20), the required matrices in this case are $A^i=\begin{bmatrix} 0_{2\times2} & I_{2\times2} \\ 0_{2\times2} & 0_{2\times2}\end{bmatrix}$ and $B^i=\begin{bmatrix} 0_{2\times2} \\ I_{2\times2}\end{bmatrix}$. However, since only the dynamics of v^i appear in the cost function (3.5) and the dynamics of r^i do not have any effect on the v^i dynamics, to verify the controllability condition we need to only consider the dynamics of v^i. That is, we have

$$A^i=0_{2\times2},\ B^i=I_{2\times2},\ i=1,\ldots,N \tag{3.14}$$

This pair is controllable and so guarantees the existence of a solution to the corresponding HJB equations. Furthermore, based on the definition of optimality provided previously, the presence of the term $\sum_{j\in N^i}\mathcal{F}^{ij}v^j$ in (2.20) does not have any effect on the matrices that are involved in the controllability condition. The above reasoning applies to both the LL and the MLF structures with integrator dynamics of agents.

In order to have a solution to the PDEs in (3.11), it can be shown that K^i and the interaction terms should be computed according to the following lemma.

Lemma 3.1. Assume that an LL team of mobile robots whose dynamics are governed by the double integrator equations operate subject to interactions among vehicles based on the neighboring sets as given by (2.20) for an undirected and connected network structure. The interaction coefficient terms and the control law proposed below will minimize the cost function (3.5) and also guarantee alignment of the team of vehicles (consensus over the velocity), where

$$u_g^{i*} = \sum_{j \in N^i} \mathcal{F}^{ij} v^j, \;\; \mathcal{F}^{ij} = 2K^i(t)^{-1} Q^{ij}, \;\; \forall i, j = 1, \dots, N \tag{3.15}$$

$$u_l^{i*} = -\frac{1}{2}(R^i)^{-1} K^i(t) v^i, \;\; i = 1, \dots, N \tag{3.16}$$

u_g^{i*}, u_l^{i*} stand for the optimal values of u_g^i, u_l^i, respectively and K^i satisfies the DRE

$$-\dot{K^i} = 2|N^i| Q^{ij} - \frac{1}{2} K^i (R^i)^{-1} K^i, \;\; K^i(T) = 2E, \;\; i = 1, \dots, N \tag{3.17}$$

Proof: The details are given in Appendix A.

Remark 3.4. It can be seen from (3.16) that the control law u_l^{i*} depends *only* on the state of agent i, and the term u_g^{i*} ensures that the effects of neighboring agents on the dynamics of agent i are present. This provides a semi-decentralized control protocol according to information exchanges that are within the neighboring sets.

The next theorem provides important properties of the proposed control strategy.

Theorem 3.1. **a) Consensus Protocol:** For the team of vehicles described in Lemma 3.1 under the condition of infinite horizon scenario (i.e., $T \longrightarrow \infty$) and for an undirected and connected network structure, the combined control law reduces to the following well-known average consensus (agreement) protocol, i.e.

$$u^{i*}(v^i, v^j) = u_l^{i*}(v^i) + u_g^{i*}(v^j) = \Gamma^i(v^i - \frac{\sum_{j \in N^i} v^j}{|N^i|}),$$

$$\Gamma^i = -\frac{1}{2}(R^i)^{-1}K^i, \quad i = 1, \ldots, N \tag{3.18}$$

b) Stability: The above protocol furthermore guarantees consensus on a constant common value, v^c, in a globally asymptotic manner, i.e.

$$v^i \rightarrow v^j \rightarrow v^c, \quad as\ t \rightarrow \infty, \quad \forall i, j = 1, \ldots, N \tag{3.19}$$

c) Consensus Value: The consensus value achieved by the team will be the following

$$v^c = (w_1 w_1^T + w_2 w_2^T) Avg(v(0)) \tag{3.20}$$

in which w_1 and w_2 are the eigenvectors of the matrix Γ^i, v^c is the consensus vector and $v(0)$ is the concatenation vector of the initial velocities of agents, i.e. $v(0) = [(v^1(0))^T \ \ldots \ (v^N(0))^T]^T$.

Proof: The details are given in Appendix A.

Remark 3.5. It follows readily from Theorem 3.1 that in the infinite horizon case the control law obtained by the proposed method is the well-known "average consensus protocol" that is proposed in [13].

3.2.2 Consensus problem in a Modified Leader-Follower (MLF) multi-vehicle team

Now consider a team of agents with an MLF structure as described in Figure 2.1(b), dynamics of the agents as given in (2.20), and the cost functions for the leader and followers as specified according to (3.6) and (3.5), respectively by replacing Y^i with v^i. The HJB equations are reduced to the following

$$-\frac{\partial V^i}{\partial t}(t, X^i) = \min_{u \in \mathcal{U}} \Lambda^i(t, X^i, u^i), \tag{3.21}$$

$$\begin{aligned}\Lambda^1(t, X^1, u^1) = &\sum_{j \in N^1} (v^1 - v^j)^T Q^{1j}(v^1 - v^j) + (v^1 - v^d)^T \Gamma (v^1 - v^d) \\ &+ (u_l^1)^T R^1 u_l^1 + \frac{\partial V^1}{\partial v^1}(u_l^1 + \sum_{j \in N^1} \mathcal{F}^{1j} v^j),\end{aligned} \tag{3.22}$$

$$\begin{aligned}\Lambda^i(t, X^i, u^i) = &\sum_{j \in N^i} (v^i - v^j)^T Q^{ij}(v^i - v^j) + (u_l^i)^T R^i u_l^i \\ &+ \frac{\partial V^i}{\partial v^i}(u_l^i + \sum_{j \in N^i} \mathcal{F}^{ij} v^j), \; i = 2, \ldots, N\end{aligned} \tag{3.23}$$

and a choice of V^i and V^1 is specified as

$$\begin{aligned}V^1 &= \frac{1}{2}(v^1)^T K^1(t) v^1 + (g^1(t))^T v^1 + \gamma^1(t), \\ V^i &= \frac{1}{2}(v^i)^T K^i(t) v^i + \gamma^i(t), \; i = 2, \ldots, N\end{aligned} \tag{3.24}$$

in which K^i and γ^i, $i = 1, \ldots, N$, and g^1 are time-varying parameters to be chosen.

In order to find a solution to the PDEs in (3.21)-(3.23), u^i and the interaction terms $\mathcal{F}^{ij}$ should be selected according to the following lemma.

Lemma 3.2. Assume a team of mobile robots whose dynamics are governed by the double integrator equations as given in (2.20) having an MLF structure.

The interaction terms and the control laws proposed below will provide a solution to the HJB equations in (3.21)-(3.23) and simultaneously minimize the cost functions (3.5) and (3.6) to guarantee the alignment of the vehicles, where

$$u_g^{i*} = \sum_{j \in N^i} \mathcal{F}^{ij} v^j = \sum_{j \in N^i} 2(K^i)^{-1} Q^{ij} v^j, \ \forall i \tag{3.25}$$

$$u_l^{i*} = -\frac{1}{2}(R^i)^{-1} K^i(t) v^i, \ i = 2, \dots, N \tag{3.26}$$

$$u_l^{1*} = -\frac{1}{2}(R^1)^{-1}(K^1(t)v^1 + g^1(t)) \tag{3.27}$$

and where the leader's parameter g^1 and the DREs for determining K^i satisfy

$$-\dot{K^i} = 2|N^i|Q^{ij} - \frac{1}{2}K^i(R^i)^{-1}K^i, \ K^i(T) = 2E^i, \ i = 2, \dots, N \tag{3.28}$$

$$-\dot{K^1} = 2(|N^1|Q^{1j} + \Gamma) - \frac{1}{2}K^1(R^1)^{-1}K^1, \ K^1(T) = 2E^1 \tag{3.29}$$

$$\dot{g^1} = 2\Gamma v^d(t) + \frac{1}{2}K^1(R^1)^{-1}g^1, \ g^1(T) = F^1 \tag{3.30}$$

Proof: The details are provided in Appendix A.

The discussions on the existence of a solution are similar to those presented in Subsection 3.2.1. The next theorem provides important properties of the proposed control strategy.

Theorem 3.2. a) Modified Consensus Protocol: For the team of vehicles described in Lemma 3.2 and associated with the infinite horizon scenario (i.e., $T \longrightarrow \infty$), the combined control law reduces to the modified agreement protocol for the MLF structure. The protocol for a follower is given by

$$u^{i*}(v^i, v^j) = u_l^{i*}(v^i) + u_g^{i*}(v^j) = \Gamma^i(v^i - \frac{\sum_{j \in N^i} v^j}{|N^i|}) \tag{3.31}$$

and for the leader is described by

$$u^{1*}(v^1, v^j) = \Gamma^1(v^1 - \frac{\sum_{j\in N^1} v^j}{|N^1|}) + \beta^1(v^1 - v^d) \tag{3.32}$$

$$\Gamma^i = -2(K^i)^{-1}|N^i|Q^{ij}, \ i = 1, \dots, N, \ \ \beta^1 = -2(K^1)^{-1}\Gamma \tag{3.33}$$

b) Stability: The above protocol is stabilizing, i.e. the error dynamics of the entire team is asymptotically stable, implying that

$$e^i(t) = v^i(t) - v^d \to 0 \ \text{ as } t \to \infty, \quad i = 1, \dots, N \tag{3.34}$$

Proof: The details are provided in Appendix A.

It is seen that the second term of the leader's control input in (3.32) guarantees command tracking even if the first term is not included. However, in Chapter 4 it will be shown that the first part keeps and maintains the team cohesion and guarantees that none of the followers are "lost" without affecting the others' and specifically the leader's behavior.

3.3 Agents with linear dynamical model

3.3.1 Consensus problem in an MLF multi-vehicle team

Assume that the agents are ground vehicles governed by linear dynamics as given in (2.21). Now consider an MLF structure as shown in Figure 2.1(b) and the cost functions for the followers and the leader as specified according to (3.5) and (3.6) by replacing Y^i with v^i, respectively. The HJB equations will consequently reduce to the following

$$-\frac{\partial V^i}{\partial t}(t, v^i) = \min_{u^i \in \mathcal{U}^i} \Lambda^i(t, v^i, u^i) \tag{3.35}$$

$$\begin{aligned}\Lambda^1(t,v^1,u^1) &= \sum_{j\in N^1}(v^1-v^j)^TQ^{1j}(v^1-v^j)+(v^1-v^d)^T\Gamma(v^1-v^d)\\ &+(u_l^1)^TR^1u_l^1+\frac{\partial V^1}{\partial v^1}(t,v^1)(A^1v^1+B^1(u_g^1+u_l^1)),\\ V^1(T,v^1) &= v^1(T)^TE^1v^1(T)+(F^1)^Tv^1(T)+G^1(T)\end{aligned} \tag{3.36}$$

$$\begin{aligned}\Lambda^i(t,v^i,u^i) &= \sum_{j\in N^i}(v^i-v^j)^TQ^{ij}(v^i-v^j)+(u_l^i)^TR^iu_l^i\\ &+\frac{\partial V^i}{\partial v^i}(t,v^i)(A^iv^i+B^i(u_g^i+u_l^i)),\\ V^i(T,v^i) &= v^i(T)^TE^iv^i(T)+(F^i)^Tv^i(T)+G^i(T),\ i=2,\ldots,N\end{aligned} \tag{3.37}$$

Consequently, a choice of V^i for the followers and the leader may be specified according to

$$V^i=\frac{1}{2}(v^i)^TK^i(t)v^i+\gamma^i(t),\ i=2,\ldots,N \tag{3.38}$$

$$V^1=\frac{1}{2}(v^1)^TK^1(t)v^1+(g^1(t))^Tv^1+\gamma^1(t) \tag{3.39}$$

where K^i,γ^i, and g^1 are the time-varying parameters to be specified.

Existence of a solution: To guarantee existence of a solution to the above minimization problem the notion of reachability should be verified for the open-loop system [133]. Based on the representation of system (2.21), and given that only the dynamics of v^i appears in the cost function (3.5), to verify the reachability condition we need to only consider the dynamics of v^i. Furthermore, based on the definition of optimality given previously, where the other agents dynamics are considered as time-varying functions only, the presence of the term $B^i\sum_{j\in N^i}\mathcal{F}^{ij}Y^j$ in (2.21) does not have any effect on the matrices that are involved in the reachability condition. Therefore, due to non-singularity of the matrix B^i, the pair (A^i,B^i) is always reachable, and so existence of a solution is always guaranteed.

Solution of the corresponding HJB equations is now provided in the following lemma.

Lemma 3.3. Assume a team of agents whose dynamics are governed by equation (2.21) with the pair (A^i, Ω^i) being observable, where $Q^{ij} = (\Omega^i)^T \Omega^i$, for an MLF structure. The leader is aware of the desired command specifications and requirements while the followers operate subject to interactions among agents based on the neighboring sets. The interaction terms and the control law proposed below will provide a solution to the HJB equations in (3.36) and (3.37) and simultaneously minimize the cost function (3.5) for the followers and the cost function (3.6) for the leader and guarantee a consensus achievement with the consensus state of $v^i = v^d, \ \forall i$, where

$$u_g^{i*} = \sum_{j \in N^i} \mathcal{F}^{ij} v^j = \sum_{j \in N^i} 2(K^i B^i)^{-1} Q^{ij} v^j, \quad i = 1, \ldots, N \tag{3.40}$$

$$u_l^{i*} = -\frac{1}{2}(R^i)^{-1}(B^i)^T K^i(t) v^i, \ i = 2, \ldots, N \tag{3.41}$$

$$u_l^{1*} = -\frac{1}{2}(R^1)^{-1}(B^1)^T (K^1(t) v^1 + g^1(t)) \tag{3.42}$$

and the leader's parameter g^1 and the Riccati equations for determining K^i satisfy:

$$\begin{aligned} &-\dot{K^i} = 2|N^i| Q^{ij} - \frac{1}{2} K^i B^i (R^i)^{-1} (B^i)^T K^i + (A^i)^T K^i + K^i A^i, \\ &K^i(T) = 2E^i, \ i = 2, \ldots, N \end{aligned} \tag{3.43}$$

$$\begin{aligned} &-\dot{K^1} = 2(|N^1| Q^{1j} + \Gamma) + (A^1)^T K^1 + K^1 A^1 - \frac{1}{2} K^1 B^1 (R^1)^{-1} (B^1)^T K^1, \\ &K^1(T) = 2E^1 \end{aligned} \tag{3.44}$$

$$\dot{g^1} = 2\Gamma v^d(t) + (\frac{1}{2} K^1 B^1 (R^1)^{-1} (B^1)^T - (A^1)^T) g^1, \quad g^1(T) = F^1 \tag{3.45}$$

Proof: The details are provided in Appendix A.

The next theorem provides the stability property of the closed-loop team of agents as well as the behavior of the team in achieving consensus.

Theorem 3.3. a) Modified Consensus Protocol: For the team of agents described in Lemma 3.3 and associated with the infinite horizon scenario (i.e., when $T \longrightarrow \infty$), the combined control law reduces to a modified average consensus protocol (agreement protocol) for the MLF structure. The protocol for the followers and the leader are given by

$$u^{i*}(v^i, v^j) = u_l^{i*}(v^i) + u_g^{i*}(v^j) = \Gamma^i(v^i - \frac{\sum_{j \in N^i} v^j}{|N^i|}) + \beta^i v^i, \ i = 2, \dots, N \tag{3.46}$$

$$\begin{aligned} u^{1*} &= u_l^{1*}(v^1) + u_g^{1*}(v^j) = \Gamma^1(v^1 - \frac{\sum_{j \in N^1} v^j}{|N^1|}) + \alpha^1(v^1 - v^d) \\ &+ \beta^1 v^1 - (K^1 B^1)^{-1}(A^1)^T g^1 \end{aligned} \tag{3.47}$$

in which $\alpha^1 = -2(K^1 B^1)^{-1}\Gamma$, $\Gamma^i = -2(K^i B^i)^{-1}|N^i| Q^{ij}$, and for $\forall i$, $\beta^i = -(K^i B^i)^{-1}(K^i A^i + (A^i)^T K^i)$.

b) Stability: The above protocol is stabilizing, i.e. the error dynamics of the entire team is asymptotically stable, implying that

$$e^i = v^i - v^d \rightarrow 0 \quad as \ t \rightarrow \infty, \quad i = 1, \dots, N \tag{3.48}$$

if parameters Γ, and R^1 in cost function (3.6) and matrix K^1 in (3.44) are determined appropriately and such that the following set of LMIs in terms of

Γ, K^1, Z^1 $(R^1 = -\frac{1}{2}(B^1)^T K^1 (Z^1)^{-1} K^1 B^1)$ are satisfied

$$\begin{cases} \Upsilon + \Upsilon^T > 0, \quad \Upsilon = 2L \otimes Q^{ij} + A^T K + 2G \\ (A^1)^T K^1 + K^1 A^1 + Z^1 + 2(|N^1| Q^{1j} + \Gamma) = 0 \\ Z^1 = -\frac{1}{2} K^1 B^1 (R^1)^{-1} (B^1)^T K^1 \\ \Gamma > 0, \ K^1 > 0, \ Z^1 < 0 \end{cases} \tag{3.49}$$

where $K = Diag\{K^i, i = 1, \ldots, N\}$, $A = Diag\{A^i, i = 1, \ldots, N\}$, and $G = Diag\{\Gamma, 0, \ldots, 0\}$.

Proof: The details are provided in Appendix A.

Remark 3.6. It follows readily from Theorem 3.3 that in the infinite horizon case the control law obtained by the proposed methodology is a modified version of the well-known "average consensus protocol" [13].

3.3.2 Consensus problem in an LL multi-vehicle team

In this case the agents dynamics are governed by (2.21) and the cost function for all the agents is given by (3.5) in a LL structure similar to Figure 2.1(a). The HJB equations defined previously will reduce to (3.35) and (3.37). Similar to the previous discussions and in order to have a decentralized solution, the minimization is performed with respect to v^i only, and a choice for V^i may be specified as in (3.38). The existence of a solution to this problem follows along the similar lines as those invoked for the MLF structure. As a matter of fact, these attributes are invariant under topological changes, that is they are valid for both the leaderless or the leader-follower architectures.

Based on the choice of V^i as in (3.38) and in order to have a solution to the PDEs in (3.35) and (3.37), it can be shown that K^i and the interaction terms should be computed according to the following lemma.

Lemma 3.4. Assume a team of agents whose dynamics are governed by equation (2.21) for an LL structure. The team operates subject to interactions among vehicles based on the neighboring sets. The interaction terms and the control law proposed below will provide a solution to the HJB equations in (3.35) and (3.37) and simultaneously minimize the cost function (3.5) and guarantee consensus achievement with the consensus state of $Y^i = v^i = v^c$, $\forall i$, with

$$u_g^{i*} = \sum_{j \in N^i} \mathcal{F}^{ij} v^j = \sum_{j \in N^i} 2(K^i B^i)^{-1} Q^{ij} v^j, \quad i = 1, \ldots, N \tag{3.50}$$

$$u_l^{i*} = -\frac{1}{2}(R^i)^{-1}(B^i)^T K^i(t) v^i, \; i = 1, \ldots, N \tag{3.51}$$

$$-\dot{K}^i = 2|N^i|Q^{ij} - \frac{1}{2}K^i B^i (R^i)^{-1}(B^i)^T K^i + (A^i)^T K^i + K^i A^i, \; K^i(T) = 2E^i \tag{3.52}$$

Proof: Follows along the similar lines as in the proof of Lemma 3.3, and is therefore omitted.

The next theorem provides the features of the proposed control strategy in terms of the already well-known behavior of cooperative teams as well as the requirement of guaranteeing team stability.

Theorem 3.4. a) Consensus Protocol: For the team of agents described in Lemma 3.4 and associated with the infinite horizon scenario (i.e., $T \longrightarrow \infty$), the combined control law reduces to the modified average consensus protocol (agreement protocol) for the LL structure. The protocol for the agents is given by

$$u^{i*}(v^i, v^j) = u_l^{i*}(v^i) + u_g^{i*}(v^j) = \Gamma^i\left(v^i - \frac{\sum_{j \in N^i} v^j}{|N^i|}\right) + \beta^i v^i \tag{3.53}$$

where $\Gamma^i = -2(K^i B^i)^{-1}|N^i|Q^{ij}$, and $\beta^i = -(K^i B^i)^{-1}(K^i A^i + (A^i)^T K^i)$, $i = 1, \ldots, N$.

b) Stability: The above protocol furthermore guarantees consensus on a constant common value, v^c, in a globally asymptotic manner, i.e.

$$v^i \to v^j \to v^c, \quad \text{as } t \to \infty, \quad \forall i, j = 1, \dots, N \tag{3.54}$$

if A^i satisfies the condition $(A^i)^T K^i + K^i A^i \geq 0$ with A^i having at least one zero eigenvalue.

c) Consensus Value: The consensus value achieved by the team, i.e. v^c, is in the null space of $(A^i)^T K^i$.

Proof: The details are provided in Appendix A.

3.4 Simulation results

3.4.1 Double integrator dynamical model

In this subsection, simulation results are presented for both finite and infinite horizon scenarios for the LL and the MLF structures. To obtain numerical solutions of the DRE in the finite horizon case, the Backward Differentiation Formula (BDF) that is described in [134] is used. The simulations are conducted for a team of four mobile robots with dynamical equations given in (2.20). Without loss of generality, the topology is assumed to be a ring topology. The objective for the team is to ensure that all agents have the same velocity in the steady state. Results shown below are conducted to capture the average behavior of the proposed control strategies through Monte Carlo simulations. The *average* team response due to 30 different *randomly* selected initial conditions are presented.

In Figures 3.1(a), 3.1(b) the x and y components of v^i, i.e. v^i_x, v^i_y are shown for the LL structure in the infinite horizon case. Figure 3.1(c) illustrates the actual average path trajectories generated by the vehicles in the $x-y$ plane. It may be observed that the vehicles are aligned and move together after the transients have died out. Figures 3.2(a), 3.2(b), and 3.2(c) depict the same trajectories but now for the finite horizon scenario.

For the MLF structure, simulations are now conducted for the same configuration as above, however the objective is to ensure that the team members have the same velocity as the desired value specified by the leader, that is v^d. In Figures 3.3(a), 3.3(b) the x and y components of v^i, i.e. v_x^i, v_y^i are shown for the finite horizon scenario. The presence of transients in the final state of the variables is due to the finite horizon formulation of the optimal control problem (a two-point boundary value problem). The desired velocity (leader command) is chosen as $v^d = [3 \quad 4]^T (m/s)$. Figure 3.3(c) shows the actual path trajectories generated by the vehicles in the $x-y$ plane. Figures 3.4(a), 3.4(b), and 3.4(c) depict the same trajectories but now for the infinite horizon scenario. In all the simulations the parameters selected are $Q^{ij} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix}\right], R^i = I_{2\times 2}, |N^i| = 2,\ E^i = 0.5 I_{2\times 2},\ \Gamma = \left[\begin{smallmatrix} 10 & 0 \\ 0 & 40 \end{smallmatrix}\right]$, and the random initial conditions, i.e. $X_0^i = [(r^i(0))^T \ (v^i(0))^T]^T$, $\forall i$, for the Monte Carlo simulations are considered as

$X_0^1 = [r(0,1) \ \ r(0,1) \ \ r(-5,0) \ \ r(-6,-1)]^T,$

$X_0^2 = [r(1,2) \ \ r(1,2) \ \ r(0,5) \ \ r(-5,5)]^T,$

$X_0^3 = [r(2,3) \ \ r(2,3) \ \ r(5,10) \ \ r(1,11)]^T,$

$X_0^4 = [r(3,4) \ \ r(3,4) \ \ r(-10,-5) \ \ r(-6,4)]^T,$

for the LL structure and

$X_0^1 = [r(0,15) \ \ r(0,25) \ \ r(-5,0) \ \ r(-6,-1)]^T,$

$X_0^2 = [r(15,30) \ \ r(25,50) \ \ r(0,5) \ \ r(-5,5)]^T,$

$X_0^3 = [r(30,45) \ \ r(50,75) \ \ r(5,10) \ \ r(1,11)]^T,$

$X_0^4 = [r(45,60) \ \ r(75,100) \ \ r(-10,-5) \ r(-6,4)]^T,$

for the MLF structure, where $r(x, y)$ stands for a random variable in the interval $[x, y]$. Also, for the LL structure the time horizon is selected as $T = 6$ and for the MLF structure $T = 15$.

In order to provide a better insight into the controller performance for different cases considered above, the average cost values obtained by running the Monte Carlo simulations corresponding to each team structure and design assumptions are provided in Table 3.1. We can conclude that the finite horizon design scenario results in a higher average cost as compared to the infinite horizon design scenario. Note that the average velocities corresponding to the LL structure are $[-1.46\ 0.108]^T$ for the infinite horizon scenario (and are $[-1.35\ 0.422]^T$ for the results corresponding to the finite horizon scenario).

3.4.2 Linear dynamical model

In this subsection, simulation results are presented for an infinite horizon scenario for the LL and the MLF structures associated with a linear model of agents (as given in (2.21)). The simulations are conducted for a team of four vehicles where the objective for the team is to ensure that all agents have the same velocity in steady state. Without loss of any generality, and for sake of

Table 3.1: The mean performance index corresponding to different team structures and control design assumptions (FH and IH stand for the finite and infinite horizon scenarios, respectively).

	Average performance index for the Monte Carlo simulations			
Team structure	Agent 1	Agent 2	Agent 3	Agent 4
LL (FH)	2488.2	2163.2	4008.5	4510.7
LL (IH)	1963.7	1716.1	3794.3	4198.4
MLF (FH)	231990	16964	20096	19287
MLF (IH)	222820	15899	17264	16822

only simulations the topology is assumed to be a ring. For the MLF structure, the objective is to ensure that the team members have the same velocity as the desired value specified by the leader, that is $v^d(t)$. In Figures 3.5(a) and 3.5(b), the x and y components of v^i are shown for $i = 1, \ldots, 4$. Figure 3.5(c) depicts the actual path trajectories generated by the vehicles in the $x - y$ plane. It may be concluded that vehicles are aligned and move together with the same desired velocity as the command provided by the leader. For simulations, the command is assumed to be a pulse-like signal with a duration of 0.4 *sec* and its value switches between $v^d = [3 \;\; 4]^T$ and $v^d = [5 \;\; -1]^T$. The initial state of the vehicles, i.e. $X_0^i = [(r^i(0))^T \; (v^i(0))^T]^T$, $\forall i$, are selected at $X_0^1 = [0.6 \;\; 1 \;\; 5 \;\; 3]^T$, $X_0^2 = [2 \;\; 1 \;\; -5 \;\; -4]^T$, $X_0^3 = [0.4 \;\; 3 \;\; -1 \;\; -2]^T$, $X_0^4 = [2 \;\; 0 \;\; 3 \;\; 4]^T$, and the other simulation parameters are chosen as $Q^{ij} = 100 I_{2\times 2}$, $|N^i| = 2$, and for $i = 2, 3, 4$, $R^i = 0.01 I_{2\times 2}$. The parameters corresponding to the agents models are chosen as $\bar{A}^i = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 6 \end{smallmatrix}\right), A^i = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$, and $B^i = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 6 \end{smallmatrix}\right)$. The matrices R^1, K^1, Γ are found by solving the set of LMIs in (3.49) as: $R^1 = 10^{-5}\left(\begin{smallmatrix} 3.28 & -0.027 \\ -0.027 & 5.13 \end{smallmatrix}\right)$, $K^1 = \left(\begin{smallmatrix} 2.013 & -0.39 \\ -0.39 & 0.323 \end{smallmatrix}\right)$, $\Gamma = 10^4\left(\begin{smallmatrix} 1.1879 & 0 \\ 0 & 1.1987 \end{smallmatrix}\right)$.

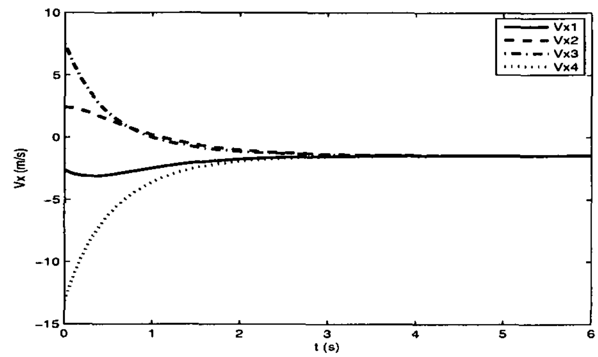
For the LL structure, simulations are conducted for the same configuration as above, however the objective is to ensure that the team members have the same velocity which is not predefined, but should be in the null space of the matrix $(A^i)^T K^i$. In Figures 3.6(a) and 3.6(b), the x and y components of v^i are shown for $i = 1, \ldots, 4$. Figure 3.6(c) depicts the actual path trajectories generated by the vehicles in the $x - y$ plane. The initial state of the vehicles and the other simulation parameters are all the same as in the MLF case. Also, the parameters corresponding to the model are chosen the same as in MLF structure except for the A^i matrix which is selected as $\left(\begin{smallmatrix} 2 & 4 \\ 1 & 2 \end{smallmatrix}\right)$. The controller gain $K^i = \left(\begin{smallmatrix} 4.45 & -0.79 \\ -0.79 & 0.59 \end{smallmatrix}\right)$ is obtained from solving the Riccati equation for the above configuration. From the simulation results, it may be clearly concluded that the vehicles are aligned and move together with the same desired velocity $v^c = [-0.5 \quad -4.3]^T \in \text{Null}((A^i)^T K^i)$.

3.5 Conclusions

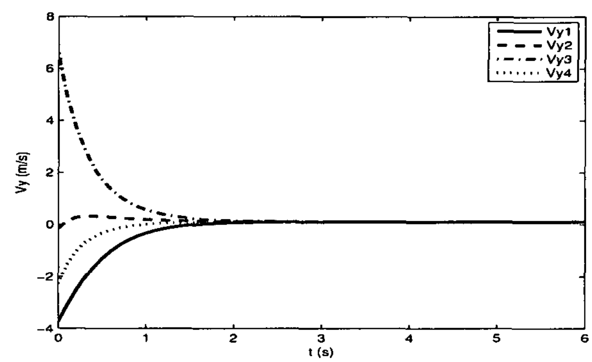
The problem of cooperation in a team of unmanned systems with the goal of consensus seeking was considered for both LL and MLF structures. A semi-decentralized optimal control strategy was designed for a team of agents using minimization of agents' individual performance cost functions subject to partial availability of local information. An unexpected and interesting outcome of the proposed theoretical work is that in an infinite horizon scenario it is shown formally that the control law results in either the well-known "average consensus protocol" strategy or a modified version of it [13] for both the LL and MLF structures. In other words, a performance index is introduced that is minimized by the consensus protocol through the proposed methodology. Corresponding to the MLF structure it was assumed that only the leader is aware of the desired command requirements and specifications and that there is a corrective feedback from the followers to the leader when agents are connected through a prespecified topology.

One of the contributions of the present work compared to the synthesis methods introduced in [9,39,135] is in introduction of interaction terms in dynamical model of agents to describe and characterize the interconnections and information exchanges among agents. This novel modelling approach provides a framework in which local and global control requirements may be partially decoupled. Another advantage of the proposed methodology is robustness of the team to uncertainties and faults in the leader or followers and adaptability of team members to these unanticipated situations as will be discussed in

Chapter 4. Moreover, while optimality of the solution is guaranteed, given that the optimal control is a multi-objective framework, the proposed method has the added potential advantage of being capable of accommodating other additional specifications, e.g. new timing constraints or limited control input availability.



(a)



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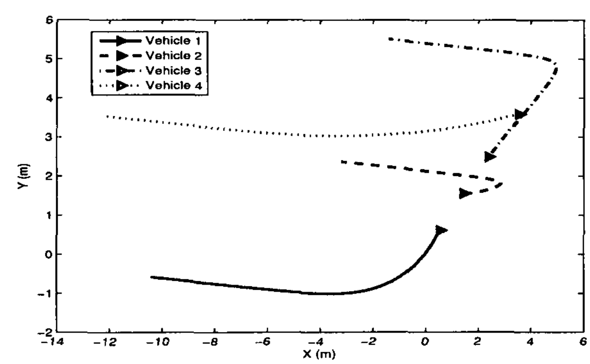
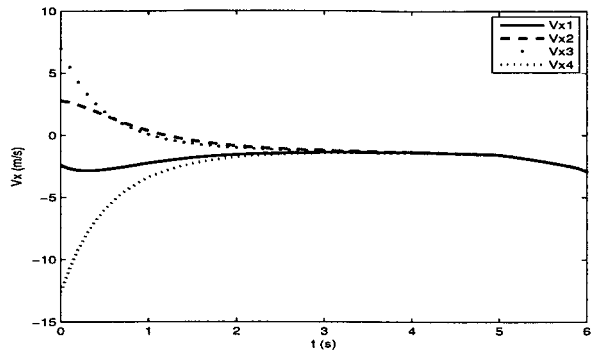


(c)

Figure 3.1: (a) The x-component of the average velocity profile, (b) the y-component of the average velocity profile, and (c) the $x - y$ path trajectories of an LL team of four agents (Monte Carlo simulation runs) resulting from the optimal control strategy in the infinite horizon scenario.



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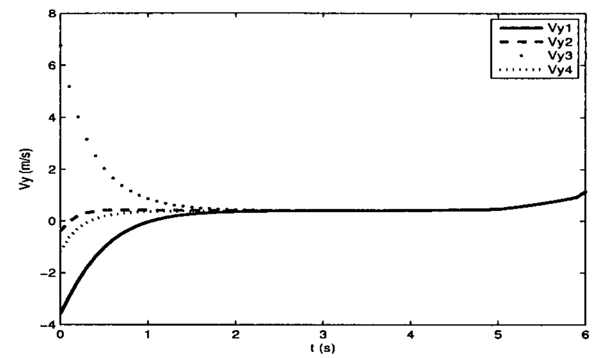

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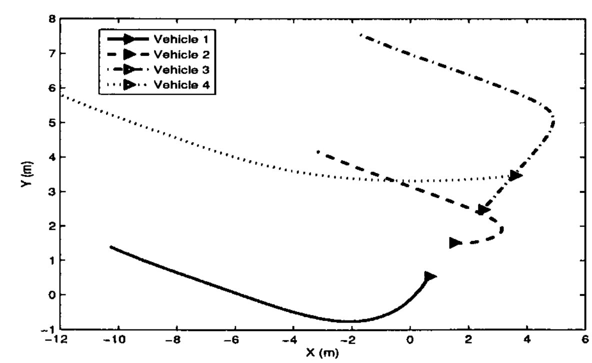
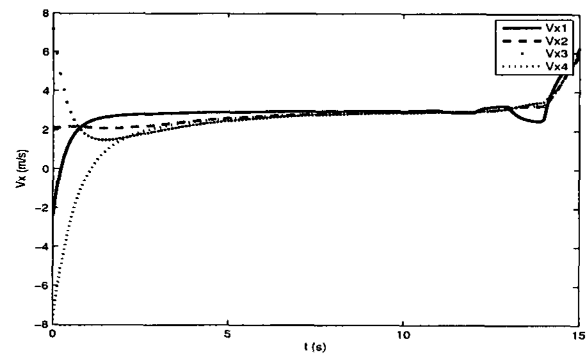


(c)

Figure 3.2: (a) The x-component of the average velocity profile, (b) the y-component of the average velocity profile, and (c) the $x-y$ path trajectories of an LL team of four agents (Monte Carlo simulation runs) resulting from the optimal control strategy in the finite horizon scenario.



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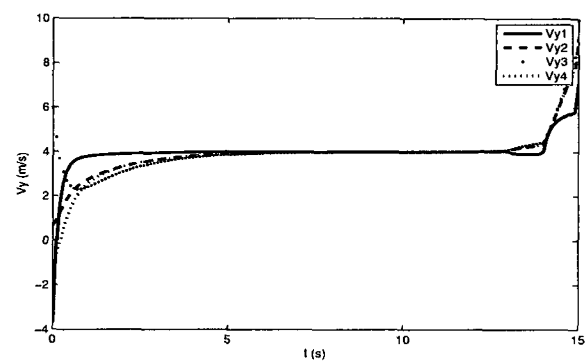

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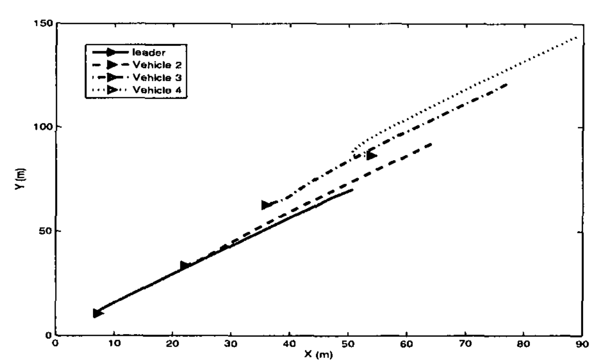
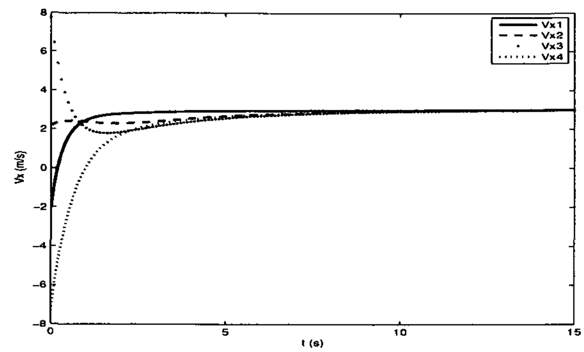
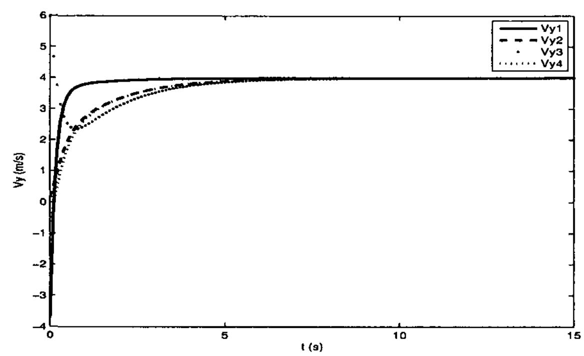


(c)

Figure 3.3: (a) The x-component of the average velocity profile, (b) the y-component of the average velocity profile, and (c) the $x - y$ path trajectories of an MLF team of four agents (Monte Carlo simulation runs) resulting from the optimal control strategy in the finite horizon scenario.



(a)



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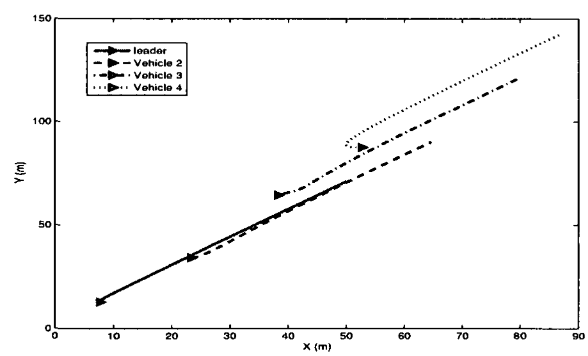
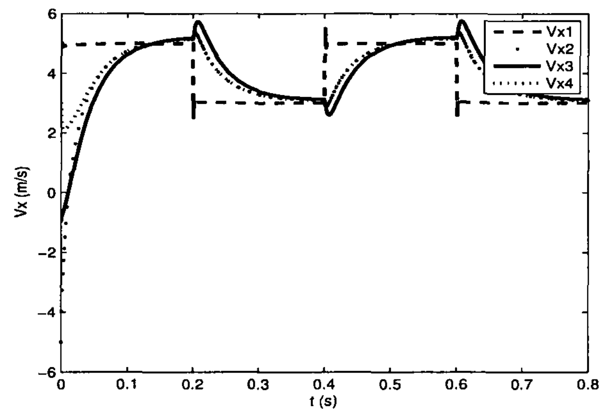
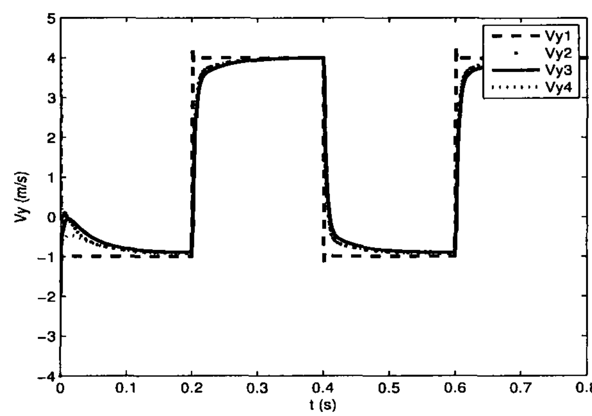


(c)

Figure 3.4: (a) The x-component of the average velocity profile, (b) the y-component of the average velocity profile, and (c) the $x - y$ path trajectories of an MLF team of four agents (Monte Carlo simulation runs) resulting from the optimal control strategy in the infinite horizon scenario.



(a)



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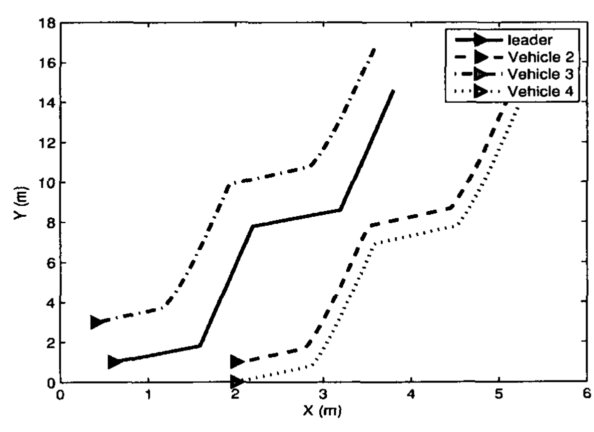
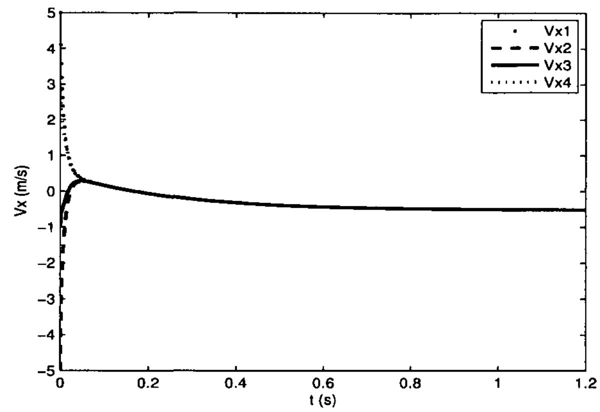
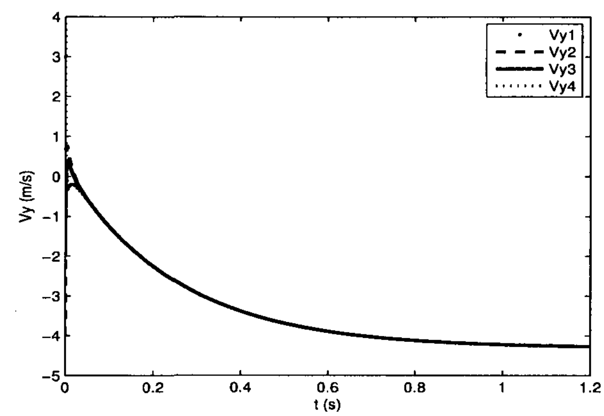


(c)

Figure 3.5: (a) The x-component of the velocity profile, (b) the y-component of the velocity profile, and (c) The $x - y$ path trajectories of an MLF team of four agents with linear dynamical model resulting from the optimal control strategy in the infinite horizon scenario.



(a)



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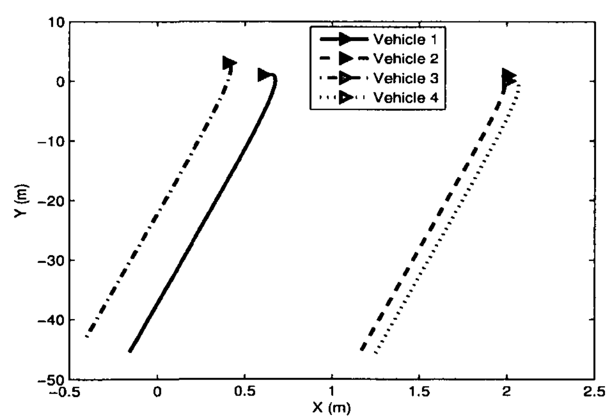


(c)

Figure 3.6: (a) The x-component of the velocity profile, (b) the y-component of the velocity profile, and (c) the $x - y$ path trajectories of an LL team of four agents with linear dynamical model resulting from the optimal control strategy in an infinite horizon scenario.

Chapter 4

Non-ideal considerations for semi-decentralized optimal team cooperation

In this chapter two non-ideal considerations are discussed for team cooperation problem. In other words, I will generalize the results obtained in the previous chapter for cases when more challenging environments are involved. I have considered two scenarios. First, the performance of the previously designed team in the presence of actuator faults is investigated. In the second part of this chapter, the control design is modified to address stability and consensus seeking in a switching network topology.

4.1 Team behavior in the presence of actuator faults

In practice, it is quite possible that some agents in a team may become unable to follow the team command due to anomalies and faults. Some sources of this problem can be due to actuator faults or saturation, faults in measurement of neighbors states, or communication links faults. As a result of these malfunctions, the faulty agent cannot follow the command provided by the team to achieve the predefined goal. This may result in permanent separation of that agent from the team which correspondingly may affect the cohesion of the team. In this section, I provide results on performance analysis of a team of agents subject to actuator faults. The team goal is to accomplish a cohesive motion using the semi-decentralized optimal control that was proposed in Chapter 3. In the following I will investigate the performance of a team with an MLF team structure subject to three types of faults introduced in Chapter 2. Any fault occurring in the LL structure is similar to leader failure case in an MLF structure.

A summary of the materials presented below is published in [130, 136, 137].

4.1.1 Team behavior subject to a Loss of Effectiveness (LOE) fault in an agent's actuator

For the team of agents that is described in Chapter 2 and for dynamical equation (2.20) in an MLF structure, assume that some of the agents, either some followers or the leader, fail to produce the team control command as described in Theorem 3.2. Specifically, due to an actuator LOE fault we now instead have $u^i_f = \alpha u^i, \quad 0 < \alpha \leq 1$, where u^i_f denotes the actual control effort that is applied by the actuator with u^i representing the designed control input. Denote the set of failed agents by $A_f = \{i = N - q + 1, \dots, N\}$, and without loss of generality assume that these are the last q agents of the team. If this is not the case, the agents' labels can be easily changed for this purpose. The concatenated velocity vector of failed agents can be defined as

$v_f = [(v^{N-q+1})^T, \ldots, (v^N)^T]^T$. Using the notion of velocity error as introduced in Theorem 3.2, let us define the total error vector corresponding to the healthy agents as e_w and the one corresponding to the faulty agents as e_f.

Now, assume that the closed-loop dynamics of the entire team is described by $\dot{e} = L_{cl}e$, where $e = [(e^1)^T \ \ldots \ (e^N)^T]^T = [e_w^T \ e_f^T]^T$ and

$$L_{cl} = \begin{bmatrix} \Gamma^1 + \beta^1 & \frac{l_{12}}{|N^1|}\Gamma^1 & \ldots & \frac{l_{1N}}{|N^1|}\Gamma^1 \\ \frac{l_{21}}{|N^2|}\Gamma^2 & \Gamma^2 & \ldots & \frac{l_{2N}}{|N^2|}\Gamma^2 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{l_{i1}}{|N^i|}\Gamma^i & \ldots & \frac{l_{ij}}{|N^i|}\Gamma^i & \ldots \\ \vdots & \vdots & \vdots & \vdots \\ \frac{l_{N1}}{|N^N|}\Gamma^N & \ldots & \frac{l_{N(N-1)}}{|N^N|}\Gamma^N & \Gamma^N \end{bmatrix} \tag{4.1}$$

where l_{ij} is an element of the Laplacian matrix L, and Γ^i and β^1 are defined in (3.33). Partition L_{cl} in order to separate the dynamics of the faulty and the healthy agents as follows

$$L_{cl} = \begin{bmatrix} (L_{11})_{m(N-q)\times m(N-q)} & \vdots & (L_{12})_{m(N-q)\times mq} \\ \cdots\cdots & \cdots & \cdots\cdots \\ (L_{21})_{mq\times m(N-q)} & \vdots & (L_{22})_{mq\times mq} \end{bmatrix}$$

where m is the dimension of v^i (assumed here to be 2). Due to the presence of faults, the closed-loop error dynamics will now be changed into

$$\dot{e} = \begin{bmatrix} L_{11} & \vdots & L_{12} \\ \cdots & \cdots & \cdots \\ \alpha L_{21} & \vdots & \alpha L_{22} \end{bmatrix} \begin{bmatrix} e_w \\ e_f \end{bmatrix} = L_f e \tag{4.2}$$

The following lemma shows that the error dynamics will remain stable despite the presence of agents' faults. Moreover, the consensus achieving goal can still be guaranteed and maintained.

Lemma 4.1. a) Stability Analysis: For the team of agents that is described in Theorem 3.2, if some agents fail to comply with the designed team control command and instead implement $u^i_f = \alpha u^i$ corresponding to an LOE fault, the closed-loop error dynamics still remains stable.

b) Consensus Achievement: Moreover, the velocity error, i.e. $e^i = v^i - v^d$, $\forall i$ will asymptotically approach to zero, and consequently the consensus will still be achieved.

Proof: The details are provided in Appendix B.

It is worth noting that although this fault does not deteriorate the stability property of the closed-loop dynamics, it affects the transient behavior of the agents. Specifically, the transient convergence rate becomes dependent on the parameter α.

4.1.2 Team behavior subject to an actuator float fault in an agent

Fault occurrence in the followers

For the team of agents with a double integrator dynamical equation that is described in Theorem 3.2, assume that a number of follower agents fail to produce the team control command as described in (3.31) and one now instead has $u^i = 0$. The concatenated velocity vector of failed agents, v_f, is a constant vector (due to u^i being zero).

Now, assume that the closed-loop dynamics of the entire team is described by $\dot{e} = L_{cl}e$, where $e = [e_w^T \; e_f^T]^T$ and L_{cl} are defined as in previous cases and e_f will be a constant vector if v^d is time-invariant. Partition L_{cl}

so as to separate the dynamics of the faulty and the healthy agents. Due to presence of faults, this dynamics is now governed by

$$\dot{e} = \begin{bmatrix} L_{11} & \vdots & L_{12} \\ \cdots\cdots & & \cdots\cdots \\ 0_{mq\times m(N-q)} & \vdots & 0_{mq\times mq} \end{bmatrix} \begin{bmatrix} e_w \\ e_f \end{bmatrix} \tag{4.3}$$

The following lemma demonstrates that the error dynamics will remain stable despite the presence of followers faults.

Lemma 4.2. a) Stability Analysis: For the team of agents that is described in Theorem 3.2, if a number of follower agents fail to comply with the team control command as described in (3.31) and instead implement $u^i = 0$ (a float fault), then the closed-loop error dynamics still remains stable and the velocity error, i.e. $e^i = v^i - v^d$, $\forall i$ will remain bounded.

b) Steady State Error: Moreover, the steady state of the velocity error e_{ss} is governed by

$$e \longrightarrow e_{ss} \ as\ t \longrightarrow \infty, \ e_{ss} = \begin{bmatrix} -L_{11}^{-1}L_{12} \\ \cdots\cdots \\ I_{mq\times mq} \end{bmatrix} e^f, \ e = \begin{bmatrix} e_w \\ e_f \end{bmatrix}, \tag{4.4}$$
$$e^f = v^f - ([1 \ \ldots \ 1]_{1\times q}^T \otimes v^d)$$

where e^f, v^f are the error and velocity vectors of the failed agents at the point when their state is frozen.

Proof: The details are provided in Appendix B.

Fault occurrence in the leader

Similar to the discussion in the previous part, if the leader has a fault, e.g. its velocity is frozen at a constant value, namely v_f, we can still achieve stability of the team error dynamics. The closed-loop system matrix can be partitioned as before. In the presence of the leader fault the closed-loop error dynamics is changed to

$$\dot{e} = \begin{bmatrix} 0_{m\times m} & \vdots & 0_{m\times(N-1)m} \\ \multicolumn{3}{c}{\dotfill} \\ L_{21} & \vdots & L_{22} \end{bmatrix} \begin{bmatrix} e_f \\ e_w \end{bmatrix} \tag{4.5}$$

The following lemma shows that the error dynamics still remains stable even if a fault has occurred in the leader.

Lemma 4.3. a) Stability Analysis: For the team of agents described in Theorem 3.2, if the leader fails to comply with the team control command as described in (3.32) and instead a zero control is implemented (a float fault), i.e. $u^1 = 0$, then the error dynamics still remains stable and the tracking error, i.e. $e^i = v^i - v^d,\ \forall i$ remains bounded.

b) Steady State Error: Moreover, the final value of the tracking error vector e_{ss} is governed by

$$e \to e_{ss},\ as\ t \to \infty,\ e_{ss} = \mathbf{1} \otimes e^f,\ e^f = v^f - v^d \tag{4.6}$$

where e^f, v^f are the error and velocity vectors of the leader at which its state is frozen and $\mathbf{1}$ is the vector of ones.

Proof: The details are provided in Appendix B.

Up to this point it is shown that the team will remain stable in presence of faults. In the conventional leader-follower structure, if a follower fails to follow the leader's command, it would be separated from the team and could be lost from the team forever. However, in the MLF structure considered here, due to existence of a feedback from followers to the leader as well as connections among the followers, if one of the followers cannot follow the command, all other followers and the leader will adapt themselves to this change until this agent is recovered. In this manner the cohesion of the team will be preserved and no member will be lost without affecting the others. In the following, I will show that the healthy agents will adapt themselves to the state of the faulty agent.

Leader and followers adaptability to fault occurrence

In order to show the adaptability property we need the following theorem from the graph theory literature [119]. The theorem provides a relationship for the second order minors of the Laplacian matrix of a tree-like graph.

Theorem 4.1. [119] Assume that L is the Laplacian matrix of an undirected graph. Denote the minors of the matrix L that is obtained by eliminating m rows and columns by $L(W|U)$, where W and U are the sets of eliminated rows and columns, respectively. Consider a tree T with N vertices and denote the path [119] between two nodes u and v by $P(u,v)$. Then, for $1 \leq i < j \leq N$ and $1 \leq k < l \leq N$ one has

$$(-1)^{i+j+k+l} det(L(i,j|k,l)) = \pm length(P(v_i,v_j) \cap P(v_k,v_l)) \tag{4.7}$$

Note that the sign of the determinant depends on the relative orientation of $P(v_i,v_j)$ and $P(v_k,v_l)$ with respect to one another in the following sense. If we orient $P(v_i,v_j)$ from v_i towards v_j, and $P(v_k,v_l)$ from v_k towards v_l, they both induce an orientation on their intersection. If these orientations agree, the sign is +1; otherwise, it is -1.

Proof: The details are provided in [119]. ■

At this point and for only proof of the next theorem I assume that the information structure is described by a tree-like graph.

Theorem 4.2. Leader and Followers Adaptability: For the team of agents that is described in Theorem 3.2, if a follower fails to produce the team control command as specified in (3.31) due to a float fault, i.e. $u^i = 0$, then all the agents will adapt themselves to this agent's change, i.e. the direction of change in the state of the faulty follower will be the same as the change in the rest of the team. This implies that the steady state error of the faulty follower and the healthy members will have the same sign, that is

$$e^k \bullet e^f > 0, \; e^f = v^f - v^d, \; e^k = v^k - v^d, \; k = 1, \ldots, N-1 \qquad (4.8)$$

in which v^f is the velocity at which the faulty agent's velocity is frozen and e^f is the corresponding error, e^k is the velocity error of the agent k, "$\bullet$" is the Hadamard product [120], and "> 0" refers to positiveness of the vector's elements.

Proof: The proof is provided in Appendix B.

4.1.3 Team behavior subject to a Lock-In-Place (LIP) fault in an agent

For the team of agents that is described in Theorem 3.2, assume that one of the followers fails to produce the team control command due to an LIP fault and the applied control input is now frozen at a constant value, that is $u^i_f = u_c$, where u_c is a constant value. In this situation, and similar to the discussion in the previous subsection, the closed-loop error dynamics can be represented by the following expression

$$\dot{e} = \begin{bmatrix} \dot{e}_w \\ \dot{e}_f \end{bmatrix} = \begin{bmatrix} (L_{11})_{m(N-1)\times m(N-1)} & \vdots & (L_{12})_{m(N-1)\times m} \\ \cdots\cdots & \cdots & \cdots\cdots \\ 0_{m\times m(N-1)} & \vdots & 0_{m\times m} \end{bmatrix} e + \begin{bmatrix} 0 \\ u_c \end{bmatrix} \quad (4.9)$$

From the above equation it can be concluded that e_f can grow without a bound, and therefore the error dynamics (4.9) is not stable. In other words, in this situation and since the open-loop matrix is not asymptotically stable, i.e. $A^i = 0$, we cannot guarantee stability. However, if at least the open-loop matrix of the faulty agent is stable, there might be a possibility to guarantee stability of the error dynamics. Hence, in this subsection I first determine the agents trajectories that are described by the dynamics (2.20). Next, I will analyze the team behavior assuming that the agents have a stable open-loop system matrix. Therefore, here I will discuss both types of agents' dynamical representation given in Chapter 2, i.e. linear and double integrator models. I will show that when the open-loop system matrix is asymptotically stable, the stability of the error dynamics is guaranteed but as in the float type of fault consensus can no longer be achieved. The following two lemmas summarize our results and an ultimate value of the velocity error vector e_{ss} is obtained.

Lemma 4.4. For the team of agents that is described in Theorem 3.2, assume that a follower agent fails to comply with the team control command as specified in (3.31) and instead implements $u_f^i = u_c$ (LIP fault), where u_c is a constant vector. The ultimate time-varying value of the velocity error vector

e_{ss} can be specified according to

$$e \longrightarrow e_{ss} \ as \ t \longrightarrow \infty, \ e_{ss} = \begin{bmatrix} -L_{11}^{-1}[L_{12}u_c t - L_{12}u_c t_f + L_{12}e^f + L_{11}^{-1}L_{12}u_c] \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ u_c t - u_c t_f + e^f \end{bmatrix} \tag{4.10}$$

where t_f denotes the time of fault injection and e^f is the faulty agent velocity error vector at t_f.

Proof: The proof is provided in Appendix B.

Based on the definition of the float fault, this fault may be considered as an LIP fault when $u_c = 0$. Clearly, the steady state error provided in (4.10) is the same as the one given in (4.4) if u_c is replaced by zero.

For the rest of our discussion, let us consider a team of agents with the governing linear dynamics (2.21) and assume that the faulty agent's open-loop matrix A^f is asymptotically stable. The error dynamics for the entire team can be written as $\dot{e} = L_{cl}e + f(v^d, g^1)$, where L_{cl} is defined as

$$L_{cl} = -K^{-1}(2L \otimes Q^{ij} + A^T K + 2G) = -K^{-1}\Upsilon \tag{4.11}$$

and K, A, G, Υ are defined in Theorem 3.3. $f(v^d, g^1)$ is defined as follows where it is decomposed corresponding to the healthy and faulty agents dynamics as

$$f(v^d, g^1) = \begin{bmatrix} f_1(v^d, g^1) \\ \cdots\cdots\cdots \\ f_2(v^d, g^1) \end{bmatrix} = \begin{bmatrix} -(K^1)^{-1}(A^1)^T(g^1 + K^1 v^d) \\ -(K^2)^{-1}(A^2)^T K^2 v^d \\ \vdots \\ -(K^i)^{-1}(A^i)^T K^i v^d \\ \vdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ -(K^f)^{-1}(A^f)^T K^f v^d \end{bmatrix} \tag{4.12}$$

where superscript f stands for the quantities that correspond to the faulty agent. The rest of parameters are defined as in Chapter 3. When an LIP fault occurs in a follower, the closed-loop dynamics is then governed by

$$\dot{e} = \begin{bmatrix} \dot{e}_w \\ \dot{e}_f \end{bmatrix} = \begin{bmatrix} (L_{11})_{m(N-1)\times m(N-1)} & \vdots & (L_{12})_{m(N-1)\times m} \\ \cdots\cdots\cdots\cdots & & \cdots\cdots\cdots\cdots \\ 0_{m\times m(N-1)} & \vdots & A^f \end{bmatrix} e + \begin{bmatrix} f_1(v^d, g^1) \\ A^f v^d + B^f u_c \end{bmatrix} \tag{4.13}$$

where L_{ij} denotes the corresponding partitioning of the matrix $-K^{-1}(2L \otimes Q^{ij} + A^T K + 2G)$. In the following lemma, we will see the results on stability and consensus seeking for the above selection of agents' dynamics.

Lemma 4.5. a) Stability Analysis: Consider a team of agents with the governing dynamics (2.21) and the control laws for the followers and the leader given by (3.46) and (3.47), respectively. When a follower fails to comply with the designed team control command and instead implements $u_f^i = u_c$, where u_c is a constant value, the closed-loop error dynamics still remains stable if the corresponding open-loop matrix of the faulty agent, i.e. A^f, is Hurwitz.

b) Steady State Error: Moreover, the ultimate value of the velocity error vector e_{ss} is governed by

$$e \longrightarrow e_{ss} \ as \ t \longrightarrow \infty, \ \ e_{ss} = \begin{bmatrix} L_{11}^{-1}[L_{12}((A^f)^{-1}B^f u_c + v^d) - f_1(v^d, g^1)] \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ -(A^f)^{-1}B^f u_c - v^d \end{bmatrix} \tag{4.14}$$

where $f_1(v^d, g^1)$ is defined as

$$f_1(v^d, g^1) = \begin{bmatrix} -(K^1)^{-1}(A^1)^T(g^1 + K^1 v^d) \\ -(K^2)^{-1}(A^2)^T K^2 v^d \\ \vdots \\ -(K^i)^{-1}(A^i)^T K^i v^d \\ \vdots \end{bmatrix} \tag{4.15}$$

and the superscript f stands for the quantities that correspond to the faulty agent.

Proof: The proof is provided in Appendix B.

4.1.4 Leaderless structure

The results corresponding to the situation in which one of the members of a leaderless team fails is quite similar to the leader failure in the MLF architecture as discussed in the previous subsections, and therefore the associated results are not included.

4.2 Switching network structure

In many situations two agents in a team may not be able to obtain the state of each other, either through communication links or by means of on board sensor measurements. This may arise due to either restrictions on their communications, e.g. due to large distances or appearance of obstacles among the team members, or it can be because of the changes that are preplanned in the mission of the team. Consequently, due to a specific mission defined for a team the communication network structure among the team members may no longer be fixed and therefore corresponds to a switching network architecture. In this situation team members have to find new neighbors in order to maintain the connectivity of the team information graph. This implies that the neighboring sets should be defined as time-varying sets, namely $N^i(t)$. These neighboring sets will result in a set of information graphs with time-varying Laplacian matrices, for which the only assumed condition is their connectivity.

In addition to the changes that may occur in the communication structure of a network, in some circumstances in the leader-follower structure, the assignment of the leader may also change during the mission. This can be either as a result of the fact that some agents are more accessible in certain stages of the mission or for safety issues some agents are more reliable or safer to be assigned as the leader during some parts of the mission. In these conditions the leader assignment can be time-varying as well. Therefore, the team structure will no longer remain fixed and consequently we have to analyze the team behavior subject to a switching topology.

The main contribution of the present work is to introduce a "design-based" strategy which can guarantee consensus achievement for a team of agents with a general underlying network graph subject to network topology as well as the leader assignment changes. In contrast to earlier work in the literature that have focused on analysis of the consensus algorithm subject to a time-varying structure, the approach pursued in the present work is mainly based on utilization of control techniques to design a switching strategy. The proposed framework can handle strongly connected, directed, and unbalanced graphs under a switching network configuration. By assigning the eigenvectors of the closed-loop matrix which corresponds to the error dynamics of the team to a desirable vector, the existence of a common Lyapunov function, and consequently the stability and consensus achievement are guaranteed.

The design strategy for the original fixed team structure is based on the semi-decentralized optimal control approach defined in Chapter 3. However, with the modification done in the present section one requires that the control gain matrices that are defined in the cost functions take on specific values. Subsequently, this results in a constraint on the optimal control law which is designed initially for the fixed network topology. It is shown that by introducing additional criteria the desirable performance specifications of the team can still be ensured and guaranteed. As a demonstration and representation of such a criterion, a performance-control effort tradeoff is considered and analyzed in details.

A summary of the materials presented in the following sections are published in [138].

4.2.1 Switching control input and stability analysis

In this part, I only discuss the MLF team structure. The LL structure can be treated similarly. Now, assume that the agents' dynamical model is given as in (2.20) and define the error for each agent as $e^i = v^i - v^d$, where the desired leader command v^d is time-invariant. The error dynamics for the entire team can be obtained as $\dot{e} = L_{cl}e$, where L_{cl} is defined in (4.1). This matrix can be further simplified as follows

$$L_{cl} = -2K^{-1}(L \otimes Q + \begin{bmatrix} \Gamma & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{bmatrix}) = -2K^{-1}(L \otimes Q + G) \quad (4.16)$$

where $K = Diag\{K^i, i = 1, \dots, N\}$ and $G = Diag\{\Gamma, 0, \dots, 0\}$. The above expression of L_{cl} will be used in the following discussion on stability analysis of switching topologies.

Now, assume that there is a team of agents that is characterized by a switching topology due to the time-varying neighboring sets $N^i(t)$ or time-varying leader assignment. Associated with this scenario a switching signal denoted by $\sigma(t) : \mathbb{R}^+ \rightarrow \mathbb{N}$ is defined which is a train of rectangular pulsed signals that has a constant integer value over each time interval τ as shown in Figure 4.1. The communication links among the agents are assumed to be directional with a Laplacian matrix denoted by L. For the case of switching networks, this matrix is a function of the switching signal $\sigma(t)$ and can be written as $L_{\sigma(t)}$, where

$$L_\sigma \in \{L | L \text{ describes the Laplacian of a strongly connected digraph}\} \quad (4.17)$$

Hence, during each time interval the Laplacian matrix describing the underlying team architecture graph belongs to the family of Laplacian matrices defined in (4.17).

Assumption 4.1. The Laplacian matrix $L_{\sigma(t)}$ is provided to all the agents of the network.

This assumption will be further used to evaluate the switching control

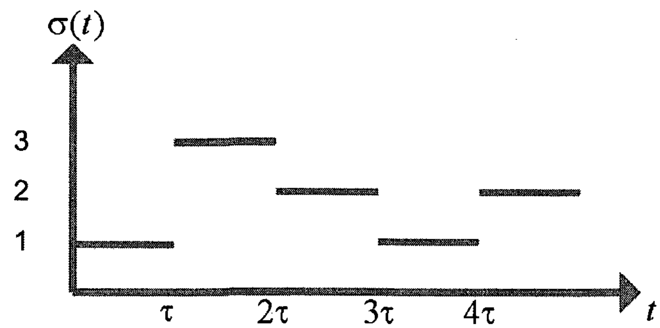


Figure 4.1: Switching signal $\sigma(t)$

signal. It is worth noting that providing this information to the individual agents does not impose an extra restriction on the semi-decentralized structure of the proposed control strategy. In fact regardless of this issue, to ensure and verify the connectedness requirement of the information exchange graph, each agent should already switch its communication links so that the entire network remains connected. Therefore, each agent should be aware of both local as well as global connections. In other words, the requirement that each agent is aware of the Laplacian of the team is not an impediment for or an extra restriction on our proposed semi-decentralized control strategy.

To emphasize that the leader assignment is time-varying, i.e. agent 1 is not necessarily the leader, we may assume that the matrix G in (4.16) is a function of the switching signal $\sigma(t)$ as well, i.e. G_σ. Depending on the agent assigned as the leader, the corresponding row in the matrix G will be non-zero. Subsequently, I denote the parameters associated with a switching by the subscript $\sigma(t)$, i.e. $(.)_\sigma$. Therefore, the closed-loop matrix defined in (4.16) is rewritten as

$$L_{cl,\sigma} = -2K_\sigma^{-1}(L_\sigma \otimes Q + G_\sigma)$$

where $L_{cl,\sigma}, K_\sigma, L_\sigma, G_\sigma$ are the matrices L_{cl}, K, L, G corresponding to the switching structure, respectively. Obviously, the controller coefficient matrix K depends on the switching state since K is a function of the neighboring sets $N^i(t)$. If $L_{cl,\sigma}$ were describing the Laplacian of a balanced graph, then it were straight forward to find a common Lyapunov function for the entire switching network and correspondingly to prove the stability. However, in the present case this matrix does not have the above mentioned property. In the following I try to design a switching controller such that the network stability can be guaranteed.

Towards this end, let us partition the matrix $L_{cl,\sigma}$ into two parts, namely $\bar{L}_\sigma = K_\sigma^{-1}(L_\sigma \otimes Q)$ and $K_\sigma^{-1}G_\sigma$. The first part, $\bar{L}_\sigma$ is itself the Laplacian of a directed weighted graph which is not necessarily balanced. However, if we could transform $\bar{L}_\sigma$ into the Laplacian of a balanced graph, then it is easy to show that a common Lyapunov function for the corresponding switching system exists. One solution to achieve the above goal is to design a **switching control** such that $\bar{L}_\sigma$ becomes the Laplacian of a balanced graph for any switching network. This implies that we need to modify the design of matrix K_σ, as given in Lemma 3.2 and Theorem 3.2, such that $\bar{L}_\sigma$ satisfies the required property. One way to design K_σ to compensate for the switching structure is by selecting different Q^{ij}'s for different nodes in each switching structure (in contrast to the assumption in Theorem 3.2). If such a control design goal can be accomplished, not only undirected graphs but also directed and unbalanced graphs can be analyzed under the switching network topology assumption.

Towards this end, let us assume that Q^{ij} is no longer equal to Q and has different values that are denoted by $Q^i_\sigma(t)$ for each agent i and each switching state $\sigma(t)$ so that $\bar{L}_\sigma$ can be written as follows

$$\bar{L}_\sigma = \begin{bmatrix} (K_\sigma^1)^{-1}Q_\sigma^1 l_{11} & (K_\sigma^1)^{-1}Q_\sigma^1 l_{12} & \dots & (K_\sigma^1)^{-1}Q_\sigma^1 l_{1N} \\ (K_\sigma^2)^{-1}Q_\sigma^2 l_{21} & (K_\sigma^2)^{-1}Q_\sigma^2 l_{22} & \dots & (K_\sigma^2)^{-1}Q_\sigma^2 l_{2N} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

where l_{ij} is the ijth entry of the matrix L_σ which is time dependent (due to assumption of switching topology) and K_σ^i is matrix K^i corresponds to the switching network structure. In order to have $\bar{L}_\sigma$ as a balanced matrix, we should have

$$(\mathbf{1}^T \otimes I_n)\bar{L}_\sigma = \mathbf{0} \rightarrow \begin{bmatrix} (K_\sigma^1)^{-1}Q_\sigma^1 l_{11} & (K_\sigma^2)^{-1}Q_\sigma^2 l_{22} & \dots & (K_\sigma^N)^{-1}Q_\sigma^N l_{NN} \end{bmatrix} \times$$
$$(\begin{bmatrix} 1 & l_{12}/l_{11} & \dots & l_{1N}/l_{11} \\ l_{21}/l_{22} & 1 & \dots & l_{2N}/l_{22} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \otimes I_n) = \mu_\sigma^T(\hat{L}_\sigma \otimes I_n) = 0 \tag{4.18}$$

where n is the dimension of the agents' output, $\hat{L}_\sigma$ is the normalized Laplacian matrix of the graph, and μ_σ is defined as follows

$$\mu_\sigma = \begin{bmatrix} (K_\sigma^1)^{-1}Q_\sigma^1 l_{11} & (K_\sigma^2)^{-1}Q_\sigma^2 l_{22} & \dots & (K_\sigma^N)^{-1}Q_\sigma^N l_{NN} \end{bmatrix}^T \tag{4.19}$$

To ensure that expression (4.18) is satisfied, Q^{ij} (Q_σ^i), R^i (R_σ^i), and Γ (Γ_σ) should be selected such that μ_σ in (4.19) will be in the left null-space of $\hat{L}_\sigma \otimes I_n$. Assume that ω_σ is a normalized vector in the left null-space of $\hat{L}_\sigma$ (the eigenvector of $\hat{L}_\sigma$ corresponds to the zero eigenvalue), then we should have

$$\mu_\sigma = \kappa\omega_\sigma \otimes I_n \tag{4.20}$$

where κ is a scaling factor that should be selected by using a specific criterion, e.g. along the lines that are provided in the next subsection. Therefore, (4.20) is the main requirement that should be satisfied by proper selection of Q^i_σ, R^i_σ, and Γ_σ. Now, I state the following lemma which is used in the subsequent discussions.

Lemma 4.6. The Laplacian matrix of any strongly connected directed graph has a left eigenvector which corresponds to the zero eigenvalue and whose entries have the same sign, i.e. they are either all positive or all negative.

Proof: The details are provided in Appendix B.

We are now in a position to summarize the previous discussions into the following theorem.

Theorem 4.3. Stability Analysis Under Switching Structure: For the team of vehicles described in Lemma 3.2, and under the assumptions of switching network and switching leader, the control laws $u^i_\sigma, i = 1, \ldots, N$ selected according to

$$u^i_\sigma(v^i, v^j) = \Gamma^i_\sigma(v^i - \frac{\sum_{j\in N^i} v^j}{|N^i(t)|}), \;\; i = 2, \ldots, N \tag{4.21}$$

$$u^1_\sigma(v^1, v^j) = \Gamma^1_\sigma(v^1 - \frac{\sum_{j\in N^1} v^j}{|N^1(t)|}) + \beta^1_\sigma(v^1 - v^d) \tag{4.22}$$

$$\Gamma^i_\sigma = -2\kappa\rho_{i,\sigma} I_n, \;\; i = 1, \ldots, N, \;\; \beta^1_\sigma = -\gamma(\kappa\rho_{1,\sigma} r^1 + \sqrt{(\kappa\rho_{1,\sigma} r^1)^2 + \gamma r^1}) I_n \tag{4.23}$$

will guarantee that the cost functions in (3.5), (3.6) are minimized if parameter Q^{ij} in these cost functions is selected as $Q^{ij} = q^i I$, $\forall i$. q^i is obtained from the following equations

$$\begin{aligned} &|N^1|^2(q^1)^2 - (4\kappa^2\rho^2_{1,\sigma}|N^1|r^1)q^1 - 4(\kappa\rho_{1,\sigma})^2\gamma r^1 = 0, \\ &q^i = \frac{4(\kappa\rho_{i,\sigma})^2 r^i}{|N^i|}, \;\; i = 2, \ldots, N \end{aligned} \tag{4.24}$$

where $\rho_{i,\sigma}$ is the ith element of the vector ω_σ, i.e. an eigenvector of the normalized Laplacian matrix of the graph $\hat{L}_\sigma$, corresponding to its zero eigenvalue. Matrices R^i and Γ used in cost functions (3.5), (3.6) are chosen as $R^i = r^i I$, $\Gamma = \gamma I$, where r^i, γ are two positive constants and κ is a design parameter. This in turn guarantees that for the family of the closed-loop error dynamics

$$\dot{e} = L_{cl,\sigma} e, \;\; L_{cl,\sigma} = -2K_\sigma^{-1}(Q_\sigma L_\sigma \otimes I_n + G_\sigma) \tag{4.25}$$

a common Lyapunov function exists. This function ensures that the closed-loop dynamics is asymptotically stable, where $e = [(e^1)^T \; \dots \; (e^N)^T]^T$, $e^i = v^i - v^d$, and $Q_\sigma = diag\{Q_\sigma^1, \dots, Q_\sigma^N\}$. Therefore, the team consensus is achieved for a switching network topology, i.e. $v^i \to v^d$, $\forall i$.

Proof: See Appendix B for the details.

Remark 4.1. It is worth mentioning that in the above theorem theoretically there are no constraints on the switching signal as it can be selected arbitrary and with any desired frequency characteristics. However, from practical considerations viewpoint the switching frequency should be selected based on the dynamic range of the actuators that are employed for implementing the corresponding switching control law. In other words, the physical constraints imposed by the practical specifications of the actuators should be considered in the selection of the switching signal.

Given that the performance of the optimal controller is now limited due to the additional constraints that are imposed on the cost function gains $Q^{ij}(Q_\sigma^i)$ as in (4.24), one may compensate the performance degradation by introducing a new criterion for selecting the design parameter κ. This parameter can be considered as a scaling factor which can play the role of defining weights given to different design specifications. Various criteria can be considered in order to guarantee a specific closed-loop behavior. One such criterion deals with a tradeoff between the control performance and the control effort, i.e. the relationship between the matrices $Q^{ij}(Q^i_\sigma)$ and R^i as discussed in the following subsection.

4.2.2 Selection criterion for κ: performance-control effort tradeoff

An issue that we now need to consider deals with defining the criterion for selecting the scaling factor κ. One such criterion may be specified by making a tradeoff between the control performance and the control effort. According to the definitions of the cost functions given in (3.5) and (3.6), Q^{ij} defines the weight that is assigned to the performance, whereas R^i is the weight that is assigned to the control effort. Therefore, depending on the specifics of an application the selected weights can change. For example, we may require a predefined ratio between the matrices Q^{ij} and R^i, i.e. we may require that $\frac{\lambda_{max}(Q^{ij})}{\lambda_{max}(R^i)} > m_i$, where m_i is the desirable value describing the tradeoff between the performance and control effort gain matrices. The following lemma provides sufficient conditions for guaranteeing this requirement.

Lemma 4.7. In a switching network topology as described in Theorem 4.3, to achieve a tradeoff between the performance-control effort in cost function (3.5) as manifested by $\frac{\lambda_{max}(Q^{ij})}{\lambda_{max}(R^i)} > m_i$, $i = 1, \ldots, N$ the design parameter κ defined in Theorem 4.3 should be selected as

$$\kappa^2 > \frac{1}{4}\max\{\frac{m_1|N^1|}{\rho^2_{1,\sigma}}, \frac{\max_{i=2,\ldots,N}(m_i|N^i|)}{\min_{i=2,\ldots,N}(\rho^2_{i,\sigma})}\} \tag{4.26}$$

where m_i is the desirable value describing the tradeoff between the performance and control effort gain matrices.

Proof: See Appendix B for the details.

4.3 Simulation results

4.3.1 Effects of actuator faults on team performance

In this part, simulation results are presented for LOE, LIP and float faults that occur in one of the vehicles in a team of four mobile robots. Without loss of generality the ring topology is considered for the MLF team.

LOE fault

Simulations are performed for the agents' dynamics as given in (2.20). The leader command is assumed to be a pulse-like signal with a duration of 50 *sec* and its value switches between $v^d = [3 \quad 4]^T$ and $v^d = [5 \quad -1]^T$. The state vector of each agent X^i is composed of position and velocity vectors, i.e. $X^i = [(r^i)^T, \quad (v^i)^T]^T$ and position and velocity vectors are two-dimensional, i.e. $r^i = [r^i_x, \quad r^i_y]^T$, $v^i = [v^i_x, \quad v^i_y]^T$. The initial state of the vehicles are selected as $X^1_0 = [6 \ 1 \ 5 \ 3]^T$, $X^2_0 = [2 \ 4 \ -5 \ -4]^T$, $X^3_0 = [4 \ 3 \ -1 \ -2]^T$, $X^4_0 = [2 \ 0 \ 3 \ 4]^T$ and the other parameters are selected to be $Q^{ij} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix}\right]$, $R^i = I_{2\times 2}$, $|N^i| = 2$, and $\Gamma = \left[\begin{smallmatrix} 10 & 0 \\ 0 & 40 \end{smallmatrix}\right]$. Figures 4.2(a) and 4.2(b) show the x and y components of the velocity profile of the agents when the fourth agent is injected with an LOE fault during the period $115 \leq t \leq 135$. In this period, the fourth agent's actuator is set to $u^4_f = 0.5u^4$. It can be seen that the occurrence of the fault affects the team performance in a short time period and soon after the team recovers its cohesion and achieves consensus. Figure 4.2(c) shows the actual path trajectories generated by the vehicles in the $x - y$ plane. The above verifies the stability and consensus results of the team subject to actuator LOE fault as obtained in Section 4.1.1.

Float fault

Results shown in this part are conducted to capture the average behavior of our proposed control strategies through Monte Carlo simulations. The average team response due to 30 different randomly selected initial conditions are presented for the agents' dynamics as given in (2.20). Figures 4.3(a) and 4.3(b) show the x and y components of the average velocity of agents when a float fault happens for the third agent during the period $20 \leq t \leq 30$. In this period, the third agent cannot follow the team command ($v^d = [3\ 4]^T$) and its velocity is frozen at $v^3 = [6\ 1]^T$. It can be seen that the other members modify their speed to keep the team cohesion. Figure 4.3(c) shows the $x - y$ path that is generated in this case. The value to which the team vehicles' average velocity error converges is $e_{ss} = [1.3 \quad -1.29 \quad 2.14 \quad -2.15 \quad 2.14 \quad -2.15 \quad 3 \quad -3]^T$, which is compatible with the result given in (4.4).

The simulation parameters selected are $Q^{ij} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix}\right]$, $R^i = I_{2\times 2}$, $|N^i| = 2$, $E^i = 0.5 I_{2\times 2}$, $\Gamma = \left[\begin{smallmatrix} 1.3 & 0 \\ 0 & 4 \end{smallmatrix}\right]$, and the random initial conditions for the Monte Carlo simulations are considered as

$X_0^1 = [r(0,15) \quad r(0,25) \quad r(-5,0) \quad r(-6,-1)]^T$,

$X_0^2 = [r(15,30) \quad r(25,50) \quad r(0,5) \quad r(-5,5)]^T$,

$X_0^3 = [r(30,45) \quad r(50,75) \quad r(5,10) \quad r(1,11)]^T$,

$X_0^4 = [r(45,60) \quad r(75,100) \quad r(-10,-5) \quad r(-6,4)]^T$,

where $r(x,y)$ stands for a random variable in the interval $[x,y]$. Figures 4.4(a) and 4.4(b) depict the average v_x^i, v_y^i trajectories when the velocity of the third agent is kept at $v^3 = [0\ 3]^T$ (the agent does not move in the x direction). Figure 4.4(c) shows the $x - y$ path that is generated in this case.

LIP fault

In this part, simulations are performed for the agents' dynamics as given in (2.20) and (2.21). For the former case, all the settings are the same as the ones used in the case of LOE fault except the command pulse duration which is selected as 20 *sec*. Figures 4.5(a) and 4.5(b) show the x and y components of the velocity profile of the agents when the third agent is injected with an LIP fault during the period $20.5 \leq t \leq 25$. During this period, the third agent's actuator is set to $u_f^3 = u^3(t = 20.5)$. It can be seen that after the occurrence of the fault the agents' velocity diverge to different values as predicted in Lemma 4.4. Figure 4.5(c) shows the actual path trajectories generated by the vehicles in the $x - y$ plane.

For the linear dynamical model (2.21), the command and the initial state of the vehicles are similar to the previous case. Other parameters are selected to be $Q^{ij} = 100I_{2\times2}$, and $R^i = 0.01I_{2\times2}$ for $i = 2, \ldots, N$. The parameters corresponding to the model are chosen as $\bar{A}^i = \begin{bmatrix} 1 & 0 \\ 2 & 6 \end{bmatrix}$, $A^i = -I_{2\times2}$, and $B^i = \begin{bmatrix} 1 & 1 \\ 2 & 6 \end{bmatrix}$. The matrices R^1, K^1, Γ are found by solving the set of LMIs in (3.49) as: $R^1 = 10^5 \begin{bmatrix} 0.10 & 0.17 \\ 0.17 & 1.27 \end{bmatrix}$, $K^1 = 10^4 \begin{bmatrix} 8.35 & -1.22 \\ -1.22 & 7.09 \end{bmatrix}$, and $\Gamma = 10^5 \begin{bmatrix} 2.35 & 0 \\ 0 & 1.27 \end{bmatrix}$. Figure 4.6(a) shows the actual path trajectories that are generated by the vehicles in the $x-y$ plane. Figures 4.6(b) and 4.6(c) show the x and y components of the velocity profile of the agents when the third agent is injected with an LIP fault during the period $20.5 \leq t \leq 25$. During this period, the third agent's actuator is set to $u_f^3 = u^3(t = 20.5)$. It can be seen that after the occurrence of the fault the agents' velocity converge to values that are different from the set-point but are finite. This verifies the stability and boundedness results of the agents' velocity subject to actuator LIP fault as obtained in Lemma 4.5.

4.3.2 Team performance in a switching network topology

Simulation results presented in this part are for a team of four agents. The team structure switches between 3 structures based on a specific switching signal pattern that is shown in Figure 4.1. It follows from this figure that the switching signal can take 3 different values at different time intervals, namely 1, 2, and 3. In other words, there are three different states for the team structure and the leader assignment during the mission. The leader assignment is changing at each switching instant and is defined to be according to agents 1, 4, and 2 corresponding to $\sigma(t) = 1, 2, 3$, respectively. Moreover, the leader command is a pulse-like signal which has the same duration as the switching signal time interval, τ. The leader command values for $\sigma(t) = 1, 2, 3$ is $v^d = [15\ 14]^T$, $[7\ 20]^T$, $[20\ 6]^T$, respectively. The graphs describing the network structure are directional and the Laplacian matrices corresponding to the three switching states are as follows

$$L_1 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \; L_2 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix},$$

$$L_3 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

The structure dynamic transition of the team can be seen in Figure 4.7.

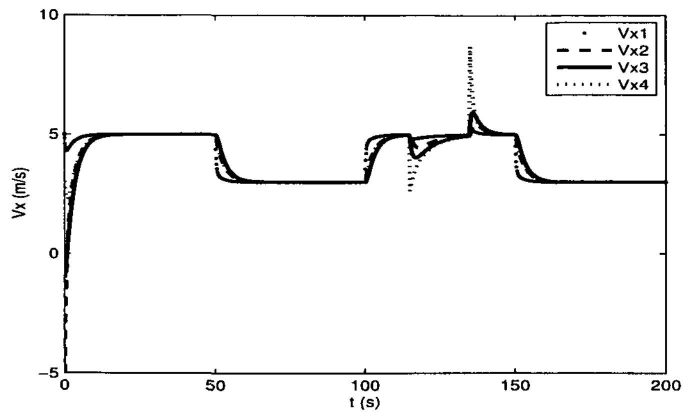
The simulation results are obtained by applying the switching control laws given in Theorem 4.3 to the agents with the dynamics governed by (2.20). In Figure 4.8(a), the $x-$component and in Figure 4.8(b), the $y-$component of the velocity profiles of the four-agent team are shown for the above configurations. Figure 4.8(c) shows the paths that are generated by the agents during the mission where the team members are switching to different structures and operating under different commands and leaders. It can be seen that the team goal, i.e. consensus achievement, is guaranteed in the presence of the switching topology and switching leader.

4.4 Conclusions

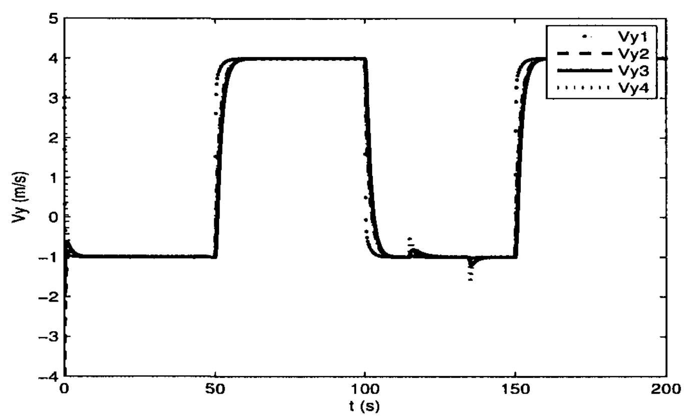
In this chapter two non-ideal considerations are analyzed for team cooperation problem, i.e. actuator faults and switching network topology.

First, I provided a formal analysis and an insight into the effects of various actuator faults on the performance of a team of agents. It was shown that appearance of an LOE fault in one of the agents does not deteriorate the stability or the consensus seeking goal of the team. This fault will only result in a different transient behavior, e.g. a change in the agent's convergence rate, without a change in the consensus value. On the other hand, if the fault in one or more of the agents is of the float type, either in the leader or the followers, the team does not maintain its consensus any longer, however the stability of the team can still be guaranteed. Moreover, the leader and healthy followers adapt themselves to the changes when a float fault occurs in one of the agents. In this manner cohesion and cooperation of the team is maintained and the team remains together until the fault is recovered. Finally, the behavior of the team in the presence of an LIP fault was also investigated. It was shown that stability of the team can be guaranteed if the open-loop system matrix is stable but the consensus cannot be achieved anymore. Under the scenario of an LIP or a float fault the steady state error is analytically obtained.

In the second part of this chapter, a semi-decentralized optimal control design strategy for consensus seeking in a team of agents with both switching structure and leader changes was presented. In contrast to the common assumptions in the literature where graphs are assumed to be balanced, here it was assumed that the graph describing the communication topology is not necessarily a balanced graph. A criterion for selecting the controller parameters was proposed to guarantee a specific performance requirement.



(a)



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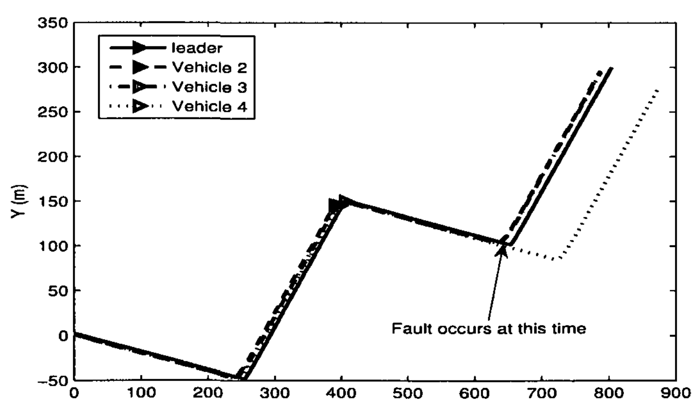
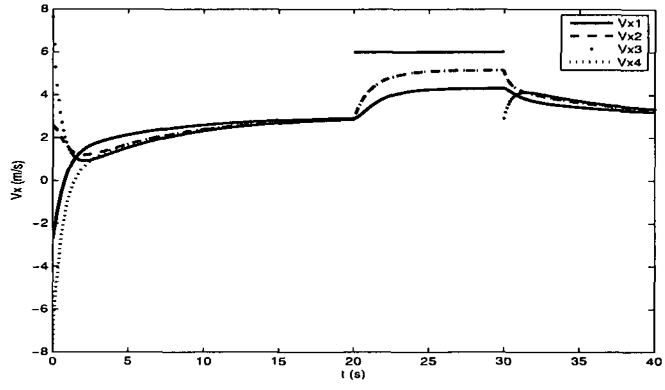
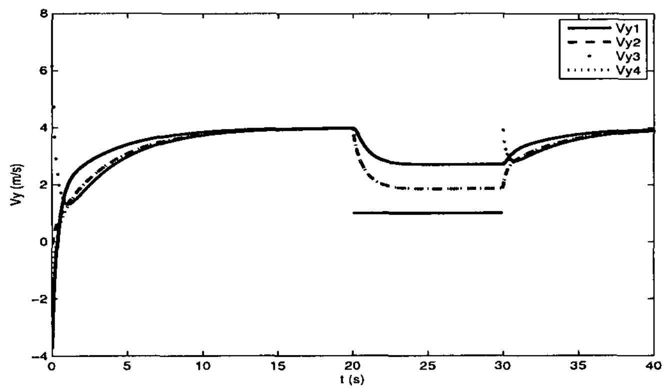


(c)

Figure 4.2: (a) The x and (b) the y components of the velocity profile and (c) the $x-y$ path trajectories of an MLF team of four agents in presence of an LOE fault in the fourth vehicle for $115 \leq t \leq 135$, where $u_f^4 = 0.5u^4$.



(a)



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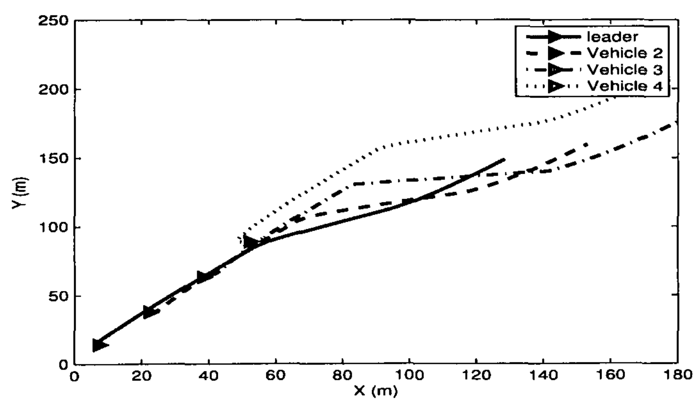
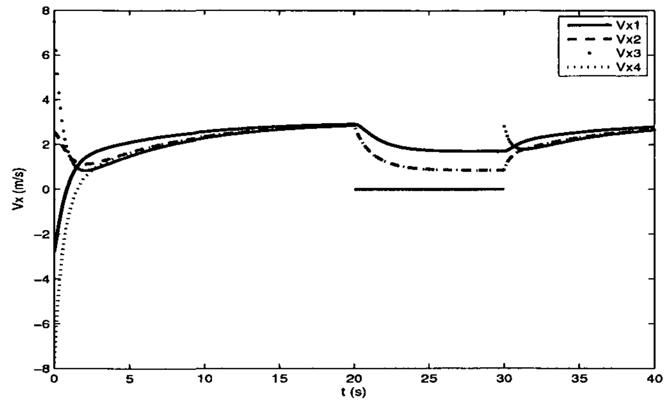
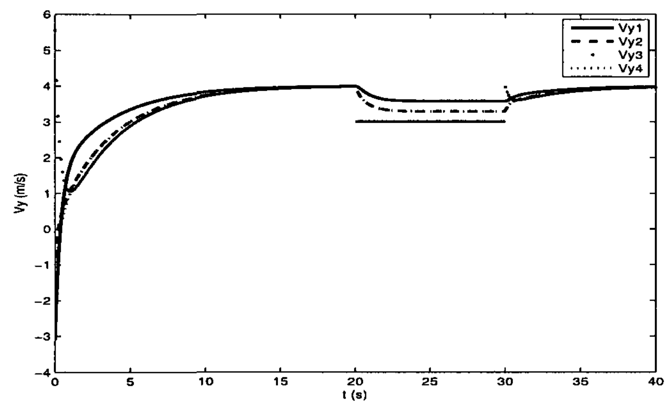


(c)

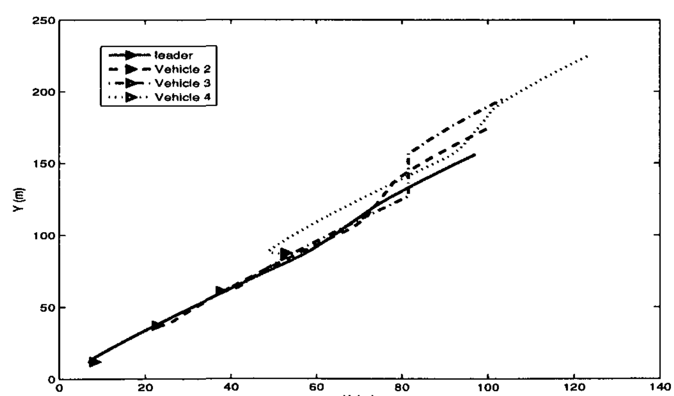
Figure 4.3: (a) The x-component, (b) the y-component, and (c) the $x - y$ path trajectories of an MLF team of four agents (Monte Carlo simulation runs) in the presence of a float fault in the third vehicle (velocity is frozen at $v^3 = [6\ 1]^T$). The jump in the velocity of agent 3 at $t = 30$ *sec* is due to the initiation of a recovery procedure in the actuator of agent 3 (following a fault injected at $t = 20$ *sec*) to the healthy and normal velocity after $t \geq 30$ *sec*.



(a)

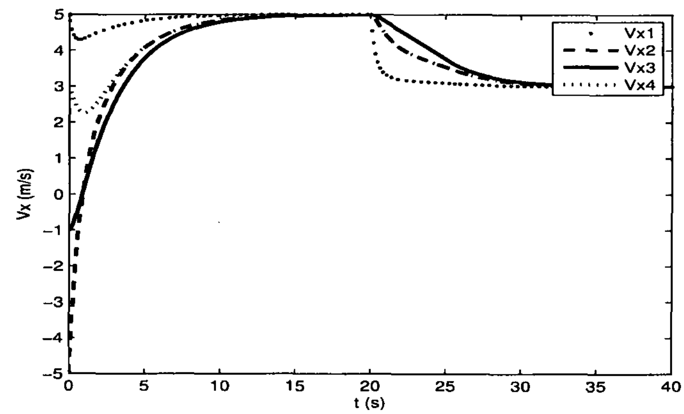


(b)

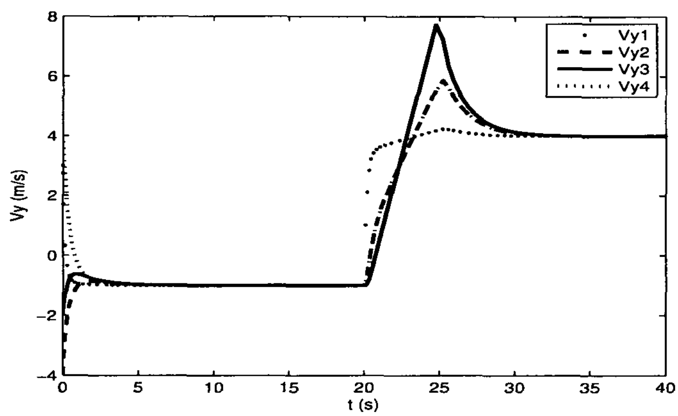


(c)

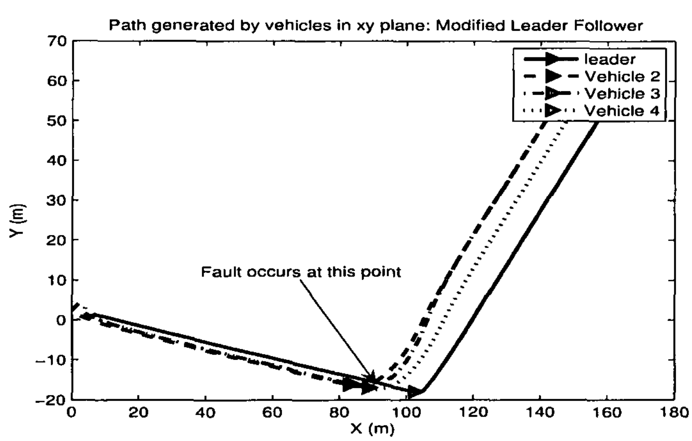
Figure 4.4: (a) The x-component of the average velocity profile, (b) the y-component of the average velocity profile, and (c) the $x-y$ path trajectories of an MLF team of four agents (Monte Carlo simulation runs) in the presence of a float fault in the third vehicle (velocity is frozen at $v^3 = [0\ 3]^T$ for $20 \leq t \leq 30$).



(a)

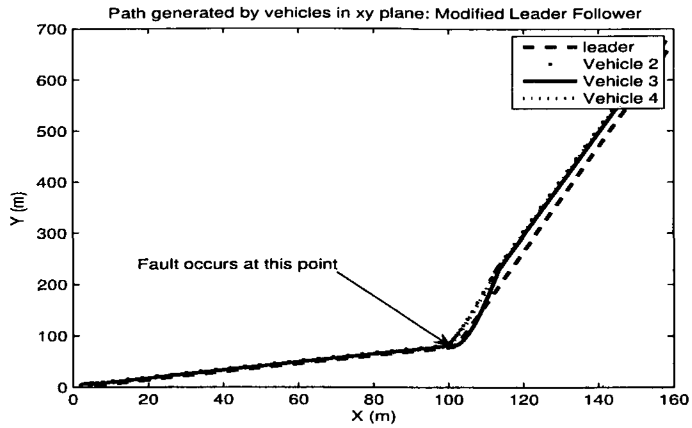


(b)

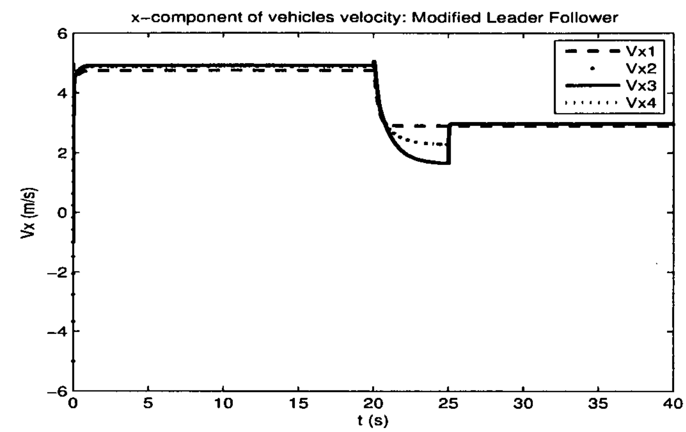


(c)

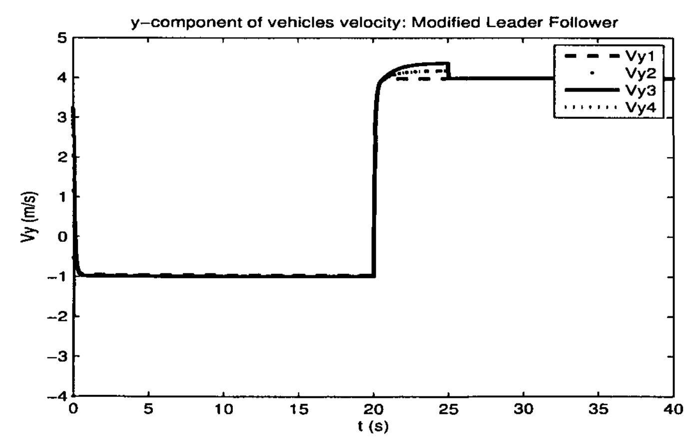
Figure 4.5: (a) The x component of the velocity profile, (b) the y component of the velocity profile, and (c) the $x - y$ path trajectories for a Modified Leader-Follower (MLF) team of four agents in presence of a Lock-In-Place (LIP) fault in the third vehicle for $20.5 \leq t \leq 25$ where $u_f^3 = u^3(t = 20.5)$.



(a)



(b)



(c)

Figure 4.6: (a) The $x-y$ path trajectories, (b) the x component of the velocity profile, and (b) the y component of the velocity profile of an MLF team of four agents with a linear model in presence of an LIP fault in the third vehicle for $20.5 \leq t \leq 25$ where $u_f^3 = u^3(t = 20.5)$.

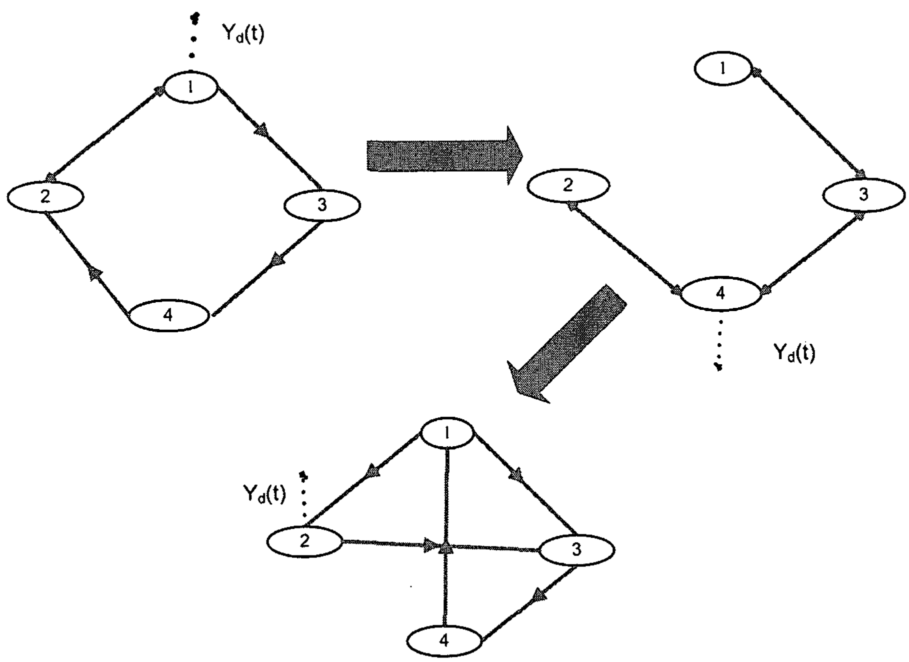
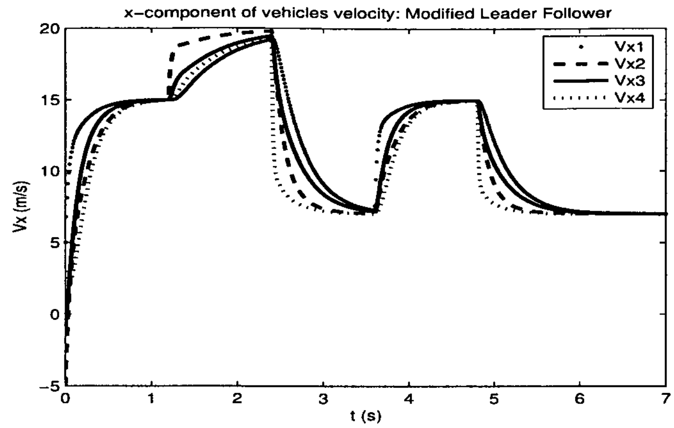
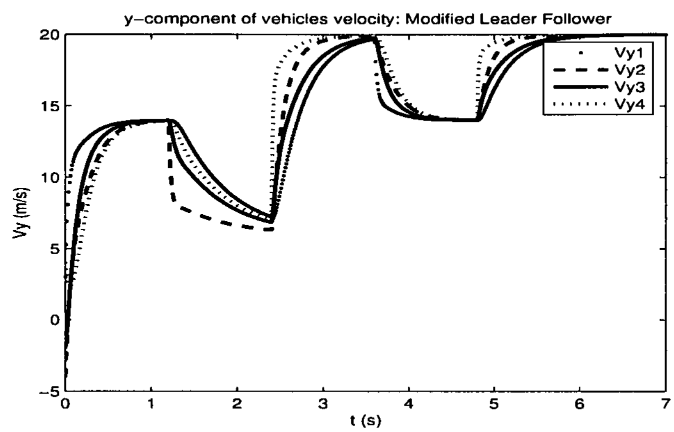


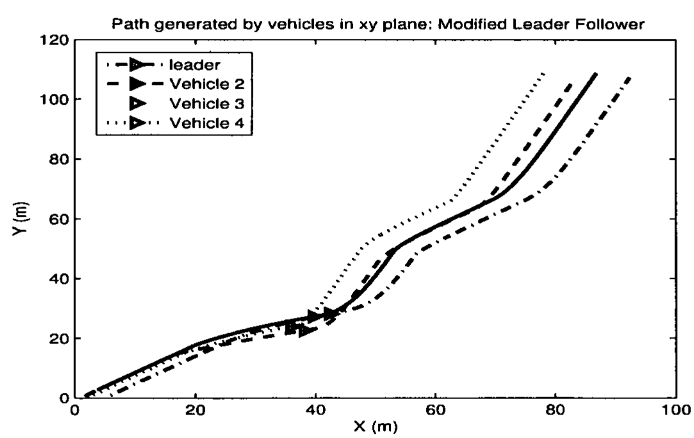
Figure 4.7: The structure dynamic transition of the team between three different switching topology and leader assignment



(a)



(b)



(c)

Figure 4.8: a) The x-component and b) the y-component of the velocity profile and c) the $x-y$ path trajectories of an MLF team of four agents with switching structure and switching leader that are obtained by the application of the proposed switching control strategy.

Chapter 5

Linear matrix inequalities in optimal control and game theory formulation of team cooperation problem

In this chapter, I will utilize the LMI formulation to solve the consensus seeking problem in two frameworks, i.e. game theory and optimal control. In both of these approaches I take the advantage of LMIs to formulate the existing constraints on the consensus seeking problem. In other words, some constraints are added to the original problem in order to provide a decentralized solution which guarantees consensus achievement. The first part of this chapter is dedicated to the game-theoretic based approach and the second part discusses the optimal approach based on the state decomposition idea as will be discussed later.

5.1 A cooperative game theory approach to consensus seeking

In this section, the methodology to solve the consensus seeking problem is based on cooperative game theory. However, in order to clarify the cooperative nature of the method and for comparison purposes, first the semi-decentralized optimal control strategy based on minimization of individual costs that is introduced in Chapter 3 is utilized. Cooperative game theory is then used to ensure team cooperation by considering a linear combination of the individual cost as a team cost function. The cooperative game theory framework has the advantage of being a multi-objective design tool which is well suited for the problem under consideration in this thesis. Moreover, this approach guarantees a "cooperative" solution when compared to other multi-objective design tools.

Minimization of the team cost function results in a set of Pareto-efficient solutions. The choice of the NBS among the set of Pareto-efficient solutions guarantees the minimum individual cost. The Nash Bargaining Solution (NBS) is obtained by maximizing the product of the difference between the costs achieved through the semi-decentralized optimal control strategy and the one obtained through the Pareto-efficient solution. The latter solution results in a lower cost for each agent at the expense of requiring full information set. To avoid this drawback some constraints are added to the structure of the controller that is suggested for the entire team using the LMI formulation of the minimization problem. Furthermore, the consensus achievement condition is added as a constraint to the set of LMIs. Consequently, although the controller is designed to minimize a unique team cost function, it only uses the available information set for each agent. A comparison between the average cost that is obtained by using the above two methods is conducted.

5.1.1 Problem Formulation

According to the discussions in Chapter 2, cooperation in a team of N players (agents), e.g. consensus seeking, can be solved in the framework of cooperative games. The goal in this part is to develop a cooperative solution that utilizes decentralized cost functions and combines them in a team cost function. This will ensure improvements in minimizing individual costs by utilizing the game-theoretic method. Towards this end, I try to find a set of Pareto optimal solutions for minimization of the team cost function through solving the minimization problem in (2.39). An NBS solution can be selected among this set of Pareto-efficient solutions by solving the maximization problem in (2.41) (or (2.40)).

Assume a leaderless team of agents where the dynamical model of each agent and the related cost functions are described in (2.6) and (3.5), respectively. The first step is to combine the individual cost functions in (3.5) into a team cost function J^c according to the following

$$J^c = \sum_{i=1}^{N} \alpha^i J^i(U) = \int_0^T [Y^T \hat{Q} Y + U^T R U] dt = \int_0^T [X^T Q X + U^T R U] dt \tag{5.1}$$

in which $\alpha = (\alpha^1, \dots, \alpha^N) \in \mathcal{A}$ as defined in Chapter 2, $J^i(U)$ is the cost function for the ith agent (player) that is defined in (3.5) and $U(\alpha) = [(u_l^1)^T \ \dots \ (u_l^N)^T]^T$ is the vector of all the agents' local input vectors. X, Y are the vectors of entire team state and output as defined in Chapter 2 and other parameters are defined as follows

$$R = Diag\{\alpha^1 R^1, \ldots, \alpha^N R^N\}, \quad \hat{Q} = [\delta_{hk}]_{N\times N}, \quad Q = C^T \hat{Q} C,$$

$$\delta_{hh} = \sum_{j\in\mathcal{N}^h} \alpha^j Q^{jh} + \alpha^h \sum_{k\in N^h} Q^{hk}, \delta_{hk} = \begin{cases} -\alpha^h Q^{hk} - \alpha^k Q^{kh} & \text{for } \ k \in N^h \\ 0 & \text{otherwise} \end{cases} \tag{5.2}$$

where N^h is the neighboring set of agent h and $\mathcal{N}^h$ denotes the set of indices of the neighboring sets to which agent h belongs and C is defined in (2.8). Each agent belongs to only those clusters in which one of the agent's neighbors exists. Therefore, the total number of these clusters is the same as the number of neighbors of that agent, i.e. $\mathcal{N}^h = N^h$.

The associated dynamical model constraint of the team is given by (2.7) and (2.8). The Pareto efficient solution for minimizing the team cost function (5.1) is achieved by invoking the following strategy

$$U^*(\alpha) = \arg\min_{U\in\mathcal{U}} \sum_{i=1}^{N} \alpha^i J^i(U) = \arg\min_{U\in\mathcal{U}} J^c(\alpha) \tag{5.3}$$

in which $U^*(\alpha)$ is the optimal value of $U(\alpha)$.

The set of solutions to the minimization problem (5.3) is a function of the parameter α which provides a set of Pareto-efficient solutions. Among these solutions, a unique solution can be obtained by using one of the methods that was mentioned in Chapter 2, e.g., the NBS. Using this method the unique solution to the problem (a unique α) is given by (2.41) in which J^i's are defined in (3.5) and are calculated by applying the solution of the minimization problem (5.3) to the system that is given in (2.7), and hence are functions of the parameter α. The terms d^i's are the values of the cost defined in (3.5) which is obtained by applying a non-cooperative approach (e.g. a decentralized optimal controller) to the individual subsystems in (2.6).

By solving the maximization problem (2.41) the parameter α can be found and substituted in the set of control strategies that are obtained in (5.3). This solution guarantees that the product of the distances between d^i's (non-cooperative solution) and J^i's (cooperative solution) is maximized, implying that the individual costs in the latter case are minimized as much as possible.

Let us first solve the minimization problem (5.3) and then apply a modified version of the algorithm given in [124] to solve the maximization problem (2.41).

A summary of the following materials is published in [139, 140].

5.1.2 Solution of the minimization problem: an LMI formulation

In order to solve the minimization problem (5.3), the cost function (5.1) should be minimized subject to the dynamical constraint (2.7). This is a standard LQR problem and its solution for an infinite horizon case (i.e. $T \to \infty$) will result in the following control law

$$U^*(\alpha, X) = -R^{-1}B^TPX, \quad Q - PBR^{-1}B^TP + PA + A^TP = 0 \qquad (5.4)$$

The control U^* can be constructed if the above algebraic Riccati equation (ARE) has a solution for P. However, some issues arise when the above control law is applied to the dynamical system (2.7). In fact, given that the matrix P is not guaranteed to be block-diagonal, the control signal U^* yields a centralized strategy in the sense that its components, i.e. u_l^{i*}, are dependent on the information from the entire team. Moreover, the solution suggested by (5.4) does not necessarily guarantee that a non-zero consensus is achieved for an arbitrary parameter selection. In other words, zero consensus is also a possible solution of (5.4).

Hence, to ensure that a desirable consensus solution is obtained that also satisfies the constraints on the availability of information, one needs to impose additional constraints on the original minimization problem. However, by adding additional constraints to the cost function (5.1), e.g. by considering a barrier function, the problem will no longer be a convex optimization problem and may not necessarily have a unique solution. To remedy this problem, the original cost function is kept unchanged, however the optimization problem is now formulated as an LMI problem so that the constraints due to the consensus and the controller structure are incorporated as convex constraints.

As was pointed out in Chapter 2, the LQR problem can be formulated as a maximization or a minimization problem subject to a set of matrix inequalities. In other words, instead of solving the ARE (5.4), as an example the following maximization problem can be solved

$$\begin{aligned} &\max trace(P) \;\; \text{s.t.} \\ &PA + A^T P - PBR^{-1}B^T P + Q \geq 0, \;\; P \geq 0 \end{aligned} \tag{5.5}$$

this is the formulation provided in (2.36). This formulation can be translated into an LMI maximization problem which can be stated as the following problem.

Problem A

The above problem can be formulated as a maximization problem subject to a set of LMIs, namely

$$\max trace(P) \text{ s.t. } \begin{bmatrix} PA + A^T P + Q & PB \\ B^T P & R \end{bmatrix} \geq 0, \;\; P \geq 0 \tag{5.6}$$

It can be shown that the above maximization problem has a solution if and only if the following ARE has a solution

$$Q - PBR^{-1}B^T P^T + PA + A^T P = 0 \tag{5.7}$$

Moreover, if $R > 0$ and $Q \geq 0$, the unique optimal solution to the maximization Problem A is the maximal solution to the ARE in (5.7) [126].

In the above discussions I showed how to formulate the optimization problem as a set of LMIs. Solutions to this set of LMIs which also minimizes the cost function (5.1) guarantees the consensus seeking, i.e. $X \to \xi\mathbf{1}$. However, among these solutions a possible solution is when $\xi = 0$, that is when the closed-loop system is asymptotically stable and converges to the origin. This solution is not desirable since it is a trivial solution of the consensus seeking problem and should be avoided. For this purpose, we may add the consensus seeking condition to Problem A, i.e. $(A - BR^{-1}B^T P)S = 0$ will be incorporated into Problem A. Here, S is the unity vector, i.e. $S = \mathbf{1}$. This constraint will guarantee that the closed-loop matrix has a zero eigenvalue and is not Hurwitz. Therefore, if the rest of eigenvalues of this matrix are negative, i.e. it is stable, then the system trajectory will move toward a constant nonzero state which is in the consensus space S. On the other hand, by adding other constraints to the LMI problem as will be discussed later on in this section, stability of the closed-loop matrix would be guaranteed as well. We now have a new formulation to our problem as stated next.

Problem B

The LQR minimization problem for consensus seeking can be formulated as a maximization problem subject to a set of LMIs, namely

$$\max trace(P) \text{ s.t.}$$
$$\begin{cases} 1. \begin{bmatrix} PA + A^T P + Q & PB \\ B^T P & R \end{bmatrix} \geq 0, \ P \geq 0 \\ 2. \ (A - BR^{-1}B^T P)S = 0 \end{cases} \tag{5.8}$$

where the optimal control law is selected as $U^* = -R^{-1}B^T PX$ and P is obtained by solving the above set of LMIs.

Consensus seeking subject to a predefined information structure

As discussed previously, the solution to the above problem as well as the one given in (5.4) requires full network information for each agent. However, each agent has only access to its neighboring set information. Therefore, one needs to impose a constraint on the controller structure in order to satisfy the corresponding availability of information. For the sake of notational simplicity assume that each agent has a one-dimensional state-space representation, i.e. A^i in (2.6) is a scalar. The case of a non-scalar system matrix can be treated similarly. The controller coefficient, i.e. $R^{-1}B^T P$ in Problem B should have the same structure as the Laplacian matrix so that the neighboring set constraint holds. However, due to their definitions both R and B are block diagonal. Therefore, it suffices to restrict P to have the same structure as the Laplacian matrix, i.e. $P(i,j) = 0$ if $L(i,j) = 0$, where $L(i,j)$ designates the ijth entry of the Laplacian matrix L. We may now solve the following problem to minimize the cost function (5.1) while simultaneously satisfying all the problem constraints, namely we now have:

Problem $\underline{C}$

$$
\max trace(P) \text{ s.t.}
\begin{cases}
1. \begin{bmatrix} PA + A^T P + Q & PB \\ B^T P & R \end{bmatrix} \geq 0, \; P \geq 0 \\
2. \; (A - BR^{-1}B^T P)S = 0 \\
3. \; P(i,j) = 0 \text{ if } L(i,j) = 0, \; \forall i,j = 1, \ldots, N
\end{cases}
\tag{5.9}
$$

This problem is an LMI maximization problem in terms of P.

Up to now, we have formulated the minimization problem (5.3) as a set of LMIs. Now, let us try to solve the maximization problem that is given by (2.41) (or (2.40)). For this purpose, we need to calculate the individual selfish agent costs J^i by utilizing a given method. For this purpose, the semi-decentralized optimal control strategy developed previously in Chapter 3 is used. These values are considered as the "non-cooperative" outcome of the team, referred to as d^i's in (2.41) (or (2.40)). We use the proposed optimal control strategy that results from the "individual" minimization of the agent cost functions (3.5) as provided in Lemma 3.1 (or Lemma 3.4).

Remark 5.1. It is worth noting that any algorithm which guarantees consensus seeking can be considered as a cooperative algorithm. However, in the context of the formulation based on game theory, the approach based on the semi-decentralized optimal control is classified as non-cooperative. The reason for such designation follows from the previous discussions and definitions where the game theoretic framework yields more characteristics of a cooperative solution when compared to the solution that is obtained by the optimal control strategy. Consequently, the cost values that are obtained using the semi-decentralized optimal control strategy are referred to as the "non-cooperative" outcomes (or threat points).

We are now in a position to develop an algorithm for determining an NBS to the cooperative game theory problem.

5.1.3 An algorithm for finding a Nash Bargaining Solution (NBS)

Up to this point, I have shown that for any given $\alpha > 0$ the maximization Problem C should first be solved. We now need an algorithm for solving the maximization problem (2.41) (or (2.40)) over different values of α so that a suitable and unique α can be found. In [124], two numerical algorithms for solving this maximization problem are given. With minor modifications made to one of these algorithms, the following algorithm will be used for the numerical simulation purposes. Namely, we have

Algorithm I

- **Step 1** Start with an initial selection for $\alpha_0 \in \mathcal{A}$ (e.g. $\alpha_0 = [1/N, \ldots, 1/N]$ is a good choice).

- **Step 2** Compute $U^*(\alpha_0) = \arg\min_{U \in \mathcal{U}} \sum_{i=1}^{N} \alpha_0^i J^i(U)$ by solving the maximization Problemc C.

- **Step 3** Verify if $J^i(U^*) \leq d^i$, $i = 1, \ldots, N$, where d^i is the optimal value of (3.5) when the control laws (3.15)-(3.17) (or controls (3.50)-(3.52)) are applied to the dynamical subsystems (2.6). If this condition is not satisfied, then there is at least one i_0 for which $J^{i_0}(U^*) > d^{i_0}$. In that case, update $\alpha_0^{i_0} = \alpha_0^{i_0} + 0.01$, $\alpha_0^i = \alpha_0^i - \frac{0.01}{N-1}$, for $i \neq i_0$ and return to Step 2 (similarly extend the update rule for more than one i_0).

- **Step 4** Calculate

$$\tilde{\alpha}^j = \frac{\prod_{i \neq j} (d^i - J^i(U^*(\alpha_0)))}{\sum_{i=1}^{N} \prod_{k \neq i} (d^k - J^k(U^*(\alpha_0)))}, \; j = 1, \ldots, N$$

- **Step 5** Apply the update rule $\alpha_0^i = 0.9\alpha_0^i + 0.1\tilde{\alpha}^i$. If $|\tilde{\alpha}^i - \alpha_0^i| < 0.01$ for $i = 1, \ldots, N$, then terminate the algorithm and set $\alpha = \tilde{\alpha}$, else return to Step 2.

The above discussions are now summarized in the following theorem.

Theorem 5.1. Consider a team of agents with individual dynamical representation (2.6) or the team dynamics (2.7), the individual cost function (3.5), and the team cost function (5.1) with the corresponding parameters (5.2). Furthermore, assume that the desirable value of the parameter α is found by using Algorithm I. Moreover, the control law U^* is designed as $U^* = -R^{-1}B^T PX$, and P is the solution to the following optimization problem

$$\max trace(P) \text{ s.t.}$$
$$\begin{cases} 1. \begin{bmatrix} PA + A^T P + Q & PB \\ B^T P & R \end{bmatrix} \geq 0, \; P \geq 0 \\ 2. \; A_c = (A - BR^{-1}B^T P), \; A_c S = 0 \\ 3. \; P(i,j) = 0 \text{ if } L(i,j) = 0, \; \forall i,j = 1, \ldots, N \end{cases} \tag{5.10}$$

where $S = \mathbf{1}$. It then follows that

(a) In the infinite horizon scenario, i.e. $T \to \infty$, the above controller solves the following min-max problem

$$U^* = \arg\min_{U \in \mathcal{U}} \sum_{i=1}^{N} \alpha^i J^i(U), \;\; \alpha \in \mathcal{A},$$

$$\mathcal{A} = \{\alpha = (\alpha^1, \dots, \alpha^N) | \alpha^i \geq 0 \text{ and } \sum_{i=1}^{N} \alpha^i = 1\},$$

$$\alpha^* = \arg\max_{\alpha} \prod_{i=1}^{N} (d^i - J^i(\alpha, U^*)), \;\; J \preceq d$$

The solution to this min-max problem guarantees consensus achieving for the proposed team of agents, i.e. in steady state $X \to \xi \mathbf{1}$, where ξ is a constant coefficient of the consensus value.

(b) In addition, the suggested control law guarantees a "stable" consensus of agents output to a common value subject to the dynamical and information structure constraints of the team in a cooperative manner, if for at least one connected subgraph of the original graph, we have

$$A_c(i,j) \neq 0 \text{ if } L_{sub}(i,j) \neq 0, \;\; \forall i,j = 1, \dots, N \tag{5.11}$$

where L_{sub} denotes the Laplacian matrix of any such arbitrary connected subgraph.

(c) Moreover, the optimal value of the cost function (5.1) has a finite infimum of $X^T(0)PX(0) - \xi^2 \sum_i \sum_j P(i,j)$, where P is obtained from (5.10). $X(0)$ is the initial value of the entire team state vector.

Proof: The proof is provided in Appendix C. ■

We can now conclude that by using the above results, the team consensus goal can be achieved in a decentralized cooperative manner while simultaneously satisfying all the given information constraints of the team.

5.2 An LMI approach to optimal consensus seeking

In this section, an optimal consensus protocol is "designed" using optimal control and LMI design tools. For this purpose, the idea of decomposing the state vector into two components as introduced in [39] is adopted for solving the optimal consensus problem. As opposed to [39], where H_2 design methodology is used for design of a robust consensus seeking algorithm, here I start with a Hamilton-Jacobi-Bellman equation. Then, I will show the difficulties that arise if this formulation is utilized. Therefore, I propose the LMI formulation of the LQR problem. After decomposing the state vector, a global cost function is suggested for the entire network to achieve a stable consensus. Minimization of this global cost function guarantees a stable consensus with an optimal control effort. The global cost function formulation provides more insight into the optimal performance of the entire network and would result in a global optimal (or suboptimal) solution.

In what follows, I first decompose the state vector of the entire team into components in consensus subspace and its orthogonal subspace. This decomposition helps to reduce the consensus seeking problem into a stabilization problem. Then, this stabilization problem is formulated and solved using optimal control technique and LMIs.

Remark 5.2. In this section only, A^* stands for the complex conjugate transpose of A, whereas in the rest of this thesis, A^* describes the optimal value of quantity A.

A summary of the materials presented below is published in [141].

5.2.1 State decomposition

Using the consensus definition given in Definition 2.3, the orthonormal basis for the subspace $\mathcal{S}$ is denoted by $S_{Nn \times 1} = \mathbf{1}$. The orthonormal complement of this matrix is denoted by $\bar{S}_{Nn \times (Nn-1)}$, which is a basis for the corresponding subspace orthonormal to $\mathcal{S}$. The following relationships are satisfied by these matrices:

$$\bar{S}^* S = 0, \ \bar{S}^* \bar{S} = I, \ S^* S = 1, \ \bar{S}\bar{S}^* + SS^* = I \tag{5.12}$$

Now, the state vector X can be decomposed into two orthogonal components in the above mentioned subspaces and can be written as [39]:

$$X = [\bar{S} \ S] \begin{bmatrix} X_{\bar{s}} \\ X_s \end{bmatrix} \tag{5.13}$$

Assuming that the control input has a state feedback structure, i.e. $U = KX$, then the dynamical equation of the system given in (2.7) will be transformed into:

$$\begin{bmatrix} \dot{X}_{\bar{s}} \\ \dot{X}_s \end{bmatrix} = \begin{bmatrix} \bar{S}^* \\ S^* \end{bmatrix} (A + BK)[\bar{S} \ S] \begin{bmatrix} X_{\bar{s}} \\ X_s \end{bmatrix} \tag{5.14}$$

This follows from the fact that $[\bar{S} \ S]^{-1} = \begin{bmatrix} \bar{S}^* \\ S^* \end{bmatrix}$.

Since the goal is to ensure consensus in subspace $\mathcal{S}$ for the closed-loop system, the following equilibrium condition is imposed on the above dynamical equation:

$$(A + BK)S = 0 \tag{5.15}$$

In other words the equilibria should lie in the consensus subspace. This condition should be incorporated in the design procedure. Then, we will have:

$$\begin{bmatrix} \dot{X}_{\bar{s}} \\ \dot{X}_s \end{bmatrix} = \begin{bmatrix} \bar{S}^* \\ S^* \end{bmatrix} (A+BK)\bar{S}X_{\bar{s}} = \begin{bmatrix} \bar{S}^*(A+BK)\bar{S} & 0 \\ S^*(A+BK)\bar{S} & 0 \end{bmatrix} \begin{bmatrix} X_{\bar{s}} \\ X_s \end{bmatrix} \quad (5.16)$$

In order to achieve consensus the final state of the system should be a vector in subspace $\mathcal{S}$. Therefore, the component $X_{\bar{s}}$ should converge to zero in steady state. This implies that this part of the system dynamics should be asymptotically stable. Moreover, the dynamics corresponding to X_s is only dependent on $X_{\bar{s}}$, and therefore we are only concerned with the dynamics corresponding to $X_{\bar{s}}$ as governed by:

$$\dot{X}_{\bar{s}} = \bar{S}^*(A+BK)\bar{S}X_{\bar{s}} = \bar{S}^*A\bar{S}X_{\bar{s}} + \bar{S}^*BK\bar{S}X_{\bar{s}} = \bar{A}X_{\bar{s}} + \bar{B}\bar{K}X_{\bar{s}} = \bar{A}X_{\bar{s}} + \bar{B}\bar{U} \quad (5.17)$$

where

$$\bar{A} = \bar{S}^*A\bar{S},\ \bar{B} = \bar{S}^*B,\ \bar{K} = K\bar{S},\ \bar{U} = \bar{K}X_{\bar{s}} \quad (5.18)$$

If this part of the dynamics is stabilized asymptotically to zero, $X_{\bar{s}}$ will go to zero and hence X_s will approach to a constant value. On the other hand, if condition (5.15) is imposed, this constant value will be in the consensus subspace. Therefore, the consensus will be achieved.

Now we may design a state feedback control strategy to guarantee the consensus achievement by the closed-loop system. Towards this end, optimal control techniques will be used below to design the controller to guarantee a stable consensus in an optimal manner. As mentioned in the above discussion the purpose of the control design is to stabilize that part of the system dynamics which corresponds to the subspace $\bar{\mathcal{S}} = span\{\bar{S}\}$. Therefore, the goal is to design the corresponding control gain, i.e. $\bar{K}$. Based on this value of $\bar{K}$

the corresponding value of K for the original system can be obtained. In the following this controller is designed using optimal control techniques.

5.2.2 Optimal control design

Optimality here refers to the situation when the dynamics of $X_{\bar{s}}$ is stabilized in an optimal manner. For characterizing optimality we need to define a formal performance index. We can define either individual performance indices or a single index (cost function) for the entire team. In Chapter 3, I proposed individual cost functions and suggested a semi-decentralized control strategy for minimizing these cost functions. Although the individual cost functions do better fit within a decentralized control structure, they cannot be utilized as an index of the team performance. In contrast, the team cost function which is used here is a good index of the team performance and its minimization can result in a globally optimal (or suboptimal) solution. However, the solution will be centralized. Fortunately, by using the LMI formulation, it will be shown that this centralized solution can be avoided by adding a constraint on the structure of the controller gain matrix.

In order to stabilize the dynamics given in (5.17), let us define the team cost function that is to be minimized as follows:

$$d = \int_0^\infty \{X_{\bar{s}}^T \hat{Q} X_{\bar{s}} + \bar{U}^T R \bar{U}\} dt, \;\; X = [\bar{S} \; S] \begin{bmatrix} X_{\bar{s}} \\ X_s \end{bmatrix} \tag{5.19}$$

where $\hat{Q}$ has a predefined structure as $\hat{Q} = \bar{S}^* Q \bar{S} > 0$ and Q and R are PD matrices (if $Q_{Nn \times Nn}$ is selected to be a PD matrix, since $rank(\bar{S}_{(Nn)\times(Nn-1)}) = Nn - 1$, then $\hat{Q}$ will also be a PD matrix [142]).

In the following, I will show that in general the minimization of this cost function using the Riccati equation does not result in a consensus for a network of agents with general dynamical representation. Therefore, in the following subsections an LMI formulation is utilized for the optimization problem which can incorporate the requirements of consensus achievement and results in an optimal consensus algorithm.

Discussion on the solution of the Riccati equation

The problem of minimizing cost function (5.19) subject to dynamical constraint (5.17) is a standard LQR problem. The solution to this LQR problem can be achieved by solving the corresponding Riccati equation as follows:

$$\bar{U} = -R^{-1}\bar{B}^* P X_{\bar{S}} \tag{5.20}$$

where P satisfies the following Riccati equation

$$P\bar{A} + \bar{A}^* P - P\bar{B}R^{-1}\bar{B}^* P + \hat{Q} = 0 \tag{5.21}$$

Therefore, $\bar{K} = K\bar{S} = -R^{-1}\bar{B}^* P$ and from the properties of matrix $\bar{S}$ one can find K as $K = -R^{-1}\bar{B}^* P\bar{S}^*$. Hence, the control input to the original system is given by:

$$U = -R^{-1}\bar{B}^* P\bar{S}^* X \tag{5.22}$$

By applying this input to system (2.7), the closed-loop dynamics can be written as:

$$\dot{X} = (A - BR^{-1}\bar{B}^* P\bar{S}^*)X \tag{5.23}$$

In order to achieve consensus for the closed-loop system, the matrix S should be in the null-space of the closed-loop matrix, i.e. $[A - BR^{-1}\bar{B}^* P\bar{S}^*]S =$

0. However, the second part of this expression is zero due to the properties of $\bar{S}$ as stated in (5.12), i.e.

$$-BR^{-1}\bar{B}^{*}P\bar{S}^{*}S = 0 \tag{5.24}$$

In other words, we should have $AS = 0$ to guarantee a stable consensus. The above discussion is formally summarized in the following lemma.

Lemma 5.1. Consider a team of agents with the team entire dynamics given in (2.7), where interaction terms are incorporated in the agents' dynamics. Assume that a state decomposition procedure is performed and the consensus seeking problem is reduced to stabilization of the dynamical equation (5.17). Then, the solution of the corresponding Riccati equation given in (5.21) which minimizes the cost function (5.19) subject to the dynamical constraint (5.17) may not result in a stable consensus algorithm unless the consensus subspace is in the null space of the open-loop matrix A, i.e.

$$AS = 0 \tag{5.25}$$

In other words, this solution may not provide a stable equilibria in the consensus subspace.

Proof: Follows from the previous constructive results. ■

In general, the condition $AS = 0$ may not be satisfied by the subsystems in the network. Therefore, for a system with arbitrary matrix A, the optimal solution obtained by solving the Riccati equation does not guarantee consensus achievement. To simply explain this observation one may note that according to the definition of the control input U given in (5.22), this control only provides a component in $\bar{\mathcal{S}}$. Hence, the term BU does not contribute to the $\mathcal{S}$ component of $\dot{X}$. Therefore, to have a stable solution where $\dot{X} = 0$, the term AX should enjoy the same property, i.e. the component of AX in $\mathcal{S}$ subspace should be zero:

$$AS = 0 \tag{5.26}$$

Since this condition is not generally satisfied by an arbitrary system matrix A, the consensus condition (5.15) in general should be imposed onto the optimal solution that is achieved through the solution of the proposed minimization problem as an extra constraint. Hence, instead of obtaining an optimal solution through the solution of the Riccati equation, in the following I try to find an optimal solution for the above minimization problem subject to the consensus constraint through solution of a set of LMIs.

LMI formulation of the optimal consensus seeking

As in the previous part, the problem of minimizing the cost function (5.19) subject to the dynamical constraint (5.17) cannot be solved as a standard LQR problem if consensus seeking is to be added as a constraint. Instead, we may use one of the LMI formulations for solving the optimal problem which was introduced in Chapter 2. In other words, instead of solving the ARE (5.21), the controller $\bar{U} = \bar{K}X_{\bar{s}}$ that minimizes the cost function (5.19) subject to (5.17) is achieved by solving for and determining the appropriate matrix P:

$$\begin{aligned} &\min trace(P) \quad s.t. \\ &P(\bar{A}+\bar{B}\bar{K}) + (\bar{A}+\bar{B}\bar{K})^*P + \hat{Q} + \bar{K}^*R\bar{K} \leq 0, \;\; P \geq 0 \end{aligned} \tag{5.27}$$

where $\bar{K} = -R^{-1}\bar{B}^*P$ yields the optimal solution. For the dynamical system (5.17) the inequality constraint (5.27) can be written as:

$$P(\bar{S}^*A\bar{S} + \bar{S}^*BK\bar{S}) + (\bar{S}^*A\bar{S} + \bar{S}^*BK\bar{S})^*P + \bar{S}^*Q\bar{S} + \bar{S}^*K^*RK\bar{S} \leq 0 \tag{5.28}$$

By multiplying both sides of this inequality by P^{-1} we get:

$$\begin{aligned} &\bar{S}^*(A + BK)\bar{S}P^{-1} + P^{-1}\bar{S}^*(A^* + K^*B^*)\bar{S} \\ &+ P^{-1}\bar{S}^*Q\bar{S}P^{-1} + P^{-1}\bar{S}^*K^*RK\bar{S}P^{-1} \leq 0 \end{aligned} \tag{5.29}$$

Now define a new variable $Z = Z^* > 0$ that satisfies the following equation [39]:

$$Z = \bar{S}\bar{S}^*Z\bar{S}\bar{S}^* + SS^*ZSS^* \tag{5.30}$$

an example of which can be in the following form [39]:

$$Z = [\bar{S}\ S]\begin{bmatrix} P^{-1} & 0 \\ 0 & M \end{bmatrix}\begin{bmatrix} \bar{S}^* \\ S^* \end{bmatrix} \tag{5.31}$$

where $M = S^*ZS$ and $P^{-1} = \bar{S}^*Z\bar{S}$ are PD matrices and P can be the same matrix as the one used in (5.29). Corresponding to this definition of Z we will have $Z\bar{S} = \bar{S}P^{-1}$. Substitute $Z\bar{S} = \bar{S}P^{-1}$ into (5.29) to get:

$$\bar{S}^*(AZ + BKZ + ZA^* + ZK^*B^* + ZQZ + ZK^*RKZ)\bar{S} \leq 0 \tag{5.32}$$

Introduce a new variable $W = KZ$, so that we have:

$$\bar{S}^*(AZ + BW + ZA^* + W^*B^* + ZQZ + W^*RW)\bar{S} \leq 0 \tag{5.33}$$

This can be written as an LMI condition using Schur complement and noting that $R > 0$ and $Q \geq 0$, namely

$$\begin{aligned}
&\begin{bmatrix} \Upsilon & \bar{S}^* Z Q^{1/2} & \bar{S}^* W^* R^{1/2} \\ Q^{1/2} Z \bar{S} & -I & 0 \\ R^{1/2} W \bar{S} & 0 & -I \end{bmatrix} \leq 0, \\
&\Upsilon = \bar{S}^*(AZ + BW + ZA^* + W^*B^*)\bar{S}
\end{aligned} \tag{5.34}$$

where Z, W are LMI parameters. Therefore, we have shown that the minimization problem in (5.27) can be written as follows:

$$\begin{aligned}
&\min trace(P) \quad s.t. \\
&\begin{bmatrix} \Upsilon & \bar{S}^* Z Q^{1/2} & \bar{S}^* W^* R^{1/2} \\ Q^{1/2} Z \bar{S} & -I & 0 \\ R^{1/2} W \bar{S} & 0 & -I \end{bmatrix} \leq 0, \\
&\Upsilon = \bar{S}^*(AZ + BW + ZA^* + W^*B^*)\bar{S}, \\
&Z = \bar{S}\bar{S}^* Z \bar{S}\bar{S}^* + SS^* Z SS^*, \;\; Z\bar{S} = \bar{S}P^{-1}
\end{aligned} \tag{5.35}$$

where $K = WZ$. In the following, I will discuss the conditions for existence of a solution to the above minimization problem and then present the main results of this section as a theorem.

Discussion on the existence of solutions

It is well-known that detectability and stabilizability conditions are sufficient for existence of a unique stabilizing solution to a linear quadratic optimal control problem. The following lemma illustrate and formulate these conditions for our specific problem.

Lemma 5.2. The minimization problem (5.27), or equivalently (5.35), subject to the dynamical constraint (5.17) has an optimal stabilizing solution if matrices A, B, Q are given such that the following inequalities have a solution for P_2:

1. *Stabilizability condition:* $\bar{S}^*(AP_2 + P_2A^* - BB^*)\bar{S} < 0,$ (5.36)

2. *Detectability condition:* $\bar{S}^*(P_2A + A^*P_2 - Q)\bar{S} < 0,$ (5.37)

where $P_2 > 0$ satisfies $P_2 = \bar{S}\bar{S}^*P_2\bar{S}\bar{S}^* + SS^*P_2SS^*$.

Proof: The details are provided in Appendix C.

Remark 5.3. For the current problem the system matrix A given in (2.8) is a function of the interaction coefficients, $\mathcal{F}^{ij}$, and therefore can be viewed as a design parameter. In case that matrices B, Q, and the matrix A with no interaction terms, i.e. $\mathcal{F}^{ij} = 0,\ \forall i, j$, satisfy the conditions (5.36) and (5.37), the existence of a solution is guaranteed. However, if these conditions are not guaranteed then we may select the coefficients $\mathcal{F}^{ij}$ so that they are satisfied. A simple approach is to take P_2 as an identity matrix and then select A such that both inequalities are satisfied. However, it should be noted that in some special conditions, and due to the special structure of matrix A, it might not be possible to find a solution for the inequalities in (5.36), (5.37) using this method. An example of this situation is when $A^i = 0,\ \forall i$, and Q has the same structure as the Laplacian matrix. In this case $A + A^* - Q$ will have a positive eigenvalue regardless of the selection of the parameters of matrix A. In this situation one possible solution is to add an internal loop for each individual controller so that the required conditions in (5.36) and (5.37) are satisfied (by adding diagonal elements to the matrix A).

The main result of this section and the conclusion from the above discussions are summarized in the following theorem.

Theorem 5.2. a. Consider a team of agents with the dynamical equation as in (2.7). Assume that the state vector is decomposed into components in consensus subspace and its orthogonal subspace. Therefore, the dynamics corresponding to this decomposition is given by (5.17) with the cost function in (5.19) to be minimized for stabilization of this dynamics. Also, assume that the matrices A, B, Q satisfy the conditions of Lemma 5.2. Moreover, assume that the control input U is selected as $U = KX$, where $K = WZ^{-1}$ and the LMI variables Γ, W and Z are obtained through the minimization problem below:

$$\min trace(\Gamma) \quad s.t.$$
$$\begin{cases} 1. \begin{bmatrix} \Gamma & I \\ I & \bar{S}^* Z \bar{S} \end{bmatrix} > 0, \\ 2. \begin{bmatrix} \Upsilon & \bar{S}^* Z Q^{1/2} & \bar{S}^* W^* R^{1/2} \\ Q^{1/2} Z \bar{S} & -I & 0 \\ R^{1/2} W \bar{S} & 0 & -I \end{bmatrix} \leq 0, \\ \Upsilon = \bar{S}^*(AZ + BW + ZA^* + W^*B^*)\bar{S}, \\ 3. \ (AZ + BW)S = 0, \\ 4. \ Z = \bar{S}\bar{S}^* Z \bar{S}\bar{S}^* + SS^*ZSS^*, \ Z > 0 \end{cases} \tag{5.38}$$

Then, the cost function (5.19) is minimized and the system (5.17) is asymptotically stabilized. This in turn makes the system in (2.7) reach a stable consensus in an optimal manner.

b. Furthermore, if the following constraints are added to the above minimization problem the controller will be semi-decentralized. In other words, only partial information available through the predefined neighboring sets is used by individual controllers, provided that:

$$\begin{cases} 1.\ Z \text{ is diagonal, i.e. } Z = \begin{pmatrix} Z_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & Z_N \end{pmatrix}, \text{ and} \\ 2.\ W(i,j) = 0 \text{ if } L(i,j) = 0 \end{cases} \tag{5.39}$$

where L is the Laplacian matrix of the graph describing the network.

Proof: The details are given in Appendix C.

5.2.3 Discussion on graph connectivity

In contrast to most of the consensus seeking approaches, in the proposed approach in this section, there has been no explicit restriction on the connectivity of the network underlying graph. In the following, I will show that the graph connectivity is a requirement to guarantee consensus achievement. First, I prove the following lemma which is required for the remainder of the discussion in this subsection.

Lemma 5.3. The closed-loop matrix of the entire network, i.e. $A + BK$ represents the Laplacian matrix of a weighted graph. The corresponding graph is a subgraph of the original network graph but with different weights assigned to its edges.

Proof: The details are given in Appendix C.

Remark 5.4. From the above lemma, it follows that even if the original graph is connected, we may not conclude that the matrix $A+BK$ represents a connected graph. However, in the following it is shown that for guaranteeing the existence of a solution to the consensus seeking problem, not only the original graph should be connected but also $A + BK$ should represent the Laplacian of a connected graph.

Theorem 5.3. a) If the graph corresponding to the entire network is not connected, the existence of a solution to the consensus problem cannot be guaranteed.

b) Moreover, if consensus seeking is guaranteed, matrix $A + BK$ will describe the Laplacian of a connected subgraph of the original graph.

Proof: The details are given in Appendix C.

5.3 Simulation results

5.3.1 Game theory approach

The simulation results that are presented in this section correspond to a team of four agents that are being controlled by using two control strategies, namely the semi-decentralized optimal control law that is given by Lemma 3.1 (or Lemma 3.4) and the cooperative game theoretic-based control law that is given by Theorem 5.1. The first set of numerical simulations corresponds to the application of the control laws (3.15)-(3.17) (or (3.50)-(3.52)) to the individual agent model (2.6). In the second set, the numerical simulation results are obtained by applying the control law $U = KX$ with $K = -R^{-1}B^TP$ to the team dynamics that is described by (2.7). It is assumed that the state vector X^i of each agent is the velocity vector of that agent, i.e. $X^i = v^i$ and that velocity vector has two components, i.e. $v^i = [v_x^i,\ \ v_y^i]^T$. The matrix K is obtained by solving the Problem C and by utilizing the maximization Algorithm I. Results shown below are conducted through Monte Carlo simulation runs to capture the average behavior of the proposed control strategies. The average team responses are due to 15 different randomly selected initial conditions.

The simulation parameters for both control approaches are selected as follows: $A^i = 0_{2\times2}$, $R^i = I_{2\times2}$, $C^i = I_{2\times2}$, $B^i = \begin{pmatrix} 4 & -3 \\ -2 & 3 \end{pmatrix}$, and $Q^{ij} = \begin{pmatrix} 10 & 3 \\ 3 & 4 \end{pmatrix}$. The random initial conditions of the velocity vector, i.e. v_0^i, for the Monte Carlo simulations are considered as $v_0^1 = [r(6,8) \;\; r(1,3)]^T$, $v_0^2 = [r(5,7) \;\; r(3,5)]^T$, $v_0^3 = [r(2,4) \;\; r(1,3)]^T$, and $v_0^4 = [r(-5,-3) \;\; r(-4,-2)]^T$, where $r(x,\ y)$ represents a random variable in the interval $[x, y]$.

In the cooperative game theory strategy the initial value for the parameter α is selected as $\alpha_0 = [1/4, \ldots, 1/4]$ and its optimal average value is obtained by using the procedure that is outlined in Algorithm I for 15 Monte Carlo simulation runs as $\alpha = [0.2276\ 0.2005\ 0.2486\ 0.3232]$. The interaction gains are selected as $\mathcal{F}^{ij} = 1.6 I_{2\times2}$.

Table 5.1 compares the average values of the cost function in (3.5) that are obtained by running the Monte Carlo simulations for the four agents under the two proposed control approaches for a period of 2 *sec*. As expected the average costs for the cooperative game theory approach are less than those that are obtained from the optimal control approach. However, it should be noted that this achievement is at the expense of an increased computational complexity. In fact, in the former method two optimization problems, namely a maximization and a minimization problem should be solved as compared to the semi-decentralized approach where only a single minimization problem needs to be solved. Therefore, there is a tradeoff between the control computational complexity and the achievable control performance. A quantitative evaluation criteria of the tradeoffs to a large extent will depend on the specific application under investigation and the practical constraints of the system.

In Figures 5.1(a) and 5.2(a) the $x-$components and in Figure 5.1(b) and 5.2(b), the $y-$components of the average velocity profiles of the four-agent team are shown for the semi-decentralized optimal strategy and the game theoretic-based strategy, respectively. Similarly, in Figures 5.3(a) and 5.3(b), the $x-$component of the average control input efforts of the four-agent team are shown for the semi-decentralized optimal and the game theoretic-based controllers, respectively.

Remark 5.5. It should be noted that the final values that are obtained for the semi-decentralized optimal control strategy are the average of the states initial values. In fact the control law provided in Theorem 3.1 is a weighted average consensus protocol which results in consensus on the average value of the initial state vector. However, the cooperative game theory protocol that is obtained by solving the set of LMIs given in (5.10) is not necessarily an average consensus protocol. Therefore, in this case the consensus value can be any arbitrary number.

5.3.2 LMI-based optimal control approach

Remark 5.6. It is worth noting that in the LMI-based optimal control approach, there is more flexibility in design of both the local controllers and the interaction terms. In other words, u_l^i in (2.6) can be a function of both the

Table 5.1: A comparative evaluation of the average value of the performance index corresponding to the two control design strategies for the cost functions defined in (3.5) for $T = 2$ *sec.*

	Average Performance Index			
Control Scheme	Agent 1	Agent 2	Agent 3	Agent 4
Semi-decentralized optimal control	14,648	14,781	11,629	12,519
Cooperative game theory	8,545	7,730	6,138	8,370

local information X^i as well as the global information $X^j,\ j \in N^i$. Consequently, the interaction term $\mathcal{F}^{ij}$ can be selected as zero even though the agent j is in the neighboring set of agent i.

In this part two examples of team configuration are presented and simulated as follows.

Example 1: Simulation results presented in this example are for a team of four agents with the team dynamical equation as in (2.7)-(2.8). The simulations presented here are done for two cases. In the first case the requirement given in Lemma 5.1 is not satisfied by the system matrix A, i.e. $AS \neq 0$. The Laplacian matrix corresponding to the connected graph describing the network structure is $L = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$. Other simulation parameters are $A^i = \begin{pmatrix} 1 & 1 \\ 0.5 & 1 \end{pmatrix}$, $B^i = 2I_{2\times2},\ c^i = I_{2\times2},\ Q^{ij} = 6I_{2\times2},\ R^i = 2I_{2\times2}$. The initial condition of the state vector is selected as $X(0) = \begin{pmatrix} 6 & 1 & 5 & 3 & 2 & 1 & -5 & -4 \end{pmatrix}^T$. The state vector X is composed of state vectors of all agents in the team and the state vector X^i of each agent is the velocity vector of that agent, i.e. $X^i = v^i$ which has two components, i.e. $v^i = [v^i_x,\ \ v^i_y]^T$. The interaction coefficients $\mathcal{F}^{ij},\ \forall i, j$ are assumed to be zero.

It can be verified that the above parameters satisfy the conditions provided in Lemma 5.2 and the corresponding matrix P_2 can be selected as the identity matrix. The simulation results are obtained by applying the control law $U = KX$ to the system (2.7). Matrix K is first evaluated through the set of LMIs given in Theorem 5.2. In Figure 5.4(a), the $x-$component and in Figure 5.4(b), the $y-$component of the velocity profiles of the four-agent team are shown for the above configuration. In Figures 5.5(a) and 5.5(b) I have applied the control strategy that is obtained through the solution of the Riccati equation and given by (5.22) to the system (2.7) with the above configuration. It can be seen that as predicted in Lemma 5.1 the closed-loop system is unstable. This is due to the fact that the open-loop matrix A does not satisfy the property $AS = 0$. Therefore, the closed-loop dynamics cannot provide a stable consensus by using the matrix K that is obtained from standard LQR-based design methodology using Riccati equation.

In the second part of the simulations, I have selected a matrix A such that $AS = 0$. The corresponding results are presented in Figures 5.6-5.7. To guarantee the condition $AS = 0$, the interaction coefficients are selected as $\mathcal{F}^{ij} = -0.25 \begin{pmatrix} 1 & 1 \\ 0.5 & 1 \end{pmatrix}$, $\forall i, j \in N^i$ for this case. Other simulation parameters are the same as in the previous case. The simulation results are obtained by applying the controllers that are designed based on the Riccati equation solutions and the solutions to the set of LMIs in (5.38). Figures 5.6(a) and 5.6(b) correspond to the latter approach, whereas Figures 5.7(a) and 5.7(b) correspond to the former approach. For comparison between these two approaches I have calculated a performance index for both methods. Since one does not have direct access to $X_{\bar{s}}$, instead of the performance index (5.19) I used the following cost function for comparison purposes:

$$PI = \int_0^T \{X^T Q X + U^T R U\} dt \tag{5.40}$$

where T is selected to be $10s$. The values obtained for the above performance index are 865.5 and 883.2 corresponding to the Riccati equation and the LMIs approaches, respectively. Also, the controller provided by the solution of the Riccati equation reaches consensus faster when compared to the controller corresponding to the LMIs solution ($1.8s$ vs. $2.6s$).

Example 2: Simulation results conducted in this example are presented for a team of six vehicles. The simulations are performed for the two case scenarios as in Example 1. The graph describing the network structure is shown in Figure 5.8. The numerical parameters selected for this network of multi-agents are $A^1 = \begin{pmatrix} 1 & 1 \\ 0.5 & 1 \end{pmatrix}$, $A^2 = \begin{pmatrix} -2 & 1 \\ 3 & 5 \end{pmatrix}$, $A^3 = \begin{pmatrix} 0 & 3 \\ 4 & -1 \end{pmatrix}$, $A^4 = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$, $A^5 = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$, and $A^6 = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}$. B^i, c^i, R^i, Q^{ij} are selected as in the previous example. The initial conditions of the velocity vectors are selected as $X^1(0) = [6\ 1]^T$, $X^2(0) = [8\ 3]^T$, $X^3(0) = [2\ 7]^T$, $X^4(0) = [-5\ -4]^T$, $X^5(0) = [12\ -6]^T$, and $X^6(0) = [-1\ -2]^T$. The simulations are obtained by applying the controllers that are designed based on the Riccati equation solutions and the solutions to a set of LMIs. In Figure 5.9(a), the $x-$component and in Figure 5.9(b), the $y-$component of the velocity profiles of the six-agent team are shown for the above settings in the latter approach. For sake of comparison in Figures 5.10(a) and 5.10(b) the former strategy is used under the above configuration. It can be seen that the closed-loop system is unstable since the open-loop system matrix A does not satisfy the property $AS = 0$.

In the second case scenario I have selected a matrix A such that $AS = 0$. The corresponding results are presented in Figures 5.11-5.12. To guarantee the condition $AS = 0$, the interaction coefficients $\mathcal{F}^{ij}$ are selected to be zero and the system matrices of all the agents are selected to be the same and given as $A^i = \begin{pmatrix} 1 & -1 \\ -5 & 5 \end{pmatrix}$, $\forall i$. The other simulation parameters are selected to be the same as in the previous scenario. The simulation results are obtained by applying the controllers that are designed based on the Riccati equation solutions and the solutions to a set of LMIs. Figure 5.11 corresponds to the latter approach and Figure 5.12 corresponds to the former approach.

As can be seen from Figures 5.11 and 5.12 both control methodologies in this case ($AS = 0$) yield an asymptotically stable closed-loop system. For comparison between the two control approaches I have calculated a performance index corresponding to each method. The numerical values that are obtained for the performance index (5.40) are given in Table 5.2. It can be seen that the cost expensed for the LMI approach is indeed higher than the one used by the Riccati equation-based approach when $AS = 0$. Obviously, for the scenario when $AS \neq 0$, the LMI approach results in a lower cost given that the Riccati equation-based approach yields an unstable system. On the other hand, the controller provided by the LMI approach indeed reaches consensus faster when compared to the controller corresponding to the Riccati equation solution ($0.6s$ vs. $1.06s$ in case of $AS = 0$). The associated numerical results are shown in Table 5.3.

5.4 Conclusions

In this chapter the LMI formulation is utilized to solve the consensus seeking problem in two frameworks, i.e. game theory and optimal control frameworks.

First, a novel design-based approach is proposed in order to address the consensus control problem using a single team cost function within a game

Table 5.2: A comparative evaluation of the performance index corresponding to the two control design strategies for the cost function (5.40) for $T = 2$ *sec* in Example 2.

	Performance Index (5.40)	
Control Scheme	$AS = 0$	$AS \neq 0$
Riccati equation-based	1079.5	1.0407e+007
LMI-based	2098.5	2012.1

theoretic framework. The advantage of minimizing a cost function that describes the total performance of the team is that it can provide a better insight into performance of the entire team when compared to the individual agent performance indices. However, the potential main disadvantage of this formulation is clearly the requirement of availability of full information set for control design purpose. In the present work this problem is alleviated and the imposed information structure of the team is taken into account by using an LMI formulation. It is shown that if a cost function describing the total performance is minimized, a lower team cost as well as lower individual costs may be achieved. A comparative study is performed between the cooperative game theory strategy and the semi-decentralized optimal control strategy introduced in Chapter 3. This comparison reveals that the former approach results in lower individual as well as team cost values as predicted. Moreover, the cooperative game theory approach results in a global optimal solution that is subject to the imposed communications constraints.

In the second part, an optimal control design strategy based on state decomposition is introduced to guarantee consensus achievement in a network of multi-agents. It is shown that the LMI optimization provides more flexibility when compared to the method based on solution of the Riccati equation. In other words, the approach based on the solution of the Riccati equation in general fails to provide a solution for a stable consensus protocol. Therefore,

Table 5.3: A comparative study of the consensus reaching time corresponding to the two control design strategies in Example 2.

	Consensus Time	
Control Scheme	$AS = 0$	$AS \neq 0$
Riccati equation-based	$1.06s$	-
LMI-based	$0.6s$	$0.8s$

the LMI formulation is used to solve the corresponding optimization problem and simultaneously to address the consensus achievement constraint. Moreover, using the LMI formulation a controller specific structure based on the neighboring sets can be imposed as an additional LMI constraint. Therefore, in the individual control design the only required information will be what has been received from the corresponding neighbors in the controller's neighboring set.

The solutions obtained in this chapter incorporate all the imposed constraints and suggest a global optimal (suboptimal) solution. Also, the proposed formulations provide a single index for describing and analyzing the total performance of the team. These frameworks have sufficient flexibility to accommodate additional constraints and design criteria in the proposed methodologies and solutions. Moreover, since both game theory and optimal control are multi-objective frameworks and with the help of the LMI formulation, the proposed methods have the advantage of being capable of addressing additional specifications, e.g. limited control input availability, specific control structure, and consensus achievement constraint.

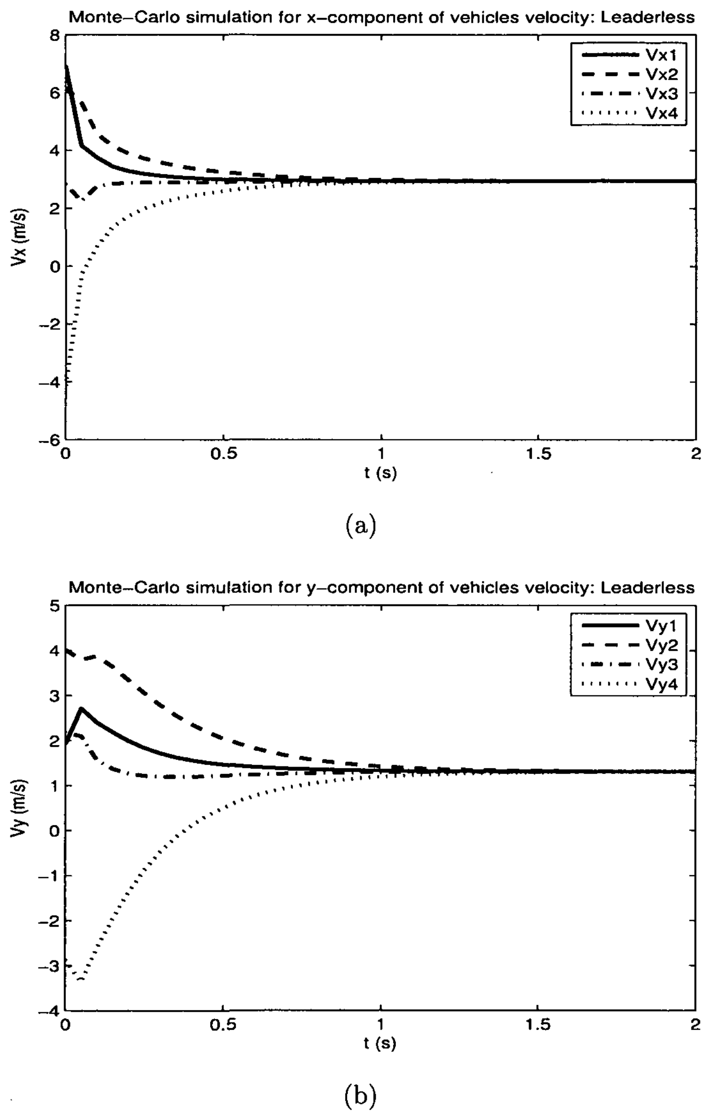
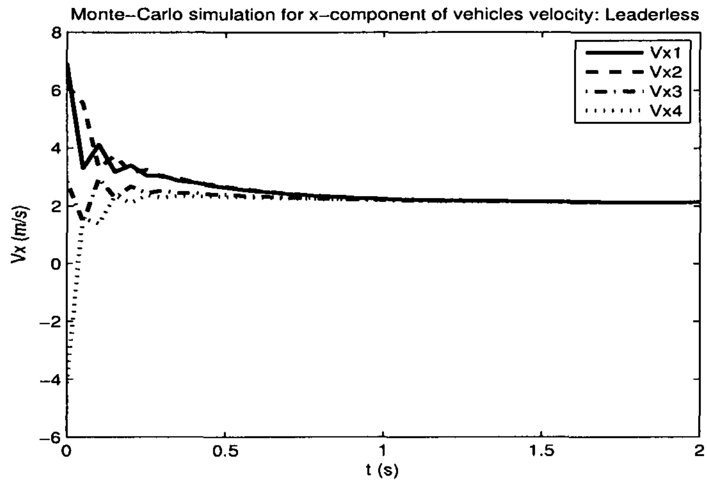
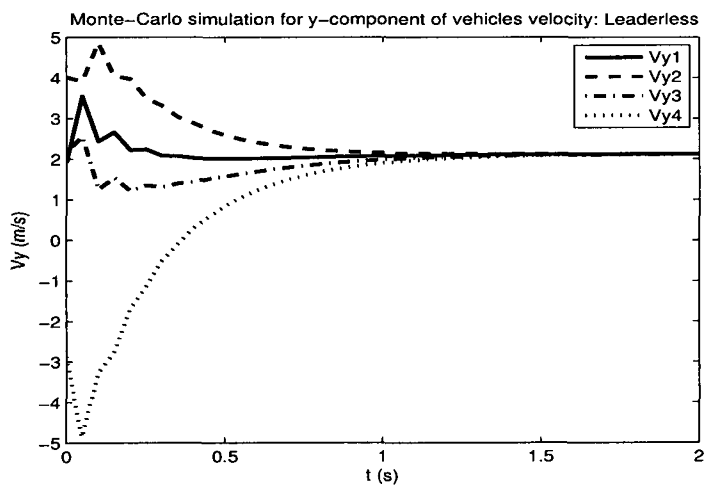


Figure 5.1: (a) The x-component and (b) the y-component of the average velocity profiles that are obtained by applying the semi-decentralized optimal control strategy to a team of four agents.

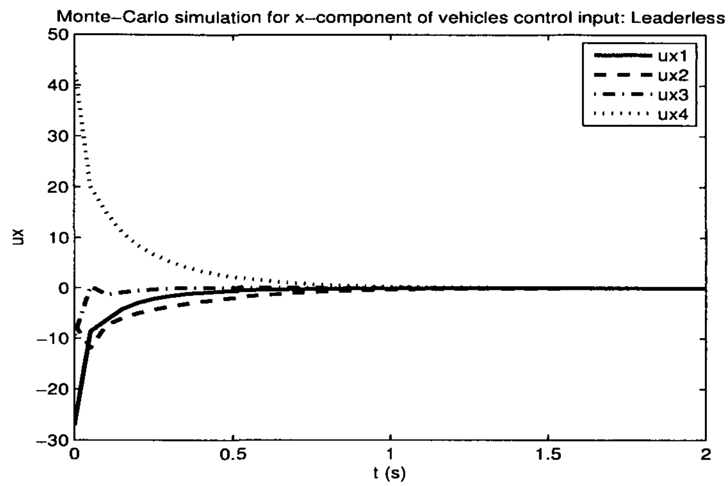


(a)

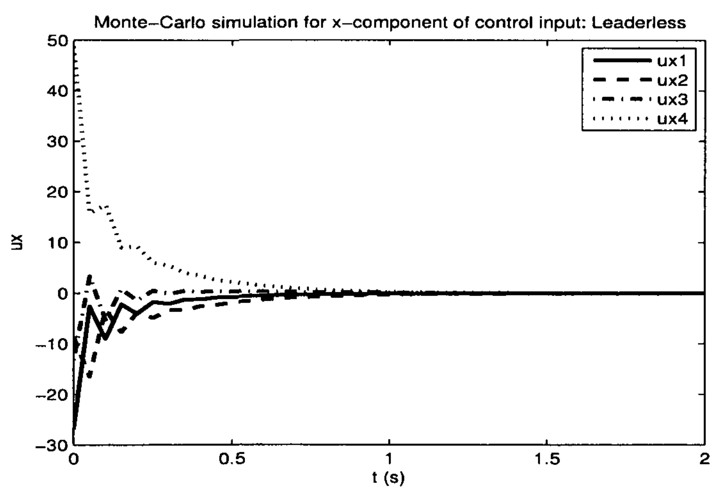


(b)

Figure 5.2: (a) The x-component and (b) the y-component of the average velocity profiles that are obtained by applying the cooperative game theory strategy to a team of four agents.

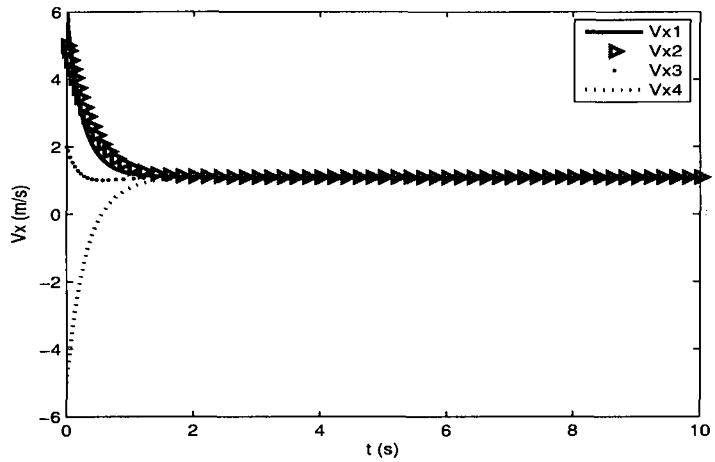


(a)

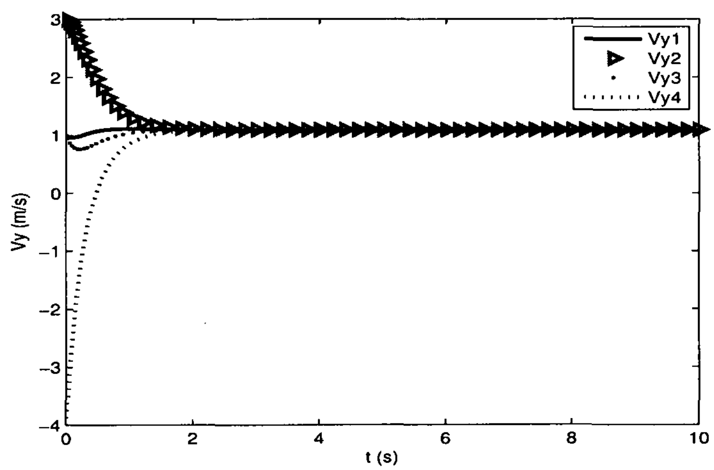


(b)

Figure 5.3: The x-component of the average control efforts that are obtained by applying (a) the semi-decentralized optimal control strategy and (b) the cooperative game theory approach to a team of four agents.

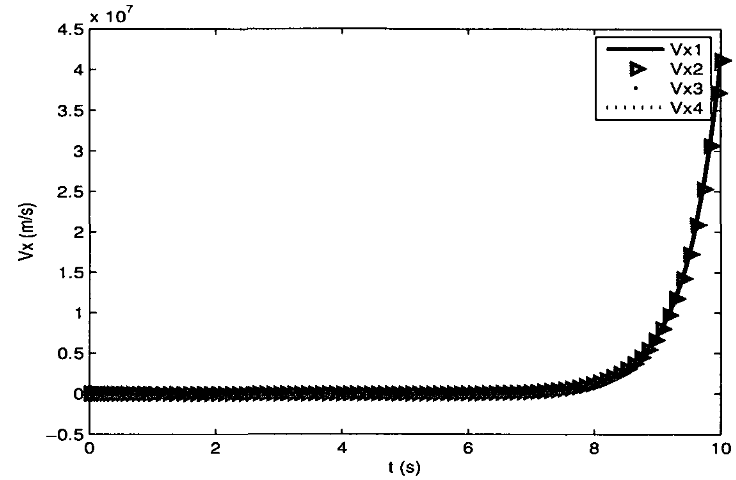


(a)

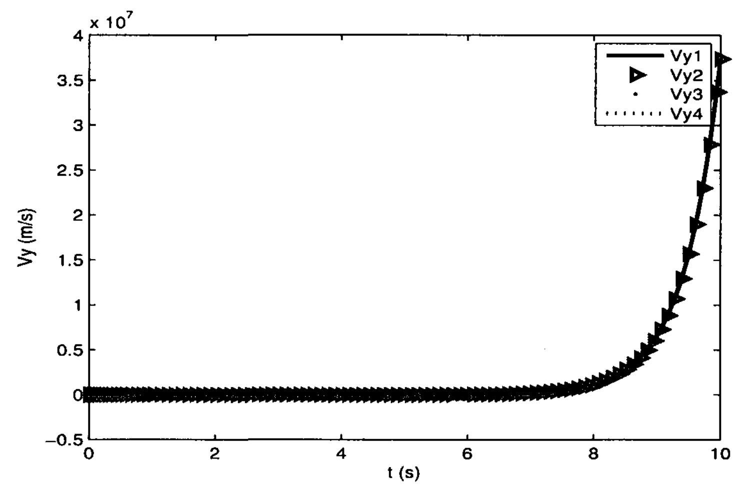


(b)

Figure 5.4: a) The x-component and b) the y-component of the velocity profile; optimal design based on the solution of the LMIs when $AS \neq 0$ in Example 1.

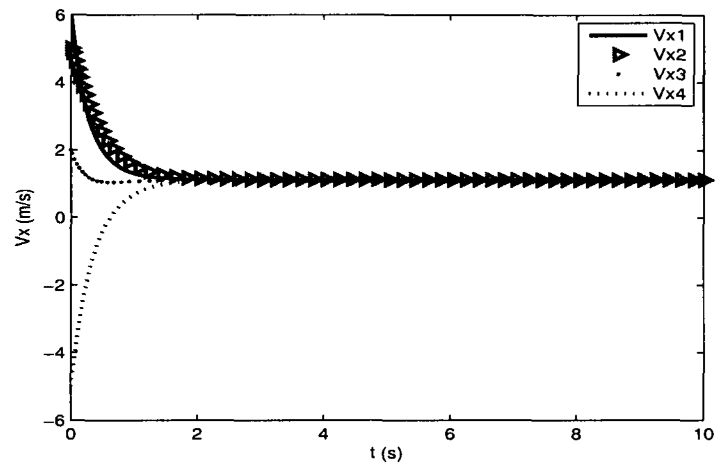


(a)

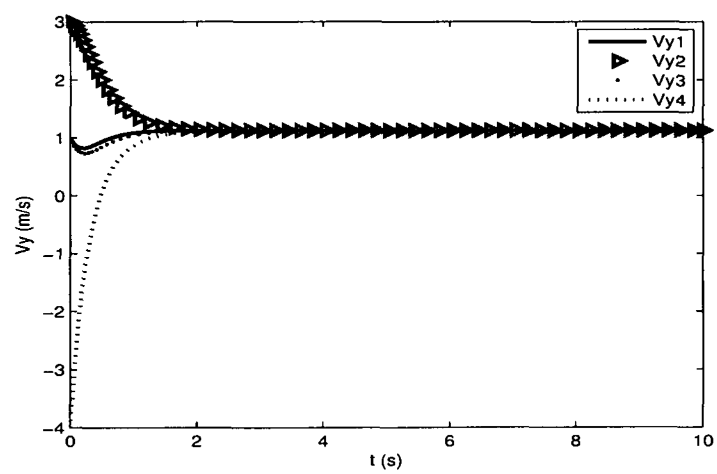


(b)

Figure 5.5: a) The x-component and b) the y-component of the velocity profile; optimal design based on the solution of the Riccati equation when $AS \neq 0$ in Example 1.

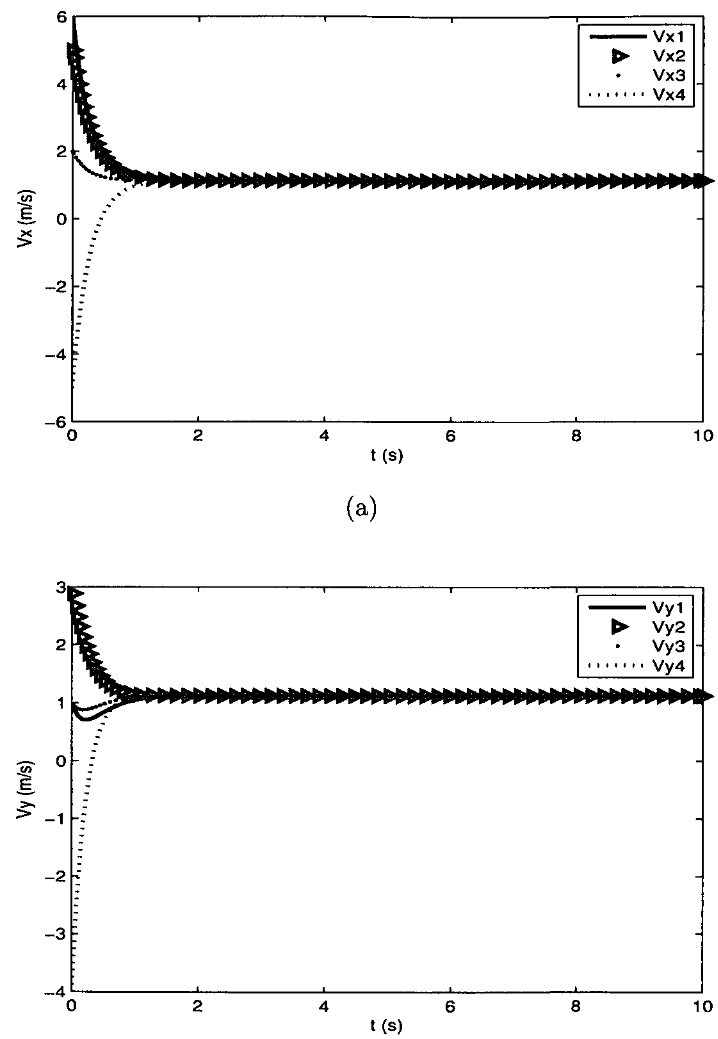


(a)



(b)

Figure 5.6: a) The x-component and b) the y-component of the velocity profile; optimal design based on the solution of the LMIs when $AS = 0$ in Example 1.



(a)

(b)

Figure 5.7: a) The x-component and b) the y-component of the velocity profile; optimal design based on the solution of the Riccati equation when $AS = 0$ in Example 1.

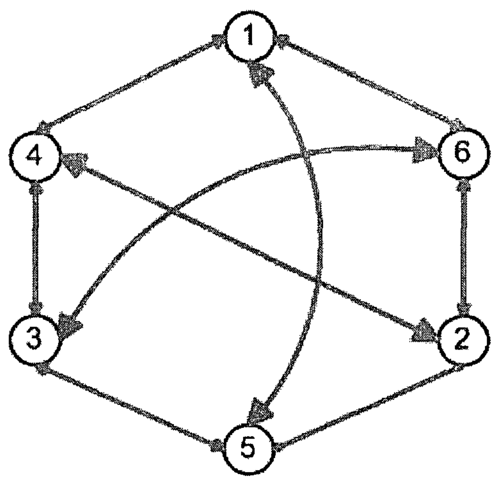
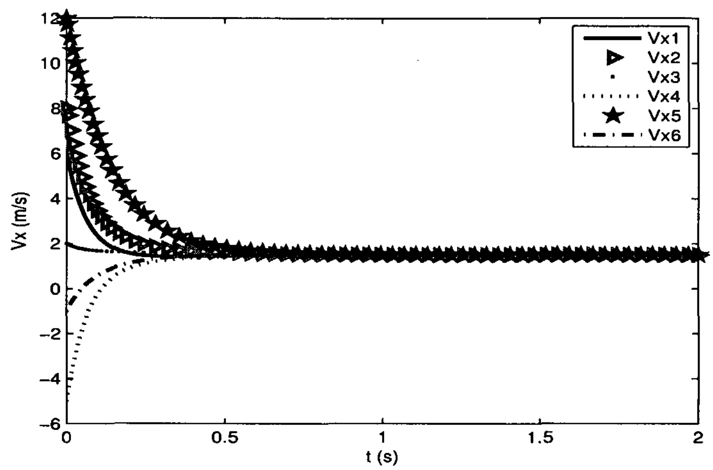
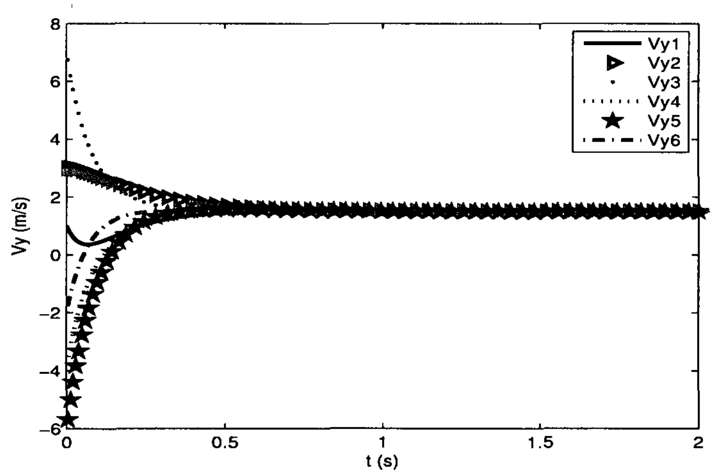


Figure 5.8: Graph describing the topology of a network of multi-agents

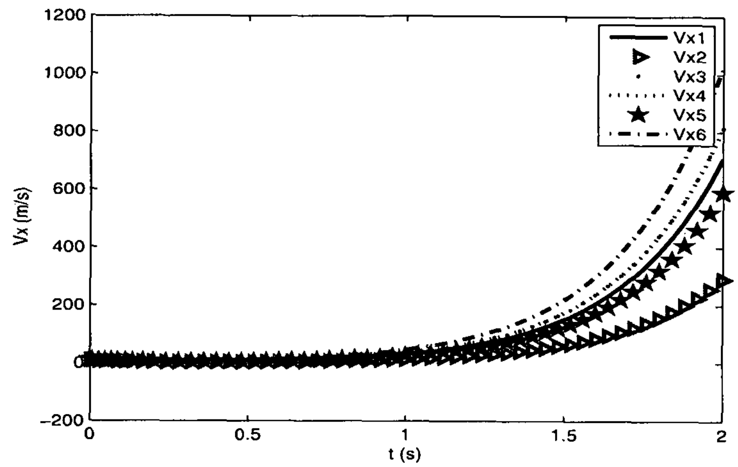


(a)

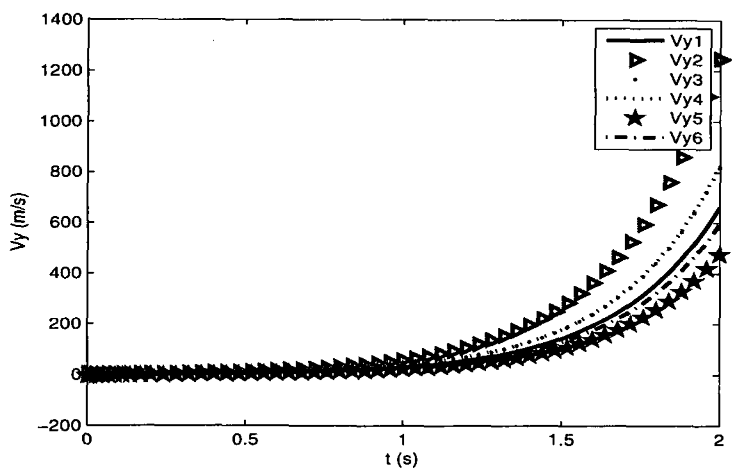


(b)

Figure 5.9: a) The x-component and b) the y-component of the velocity profiles corresponding to an optimal control design based on the solution of the LMIs when $AS \neq 0$ in Example 2.

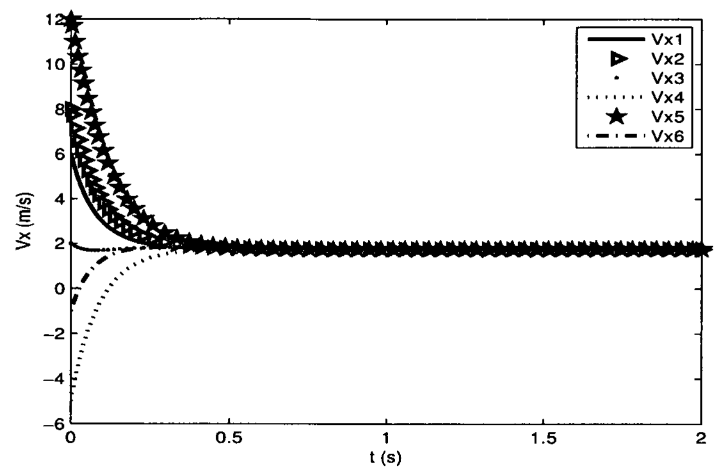


(a)

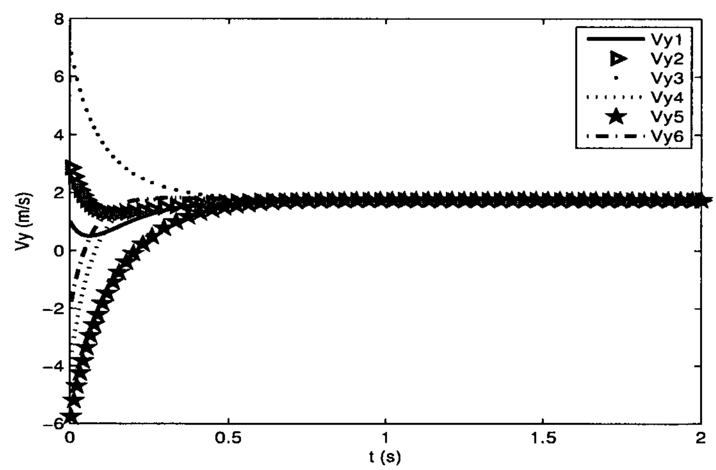


(b)

Figure 5.10: a) The x-component and b) the y-component of the velocity profiles corresponding to an optimal control design based on the solution of the Riccati equation when $AS \neq 0$ in Example 2.

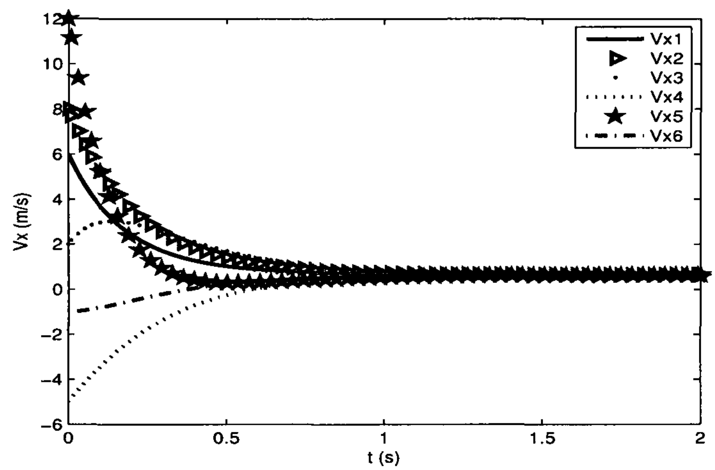


(a)

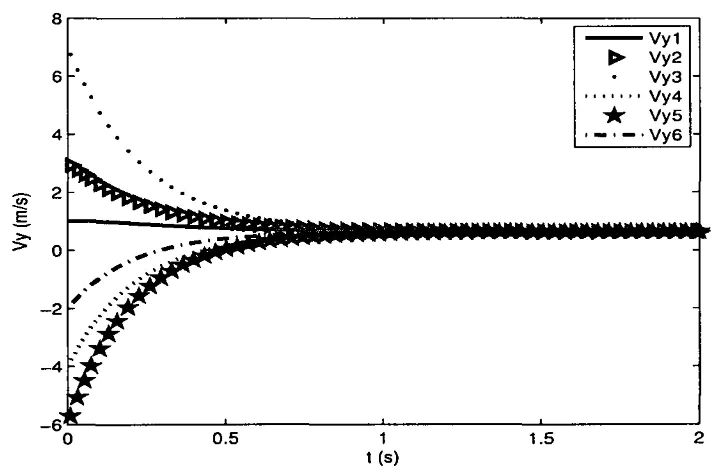


(b)

Figure 5.11: a) The x-component and b) the y-component of the velocity profiles corresponding to an optimal control design based on the solution of the LMIs when $AS = 0$ in Example 2.



(a)



(b)

Figure 5.12: a) The x-component and b) the y-component of the velocity profiles corresponding to an optimal control design based on the solution of the Riccati equation when $AS = 0$ in Example 2.

Chapter 6

Conclusions and future work

6.1 Conclusions

The main concentration of this research has been on the coordination and collaboration in a network of multi-agents. The main focus was on how to design a control law to satisfy several goals in a team of agents such as stability of the consensus with partial team information and presence of faults or topology changes. The problem of team collaboration, which was of considerable importance, played the key role in the proposed methodologies. The goals in this research were to develop innovative and novel concepts, techniques, and solutions to meet the stringent requirements that are envisaged for the network of unmanned vehicles to be employed in different applications.

First, I have solved the problem of team cooperation for the LL and the MLF structures by using optimal control theory technique based on the solution of HJB equations. The consensus problem is solved for a team of agents having a general linear dynamical model characteristics or a point-mass model. I have also introduced interaction terms in the dynamical representation of the agents. Stability of the team was guaranteed using modified consensus algorithms achieved by minimizing a set of individual cost functions. In another approach to find an optimal consensus algorithm, the idea of state decomposition was used to reduce the consensus seeking problem into a stabilization problem. In another methodology, the game theory was used to formulate the consensus seeking problem in a "more" cooperative framework. For this purpose a team cost function was defined and a min-max problem was solved to obtain a cooperative optimal solution for the consensus seeking problem. It was shown that the result obtained by this approach results in a lower cost values when compared to the values obtained by the optimal control technique. In this approach and the optimal control approach based on state decomposition, linear matrix inequalities were used to impose both the decentralized nature of the problem and the consensus constraint on the designed controllers.

Moreover, I have analyzed the performance of the previously designed cooperative team in presence of actuator anomalies for three types of faults. It was shown that depending on the fault type, the steady state error of the members output may be zero, bounded or time-varying. The steady state behavior of the team members was discussed and the final value to which each agent converges was predicted for all three types of faults. Also, adaptability of team members to these unanticipated situations and circumstances was discussed and verified. Later, the assumption of having a fixed and undirected network topology was relaxed to reflect a more realistic problem. Therefore, I have considered the switching topology for the team and assumed that the links among the agents can be bidirectional and weighted as well as time-varying. Even the leader assignment was assumed to be flexible and time-varying. It was shown that if the team system matrix corresponding to the error dynamics of the team is designed appropriately, a common Lyapunov function can be found for the team. Therefore, the stability and consensus achievement of the network with a switching structure and switching leader assignment can be guaranteed. For this purpose, some constraints should be imposed on the optimal controller coefficients designed initially for the fixed network topology. It was shown that by introducing additional criteria, the desirable performance specifications of the team can still be ensured and guaranteed. As a demonstration of such a criterion, performance-control effort tradeoff was considered and analyzed in details.

As a conclusion, in this research the team cooperation problem was formulated in a framework which is broad and flexible enough to address a wide range of problems that arise in cooperative control with restricted information exchange structures and agents' dynamics while considering all the given limiting constraints. This work has provided novel advances and improvements in the existing literature on cooperative control toward addressing more realistic and challenging problems.

6.2 Future work

Some of the future extensions of the present research are as follows:

- Generalization of the proposed methodologies to heterogenous types of agents and investigation of the consensus protocols for the agents with nonlinear dynamics.
- Design of compensating controllers (recovery strategies) that can avoid deterioration of the team performance in presence of members' faults.
- Incorporation of stochastic actuator faults into the proposed framework, i.e. fault happening is stochastic in terms of the variation of α.

- Introduction of a quantified index to measure the effects of decentralization of information on the increase of the proposed team cost. In other words, to quantify the effects of the connectedness of the information graph on the performance of the team. This quantization criterion may provide an insight into the tradeoffs that exist between the availability of information and the team cost for the proposed methods in this research.

- Obtaining a solution that ensures the required stability for a general switching network structure while the restrictions that are imposed on the optimal performance of the controller are minimized.

- Considering the evolving (dynamic) networks where agents are added or removed from the team.

- Combination of parallel estimation techniques with the common consensus protocols. In many applications the agents need to estimate the required information due to incomplete measurement, and missed or partial information. Hence, either an output of the entire group is available and the state vector of the group should be estimated or part of neighbors' state is available and the rest should be estimated. Based on this, an estimation of the state of the entire group (or some of the agents) should be found. An important issue is to find the minimum information which should be available for each member to be able to estimate the required information and remove the disagreement dynamics (observability definition).

- Considering the effect of noisy communication channels on the proposed methodologies.

- Considering interaction of several teams while some of them act as an adversary in an environment with dynamic obstacles and popup threats.

Any of the above mentioned directions can be investigated in detail as a challenging research topic which can significantly improve the current results on team cooperation and specifically on the consensus seeking problem.

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Appendix A

Proofs of the lemmas and theorems of Chapter 3

<u>**Proof of Lemma 3.1**</u>

First note that by assuming $V^i = \frac{1}{2}(v^i)^T K^i(t) v^i + \gamma^i$, Λ^i in (3.11) can be simplified as:

$$\begin{aligned}\Lambda^i(t, X^i, u^{i*}) &= \sum_{j\in N^i} (v^i - v^j)^T Q^{ij}(v^i - v^j) + (u_l^{i*})^T R^i u_l^{i*} \\ &+ \frac{\partial V^i}{\partial v^i}(u_l^{i*} + \sum_{j\in N^i} \mathcal{F}^{ij} v^j) = -\frac{\partial V^i}{\partial t}(t, X^i)\end{aligned} \tag{A.1}$$

Now by replacing $V^i, \mathcal{F}^{ij}$ and u_l^{i*} according to (3.13), (3.15), and (3.16), respectively, we obtain

$$\begin{aligned}&-(v^i)^T \frac{\dot{K}^i}{2} v^i - \dot{\gamma}^i = \sum_{j\in N^i} (v^i - v^j)^T Q^{ij}(v^i - v^j) \\ &-(v^i)^T \frac{K^i}{4}(R^i)^{-1} K^i v^i + (v^i)^T K^i \sum_{j\in N^i} 2(K^i)^{-1} Q^{ij} v^j\end{aligned} \tag{A.2}$$

By equating the corresponding terms in $(v^i)^T v^i$, and v^i, the Riccati equation in (3.17) may be obtained. This implies that the HJB equation (3.11) has a solution which satisfies the boundary conditions and so using the results presented in [123], it provides the optimal strategy (see [123], page 222). Moreover, the resulting Riccati equation in (3.17) has a solution since it describes the equation corresponding to the LQR problem in a linear system with the pair (A^i, B^i) described in (3.14). Due to controllability of this pair the solution exists. This completes the proof. ∎

Proof of Theorem 3.1

a) Consensus protocol: Note that in the infinite horizon case, $\dot{K}^i = 0$ and so the differential Riccati equation (3.17) will reduce to an algebraic equation given by

$$2|N^i|Q^{ij} - \frac{1}{2}K^i(R^i)^{-1}K^i = 0 \Rightarrow 2|N^i|(K^i)^{-1}Q^{ij} = \frac{1}{2}(R^i)^{-1}K^i \tag{A.3}$$

and consequently for $i = 1, \ldots, N$, we obtain

$$u^{i*}(v^i, v^j) = u_l^{i*} + u_g^{i*} = -\frac{1}{2}(R^i)^{-1}K^i(v^i - \frac{\sum_{j\in N^i} v^j}{|N^i|}) = \Gamma^i(v^i - \frac{\sum_{j\in N^i} v^j}{|N^i|}) \tag{A.4}$$

b) and c) By applying the control law (3.18) to the dynamical equations of each agent in (2.20), the closed-loop velocity dynamics of the entire team will be found as $\dot{v} = L_{cl}v$, in which

$$
L_{cl} = \begin{bmatrix}
\Gamma^1 & \frac{l_{12}}{|N^1|}\Gamma^1 & \dots & \frac{l_{1N}}{|N^1|}\Gamma^1 \\
\frac{l_{21}}{|N^2|}\Gamma^2 & \Gamma^2 & \dots & \frac{l_{2N}}{|N^2|}\Gamma^2 \\
\vdots & \vdots & \vdots & \vdots \\
\frac{l_{i1}}{|N^i|}\Gamma^i & \dots & \frac{l_{ij}}{|N^i|}\Gamma^i & \dots \\
\vdots & \vdots & \vdots & \vdots \\
\frac{l_{N1}}{|N^N|}\Gamma^N & \dots & \frac{l_{N(N-1)}}{|N^N|}\Gamma^N & \Gamma^N
\end{bmatrix},
$$

$v = [(v^1)^T \ \dots \ (v^N)^T]^T$, and l_{ij} is the ijth element of the Laplacian matrix L. For sake of simplicity and without loss of generality, assume that $|N^i| = \bar{N}, \forall i$, and $R^i = R$, $Q^{ij} = Q \ \forall i, j$ are diagonal PD matrices. Hence, $K^i = K^j, \forall i, j$ and so $L_{cl} = \frac{1}{\bar{N}} L \otimes \Gamma$, where $\Gamma = \Gamma^i, \forall i$. It is known from the graph theory that matrix L is always PSD and for undirected connected graphs it has a single zero eigenvalue associated with a unit eigenvector $[1 \ 1 \ \dots \ 1]^T$. Also, using the properties of Riccati equations and given the fact that the pair (A^i, B^i) defined in (3.14) is reachable and (A^i, Ω^i) is observable, where $Q^{ij} = (\Omega^i)^T \Omega^i$, by putting $\dot{K}^i = 0$ in (3.17), we conclude that the solution K^i is PD [133].

This implies that Γ is diagonal and Negative Definite (ND). Hence, all the eigenvalues of $L \otimes \Gamma$ are negative except for two (or the size of Γ) zero eigenvalues associated with eigenvectors: $w(L \otimes \Gamma) = \frac{1}{\sqrt{N}}[1 \ 1 \ \dots \ 1]^T \otimes w(\Gamma)$, i.e. γ_1 and γ_2. Here $w(\Gamma)$ denotes any eigenvector of matrix Γ, i.e. w_1, w_2. Therefore due to the symmetry of $L \otimes \Gamma$ it is Negative Semi-Definite (NSD) with distinct eigenvectors [143].

Now assume that the matrix Δ is the similarity transformation matrix consisting of all eigenvectors of matrix L_{cl} and define the new state vector to be $\bar{v} = \Delta^{-1} v(t)$. The transformed closed-loop system will be $\dot{\bar{v}} = J\bar{v}$, in which J is the Jordan form of L_{cl} which is fully diagonal with the first two (or size of Γ) diagonal elements being zero and the rest being negative. Therefore, the final state of the closed-loop system in the Jordan canonical form will be $\bar{v}(\infty) = [\bar{v}_1(0)\ \bar{v}_2(0)\ 0\ \ldots\ 0]^T$, and consequently in the original coordinates the steady state value of the vector v will be a linear combination of the first two eigenvectors of L_{cl} corresponding to the zero eigenvalues, i.e. $v(\infty) = \gamma_1\bar{v}_1(0) + \gamma_2\bar{v}_2(0)$. Also, since $\bar{v}(0) = \Delta^{-1}v(0)$, we will have $\bar{v}_1(0) = \gamma_1^T v(0)$ and $\bar{v}_2(0) = \gamma_2^T v(0)$. Hence

$$v(\infty) = \gamma_1\bar{v}_1(0) + \gamma_2\bar{v}_2(0) = (\gamma_1\gamma_1^T + \gamma_2\gamma_2^T)v(0)$$

$$= \frac{1}{N}(\begin{bmatrix} 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \ldots & 1 \end{bmatrix} \otimes (w_1w_1^T + w_2w_2^T))v(0)$$

$$= [1\ 1\ \ldots\ 1]^T \otimes (w_1w_1^T + w_2w_2^T)Avg(v(0))$$

In other words, the final state will be a constant vector of the form $[v^c\ v^c\ \ldots\ v^c]^T$ in which v^c is given by (3.20), and this will lead to the consensus achievement. Moreover, if the matrix Γ has the same diagonal entries, $w_1w_1^T + w_2w_2^T$ is identity and so the consensus value is $Avg(v(0))$, i.e. an average consensus is achieved. ■

Proof of Lemma 3.2

I only prove the leader case and the followers case is similar. Note that Λ^1 in (3.22) can be simplified by replacing V^1, u_g^{1*} and u_l^{1*} according to (3.24), (3.25), and (3.27), respectively:

$$\begin{aligned}\Lambda^1 &= \sum_{j\in N^1}(v^1-v^j)^TQ^{1j}(v^1-v^j)+(v^1-v^d)^T\Gamma(v^1-v^d)-(v^1)^T\frac{K^1}{4}(R^1)^{-1}K^1v^1\\&-(g^1)^T\frac{(R^1)^{-1}}{2}K^1v^1-(g^1)^T\frac{(R^1)^{-1}}{4}g^1+((v^1)^TK^1+(g^1)^T)\sum_{j\in N^1}2(K^1)^{-1}Q^{1j}v^j\\&=-(v^1)^T\frac{\dot{K}^1}{2}v^1-(\dot{g}^1)^Tv^1-\dot{\gamma}^1\end{aligned}$$

by equating the corresponding terms in $(v^1)^Tv^1$, and v^1, the equations in (3.29) and (3.30) can be obtained. ■

Proof of Theorem 3.2

a) Modified Consensus Protocol. The proof is similar to the proof of part a) of Theorem 3.1 and therefore is omitted.

b) Stability Analysis: To prove this part of the theorem let us first illustrate the following fact:

Fact A.1. For any PD matrix B and matrix A satisfying $A^T+A<0$ ($A^T+A\leqq 0$), the product $C=BA$ is Hurwitz.

Proof: We have to show that C satisfies the Lyapunov equation for some PD matrices P and Q, i.e. $C^TP+PC=-Q$. Let's take $P=B^{-1}$. Since B is symmetric, we will have $C^TP+PC=A^TB^TB^{-1}+B^{-1}BA=A+A^T<0$. This clearly shows that matrix C is Hurwitz. ■

Let us now assume that the desired leader command v^d is time-invariant and define the tracking error for each agent as $e^i=v^i-v^d$. The error dynamics for the entire team can be found by using the agents' dynamical equations and the input commands for the leader and followers as given by (2.20), (3.32), and (3.31), respectively, to obtain $\dot{e} = L_{cl}e$ with

$$L_{cl} = \begin{bmatrix} \Gamma^1 + \beta^1 & \frac{l_{12}}{|N^1|}\Gamma^1 & \dots & \frac{l_{1N}}{|N^1|}\Gamma^1 \\ \frac{l_{21}}{|N^2|}\Gamma^2 & \Gamma^2 & \dots & \frac{l_{2N}}{|N^2|}\Gamma^2 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{l_{i1}}{|N^i|}\Gamma^i & \dots & \frac{l_{ij}}{|N^i|}\Gamma^i & \dots \\ \vdots & \vdots & \vdots & \vdots \\ \frac{l_{N1}}{|N^N|}\Gamma^N & \dots & \frac{l_{N(N-1)}}{|N^N|}\Gamma^N & \Gamma^N \end{bmatrix} \tag{A.5}$$

where $e = [(e^1)^T \ \dots \ (e^N)^T]^T$ and the rest of parameters are defined as before.

For sake of simplicity, let us assume that $R^i = R, \ Q^{ij} = Q \ \forall i, j$. Hence

$$L_{cl} = -2K^{-1}(L \otimes Q + G) = -2K^{-1}(L \otimes Q + \begin{bmatrix} \Gamma & \vdots & 0 \\ \dots & \dots & \dots \\ 0 & \vdots & 0 \end{bmatrix})$$

where $K = Diag\{K^i, i = 1, \dots, N\}$ and $G = \begin{bmatrix} \Gamma & \vdots & 0 \\ \dots & \dots & \dots \\ 0 & \vdots & 0 \end{bmatrix}$. Then, $L \otimes Q$ is PSD with two (or the size of Q) zero eigenvalues associated with the eigenvectors: $w(L \otimes Q) = [1\ 1\ \dots\ 1]^T \otimes w(Q)$, in which $w(Q)$ denotes any eigenvector of the matrix Q. On the other hand, G is PSD because Γ is PD. Hence, $L \otimes Q + G$ is PSD. However, it can be verified that the null-spaces of $L \otimes Q$ and G do not have an intersection, i.e. $\{w | w = [w(Q)^T \ w(Q)^T \ \dots \ w(Q)^T]^T\} \cap \{w | w = [0 \ \ \bar{w}]^T\} = \{0\}$, and so $L \otimes Q + G$ is PD. Also, based on the discussions in the proof of Theorem 3.1, the solution of Riccati equations obtained by setting $\dot{K}^i = 0$ in (3.28) and $\dot{K}^1 = 0$ in (3.29) are PD and symmetric, and hence K is PD and symmetric. Finally, using Fact A.1, L_{cl} is Hurwitz, and consequently the error dynamics is asymptotically stable. ■

Proof of Lemma 3.3

For the leader case, Λ^1 in (3.36) can be simplified by replacing V^1, u_g^{1*} and u_l^{1*} according to (3.39), (3.40), and (3.42), respectively. By equating the corresponding terms in $(v^1)^T v^1$, $(v^1)^T v^j$, and v^1, equations (3.44) and (3.45) may be derived. The follower case is shown along the similar lines. In other words, the HJB equations in (3.35)-(3.37) have a solution that satisfies the boundary conditions so that by using the results developed in [123], an optimal strategy is achieved (see [123], page 222). Moreover, the Riccati equation in (3.43) (or (3.44)) has a solution since it describes the equation corresponding to an LQR problem for a linear system with a reachable pair (A^i, B^i). Therefore, the solution to equation (3.43) (or (3.44)) is guaranteed. This therefore completes the proof of this lemma. ■

Proof of Theorem 3.3

a) Let us first start with the followers case. The leader case may be proved in a similar manner. Note that in the infinite horizon case solutions are achieved by equating $\dot{K}^i = 0$, $\dot{K}^1 = 0$ and $\dot{g}^1 = 0$ in equations (3.43), (3.44), and (3.45), respectively. Consequently, the differential Riccati equation (3.43) reduces to the following algebraic equation

$$
\begin{aligned}
&2|N^i|Q^{ij} - \frac{1}{2}K^iB^i(R^i)^{-1}(B^i)^TK^i + (A^i)^TK^i + K^iA^i = 0 \Rightarrow \\
&\frac{1}{2}(R^i)^{-1}(B^i)^TK^i = 2|N^i|(K^iB^i)^{-1}Q^{ij} + (K^iB^i)^{-1}((A^i)^TK^i + K^iA^i)
\end{aligned}
$$

which after some algebraic manipulations yields

$$
u^{i*} = -2|N^i|(K^iB^i)^{-1}Q^{ij}(v^i - \frac{\sum_{j\in N^i} v^j}{|N^i|}) - (K^iB^i)^{-1}((A^i)^TK^i + K^iA^i)v^i
$$

b) Let us assume that the desired leader command v^d is time-invariant and the tracking error for each agent is given by $e^i = v^i - v^d$. Given the definition of e, the error dynamics is now derived by using agents dynamical equations, the input command for the leader, and the input command for the followers as given in (2.21), (3.47), and, (3.46), respectively, according to

$$\begin{aligned}\dot{e}^i &= B^i\Gamma^i(e^i - \frac{\sum_{j\in N^i} e^j}{|N^i|}) - (K^i)^{-1}(A^i)^T K^i(e^i + v^d), \quad i = 2,\ldots,N \\ \dot{e}^1 &= B^1\Gamma^1(e^1 - \frac{\sum_{j\in N^1} e^j}{|N^1|}) + B^1\alpha^1 e^1 - (K^1)^{-1}(A^1)^T(g^1 + K^1e^1 + K^1v^d)\end{aligned} \tag{A.6}$$

Hence, the error dynamics for the entire team can be written as $\dot{e} = L_{cl}e + f(v^d, g^1)$, where $e = [(e^1)^T \ \ldots \ (e^N)^T]^T$, L_{cl} is given by

$$L_{cl} = -K^{-1}(2L \otimes Q^{ij} + A^TK + 2G)$$

and matrices K, A, G are defined in the statement of Theorem 3.3. $f(v^d, g^1)$ is the part of error dynamics in (A.6) that is a function of v^d and g^1 which does not affect the stability analysis and is given by

$$f(v^d, g^1) = \begin{bmatrix} -(K^1)^{-1}(A^1)^T(g^1 + K^1v^d) \\ \vdots \\ -(K^i)^{-1}(A^i)^TK^iv^d \\ \vdots \\ -(K^N)^{-1}(A^N)^TK^Nv^d \end{bmatrix} \tag{A.7}$$

Stability of this matrix can be guaranteed only if matrices Q^{ij}, R^i, and Γ are selected properly such that the matrix L_{cl} is Hurwitz. Intuitively, to achieve a good tracking of the desired output by the followers, Q^{ij} and Γ

should be selected sufficiently large when compared to R^i. However, in order to obtain a formal solution, one method for finding suitable values for Q^{ij}, R^i, and Γ is to use the LMI technique to ensure stability of the closed-loop system. Here, I assume that Q^{ij}, $\forall i,j$ and R^i, $i = 2, \dots, N$ are predefined and so one tries to find R^1 and Γ such that L_{cl} is Hurwitz. For this purpose, let the following LMI in variables Γ, K^1 be satisfied

$$\Upsilon + \Upsilon^T > 0, \quad \Upsilon = -KL_{cl} = 2L \otimes Q^{ij} + A^T K + 2G$$

As it was discussed previously the pair (A^i, B^i) used in (2.21) is reachable, and $Q^{ij} = (\Omega^i)^T \Omega^i > 0$ may always be selected so that the matrix Ω^i is full rank, and hence the pair (A^i, Ω^i) will be observable. Therefore, the solution of Riccati equations obtained by setting $\dot{K^i} = 0$ in (3.43) and $\dot{K^1} = 0$ in (3.44), will be PD, leading to K to be also PD. Now by invoking Fact A.1, we may conclude that $L_{cl} = -K^{-1}\Upsilon$ is Hurwitz. On the other hand, K^1 has to satisfy the Riccati equation (3.44) and therefore this equation should be added as a constraint. Hence, the following set of LMIs may be considered in which the unknowns to be determined are the elements Γ, K^1, and $Z^1 = -\frac{1}{2}K^1 B^1 (R^1)^{-1} (B^1)^T K^1$, governed by expressions

$$\begin{cases} \Upsilon + \Upsilon^T > 0, \quad \Upsilon = 2L \otimes Q^{ij} + A^T K + 2G \\ (A^1)^T K^1 + K^1 A^1 + Z^1 + 2(|N^1| Q^{1j} + \Gamma) = 0, \\ \Gamma > 0, \quad K^1 > 0, \quad Z^1 < 0 \end{cases}$$

and from which Γ, K^1, and Z^1, and hence $R^1 = -\frac{1}{2}(B^1)^T K^1 (Z^1)^{-1} K^1 B^1$ may be calculated.

Note that the matrix Q^{1j} is assumed to be predefined and is set equal to $Q^{ij}, i = 2, \dots, N$. However, Q^{1j} can also be selected and designed formally by adding it to the set of LMI parameters. ■

Proof of Theorem 3.4

a) The proof for this part is similar to the proof of part a) of Theorem 3.3.

b), c) We first note that for consensus achievement, one should determine the closed-loop dynamics corresponding to the team velocity. Using the results of part a) and the combined control law given in (3.53), the dynamics of each agent's velocity will be governed by

$$\begin{aligned}\dot{v}^i &= A^i v^i - 2(K^i)^{-1}|N^i|Q^{ij}(v^i - \frac{\sum_{j\in N^i} v^j}{|N^i|}) - (K^i)^{-1}(K^i A^i + (A^i)^T K^i)v^i \\ &= -2(K^i)^{-1}|N^i|Q^{ij}(v^i - \frac{\sum_{j\in N^i} v^j}{|N^i|}) - (K^i)^{-1}(A^i)^T K^i v^i\end{aligned}$$

Consequently, the closed-loop system team matrix will become

$$L_{cl} = -K^{-1}(2L \otimes Q^{ij} + A^T K)$$

where $\dot{v} = L_{cl}v$, $v = [(v^1)^T \ (v^2)^T \ \ldots \ (v^N)^T]^T$, $K = Diag\{K^i, i = 1, \ldots, N\}$, and $A = Diag\{A^i, i = 1, \ldots, N\}$. Now in order to have consensus, L_{cl} should be stable and moreover we should have

$$L_{cl}w = 0, \ w = [(v^c)^T \ (v^c)^T \ \ldots \ (v^c)^T]^T \tag{A.8}$$

which implies that $2L \otimes Q^{ij}w + A^T K w = 0$. The first term is always zero due to the properties of the Laplacian matrix, i.e. $2L \otimes Q^{ij}w = 0$, and so we should have

$$A^T K w = 0 \Rightarrow \ (A^i)^T K^i v^c = 0 \tag{A.9}$$

Due to the non-singularity of K^i, the above expression implies that the matrix A^i should be rank deficient, i.e. $|(A^i)^T| = 0$ and v^c should be the eigenvector of $(A^i)^T K^i$ corresponding to the zero eigenvalue. This means that v^c should be in the null space of $(A^i)^T K^i$.

Moreover, in order to have a stable consensus, $L_{cl} = -K^{-1}(2L \otimes Q^{ij} + A^T K)$ should be stable, i.e. all its eigenvalues should be negative except the zeros corresponding to the eigenvector w. For this to hold and since K^{-1} is PD it suffices to have $(2L \otimes Q^{ij} + A^T K) + (2L \otimes Q^{ij} + A^T K)^T \geq 0$ (see Fact A.1). Equivalently, we should have $2L \otimes Q^{ij} + (2L \otimes Q^{ij})^T + A^T K + KA \geq 0$. However, $2L \otimes Q^{ij} + (2L \otimes Q^{ij})^T \geq 0$ holds by the definition of Laplacian matrix and connectivity of the graph. Therefore, it is enough to have $A^T K + KA \geq 0$ or $(A^i)^T K^i + K^i A^i \geq 0$. ■

Appendix B

Proofs of the lemmas and theorems of Chapter 4

Proof of Lemma 4.1

a, b) Here, I only discuss the followers failure. It should be noted that with a minor modification the proof presented here can accommodate LOE faults in the leader as well.

Based on the discussions given in the proof of Theorem 3.2, the error dynamics closed-loop matrix L_{cl} can be written as

$$L_{cl} = -2K^{-1}(L \otimes Q + G) = -2K^{-1}(L \otimes Q + \begin{bmatrix} \Gamma & \vdots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \vdots & 0 \end{bmatrix}) \tag{B.1}$$

When an LOE fault is injected in some followers the closed-loop dynamics of the system is governed by (4.2), with the closed-loop matrix given as

$$L_f = \begin{bmatrix} L_{11} & \vdots & L_{12} \\ \cdots & \cdots & \cdots \\ \alpha L_{21} & \vdots & \alpha L_{22} \end{bmatrix} = I_f L_{cl} = -2I_f K^{-1}(L \otimes Q + G) =$$

$$-2I_f \begin{bmatrix} (K_{11}^{-1})_{m(N-q)\times m(N-q)} & \vdots & 0_{m(N-q)\times mq} \\ \cdots & \cdots & \cdots \\ 0_{mq\times m(N-q)} & \vdots & (K_{22}^{-1})_{mq\times mq} \end{bmatrix} \times \tag{B.2}$$

$$\begin{bmatrix} (\bar{L}_{11} + G_{11}) & \vdots & (\bar{L}_{12} + G_{12}) \\ \cdots & \cdots & \cdots \\ (\bar{L}_{21} + G_{21}) & \vdots & (\bar{L}_{22} + G_{22}) \end{bmatrix}$$

where $K_{ij}^{-1}, \bar{L}_{ij}, G_{ij}$ denote the partitions of matrices K^{-1}, $L \otimes Q$, and G, respectively which correspond to the faulty and healthy agents dynamics. I_f is defined as

$$I_f = \begin{bmatrix} I_{m(N-q)\times m(N-q)} & \vdots & 0_{m(N-q)\times mq} \\ \cdots & \cdots & \cdots \\ 0_{mq\times m(N-q)} & \vdots & \alpha I_{mq\times mq} \end{bmatrix} \tag{B.3}$$

This matrix is PD and diagonal for $0 < \alpha \leq 1$, and therefore $I_f K^{-1}$ is also PD. In addition, $L \otimes Q + G$ can be shown to be a PD matrix. Hence, by invoking Fact A.1, it can be seen that L_f is asymptotically stable, and consequently the error asymptotically approaches to zero. This implies that the consensus achievement is guaranteed and all the agents' output will converge to the command provided by the leader. ■

Proof of Lemma 4.2

a) As discussed previously, the error dynamics closed-loop matrix L_{cl} can be written as

$$L_{cl} = -2K^{-1}(L \otimes Q + G)$$

In case of faults in some followers the closed-loop dynamics will be governed by (4.3), where the closed-loop matrix can be written as

$$\begin{aligned}
L_f &= \begin{bmatrix} L_{11} & \vdots & L_{12} \\ \cdots & \cdots & \cdots \\ 0_{mq\times m(N-q)} & \vdots & 0_{mq\times mq} \end{bmatrix} = I_f L_{cl} = -2I_f K^{-1}(L \otimes Q + G) = \\
&- 2I_f \begin{bmatrix} (K_{11}^{-1})_{m(N-q)\times m(N-q)} & \vdots & 0_{m(N-q)\times mq} \\ \cdots & \cdots & \cdots \\ 0_{mq\times m(N-q)} & \vdots & (K_{22}^{-1})_{mq\times mq} \end{bmatrix} \\
&\times \begin{bmatrix} (\bar{L}_{11} + G_{11}) & \vdots & (\bar{L}_{12} + G_{12}) \\ \cdots & \cdots & \cdots \\ (\bar{L}_{21} + G_{21}) & \vdots & (\bar{L}_{22} + G_{22}) \end{bmatrix} \\
&= \begin{bmatrix} (L_{11})_{m(N-q)\times m(N-q)} & \vdots & (L_{12})_{m(N-q)\times mq} \\ \cdots & \cdots & \cdots \\ 0_{mq\times m(N-q)} & \vdots & 0_{mq\times mq} \end{bmatrix}
\end{aligned} \tag{B.4}$$

where $K_{ij}^{-1}, \bar{L}_{ij}, G_{ij}$ denote the corresponding partitions of matrices K^{-1}, $L \otimes Q$, and G, respectively, which correspond to the faulty and healthy agents dynamics. I_f is defined as follows

$$I_f = \begin{bmatrix} I_{m(N-q)\times m(N-q)} & \vdots & 0_{m(N-q)\times mq} \\ \cdots\cdots & \cdots & \cdots\cdots \\ 0_{mq\times m(N-q)} & \vdots & 0_{mq\times mq} \end{bmatrix} \tag{B.5}$$

This matrix is PSD and so $I_f K^{-1}$ is PSD. Also, $L \otimes Q + G$ is shown to be a PD matrix. Hence, by invoking Fact A.1, it can be seen that L_f will be stable and so the error remains bounded.

b) It can be readily verified that the eigenvalues of L_f, consist of eigenvalues of matrix L_{11} and mq zeros. Using (B.4), L_{11} can be written as $L_{11} = -2K_{11}^{-1}(\bar{L}_{11} + G_{11})$ and due to the positive definiteness of $L \otimes Q + G$ and K^{-1}, $\bar{L}_{11} + G_{11}$ and K_{11}^{-1} are both PD, and therefore L_{11} is Hurwitz (see Fact A.1). Hence, its eigenvalues are all negative and therefore the only zero eigenvalues of L_f are due to the zero rows of matrix L_f. Moreover, the eigenvectors of L_f correspond to zero eigenvalues, i.e. $w(L_f)$ are as follows

$$w(L_f) \in \{w | w \in Null\{[L_{11} \vdots L_{12}]\}\} \tag{B.6}$$

which provides mq distinct eigenvectors corresponding to zero eigenvalues. This is due to the fact that the rows of matrix $[L_{11} \vdots L_{12}]$ are linearly independent (L_{cl} is full rank) and so the rank of $Null\{[L_{11} \vdots L_{12}]\}$ is mq, which can provide mq independent eigenvectors for L_f corresponding to zero eigenvalues. This results in diagonal Jordan blocks corresponding to zero eigenvalues, and hence the final value of the error vector will be a linear combination of these eigenvectors and therefore is in the $Null\{[L_{11} \vdots L_{12}]\}$, i.e.

$$e_{ss} \in \{w | w = [w_1^T \ \ w_2^T]^T \in Null\{[L_{11} \vdots L_{12}]\}\} \Rightarrow$$

$$[L_{11} \vdots L_{12}] e_{ss} = L_{11} e_{wss} + L_{12} e_{fss} = 0 \Rightarrow \ e_{wss} = -L_{11}^{-1} L_{12} e_{fss} = -L_{11}^{-1} L_{12} e^f$$

$$\Rightarrow \ e_{ss} = \begin{bmatrix} -L_{11}^{-1} L_{12} \\ \cdots\cdots\cdots \\ I_{mq \times mq} \end{bmatrix} e^f \tag{B.7}$$

■

Proof of Lemma 4.3

a, b) In case of leader fault, the closed-loop dynamics will be governed by (4.5) and the closed-loop matrix can be written as:

$$L_f = \begin{bmatrix} 0_{m \times m} & \vdots & 0_{m \times (N-1)m} \\ \cdots & \cdots & \cdots \\ L_{21} & \vdots & L_{22} \end{bmatrix} = I_f L_{cl} = -2 I_f K^{-1} (L \otimes Q + G) =$$

$$-2K^{-1}((I_{ff} L) \otimes Q) = \frac{1}{|N^i|} (I_{ff} L) \otimes \Gamma^i = \tag{B.8}$$

$$\begin{bmatrix} 0_{m \times m} & \vdots & 0_{m \times (N-1)m} \\ \cdots & \cdots & \cdots \\ (L_{21})_{(N-1)m \times m} & \vdots & (L_{22})_{(N-1)m \times (N-1)m} \end{bmatrix}$$

where Γ^i is defined as in (3.33), and I_f and I_{ff} are defined as follows:

$$I_f = \begin{bmatrix} 0_{m \times m} & \vdots & 0_{m \times m(N-1)} \\ \cdots & \cdots & \cdots \\ 0_{m(N-1) \times m} & \vdots & I_{m(N-1) \times m(N-1)} \end{bmatrix}, \ I_{ff} = \begin{bmatrix} 0 & \vdots & 0_{1 \times (N-1)} \\ \cdots & \cdots & \cdots \\ 0_{(N-1) \times 1} & \vdots & I_{(N-1) \times (N-1)} \end{bmatrix} \tag{B.9}$$

It can be shown that multiplying the Laplacian matrix by any matrix of kind I_{ff} will not change the spectrum of the Laplacian matrix [120]. Namely, it will have the same number of zero eigenvalues (the same inertia). Moreover, it is known from graph theory that L is always PSD and for undirected connected graphs it has a single zero eigenvalue associated with a unit eigenvector, i.e.: $[1\ 1\ \ldots\ 1]^T$. This implies that $I_{ff}L$ has one zero eigenvalue and $N-1$ positive eigenvalues. Also, without loss of generality assume that both Q^{ij} and R^i are diagonal. On the other hand, using properties of Riccati equations and due to the observability and reachability conditions, the solution of the Riccati equation, K^i, obtained by putting $\dot{K}^i = 0$ in (3.28) will be diagonal and PD (see [133]). This implies that Γ^i is diagonal and negative definite.

Using properties of Kronecker product, all the eigenvalues of $(I_{ff}L) \otimes \Gamma^i$ are negative except for two (or the size of Γ^i) zero eigenvalues associated with distinct eigenvectors: $\frac{1}{\sqrt{N}}[1\ 1\ \ldots\ 1]^T \otimes w(\Gamma^i)$. Here $w(\Gamma^i)$ denotes any eigenvector of matrix Γ^i. Hence, $(I_{ff}L) \otimes \Gamma^i$ is stable. Now, using similar discussion as in the proof of Lemma 4.2, e_{ss} will be in the $Null\{[L_{21} \vdots L_{22}]\}$ which has rank m (here 2). It can be seen that any element in this space is of the form $[(e^f)^T\ (e^f)^T\ \ldots\ (e^f)^T]^T$, i.e. $e_{ss} = [(e^f)^T\ (e^f)^T\ \ldots\ (e^f)^T]^T$. ∎

<u>**Proof of Theorem 4.2**</u>

Leader Adaptability: Without loss of generality, assume that $m = 1$ or e_f (e^f) is a scalar and $q = 1$, i.e. single input-single output subsystems. In order to show that $e^1 \bullet e^f > 0$, it is sufficient to show that $[e^f\ 0\ \ldots\ 0]_{1\times(N-1)} e_{wss} > 0$, where e_{wss} is the steady state value of vector e_w. In the following we try to simplify this condition as much as possible to finally verify its correctness. Using (4.4) this condition reduces to

$$\begin{aligned}&[e^f\ 0\ \dots\ 0]_{1\times(N-1)}e_{wss} > 0 \leftrightarrow -(e^f)^2[1\ 0\ \dots\ 0]L_{11}^{-1}L_{12} > 0\\&\leftrightarrow [1\ 0\ \dots\ 0]L_{11}^{-1}L_{12} < 0\end{aligned} \tag{B.10}$$

Also, based on the definition of L_{cl} given in (4.1), I_f given in (B.5) and L_f given in (B.4), we have

$$\begin{aligned}&L_f[1\ 1\ \dots\ 1]^T_{1\times N} = I_f L_{cl}[1\ 1\ \dots\ 1]^T_{1\times N} = [\beta^1\ 0\ \dots\ 0]^T_{1\times N} \leftrightarrow\\&L_{11}[1\ 1\ \dots\ 1]^T_{1\times(N-1)} + L_{12} = [\beta^1\ 0\ \dots\ 0]^T_{1\times(N-1)} \leftrightarrow\\&L_{12} = -L_{11}[1\ 1\ \dots\ 1]^T + [\beta^1\ 0\ \dots\ 0]^T\end{aligned}$$

By replacing this value for L_{12} in (B.10) and given that $\beta^1 < 0$, we get

$$[1\ 0\ \dots\ 0]L_{11}^{-1}L_{12} = [1\ 0\ \dots\ 0]L_{11}^{-1}[\beta^1\ 0\ \dots\ 0]^T - 1 < 0 \leftrightarrow L_{11}^{-1}(1,1) > \frac{1}{\beta^1} \tag{B.11}$$

where $L_{11}^{-1}(i,j)$ is the ijth element of the matrix L_{11}^{-1}. Now, in order to show that (B.11) is satisfied we can use the properties of inverse of a matrix to describe this element as

$$L_{11}^{-1}(1,1) = \frac{C_{11}}{det(L_{11})} = \frac{C_{11}}{L_{11}(1,1)C_{11} + \ldots + L_{11}(1,N-1)C_{1(N-1)}}$$

$$= \frac{C_{11}}{(\Gamma^1+\beta^1)C_{11} + \ldots + L_{11}(1,N-1)C_{1(N-1)}} = \frac{C_{11}}{\beta^1 C_{11} + det(-2K_{11}^{-1}\bar{L}_{11})}$$

where $L_{11}(i,j)$ is the ijth element of the matrix L_{11} and C_{ij} stands for the ijth cofactor of L_{11}. $K_{ij}^{-1}, \bar{L}_{ij}$ denote the corresponding partitions of matrices K^{-1}, $L \otimes Q$, respectively, which correspond to the faulty and healthy agents dynamics. Γ^1, β^1 are defined in (4.1). Without loss of any generality, assume that N, i.e. the total number of agents, is an odd number. Similar reasoning may be used for an even N. Since L_{11} is $(N-1) \times (N-1)$, $N-1$ is even, and both K_{11}^{-1} and $\bar{L}_{11} + G_{11}$ are PD matrices and $N-1$ is even, then the determinant of the matrix $L_{11} = -2K_{11}^{-1}(\bar{L}_{11} + G_{11})$ is a positive number. Hence, $\beta^1 C_{11} + det(-2K_{11}^{-1}\bar{L}_{11})$ is positive. Similarly, $det(-2K_{11}^{-1}\bar{L}_{11})$ is also positive, since both K_{11}^{-1} and $\bar{L}_{11}$ are PD matrices. Therefore, $\beta^1 C_{11} + det(-2K_{11}^{-1}\bar{L}_{11}) > \beta^1 C_{11}$ and so $\frac{\beta^1 C_{11}}{\beta^1 C_{11} + det(-2K_{11}^{-1}\bar{L}_{11})} < 1$ or equivalently $\frac{C_{11}}{\beta^1 C_{11} + det(-2K_{11}^{-1}\bar{L}_{11})} > \frac{1}{\beta^1}$. The correctness of this inequality guarantees that the initial inequality, i.e. $e^1 \bullet e^f > 0$, is true. This completes the proof of this part.

Followers Adaptability: In order to show that $e^k \bullet e^f > 0$, $k = 2, \ldots, N$ it is sufficient to show that $[0 \ \ldots \ (e^f)_k \ \ldots \ 0]e_w > 0$. Using (4.4) this condition reduces to

$$-(e^f)^2[0 \ \ldots \ (1)_k \ \ldots \ 0]L_{11}^{-1}L_{12} > 0 \rightarrow [0 \ \ldots \ (1)_k \ \ldots \ 0]L_{11}^{-1}L_{12} < 0 \tag{B.12}$$

Similar to the leader's case we have

$$L_{12} = -L_{11}[1 \ 1 \ \ldots \ 1]^T + [\beta^1 \ 0 \ \ldots \ 0]^T$$

By replacing the above value for L_{12} in (B.12) and given that $\beta^1 < 0$, we obtain

$$[0 \ \ldots \ (1)_k \ \ldots \ 0]L_{11}^{-1}L_{12} = [0 \ \ldots \ (1)_k \ \ldots \ 0]L_{11}^{-1} \times [\beta^1 \ 0 \ \ldots \ 0]^T - 1 < 0$$
$$\leftrightarrow L_{11}^{-1}(k,1) > \frac{1}{\beta^1}$$

By using the properties of an inverse of a matrix we have

$$L_{11}^{-1}(k,1) = \frac{C_{1k}}{det(L_{11})} = \frac{C_{1k}}{L_{11}(1,1)C_{11} + L_{11}(1,2)C_{12} + \ldots + L_{11}(1,N-1)C_{1(N-1)}}$$

$$= \frac{C_{1k}}{(\Gamma^1 + \beta^1)C_{11} + \ldots + L_{11}(1,N-1)C_{1(N-1)}} = \frac{C_{1k}}{\beta^1 C_{11} + det(-2K_{11}^{-1}\bar{L}_{11})} > \frac{1}{\beta^1} \tag{B.13}$$

where C_{ij}, K_{11}^{-1}, $\bar{L}_{11}$, Γ^1, β^1 are defined as before. As was discussed in the first part of the proof both $\beta^1 C_{11} + det(-2K_{11}^{-1}\bar{L}_{11})$ and $det(-2K_{11}^{-1}\bar{L}_{11})$ are positive. On the other hand, $\beta^1 C_{11}$ is a function of Γ and can take on arbitrary large values regardless of the magnitude of $det(-2K_{11}^{-1}\bar{L}_{11})$. Therefore, in order to maintain the two conditions $det(-2K_{11}^{-1}\bar{L}_{11}) > 0$ and $\beta^1 C_{11} + det(-2K_{11}^{-1}\bar{L}_{11}) > 0$, the term $\beta^1 C_{11}$ has to be positive, and therefore C_{11} should be negative.

Now, in order to prove that (B.13) holds and given that $\beta^1 < 0$, we should show the following inequality

$$\beta^1 C_{1k} < \beta^1 C_{11} + det(-2K_{11}^{-1}\bar{L}_{11})$$

For this to hold it suffices to have $\beta^1 C_{1k} < \beta^1 C_{11}$. Since, $C_{11} < 0$, if we can show that $|C_{1k}| < |C_{11}|$, then (B.13) will be guaranteed. In the following we show this property.

First, note that for any $k > 1$, we have the following property

$$C_{1k}(L_{11}) = C_{1k}(-2K_{11}^{-1}(\bar{L}_{11} + G_{11})) = C_{1k}(-2K_{11}^{-1}\bar{L}_{11}) \tag{B.14}$$

The last equality holds as a result of the definition of matrix G_{11}. Also, $C_{1k}(-2K_{11}^{-1}\bar{L}_{11}) = (-1)^{1+k}M_{1k}(-2K_{11}^{-1}\bar{L}_{11})$. However, $M_{1k}(-2K_{11}^{-1}\bar{L}_{11})$ is equal to the 2×2 minor of matrix $\bar{L} = -2K^{-1}(L \otimes Q)$, resulting from the deletion of rows $1, N$ and columns k, N. Similarly $M_{11}(-2K_{11}^{-1}\bar{L}_{11})$ is the 2×2 minor resulting from the deletion of rows and columns $1, N$ of matrix $\bar{L}$. In other words,

$M_{1k}(-2K_{11}^{-1}\bar{L}_{11}) = det(\bar{L}(1, N|k, N))$ and $M_{11}(-2K_{11}^{-1}\bar{L}_{11}) = det(\bar{L}(1, N|1, N))$.

Also, $-\bar{L}$ can be considered as the Laplacian matrix of a weighted graph. Therefore, based on the assumption that the information structure is described by a tree-like graph, the result of Theorem 4.1 may be used. For the general proof when the graph consists of some cycles, the general form of the matrix tree theorem presented in [144] should be used. Using Theorem 4.1, we may describe $|det(\bar{L}(1, N|1, N))|$ and $|det(\bar{L}(1, N|k, N))|$ as follows

$$
\begin{aligned}
|det(\bar{L}(1, N|1, N))| &= length(P(v_1, v_N) \cap P(v_1, v_N)) = length(P(v_1, v_N)), \\
|det(\bar{L}(1, N|k, N))| &= length(P(v_1, v_N) \cap P(v_k, v_N))
\end{aligned}
\tag{B.15}
$$

The second path is the intersection of two paths $P(v_1, v_N)$ and $P(v_k, v_N)$ which is obviously a subset of $P(v_1, v_N)$ and so its length is smaller. Hence, $|det(\bar{L}(1, N|k, N))|$ is smaller than $|det(\bar{L}(1, N|1, N))|$. This in turn means that $|C_{1k}| < |C_{11}|$. ■

Proof of Lemma 4.4

When an LIP fault occurs, the dynamics of the faulty and healthy agents in the team are governed by the following equations

$$
\begin{cases}
\dot{e}_f = u_c \Rightarrow e_f = u_c t - u_c t_f + e^f \\
\dot{e}_w = L_{11} e_w + L_{12} e_f = L_{11} e_w + L_{12}(u_c t - u_c t_f + e^f)
\end{cases}
\tag{B.16}
$$

Therefore, e_w can be obtained as

$$
\begin{aligned}
e_w(t) &= e^{L_{11}(t-t_f)}e_w(t_f) + \int_{t_f}^{t} e^{L_{11}(t-\tau)}L_{12}(u_c\tau - u_c t_f + e^f)d\tau \\
&= e^{L_{11}(t-t_f)}e_w(t_f) - [e^{L_{11}(t-\tau)}L_{11}^{-1}L_{12}(u_c\tau - u_c t_f + e^f) + e^{L_{11}(t-\tau)}L_{11}^{-2}L_{12}u_c]_{t_f}^{t} \\
&= e^{L_{11}(t-t_f)}[e_w(t_f) + L_{11}^{-1}L_{12}e^f - L_{11}^{-2}L_{12}u_c] - [L_{11}^{-1}L_{12}e^f + L_{11}^{-2}L_{12}u_c] \\
&- L_{11}^{-1}L_{12}u_c t + L_{11}^{-1}L_{12}u_c t_f
\end{aligned}
\tag{B.17}
$$

Similar to the discussion that is presented in the proof of Lemma 4.2, it can be shown that L_{11} is Hurwitz. Therefore, in steady state the effects of the first term in the above expression vanishes asymptotically, since $e^{L_{11}(t-t_f)} \rightarrow 0$ as $t \longrightarrow \infty$. Therefore, the dominant solution of $e_w(t)$ as $t \rightarrow \infty$ is governed by

$$
e_w(t) \rightarrow -[L_{11}^{-1}L_{12}e^f + L_{11}^{-2}L_{12}u_c] - L_{11}^{-1}L_{12}u_c t + L_{11}^{-1}L_{12}u_c t_f \tag{B.18}
$$

and therefore e_{ss} converges to the value that is given in (4.10). Obviously, the error dynamics is not stable in this case. ■

Proof of Lemma 4.5

According to the discussion given in Subsection 4.1.3, the error dynamics for the entire team can be written as $\dot{e} = L_{cl}e + f(v^d, g^1)$, where L_{cl} is defined as

$$
L_{cl} = -K^{-1}(2L \otimes Q^{ij} + A^T K + 2G) = -K^{-1}\Upsilon \tag{B.19}
$$

and $K,\ A,\ G,\ \Upsilon,\ f(v^d, g^1)$ are defined as before. When an LIP fault occurs in a follower, the closed-loop dynamics is then governed by

$$\dot{e} = \begin{bmatrix} \dot{e}_w \\ \dot{e}_f \end{bmatrix} = \begin{bmatrix} (L_{11})_{m(N-1)\times m(N-1)} & \vdots & (L_{12})_{m(N-1)\times m} \\ \cdots\cdots & & \cdots\cdots \\ 0_{m\times m(N-1)} & \vdots & A^f \end{bmatrix} e + \begin{bmatrix} f_1(v^d, g^1) \\ A^f v^d + B^f u_c \end{bmatrix} \tag{B.20}$$

The above dynamical system is stable if both matrices L_{11} and A^f are stable. The latter is true by assumption and the former can be shown as follows. Towards this end, we should note that L_{11} can be written as follows

$$L_{11} = -K_{11}^{-1}\Upsilon_{11} \tag{B.21}$$

where $K_{ij}^{-1}, \Upsilon_{ij}$ denote the corresponding partitions of matrices K^{-1}, Υ, respectively, which correspond to the faulty and healthy agents dynamics. Now, from Theorem 3.3 we know that $\Upsilon + \Upsilon^T > 0$, and therefore its partition Υ_{11} enjoys the same property, i.e. $\Upsilon_{11} + \Upsilon_{11}^T > 0$. This is due to the fact that $\Upsilon_{11} + \Upsilon_{11}^T$ is a principal minor of $\Upsilon + \Upsilon^T$ and since $\Upsilon + \Upsilon^T$ is PD, any of its principal minors is also PD. Moreover, K_{11}^{-1} is PD for similar reason. Now, invoking Fact A.1, we can conclude that L_{11} is stable. Hence, the entire error dynamics is stable. Given that A^f is Hurwitz, $(A^f)^{-1}$ is defined and hence in the steady state we have

$$\begin{cases} \dot{e}_f = A^f(e_f + v^d) + B^f u_c = 0 \Rightarrow (e_f)_{ss} = -(A^f)^{-1}B^f u_c - v^d \\ \dot{e}_w = L_{11}e_w + L_{12}e_f + f_1(v^d, g^1) \Rightarrow \\ (e_w)_{ss} = L_{11}^{-1}[L_{12}((A^f)^{-1}B^f u_c + v^d) - f_1(v^d, g^1)] \end{cases} \tag{B.22}$$

This completes the proof of this lemma. ∎

Proof of Lemma 4.6

Since the normalized adjacency matrix, $\hat{\mathbf{A}}$, is nonnegative by its definition, it will satisfy all the properties that are stated in the Perron-Frobenius Theorem for nonnegative matrices. In other words, it has an algebraically simple eigenvalue which is equal to the spectral radius $\rho(\hat{\mathbf{A}})$ and the corresponding eigenvector is a positive vector. Also, from graph theory and using Perron-Frobenius Theorem, we know that $\rho(\hat{\mathbf{A}}) = 1$ for a normalized adjacency matrix. Hence, 1 is an eigenvalue of $\hat{\mathbf{A}}$ and the corresponding eigenvector will have positive entries. This applies to both the right and the left eigenvectors of $\hat{\mathbf{A}}$. Using the relationship between the normalized Laplacian matrix $\hat{L}$ and the normalized adjacency matrix $\hat{\mathbf{A}}$ [118], that is $\hat{L} = I - \hat{\mathbf{A}}$, the zero eigenvalue of $\hat{L}$ corresponds to the 1 eigenvalue of $\hat{\mathbf{A}}$, and the corresponding left and right eigenvectors of $\hat{L}$ are the same as those of $\hat{A}$. The right eigenvector is $\mathbf{1}$ but the left eigenvector is not $\mathbf{1}$ for directed graphs in general, unless they are balanced. However, this vector has entries with the same sign. This completes the proof. $\blacksquare$

Proof of Theorem 4.3

First, we should show that if the control laws defined in (3.31)-(3.33) are modified according to the switching control laws given in (4.21)-(4.23), the matrix $\bar{L}_\sigma$ can be transformed into a balanced matrix. Then, we will use this property to find a common Lyapunov function for the overall switching system. Using the results that are obtained in Lemma 3.2, we know that the following relationships hold between K^i (K^i_σ) and Q^{ij} (Q^i_σ) for an infinite horizon problem

$$\begin{aligned} &2|N^i|Q^{ij} - \frac{1}{2}K^i(R^i)^{-1}K^i = 0, \ i = 2, \dots, N, \\ &2(|N^1|Q^{1j} + \Gamma) - \frac{1}{2}K^1(R^1)^{-1}K^1 = 0 \end{aligned} \tag{B.23}$$

For sake of notational simplicity let us assume that all the design parameter matrices are diagonal as follows

$$Q^{ij} = q^i I, \ R^i = r^i I, \ \Gamma = \gamma I \tag{B.24}$$

where q^i, r^i, and γ are positive scalars. Then the solutions to (B.23) are given by

$$\begin{aligned} &K^i = 2\sqrt{|N^i|q^i r^i} I, \ i = 2, \dots, N, \\ &K^1 = 2\sqrt{(|N^1|q^1 + \gamma) r^1} I \end{aligned} \tag{B.25}$$

We have seen in Section 4.2.1 that in order to make the matrix $\bar{L}_\sigma$ balanced, equation (4.20) should be satisfied. This implies that the optimal design parameters $Q^{ij}(Q^i_\sigma), R^i(R^i_\sigma)$, and $\Gamma(\Gamma_\sigma)$ should be selected appropriately so that (4.20) is guaranteed.

Let us denote the ith element of the vector ω_σ as $\rho_{i,\sigma}$. Using the definition of μ_σ given in (4.19), we should have

$$\begin{aligned} \mu_\sigma^T &= \begin{pmatrix} (K^1_\sigma)^{-1}Q^1_\sigma l_{11} & (K^2_\sigma)^{-1}Q^2_\sigma l_{22} & \dots & (K^N_\sigma)^{-1}Q^N_\sigma l_{NN} \end{pmatrix} \\ &= \begin{pmatrix} \frac{|N^1|q^1}{2\sqrt{(|N^1|q^1+\gamma)r^1}} & \frac{|N^2|q^2}{2\sqrt{|N^2|q^2r^2}} & \dots & \frac{|N^N|q^N}{2\sqrt{|N^N|q^N r^N}} \end{pmatrix} \otimes I_n \\ &= \kappa \begin{bmatrix} \rho_{1,\sigma} & \rho_{2,\sigma} & \dots & \rho_{N,\sigma} \end{bmatrix} \otimes I_n \end{aligned} \tag{B.26}$$

Therefore, the following relationships should be satisfied

$$\kappa\rho_{1,\sigma} = \frac{|N^1|q^1}{2\sqrt{(|N^1|q^1+\gamma)r^1}}, \quad \kappa\rho_{i,\sigma} = \frac{|N^i|q^i}{2\sqrt{|N^i|q^ir^i}}, \ i = 2,\ldots,N \tag{B.27}$$

In the first equation of (B.27), $\rho_{1,\sigma}$ and $|N^1|$ are given and κ, q^1, r^1, γ are parameters to be selected. Similarly, in the second equation of (B.27), κ, q^i, r^i are to be selected. It is assumed that r^i and γ are set to fixed values and one then tries to find q^i that satisfies the above equations. Design of κ is discussed in Subsection 4.2.2. Therefore, the following equations in terms of q^i should be satisfied

$$\begin{aligned} &|N^1|^2(q^1)^2 - (4\kappa^2\rho_{1,\sigma}^2|N^1|r^1)q^1 - 4(\kappa\rho_{1,\sigma})^2\gamma r^1 = 0, \\ &q^i = \frac{4(\kappa\rho_{i,\sigma})^2 r^i}{|N^i|}, \ i = 2,\ldots,N \end{aligned} \tag{B.28}$$

It is not difficult to show that the first equation in (B.28) always has a positive solution $q^1 = \frac{2\kappa\rho_{1,\sigma}}{|N^1|}(\kappa\rho_{1,\sigma}r^1 + \sqrt{(\kappa\rho_{1,\sigma}r^1)^2+\gamma r^1})$. Also, from the second equation of (B.28), it is obvious that $q^i, i = 2,\ldots,N$ is always positive.

It should be noted that for the above results to hold one should ensure a property in the left null space of $\hat{L}_\sigma$. Namely, due to the positive definiteness of $(K_\sigma^i)^{-1}Q_\sigma^i l_{ii}$, all the elements of the vector μ_σ are of the same sign, i.e. positive, which implies that the null space of $\hat{L}_\sigma$ should also enjoy this property. This can be shown by using the results that are provided in Lemma 4.6.

We are now in a position to use the above relationships to determine the switching control law. From Lemma 3.2, the control inputs can be calculated by using (3.31)-(3.33). By replacing q^i from (B.28), and K^i from (B.25) we obtain

$$
\begin{aligned}
\Gamma^1 &= -2\frac{|N^1|q^1}{2\sqrt{(|N^1|q^1+\gamma)r^1}}I_n = -2\kappa\rho_{1,\sigma}I_n, \\
\Gamma^i &= -2\frac{|N^i|q^i}{2\sqrt{|N^i|q^i r^i}}I_n = -2\kappa\rho_{i,\sigma}I_n, \quad i = 2,\dots,N \\
\beta^1 &= -2\frac{\gamma}{2\sqrt{(|N^1|q^1+\gamma)r^1}}I_n = -\gamma(\kappa\rho_{1,\sigma}r^1 + \sqrt{(\kappa\rho_{1,\sigma}r^1)^2+\gamma r^1})I_n
\end{aligned} \tag{B.29}
$$

where $\rho_{i,\sigma}$ is found from the Laplacian matrix of the network at each switching stage. The control laws provided in (4.21)-(4.23) with parameters as in (B.29), guarantee that $\mathbf{1}^T \otimes I_n$ is in the left null space of $\bar{L}_\sigma$ and therefore $\bar{L}_\sigma$ is the Laplacian of a balanced graph.

Now, to show the stability of the closed-loop switching system we should select a common Lyapunov function candidate which is valid for all the switching states. Let us select the Lyapunov function candidate as $V = \frac{1}{2}e^T e$. Its time derivative along the trajectories of (4.25) is given by $\dot{V} = \frac{1}{2}e^T(L_{cl,\sigma} + L_{cl,\sigma}^T)e = -e^T(\bar{L}_\sigma + \bar{L}_\sigma^T + K_\sigma^{-1}G_\sigma + G_\sigma^T K_\sigma^{-1})e$. Based on the previous discussions $\bar{L}_\sigma = K_\sigma^{-1}Q_\sigma L_\sigma \otimes I_n$ can be considered as the Laplacian matrix of a weighted balanced graph, and by using the results provided in [13], $\bar{L}_\sigma + \bar{L}_\sigma^T$ is also a valid Laplacian matrix representing an undirected (due to its symmetry) and connected graph. Hence, it is a PSD matrix. Moreover, the second term, i.e. $K_\sigma^{-1}G_\sigma$ is a diagonal matrix with one non-zero element and so is PSD. Hence, $L_{cl,\sigma} + L_{cl,\sigma}^T$ is at least NSD. Also, similar to the discussion given in the proof of Theorem 3.2, the null spaces of the two matrices $\bar{L}_\sigma + \bar{L}_\sigma^T$ and $K_\sigma^{-1}G_\sigma$ do not have any common intersection, and hence their summation is a PD matrix and so $\dot{V} < 0$. Consequently, we can guarantee consensus achievement and therefore the proof is complete. ■

Remark B.1. For evaluating the control laws (4.21) and (4.22) at each switching interval, each agent is required to compute an eigenvector of the network Laplacian matrix. In other words, the Laplacian matrix should be known to all the agents, which is guaranteed by Assumption 4.1.

Proof of Lemma 4.7

Similar to the proof of Theorem 4.3, and without loss of generality, assume that all the matrices involved are diagonal matrices. For the followers' case, we will then have $\frac{\lambda_{max}(Q^{ij})}{\lambda_{max}(R^i)} = \frac{q^i}{r^i} = \frac{4(\kappa\rho_{i,\sigma})^2}{|N^i|} > m_i$, $i = 2, \ldots, N$, and given that $\forall i$, $\rho_{i,\sigma} \neq 0$ (Lemma 4.6), we have $\kappa^2 > \frac{\max_{i=2,\ldots,N}(m_i|N^i|)}{4\min_{i=2,\ldots,N}(\rho_{i,\sigma}^2)}$. On the other hand for the leader agent, we have the following relationship

$$q^1 = \frac{2\kappa\rho_{1,\sigma}}{|N^1|}(\kappa\rho_{1,\sigma}r^1 + \sqrt{(\kappa\rho_{1,\sigma}r^1)^2 + \gamma r^1}) > \frac{4\kappa^2\rho_{1,\sigma}^2 r^1}{|N^1|} \tag{B.30}$$

to satisfy the condition $\frac{q^1}{r^1} > m_1$, it is sufficient to select κ so that $\kappa^2 > \frac{m_1|N^1|}{4\rho_{1,\sigma}^2}$. Consequently, κ should satisfy the following inequality, namely

$$\kappa^2 > \frac{1}{4}\max\{\frac{m_1|N^1|}{\rho_{1,\sigma}^2}, \frac{\max_{i=2,\ldots,N}(m_i|N^i|)}{\min_{i=2,\ldots,N}(\rho_{i,\sigma}^2)}\} \tag{B.31}$$

■

Appendix C

Proofs of the lemmas and theorems of Chapter 5

__Proof of Theorem 5.1:__

(a) Follows from the constructive results that are derived in Subsections 5.1.2 and 5.1.3.

(b) For stability analysis of the closed-loop system we should note that Condition 3 in (5.10) guarantees that matrix P has at least the same zeros as the Laplacian matrix of the information graph, L(i.e. it may have more zeros, too). Also, since both B and R are block diagonal matrices, the term $BR^{-1}B^TP$ has at least the same zero elements as L. Also, based on its definition as given in (2.8), matrix A has this property, as well. Therefore, the closed-loop matrix A_c has a structure similar to Laplacian of a subgraph of the original graph but this subgraph may not be in general a connected one (due to the extra zero entries that may appear in A_c). However, condition given in (5.11) guarantees that A_c does not have any other zero besides the ones that exist in the Laplacian matrix of one of the connected subgraphs of the original graph. Therefore, A_c has the minimum required non-zero elements to describe the Laplacian matrix of a "strongly connected" graph. Moreover, since A_c has the "structure" of a Laplacian matrix and it also satisfies condition 2 in (5.10), it is in fact the Laplacian matrix of a weighted and strongly connected graph. From the graph theory, and in particular as shown in [13], it is known that the Laplacian matrix of any strongly connected graph has one and only one zero eigenvalue and $N-1$ negative eigenvalues [118]. Therefore, it is a stable matrix with one zero eigenvalue.

(c) Assume that P is obtained from the optimization problem (5.10). Then we have,

$$\begin{aligned}&\int_0^T \frac{d}{dt}(X^TPX)dt = \int_0^T [(AX+BU)^TPX + X^TP(AX+BU)]dt\\&= X^T(T)PX(T) - X^T(0)PX(0)\end{aligned} \tag{C.1}$$

so that we have

$$\int_0^T [(AX+BU)^TPX+X^TP(AX+BU)]dt+X^T(0)PX(0)-X^T(T)PX(T) = 0 \tag{C.2}$$

By adding the above expression to the cost function (5.1) we get

$$\begin{aligned}J^c &= \int_0^T [X^TQX + U^TRU + (AX+BU)^TPX + X^TP(AX+BU)]dt\\&+ X^T(0)PX(0) - X^T(T)PX(T)\\&= \int_0^T [X^T(Q+A^TP+PA)X + U^TRU + U^TB^TPX + X^TPBU]dt\\&+ X^T(0)PX(0) - X^T(T)PX(T)\end{aligned} \tag{C.3}$$

Since P is a solution to (5.10), it satisfies (5.5) as well and therefore one gets

$$PA + A^TP - PBR^{-1}B^TP + Q \geq 0 \tag{C.4}$$

Hence,

$$
\begin{aligned}
J^c &\geq \int_0^T [X^T(PBR^{-1}B^TP)X + U^TRU + U^TB^TPX + X^TPBU]dt \\
&+ X^T(0)PX(0) - X^T(T)PX(T) \\
&= \int_0^T [(U + R^{-1}B^TPX)^TR(U + R^{-1}B^TPX)]dt \\
&+ X^T(0)PX(0) - X^T(T)PX(T)
\end{aligned}
\tag{C.5}
$$

In order to minimize the integral part of the above cost function one may select the control input as $U^* = -R^{-1}B^TPX$, which is already satisfied due to the definition of the control law.

Moreover, since P is obtained through the optimization problem (5.10), then it is guaranteed that in steady state, consensus will be achieved. In other words, if we assume that T is large enough (or $T \to \infty$) to let the system obtain a steady state, then $X(T) = \xi[1\ 1\ \ldots\ 1]^T$. Correspondingly, it can be shown that $X^T(T)PX(T) = \xi^2 \sum_i \sum_j P(i,j)$, where $P(i,j)$ represents the ijth entry of the matrix P. Therefore, when $T \to \infty$ the optimal cost has a lower bound that is given by

$$J^{c*} \geq X^T(0)PX(0) - \xi^2 \sum_i \sum_j P(i,j) \tag{C.6}$$

Therefore, $X^T(0)PX(0) - \xi^2 \sum_i \sum_j P(i,j)$ is the finite infimum of J^c. ■

Proof of Lemma 5.2

An optimal stabilizing solution for the minimization problem (5.27), or (5.35), exists if the pair $(\bar{A}, \bar{B})$ is stabilizable and the pair $(\bar{A}, \hat{Q})$ (or $(\bar{A}, \Omega)$, $\hat{Q} = \Omega\Omega^*$) is detectable [145]. Each of these conditions can be checked through an LMI [121]:

1. $(\bar{A}, \bar{B})$ stabilizable $\Leftrightarrow$ $\bar{A}P_1 + P_1\bar{A}^* < \bar{B}\bar{B}^*$ has a PD solution for P_1.

2. $(\bar{A}, \hat{Q})$ detectable $\Leftrightarrow$ $\bar{A}^*P_1 + P_1\bar{A} < \hat{Q}$ has a PD solution for P_1.

Let us define a new variable $P_2 > 0$ such that:

$$P_2 = \bar{S}\bar{S}^*P_2\bar{S}\bar{S}^* + SS^*P_2SS^* \tag{C.7}$$

An example of such a matrix can be in the following form:

$$P_2 = [\bar{S}\ S] \begin{bmatrix} P_1 & 0 \\ 0 & M_1 \end{bmatrix} \begin{bmatrix} \bar{S}^* \\ S^* \end{bmatrix}, \ M_1 > 0 \tag{C.8}$$

Now we have $P_2\bar{S} = \bar{S}P_1$. Using the definition of matrices $\bar{A}, \bar{B}$, the following conditions should be satisfied to verify stabilizability and detectability conditions:

$$\begin{aligned} &1.\ \bar{A}P_1 + P_1\bar{A}^* < \bar{B}\bar{B}^* \ \Leftrightarrow\ \bar{S}^*A\bar{S}P_1 + P_1\bar{S}^*A^*\bar{S} < \bar{S}^*BB^*\bar{S} \ \Leftrightarrow \\ &\bar{S}^*(AP_2 + P_2A^*)\bar{S} < \bar{S}^*BB^*\bar{S} \Leftrightarrow\ \bar{S}^*(AP_2 + P_2A^* - BB^*)\bar{S} < 0 \\ &2.\ P_1\bar{A} + \bar{A}^*P_1 < \hat{Q} \ \Leftrightarrow P_1\bar{S}^*A\bar{S} + \bar{S}^*A^*\bar{S}P_1 < \bar{S}^*Q\bar{S} \ \Leftrightarrow \\ &\bar{S}^*(P_2A + A^*P_2)\bar{S} < \bar{S}^*Q\bar{S} \ \Leftrightarrow \bar{S}^*(P_2A + A^*P_2 - Q)\bar{S} < 0 \end{aligned} \tag{C.9}$$

which are the same conditions as the ones given in (5.36) and (5.37). ■

<u>Proof of Theorem 5.2</u>

a. First, note that from the previous discussions we have $P^{-1} = \bar{S}^*Z\bar{S}$. Since $rank(\bar{S}) = Nn - 1$ and $Z > 0$ we have $\bar{S}^*Z\bar{S} > 0$, and hence $P = (\bar{S}^*Z\bar{S})^{-1}$. Therefore:

$$\min trace(P) = \min trace((\bar{S}^*Z\bar{S})^{-1}) \tag{C.10}$$

This in turn can be written equivalently by introducing an auxiliary variable Γ such that [127]:

$$\min trace(\Gamma) \; s.t. \; \begin{bmatrix} \Gamma & I \\ I & \bar{S}^* Z \bar{S} \end{bmatrix} > 0 \tag{C.11}$$

which is the first condition to be considered. Moreover, by twice applying the Schur complement to (5.33) the second inequality follows. Furthermore, by using the definition of matrix Z [39], it can be shown that Condition 3 is equivalent to the consensus constraint given in (5.15). Finally, the last condition is an assumption on the structure of the new variable Z which has already been discussed.

In summary, the minimization formulation, together with the first two inequalities in (5.38) and Condition 4 are used to design an optimal stabilizing controller for system (5.17) as was discussed previously. The third equality is used to guarantee that consensus is achieved.

b. The matrix $K = WZ^{-1}$ will have the same structure as W if Z is selected to be diagonal [39]. Therefore, we may transform any required constraint on the control gain matrix to that of the matrix W by considering Z to be diagonal. Therefore, if the individual controllers are to be designed based on information received from the neighbors of each agent, the structure for W may be chosen as the Laplacian matrix, so that the members information in the neighboring sets are sufficient for design of each agent's control signal. ■

Proof of Lemma 5.3

Based on the definition of matrices A, B given in (2.8) and the restriction on the structure of K as provided in Theorem 5.2, the matrix $A + BK$ has a structure similar to that of the Laplacian matrix of the entire network, with the possibility of some additional zero elements. This matrix should also satisfy the condition in (5.15) and so it can be considered as a Laplacian matrix [118]. Moreover, the graph corresponding to this matrix has at most the same edges as the graph of the entire network. Therefore, it represents a subgraph of the network graph with different edges' weights. ■

Proof of Theorem 5.3

a) As was discussed in Subsection 5.2.1, two conditions should be satisfied to guarantee consensus achievement. One of these conditions is (5.15) and the other one is the design of matrix K to assure that the system in (5.17) is asymptotically stable. In the following, we show that connectedness of the network underlying graph is necessary for satisfying the latter condition.

To satisfy the asymptotic stability condition, the matrix $\bar{S}^*(A+BK)\bar{S}$ in (5.17) should be Hurwitz. In other words, it should have no zero eigenvalue. Now, we show that this is violated if the network graph is not connected. Based on its definition, matrix $\bar{S}_{Nn\times(Nn-1)}$ consists of $Nn-1$ independent column vectors. Denote these vectors by $\bar{S}_1, \ldots, \bar{S}_{Nn-1}$, i.e.

$$\bar{S} = [\bar{S}_1,\ \bar{S}_2,\ \ldots, \bar{S}_{Nn-1}] \tag{C.12}$$

Then, we will have

$$\bar{S}^*(A+BK)\bar{S} = [\bar{S}_1,\ \bar{S}_2,\ \ldots, \bar{S}_{Nn-1}]^*(A+BK)[\bar{S}_1,\ \bar{S}_2,\ \ldots, \bar{S}_{Nn-1}] =$$
$$\begin{bmatrix} \bar{S}_1^*(A+BK)\bar{S}_1 & \ldots & \bar{S}_1^*(A+BK)\bar{S}_{Nn-1} \\ \ddots & & \\ \bar{S}_{Nn-1}^*(A+BK)\bar{S}_1 & \ldots & \bar{S}_{Nn-1}^*(A+BK)\bar{S}_{Nn-1} \end{bmatrix} \tag{C.13}$$

Now, we assume that the network underlying graph is not connected. Then, the Laplacian matrix of the graph, L may have more than one zero eigenvalue. Correspondingly, $A+BK$ represents a disconnected graph and therefore has more than one zero eigenvalue. Hence, $A+BK$ has an eigenvector corresponding to one of its zero eigenvalues which is not necessarily in the $\mathcal{S}$ subspace. Let us denote this eigenvector by w, i.e. $(A+BK)w=0$. This vector in general may have components in both subspaces $\mathcal{S}$ and $\bar{\mathcal{S}}$, i.e.

$$w = \alpha_1 \bar{S}_1 + \alpha_2 \bar{S}_2 + \ldots + \alpha_{(Nn-1)} \bar{S}_{Nn-1} + \gamma S \tag{C.14}$$

with at least one nonzero α_i. Now, assume that α_1 is nonzero and since $(A+BK)S=0$ and $(A+BK)w=0$ we will have

$$\begin{aligned}
&(A+BK)(\alpha_1 \bar{S}_1 + \alpha_2 \bar{S}_2 + \ldots + \alpha_{Nn-1} \bar{S}_{Nn-1} + \gamma S) = 0 \Rightarrow \\
&(A+BK)(\alpha_1 \bar{S}_1 + \alpha_2 \bar{S}_2 + \ldots + \alpha_{Nn-1} \bar{S}_{Nn-1}) = 0 \Rightarrow \\
&(A+BK)\bar{S}_1 = \frac{1}{\alpha_1}(-\alpha_2 (A+BK)\bar{S}_2 - \ldots - \alpha_{Nn-1}(A+BK)\bar{S}_{Nn-1}) \Rightarrow \\
&\begin{cases}
\bar{S}_1^*(A+BK)\bar{S}_1 = \frac{1}{\alpha_1}(-\alpha_2 \bar{S}_1^*(A+BK)\bar{S}_2 - \ldots - \alpha_{Nn-1}\bar{S}_1^*(A+BK)\bar{S}_{Nn-1}) \\
\bar{S}_2^*(A+BK)\bar{S}_1 = \frac{1}{\alpha_1}(-\alpha_2 \bar{S}_2^*(A+BK)\bar{S}_2 - \ldots - \alpha_{Nn-1}\bar{S}_2^*(A+BK)\bar{S}_{Nn-1}) \\
\vdots \\
\bar{S}_{Nn-1}^*(A+BK)\bar{S}_1 = \\
\frac{1}{\alpha_1}(-\alpha_2 \bar{S}_{Nn-1}^*(A+BK)\bar{S}_2 - \ldots - \alpha_{Nn-1}\bar{S}_{Nn-1}^*(A+BK)\bar{S}_{Nn-1})
\end{cases}
\end{aligned} \tag{C.15}$$

If at least one of α_i's besides α_1 is nonzero then from the above equations we can conclude that at least one column of matrix $\bar{S}^*(A+BK)\bar{S}$ can be written as a linear combination of other columns, i.e. $\bar{S}^*(A+BK)\bar{S}$ is rank deficient and so has a zero eigenvalue. This is contradictory with asymptotic stability condition of system (5.17). On the other hand, if all α_i's are zero except α_1 then $w = \alpha_1 \bar{S}_1 + \gamma S$.

Therefore, $(A + BK)(\alpha_1 \bar{S}_1 + \gamma S) = 0$ and so $(A + BK)\bar{S}_1 = 0$. This means that the first column of matrix $\bar{S}^*(A + BK)\bar{S}$ is zero which results in singularity of this matrix. Therefore, we may conclude that if the underlying graph is not connected, the asymptotic stability of system (5.17) cannot be guaranteed. This implies that consensus achievement cannot be guaranteed. In other words connectivity of the underlying network graph is a necessary condition for consensus achievement.

b) We have previously shown that $A + BK$ should have the same structure as the L matrix with some possible additional zero elements. Therefore, it describes a subgraph of the original graph. On the other hand, based on the discussions in the previous part in order to guarantee consensus, $A + BK$ should describe the Laplacian of a connected graph. This means that if a matrix K satisfies the LMI conditions provided in (5.38), then the additional zero elements are such that $A + BK$ represents a connected network. Therefore, $A + BK$ describes the Laplacian matrix of a connected subgraph of the original connected network underlying graph. ■