# Optimal Pricing and Seat Allocation in the Airline Industry Under Market Competition

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*Dedicated to my grand mother, Syeda Waheeda Khatoon,*

*and to my mother, Syeda Afifa Khatoon*

# ABSTRACT

Optimal Pricing and Seat Allocation in the Airline Industry Under Market Competition


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The current practice of revenue management is either quantity based or price based. A quantity based revenue management is most commonly observed in the airline industry; whereas a price based revenue management is practiced in retail enterprises. Recent improvement of information technology has not only increased the market size, but also has increased market competition. In a competitive environment customers choose among substitutable products depending on several rationalities, however a paramount factor in most selections is price. This thesis investigates pricing issue in revenue management and makes three contributions.

First, price based revenue management is studied in the airline industry in a competitive market. Airlines compete for customers using their fare pricing strategies while having fixed capacity allocated in each fare class. The demand for each fare class of an airline is dependent on its fare price and the fare price offered by rival airline(s). A game theoretic approach is used to address the problem assuming both the deterministic and stochastic price sensitive customer demand for each fare class. The existence and uniqueness of Nash equilibrium for the game is shown for both deterministic and stochastic demands. A sensitivity analysis is carried out to

determine fare pricing in each fare class considering various situations in the case of deterministic demand. The analysis is further extended to stochastic price sensitive demand, and a sensitivity analysis of the fare prices for each fare class is also reported.

Second, an integrated approach to price and quantity based revenue management with an application to the airline industry is presented. The models proposed enable joint control of fare pricing and seat allocation in a duopoly competitive market. Both non cooperative and cooperative bargaining games are studied. Numerical experimentation is performed to study both competitive and cooperative fare pricing along with seat inventory control assuming a nested control on booking limits. In the case of a non cooperative game, Nash equilibrium for the competing airlines is determined assuming both symmetric and asymmetric market competition. A sensitivity analysis based on a statistical design of experiments is also presented to study the behavior of the game. Statistical evidence is established which shows that cooperation improves the revenue to the competing airlines.

Lastly, a distribution free approach for pricing in revenue management is explored. The approach assumes the worst possible demand distribution and optimizes the lower bound estimate on revenue, while jointly controlling the price and capacity. The approach is first addressed to revenue management's most commonly observed standard newsvendor problem. Extensions to the problem are identified which can be applied to airline industry. Later the analysis is extended to consider the following cases: a shortage cost penalty; a holding and shortage cost; a recourse cost, with a second purchasing opportunity; and the case of random yields. An application of the approach is also suggested to capacity constrained industries facing restrictions such as limited budget. A numerical study reveals that the approach results in a near optimal estimate on revenue. Using a statistical comparison it is also shown that the outcomes of the standard newsvendor problem are significantly different than its extensions.

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# Chapter 1

# Revenue Management

Revenue Management (RM), also known as Yield Management (YM), deals with controlling the availability and the price of perishable assets to maximize revenue. RM is used by many service industries to implement techniques to gain the best return from an investment in perishable assets. There are many applications of RM, however, it is mostly practiced in capacity-constrained service industries. The commercial aviation industry is the largest user of RM. Airline RM started in the 1970's when deregulation was implemented in the US. The successful practice of RM is also observed in other service industries such as hotels, restaurants, car rental agencies and cruise liners. The RM practice in the aviation industry is divided into four distinct but closely related categories: i) Forecasting; ii) Overbooking; iii) Seat inventory control; and iv) Pricing. In general, RM practice is classified into Quantity-based RM and Price-based RM. In quantity-based RM, the revenue of a firm is optimized by adjusting the availability of capacity for pre-determined prices. This practice is mostly observed in the aviation industry in the form of a booking control on fare classes. A price-based RM is more commonly observed in retail industries. In this practice, the prices are controlled for a fixed capacity of perishable assets to maximize the gain on investment. Several airlines have adopted RM practices and have been able to significantly improve their revenues. Boyd (1998) reported that the use of RM tools had helped North American service industries of increase profits, noting that American Airlines had been able to improve its revenue by $500 million per year due to the use of RM techniques. Delta Airlines also reported that the use of RM practices has improved its revenue by $300 million per year. Non-airline based industries such as hotels also benefited from the RM tools. For example, Marriott hotels increased their revenues by $100 million per year using RM techniques with a marginal increase in the capacity and operating cost.

## 1.1 A Brief History of Revenue Management

The RM literature has been developed for the last four decades. During the initial period, airline RM research was more focused on *Overbooking*. Overbooking is a strategy used to compensate for *No shows* and *Cancellations*. The development in information technology improved RM practices. Overbooking practices in airline RM later brought further dimensions to the problem, such as the forecasting of cancelations and no shows.

In the 70s, some airlines started offering *discounted fares* to their customers. This policy attracted customers and resulted in a boost in the revenue of the carriers. For example, BOAC (presently known as British Airways) offered early bird bookings to customers who booked 21 days before the flight departure. This offer resulted in a significant gain in revenue from the seat that would otherwise remain empty. This strategy invoked another problem which was to determine the number of seats to protect for full fare customers. If the number of seats protected are less than the required level, the airline would *spill* full fare customers, who usually purchase tickets at the last minute, paying the highest fares. On the other hand if too many seats are protected, then the flight departs with more empty seats. There is no simple rule to estimate the seat protection level as airline customer behavior is unpredictable and varies widely depending on fare combinations, itineraries, seasons, travel day and time along with many other factors. The development of effective control rules requires detailed information about the customer behavior with fare changes, booking histories and competitors in the market. Littlewood (1972) proposed a simple rule to overcome these problems. According to this rule, a discounted fare booking should be accepted as long as its expected revenue exceeds the full fare customer. This rule gave birth to the aforementioned RM or YM. The deregulation of U.S domestic and international airlines gave birth to RM in North America. The first step was made by American Airlines in offering super saver fare in 1977.

The objective of RM is to maximize the profits, however, airline short-term costs are largely fixed, and the variable costs per passenger are relatively small. Thus in most situations, it is sufficient to determine the *booking policies* that maximize revenues. During the past two decades the practice of RM has significantly improved starting from *single leg control* to *segment control* finally to *origin-destination control.*

Previous research surveys of the airline Operations Research (OR) and RM are available in Belobaba (1987) and Etschmaier and Rothstein (1974). A general review of *Perishable Asset Revenue Management* problems is provided in Weatherford and Bodily (1992). There are several master's theses and doctoral dissertations presenting a very good literature review in the area. The relevant master theses are Sa

(1987), Williamson (1988), and Sun (1992). Some doctoral dissertations are Rothstein (1968), Belobaba (1987), McGill (1989), Williamson (1992), Chatwin (1993). The most recent survey of RM can be found in McGill and Ryzin (1999). Although the major application of RM has been mostly studied in the airline industry, RM is also equally applicable to automobile rentals, restaurants, railways etc. (see McGill and Ryzin (1999)). The remainder of this chapter, identifies the contribution made in published works that contains journal articles, conference proceedings, Master's and PhD Theses and working papers. This work contains only a brief overview of research work more relevant to the proposed study which will be addressed in this doctoral dissertation.

Airline RM has several components, of which four are considered to be primary and distinct. These four major components are *Forecasting*, *Overbooking*, *Seat inventory allocation/control*, and *Pricing*. As mentioned earlier, overbooking has received the most attention. Forecasting has been primarily considered as a statistical tool, whereas the Seat inventory control and pricing have been studied independently. Figure 1.1 shows a typical airline RM framework. In the figure, the potential need for integration between fare pricing and seat inventory control decisions is identified. The majority of the studies conducted in OR do not include market competition, thus do not lead to realistic decisions to maximize on airline's revenue in today's competitive market.

### 1.1.1 Forecasting

Forecasting is a tool for planning. It is extremely important for industries dealing with precarious demand situations, like airlines, car rentals, restaurants, hotels etc. In the airline industry the aim of forecasting is to estimate the demand, no shows,

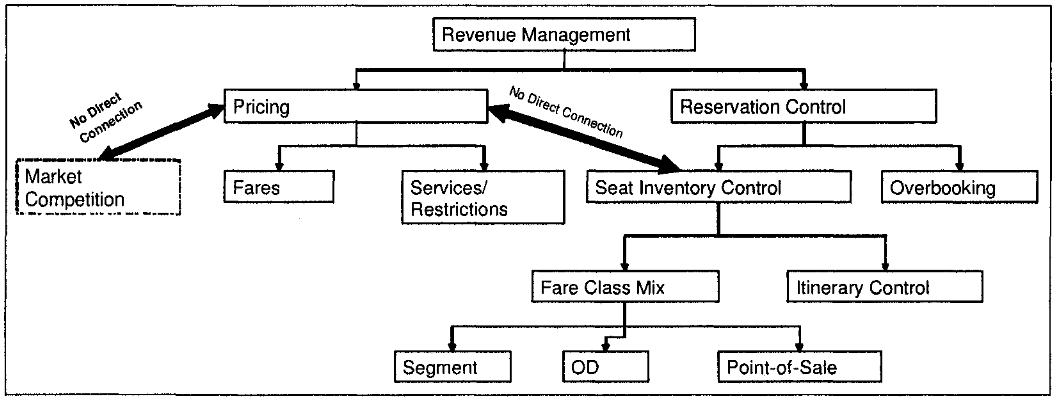


Figure 1.1: General Airline RM framework without market competition

overbooking, and spills. The forecasting decision immediately effects the overbooking calculations and hence seat allocations. Figure 1.2 shows the classification of airline customers based on their behaviors. The preferences of leisure customers are completely different from those of business customers. A business customer is more time conscious, whereas a leisure customer is more price conscious. Distinct behaviors of the two categories of customers can be also observed in their booking request as shown in Figure 1.2. Research publication in forecasting is concurrent with the

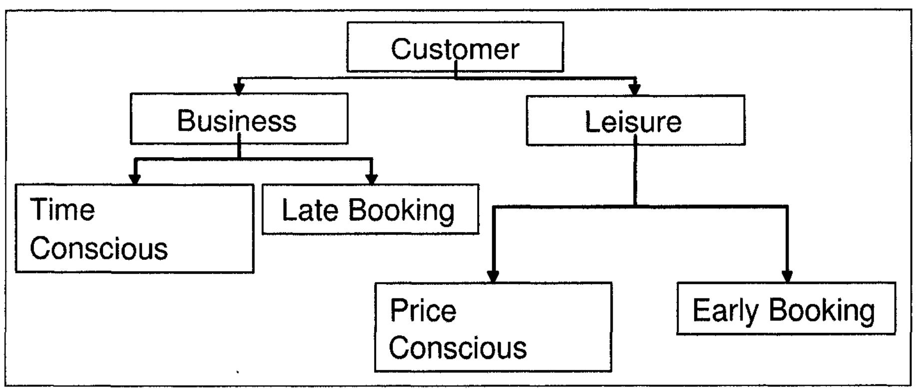


Figure 1.2: Customer classification

literature on overbooking because the overbooking calculations are dependent on forecasts of the estimates of demand, cancellations, and no-shows. The disaggregation approach to forecasting customer demand is found difficult. In following, some literature review pertinent to demand distribution, demand arrival process, aggregate versus disaggregate demand forecasting and current practice in forecasting is presented.

The first paper describing statistical models of passenger bookings, cancellations, and no-shows was Backmann and Bobkowski (1958). In the paper, Poisson, negative binomial and gamma models for passenger arrival were compared with a gamma distribution providing the best fit to the model. Beckmann (1958) also used a gamma distribution to model the components of show-ups and estimated the optimality condition for overbooking. Taylor (1962) determined empirical probability-generating functions for booking behaviors that determine show-ups. Allowance was made for single and batch bookings, cancellations, and no shows. The generating function was then used to estimate the parameters of a distribution for final show-ups. Lyle (1970) modeled the demand as composed of a gamma systematic component with Poisson random errors. This model led to a negative binomial distribution for total demand, as in Backmann and Bobkowski (1958), which was then truncated for demand censorship. Martinez and Sanchez (1970) proposed a convolution methodology similar to Taylor's (1962) method to determine the demand and cancellation probability distribution. This study revealed that the normal probability distribution gives a good continuous approximation to aggregate airline demand distributions, see for example, Belobaba (1987)and Shlifer and Vardi (1975). Given the central limit theorem and the role of the normal distribution as the limiting distribution for both binomial and Poisson distributions, this is not surprising. However, many researchers have pointed out that the normal distribution becomes increasingly inappropriate at greater levels of disaggregation.

Data contained in historical booking records are censored by the presence of booking and capacity limits on past demands. Swam (1990) addresses the downward bias of censoring on late booking data and suggests simple statistical remedial measures. An earlier spill formula developed by Swan has been used for many years by practitioners to unconstrain demand. Lee (1990) presents a detailed stochastic model of passenger arrivals based on a censored Poisson process and develops maximum likelihood methods for estimating the parameters of these models. McGill (1995) develops a multivariate multiple regression methodology for removing the effects of censorship in multiple booking classes, and describes a bootstrapping approach to testing for correlations between fare class demands.

### 1.1.2 Seat Inventory Control

The research in airline RM is more focused on seat inventory control, either on a single leg or in a network. A single leg model is a direct flight from an origin to a destination airport. There can be many possible topologies of an airline network. A popular topology is the Hub and Spoke network (see Figure 1.3), in which several airports (Origins) are connected to a major airport city (Hub) and then the Hub is also connected to some other destination cities (Spokes). Now we discuss the relevant literature in each of the two categories: Single leg seat allocation and Network allocation.

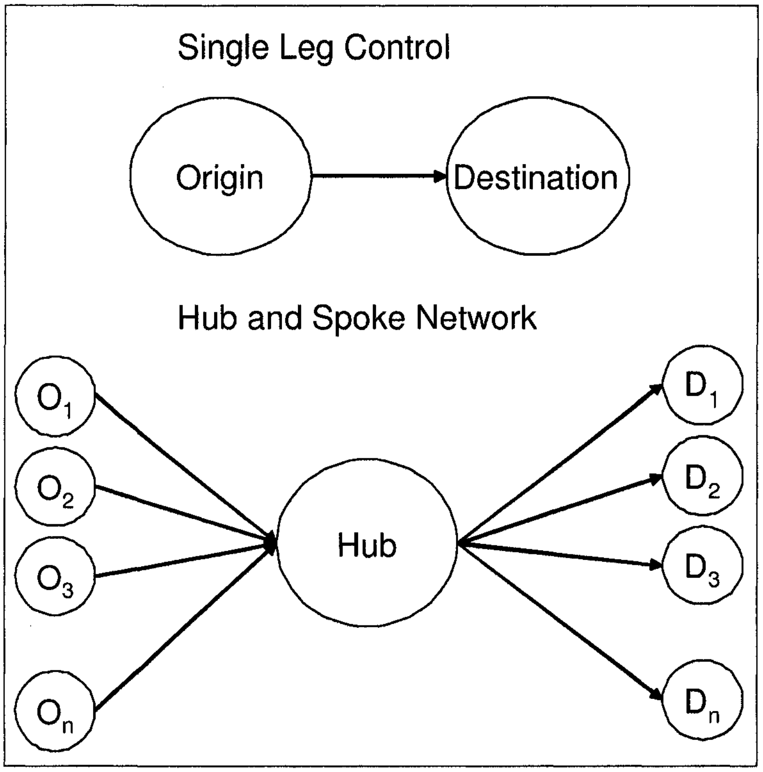

Figure 1.3: Single leg and hub and spoke airline network

### Single Leg Seat Allocation

The seat inventory control for a single leg is widely studied in the airline RM research. The most commonly used assumptions are that each customer belongs to a specific booking class, and if the request of a customer is declined the customer is lost. More assumptions are identified by McGill and Ryzin (1999). Two versions, namely *static* and *dynamic*, of this problem have been studied by researchers. In static problems, it is typically assumed that the demand for the different fare classes arrives sequentially. It is also assumed that the demand for the lower fare classes arrives before the higher fare classes. Seat inventory control is implemented by using booking limits on the number of requests for each fare class that the airline accepts. Since the arrival order of booking requests is known, the booking limit calculations require a knowledge of the demand distributions for each fare class. Littlewood (1972) made the first study in discount allocation i.e., seat allocation in a single leg. Belobaba (1987, 1989) extended the work of Littlewood and developed the Expected Marginal Seat Revenue (EMSR) heuristic, the most commonly used in practice and known as *Nested Booking Limits*. A simple illustration of nesting is presented in Figure 1.4, where the seat allocation for four fare classes is presented. Fare class are numbered from 1 to 4 with respect to their fare prices, $f_1 \geq f_2 \geq f_3 \geq f_4$ and $C$ is capacity of the aircraft. Using the EMSR, the booking limit for fare class $f_1$ is $BL_1 = C$. Similarly for fare class 2 with fare price $f_2$, the booking limit is $BL_2 = BL_1 - PL_1$, where $PL_1$ is the protection level which is the seat protected for the fare class $f_1$ from the rest of the fare classes. Using the same analogy the booking limits for the rest of the fare classes can be estimated. Some other works on

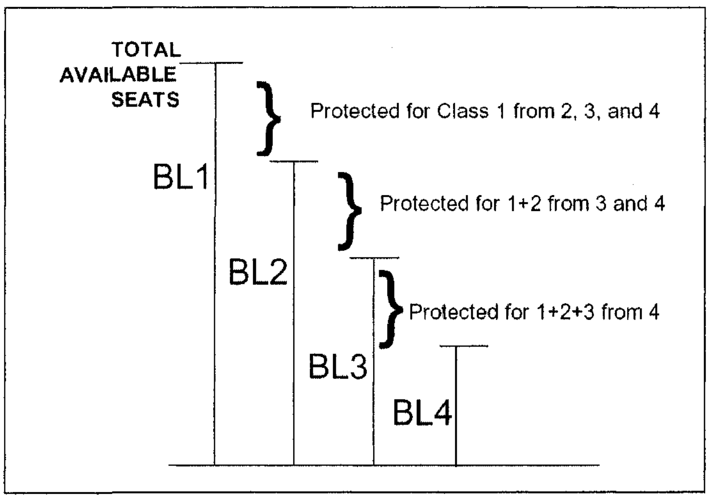


Figure 1.4: Nested booking limits and protection level for fare classes (overbooking curves)

booking limit estimation are Curry (1990), Wollmer (1992), Brumelle and McGill (1993), Robinson (1995), and Li and Oum (2002).

In dynamic problems, the customer arrival is modeled using stochastic process. The airline makes the decision whether to accept or refuse a customer. The decision is based on the unsold seats in each fare class and the time remaining to departure. Usually the problem is solved using a Dynamic Programming (DP) approach and properties of the value function are studied to determine the optimal seat allocation policy. Lee and Hersh (1993) were first to use a DP approach to solve the problem. Their basic model has been reconsidered and studied by several other researchers. Recently Subramanian et al. (1999) extended the discrete time DP model by incorporating no shows and cancellations in the model. You (2001) deals with a single leg airline RM problem with multiple fare classes, considering a free upgrading of economy class passengers to business class. In this work, a discrete time DP model is developed and an optimal booking policy is identified. Gosavi et al. (2002) used a reinforcement learning approach to a single leg airline RM problem with multi-fare classes and overbooking. The single leg airline RM problem is solved using Semi-Markov Decision Process (SMDP) with the implementation of a simulation based reinforcement learning technique. The approach outperforms the most commonly used nesting heuristic EMSR. Ringbom and Shy (2002) studied the "adjustable-curtain" strategy where the airline adjusts the size of the business-class section of the aircraft shortly before boarding takes place. A simple method was developed for computing the revenue maximization policy observing this strategy. Brumelle and Walczak (2003) developed a dynamic model for a single leg flight with multiple fare classes and customer arrival following a semi-markov process. In the model, a customer can request multiple seats and overbooking is permitted. The structural properties of the optimal control policy was developed. Talluri and Ryzin (2004) considered RM under customer choice behavior, such as buy-up and buy-down, and provided an exact and quite general analysis of the problem. An estimation procedure based on the expectation maximization method was developed. Karaesmen and Ryzin (2004) considered the problem with multiple reservation and inventory classes, in which the multiple inventory classes was used as substitutes to satisfy the demand of a given reservation class when needed. The problem was modeled as a two-period optimization problem. In the first period, reservations are accepted with the probabilistic knowledge of cancellations. In the second period, cancellations are realized and the remaining customers are assigned to various inventory classes to maximize the net revenue. A stochastic gradient algorithm was proposed to find the optimal overbooking limits. Ryzin and McGill (2000) investigated a simple adaptive approach to optimize the seat allocation that used only historical observations. Stochastic approximation theory was used to prove the convergence of the simple adaptive approach. The adaptive approach was compared with the $\text{EMSR}_\text{b}$ that combined with a censored forecasting method. Slyke and Young (2000) modeled a finite horizon stochastic knapsack problem using a continuous time, discrete state, and finite horizon DP. The main focus of the study was to identify the optimal return function and the optimal acceptance strategy. Pak and Piersma (2002) provided an overview of the solution methods presented in the RM literature. In Barnhart et al. (2003), a historical perspective of OR contributions in airline RM is studied in detail. Pulugurtha and Nambisan (2003) developed a decision support tool for airline YM using a genetic algorithm. The decision support tool considers the effect of time-dependent demand, ticket cancelations, and overbooking policy. Silver (2004) reviewed a wide range of heuristic procedures to solve real world problems. The paper discussed several heuristic performance evaluation procedures. Bertsimans and

Shioda (2003) developed two classes of optimization models to maximize revenue in restaurants. The model first decides dynamically when to accept a reservation from an incoming party. The model is then solved using integer programming, stochastic programming and approximate DP methods. The second model addresses reservations for future arrivals. A two step procedure using stochastic gradient algorithm was proposed to solve the problem. Petruzzi and Monahan (2003) studied the problem of determining an optimal control policy for a retailer i.e., when the retailer should terminate the primary selling season by selling remaining inventory to a secondary market. Anderson and Wilson (2003) indicate that the use of standard YM approaches to pricing by airlines can result in significantly reduced revenues when buyers are using an informed and strategic approach to purchasing.

### Network Seat Allocation

In practice, Belobaba's (1989) EMSR heuristic is found to be very profitable for single leg RM. In the early 90's airline companies were focussed on the optimization of network revenue. The decision making in a network is much more complex. Take an example of a booking request for a higher fare class on a flight from Beijing to London. Likewise a request of a low fare class can be made from Beijing to Chicago via London. If the direct flight from London to Chicago has empty seats, then it is more profitable to accept the low fare class request. This vision of trade off between accepting a booking in a different fare class considering origin and final destination is not possible without studying the network RM.

Williamson (1992) used simulations and showed that including the network aspects in RM was superior to the leg based methods and generated higher profits. Although Gallego and Ryzin (1997) were able to capture the network effects in the DP model, in practice it is infeasible, as the DP model becomes intractable. In the DP model, the state variables are the number of seats available in a network, the number of fare classes, and the number of possible itineraries.

Mathematical Programming (MP) models can better capture the network effects in the airline RM problem. The solution to a MP model is reported using two methods. The first method is the *booking limit* and other is *bid prices*. MP using these two ways were studied by Glover et al. (1982), Wollmer (1986), Williamson (1992), and Talluri and Ryzin (1999). Some other methods are proposed by Curry (1990), Bratu (1999), Bertsimas and Popescu (2003). Boer et al. (2002) studied the network seat inventory control using a stochastic programming model. A deterministic model was also studied as a special case of the stochastic programming model. The study carefully examined the trade-off between computation time and the aggregation level of demand uncertainty, with examples of a multi-leg flight and a single hub network. In Bertsimas and Popescu (2003), a new algorithm was developed based on an approximate DP to solve RM problem in a network environment. The algorithm was extended to incorporate no-shows and cancelation. A comparative study was made of the proposed algorithm with bid-price control method. Chen et al. (2003) addressed the RM problem for hub-and-spoke network. The proposed method was a variant of the orthogonal array experimental designs and multivariate adaptive regression splines stochastic DP method. The study was made using a real world problem with 20 cities and 31 legs. Gosavi (2004) proposed a model free reinforcement learning algorithm to solve real world YM problems in the airline industry. The convergence analysis of the algorithm was presented using an ordinary differential equation method. Bish et al. (2004) studied the benefits of Demand Driven Swapping (DDS), defined as the advantage of a system flexibility to swap the aircraft dynamically as the departure nears and more accurate information about the demand is available. The conditions are determined when the DDS is beneficial.

### 1.1.3 Overbooking

Overbooking has the longest research history in airline RM. Prior to Littlewood's (1972) work in seat inventory control, almost all quantitative research on airline RM was focused on developing effective overbooking policies. Overbooking is an estimation of the right balance between the opportunity cost of empty seats for which potential customers are declined and the cost of "denied boarding", more commonly known as *spill.* The spill happens when a confirmed passenger is denied boarding when the seats on a flight are oversold. The cost of the denied boarding includes direct compensation offered to these passengers like cash, flight vouchers, but also of lost goodwill, which may lead passengers to choose different airlines in the future. Figure 1.5 shows the relationship between the OR tools and the main components of airline RM. As shown in the figure overbooking models are interrelated with forecasting and OR tools. This integration results in an optimal booking limits for each fare class.

The overbooking research can again be classified into two: *static* models and *dynamic* models. Some previous work using the static approach includes Beckmann (1958), Backmann and Bobkowski (1958), Littlewood (1972), and Belobaba (1987).

Dynamic models includes Rothstein (1968, 1971), Alstrup et al. (1986), Chatwin (1993, 1996, 1999). In static models, the airline sets a fixed overbooking level at the beginning of the booking period, which determines up to how many reservations it can accept at any given time. The dynamic models focus on individual booking

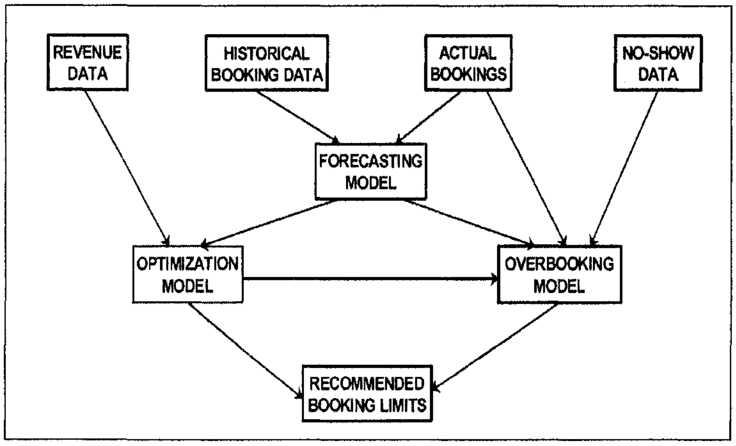


Figure 1.5: Links between seat allocation, forecasting and OR tools

requests over time. In most of the models, it is typically assumed that there are no cancellations or no shows. In such a case, there is no need to consider overbooking to avoid such problems.

### 1.1.4 Pricing

Similar to the seat allocation models, the pricing models are also divided mainly into static and dynamic models. There are also some models that deal with joint resource (seat) allocation and pricing. Static models are based on aggregate demand distributions. On the other hand, dynamic models consider demand as a controllable stochastic process.

#### Static Pricing Models

The static airline seat pricing problem can be considered as the case of the multi-product newsvendor problem, where the production costs are fixed and the products are perishable without any salvage value. Several researchers have studied different extensions to the classical single product newsvendor problem with price as the decision variable. This includes the works of Mills (1959, 1962), Karlin and Carr (1962), Nevins (1966), Zabel (1972), Thomas (1974), Thowsen (1975), Petruzzi and Dada (1999), and Federgruen and Heching (1999). All of these researchers considered both the single and the multiple period combined pricing and inventory control problem, which is typically solved using DP.

### Dynamic Pricing Models

Gallego and Ryzin (1994), and Zhao and Zheng (2000) considered the problem of optimal pricing of a single product inventory over a finite planning period before it perished i.e., no salvage value. The demand in these studies was modeled as a controllable continuous time stochastic process in which the demand arrival intensity is known and is a decreasing function of the price. Chatwin's (2000) work is a special case of Zhao and Zheng's (2000) model where only a finite set of prices were considered. Feng and Gallego (2000) extended the model by allowing demand intensities depend on sale-to-date. You (1999) extended the model by allowing group booking.

Gallego and Ryzin (1997) built their earlier model considering a finite horizon joint pricing and resource allocation problem. The continuous time pricing problem cannot be solved exactly, but they proposed solution using two heuristics. Similar to Gallego and Ryzin's work, Paschalidis and Tsitsiklis (2000) addressed the dynamic pricing problem of network services, which they formulated as a finite-state, continuous time, infinite-horizon average reward problem. They showed that a static, deterministic model can be used to determine an asymptotically optimal pricing policy. Kleywegt (2001) determined an optimal control formulation for multi-product dynamic pricing. Elmaghraby and Keskinocak (2003) reviewed the literature and current practices in dynamic pricing, focusing on dynamic pricing in the presence of inventory considerations. You (2003) extended the newsvendor problem to consider the demand as price dependent and that customers may cancel their orders. An optimal decision policy was developed to maximize the total expected profit. Lin (2004) considered a sequential dynamic pricing model where sellers sell a stock to a random number of customers, one at a time with a single request. The problem was formulated using stochastic DP. Tight bounds were developed on optimal expected revenue and an asymptotically optimal heuristic policy was also proposed. Kuyumcu and Garcia-Diaz (2000) developed a new analytical procedure for the joint pricing and seat allocation problem considering demand forecasts, the number of fare classes, and aircraft capacities. The proposed polyhedral graph approach utilizes split graphs and cutting planes, thus results significantly save CPU time when compared to commercially available general purpose integer programming softwares.

It is clearly evident that OR has been one of the principal contributors to the enormous growth that the air transport sector has experienced during the past 50 years. In the best tradition of OR, the development of models and solutions has been motivated by issues and problems encountered in practice and has led, in several instances, to insights of a general nature and to important methodological advances in the OR field at large. At this point, OR models and algorithms are diffused throughout the sector and constitute an integral part of the standard practices of airlines, airports, and airport transport management service providers. In view of the numerous challenges facing the entire air transport industry, it is safe to expect a continuing central role for OR. As indicated in this thesis, there are many promising topics for future research in each of the areas examined. At the most fundamental level, and in general terms, the frontiers can be summarized as follows: (i) Relaxing the boundaries between the successive stages of aircraft and crew schedule planning, so that schedule design, fleet assignment, aircraft maintenance routing, and crew scheduling might eventually be performed in an integrated way, rather than solved sequentially as interrelated, but distinct subproblems; (ii) Including pricing decisions in RM, instead of treating fares and fare classes as fixed, externally specified inputs; (iii) Developing fast decision support tools that increase the safety and efficiency of air transport operations by taking advantage of the massive, real-time data flows in the increasingly info-centric aviation infrastructure.

### 1.1.5 Airline RM and Newsvendor Problem

In its simplest form, RM can be studied in the context of the newsvendor problem, which has a simple yet an elegant structure. The problem is considered as a building block in stochastic inventory control. Petruzzi and Dada (1999) identified the problem as an excellent tool for examining how operational problems interact with marketing issues to influence decision-making processes at the firm level. A standard newsvendor problem assumes a single perishable commodity in a single selling period, with capacity controlled for a fixed price such that the revenue (profit) is maximized. McGill and Ryzin (1999) show that a single leg flight with two fare classes RM problem is essentially equivalent to a single period inventory or newsvendor problem, as in the problem the commodity is considered perishable like the seats in airline RM. A single period newsvendor problem also best represents the sale of fashion products. Fisher and Raman (1996) identified that most fashion apparel companies design and introduce completely new products every season, to be sold in a single period. At the end of the season, it is not uncommon to observe the cost of sold out excess inventory and the cost of lost sales to be below the purchase price. Thus, studies in a single-period newsvendor problem have a direct impact on RM research.

A single period newsvendor problem is a building block in stochastic inventory control. It incorporates the fundamental techniques for stochastic decision making that can be applied to a much broader scope. The problem is well researched and has a rich history which can be traced back to Edgeworth (1888) where it first appeared in the context of banking. The research in this area boomed in the 1950's due to the war effects, while the problem was formulated in inventory theory. Arrow et al. (1951) showed that it was critical to have optimal buffer stocks in an inventory control system. Porteus (1990) and Lee and Nahmias (1990) presented a thorough review on the newsvendor problem with stochastic demand. In most studies, the pricing was considered as a fixed parameter rather than a decision variable. Whitin (1955) was the first to discuss the pricing issues in inventory control theory. Mills (1959) extended Whitin's work by modeling the uncertainty in the price sensitive demand. He suggested an additive form for the study and assumed stochastic demand was the summation of riskless demand and a random factor. The riskless demand was considered a deterministic function of price. The most evident benefit of such modeling is that the random behavior of the demand is captured using standard distributions independent from the pricing. Karlin and Carr (1962) presented a multiplicative form for the demand. In this model, the random demand is considered as the product of a riskless demand function and a random factor. Both the additive and multiplicative models were considered fundamental in the pricing problem. Some subsequent contributions to the additive model were made by Ernst (1970), Young (1978), Lau and Lau (1988) and Petruzzi and Dada (1999). Contributions to the multiplicative model came from Nevins (1966), Zabel (1970), Young (1978) and Petruzzi and Dada (1999). Mieghem and Dada (1999) studied the capacity and pricing in price versus production postponement in a competitive market. A coordination of a dynamic joint pricing and production in a supply chain was studied by Zhao and Wang (2002) using a leader follower game. Optimal control policies were identified for the channel coordination. Bish and Wang (2004) studied the optimal resource investment decision faced by a two-product, price-setting firm that operates in a monopolistic setting and employs a postponed pricing scheme. The principles on the firm's optimal resource investment decision were provided. Gupta et al. (2006) developed a pricing model and heuristic solution procedures for clearing end-of-season inventory.

Chen et al. (2006) addressed dynamically adjusting the production rate and sale price to maximize the long run discounted profit. They proposed algorithm to compute the base stock level and the price switch threshold, along with an extension to multiple price choices. They also studied an optimal pricing and inventory control policy in periodic-review systems with fixed ordering cost and lost sales. Bell and Zhang (2006) examined different decisions surrounding the implementation of aggressive RM pricing in the context of a firm facing a single period stochastic pricing and stocking problem with identified decisions having large financial effects. Bhargava et al. (2006) studied the optimal stockout compensation in the electronic retailing industry, considering price as a decision tool.

### 1.1.6 Distribution Free Approach in RM

Gallego and Moon (1993) were the first to study the distribution free model of the newsvendor problem. They proved the optimality of Scarf's (1952) ordering rule to the problem. The model determined the optimal order quantity with a given mean and standard deviation/error which maximized the expected profit against the worst possible demand distribution. In their work, the distribution free approach was also studied for a variety of cases such as: the recourse case where there is more than one chance of ordering after an initial order is placed, the fixed ordering cost case, the random yield case and the multi-item case. Several papers have appeared following Gallego and Moon's (1993) work considering new aspects to their study. Moon and Gallego (1994) considered a distribution free approach for optimal determination of lead times and backorders for both continuous and periodic review models. Moon and Choi (1994) studied a distribution free approach for optimal quantity determination in a continuous review inventory system with a service level constraint. Extending their work, Moon and Choi (1995) also studied the distribution free approach to the newsvendor problem with balking. Several manufacturing policies such as make-to-order, make-in-advance and some composite policies were investigated by Moon and Choi (1997) using a distribution free approach.

Some other closely related works were Moon and Silver (2000), Wu and Ouyang (2001), Ouyang and Chang (2002) and AlFares and Elmorra (2005). All of these papers mainly focus on a quantity decision for a fixed price and thus do not provide an integrated frame work to optimally control pricing and capacity jointly.

The proposed research utilizes a distribution free approach and focuses on two issues: Firstly, it combines the practice of quantity-based RM and price-based RM in an integrated framework, where both the price and quantity are optimized jointly; Secondly, it explores the possibility of using simple parameters of demand, such as the mean and the variance, instead of using the precise distributional information for RM. As discussed in McGill (1995), in most passenger transportation systems, the historical data represents the tickets sales not the actual demand. Hence, the distribution free approach is a good tool to best capture the stochastic behavior in a deterministic approximation. Such approximations are derived, whereas finding a solution by using standard optimization techniques is computationally straightforward.

## 1.2 Contributions and Organization of Thesis

The objective of this thesis is to study an integrated framework towards price-based and quantity-based RM. The approach enables jointly controlling the pricing and capacity such that the revenue is maximized. In order to present a competitive analysis, a duopoly environment is considered in this research which could also be directly extended to oligopoly competition. The use of competitive pricing strategies is emphasized to maximize the revenue. Furthermore, the lower bounds on revenue estimates are developed using the distribution free approach. This approach maximizes the revenue while not knowing the distribution. The simple parameters such as mean and variance of stochastic demand are used in the analysis. The approach develops a lower bound estimate while assuming several standard problems observed in RM. The implication of this study on airline RM is also identified.

In Chapter 1, a general introduction of RM with more focus on the airline industry is presented. A literature review in the area is presented and current problems with research possibilities are identified. Chapter 2 discusses the price-based RM with application to the airline industry in a duopoly competition. Two airlines compete for price rational customers in each fare class. We show the uniqueness of Nash equilibrium for the game assuming the seat capacity is known and fixed. A sensitivity analysis is carried out to determine expressions of unique Nash equilibrium under various situations of deterministic and stochastic demand. The analysis is further extended to fare pricing competition that provides an integrated approach for joint control of fare pricing and seat inventory control assuming nested booking limits. The fare pricing competition models are developed using additive and multiplicative approaches. A numerical study is carried out on the three suggested games: i) non cooperative; ii) cooperative bargaining game with no side payments; and iii) cooperative bargaining game with side payments. A sensitivity analysis is also reported and statistical tools to establish useful inferences about the games are also presented. A lower bounding scheme on revenue using the distribution free approach is presented in Chapter 3. The approach enables maximizing the revenue against the worst possible demand distribution. The most commonly observed problem in RM, the standard newsvendor problem, is studied while using both the additive and multiplicative models. Later the analysis is extended to consider the standard newsvendor problem with a holding and shortage cost. In both of these studies, the lower bound estimate on revenue is shown to be quasi-concave. A numerical study shows that the lower bound is near optimal. The distribution free approach is also applied to several other extensions of the standard newsvendor problem and reported in Chapter 4. The extensions include: (i) the standard newsvendor problem with a shortage cost penalty; (ii) the recourse cost case, where there is a second purchasing opportunity; (iii) the random yield case, where non-conforming products are considered; and (iv) the multiple item case, when more than one product competes for a scarce resource such as a budget allocation or a restricted total available capacity. In most cases, the lower bound estimate on revenue is shown to be quasi-concave. A numerical study for each case, demonstrates that for each extension the lower bound estimate is quite comparable to the revenue estimate when two standard distributions, uniform and normal, are assumed. Links between research in airline RM and studies in relation to extensions on the newsvendor problem are identified. In Chapter 5, a summary of this research is presented. Finally, the suggestions for future research are outlined.

# Chapter 2

# Competitive Airline Revenue Management

In most markets customers select among substitutable products using several rationalities. The price is perhaps the paramount factor in the selection among the products. As the fare pricing could better calibrate the customer behavior, therefore a price based RM could efficiently evaluate the competitive behavior of the market. Airline RM is mostly quantity based, but knowing that the aviation markets are quite competitive, the airlines must use efficient fare pricing strategies to maximize their revenues. The study presented here first considers fare pricing duopoly competition while assuming the seat allocation is pre-committed (fixed). Airlines compete for customers using their fare pricing strategies in each fare class. The demand in each fare class to an airline is dependent on its fare price and fare price offered by its rival airline(s). A game theoretic approach is used to address the problem assuming both the deterministic and stochastic price sensitive customer demand in each fare class.

Nash Equilibrium (NE) is an important solution concept in many non-cooperative games. In a game, a set of feasible actions that can be adopted by a player in the game is called its strategy set. An outcome observed as a result of a strategy played by a player in the game is called the player's payoff. In a non-cooperative game the players are allowed to choose their strategies simultaneously. A formal definition of NE established in Nash's (1950b) work is as follows:

**Definition 2.1** *A pair of strategies $(x_1^{NE}, x_2^{NE})$ is said to constitute a NE if the following pair of inequalities are satisfied for all $x_1 \in X_1$ and for all $x_2 \in X_2$:*

$$f_1(x_1^{NE}, x_2^{NE}) \geq f_1(x_1, x_2^{NE}) \quad \textit{and} \quad f_2(x_1^{NE}, x_2^{NE}) \geq f_2(x_1^{NE}, x_2)$$

*Where $X_1$ and $X_2$ represent the strategy sets of players 1 and 2 respectively.*

An alternative interpretation is, $x_1^{NE}$ and $x_2^{NE}$ solve $\max\limits_{x_1 \in X_1} \; f_1(x_1, x_2)$ and $\max\limits_{x_2 \in X_2}$ $f_2(x_1, x_2)$.

Assuming continuity and differentiability also $(x_1, x_2) \in \mathbf{R^2}$. The definition implies that if the pair $(x_1^{NE}, x_2^{NE})$ is in NE, then the player's decision must satisfy the first optimality conditions:

$$\frac{\partial f_1(x_1, x_2^{NE})}{\partial x_1}|_{x_1 = x_1^{NE}} = 0 \quad \text{and} \quad \frac{\partial f_2(x_1, x_2^{NE})}{\partial x_2}|_{x_2 = x_2^{NE}} = 0$$

## 2.1 Fare Pricing Competition

The problem is illustrated with a single flight leg competition between two airlines in a duopoly market where two airlines are offering two fare classes to their customers. The market is also divided into only two customer (low and high fare) classes and segmentation is considered perfect i.e., a high fare customer does not request a low fare ticket and vice versa. Customer diversion is also not considered in this model. Customers are considered rational and the fare price is the only factor to calibrate the rationality of the customers in each fare class. First scenario is assumed when two airlines arbitrarily pre-commit the seat allocation (booking limits) for each fare class for a known flight capacity. Once the booking limits are pre-committed both airlines compete for customers in each fare class in a non-cooperative duopoly environment using their pricing strategies. The problem is modeled using game theoretic approach. Other assumptions, used in the model are: i) Competing airlines offer single non-stop flight and they do not cooperate for a joint revenue maximization; ii) flight capacities offered by two airlines are known and fixed; iii) customer demand is observed sequentially, i.e., the low fare class demand is observed before the high fare class demand. A duopoly fare pricing with allocation competition model is presented in Figure 2.1.

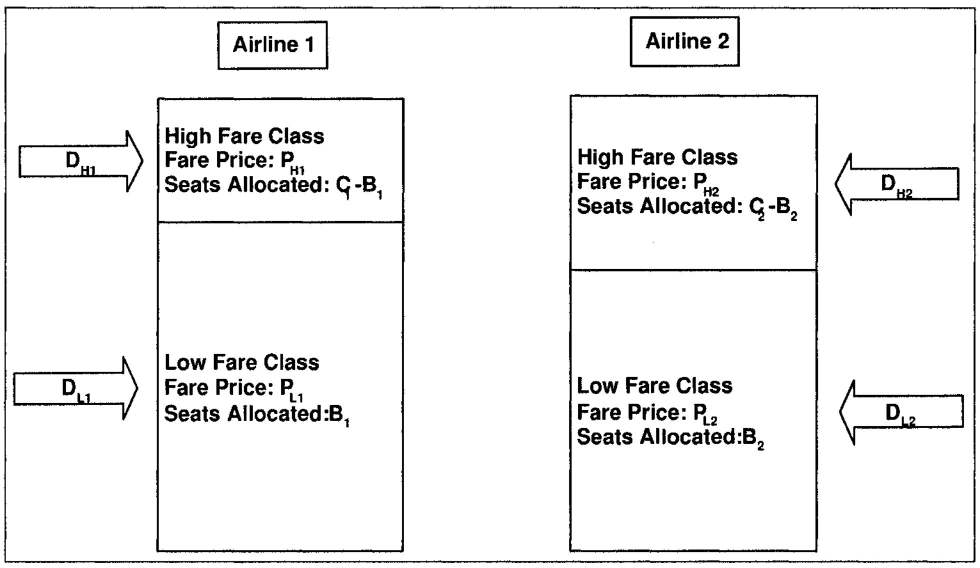


Figure 2.1: Two airlines competition for two fare classes

Now we define notations which are used in the modeling of this problem:

| | | |
|---|---|---|
| $C_i$ | : | Total flight capacity of airline $i = 1, 2$ |
| $B_i$ | : | Booking limit for low fare price committed by airline $i = 1, 2$ |
| $\Pi_i$ | : | Total revenue generated by airline $i = 1, 2$ |
| $P_{ci}$ | : | price offered by airline $i = 1, 2$, in fare class $c$ |
| $D_{ci}$ | : | Total demand airline $i$ is experiencing from customer fare class $c$ |

In the study, two fare classes, Low (L) and High (H) are considered, thus $c = \{L, H\}$. $D_{Li}$ can be more precisely written as $D_{Li}(P_{L1}, P_{L2})$ and is a function of the low fare price $P_{Li}$, offered by airline $i$, and the fare price $P_{Lj}$, $\forall j \neq i$ offered by the rival airline $j$. $D_{Li}$ is continuous and twice differentiable function bounded in $P_{Li} \in [\underline{P}_{Li}, \overline{P}_{Li}]$. Similarly $D_{Hi}$ is assumed with bounds, $P_{Hi} \in [\underline{P}_{Hi}, \overline{P}_{Hi}]$.

In following we present some existing results that help us identifying the uniqueness of Nash equilibrium. Most of these results are established in the work by Topkis (1978, 1979).

**Definition 2.2** *A function $f(x_1, x_2)$ is submodular in $(x_1, x_2)$, if $f(x_1^l, x_2) - f(x_1^h, x_2)$ is non-decreasing in $x_2$ for all $x_1^l \leq x_1^h$. A function $f(x_1, x_2)$ is supermodular if $-f(x_1, x_2)$ is submodular, where $l$ is low and $h$ is high.*

**Lemma 2.1** *Suppose $f(x_1, x_2)$ is twice differentiable, then $f(x_1, x_2)$ is submodular in $(x_1, x_2)$, if and only if* $\dfrac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2} \leq 0$

**Lemma 2.2** *A function $f(x_1, x_2)$ is supermodular (submodular) in $(x_1, x_2)$ if and only if it is submodular (supermodular) in $(x_1, -x_2)$*

### 2.1.1 Pre-Commitment of Booking Limits

Once the booking limits are known to each airline in competition, airlines compete for customers in each fare class using the fare pricing strategies. Now the revenue function of an airline under pre-committed booking limit scenario can be written as:

$$\Pi_i = P_{Li}\min\{B_i, D_{Li}\} + P_{Hi}\min\{D_{Hi}, C_i - B_i\} \tag{2.1}$$

An alternative form of the revenue function is

$$\Pi_i = P_{Li}B_i - P_{Li}[B_i - D_{Li}]^+ + P_{Hi}(C_i - B_i) - P_{Hi}[C_i - B_i - D_{Hi}]^+ \tag{2.2}$$

where, $[x]^+ = \max\{0, x\}, \forall\ x \in \mathbf{R}$

For each fare class following assumptions for the demand functions are made:

1. *Assumption 1:* $\dfrac{\partial D_{Li}}{\partial P_{Li}} < 0, \dfrac{\partial D_{Hi}}{\partial P_{Hi}} < 0, \quad \forall\ i = \{1, 2\}.$
2. *Assumption 2:* $\dfrac{\partial D_{Li}}{\partial P_{Lj}} > 0, \dfrac{\partial D_{Hi}}{\partial P_{Hj}} > 0, \quad \forall\ i, j = \{1, 2\}, i \neq j.$
3. *Assumption 3:* $-D_{Li}$ and $-D_{Hi}$ are submodular in $(P_{L1}, P_{L2})$ and $(P_{H1}, P_{H2})$ $i = \{1, 2\}$ respectively.

In Assumption 1, it is stated that the demand has increasing price elasticity, i.e., demand decreases with an increase in price. In Assumption 2, it is stated that the demand for a fare class increases with the increase in the fare pricing of its rival airline. To explain the assumption 3, we take low fare class example, $D_{L1}(P^s_{L1}, P_{L2}) - D_{L1}(P^l_{L1}, P_{L2})$ increases as $P_{L2}$ becomes smaller. It means that the reduction of demand in a fare class increases when the competing airline offer lower fares in the same fare class. These assumptions are first stated by Topkis (1979) and are commonly observed in price competition research of substitutable services/product.

Some more related works using the similar assumptions are Bernstein and Federgruen (2004a, 2004b, 2005) and also in Dai et al. (2005).

Using a variable pricing transformation as suggested in Lippman and McCardle (1997) and the results stated in Lemma 2.2, the generated revenue for an airline can be easily analyzed. It is assumed, $Z_{Li} = -P_{Li} \quad \forall\, i = \{1,2\}$ and $Z_{Hi} = -P_{Hi} \quad \forall\, i = \{1,2\}$. Previously stated assumptions are restated in the following forms:

1. *Assumption* 1′*:* $\dfrac{\partial D_{Li}}{\partial P_{Li}} < 0, \dfrac{\partial D_{Hi}}{\partial P_{Hi}} < 0, \quad \forall\, i = \{1,2\}.$
2. *Assumption* 2′*:* $\dfrac{\partial D_{Li}}{\partial Z_{Lj}} < 0, \dfrac{\partial D_{Hi}}{\partial Z_{Hj}} < 0, \quad \forall\, i,j = \{1,2\}, i \neq j.$
3. *Assumption* 3′*:* $-D_{Li}(P_{Li}, Z_{Li})$ and $-D_{Hi}(P_{Hi}, Z_{Hi})$ are supermodular in $(P_{L1}, Z_{L2})$ and $(P_{H1}, Z_{H2})$ $i = \{1,2\}$ respectively.

For an airline $i$ with pre-committed booking limits, the following Lemma, as suggested in Topkis (1978), enables us to identify the supermodularity of the competing airline's revenue function.

**Lemma 2.3** *(Topkis 1978) Suppose $g(x_1, x_2)$ is monotone in both $x_1$ and $x_2$ and is a supermodular function in $(x_1, x_2)$. Furthermore $f(.)$ is an increasing convex function. Then $f(g(x_1, x_2))$ is a supermodular function in $(x_1, x_2)$.*

In Theorem 2.1 we show that the total revenue of an airline in competition under pre-committed booking limit is supermodular. Later for the same problem we show that the Nash equilibrium is also unique.

**Theorem 2.1** *For a pre-commitment case of seat allocation, the revenue function to an airline is supermodular function of the fare prices.*

**Proof:** This argument is proved in duopoly competition. Let us consider the revenue of airline 1 as:

$$\Pi_1 = P_{L1}B_1 - P_{L1}[B_1 - D_{L1}]^+ + P_{H1}(C_1 - B_1) - P_{H1}[C_1 - B_1 - D_{H1}]^+ \quad (2.3)$$

Once the booking limit is known, the total revenue function is decomposed to revenue generated from low fare class ($\Pi_{L1}$), and high fare class ($\Pi_{H1}$), such that

$$\Pi_{L1} = P_{L1}B_1 - P_{L1}[B_1 - D_{L1}]^+ \quad (2.4)$$

$$\Pi_{H1} = P_{H1}(C_1 - B_1) - P_{H1}[C_1 - B_1 - D_{H1}]^+ \quad (2.5)$$

Hence, $\Pi_{L1}$ is supermodular function in $(P_{L1}, P_{L2})$. The similar analogy is used to draw the same conclusion for $\Pi_{H1}$.

$$\frac{\partial \Pi_{L1}}{\partial P_{L1}} = B_1 - [B_1 - D_{L1}]^+ - P_{L1}\frac{\partial [B_1 - D_{L1}]^+}{\partial P_{L1}} \quad (2.6)$$

$$\frac{\partial^2 \Pi_{L1}}{\partial P_{L1}\partial Z_{L2}} = -\frac{\partial [B_1 - D_{L1}]^+}{\partial Z_{L2}} - P_{L1}\frac{\partial^2 [B_1 - D_{L1}]^+}{\partial P_{L1}\partial Z_{L2}} \quad (2.7)$$

Form Assumption 2$'$, $\frac{\partial D_{L1}}{\partial Z_{L2}} < 0$, thus $\frac{\partial [B_1 - D_{L1}]^+}{\partial Z_{L2}} \geq 0$. Also from Assumptions 2$'$ and 3$'$, we know that $B_1 - D_{L1}$ is increasing in both $P_{L1}$ and $Z_{L2}$, hence it is a submodular function. Moreover, as $[x]^+ = \max\{x, 0\}$ is a convex increasing function. Therefore, by using Lemma 2.3, we can conclude that $\frac{\partial^2 [B_1 - D_{L1}]^+}{\partial P_{L1}\partial Z_{L2}} \geq 0$, and we obtain $\frac{\partial^2 \Pi_{L1}}{\partial P_{L1}\partial Z_{L2}} \leq 0$ which shows that it is a supermodular function in $(P_{L1}, P_{L2})$.

It is known from a theorem given in Topkis (1979) that if the strategy space is a complete lattice, the joint payoff function is upper-semicontinuous, and each player's payoff is supermodular. Therefore each player's best response is increasing in the other player's strategy. This means a strategy resulting an increase in the payoff of one player also impact a gain in the payoff of the other player as it is a non-zero sum game. When the best response exhibit this monotonicity property, the player's strategies are said to be strategic complements, and the existence of a Nash equilibrium is easy to establish (see Lippman and McCardle (1997)).

### 2.1.2 Deterministic Demand with Pre-Committed Booking Limits

Let us first assume that the demand is deterministic. Using the Theorem 2.1, we can show that Nash equilibrium is unique.

**Theorem 2.2** *In two airline's pricing game when the demand is considered deterministic there exists a unique Nash equilibrium if* $\left| \frac{\partial^2 \Pi_{Li}}{\partial P_{Li} \partial P_{Lj}} \right| < \left| \frac{\partial^2 \Pi_{Li}}{\partial {P_{Li}}^2} \right|$ *and* $\left| \frac{\partial^2 \Pi_{Hi}}{\partial P_{Hi} \partial P_{Hj}} \right| < \left| \frac{\partial^2 \Pi_{Hi}}{\partial {P_{Hi}}^2} \right|$.

**Proof:** Since the revenue function of each airline is supermodular (Theorem 2.1), a unique Nash equilibrium exists if $\left| \frac{\partial^2 \Pi_{L1}}{\partial P_{L1}^2} \right| \geq \left| \frac{\partial^2 \Pi_{L1}}{\partial P_{L1} \partial Z_{L2}} \right|$ (Topkis 1979) where:

$$\left| \frac{\partial^2 \Pi_{L1}}{\partial P_{L1}^2} \right| = \left| 2\frac{\partial [B_1 - D_{L1}]^+}{\partial P_{L1}} + P_{L1} \frac{\partial^2 [B_1 - D_{L1}]^+}{\partial {P_{L1}}^2} \right| \tag{2.8}$$

$$\left| \frac{\partial^2 \Pi_{L1}}{\partial P_{L1} \partial Z_{L2}} \right| = \left| \frac{\partial [B_1 - D_{L1}]^+}{\partial Z_{L1}} + P_{L1} \frac{\partial^2 [B_1 - D_{L1}]^+}{\partial P_{L1} \partial Z_{L2}} \right| \tag{2.9}$$

Moulin (1986) also suggests that the condition given above is sufficient for the uniqueness of Nash equilibrium, which is the slopes of the best response of each player never exceed 1 in absolute value.

In literature, linear and logit (see Chen et al. (2004)) are the most commonly used techniques to model price sensitive demand. In this work, we also propose a linear model to represent price sensitive demand. Linear functions are defined to model deterministic and price sensitive demands for each fare class as follows:

$$D_{Li} = \alpha_{Li} - \beta_{Li} P_{Li} + \theta_{Lij} P_{Lj}, \ \forall \ \beta_{Li} > \theta_{Lij}, \beta_{Li}, \theta_{Lij} \geq 0, i \neq j, i, j = \{1, 2\}$$

$$D_{Hi} \;=\; \alpha_{Hi} - \beta_{Hi}P_{Hi} + \theta_{Hij}P_{Hj},\, \forall\, \beta_{Hi} > \theta_{Hij}, \beta_{Hi}, \theta_{Hij} \geq 0, i \neq j, i, j = \{1, 2\}$$

where, $\alpha_{Li}$ is the average price insensitive demand for an airline $i$, and $\beta_{Li}$ is the mean impact of price variation on demand for an airline $i$ with a unit change in the price. The mean impact on airline $i$'s demand due to a unit variation in the price of its rival airline $j$ is given by $\theta_{Lij}$. Here it is also assumed that $\beta_{Li} > \theta_{Lij}$ and $\beta_{Hi} > \theta_{Hij}$; otherwise an airline can increase demand while still increasing the fare price. Likewise a similar interpretation is derived for the high fare class demand. It is easy to verify that the condition stated in Theorem 2.2 is always true for previously mentioned linear demand function in each fare class; hence there exist a unique Nash equilibrium. As it is mentioned earlier that under pre-committed booking limit assumption, the total revenue function is decomposed into two separate revenue functions: i) the revenue generated from low fare class $\Pi_{Li}$; and ii) revenue generated from high fare class $\Pi_{Hi} \quad \forall\, i = \{1, 2\}$. In the next section we perform sensitivity analysis for each fare classes. First we consider low fare class pricing competition followed by a high fare class pricing competition.

### Low Fare Pricing Competition

Due to the pre-committed booking limits, the booking limit of airline 1 ($B_1$ ) is known. Hence the low fare revenue function of airline 1 is:

$$\Pi_{L1} \;=\; \begin{cases} P_{L1}D_{L1}, & \text{When } D_{L1} < B_1 \\ P_{L1}B_1, & \text{Otherwise} \end{cases} \tag{2.10}$$

From Equation 2.10 we establish two distinct response functions of airline 1. When demand $D_{L1} < B_1$ then the payoff to airline 1 would be:

$$\Pi_{L1} = P_{L1}\,(\alpha_{L1} - P_{L1}\,\beta_{L1} + P_{L2}\,\theta_{L12}) \tag{2.11}$$

where $\Pi_{L1}$ is concave for a given $P_{L2}$. Now, we can determine the best response function of airline 1 under the defined situation by applying first order optimality condition as:

$$\alpha_{L1} - 2\,P_{L1}\,\beta_{L1} + P_{L2}\,\theta_{L12} = 0 \tag{2.12}$$

In graphical representation, plane $(P_{L1}, P_{L2})$, the line presented in Equation 2.12 has slope $\frac{2\,\beta_{L1}}{\theta_{L12}} > 2$, and it passes through point $(\frac{\alpha_{L1}}{2\,\beta_{L1}}, 0)$. This is the best response function of airline 1 when its low fare demand is less than its booking limit, $B_1$. This is called as "the Low fare class Capacity Greater than the Demand for airline 1" (LCGD1).

A contrary case to LCGD1 is "the Low fare class Capacity Less than the Demand for airline 1" (LCLD1), i.e., $D_{L1} \geq B_1$. The revenue function under this condition is $\Pi_{L1} = P_{L1}B_1$. The best response function becomes $B_1 = \alpha_{L1} - \beta_{L1}P_{L1} + \theta_{L12}P_{L2}$. Let $P'_{L1}$ is the price such that $D_{L1} = B_1$, then the price is:

$$P'_{L1} = \frac{\alpha_{L1} - B_1 + P_{L2}\,\theta_{L12}}{\beta_{L1}} \tag{2.13}$$

The slope of the best response function is $\frac{\beta_{L1}}{\theta_{L1}} > 1$, which passes through $(\frac{\alpha_{L1} - B_1}{\beta_{L1}}, 0)$. Based on $P'_{L1}$ we identify LCGD1 and LCLD1 situation when low fare pricing offered by airline 1 are $P_{L1} > P'_{L1}$ and $P_{L1} \leq P'_{L1}$ respectively. Depending upon booking limit $B_1$ the LCGD1 and LCLD1 situations can also be identified. When $\frac{-B_1 + \alpha_{L1}}{\beta_{L1}} \leq \frac{\alpha_{L1}}{2\,\beta_{L1}}$ i.e., $B_1 \geq \frac{\alpha_{L1}}{2}$, then airline 1's best response includes both LCGD1 and LCLD1. However for $B_1 < \frac{\alpha_{L1}}{2}$, the best response function of airline 1 contains only LCLD1.

A similar analysis can be done for the airline 2 (competitor). When airline 2 pre-commits a booking limit $B_2$, its low fare revenue function is:

$$\Pi_{L2} = \begin{cases} P_{L2}D_{L2}, & \text{When } D_{L2} < B_2 \\ P_{L2}B_2, & \text{Otherwise} \end{cases} \tag{2.14}$$

Likewise airline 1, airline 2 also has two situations in its low fare class demand. The situation when "the Low fare class Capacity Greater than the Demand for airline 2" (LCGD2) the best response function would be $\alpha_{L2} - 2\,P_{L2}\,\beta_{L2} + P_{L1}\,\theta_{L21} = 0$. For the case of "the Low fare class Capacity Less than the Demand for airline 2" (LCLD2) the best response function would be $B_2 = \alpha_{L2} - \beta_{L2}P_{L2} + \theta_{L21}P_{L1}$. In following we present various cases that airlines may experience in this deterministic low fare pricing competition.

**Case 2.1** $D_{L1} > B_1$ *and* $D_{L2} > B_2$

In this case both airlines experience demand greater than their capacities. Thus the case is LCLD1 and LCLD2. The pricing strategies at Nash equilibrium are determined by the intersection of the best response function of competing airline under LCLD1 and LCLD2 and following pricing strategies are resulted:

$$P_{L1} = \frac{(-B_1 + \alpha_{L1})\,\beta_{L2} + (-B_2 + \alpha_{L2})\,\theta_{L12}}{\beta_{L1}\,\beta_{L2} - \theta_{L12}\,\theta_{L21}} \tag{2.15}$$

$$P_{L2} = \frac{(-B_2 + \alpha_{L2})\,\beta_{L1} + (-B_1 + \alpha_{L1})\,\theta_{L21}}{\beta_{L1}\,\beta_{L2} - \theta_{L12}\,\theta_{L21}} \tag{2.16}$$

**Case 2.2** $D_{L1} \leq B_1$ *and* $D_{L2} > B_2$

Airline 2 is still facing LCLD2 but airline 1 is experiencing two responses, LCLD1 and LCGD1. The pricing at Nash equilibrium is determined such that it is the intersection of the three response functions and the pricing strategies are:

$$P_{L1} = \frac{\alpha_{L1}\,\beta_{L2} + (-B_2 + \alpha_{L2})\,\theta_{L12}}{2\,\beta_{L1}\,\beta_{L2} - \theta_{L12}\,\theta_{L21}} \tag{2.17}$$

$$P_{L2} = \frac{2\,(-B_2 + \alpha_{L2})\,\beta_{L1} + \alpha_{L1}\,\theta_{L21}}{2\,\beta_{L1}\,\beta_{L2} - \theta_{L12}\,\theta_{L21}} \tag{2.18}$$

with the following condition

$$\begin{aligned} & B_1\left(2\,\beta_{L1}\,{\beta_{L2}}^2 + 2\,\beta_{L1}\,\beta_{L2}\,\theta_{L21} - \beta_{L2}\,\theta_{L12}\,\theta_{L21} - \theta_{L12}\,{\theta_{L21}}^2\right) \\ + \; & B_2\,(\beta_{L1}\,\beta_{L2}\,\theta_{L12} + \beta_{L1}\,\theta_{L12}\,\theta_{L21}) \geq \lambda_1 \end{aligned} \tag{2.19}$$

where

$$\lambda_1 = \alpha_{L1}\,\beta_{L1}\,{\beta_{L2}}^2 - \alpha_{L2}\,\beta_{L1}\,\beta_{L2}\,\theta_{L12} + \alpha_{L1}\,\beta_{L1}\,\beta_{L2}\,\theta_{L21} + \alpha_{L2}\,\beta_{L1}\,\theta_{L12}\,\theta_{L21} \tag{2.20}$$

**Case 2.3** *$D_{L1} > B_1$ and $D_{L2} \leq B_2$*

The situation is contrary to Case 2.2, airline 2 is experiencing both LCLD2 and LCGD2. However, airline 1 is experiencing only LCLD2. The pricing strategy representing the intersection of these three best response functions is:

$$P_{L1} = \frac{2\,(-B_1 + \alpha_{L1})\,\beta_{L2} + \alpha_{L2}\,\theta_{L12}}{2\,\beta_{L1}\,\beta_{L2} - \theta_{L12}\,\theta_{L21}} \tag{2.21}$$

$$P_{L2} = \frac{\alpha_{L2}\,\beta_{L1} + (-B_1 + \alpha_{L1})\,\theta_{L21}}{2\,\beta_{L1}\,\beta_{L2} - \theta_{L12}\,\theta_{L21}} \tag{2.22}$$

having the following condition

$$\begin{aligned} & B_1\,(-\,(\beta_{L1}\,\beta_{L2}\,\theta_{L21}) - \beta_{L2}\,\theta_{L12}\,\theta_{L21}) \\ + \; & B_2\left(-2\,{\beta_{L1}}^2\,\beta_{L2} - 2\,\beta_{L1}\,\beta_{L2}\,\theta_{L12} + \beta_{L1}\,\theta_{L12}\,\theta_{L21} + {\theta_{L12}}^2\,\theta_{L21}\right) \geq \lambda_2 \end{aligned} \tag{2.23}$$

where

$$\lambda_2 = -\alpha_{L2}\,{\beta_{L1}}^2\,\beta_{L2} - \alpha_{L2}\,\beta_{L1}\,\beta_{L2}\,\theta_{L12} - \alpha_{L1}\,\beta_{L1}\,\beta_{L2}\,\theta_{L21} - \alpha_{L1}\,\beta_{L2}\,\theta_{L12}\,\theta_{L21} \tag{2.24}$$

**Case 2.4** $D_{L1} \leq B_1$ *and* $D_{L2} \leq B_2$

In this case each airline faces two situations. Airline 1 faces both LCLD1 and LCGD1 situations. The intersection of these two distinct responses is $\left(\frac{B_1}{\beta_{L1}}, \frac{2B_1 - \alpha_{L1}}{\theta_{L12}}\right)$. A condition is established such that the best response function of airline 2 when facing LCGD2 passes through this intersection. This condition is achieved by modifying the pre-committed booking limit $B_1$ to a new value $B_1^{'}$. The modified value is:

$$B_1^{'} = \frac{\beta_{L1}(2\alpha_{L1}\beta_{L2} + \alpha_{L2}\theta_{L12})}{4\beta_{L1}\beta_{L2} - \theta_{L12}\theta_{L21}} \tag{2.25}$$

A similar analysis is done for airline 2 when it is experiencing both LCLD2 and LCGD2. The intersection of the best response functions of airline facing LCLD2 and LCGD2 is $\left(\frac{2B_2 - \alpha_{L2}}{\theta_{L21}}, \frac{B_2}{\beta_{L2}}\right)$. A modified value of $B_2^{'}$ is computed such that the best response function of airline 1 passes through aforementioned intersection point when it is facing LCGD1. The modified value of $B_2^{'}$ in this situation is

$$B_2^{'} = \frac{\beta_{L2}(2\alpha_{L2}\beta_{L1} + \alpha_{L1}\theta_{L21})}{4\beta_{L1}\beta_{L2} - \theta_{L12}\theta_{L21}} \tag{2.26}$$

It is easy to verify that $B_1^{'} > \frac{\alpha_{L1}}{2}$ and similarly $B_2^{'} > \frac{\alpha_{L2}}{2}$. Depending upon $B_1$ and $B_2$ there can be four sub-cases:

**Case 2.4.1** $\frac{\alpha_{L1}}{2} \leq B_1 < B_1^{'}$ *and* $\frac{\alpha_{L2}}{2} \leq B_2 < B_2^{'}$

In this case airline 1 and airline 2 face LCLD1 and LCLD2 respectively which is aforementioned Case 2.1. The pricing strategies at Nash equilibrium are already established in Equations 2.15-2.16.

**Case 2.4.2** $B_1 \geq B_1^{'}$ *and* $\frac{\alpha_{L2}}{2} \leq B_2 < B_2^{'}$

There are two possibilities. Similar to Case 2.2, we know if the Equation 2.19 holds (where $\lambda_1$ is derived in Equation 2.20), then the pricing strategies at Nash equilibrium are identified in Equations 2.17-2.18. Otherwise airlines will face LCLD1 and LCLD2 and pricing at Nash equilibrium is identified in Equations 2.15-2.16.

**Case 2.4.3** $\frac{\alpha_{L1}}{2} \leq B_1 < B_1^{'} \, and \, B_2 \geq B_2^{'}$

Again there are two possibilities. Similar to the Case 2.3, we know if Equation 2.23 for the given $\lambda_2$ holds, then the pricing strategies at Nash equilibrium are identified in aforementioned Case 2.3 with Equations 2.21-2.22. Otherwise airlines observe LCLD1 and LCLD2.

**Case 2.4.4** $B_1 \geq B_1^{'} \, and \, B_2 \geq B_2^{'}$

In this case airlines experience LCGD1 and LCGD2. The pricing strategies at Nash equilibrium resulted by solving the best response functions of the two airlines in this situation are following:

$$P_{L1} = \frac{2\alpha_{L1}\beta_{L2} + \alpha_{L2}\theta_{L12}}{4\beta_{L1}\beta_{L2} - \theta_{L12}\theta_{L21}} \tag{2.27}$$

$$P_{L2} = \frac{2\alpha_{L2}\beta_{L1} + \alpha_{L1}\theta_{L21}}{4\beta_{L1}\beta_{L2} - \theta_{L12}\theta_{L21}} \tag{2.28}$$

### High Fare Pricing Competition

Under the pre-committed booking limit assumption, the allocated capacity for the high fare class of an airline $i$ is $C_i - B_i$, $\forall i = \{1, 2\}$. The analysis is extended for high fare class competition. It is assumed that airline 1 is observing two distinct customer demand behavior. The first behavior is "High fare Capacity Less than the Demand for airline 1" (HCLD1) and the second behavior is "High fare Capacity Greater than the Demand for airline 1" (HCGD1). In the case of HCLD1 the best response function of airline 1 is $C_1 - D_{H1} = \alpha_{H1} - P_{H1}\,\beta_{H1} + P_{H2}\,\theta_{H12}$. However in the case of HCGD1 the best response function becomes $\alpha_{H1} - 2\,P_{H1}\,\beta_{H1} + P_{H2}\,\theta_{H12} = 0$. For airline 2 the best response functions facing HCLD2 and HCGD2 are $C_2 - D_{H2} = \alpha_{H2} - P_{H2}\,\beta_{H2} + P_{H1}\,\theta_{H21}$ and $\alpha_{H2} - 2\,P_{H2}\,\beta_{H2} + P_{H1}\,\theta_{H21} = 0$ respectively. These response functions are used to drive pricing expressions at Nash equilibrium for the following cases.

**Case 2.5** $D_{H1} > (C_1 - B_1)$ *and* $D_{H2} > (C_2 - B_2)$

The effects of this condition is same as discussed in Case 2.1 for the low fare price competition. The high fare class prices at Nash equilibrium are:

$$P_{H1} = \frac{(B_1 - C_1 + \alpha_{H1})\,\beta_{H2} + (B_2 - C_2 + \alpha_{H2})\,\theta_{H12}}{\beta_{H1}\,\beta_{H2} - \theta_{H12}\,\theta_{H21}} \tag{2.29}$$

$$P_{H2} = \frac{(B_2 - C_2 + \alpha_{H2})\,\beta_{H1} + (B_1 - C_1 + \alpha_{H1})\,\theta_{H21}}{\beta_{H1}\,\beta_{H2} - \theta_{H12}\,\theta_{H21}} \tag{2.30}$$

**Case 2.6** $D_{H1} \leq (C_1 - B_1)$ *and* $D_{H2} > (C_2 - B_2)$

This case resembles Case 2.2 in the low fare price competition. In this case the high fare pricing strategies at Nash equilibrium are:

$$P_{H1} = \frac{\alpha_{H1}\,\beta_{H2} + (B_2 - C_2 + \alpha_{H2})\,\theta_{H12}}{2\,\beta_{H1}\,\beta_{H2} - \theta_{H12}\,\theta_{H21}} \tag{2.31}$$

$$P_{H2} = \frac{2\,(B_2 - C_2 + \alpha_{H2})\,\beta_{H1} + \alpha_{H1}\,\theta_{H21}}{2\,\beta_{H1}\,\beta_{H2} - \theta_{H12}\,\theta_{H21}} \tag{2.32}$$

having the following condition

$$\begin{aligned} & B_1\left(2\,\beta_{H1}\,{\beta_{H2}}^2 + 2\,\beta_{H1}\,\beta_{H2}\,\theta_{H21} - \beta_{H2}\,\theta_{H12}\,\theta_{H21} - \theta_{H12}\,{\theta_{H21}}^2\right) \\ & \qquad + B_2\,(\beta_{H1}\,\beta_{H2}\,\theta_{H12} + \beta_{H1}\,\theta_{H12}\,\theta_{H21}) \geq \lambda_3 \end{aligned} \tag{2.33}$$

where

$$\begin{aligned} \lambda_3 &= 2\,C_1\,\beta_{H1}\,{\beta_{H2}}^2 - \alpha_{H1}\,\beta_{H1}\,{\beta_{H2}}^2 + C_2\,\beta_{H1}\,\beta_{H2}\,\theta_{H12} - \alpha_{H2}\,\beta_{H1}\,\beta_{H2}\,\theta_{H12} + 2\,C_1\,\beta_{H1}\,\beta_{H2}\,\theta_{H21} \\ &- \alpha_{H1}\,\beta_{H1}\,\beta_{H2}\,\theta_{H21} + C_2\,\beta_{H1}\,\theta_{H12}\,\theta_{H21} - \alpha_{H2}\,\beta_{H1}\,\theta_{H12}\,\theta_{H21} - C_1\,\beta_{H2}\,\theta_{H12}\,\theta_{H21} \\ &- C_1\,\theta_{H12}\,{\theta_{H21}}^2 \end{aligned} \tag{2.34}$$

**Case 2.7** $D_{H1} > (C_1 - B_1)$ *and* $D_{H2} \leq (C_2 - B_2)$

The Case 2.7 is the opposite of Case 2.6 and resembles the Case 2.3 of the low fare price competition. The pricing strategies at Nash equilibrium are:

$$P_{H1} = \frac{2\,(B_1 - C_1 + \alpha_{H1})\,\beta_{H2} + \alpha_{H2}\,\theta_{H12}}{2\,\beta_{H1}\,\beta_{H2} - \theta_{H12}\,\theta_{H21}} \tag{2.35}$$

$$P_{H2} = \frac{\alpha_{H2}\,\beta_{H1} + (B_1 - C_1 + \alpha_{H1})\,\theta_{H21}}{2\,\beta_{H1}\,\beta_{H2} - \theta_{H12}\,\theta_{H21}} \tag{2.36}$$

Nash equilibrium prices are the results of the intersection of three response functions while satisfying the following condition.

$$\begin{aligned}\lambda_4 \;\leq\;& B_1\,(\beta_{H1}\,\beta_{H2}\,\theta_{H21} + \beta_{H2}\,\theta_{H12}\,\theta_{H21}) + \\ & B_2\left(2\,{\beta_{H1}}^2\,\beta_{H2} + 2\,\beta_{H1}\,\beta_{H2}\,\theta_{H12} - \beta_{H1}\,\theta_{H12}\,\theta_{H21} - {\theta_{H12}}^2\,\theta_{H21}\right)\end{aligned} \tag{2.37}$$

where

$$\begin{aligned}\lambda_4 \;=\;& 2\,C_2\,{\beta_{H1}}^2\,\beta_{H2} - \alpha_{H2}\,{\beta_{H1}}^2\,\beta_{H2} + 2\,C_2\,\beta_{H1}\,\beta_{H2}\,\theta_{H12} - \alpha_{H2}\,\beta_{H1}\,\beta_{H2}\,\theta_{H12} + C_1\,\beta_{H1}\,\beta_{H2}\,\theta_{H21} \\ -\;& \alpha_{H1}\,\beta_{H1}\,\beta_{H2}\,\theta_{H21} - C_2\,\beta_{H1}\,\theta_{H12}\,\theta_{H21} - C_1\,\beta_{H2}\,\theta_{H12}\,\theta_{H21} - \alpha_{H1}\,\beta_{H2}\,\theta_{H12}\,\theta_{H21} \\ -\;& C_2\,{\theta_{H12}}^2\,\theta_{H21}\end{aligned} \tag{2.38}$$

**Case 2.8** $D_{H1} \leq (C_1 - B_1)$ *and* $D_{H2} \leq (C_2 - B_2)$

Similar to Case 2.4, we extend our analysis for high fare class competition. Airline 1 in competition faces both HCLD1 and HCGD1 situations and likewise airline 1, the airline 2 also faces HCLD2 and HCGD2. Again a condition is determined such that the best response function of airline 2 when facing HCGD2 passes through the point of intersection of the best response functions under at HCLD1 and HCGD1 which is $\left(\dfrac{C_1 - B_1}{\beta_{H1}}, -\dfrac{2B_1 - 2C_1 + \alpha_{H1}}{\theta_{H12}}\right)$. A modified value of $B_1'$ is calculated such that the best response of airline 2 when it is experiencing HCGD2 passes through the previously determined intersection point:

$$B_1' = C_1 + \frac{\beta_{H1}(2\alpha_{H1}\beta_{H2} + \alpha_{H2}\theta_{H12})}{\theta_{H12}\theta_{H21} - 4\beta_{H1}\beta_{H2}} \tag{2.39}$$

Similarly we determine a new value of $B_2^{'}$ such that the best response function of airline 1 passes through the intersection point of the best response of airline facing HCLD2 and HCGD2. The intersection point is $\left(-\frac{2B_2 - 2C_2 + \alpha_{H2}}{\theta_{H21}}, \frac{C_2 - B_2}{\beta_{H2}}\right)$ and the modified value $B_2^{'}$ is:

$$B_2^{'} = \frac{2(2C_2 - \alpha_{H2})\beta_{H1}\beta_{H2} - (\alpha_{H1}\beta_{H2} + C_2\theta_{H12})\theta_{H21}}{4\beta_{H1}\beta_{H2} - \theta_{H12}\theta_{H21}} \tag{2.40}$$

Similar to Case 2.4, we can divide the Case 2.8 into four distinct sub-cases. Only a situation is analyzed when airlines experience HCGD1 and HCGD2. The prices at Nash equilibrium are determined by solving the best response function of airlines as follows:

$$P_{H1} = \frac{2\alpha_{H1}\beta_{H2} + \alpha_{H2}\theta_{H12}}{4\beta_{H1}\beta_{H2} - \theta_{H12}\theta_{H21}} \tag{2.41}$$

$$P_{H2} = \frac{2\alpha_{H2}\beta_{H1} + \alpha_{H1}\theta_{H21}}{4\beta_{H1}\beta_{H2} - \theta_{H12}\theta_{H21}} \tag{2.42}$$

### 2.1.3 Stochastic Demand with Pre-Committed Booking Limits

The goal in this section is to investigate pricing strategies of competing airlines for stochastic demand with pre-committed booking limits. Although most of the notations described in the deterministic demand situation are still applicable to the stochastic demand scenario, we still need to redefine the demand as a random variable with a price sensitive probability distribution function. Formally the random demand functions are written in the following ways:

$D_{Li}$ = Stochastic demand experienced by airline $i$ $\forall\, i = \{1, 2\}$ following a Cumulative demand Distribution Function (CDF) $F_{Li}(x)$

$D_{Hi}$ = Stochastic demand experienced by airline $i$ $\forall\, i = \{1, 2\}$ following a CDF $F_{Hi}(x)$

More precisely $F_{Li}(x)$ and $F_{Hi}(x)$ can be written as $F_{Li}^{(P_{L1},P_{L2})}(x)$ and $F_{Hi}^{(P_{H1},P_{H2})}(x)$ CDF of price dependent stochastic demands for low and high fare classes respectively.

The concept of stochastic ordering is also utilized which can be interpreted as the demand in a fare class at airline 1 increases (decreases) when other airline decreases (increases) the fare prices in that fare class stochastically. This is more often used to compare two random variables. More details on this concept can be found in Ross (1983). Let $a$ and $b$ are two random variables. Variable $a$ is higher in stochastic order then variable $b$ given that random variables $a$ and $b$ satisfy the following probabilistic condition:

$$P(a > x) \geq P(b > x) \quad \forall\, x \tag{2.43}$$

The three main assumptions stated in Section 2.1.1 are also observed in here. The assumptions are only verified for airline 1, however a simple analogy can be used to validate these assumptions for airline 2 as well. The Assumption 1 is written as follows:

$$D_{L1}^{P_{L1}^h,P_{L2}} \leq_{\text{st}} D_{L1}^{P_{L1}^l,P_{L2}} \quad \forall\, P_{L1}^h \geq P_{L1}^l \tag{2.44}$$

$$D_{H1}^{P_{H1}^h,P_{H2}} \leq_{\text{st}} D_{H1}^{P_{H1}^l,P_{H2}} \quad \forall\, P_{H1}^h \geq P_{H1}^l \tag{2.45}$$

In Equations 2.44-2.45, the symbol $\leq_{\text{st}}$ represents the stochastic behavior of demand and replaces the symbol $\leq$.

Based on stochastic ordering notion we can write:

$$P\left\{D_{L1}^{P_{L1}^h,P_{L2}} > x\right\} \leq P\left\{D_{L1}^{P_{L1}^l,P_{L2}} > x\right\}, \quad P_{L1}^h \geq P_{L1}^l \tag{2.46}$$

$$P\left\{D_{H1}^{P_{H1}^h,P_{H2}} > x\right\} \leq P\left\{D_{H1}^{P_{H1}^l,P_{H2}} > x\right\}, \quad P_{H1}^h \geq P_{H1}^l \tag{2.47}$$

Now, denote $1 - F_{L1}\ (x) = \overline{F}_{L1}\ (x)$ and $1 - F_{H1}\ (x) = \overline{F}_{H1}\ (x)$. Hence the Assumption 1 identified in Section 2.1.1 is rewritten as:

$$\overline{F}_{L1}^{(P_{L1}^h, P_{L2})}\ (x) \leq \overline{F}_{L1}^{(P_{L1}^l, P_{L2})}\ (x), \quad \forall\ P_{L1}^h \geq P_{L1}^l \tag{2.48}$$

$$\overline{F}_{H1}^{(P_{H1}^h, P_{H2})}\ (x) \leq \overline{F}_{H1}^{(P_{H1}^l, P_{H2})}\ (x), \quad \forall\ P_{H1}^h \geq P_{H1}^l \tag{2.49}$$

According to Assumption 2 mentioned in Section 2.1.1, we know that the demand to one airline increases (decreases) with an increase (decrease) in the price of the other airline. The interpretation of this assumption using the concept of stochastic ordering is as follows:

$$\overline{F}_{L1}^{(P_{L1}, P_{L2}^l)}\ (x) \leq \overline{F}_{L1}^{(P_{L1}, P_{L2}^h)}\ (x), \quad \forall\ P_{L1}^h \geq P_{L1}^l \tag{2.50}$$

$$\overline{F}_{H1}^{(P_{H1}, P_{H2}^l)}\ (x) \leq \overline{F}_{H1}^{(P_{H1}, P_{H2}^h)}\ (x), \quad \forall\ P_{H1}^h \geq P_{H1}^l \tag{2.51}$$

Assumption 3 in Section 2.1.1 states that for low fare class customer demand, $D_{L1}(P_{L1}^l, P_{L2}) - D_{L1}(P_{L1}^h, P_{L2})$ becomes larger as $P_{L2}$ gets smaller when condition $P_{L1}^l < P_{L1}^h$ is true. This enables us to conclude that $\overline{F}_{L1}^{(P_{L1}, P_{L2})}\ (x)$ and $\overline{F}_{H1}^{(P_{H1}, P_{H2})}\ (x)$ are supermodular functions in $(P_{L1}, P_{L2})$ and $(P_{H1}, P_{H2})$. Same conclusions can be obtained for the airline 2 using similar argument.

**Proposition 2.1** *Under the assumption of pre-committed booking limits the Nash equilibrium is unique for fare pricing game under consideration.*

**Proof:** First we identify that the revenue function in each fare class is supermodular function of the fare prices. Later we show that the revenue function also results fare pricing that leads to unique Nash equilibrium. The proof of this proposition is shown for airline 1 only, similarly airline 2 can be analyzed.

$$\Pi_{L1} = P_{L1}\, E\left[\min\{B_1, D_{L1}\}\right]$$

$$
\begin{aligned}
&= P_{L1}B_1 - P_{L1}\int_0^{B_1}(B_1 - x)d(F_{L1}(x)) \\
&= P_{L1}\int_0^{B_1}\overline{F}_{L1}(x)dx. \qquad (2.52)
\end{aligned}
$$

As it is already shown $\overline{F}_{L1}(x)$ is a supermodular function of $(P_{L1}, P_{L2})$.

$$
\frac{\partial \Pi_{L1}}{\partial P_{L1}} = \int_0^{B_1}\overline{F}_{L1}(x)dx + P_{L1}\int_0^{B_1}\frac{\partial \overline{F}_{L1}(x)}{\partial P_{L1}} \qquad (2.53)
$$

$$
\frac{\partial^2 \Pi_{L1}}{\partial P_{L1}\partial P_{L2}} = P_{L1}\int_0^{B_1}\frac{\partial^2 \overline{F}_{L1}(x)}{\partial P_{L1}\partial P_{L2}} + \int_0^{B_1}\frac{\partial \overline{F}_{L1}(x)}{\partial P_{L2}} \qquad (2.54)
$$

Hence we can show that $\frac{\partial \overline{F}_{L1}(x)}{\partial P_{L2}} \geq 0$ and $\frac{\partial^2 \overline{F}_{L1}(x)}{\partial P_{L1}\partial P_{L2}} \geq 0$, thus we can conclude that $\frac{\partial^2 \Pi_{L1}}{\partial P_{L1}\partial P_{L2}} \geq 0$ which proves the supermodularity of $\Pi_{L1}$. Furthermore the condition $\frac{\partial^2 \Pi_{L1}}{\partial P_{L1}^2} \geq \frac{\partial^2 \Pi_{L1}}{\partial P_{L1}\partial P_{L2}}$ proves the uniqueness of Nash equilibrium.

Similarly the high fare revenue for airline 1 for the pre-committed booking limits is given by:

$$
\begin{aligned}
\Pi_{H1} &= P_{H1}\,E\left[\min\{C_1 - B_1, D_{H1}\}\right] \\
&= P_{H1}(C_1 - B_1) - P_{H1}\int_0^{C_1 - B_1}(C_1 - B_1 - x)d(F_{H1}(x)) \\
&= P_{H1}\int_0^{C_1 - B_1}\overline{F}_{H1}(x)dx. \qquad (2.55)
\end{aligned}
$$

Similar to the discussion made for low fare class revenue $\Pi_{L1}$, we can also show that the revenue from the high fare class, $\Pi_{H1}$ is supermodular and leads high fare pricing resulting unique Nash equilibrium.

**Low Fare Class Pricing Competition under Stochastic Demands with Pre-Commitment**

In this section, a sensitivity analysis is conducted with stochastic demand at low fare class. The best response function at Nash equilibrium must follow the first order optimality condition:

$$\frac{\partial \Pi_{L1}}{\partial P_{L1}} = \int_0^{B_1} \overline{F}_{L1}(x)dx + P_{L1}\int_0^{B_1} \frac{\partial \overline{F}_{L1}}{\partial P_{L1}}(x)dx \tag{2.56}$$

$$\frac{\partial \Pi_{L2}}{\partial P_{L2}} = \int_0^{B_2} \overline{F}_{L2}(x)dx + P_{L2}\int_0^{B_2} \frac{\partial \overline{F}_{L2}}{\partial P_{L2}}(x)dx \tag{2.57}$$

where $\frac{\partial \Pi_{L1}}{\partial P_{L1}} = 0$ and $\frac{\partial \Pi_{L2}}{\partial P_{L2}} = 0$. Let us assume $G_{L1}(.) = \frac{\partial \Pi_{L1}}{\partial P_{L1}}$ and $G_{L2}(.) = \frac{\partial \Pi_{L2}}{\partial P_{L2}}$

Now, with the low fare pricing and booking limits by using the implicit function theory, the following sensitivity analysis is performed:

$$\begin{bmatrix} \frac{\partial P_{L1}}{\partial B_1} & \frac{\partial P_{L1}}{\partial B_2} \\ \frac{\partial P_{L2}}{\partial B_1} & \frac{\partial P_{L2}}{\partial B_2} \end{bmatrix} = - \begin{bmatrix} \frac{\partial G_{L1}}{\partial P_{L1}} & \frac{\partial G_{L1}}{\partial P_{L2}} \\ \frac{\partial G_{L2}}{\partial P_{L1}} & \frac{\partial G_{L2}}{\partial P_{L2}} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial G_{L1}}{\partial B_1} & \frac{\partial G_{L1}}{\partial B_2} \\ \frac{\partial G_{L2}}{\partial B_1} & \frac{\partial G_{L2}}{\partial B_2} \end{bmatrix}$$

$$= \frac{1}{\frac{\partial G_{L1}}{\partial P_{L1}}\frac{\partial G_{L2}}{\partial P_{L2}} - \frac{\partial G_{L1}}{\partial P_{L2}}\frac{\partial G_{L2}}{\partial P_{L1}}} \begin{bmatrix} -\frac{\partial G_{L2}}{\partial P_{L2}}\frac{\partial G_{L1}}{\partial B_1} & \frac{\partial G_{L1}}{\partial P_{L2}}\frac{\partial G_{L2}}{\partial B_2} \\ \frac{\partial G_{L2}}{\partial P_{L1}}\frac{\partial G_{L1}}{\partial B_1} & -\frac{\partial G_{L1}}{\partial P_{L1}}\frac{\partial G_{L2}}{\partial B_2} \end{bmatrix}$$

where

$$\frac{\partial G_{L1}}{\partial P_{L1}} = 2\int_0^{B_1} \frac{\partial \overline{F}_{L1}}{\partial P_{L1}}(x)dx + P_{L1}\int_0^{B_1} \frac{\partial^2 \overline{F}_{L1}}{\partial P_{L1}^2}(x)dx \tag{2.58}$$

$$\frac{\partial G_{L2}}{\partial P_{L1}} = \int_0^{B_2} \frac{\partial \overline{F}_{L2}}{\partial P_{L1}}(x)dx + P_{L2}\int_0^{B_2} \frac{\partial^2 \overline{F}_{L2}}{\partial P_{L1}\partial P_{L2}}(x)dx \tag{2.59}$$

$$\frac{\partial G_{L1}}{\partial P_{L2}} = \int_0^{B_1} \frac{\partial \overline{F}_{L1}}{\partial P_{L2}}(x)dx + P_{L1}\int_0^{B_1} \frac{\partial^2 \overline{F}_{L1}}{\partial P_{L1}\partial P_{L2}}(x)dx \tag{2.60}$$

$$\frac{\partial G_{L2}}{\partial P_{L2}} = 2\int_0^{B_2} \frac{\partial \overline{F}_{L2}}{\partial P_{L2}}(x)dx + P_{L2}\int_0^{B_2} \frac{\partial^2 \overline{F}_{L2}}{\partial P_{L2}^2}(x)dx \tag{2.61}$$

$$\frac{\partial G_{L1}}{\partial B_1} = \overline{F}_{L1}(B_1) + P_{L1}\frac{\partial \overline{F}_{L1}(B_1)}{\partial P_{L1}} \tag{2.62}$$

$$\frac{\partial G_{L2}}{\partial B_2} = \overline{F}_{L2}(B_2) + P_{L2}\frac{\partial \overline{F}_{L2}(B_2)}{\partial P_{L2}} \tag{2.63}$$

**Theorem 2.3** *At the equilibrium, a small change in $B_i$ from its pre-committed value by airline $i$ changes low fare prices at equilibrium such that* $\frac{\partial P_{Li}}{\partial B_i}/\frac{\partial P_{Lj}}{\partial B_i} = \frac{-2\int_0^{B_j} \frac{\partial \overline{F}_{Lj}}{\partial P_{Lj}}(x)dx - P_{Lj}\int_0^{B_j} \frac{\partial^2 \overline{F}_{Lj}}{\partial P_{Lj}^2}(x)dx}{\int_0^{B_j} \frac{\partial \overline{F}_{Lj}}{\partial P_{Li}}(x)dx + P_{Lj}\int_0^{B_j} \frac{\partial^2 \overline{F}_{Lj}}{\partial P_{Li}\partial P_{Lj}}(x)dx}$, *where $i \neq j$, $i, j = \{1, 2\}$.*

**Proof:** From the previously developed implicit function, we can easily show that $\frac{\partial P_{Li}}{\partial B_i} / \frac{\partial P_{Lj}}{\partial B_i} = -\frac{\partial G_{Lj}}{\partial P_{Lj}} / \frac{\partial G_{Lj}}{\partial P_{Li}}$.

Theorem 2.3 presents a relative impact of a small change in the airline $i$'s booking limit from its pre-committed value. The sign in the ratio is a measure of the directional change in the fare prices and the magnitude of the ratio is quantitative measure of the variation. More insights can be obtained while assuming the concavity of the revenue function which is a stronger condition than the supermodularity.

### High Fare Class Pricing Competition under Stochastic Demands with Pre-Commitment

The approach presented in Section 2.1.3 is followed here which allows us to derive similar mathematical expressions as follows:

$$\frac{\partial \Pi_{H1}}{\partial P_{H1}} = \int_0^{C_1-B_1} \overline{F}_{H1}(x)dx + P_{H1} \int_0^{C_1-B_1} \frac{\partial \overline{F}_{H1}}{\partial P_{H1}}(x)dx \tag{2.64}$$

$$\frac{\partial \Pi_{H2}}{\partial P_{H2}} = \int_0^{C_2-B_2} \overline{F}_{H2}(x)dx + P_{H2} \int_0^{C_2-B_2} \frac{\partial \overline{F}_{H2}}{\partial P_{H2}}(x)dx \tag{2.65}$$

that is $\frac{\partial \Pi_{H1}}{\partial P_{H1}} = 0$ and $\frac{\partial \Pi_{H2}}{\partial P_{H2}} = 0$. Let us assume $G_{H1}(.) = \frac{\partial \Pi_{H1}}{\partial P_{H1}}$ and $G_{H2}(.) = \frac{\partial \Pi_{H2}}{\partial P_{H2}}$

In order to perform the sensitivity analysis for the high fare pricing competition, the implicit function theory is again used in the following form:

$$\begin{bmatrix} \frac{\partial P_{H1}}{\partial B_1} & \frac{\partial P_{H1}}{\partial B_2} \\ \frac{\partial P_{H2}}{\partial B_1} & \frac{\partial P_{H2}}{\partial B_2} \end{bmatrix} = - \begin{bmatrix} \frac{\partial G_{H1}}{\partial P_{H1}} & \frac{\partial G_{H1}}{\partial P_{H2}} \\ \frac{\partial G_{H2}}{\partial P_{H1}} & \frac{\partial G_{H2}}{\partial P_{H2}} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial G_{H1}}{\partial B_1} & \frac{\partial G_{H1}}{\partial B_2} \\ \frac{\partial G_{H2}}{\partial B_1} & \frac{\partial G_{H2}}{\partial B_2} \end{bmatrix} \tag{2.66}$$

$$= \frac{1}{\frac{\partial G_{H1}}{\partial P_{H1}}\frac{\partial G_{H2}}{\partial P_{H2}} - \frac{\partial G_{H1}}{\partial P_{H2}}\frac{\partial G_{H2}}{\partial P_{H1}}} \begin{bmatrix} -\frac{\partial G_{H2}}{\partial P_{H2}}\frac{\partial G_{H1}}{\partial B_1} & \frac{\partial G_{H1}}{\partial P_{H2}}\frac{\partial G_{H2}}{\partial B_2} \\ \frac{\partial G_{H2}}{\partial P_{H1}}\frac{\partial G_{H1}}{\partial B_1} & -\frac{\partial G_{H1}}{\partial P_{H1}}\frac{\partial G_{H2}}{\partial B_2} \end{bmatrix} \tag{2.67}$$

where

$$\frac{\partial G_{H1}}{\partial P_{H1}} = 2\int_0^{C_1-B_1} \frac{\partial \overline{F}_{H1}}{\partial P_{H1}}(x)dx + P_{H1}\int_0^{C_1-B_1} \frac{\partial^2 \overline{F}_{H1}}{\partial P_{H1}^2}(x)dx \tag{2.68}$$

$$\frac{\partial G_{H2}}{\partial P_{H1}} = \int_0^{C_2-B_2} \frac{\partial \overline{F}_{H2}}{\partial P_{H1}}(x)dx + P_{H2}\int_0^{C_2-B_2} \frac{\partial^2 \overline{F}_{H2}}{\partial P_{H1}\partial P_{H2}}(x)dx \tag{2.69}$$

$$\frac{\partial G_{H1}}{\partial P_{H2}} = \int_0^{C_1-B_1} \frac{\partial \overline{F}_{H1}}{\partial P_{H2}}(x)dx + P_{H1}\int_0^{C_1-B_1} \frac{\partial^2 \overline{F}_{H1}}{\partial P_{H1}\partial P_{H2}}(x)dx \tag{2.70}$$

$$\frac{\partial G_{H2}}{\partial P_{H2}} = 2\int_0^{C_2-B_2} \frac{\partial \overline{F}_{H2}}{\partial P_{H2}}(x)dx + P_{H2}\int_0^{C_2-B_2} \frac{\partial^2 \overline{F}_{H2}}{\partial P_{H2}^2}(x)dx \tag{2.71}$$

$$\frac{\partial G_{H1}}{\partial B_1} = -\overline{F}_{H1}(C_1-B_1) - P_{H1}\frac{\partial \overline{F}_{H1}(C_1-B_1)}{\partial P_{H1}} \tag{2.72}$$

$$\frac{\partial G_{H2}}{\partial B_2} = -\overline{F}_{H2}(C_2-B_2) - P_{H2}\frac{\partial \overline{F}_{H2}(C_2-B_2)}{\partial P_{H2}} \tag{2.73}$$

**Theorem 2.4** *At the equilibrium, a small change in $B_i$ from its pre-committed value by an airline $i$ changes the high fare prices, such that* $\frac{\partial P_{Hi}}{\partial B_i}/\frac{\partial P_{Hj}}{\partial B_i} = \frac{-2\int_0^{C_j-B_j}\frac{\partial \overline{F}_{Hj}}{\partial P_{Hj}}(x)dx - P_{Hj}\int_0^{C_j-B_j}\frac{\partial^2 \overline{F}_{Hj}}{\partial P_{Hj}^2}(x)dx}{\int_0^{C_j-B_j}\frac{\partial \overline{F}_{Hj}}{\partial P_{Hi}}(x)dx + P_{Hj}\int_0^{C_j-B_j}\frac{\partial^2 \overline{F}_{Hj}}{\partial P_{Hi}\partial P_{Hj}}(x)dx}$, *where $i \neq j$, $i,j = \{1,2\}$.*

**Proof:** Again from the previously developed implicit function, $\frac{\partial P_{Hi}}{\partial B_i}/\frac{\partial P_{Hj}}{\partial B_i} = -\frac{\partial G_{Hj}}{\partial P_{Hj}}/\frac{\partial G_{Hj}}{\partial P_{Hi}}$.

### 2.1.4 Conclusion

The problem of fare pricing strategies is considered for airlines competing for customer in multiple fare classes while providing the same service in competition for a common pool of customers. The capacity offered in each fare class is fixed and the demand in each fare class depends upon the fare price offered by the airline and its rival(s). A game theoretic analysis of the system is presented when the airlines are either facing deterministic or stochastic price sensitive demand. It is identified that the game is supermodular and there also exist unique Nash equilibrium to this fare pricing game. Fare pricing at Nash equilibrium identifies pricing strategies that would be played by the competing airlines in order to maximize their revenue. Several case studies are presented while establishing unique fare pricing at Nash equilibrium assuming a linear deterministic price sensitive demand function. An analysis is further extended to stochastic price sensitive demand, and a sensitivity analysis is conducted of the equilibrium prices and seat allocation in each fare class.

In the next section, the fare pricing competition is further investigated while having a joint control on fare pricing and seat inventory. Mathematical models of fare pricing competition are proposed assuming a nested control on seat inventory control for the competing airlines.

## 2.2 Fare Pricing Competition with Seat Inventory Control

Competition models for joint determination of fare pricing and seat allocation with a nested control on booking limits are presented. A situation when two airlines are offering two fare classes to customers in the duopoly market is considered. The market is also segmented into only two customer classes and the segmentation is considered perfect i.e., high fare customer does not request and low fare ticket and vice versa. In each fare class the demand is price sensitive which represents the customer's rationality in each fare class. It is also assumed that the flight capacities offered by two airlines are known and fixed. Customer demand is observed sequentially, i.e., the low fare class demand is observed before the high fare class demand. A duopoly fare pricing with allocation competition model is illustrated in Figure 2.2. A numerical study is carried out to compare the non-cooperative and cooperative games. A statistical evidence is established from an empirical study that the cooperation model results superior revenue gain to the competing airlines. The statistical analysis is extended to analyze the impact of the market related factors on revenue gain under price cooperation.

### 2.2.1 The Models

Let $S_{ci}$ be the random demand to an airline $i$ for its fare class $c$. There are two fare classes, Low(L) and High(H), and thus $c = \{L, H\}$. More precisely $S_{ci}$ is $S_{ci}(D_{ci}, \xi_{ci})$, which is the function of riskless (deterministic) demand in the fare class $c$, $D_{ci}$, and a random demand factor $\xi_{ci}$, $c = \{L, H\}$.

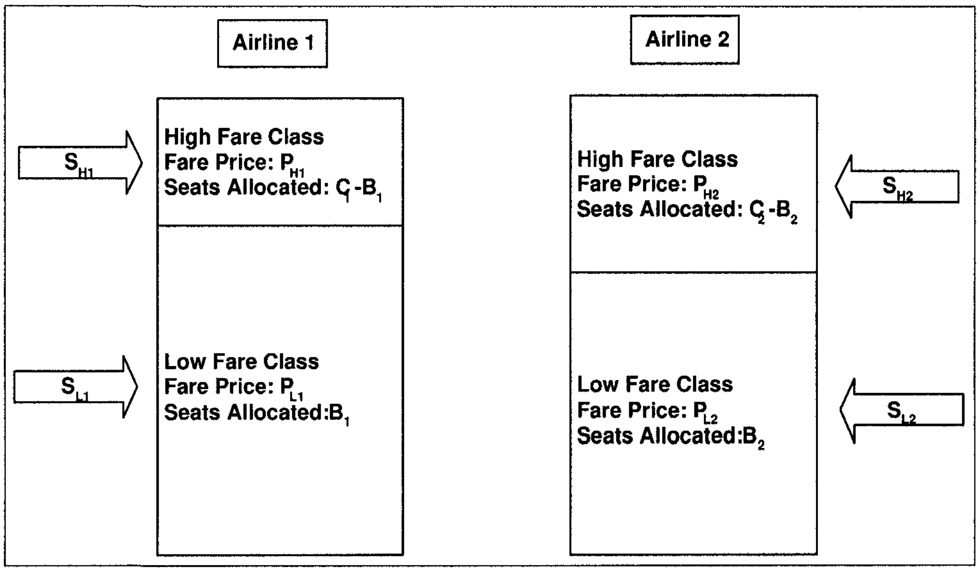


Figure 2.2: Two airlines competition for two fare classes

Now some notations are defined that will be used to model this problem:

| | | |
|---|---|---|
| $P_{ci}$ | : | Fare price offered in the fare class $c = \{L, H\}$ by airline $i = 1, 2$ |
| $C_i$ | : | Total flight capacity of airline $i = 1, 2$ |
| $B_i$ | : | Booking limit for the low fare price committed by airline $i = 1, 2$ |
| $\Pi_i$ | : | Total revenue generated by airline $i = 1, 2$ |

$D_{ci}$ is the riskless demand, a function of fare price $P_{ci}$, offered by airline $i$ and the fare price $P_{cj}$, $\forall j \neq i, i = \{1, 2\}$ of its rival airline $j$ in fare class $c$. $D_{ci}$ is a continuous and twice differentiable function. It is bounded in $P_{ci} \in [\underline{P}_{ci}, \overline{P}_{ci}]$ and $P_{cj} \in [\underline{P}_{cj}, \overline{P}_{cj}]$. Also $B_i \in [0, C_i], \forall i = \{1, 2\}$. It is also not uncommon to assume $D_{ci}$ is a supermodular function of fare prices (Topkis 1978), which is also observed in this modeling. An essential assumption to model the fare pricing game jointly with seat allocation is that the random variables, $\xi_{ci}$ are independent of fare prices. A $\xi_{ci}$ follows continuous probability distribution function $\phi_{ci}$ and cumulative distribution function,$\Phi_{ci}$, $\forall\, c = \{L, H\}$. The $\xi_{ci}$ are uncorrelated, and the expected values of $\xi_{ci}$ may vary depending upon the modeling situation. In literature mostly two types of models are used: Additive and Multiplicative. In the additive model, the random demand is additive function of riskless demand and a random factor. However for the multiplicative model, the random demand function is the multiplication of riskless demand function and the random factor.

### Additive Model

In additive model, random demand is the sum of price sensitive demand and the random demand factor.

$$S_{ci} = D_{ci} + \xi_{ci}, \forall\, c = \{L, H\}$$

The random demand factors, $\xi_{ci}$ are drawn from two distinct distributions such that $E[\xi_{ci}] = 0$ and $\xi_{ci} \in [\underline{\xi}_{ci}, \overline{\xi}_{ci}]$. The payoff to airline $i$ is $\Pi_i(B_i, \mathbf{P_L}, \mathbf{P_H})$, where $\mathbf{P_L} = (P_{Li}, P_{Lj})$ and $\mathbf{P_H} = (P_{Hi}, P_{Hj})$, $\forall\, i,j = \{1,2\}, i \neq j$. For brevity, it is written as $\Pi_i = \Pi_i(B_i, \mathbf{P_L}, \mathbf{P_H})$.

The total expected revenue generated by airline $i$ which is offering only two fare classes is given by:

$$\Pi_i = P_{Li} \min\{S_{Li}, B_i\} + P_{Hi} \min\{S_{Hi}, C_i - \min\{S_{Li}, B_i\}\} \tag{2.74}$$

The revenue function can be partitioned for the two distinct fare classes. Let $\Pi_{Li}$ be the revenue generated from the low fare class and $\Pi_{Hi}$ the revenue generated form the high fare class respectively. The total revenue that the low fare class generates can be expressed as follows:

$$\begin{aligned}
\Pi_{Li} &= E_{\xi_{Li}}\left[P_{Li} \min(S_{Li}, B_i)\right] \\
&= P_{Li} E_{\xi_{Li}}[S_{Li}] - P_{Li} E_{\xi_{Li}}[S_{Li} - B_i]^+ \\
&= P_{Li} D_{Li} - P_{Li} \int_{B_i - D_{Li}}^{\overline{\xi}_{Li}} (D_{Li} + \xi_{Li} - B_i)\phi_{Li}(\xi_{Li})\, d\xi_{Li} \qquad (2.75)
\end{aligned}$$

Finally, the revenue generated by the low fare class is:

$$\Pi_{Li} = P_{Li} B_i - P_{Li} \int_{\underline{\xi}_{Li}}^{B_i - D_{Li}} \Phi_{Li}(\xi_{Li})\, d\xi_{Li} \tag{2.76}$$

Assuming condition, $\underline{\xi}_{Li} \leq B_i - D_{Li} \leq \overline{\xi}_{Li}$ is ture

Similarly the revenue generated from the high fare class $\Pi_{Hi}$ is:

$$\begin{aligned}
\Pi_{Hi} &= E_{\xi_{Hi}} E_{\xi_{Li}}\left[P_{Hi} \min(S_{Hi}, C_i - \min(S_{Li}, B_i))\right] \\
&= P_{Hi} D_{Hi} - P_{Hi}\, E_{\xi_{Hi}} E_{\xi_{Li}} \left[S_{Hi} - C_i + S_{Li} - [S_{Li} - B_i]^+\right]^+ \\
&= P_{Hi} D_{Hi} - P_{Hi} E_{\xi_{Hi}} \left[\xi_{Hi} - \left(C_i + \int_{\underline{\xi}_{Li}}^{B_i - D_{Li}} \Phi_{Li}(\xi_{Li}) d\xi_{Li} - D_{Hi} - B_i\right)\right]^+ \\
&= P_{Hi} D_{Hi} - P_{Hi} \int_{y_i}^{\overline{\xi}_{Hi}} (\xi_{Hi} - y_i)\, \phi_{Hi}(\xi_{Hi}) d\xi_{Hi} \qquad (2.77)
\end{aligned}$$

where,

$$y_i = C_i + \int_{\underline{\xi}_{Li}}^{B_i - D_{Li}} \Phi_{Li}(\xi_{Li}) d\xi_{Li} - D_{Hi} - B_i$$

Assuming $y_i$ is bounded as, $\underline{\xi}_{Hi} \leq y_i \leq \overline{\xi}_{Hi}$, then the revenue expression for the high fare class can be rewritten as:

$$\Pi_{Hi} = P_{Hi} \left( C_i - B_i + \int_{\underline{\xi}_{Li}}^{B_i - D_{Li}} \Phi_{Li}(\xi_{Li})\, d\xi_{Li} - \int_{\underline{\xi}_{Hi}}^{y_i} \Phi_{Hi}(\xi_{Hi}) d\xi_{Hi} \right) \tag{2.78}$$

The total expected revenue generated from both fare classes is:

$$\begin{aligned} \Pi_i &= \Pi_{Li} + \Pi_{Hi} \\ &= P_{Hi} C_i - (P_{Hi} - P_{Li}) B_i + (P_{Hi} - P_{Li}) \int_{\underline{\xi}_{Li}}^{B_i - D_{Li}} \Phi_{Li}(\xi_{Li}) d\xi_{Li} \\ &- P_{Hi} \int_{\underline{\xi}_{Hi}}^{C_i + \int_{\underline{\xi}_{Li}}^{B_i - D_{Li}} \Phi_{Li}(\xi_{Li})\, d\xi_{Li} - D_{Hi} - B_i} \Phi_{Hi}(\xi_{Hi}) d\xi_{Hi} \end{aligned} \tag{2.79}$$

In Equation 2.79, the first two terms are the riskless revenue gain to airline $i$ when $B_i$ seats are allocated for its low fare class. The third term is the expected revenue gain that may incur when the low fare class demand is observed less than the allocated seat capacity $B_i$ and this capacity is assigned to high fare class. The last term is the expected loss in revenue when demand in high fare class is observed less than its capacity after the low fare class demand is being observed.

### Multiplicative Model

In the case of multiplicative model, demands in low fare and high fare classes are modeled as follows:

$$S_{ci} = D_{ci}\, \xi_{ci}$$

Similar to the additive model, the total expected revenue generated by airline $i$ offering only two fare classes is again given by:

$$\Pi_i = P_{Li} \min \{S_{Li}, B_i\} + P_{Hi} \min \{S_{Hi}, C_i - \min \{S_{Li}, B_i\}\}$$

The revenue function can be further partitioned into the revenue function for both fare classes. Similar to the additive model, the total revenue generated in the low fare class is given by:

$$\begin{aligned}
\Pi_{Li} &= E_{\xi_{Li}}\left[P_{Li}\min\left(S_{Li}, B_i\right)\right] \\
&= P_{Li}E_{\xi_{Li}}\left[S_{Li}\right] - P_{Li}E_{\xi_{Li}}\left[S_{Li} - B_i\right]^+ \\
&= P_{Li}D_{Li} - P_{Li}D_{Li}\int_{\frac{B_i}{D_{Li}}}^{\overline{\xi}_{Li}}(\xi_{Li} - \frac{B_i}{D_{Li}})\phi_{Li}(\xi_{Li})d\xi_{Li} \qquad (2.80)
\end{aligned}$$

Assuming $\underline{\xi}_{Li} \leq \frac{B_i}{D_{Li}} \leq \overline{\xi}_{Li}$, ia true, the revenue function becomes:

$$\Pi_{Li} = B_i P_{Li} - P_{Li}D_{Li}\int_{\underline{\xi}_{Li}}^{\frac{B_i}{D_{Li}}} \Phi_{Li}(\xi_{Li})d\xi_{Li} \qquad (2.81)$$

Similarly the revenue generated from the high fare class $\Pi_{Hi}$ is:

$$\begin{aligned}
\Pi_{Hi} &= E_{\xi_{Hi}}E_{\xi_{Li}}\left[P_{Hi}\min\left(S_{Hi}, C_i - \min\left(S_{Li}, B_i\right)\right)\right] \\
&= P_{Hi}D_{Hi} - P_{Hi}\,E_{\xi_{Hi}}E_{\xi_{Li}}\left[S_{Hi} - C_i + S_{Li} - \left[S_{Li} - B_i\right]^+\right]^+ \\
&= P_{Hi}D_{Hi} - P_{Hi}\,E_{\xi_{Hi}}\left[S_{Hi} - C_i + B_i - D_{Li}\int_{\underline{\xi}_{Li}}^{\frac{B_i}{D_{Li}}} \Phi_{Li}(\xi_{Li})\ d\xi_{Li}\right]^+ \\
&= P_{Hi}D_{Hi} - P_{Hi}D_{Hi}\int_{y_i/D_{Hi}}^{\overline{\xi}_{Hi}} \left(\xi_{Hi} - y_i/D_{Hi}\right)\phi_{Hi}(\xi_{Hi})d\xi_{Hi} \qquad (2.82)
\end{aligned}$$

where,

$$y_i = C_i - B_i + D_{Li}\int_{\underline{\xi}_{Li}}^{\frac{B_i}{D_{Li}}} \Phi_{Li}(\xi_{Li})\,d\xi_{Li} \qquad (2.83)$$

Assuming $\underline{\xi}_{Hi} \leq \frac{y_i}{D_{Hi}} \leq \overline{\xi}_{Hi}$ holds, the expected revenue becomes:

$$\Pi_{Hi} = P_{Hi}\left(C_i - B_i + D_{Li}\int_{\underline{\xi}_{Li}}^{\frac{B_i}{D_{Li}}} \Phi_{Li}(\xi_{Li})\,d\xi_{Li}\right) - P_{Hi}D_{Hi}\int_{\underline{\xi}_{Hi}}^{\frac{y_i}{D_{Hi}}} \Phi_{Hi}(\xi_{Hi})d\xi_{Hi} \qquad (2.84)$$

The total expected revenue generated from both fare classes is:

$$\Pi_i = \Pi_{Li} + \Pi_{Hi}$$

$$
\begin{aligned}
&= P_{Hi}C_i - (P_{Hi} - P_{Li})B_i + (P_{Hi} - P_{Li})D_{Li}\int_{\underline{\xi}_{Li}}^{\frac{B_i}{D_{Li}}} \Phi_{Li}(\xi_{Li})d\xi_{Li} \\
&- P_{Hi}D_{Hi}\int_{\underline{\xi}_{Hi}}^{(C_i - B_i + D_{Li}\int_{\underline{\xi}_{Li}}^{B_i/D_{Li}} \Phi_{Li}(\xi_{Li})d\xi_{Li})/D_{Hi}} \Phi_{Hi}(\xi_{Hi})d\xi_{Hi} \qquad (2.85)
\end{aligned}
$$

Equation 2.85 has similar structure as identified in aforementioned Equation 2.79, however the modeling scenario is multiplicative.

The models are analyzed under both the non-cooperative and cooperative game settings. A bargaining game is classified as a cooperative game, in a cooperative game communication between player is allowed (or possible). With cooperation their players are expected to improve their payoffs better than the non-cooperative payoffs. In most cases a cooperative game between three or more players is formulated using characteristic functions, which specify the payoff of each coalition. The solution concepts employed in these game are Shapley value (Shapley 1953) and Nucleolus (Schmeidler 1969). Cooperative games with only two players are usually analyzed with Nash arbitration scheme (Nash 1950a). For cooperative games with no characteristic function, the Nash arbitration scheme may provide an acceptable solution. A Nash arbitration scheme can be explored with two options: i) An arbitration scheme with no side payment option; and ii) An arbitration scheme with a side payment option. In a cooperative bargaining game with no side payments option, it is assumed that the players communicate once at the beginning of the game. However in the cooperative games with side payments, it is possible for one player to make side payments to the other player. One player can optimize the total payoff as if it is a joint profit maximization monopoly and this situation is referred as Full Cooperation (FC).

Nash arbitration scheme, also called Nash Bargaining Solution (NBS), is based on four axioms: (i) Rationality; (ii) Linear invariance; (iii) Symmetry; and (iv) Independence of irrelevant alternatives. In this analysis, two distinct bargaining situations are considered. First a bargaining game with No Side Payment (NSP) is considered. In this case the optimization problem becomes:

$$\begin{aligned}
&\max\,(\Pi_1 - \Pi_1^{NE})(\Pi_2 - \Pi_2^{NE}) \\
&\text{subject to} \\
&\quad \Pi_1 \geq \Pi_1^{NE} \\
&\quad \Pi_2 \geq \Pi_2^{NE}
\end{aligned}$$

This is a non-linear optimization problem where non-cooperative payoff to each airline is assumed as a status-quo which is considered as player's "security" level. A status-quo is a payoff to each player before they agree to cooperate. The second situation is the bargaining game with Side Payment (SP) option. SP option is a situation when two airlines completely cooperate with each other with patience. One airline controls the fare pricing and booking limits of both airlines such that the total payoff is maximized. Later the gain of full cooperation is shared among the airlines according to the pre-defined cooperation terms. SP based on aforementioned basic axioms and is determined as follows:

$$\mathrm{SP} = \frac{1}{2}\left(\Pi_1^{\mathrm{FC}} - \Pi_1^{\mathrm{NE}} + \Pi_2^{\mathrm{NE}} - \Pi_2^{\mathrm{FC}}\right) \tag{2.86}$$

The payoff to each of the airlines would be:

$$\Pi_1^{SP} = \Pi_1^{FC} - \mathrm{SP} = \Pi_1^{\mathrm{NE}} + \frac{(\Pi_1^{\mathrm{FC}} + \Pi_2^{\mathrm{FC}}) - (\Pi_1^{\mathrm{NE}} + \Pi_2^{\mathrm{NE}})}{2} \tag{2.87}$$

$$\Pi_2^{SP} = \Pi_2^{FC} + \mathrm{SP} = \Pi_2^{\mathrm{NE}} + \frac{(\Pi_1^{\mathrm{FC}} + \Pi_2^{\mathrm{FC}}) - (\Pi_1^{\mathrm{NE}} + \Pi_2^{\mathrm{NE}})}{2} \tag{2.88}$$

### 2.2.2 Numerical Analysis

In this section, a numerical study of the proposed model is presented. The purpose is to analyze the impact of fare price cooperation. The analysis first addresses the impact of cooperation under the symmetric market condition. The symmetric market condition is observed when both competing airlines are facing same market conditions. The analysis is extended to a scenario when the rival airline (airline 2) assumes a variation in its market situation. These variations are referred as an asymmetric market situation.

**Symmetric Market Situation**

In this analysis, the demand randomness factors are assumed to be uniformly distributed. The study comprise both the additive and multiplicative models. In the case of additive model, $\xi_{Li}$ and $\xi_{Hi}$ $\forall\, i = \{1, 2\}$ are represented with symmetric market randomization condition with a mean at 0 and truncation bounds at $[\underline{\xi}_{Li}, \overline{\xi}_{Li}] = [\underline{\xi}_{Hi}, \overline{\xi}_{Hi}] = [-30, 30]$. For the multiplicative model, random demand factors, $\xi_{Li}$ and $\xi_{Hi}$ $\forall\, i = \{1, 2\}$ are also represented with symmetric market randomization condition with a mean at 1 and truncation bounds at $[\underline{\xi}_{Li}, \overline{\xi}_{Li}] = [\underline{\xi}_{Hi}, \overline{\xi}_{Hi}] = [0, 2]$. For both the additive and multiplicative models, linear riskless demand functions $D_{Li}$ and $D_{Hi}$ are assumed. Thus, $D_{Li} = \alpha_{Li} - \beta_{Li}\, P_{Li} + \theta_{Lij}\, P_{Lj}$ and $D_{Hi} = \alpha_{Hi} - \beta_{Hi}\, P_{Hi} + \theta_{Hij}\, P_{Hj}$, $\forall\, i, j = 1, 2,\ i \neq j$. As mentioned previously, under symmetric competition conditions in which both airlines have the same capacities and market conditions. In this numerical study, the symmetric competition includes; $C_i = 100$, $\underline{P}_{Li} = 0$, $\overline{P}_{Li} = 100$, $\underline{P}_{Hi} = 100$, $\overline{P}_{Hi} = 200$, $\alpha_{Li} = 60$, $\alpha_{Hi} = 40$, $\beta_{Li} = 0.25$, $\beta_{Hi} = 0.15$ $\forall\, i = \{1, 2\}$. Also $\theta_{Lij} = 0.15, \theta_{Hij} = 0.10$ $\forall\, i, j = \{1, 2\},\ i \neq j$. For a fare pricing and booking limit control applied by the rival airline the fare prices in each fare class and booking limit are searched numerically for the competing airline using a commercially available optimization routine i.e., FMINCON in MATLAB can be used to minimize $-\Pi$ and thus maximize $\Pi$. Thus Equations 2.79-2.85 are solved numerically using aforementioned method for additive model and multiplicative models respectively. In the non cooperative game, the NE is assumed when both airlines are unable to improve more than an absolute value of $10^{-3}$ in their payoffs. In cooperative games the search also terminates when joint payoff of the airlines can not be improved beyond an absolute value of $10^{-3}$. In both bargaining games, it is assumed that non cooperative payoffs to each of the airlines is status-quo for the bargaining game.

In Table 2.1, a comparative study with additive model under the symmetric market competition condition is presented. The comparison report measures the improvement in the payoff and variation in fare pricing and booking limit strategies compared to their values under no cooperation. In the case of cooperation with no side payments, airline 1 reduces its seating capacity allocated for the low fare class about 25% and uses it to satisfy the high fare class demand. The high fare class prices are improved by about 25%, however the low fare class prices are reduced by about 4%. Airline 2 also reduces its capacity for the low fare class demand about 25%. The low fare class prices are reduced by 12% and the high fare class prices are improved by about 26%. Both airlines do not improve from their non cooperative outcomes but are able to improve their revenue gain from the high fare class demand. In cooperation with SP option, both airlines play same booking limit and fare pricing strategies and also observe the same revenue gains. The capacity allocated to the low fare class demand is reduced by about 3%. Both the low and high fare class prices are improved by about 13% and 80% respectively. Improvement observed in the payoffs to each of the airlines is about 16%.

The analysis is extended to the case of multiplicative model and presented in Table 2.2. The numerical experiments show that the outcomes of the cooperation with NSP and SP options present similar trends. Both airlines are able to allocate about 6% more seats to satisfy the high fare class demand. The prices for the low and high fare classes are raised by about 14 and 5% respectively. Finally, the revenue gain is about 12%.

| | Airline 1 | | | | Airline 2 | | | |
|---|---|---|---|---|---|---|---|---|
| | $B_1$ | $P_{L1}$ | $P_{H1}$ | $\Pi_1$ | $B_2$ | $P_{L2}$ | $P_{H2}$ | $\Pi_2$ |
| NE | 72.35 | 176.53 | 205.18 | 13570.21 | 72.35 | 176.53 | 205.18 | 13570.21 |
| NBS with NSP | 54.05 | 169.57 | 255.81 | 13570.21 | 55.11 | 156.04 | 259.43 | 13570.21 |
| NBS with SP | 70 | 200 | 368.28 | 15732.42 | 70 | 200 | 368.28 | 15732.42 |
| Gain of cooperation | 2162.2 | | | | 2162.2 | | | |

Table 2.1: Comparison of cooperative and non cooperative games in additive model

| | Airline 1 | | | | Airline 2 | | | |
|---|---|---|---|---|---|---|---|---|
| | $B_1$ | $P_{L1}$ | $P_{H1}$ | $\Pi_1$ | $B_2$ | $P_{L2}$ | $P_{H2}$ | $\Pi_2$ |
| NE | 84.90 | 175.50 | 208.32 | 13608.25 | 84.90 | 175.50 | 208.32 | 13608.25 |
| NBS with NSP | 80 | 200 | 310.10 | 15212.24 | 80 | 200 | 310.10 | 15212.24 |
| NBS with SP | 80 | 200 | 310.10 | 15212.24 | 80 | 200 | 310.10 | 15212.24 |
| Gain of cooperation | 1604 | | | | 1604 | | | |

Table 2.2: Comparison of cooperative and non cooperative games in multiplicative model

### Asymmetric Market Situation

The numerical analysis is extended from two airlines having the same market conditions to a situation where airline 2 experiences variation in its market conditions. The price related parameters are generated randomly using uniform distribution as shown in Table 2.6. The empirical study is conducted based on a sample size of about 250. The findings reported may vary with the changes in the experimental setup, such as variation in the parameter reported in the Table 2.6. For each sample data, the outcomes of the three games: i) Non-cooperation which results fare pricing and booking limits at NE; ii) NBS with NSP option; and iii) NBS with SP option are compared.

| Parameters | Values |
|---|---|
| $C_2$ | U[80, 120] |
| $\alpha_{L2}$ | U[55, 65] |
| $\alpha_{H2}$ | U[35, 45] |
| $\beta_{L2}$ | U[0.2, 0.3] |
| $\beta_{H2}$ | U[0.1, 0.2] |
| $\theta_{L21}$ | U[0.1, 0.2] |
| $\theta_{H21}$ | U[0.05, 0.15] |
| Model Type (I) | [Additive, Multiplicative] |

Table 2.3: Factors variation under asymmetric situation

### Impact on Airline 1

In Table 2.4, a pair-wise comparison of payoffs of the airlines along with the control of booking limits and fare pricing for airline 1 using t-test (see Kutner et al. (2005) for details) is presented.

In this analysis following observations were made:

- Airline 1 is able to improve its payoff when it agrees to cooperate with its rival airline. The relative improvement in payoffs with NSP option compared to non cooperative payoff i.e., $\frac{\Pi_1^{NE} - \Pi_1^{NSP}}{\Pi_1^{NE}}$ , is about 5.4% . The gain further improves to 6.0% when cooperation with SP option is compared with NSP option.

- About 1.9% more seats are allocated for high fare class customers when airline 1 assumes non cooperative game situation compared to non cooperative control, although it is not found significant at 5% level. When airline 1 accepts cooperation with NSP option, it allocates 1.9% more seats for high fare customers compared to cooperation with SP option and it is found significant.

- Airline 1 is able to raise its low fare prices by about 4.8% in cooperative scenario with NSP option compared to non cooperative scenario. The low fare prices further raised by about 6.7% when airline 1 accepts cooperation with SP option. Both improvements in the low fare prices are found significant at 5% level. Similar to the trend observed in the low fare class prices, airline 1 is also able to raise its prices for higher fare class.

- Airline 1 increases its high fare class price by about 35% in the case of cooperative bargaining with NSP option compared to the non cooperative outcomes and it is found significant. This gain further improves significantly by about 8.6% in the case of cooperation with SP option when compared with cooperation with NSP.

| | Paired Differences | | | | | t | df | Sig. |
|---|---|---|---|---|---|---|---|---|
| | Mean | Std. | Std. | 95% CI | | | | (2-tailed) |
| Comparison | | Deviation | Error Mean | Lower | Upper | | | |
| $\frac{B_1^{NE} - B_1^{NSP}}{B_1^{NE}}$ | 0.019 | 0.160 | 0.010 | -0.002 | 0.039 | 1.819 | 243 | 0.07 |
| $\frac{B_1^{NSP} - B_1^{SP}}{B_1^{NSP}}$ | 0.019 | 0.019 | 0.003 | 0.014 | 0.025 | 6.644 | 243 | 0 |
| $\frac{P_{L1}^{NE} - P_{L1}^{NSP}}{P_{L1}^{NE}}$ | -0.048 | 0.120 | 0.008 | -0.064 | -0.033 | -6.283 | 243 | 0 |
| $\frac{P_{L1}^{NSP} - P_{L1}^{SP}}{P_{L1}^{NSP}}$ | -0.067 | 0.130 | 0.008 | -0.083 | -0.051 | -8.039 | 243 | 0 |
| $\frac{P_{H1}^{NE} - P_{H1}^{NSP}}{P_{H1}^{NE}}$ | -0.355 | 0.230 | 0.015 | -0.384 | -0.326 | -24.174 | 243 | 0 |
| $\frac{P_{H1}^{NSP} - P_{H1}^{SP}}{P_{H1}^{NSP}}$ | -0.086 | 0.141 | 0.009 | -0.103 | -0.068 | -9.484 | 243 | 0 |
| $\frac{\Pi_1^{NE} - \Pi_1^{NSP}}{\Pi_1^{NE}}$ | -0.054 | 0.065 | 0.004 | -0.062 | -0.045 | -12.934 | 242 | 0 |
| $\frac{\Pi_1^{NSP} - \Pi_1^{SP}}{\Pi_1^{NSP}}$ | -0.060 | 0.059 | 0.004 | -0.067 | -0.052 | -15.878 | 242 | 0 |

Table 2.4: Comparison of booking limits, fare pricing and payoff of airline 1

### Impact on Airline 2

Extending the analysis with airline 2, the following observations are reported:

- A significant increase in the payoff to airline 2 by about 6.5% is observed, when airline 2 cooperates with NSP option compared to its non cooperative payoff. This improvement further augments to 5.2% in the case of cooperation with SP option, when compared with NSP option.

- Airline 2 is able to allocate more seats to satisfy high fare class demand when it assumes the cooperative game with NSP compared to the non cooperative game. This improvement is measured by about 5.3%. The variation in the seat allocation strategy in the case of two cooperation games is not significant.

- Similar to airline 1, airline 2 also increases its low fare prices. The increments in the low and high fare prices are 9.6% and 6.3% respectively. The first comparison is between cooperation with NSP and non cooperation. The second comparison is between cooperation with SP and NSP options. The increment observed in the case of high fare class pricing is about 28.4% and 5.8% respectively.

| | Paired Differences | | | | | t | df | Sig. |
|---|---|---|---|---|---|---|---|---|
| | Mean | Std. | Std. | 95% CI | | | | (2-tailed) |
| Comparison | | Deviation | Error Mean | Lower | Upper | | | |
| $\frac{B_2^{NE} - B_2^{NSP}}{B_2^{NE}}$ | 0.053 | 0.177 | 0.011 | 0.031 | 0.075 | 4.676 | 243 | 0 |
| $\frac{B_2^{NSP} - B_2^{SP}}{B_2^{NSP}}$ | -0.016 | 0.255 | 0.016 | -0.048 | 0.016 | -0.994 | 243 | 0.321 |
| $\frac{P_{L2}^{NE} - P_{L2}^{NSP}}{P_{L2}^{NE}}$ | -0.096 | 0.171 | 0.011 | -0.118 | -0.074 | -8.753 | 243 | 0 |
| $\frac{P_{L2}^{NSP} - P_{L2}^{SP}}{P_{L2}^{NSP}}$ | -0.063 | 0.120 | 0.008 | -0.078 | -0.048 | -8.235 | 243 | 0 |
| $\frac{P_{H2}^{NE} - P_{H2}^{NSP}}{P_{H2}^{NE}}$ | -0.284 | 0.231 | 0.015 | -0.313 | -0.255 | -19.224 | 243 | 0 |
| $\frac{P_{H2}^{NSP} - P_{H2}^{SP}}{P_{H2}^{NSP}}$ | -0.058 | 0.126 | 0.008 | -0.074 | -0.043 | -7.222 | 243 | 0 |
| $\frac{\Pi_2^{NE} - \Pi_2^{NSP}}{\Pi_2^{NE}}$ | -0.065 | 0.093 | 0.006 | -0.077 | -0.054 | -10.978 | 243 | 0 |
| $\frac{\Pi_2^{NSP} - \Pi_2^{SP}}{\Pi_2^{NSP}}$ | -0.052 | 0.099 | 0.006 | -0.065 | -0.040 | -8.211 | 243 | 0 |

Table 2.5: Comparison of booking limits, fare pricing and payoff of airline 2

**Experimental Design for Non Cooperative Game Analysis**

We extend the study in which the model related parameters for airline 2 are subject to variation and their impact is studied. The impact is studied on payoff, booking limits, and fare pricing of both of the competing airlines in a non-cooperative game. The suggested study uses Statistical Design of Experiment(s) (DOE) by considering a fractional factorial design with six factors each at two levels. The factors with their levels are identified in Table 2.6.

Table 2.6: Factors for two level factorial design

| Parameters | Levels |
|---|---|
| $C_2$ | [80, 120] |
| $\beta_{L2}$ | [0.2, 0.3] |
| $\beta_{H2}$ | [0.1, 0.2] |
| $\theta_{L21}$ | [0.1, 0.2] |
| $\theta_{H21}$ | [0.05, 0.15] |
| Model Type (I) | [Additive, Multiplicative] |

The DOE analysis with $\Pi_1$ is presented with ANOVA in Table 2.7. The significant main and two-way interaction effects can be identified from the table. We used the ANOVA table to develop the first-order regression equation (Equation 2.89). The equation is established by considering the factors that have significant main and two way interaction effect also the level significance is 5 % ($\alpha = 0.05$). When coded units are used in a regression equation then the factor at low level is replaced by -1 and similarly a factor at its high level is replaced by +1 (see Montgomery (1991)). Rest of the regression equations presented in this paper also use same principal mentioned for development of Equation 2.89. Form Equation 2.89 we conclude that an increase in the capacity of airline 2 ($C_2$) results a decrease in the payoff of airline 1. An increase of $\beta_{L2}$ and $\beta_{H2}$ for airline 2 also results a decrease in the payoff of airline 1, however increasing $\theta_{H21}$ improves the payoff to airline 1. The interaction effect of $\beta_{H2}$ and $\theta_{H21}$ is also significant. A simultaneous increase in $\beta_{H2}$ and $\theta_{H21}$ results a decrease in the payoff of airline 1.

$$\hat{\Pi}_1 = 14861 - 598C_2 - 537\beta_{L2} - 1337\beta_{H2} + 867\theta_{H21} - 922\beta_{H2}\theta_{H21} \quad (2.89)$$

Table 2.7: ANOVA with $\Pi_1$ as response

| Term | Effect | Coef | SE Coef | T | p-value |
|---|---|---|---|---|---|
| Constant | | 14861 | 176.8 | 84.04 | 0 |
| $C_2$ | -1196 | -598 | 176.8 | -3.38 | 0.007 |
| $\beta_{L2}$ | -1073 | -537 | 176.8 | -3.03 | 0.013 |
| $\beta_{H2}$ | -2674 | -1337 | 176.8 | -7.56 | 0 |
| $\theta_{L21}$ | 752 | 376 | 176.8 | 2.13 | 0.059 |
| $\theta_{H21}$ | 1733 | 867 | 176.8 | 4.9 | 0.001 |
| I | 393 | 197 | 176.8 | 1.11 | 0.292 |
| $C_2\,\beta_{L2}$ | -69 | -35 | 176.8 | -0.2 | 0.849 |
| $C_2\,\beta_{H2}$ | 369 | 184 | 176.8 | 1.04 | 0.322 |
| $C_2\,\theta_{L21}$ | -165 | -82 | 176.8 | -0.47 | 0.651 |
| $C_2\,\theta_{H21}$ | 282 | 141 | 176.8 | 0.8 | 0.444 |
| $C_2\,$I | 44 | 22 | 176.8 | 0.12 | 0.903 |
| $\beta_{L2}\,\beta_{H2}$ | 32 | 16 | 176.8 | 0.09 | 0.93 |
| $\beta_{L2}\,\theta_{L21}$ | -158 | -79 | 176.8 | -0.45 | 0.665 |
| $\beta_{L2}\,\theta_{H21}$ | 95 | 47 | 176.8 | 0.27 | 0.795 |
| $\beta_{L2}\,$I | 60 | 30 | 176.8 | 0.17 | 0.868 |
| $\beta_{H2}\,\theta_{L21}$ | -95 | -47 | 176.8 | -0.27 | 0.794 |
| $\beta_{H2}\,\theta_{H21}$ | -1845 | -922 | 176.8 | -5.22 | 0 |
| $\beta_{H2}\,$I | 565 | 282 | 176.8 | 1.6 | 0.141 |
| $\theta_{L21}\,\theta_{H21}$ | -77 | -39 | 176.8 | -0.22 | 0.832 |
| $\theta_{L21}\,$I | -64 | -32 | 176.8 | -0.18 | 0.861 |
| $\theta_{H21}\,$I | -396 | -198 | 176.8 | -1.12 | 0.289 |

Similarly the DOE analysis is extended to study the payoff of competing airline 2. Later the booking limits and fare pricing are also studied for both airlines in a similar way. A DOE analysis with $\Pi_2$ as response has resulted in the first-order regression Equation 2.90. An increase in $\beta_{L2}$ and $\beta_{H2}$ decreases the payoff to airline 2 but an increase in $\theta_{L21}$ and $\theta_{H21}$ has opposite impact.

$$\hat{\Pi}_2 = 15234 - 1182\,\beta_{L2} - 3080\,\beta_{H2} + 1182\,\theta_{L21} + 2821\,\theta_{H21} - 1724\,\beta_{H2}\,\theta_{H21} \tag{2.90}$$

A regression analysis with booking limit $B_1$, as response is reported in Equation 2.91. An increase in $\beta_{H2}$ and use of multiplicative model results an increase in the booking limit to airline 1, $B_1$. A simultaneous increase or decrease in $\beta_{H2}$ and $\theta_{H21}$ also increases the booking limit $B_1$.

$$\hat{B}_1 = 76.061 - 1.329\,\beta_{L2} + 1.019\,\beta_{H2} + 5.127\,\mathrm{I} + 0.858\,\beta_{\mathrm{H2}}\,\theta_{\mathrm{H21}} \tag{2.91}$$

Likewise DOE analysis with $B_1$, a DOE analysis with $B_2$ as response has resulted regression reported in Equation 2.92. From the Equation 2.92 we conclude that an increase in the capacity of airline 2 significantly increases its booking limit. Interaction effects $\beta_{H2}\,\theta_{H21}$ and $\mathrm{I}\,\theta_{\mathrm{H21}}$ also found significant. A simultaneous increase in factors $\beta_{H2}$ and I results a reduction in $B_2$. The impact of simultaneous increase in the factors $\theta_{H21}$ and I causes an increase in $B_2$.

$$\hat{B}_2 = 69.998 + 10.033\,C_2 + 9.677\,\beta_{H2}\,\theta_{H21} - 7.946\,\beta_{H2}\,\mathrm{I} + 7.605\,\theta_{\mathrm{H21}}\,\mathrm{I} \tag{2.92}$$

We also carried out DOE with the low fare price of airline 1 and regression equation based on significant factors is shown in Equation 2.93. We can infer that an increase in capacity of airline 2, $\beta_{L2}$ and $\beta_{H2}$ significantly reduces the low fare price $P_{L1}$. However an increase in $\theta_{L21}$ and $\theta_{H21}$ results an increase in $P_{L1}$.

$$\hat{P}_{L1} = 178.343 - 6.050\,C_2 - 7.156\,\beta_{L2} - 6.256\,\beta_{H2} + 4.449\,\theta_{L21} + 4.882\,\theta_{H21} \tag{2.93}$$

In Equation 2.94, the first-order regression analysis from DOE in which the response in low fare price offered by airline 2 is presented. An increase in $C_2$ and $\beta_{L2}$ results a significant decrease in $P_{L2}$. On the other hand $\theta_{L21}$ is positively correlated to $P_{L2}$.

$$\hat{P}_{L2} = 169.00 - 13.63\,C_2 - 18.33\,\beta_{L2} + 10.83\,\theta_{L21} \tag{2.94}$$

Similarly a DOE considering $P_{H1}$ as response is presented using a regression in Equation 2.95. An increase in $C_2$ and $\beta_{H2}$ results a decrease in $P_{H1}$, however an increase in $\theta_{H21}$ and I results in increase of $P_{H1}$ significantly. A simultaneous increase or decrease in $\beta_{H2}$ and $\theta_{H21}$ results a decrease in $P_{H1}$.

$$\hat{P}_{H1} = 235.63 - 6.83\,C_2 - 23.59\,\beta_{H2} + 13.78\,\theta_{H21} + 7.99\,\mathrm{I} - 18.05\,\beta_{\mathrm{H2}}\,\theta_{\mathrm{H21}} \tag{2.95}$$

High fare price $P_{H2}$ decreases significantly with an increase in $C_2$ and $\beta_{H2}$ as revealed in Equations 2.96. $\theta_{H21}$ and interaction of $\beta_{H2}$ and $\theta_{H21}$ also have significant impact on $P_{H2}$.

$$\hat{P}_{H2} = 273.53 - 12.85\,C_2 - 54.36\,\beta_{H2} + 30.74\,\theta_{H21} - 40.48\,\beta_{H2}\,\theta_{H21} \tag{2.96}$$

### Regression Based Analysis on the Revenue Gain in Cooperation

A regression analysis is used to study the impact of various market related parameters on the revenue gain improvement observed due to cooperation. Both the additive and multiplicative model are used in this statistical analysis. In Equation 2.97, a regression analysis is presented where response $\hat{y}$ is the gain observed by airline 1 in the cooperation with SP option compared to the non cooperative payoff i.e., $\hat{y} = \frac{\Pi_1^{NSP}}{\Pi_1^{NE}}$. The regression uses standardized predictor variables. The indicator variable for model type, I, is not considered for regression with standardized variables. In the regression analysis, the impact of $C_2$ and $\alpha_{L2}$ are significant at 5% level. As an interpretation, unit standard deviation increment in $C_2$ improves revenue gain by about 18%. A unit standard deviation increment in $\alpha_{L2}$ reduces the gain of cooperation by about 16.7%.

$$\begin{aligned} \hat{y} &= 0.180\,C_2 - 0.167\,\alpha_{L2} - 0.072\,\alpha_{H2} + 0.102\,\beta_{L2} - 0.022\,\beta_{H2} - 0.077\,\theta_{L21} \\ &+ 0.035\,\theta_{H21}\,(R^2 = 0.091) \end{aligned} \tag{2.97}$$

Similar to previous regression analysis, Equation 2.98 reports the regression analysis considering the relative improvement of cooperation with SP option over cooperation with NSP option for airline 1. Thus the response is $\hat{y} = \frac{\Pi_1^{SP}}{\Pi_1^{NSP}}$. $C_2$ and $\theta_{L21}$ are found to be significant at 5% level. A unit standard deviation increase in $C_2$ improves the revenue by about 21% and a unit standard deviation increase in $\theta_{L21}$ improves the revenue by about 28%.

$$\begin{aligned} \hat{y} &= 0.211\,C_2 + 0.044\,\alpha_{L2} - 0.032\,\alpha_{H2} + 0.067\,\beta_{L2} - 0.044\,\beta_{H2} - 0.005\,\theta_{L21} \\ &+ 0.285\,\theta_{H21}\,(R^2 = 0.167) \end{aligned} \tag{2.98}$$

The regression based analysis about the impact of market related parameters on the revenue gain for airline 1 is also carried out. In Equation 2.99, $\hat{y} = \frac{\Pi_2^{NSP}}{\Pi_2^{NE}}$ which measures the improvement observed by airline 2 in cooperation with NSP option compared to non cooperative analysis. Four parameters, $C_2$, $\alpha_{L2}$, $\theta_{L21}$ and $\theta_{H21}$ are found to have significant effect at 5% level.

$$\begin{aligned} \hat{y} &= 0.202\, C_2 - 0.137\, \alpha_{L2} - 0.055\, \alpha_{H2} + 0.11\, \beta_{L2} + 0.051\, \beta_{H2} - 0.129\, \theta_{L21} \\ &+ 0.150\, \theta_{H21}\, (R^2 = 0.119) \end{aligned} \tag{2.99}$$

A similar regression analysis with a response $\hat{y} = \frac{\Pi_2^{SP}}{\Pi_2^{NSP}}$ is reported in Equation 2.100. It represents the revenue gain airline 2 is experiencing for adopting cooperation with SP option compared to cooperation with NSP option. None of the parameters are found to be significant at 5% level.

$$\begin{aligned} \hat{y} &= 0.050\, C_2 + 0.025\, \alpha_{L2} - 0.076\, \alpha_{H2} + 0.072\, \beta_{L2} + 0.026\, \beta_{H2} + 0.03\, \theta_{L21} \\ &- 0.005\, \theta_{H21}\, (R^2 = 0.016) \end{aligned} \tag{2.100}$$

### 2.2.3 Conclusions

The problem of joint control of nested booking limit and fare pricing in a duopoly market competition is addressed. A mathematical framework is developed and both the non cooperative and cooperative bargaining games are analyzed. The bargaining games are further segregated into two classes: i) bargaining game with NSP option; and ii) bargaining game with SP option. It is shown in a numerical study that cooperation results a better payoff to each of the two airlines. In the case of cooperative bargaining both airlines are able to sell at a higher price in each fare class while still able to maintain the customer demand. The bargaining game with the side payment option is observed to be superior to the bargaining game with no side payments.

Statistical analysis are carried out to study, the impact of cooperation. The analysis is extended to further analyze the impact of various model related parameters on payoffs, fare pricing and booking limit controls of the bargaining airline with both the side payment and no side payment option.

## 2.3 Synthesis

In this chapter, fare pricing competition is analyzed in depth under market competition. The research has implications in setting the best fare prices in a competitive aviation market. The pricing expressions are developed for duopoly market competition. The expressions determine best fare price offered by an airline given fare price offered by its rival airline is known. The research also identifies several customer demand scenarios which may be faced by airlines in a competitive market and suggests a pricing strategy under competition such the revenue is maximized. This chapter also addresses the problem of jointly controlling the fare pricing and seat allocation under competitive market conditions. The approach integrates the practice of price based and the quantity based RM into an integrated framework. Competition models are developed to study the competition. A simulation based numerical study is presented which determines the impact of various market related parameters on the payoffs, seat inventory control and fare pricing strategies of the competing airlines in the game. This analysis enables an airline to determine how variation in market conditions could impact its revenue gain and pricing strategies. In airline alliance, code sharing is commonly observed. In code sharing, airlines share their seats with other competing airlines in the market. The concept is further explored for a cooperation game of both the fare pricing and seat allocation among the airlines in a duopoly market. The model of joint fare pricing and seat inventory control is studied assuming a bargain game. It would be considered as an airline alliance while sharing both the prices and seat inventory. The study shows, if permitted, the airline alliance improves the revenue to the airlines.

In the next chapter, we present a distribution approach for modeling and analysis of RM problem while jointly controlling the pricing and capacity in monopoly. The approach does not assume any distribution for modeling the demand and thus present an analysis under worst possible demand distribution.

# Chapter 3

# Distribution Free Approach for Pricing in Revenue Management

As mentioned earlier in Chapter 1 that newsvendor problem research has a direct implication on RM. While addressing the newsvendor problem, a large number of RM applications can be studied. The problem has a simple but an elegant structure. It is also noted earlier that its application is not only limited to stochastic inventory control, but historically the problem has also been studied in the context of banking and in many other practical situations. The scope of this chapter is to study the integration of price based and quantity based RM practice on the problem. This develops a joint control of pricing and capacity for the problem. The use of the distribution free approach would also be very interesting to the problem: firstly it enables us to optimize the revenue under worst possible demand distribution; and secondly, its utilization could be numerous where RM is practiced for single period perishable products such as fashion appraisal, technology items etc. These products not only have short life cycle but also do not have any customer demand history.

The approach can also help airlines improve their revenues under worst possible customer demand behavior. A multi-product newsvendor problem resembles airline RM problem when multiple products compete for budget and/or capacity. This situation is commonly observed in airline RM when multiple fare classes compete for the cabin capacity allocation. The distribution free approach is first addressed to the standard newsvendor problem and later also tested on extended to the standard newsvendor problem with shortage and holding cost.

## 3.1 Standard Newsvendor Problem

The standard newsvendor problem is analyzed in a monopolistic situation. The commodity/service has marginal cost $C$ and there is no fixed cost associated with the commodity. In this study, the price $P$ and the quantity $Q$ are determined jointly, such that the total revenue is maximized in a monopolistic market. It is an extension of the standard newsvendor problem since both the price $P$ and quantity $Q$ are considered as the decision variables. The firm faces random demand $\mathcal{D}$, which is a function of the riskless demand $D$ and the stochastic random factor $\xi$. More precisely the random demand is $\mathcal{D} = \mathcal{D}(P, \xi)$. Assuming no shortage and holding costs, the objective function of a firm in a monopolistic market is written as follows:

$$\Pi\left(P, Q\right) = E_{\xi}\left[P \min\{Q, \mathcal{D}\} - C\, Q\right] \tag{3.1}$$

As identified in the literature review earlier in Chapter 1, two types of modeling scenarios are usually considered to incorporate the randomness: i) additive; and ii) multiplicative. In the case of additive model, the random demand $\mathcal{D}$ is given as:

$$\mathcal{D} = D + \xi \tag{3.2}$$

where $\xi$ has an expected mean of $\mu = E[\xi] = 0$ and a standard deviation of $\sigma$.

In the multiplicative model, $\mathcal{D}$ is the product of $D$ and $\xi$ such that:

$$\mathcal{D} = D\,\xi \tag{3.3}$$

where $\xi$ has an expected mean $\mu = E[\xi] = 1$ and standard deviation $\sigma$.

The random variable $\xi$ is assumed to be independent of the price, and has a probability distribution function $\phi$ and a cumulative probability distribution function $\Phi$. Moreover, $\xi$ is assumed to be bounded in $[\underline{\xi}, \overline{\xi}]$ and follows an Increasing Generalized Failure Rate (IGFR). For a random variable $x$, a Generalized Failure Rate (GFR) function is defined as $\Psi(x) = \dfrac{x\,\phi(x)}{1-\Phi(x)}$, where $\phi(x)$ and $\Phi(x)$ are probability and cumulative probability distribution functions respectively. The concept of IGFR is commonly observed in the stochastic inventory control theory (Lariviere and Porteus 2001) which roughly measures the percentage decrease in the probability of stock-out when the stocking quantity is increased by 1%. The uniform and normal distributions are strictly IGFR. The demand $D$ is assumed to be continuous, nonnegative, twice differentiable and defined between $[C, \overline{P}]$, where $\overline{P}$ is the maximum admissible value of $P$ and $D_{P=\overline{P}} = 0$. It is also assumed that $D$ has an Increasing Price Elasticity (IPE) without however, being restricted to a strictly IPE characteristics. The price elasticity function $e$ is defined as $e = -\dfrac{P\,D'}{D}$, where $D' = \dfrac{\partial D}{\partial P}$. Thus, the IPE assumption results in $D' < 0$.

### 3.1.1 Additive Model

Using the additive model, the analysis is extended, and Equation 3.1 can be written as:

$$\Pi\,(P,Q) \;=\; E_{\xi}\left[P\,Q - P\,[Q-\mathcal{D}]^{+} - C\,Q\right]$$

$$= \quad (P - C)\,Q - P\,E_\xi\,[Q - \mathcal{D}]^+ \tag{3.4}$$

where, $[x]^+ = \max\{0, x\}\,, x \in \mathrm{R}$

Considering $E_\xi\,[Q - \mathcal{D}]^+$ in the case of the additive model

$$\begin{aligned} E_\xi\,[Q - \mathcal{D}]^+ &= E_\xi\,[Q - D - \xi]^+ \\ &= \int_{\underline{\xi}}^{Q-D} (Q - D - \xi)\,\phi(\xi)\,d\xi \\ &= \int_{\underline{\xi}}^{Q-D} \Phi(\xi)\,d\xi \end{aligned}$$

Hence

$$\Pi\,(P^*, Q^*) = \max_{P,Q}\,(P - C)\,Q - P\int_{\underline{\xi}}^{Q-D} \Phi(\xi)\,d\xi \tag{3.5}$$

The solution procedure suggested in Yao (2002) is followed. The procedure first determines the optimal quantity/capacity $Q^*$ for a given $P$. Later, $Q^*$ is substituted in Equation 3.5 to determine the optimal price $P^*$. Using the first order optimality condition, $Q^*$ is determined as follows:

$$Q^* = D + \Phi^{-1}\,(\varrho) \tag{3.6}$$

where, $\varrho = \dfrac{P - C}{P}$.

After substituting the optimal quantity $Q^*$ in Equation 3.5, the payoff can be rewritten:

$$\Pi\,(P^*) \quad = \quad \max_{P}\,(P - C)\left(D + \Phi^{-1}(\varrho)\right) - P\int_{\underline{\xi}}^{\Phi^{-1}(\varrho)} \Phi(\xi)d\xi \tag{3.7}$$

For brevity, both $\Pi(P, Q)$ and $\Pi(P)$ are represented with $\Pi$ only. The $\Pi$ function is shown quasi-concave in Yao (2002). Using the first order optimality condition, $\dfrac{\partial \Pi}{\partial P}|_{P=P^*} = 0$, such that $P^* \in [C, \overline{P}]$ results in a unique $P^*$ that maximizes $\Pi$. Now, the expressions for the optimal price $P^*$ and the capacity $Q^*$ are derived by considering $\xi$ as uniformly distributed in a symmetric market randomness. Thus, $\underline{\xi} = -\psi$ and $\overline{\xi} = \psi$. A linear riskless demand function $D = \alpha - \beta\, P$ is considered, as well as $D' = \dfrac{\partial D}{\partial P} = -\,\beta$ and $D'' = \dfrac{\partial^2 D}{\partial P^2} = 0$ . Thus, the resulting expression for the optimal price is:

$$P^* = \frac{C(\,-\,C\beta + 4\psi + \sqrt{C^2\beta^2 - 5C\psi\beta - 2\psi^2 + 3\alpha\psi})}{3\psi} \tag{3.8}$$

and the optimal quantity is:

$$\begin{aligned} Q^* \;=\; & \\ & \frac{1}{3\psi} \\ & \Big(C^2\beta^2 - C\beta\Big(6\psi + \sqrt{C^2\beta^2 + 3\alpha\psi - 5C\beta\psi - 2\psi^2}\Big) + \\ & \quad \psi\Big(3\alpha - \psi + 2\sqrt{C^2\beta^2 + 3\alpha\psi - 5C\beta\psi - 2\psi^2}\Big)\Big) \end{aligned} \tag{3.9}$$

Numerically optimal price $P^*$ could also be determined using a line search method (Bertsekas 1999). A built-in routine in MATLAB, FMINBND can be used to minimize $-\Pi(P)$, such that $P \in [C, \overline{P}]$, and thus results $P^*$ maximizing $\Pi(P)$.

The distribution free approach to the aforementioned standard newsvendor problem is now investigated. The proposed distribution free approach uses the max-min scheme suggested in Scarf (1952) to develop a Lower Bound (LB) estimate on the revenue function, $\Pi_{LB}$. In this case, a distribution free estimate on revenue is the revenue generated under the worst possible distribution of the random demand.

First, the following relationships are established:

$$\begin{aligned} \min\{Q, \mathcal{D}\} &= \mathcal{D} - [\mathcal{D} - Q]^+ \\ [Q - \mathcal{D}]^+ &= Q - \mathcal{D} + [\mathcal{D} - Q]^+ \\ [\mathcal{D} - Q]^+ &= \mathcal{D} - Q + [Q - \mathcal{D}]^+ \end{aligned}$$

Next, using the Scarf's (1952) rule, a lower bound estimate on the profit function $\Pi_{LB}$ is written for the additive model as follows:

$$E_\xi\left[\mathcal{D}-Q\right]^+ \le \frac{(\sigma^2+(Q-D)^2)^{1/2}-(Q-D)}{2} \tag{3.10}$$

$$E_\xi\left[Q-\mathcal{D}\right]^+ \le \frac{(\sigma^2+(D-Q)^2)^{1/2}-(D-Q)}{2} \tag{3.11}$$

The lower bound estimate $\Pi_{LB}$ with the corresponding estimate on price $P_{LB}$ and quantity $Q_{LB}$ is:

$$\Pi_{LB}(P_{LB},Q_{LB}) = (P_{LB}-C)\,Q_{LB} - P_{LB}\,\frac{(\sigma^2+(D-Q_{LB})^2)^{1/2}-(D-Q_{LB})}{2} \tag{3.12}$$

Let $Q^*_{LB}$ be the optimal quantity and $P^*_{LB}$, the optimal pricing policy, such that

$$\Pi_{LB}(P^*_{LB},Q^*_{LB}) = \max_{P_{LB},Q_{LB}}\,(P_{LB}-C)\,Q_{LB} - P_{LB}\,\frac{(\sigma^2+(D-Q_{LB})^2)^{1/2}-(D-Q_{LB})}{2} \tag{3.13}$$

For brevity $\Pi_{LB}(P^*_{LB},Q^*_{LB})$ is represented by $\Pi_{LB}$. Once again, using the aforementioned conditions, following two assumptions are made: $P_{LB} \in [C,\overline{P}]$ and $\lambda_1 = \sigma^2 + (Q_{LB}-D)^2$ for $\sigma \ge 0$, $\sqrt{\lambda_1} \ge |Q_{LB}-D|$. Next, the behavior of $\Pi_{LB}$ is studied. For a given price $P_{LB}$, the first order optimality condition, $\frac{\partial \Pi_{LB}}{\partial Q_{LB}} = 0$, results in the optimal quantity $Q^*_{LB}$ . This quantity is also a unique maximizer of $\Pi_{LB}$, since the second order derivative can easily be shown to be negative i.e., $\frac{\partial^2\Pi_{LB}}{\partial Q^2_{LB}} = -\frac{\sigma^2 P_{LB}}{2\lambda_1^{3/2}} \le 0$. Likewise, in the previous analysis, for a given $Q_{LB}$, $\frac{\partial^2\Pi_{LB}}{\partial P^2_{LB}}$ is studied.

$$\frac{\partial^2\Pi_{LB}}{\partial P^2_{LB}} = \frac{(-D+Q_{LB}+\sqrt{\lambda_1})(D'(2\lambda_1 - P_{LB}(D-Q_{LB}+\sqrt{\lambda_1})D') + P_{LB}\lambda_1 D'')}{2\lambda_1^{3/2}} \tag{3.14}$$

For a demand function that is linear in price $P_{LB}$, Equation 3.14 can be simplified. Thus $D'' = 0$

$$\frac{\partial^2 \Pi_{LB}}{\partial P_{LB}^2} = \frac{(-D + Q_{LB} + \sqrt{\lambda_1})D'(2\lambda_1 - P_{LB}(D - Q_{LB} + \sqrt{\lambda_1})D')}{2\lambda_1^{3/2}} \tag{3.15}$$

As it is stated earlier, $\sqrt{\lambda_1} \geq |Q_{LB} - D|$ and $D' < 0$. Thus from Equation 3.15, it is evident that $\dfrac{\partial^2 \Pi_{LB}}{\partial P_{LB}^2} \leq 0$. Considering the current analysis of $\Pi_{LB}$, it can be concluded that when one of the decision variables is known (fixed), the $\Pi_{LB}$ is quasi-concave on the other variable. To show that $\Pi_{LB}$ is quasi-concave both in $P_{LB}$ and $Q_{LB}$, the Hessian matrix is studied and presented as follows:

$$\mathbf{H}(P_{LB}, Q_{LB}) = \begin{bmatrix} \frac{\partial^2 \Pi_{LB}}{\partial P_{LB}^2} & \frac{\partial^2 \Pi_{LB}}{\partial P_{LB}\, \partial Q_{LB}} \\ \frac{\partial^2 \Pi_{LB}}{\partial P_{LB}\, \partial Q_{LB}} & \frac{\partial^2 \Pi_{LB}}{\partial Q_{LB}^2} \end{bmatrix} \tag{3.16}$$

For brevity, the Hessian matrix $\mathbf{H}(P_{LB}, Q_{LB})$ is referred by $\mathbf{H}$ only. For the Hessian matrix $\mathbf{H}$, the first principal minors are the diagonal elements of the matrix and the second principle minor is the determinant of the matrix $\mathbf{H}$, $|\mathbf{H}| = \frac{\partial^2 \Pi_{LB}}{\partial P_{LB}^2} \frac{\partial^2 \Pi_{LB}}{\partial Q_{LB}^2} - \left(\frac{\partial^2 \Pi_{LB}}{\partial P_{LB}\, \partial Q_{LB}}\right)^2$. As mentioned in Winston (2004), $\Pi_{LB}$ is quasi-concave if its Hessian matrix satisfies two conditions: i) first principal minors must be non-positive; and ii) second principal minor must be non-negative. The first principal minors of $\mathbf{H}$ are already identified as negative and now, the second principal minor is also shown as non-negative, hence proving the concavity of $\Pi_{LB}$.

Here

$$\frac{\partial^2 \Pi_{LB}}{\partial P_{LB}\, \partial Q_{LB}} = \frac{1}{2}\left(1 + \frac{(D - Q_{LB})\lambda_1 + \sigma^2 P_{LB} D'}{\lambda_1^{3/2}}\right) \tag{3.17}$$

Thus,

$$
\begin{aligned}
|\mathbf{H}| &= \\
& -\frac{1}{4\lambda_1^3} \\
& \quad (\sigma^2 P_{LB}( - D + Q_{LB} + \sqrt{\lambda_1}) \\
& \quad\quad ( D'' P_{LB}\lambda_1 + D' ( - D' P_{LB}( D - Q_{LB} + \sqrt{\lambda_1}) + 2\lambda_1)))- \\
& \frac{1}{4}\left(1 + \frac{D' \sigma^2 P_{LB} + (D - Q_{LB})\lambda_1}{\lambda_1^{3/2}}\right)^2
\end{aligned} \tag{3.18}
$$

The linear riskless demand function $D$ for price $P_{LB}$ is again considered. The above expression reduces to:

$$
|\mathbf{H}| = \frac{-4 D' \sigma^2 P_{LB} - ( D - Q_{LB} + \sqrt{\lambda_1})^2 \sqrt{\lambda_1}}{4\lambda_1^{3/2}} \tag{3.19}
$$

Again knowing the inequality, $\sqrt{\lambda_1} \geq |Q_{LB} - D|$ holds, it can be concluded from the Equation 3.19 that $|\mathbf{H}| \geq 0$, such that conditions are satisfied: i) $P_{LB} \geq -\frac{\sigma}{4 D'}$; and ii) $Q_{LB} \geq D$ . A less likely condition is $\overline{P} < -\frac{\sigma}{4 D'}$, where $\Pi_{LB}$ can not be shown quasi-concave for $P_{LB} \in [C, \overline{P}]$. The first condition is redundant if $-\frac{\sigma}{4 D'} \leq C$, otherwise if $C < -\frac{\sigma}{4 D'} < \overline{P}$, then $P_{LB} \in \left[-\frac{\sigma}{4 D'}, \overline{P}\right]$. With the linear $D$, the second condition can be used as a single linear constraint with the $\Pi_{LB}$ as an objective function. Winston (2004) has identified in a lemma that a quasi-concave objective function subject linear constraints results global optimal solution. There can be several methods to achieve this, for instance, one could be Lagrangian relaxation, among other methods are Frank-Wolf (Bertsekas 1999). Currently, there are several commercially available routines to perform this task, such as a built-in function in MATLAB, FMINCON. This function is utilized to minimize $-\Pi_{LB}$ and therefore maximizes $\Pi_{LB}$. This completes the discussion on quasi-concavity of $\Pi_{LB}$ and the computational procedure for the determination of optimal control on $P_{LB}$

and $Q_{LB}$. For a linear demand function, $D = \alpha - \beta\, P_{LB}$, we have $D' = -\beta$ and this results, $P_{LB} \geq \frac{\sigma}{4\,\beta}$. It may be of interest to compare the outcomes of distribution free approach to the outcomes when uniform distribution is assumed for demand randomness with symmetric market conditions. In this case, $\sigma = \frac{\psi}{\sqrt{3}}$, with two other conditions: i) $Q_{LB} \geq D$; and ii) $P_{LB} \geq -\frac{\sigma}{4\,D'}$. The first condition states that the optimal lower bound estimate on the capacity $Q^*_{LB}$ must satisfy the riskless demand $D$. If $-\frac{\sigma}{4\,D'} \leq C$, the second condition is redundant , otherwise if $C < -\frac{\sigma}{4\,D'} < \overline{P}$, then $P_{LB} \in \left[-\frac{\sigma}{4\,D'}, \overline{P}\right]$. For a linear demand function, $D = \alpha - \beta\, P_{LB}$, we have $D' = -\beta$ which results in $P_{LB} \geq \frac{\sigma}{4\,\beta}$. As mentioned earlier $\sigma = \frac{\psi}{\sqrt{3}}$, and thus, $P_{LB} \geq \frac{\psi}{4\sqrt{3}\,\beta}$.

### 3.1.2 Multiplicative Model

Similar to the additive model, the revenue function for the multiplicative model is:

$$\begin{aligned}
\Pi(P,\,Q) &= (P-C)Q - P\,E_{\xi}\,[Q-\mathcal{D}]^{+} \\
&= (P-C)Q - P\,D\,E_{\xi}\,[Q/D-\xi]^{+} \\
&= (P-C)\,Q - P\,D\int_{\underline{\xi}}^{Q/D}\Phi(\xi)d\,\xi
\end{aligned} \tag{3.20}$$

For a given price $P$, the optimal quantity $Q^*$ is

$$Q^* = D\,\Phi^{-1}(\varrho) \tag{3.21}$$

Substituting the $Q^*$ in the revenue function

$$\Pi(P) = P\,D\int_{\underline{\xi}}^{\Phi^{-1}(\varrho)}\xi\,\phi(\xi)d\,\xi \tag{3.22}$$

Taking the logarithm of the $\Pi = \Pi(P)$,

$$\ln \Pi = \ln \left\{ P\,D \int_{\underline{\xi}}^{\Phi^{-1}(\varrho)} \xi\,\phi(\xi) d\xi \right\} \tag{3.23}$$

For brevity, $\Pi$ replaces $\ln \Pi$. $\Pi$ is shown concave in Yao (2002). The optimal price $P^*$ is determined using the first order optimality condition, $\frac{\partial \Pi}{\partial P}|_{P=P^*} = 0$, such that $P \in [C, \overline{P}]$. This enables a unique price $P^*$ and a quantity $Q^*$ to be determined, such that $\Pi$ is maximized. Similar to the additive model, the expressions for the optimal pricing $P^*$ and for the capacity $Q^*$ could be derived for the multiplicative model, assuming that $\xi$ follows a uniform distribution and is bounded such that $\xi \in [1-\psi, 1+\psi]$, where $0 < \psi \leq 1$, but it is not presented for brevity. The riskless demand function $D$ is linear such that $D = \alpha - \beta P$. The optimal pricing $P^*$ becomes the maximum values of $P \in [C, \overline{P}]$ which satisfies Equation 3.24.

$$-\frac{\alpha \psi C^2}{P^2} + \alpha + \beta(\psi C + C - 2P) = 0 \tag{3.24}$$

The corresponding optimal quantity $Q^*$ becomes $(\alpha - \beta P^*)((2\,\varrho - 1)\,\psi + 1)$.

Considering the distribution free approach for the multiplicative model, Scarf's (1952) rule is used again, resulting in the following inequality:

$$E_\xi[Q/D - \xi]^+ \leq \frac{(\sigma^2 + (1 - Q/D)^2)^{1/2} - (1 - Q/D)}{2} \tag{3.25}$$

The lower bound estimate of the profit function $\Pi_{LB}$ with corresponding optimal price $P^*_{LB}$ and quantity $Q^*_{LB}$ is:

$$\Pi_{LB}\,(P^*_{LB}, Q^*_{LB}) = \max_{P_{LB}, Q_{LB}}\,(P_{LB} - C)Q_{LB} - \frac{1}{2} P_{LB}\,D\left(\frac{Q_{LB}}{D} + \sqrt{\left(1 - \frac{Q_{LB}}{D}\right)^2 + \sigma^2} - 1\right) \tag{3.26}$$

Next, a set of conditions is determined, such that Equation 3.26 is a quasi-concave function. In Equation 3.27, the second order derivative of $\Pi_{LB}$, with respect to quantity $Q_{LB}$, is presented. Furthermore, in the equation, $\lambda_2 = \left(1 - \frac{Q_{LB}}{D}\right)^2 + \sigma^2$. For $0 < \sigma < 1$ and $D \leq Q_{LB} \leq 2\,D$, it is shown that $\sigma^2 \leq \lambda_2 \leq 1 + \sigma^2$. From Equation 3.27, it is easy to conclude that for any given price $P_{LB} \in [C, \overline{P}]$, the profit function $\Pi_{LB}$ is quasi-concave in the quantity $Q_{LB}$.

$$\frac{\partial^2 \Pi_{LB}}{\partial Q_{LB}^2} = -\frac{\sigma^2 P_{LB}}{2\,D\,\lambda_2^{3/2}} \tag{3.27}$$

In Equation 3.28, $\frac{\partial^2 \Pi_{LB}}{\partial P_{LB}^2}$ is determined.

$$\begin{aligned}\frac{\partial^2 \Pi_{LB}}{\partial P_{LB}^2} \quad &= \\ &\frac{1}{2D^3\lambda_2^{3/2}}\big( - \sigma^2 P_{LB} Q_{LB}^2 (D')^2 - D^2(\,D + D\,\sigma^2 - Q_{LB})\lambda_2(2D' + P_{LB}D'') + \\ &D^3\lambda_2^{3/2}(2D' + P_{LB}D'')\big)\end{aligned} \tag{3.28}$$

Again, assuming the linear riskless demand function $D$ in $P_{LB}$:

$$\frac{\partial^2 \Pi_{LB}}{\partial P_{LB}^2} = \frac{D'(2D^2(Q_{LB} + D\,( - \sigma^2 + \sqrt{\lambda_2} - 1))\lambda_2 - \sigma^2 P_{LB} Q_{LB}^2 D')}{2D^3\lambda_2^{3/2}} \tag{3.29}$$

$\frac{\partial^2 \Pi_{LB}}{\partial P_{LB}^2} \leq 0$ is shown, if $D$ has IPE for any given $Q_{LB} \in [D,\, 2D]$.

Furthermore, the quasi-concavity of $\Pi_{LB}$ is investigated by studying the Hessian matrix $\mathbf{H}$ as described in Equation 3.16. As reported in Equations 3.27-3.28 respectively both $\frac{\partial^2 \Pi_{LB}}{\partial Q_{LB}^2}$ and $\frac{\partial^2 \Pi_{LB}}{\partial P_{LB}^2}$ are non-positive and represent the diagonal elements of the Hessian matrix $\mathbf{H}$. To further prove that $\Pi_{LB}$ is quasi-concave, the second principal minor of Hessian matrix $\mathbf{H}$, which is the determinant of matrix $\mathbf{H}$ is studied.

Thus,

$$\frac{\partial^2 \Pi_{LB}}{\partial P_{LB}\,\partial Q_{LB}} = \frac{D\,(\,D - Q_{LB} + D\,\sqrt{\lambda_2})\lambda_2 + \sigma^2 P_{LB} Q_{LB} D'}{2D^2\lambda_2^{3/2}} \tag{3.30}$$

$$
\begin{aligned}
|\mathbf{H}| &= \\
&-\frac{1}{4D^4\lambda_2^3} \\
&\Big((\,D\,(\,D - Q_{LB} + D\,\sqrt{\lambda_2})\lambda_2 + \sigma^2 P_{LB} Q_{LB} D')^2 + \\
&\quad \sigma^2 P_{LB} \\
&\qquad (\,-\,\sigma^2 P_{LB} Q_{LB}^2 (D')^2 - D^2(\,D + D\,\sigma^2 - Q_{LB})\lambda_2(2D' + P_{LB}D'') + \\
&\qquad\quad D^3\lambda_2^{3/2}(2D' + P_{LB}D''))\Big)
\end{aligned}
\tag{3.31}
$$

For linear riskless demand function in $P_{LB}$, the above expression reduces to:

$$
\begin{aligned}
|\mathbf{H}| &= \\
&\frac{1}{4D^3\lambda_2^2} \\
&\Big(-\,D\,(\,D - Q_{LB} + D\,\sqrt{\lambda_2})^2\lambda_2 + \\
&\quad 2\sigma^2 P_{LB}(D^2(1+\sigma^2) + Q_{LB}(-2\,D + Q_{LB}) - D(D + Q_{LB})\sqrt{\lambda_2})D'\Big)
\end{aligned}
\tag{3.32}
$$

$|\mathbf{H}| \geq 0$ given that: i) $P_{LB} \geq \dfrac{D\sigma}{2\,D'(\sigma-2)}$; and ii) $D \leq Q_{LB} \leq 2\,D$, along with two previously stated conditions, that is, $D$ is linear and follows IPE and $0 < \sigma < 1$. $P_{LB} \geq \dfrac{D\sigma}{2\,D'(\sigma-2)}$ is redundant if $\dfrac{D\sigma}{2\,D'(\sigma-2)} \leq C$. However, if $C < \dfrac{D\sigma}{2\,D'(\sigma-2)} < \overline{P}$, then the modified bound on $P_{LB}$ becomes $P_{LB} \in \left[\dfrac{D\sigma}{2\,D'(\sigma-2)}, \overline{P}\right]$. The condition (ii) provides a set of linear constraints, likewise previous analysis, it enables us to determine global optimal for $\Pi_{LB}$, again using a lemma stated in Winston (2004). This is a non-linear optimization problem subject to linear constraints that may be solved using several standard optimization techniques, one could be Frank-Wolf method (Bertsekas 1999), a commercially available routine in MATLAB, FMINCON could also be used. This completes discussion on the quasi-concavity of $\Pi_{LB}$ and computational options to determine the optimal solution. Extending this analysis to a linear riskless demand function $D = \alpha - \beta\, P_{LB}$, $P_{LB} \geq \dfrac{\alpha\, \sigma}{\beta(4-\sigma)}$. Moreover, assuming that the demand randomness is uniformly distributed with symmetric market condition, $P_{LB} \geq \dfrac{\sqrt{3}\,\alpha\,\psi}{12\,\beta - \sqrt{3}\,\beta\,\psi}$.

## 3.2 Extension to Holding and Shortage Costs

In this section, the holding and shortages cost are incorporated to the standard newsvendor problem. Let $G$ be the holding cost and $S$ be the shortage cost associated with the aforementioned newsvendor problem. The revenue function of this extended newsvendor problem becomes:

$$\begin{aligned} \Pi\,(P,Q) &= E_\xi\left\{P\min\{Q,\mathcal{D}\} - C\,Q - G\,[Q-\mathcal{D}]^+ - S\,[\mathcal{D}-Q]^+\right\} \\ &= E_\xi\left\{(P-C)\,Q - (P+G)[Q-\mathcal{D}]^+ - S(\mathcal{D}-Q) - S[Q-\mathcal{D}]^+\right\} \\ &= (P+S-C)\,Q - (P+S+G)E_\xi\,[Q-\mathcal{D}]^+ - S\,D \qquad (3.33) \end{aligned}$$

Similar to the standard newsvendor problem, two modeling approaches are considered: i) additive; and ii)multiplicative.

### 3.2.1 Additive Model

Yao (2002) showed that the revenue function is quasi-concave given that the random factor $\xi$ follows IGFR and that the demand $D$ has IPE as well as some regulatory conditions such that, the incurred holding cost $G$ must be less than the cost $C$.

$$\Pi(P,Q) = (P+S-C)\,Q - (P+S+G)E_\xi\,[Q-D-\xi]^+ - S\,D \qquad (3.34)$$

Using the analysis previously presented for the additive model in the standard newsvendor problem:

$$\Pi\,(P^*,Q^*) = \max_{P,Q}\,(P+S-C)\,Q - (P+S+G)\int_{\underline{\xi}}^{Q-D}\Phi(\xi)\,d\xi - S\,D \qquad (3.35)$$

is obtained. The optimal quantity is determined using the first order optimality condition:

$$Q^* = D + \Phi^{-1}(\varrho) \qquad (3.36)$$

Here, $\varrho$ is redefined as: $\varrho = \dfrac{P+S-C}{P+S+G}$

Substituting $Q^*$ into Equation 3.35 results in:

$$\Pi(P^*) \;=\; \max_P\,(P+S-C)(D+\Phi^{-1}(\varrho)) - (P+S+G)\int_{\underline{\xi}}^{\Phi^{-1}(\varrho)} \Phi(\xi)\,d\xi - S\,D$$

Using the first order optimality condition, $\dfrac{\partial \Pi}{\partial P}|_{P=P^*} = 0$, the optimum price $P^* \in [C, \overline{P}]$ maximizes $\Pi$. $P^*$ is the maximum value of $P$ that satisfies Equation 3.37.

$$-\frac{\psi(C+G)^2}{(G+P+S)^2} + \alpha + (C-2P)\beta = 0 \tag{3.37}$$

The corresponding optimal quantity becomes $Q^* = \psi(2\,\varrho - 1) + \alpha - \beta\,P^*$.

The analysis is extended to develop a lower bound estimate on the problem using the aforementioned distribution free approach. The lower bound estimate is established in Equation 3.38.

$$\begin{aligned}\Pi_{LB}(P^*_{LB}, Q^*_{LB}) \;&=\; \max_{P_{LB},Q_{LB}} Q_{LB}(P_{LB}+S-C) - SD \\ &-\; \frac{1}{2}(G+P_{LB}+S)(Q_{LB}-D+\sqrt{\sigma^2+(Q_{LB}-D)^2})\end{aligned} \tag{3.38}$$

Likewise to the previous analysis, this analysis is extended to show that the $\Pi_{LB}$ is quasi-concave. The second order derivative of $\Pi_{LB}$ with respect to $Q_{LB}$ is reported in Equation 3.39. For a given $P_{LB}$, it results in the optimal quantity $Q^*_{LB}$ since $\dfrac{\partial^2 \Pi_{LB}}{\partial Q^2_{LB}} \leq 0$.

$$\frac{\partial^2 \Pi_{LB}}{\partial Q^2_{LB}} = -\frac{\sigma^2(G+S+P_{LB})}{2\lambda_1^{3/2}} \tag{3.39}$$

The second order derivative of $\Pi_{LB}$ with respect to $P_{LB}$ is studied below:

$$\begin{aligned}\frac{\partial^2 \Pi_{LB}}{\partial P^2_{LB}} \;&=\; \\ &\frac{1}{2}\Bigg(-\frac{2(D-Q_{LB})D'}{\sqrt{\lambda_1}} + 2D' - 2SD'' + \\ &\qquad \frac{(G+S+P_{LB})((-D+Q_{LB}+\sqrt{\lambda_1})\lambda_1 D'' - \sigma^2(D')^2)}{\lambda_1^{3/2}}\Bigg)\end{aligned} \tag{3.40}$$

The above mentioned suggestion assumes a linear $D$ which leads to:

$$\frac{\partial^2 \Pi_{LB}}{\partial P_{LB}^2} = \frac{D'(2(-D + Q_{LB} + \sqrt{\lambda_1})\lambda_1 - \sigma^2(G + S + P_{LB})D')}{2\lambda_1^{3/2}} \tag{3.41}$$

It is evident from Equation 3.41 that for an IPE $D$, the second derivative is negative: $\frac{\partial^2 \Pi_{LB}}{\partial P_{LB}^2} \leq 0$. Next, $|\mathbf{H}|$ is investigated to determine the condition(s) enabling $|\mathbf{H}| \geq 0$, and hence proving the quasi-concavity of $\Pi_{LB}$.

$|\mathbf{H}|$ is computed as follows:

$$\frac{\partial^2 \Pi_{LB}}{\partial P_{LB}\,\partial Q_{LB}} = \frac{1}{2}\Big(\frac{(G + S + P_{LB})D'\sigma^2 + (D - Q_{LB})\lambda_1}{\lambda_1^{3/2}} + 1\Big) \tag{3.42}$$

Thus,

$$\begin{aligned} |\mathbf{H}| \;=\; & \\ & \frac{1}{4}\Big( - \frac{1}{\lambda_1^2}( - (G + S + P_{LB})^2 (D - Q_{LB})D''\sigma^2 + \\ & \qquad (G + S + P_{LB})\sqrt{\lambda_1}(4D' + (G - S + P_{LB})D'')\sigma^2 + \\ & \qquad 2(D - Q_{LB})\lambda_1^{3/2} + (D - Q_{LB})^2\lambda_1) - 1\Big) \end{aligned} \tag{3.43}$$

Let $D$ be linear IPE, then:

$$|\mathbf{H}| = -\frac{(G + S + P_{LB})D'\sigma^2}{\lambda_1^{3/2}} - \frac{(D - Q_{LB} + \sqrt{\lambda_1})^2}{4\lambda_1} \tag{3.44}$$

Since $\sqrt{\lambda_1} \geq |Q_{LB} - D|$, and the two other conditions: i) $Q_{LB} \geq D$; and ii) $P_{LB} \geq (G + S) - \frac{\sigma}{4\,D'}$ are reached, we conclude from Equation 3.44 that $|\mathbf{H}| \geq 0$. Following the previous analysis, the first condition states that the optimal lower bound estimate on capacity $Q_{LB}^*$ must satisfy the riskless demand $D$. The second condition is redundant if $(G + S) - \frac{\sigma}{4\,D'} \leq C$. For $C < (G + S) - \frac{\sigma}{4\,D'} \leq \overline{P}$, $P_{LB} \in [(G + S) - \frac{\sigma}{4\,D'}\,, \overline{P}]$ is obtained. This proves the quasi-concavity of $\Pi_{LB}$. For a linear riskless demand $D = \alpha - \beta\, P_{LB}$, $D' = -\beta$, which ensures that $P_{LB} \geq (G + S) + \frac{\sigma}{4\,\beta}$.

Extending the distribution free analysis to compare it to a situation where the symmetric market randomness follows a uniform distribution i.e., $\sigma = \dfrac{\psi}{\sqrt{3}}$, then the LB for price is bounded by a condition, $P_{LB} \geq (G+S) + \dfrac{\psi}{4\sqrt{3}\,\beta}$.

### 3.2.2 Multiplicative Model

In this section, the random demand $\mathcal{D}$ follows the multiplicative model, i.e., $\mathcal{D} = D\,\xi$. The payoff function of the extended newsvendor problem with shortage and holding costs is:

$$\begin{aligned}\Pi(P,Q) &= (P+S-C)\,Q - (P+S+G)E_{\xi}\,[Q-\mathcal{D}]^{+} - S\,D \\ &= (P+S-C)\,Q - (P+S+G)D\int_{\underline{\xi}}^{Q/D}(Q/D-\xi)\,\phi(\xi)\,d\xi \\ &- SD \end{aligned} \tag{3.45}$$

Following a solution method presented in Yao (2002), the optimal quantity $Q^{*}$ for a given price $P$ is determined as follows:

$$Q^{*} = D\,\Phi^{-1}(\varrho) \tag{3.46}$$

Substituting $Q^{*}$ into Equation 3.45 results in:

$$\begin{aligned}\Pi(P) &= D\left\{(P+S-C)\,\Phi^{-1}(\varrho) - (P+S+G)\int_{\underline{\xi}}^{\Phi^{-1}(\varrho)}(\Phi^{-1}(\varrho)-\xi)\,\phi(\xi)\,d\xi - S\right\} \\ &= D\left\{(P+S+G)\int_{\underline{\xi}}^{\Phi^{-1}(\varrho)}\xi\,\phi(\xi)\,d\xi - S\right\}\end{aligned} \tag{3.47}$$

Now, the problem is addressed by the distribution free approach. The lower bound estimate is:

$$\begin{aligned}\Pi_{LB}(P^{*}_{LB},Q^{*}_{LB}) &= \max_{P_{LB},Q_{LB}} \\ &\quad (P_{LB}+S-C)Q_{LB} - SD- \\ &\quad \frac{1}{2}(G+S+P_{LB})\left(\sqrt{\sigma^{2}+\left(\frac{Q_{LB}}{D}-1\right)^{2}}\,D - D + Q_{LB}\right)\end{aligned} \tag{3.48}$$

The analysis shows that $\Pi_{LB}$ is quasi-concave and leads to the study of the Hessian matrix $\mathbf{H}$. From Equation 3.49, $\frac{\partial^2 \Pi_{LB}}{\partial Q_{LB}^2} \leq 0$. Since all the parameters in the equation are non-negative as can be noticed below:

$$\frac{\partial^2 \Pi_{LB}}{\partial Q_{LB}^2} = -\frac{\sigma^2 (G + S + P_{LB})}{2\, D\, \lambda_2^{3/2}} \tag{3.49}$$

and also

$$\begin{aligned} \frac{\partial^2 \Pi_{LB}}{\partial P_{LB}^2} &= \\ & \frac{1}{2 D^5 \lambda_2^{3/2}} \\ & (D'((G + S + P_{LB}) Q_{LB}^2 ((D - Q_{LB})^2 - D^2 \lambda_2) D' - \\ & \qquad 2 D^3 \lambda_2 ((\lambda_2 - \sqrt{\lambda_2}) D^2 + Q_{LB}\, D - Q_{LB}^2)) - \\ & \quad D^3 \lambda_2 ((G + S + P_{LB}) \lambda_2 D^2 - (G - S + P_{LB}) \sqrt{\lambda_2} D^2 + \\ & \qquad (G + S + P_{LB})(D - Q_{LB}) Q_{LB}) D'') \end{aligned} \tag{3.50}$$

Once again, $D$ is assumed to be linear and IPE, then

$$\begin{aligned} \frac{\partial^2 \Pi_{LB}}{\partial P_{LB}^2} &= \\ & \frac{1}{2 D^3 \lambda_2^{3/2}} \\ & (D'( - 2\, D\, \lambda_2 ((D - Q_{LB}) Q_{LB} - D^2 \sqrt{\lambda_2} + D^2 \lambda_2) - \\ & \qquad \sigma^2 (G + S + P_{LB}) Q_{LB}^2 D')) \end{aligned} \tag{3.51}$$

$\frac{\partial^2 \Pi_{LB}}{\partial P_{LB}^2} \leq 0$, as $D$ is IPE, $D \leq Q_{LB} < 2\,D$ and $0 < \sigma < 1$. This proves that the first principal minors of $\mathbf{H}$ are negative. A non-negative determinant of $\mathbf{H}$ guarantees the quasi-concavity of $\Pi_{LB}$. Hence, $|\mathbf{H}|$ is shown as non-negative below.

$$\frac{\partial^2 \Pi_{LB}}{\partial P_{LB}\, \partial Q_{LB}} = \frac{D\,(\,D - Q_{LB} + D\,\sqrt{\lambda_2}) \lambda_2 + \sigma^2 (G + S + P_{LB}) Q_{LB} D'}{2 D^2 \lambda_2^{3/2}} \tag{3.52}$$

Thus,

$$
\begin{aligned}
|\mathbf{H}| \; = \; & \\
& -\frac{1}{4D^3\lambda_2^2} \\
& \Big( D\,(\,D - Q_{LB} + D\,\sqrt{\lambda_2})^2 \lambda_2 - \\
& \quad 2\,D\,\sigma^2 (G+S+P_{LB})(\,-\,Q_{LB} + D\,(\,-\,1 + \sqrt{\lambda_2}))\sqrt{\lambda_2} D' - \\
& \quad \sigma^2 (G+S+P_{LB}) \\
& \quad\quad ((G+S+P_{LB})(D - Q_{LB})Q_{LB} - D^2(G-S+P_{LB})\sqrt{\lambda_2} + \\
& \quad\quad\quad D^2(G+S+P_{LB})\lambda_2)D'' \Big)
\end{aligned}
\tag{3.53}
$$

Assuming that $D$ is linear, then

$$
\begin{aligned}
|\mathbf{H}| \; = \; & \\
& \frac{1}{4D^2\lambda_2^{3/2}} \\
& \Big(-\,(\,D - Q_{LB} + D\,\sqrt{\lambda_2})^2\sqrt{\lambda_2} - \\
& \quad 2\sigma^2(G+S+P_{LB})(\,D + Q_{LB} - D\,\sqrt{\lambda_2})D' \Big)
\end{aligned}
\tag{3.54}
$$

From Equation 3.54, $|\mathbf{H}| \geq 0$ given that: i) $P_{LB} \geq \dfrac{D\sigma}{2\,D'(\sigma-2)} - (G+S)$; and ii) $D \leq Q_{LB} \leq 2\,D$, furthermore $D$ is assumed linear IPE and the standard deviation is bounded as $0 < \sigma < 1$. This proves the quasi-concavity of $\Pi_{LB}$. Notice that if $\dfrac{D\sigma}{2\,D'(\sigma-2)} - (G+S) \leq C$, than there are no additional constraints on price, i.e., $P_{LB} \in [C, \overline{P}]$. However, when $C < \dfrac{D\sigma}{2\,D'(\sigma-2)} - (G+S) < \overline{P}$ is true, then $\Pi_{LB}$ is quasi-concave only if $P_{LB} \in \left[\dfrac{D\sigma}{2\,D'(\sigma-2)} - (G+S), \overline{P}\right]$. Extending this analysis to a linear riskless demand function $D = \alpha - \beta\,P_{LB}$, $P_{LB} \geq \dfrac{\alpha\,\sigma - G - S}{\beta(4-\sigma)}$ is obtained. Moreover, assuming that the demand randomness is uniformly distributed with symmetric market condition, the LB of the price is bounded by a condition, $P_{LB} \geq \dfrac{3G + 3S - \sqrt{3}\alpha\psi}{\sqrt{3}\beta\psi - 12\beta}$.

## 3.3 Numerical Analysis

In this section, numerical results are presented to demonstrate the performance of the proposed distribution free approach. In both additive and multiplicative, it is assumed that the linear riskless demand function is $D = \alpha - \beta\, P_{LB}$. The study is reported based on over 100 randomly generated problems. The data is generated following the uniform distribution, such that $C \sim$ U$[20, 100]$, $\alpha \sim$ U$[100, 200]$ and $\beta \sim$ U$[0.05, 0.3]$. For the additive model the market randomness $\xi$ is bounded in $[-\psi, \psi]$ with $\psi \sim$ U$(0, 30]$. However, in the case of the multiplicative model, the market randomness, $\xi$ is bounded such that $[1 - \psi, 1 + \psi]$ and $\psi \sim$ U$(0, 1]$.

### 3.3.1 Standard Newsvendor Problem

| | | | Uniform | | | Normal | | |
|---|---|---|---|---|---|---|---|---|
| | $\sigma$ | $\overline{\xi}$ | $\frac{Q^*}{Q^*_{LB}}$ | $\frac{P^*}{P^*_{LB}}$ | $\frac{\Pi(P^*,Q^*)}{\Pi(P^*_{LB},Q^*_{LB})}$ | $\frac{Q^*}{Q^*_{LB}}$ | $\frac{P^*}{P^*_{LB}}$ | $\frac{\Pi(P^*,Q^*)}{\Pi(P^*_{LB},Q^*_{LB})}$ |
| Avg. | 8.68 | 15.03 | 0.8773 | 1.0085 | 1.0025 | 0.9704 | 1.0081 | 1.0013 |
| Min. | 0.68 | 1.18 | 0.6016 | 1.0007 | 1.0000 | 0.8373 | 1.0007 | 1.0000 |
| Max. | 16.94 | 29.34 | 1.0064 | 1.0247 | 1.0336 | 0.9992 | 1.0237 | 1.0128 |
| Std. Err. | | 8.34 | 0.0823 | 0.0056 | 0.0050 | 0.0317 | 0.0053 | 0.0023 |

Table 3.1: The additive model and the standard newsvendor problem

In Tables 3.1 and 3.2, the results of the numerical study are reported. The terms $\dfrac{Q^*}{Q^*_{LB}}$ and $\dfrac{P^*}{P^*_{LB}}$ calibrate the relative measure of variation in the optimal and the lower bound estimates of quantity and price respectively. If $\dfrac{Q^*}{Q^*_{LB}}$ is unity, then the optimal $Q^*$ and the corresponding lower bound estimate on quantity $Q^*_{LB}$ are

| | $\sigma$ | $\overline{\xi}$ | Uniform $\frac{Q^*}{Q^*_{LB}}$ | $\frac{P^*}{P^*_{LB}}$ | $\frac{\Pi(P^*,Q^*)}{\Pi(P^*_{LB},Q^*_{LB})}$ | Normal $\frac{Q^*}{Q^*_{LB}}$ | $\frac{P^*}{P^*_{LB}}$ | $\frac{\Pi(P^*,Q^*)}{\Pi(P^*_{LB},Q^*_{LB})}$ |
|---|---|---|---|---|---|---|---|---|
| Avg. | 0.30 | 0.52 | 1.0059 | 0.9979 | 1.0075 | 0.9987 | 0.9971 | 1.0060 |
| Min. | 0.02 | 0.03 | 0.7770 | 0.9891 | 1.0000 | 0.7714 | 0.9889 | 1.0000 |
| Max. | 0.58 | 1.00 | 1.1070 | 1.0023 | 1.1005 | 1.0840 | 1.0013 | 1.0737 |
| Std. Err. | | 0.28 | 0.0713 | 0.0033 | 0.0160 | 0.0687 | 0.0033 | 0.0128 |

Table 3.2: The multiplicative model and the standard newsvendor problem

equal. $\frac{Q^*}{Q^*_{LB}} > 1$ stands for a situation of under-stocking while distribution of random demand is not known, whereas $\frac{Q^*}{Q^*_{LB}} < 1$ stands for over-stocking, which is exactly the opposite of under-stocking. The same interpretation is used for under-pricing $\frac{P^*}{P^*_{LB}} > 1$ and over-pricing $\frac{P^*}{P^*_{LB}} < 1$. $\frac{\Pi(P^*, Q^*)}{\Pi(P^*_{LB}, Q^*_{LB})}$ is a relative measure of the Expected Value of Additional Information (EVAI). EVAI is an improvement in the profit observed when the demand distribution is known as well as its mean and standard error estimates (Gallego and Moon (1993)). By observing the study of the additive model reported in Table 3.1, a trend of over-stocking of about 12% is observed when the stochastic demand follows a uniform distribution, whereas a trend of under-pricing of about 0.85% is observed when the distribution information is missing. In this case, the EVAI improves the revenue gain by about 0.25%. Assuming that the demand randomness is normally distributed, there is a trend of about 3% in over-stocking and 0.81% in under-pricing. While assuming a normal distribution, the EVAI improves the revenue by approximately 0.13%. In Table 3.2, the numerical study is extended to multiplicative model using the same data. Assuming a uniformly distributed random demand in a symmetric market, a trend of under-stocking of about 0.59% and a trend of over-pricing of about 0.20% exists when the distributional information is replaced with a distribution free lower bound estimate. The EVAI improves the revenue by about 0.75% when the demand randomness follows a uniform distribution. Using a normal distribution in a similar study shows that there is a trend of over-stocking and of under-pricing when the distribution free approach is applied. Similar to the uniform distribution, the EVAI improves the revenue by about 0.60% when the normal distribution is assumed.

### 3.3.2 Extension to Holding and Shortage Cost

Likewise the standard newsvendor problem with shortage cost penalty, a numerical study is conducted on the extended newsvendor problem subjected to holding and shortage costs. In the experiments, the data generated for the standard newsvendor problem are used simultaneously with two new parameters, the holding cost $G$ and the shortage cost $S$. Both are randomly generated such that $G \sim \mathrm{U}[0.2, 0.3] \times \mathrm{C}$, and $S \sim \mathrm{U}[0.1, 0.2] \times \mathrm{C}$. In Table 3.3, the results of the numerical study are reported based on 100 randomly generated problems of the additive model using both the uniform and normal distributions. The study based on the uniform distribution shows that there is a slight under-stocking and under-pricing of about 0.48% and 0.89% respectively. Also, EVAI improves the revenue gain by about 0.20%. When normal distribution is assumed, the study shows that there is a trend of over-stocking and under-pricing of about 2% and 0.85% respectively. EVAI improves the revenue by only about 0.1%.

| | | | Uniform | | | Normal | | |
|---|---|---|---|---|---|---|---|---|
| | $\sigma$ | $\overline{\xi}$ | $\frac{Q^*}{Q^*_{LB}}$ | $\frac{P^*}{P^*_{LB}}$ | $\frac{\Pi(P^*,Q^*)}{\Pi(P^*_{LB},Q^*_{LB})}$ | $\frac{Q^*}{Q^*_{LB}}$ | $\frac{P^*}{P^*_{LB}}$ | $\frac{\Pi(P^*,Q^*)}{\Pi(P^*_{LB},Q^*_{LB})}$ |
| Avg. | 8.68 | 15.03 | 1.0048 | 1.0089 | 1.0020 | 0.9771 | 1.0085 | 1.0009 |
| Min. | 0.68 | 1.18 | 0.8893 | 1.0008 | 1.0000 | 0.8649 | 1.0007 | 1.0000 |
| Max. | 16.94 | 29.34 | 1.0548 | 1.0256 | 1.0181 | 0.9992 | 1.0243 | 1.0099 |
| Std. Err. | | 8.34 | 0.0287 | 0.0058 | 0.0028 | 0.0258 | 0.0056 | 0.0017 |

Table 3.3: The additive model and the standard newsvendor problem with shortage and holding costs

The numerical experiments extended to the multiplicative model of the same problem are reported in Table 3.4. Assuming that the demand randomness is uniformly distributed, the study shows that there exists a trend of under-stocking and over-pricing of about 2.45% and 0.20% respectively. The knowledge of distribution, i.e. EVAI, improves the revenue by about 0.48%. When the normal distribution is considered, the study shows similar trends of under-stocking and over-pricing of about 1.68% and 0.30% respectively. Also, EVAI improves the revenue by about 0.38%.

| | | | Uniform | | | Normal | | |
|---|---|---|---|---|---|---|---|---|
| | $\sigma$ | $\overline{\xi}$ | $\frac{Q^*}{Q^*_{LB}}$ | $\frac{P^*}{P^*_{LB}}$ | $\frac{\Pi(P^*,Q^*)}{\Pi(P^*_{LB},Q^*_{LB})}$ | $\frac{Q^*}{Q^*_{LB}}$ | $\frac{P^*}{P^*_{LB}}$ | $\frac{\Pi(P^*,Q^*)}{\Pi(P^*_{LB},Q^*_{LB})}$ |
| Avg. | 0.30 | 0.51 | 1.0245 | 0.9981 | 1.0048 | 1.0168 | 0.9970 | 1.0038 |
| Min. | 0.02 | 0.03 | 0.8186 | 0.9890 | 1.0000 | 0.8117 | 0.9887 | 1.0000 |
| Max. | 0.58 | 1.00 | 1.1163 | 1.0016 | 1.0593 | 1.0935 | 1.0009 | 1.0472 |
| Std. Err. | | 0.28 | 0.0583 | 0.0030 | 0.0080 | 0.0552 | 0.0031 | 0.0067 |

Table 3.4: The multiplicative model and the standard newsvendor problem with shortage and holding costs

## 3.4 Conclusions

In the context of RM, we presented a distribution free approach to solve the capacity and pricing problem jointly. The approach was used to develop deterministic lower bound for a stochastic optimization problem. The approach was first applied to standard newsvendor problem considering both the additive and multiplicative demand models. Next, the newsvendor problem with holding and shortage costs is also investigated. The developed lower bounds are shown to be quasi-concave, thus resulting in global optimal estimates on the capacity and the price, such that the revenue is maximized. The numerical experiments were performed using the two commonly observed distributions: uniform and normal. The results were conclusive that the distribution free approach results in near optimal estimates on the capacity and the price.

The work presented in this chapter has direct implications on the revenue management practice since it provides an integrated framework to jointly optimize the price and quantity of perishable assets using minimal information regarding the behavior of the customer demand. The use of distribution approach can be very beneficial for applications in RM. In many RM practices, the customer demand distribution is not known precisely. It would be of interest to management that the revenue under worst possible demand behavior is analyzed.

In the next chapter, we present several extensions to standard newsvendor problem and identify their applications into both the airline and retail industry RM.

# Chapter 4

# Distribution Free Approach for Pricing: Some Extensions with Applications

The work presented in the previous chapter is further elaborated to study the standard newsvendor problem with: shortage cost penalty; recourse cost, where there is a second purchasing opportunity; random yield case in which non-conforming items are considered; and finally, the capacity or the budget constraint. These extensions find applications in both the airline and retail RM practices. The recourse cost case could be analogous to reopening of a fare class, however; there is a need to incorporate appropriate modifications in the model when applied to airline RM. The random yield could also be applicable to solve the airline RM problem with a consideration of overbooking and no shows. The capacity constraint is a typical situation observed in the airline revenue management while the capacity is allocated either using nested or non-nested control on booking limits. Unlike previous study, only additive model is used to represent the stochastic demand.

## 4.1 Extension to Shortage Cost Penalty

A direct extension is the standard newsvendor problem with the shortage cost as penalty in the previous analysis. Thus the corresponding revenue function is:

$$\begin{aligned}
\Pi\,(P^{*},Q^{*}) &= \max_{P,Q} E_{\xi}\left\{P\min\{Q,\mathcal{D}\} - C\,Q - S\,[\mathcal{D}-Q]^{+}\right\} \\
&= (P+S-C)\,Q - S\,D - (P+S)E_{\xi}[Q-\mathcal{D}]^{+} \\
&= (P+S-C)\,Q - S\,D - (P+S)\int_{\underline{\xi}}^{Q-D} \Phi(\xi)\,d\xi \qquad (4.1)
\end{aligned}$$

The solution procedure suggested in Yao (2002) is followed, and the procedure first determines the optimal quantity (capacity) $Q^{*}$ for a given price $P$. Later $Q^{*}$ is substituted in Equation 4.1 to determine optimal price $P^{*}$. The $Q^{*}$ is determined using the first order optimality condition as follows:

$$Q^{*} = D + \Phi^{-1}\,(\varrho) \qquad (4.2)$$

where, $\varrho = \dfrac{P+S-C}{P+S}$

After substituting the optimal quantity $Q^{*}$ in Equation 4.1.

$$\begin{aligned}
\Pi\,(P^{*}) &= \max_{P}\,(P+S-C)\left(D+\Phi^{-1}(\varrho)\right) - (P+S)\int_{\underline{\xi}}^{\Phi^{-1}(\varrho)} \Phi(\xi)d\xi \\
&- SD \qquad (4.3)
\end{aligned}$$

The revenue function of standard newsvendor problem with both the shortage and holding cost is identified as quasi-concave in Yao (2002) following the assumption stated in standard newsvendor problem and some restrictions on holding cost which are not considered in this extension but will be discussed in a following case. In this thesis, the proof of quasi-concavity of the payoff function for standard newsvendor problem with shortage penalty is out of the scope as this research only investigates the distribution free approach. A numerical procedure, FMINBND available in MATLAB is used to minimize $-\Pi$, and it finds $P^*$ and $Q^*$ that maximize $\Pi$. The present case is a reduced version of aforementioned extended problem and thus may also be proved to be quasi-concave, however, the proof is not presented in this paper. The computation procedure to optimize $\Pi$ uses FMINBND as mentioned in an earlier section.

Using the distribution free approach, a LB on the revenue function with shortage cost penalty, $\Pi_{LB}\left(P_{LB}, Q_{LB}\right)$, for brevity $\Pi_{LB}$, is given in Equation 4.4.

$$\begin{aligned}\Pi_{LB}\left(P^*_{LB}, Q^*_{LB}\right) &= \max_{P_{LB}, Q_{LB}} \left(P+S-C\right)Q - \left(P+S\right)\frac{\left(\sigma^2+(D-Q)^2\right)^{1/2}-(D-Q)}{2} \\ &- \; SD \end{aligned} \tag{4.4}$$

Similar to the standard newsvendor problem, we prove the quasi-concavity of $\Pi_{LB}$. For $P_{LB} \in [C, \overline{P}]$ and $\theta = \sigma^2 + (Q_{LB} - D)^2$, it is evident from Equation 4.5 that $\dfrac{\partial^2 \Pi_{LB}}{\partial Q_{LB}^2} \leq 0$.

$$\frac{\partial^2 \Pi_{LB}}{\partial Q_{LB}^2} = -\frac{\sigma^2\left(S + P_{LB}\right)}{2\,\theta^{3/2}} \tag{4.5}$$

It is also shown that $\dfrac{\partial^2 \Pi_{LB}}{\partial P_{LB}^2} \leq 0$, for $\sigma \geq 0$, $(-D + Q_{LB} + \sqrt{\theta}) \geq 0$ and $D$ is linear and follows IPE.

$$\frac{\partial^2 \Pi_{LB}}{\partial P_{LB}^2} = \frac{D'\left(2\,\theta\left(-D + Q_{LB} + \sqrt{\theta}\right) - \sigma^2\left(S + P_{LB}\right)D'\right)}{2\,\theta^{3/2}} \tag{4.6}$$

Above analysis proves that $\Pi_{LB}$ is quasi-concave in one of the control variables while among two variables one is deterministically known. To prove $\Pi_{LB}$ is quasi-concave in both $P_{LB}$ and $Q_{LB}$, we further study the Hessian matrix, $\mathbf{H}$. In order to prove the quasi-concavity of $\Pi_{LB}$, we need to show $|\mathbf{H}| \geq 0$. Thus,

$$\frac{\partial^2 \Pi_{LB}}{\partial P_{LB}\, \partial Q_{LB}} = \frac{\theta\,(\,D + \sqrt{\theta} - Q_{LB}) + \sigma^2\,(S + P_{LB})\,D'}{2\,\theta^{3/2}} \tag{4.7}$$

Since, $|\mathbf{H}| = \dfrac{\partial^2 \Pi_{LB}}{\partial P_{LB}^2} \dfrac{\partial^2 \Pi_{LB}}{\partial Q_{LB}^2} - \left(\dfrac{\partial^2 \Pi_{LB}}{\partial P_{LB}\, \partial Q_{LB}}\right)^2$, therefore after simplification:

$$|\mathbf{H}| = -\frac{(S + P_{LB})\,D'\,\sigma^2}{\theta^{3/2}} - \frac{(\,D - Q_{LB} + \sqrt{\theta})^2}{4\,\theta} \tag{4.8}$$

$|\mathbf{H}| \geq 0$, given that: i) $Q_{LB} \geq D$; ii) $P_{LB} \geq S - \dfrac{\sigma}{4D'}$, and along with a condition that $D$ is linear and follows IPE. This proves the quasi-concavity of $\Pi_{LB}$. Two conditions are very similar to the two conditions established for standard newsvendor problem, except the addition of shortage cost, $S$ in the first condition. Again, FMINCON can be used with a modified bound on $P_{LB}$.

## 4.2 The Recourse Case

In some situations, there may exist an opportunity of placing a second order to satisfy the part of the demand not covered by the first order. If the observed random demand $\mathcal{D}$ is found greater than the ordered quantity $Q$, the second order of $\mathcal{D} - Q$ units can be placed. The second purchase cost, $\mathcal{C}$ is charged for per unit re-ordered. It is not uncommon practice to observe $\mathcal{C} > C$. Hence the expected revenue for a newsvendor problem with second order option is established as follows:

$$\begin{aligned} \Pi\,(P^*, Q^*) &= \max_{P,Q}\,(P - C)\,Q - P\,E_\xi\,[Q - \mathcal{D}]^+ - \mathcal{C}\,E_\xi\,[\mathcal{D} - Q]^+ \\ &= (P - C + \mathcal{C})Q - \mathcal{C}D - (P + \mathcal{C})E_\xi\,[Q - \mathcal{D}]^+ \\ &= (P - C + \mathcal{C})Q - \mathcal{C}D - (P + \mathcal{C}) \int_{\underline{\xi}}^{Q-D} \Phi(\xi)\,d\xi \end{aligned} \tag{4.9}$$

The first order optimality condition would result $Q^*$:

$$Q^* = D + \Phi^{-1}(\varrho) \tag{4.10}$$

where $\varrho = \dfrac{P + \mathcal{C} - C}{P + \mathcal{C}}$

$$\Pi(P^*, Q^*) \;=\; \max_P \,(P - C + \mathcal{C})(D + \Phi^{-1}(\varrho)) - \mathcal{C}D - (P + \mathcal{C}) \int_{\underline{\xi}}^{\Phi^{-1}(\varrho)} \Phi(\xi)\, d\xi$$

The computational procedure to obtain $P^*$ and $Q^*$ is the same as in the standard newsvendor problem extended to shortage cost penalty.

Extending the distribution free approach to revenue function described in Equation 4.11. Again using the aforementioned conditions we assume $P_{LB} \in [C, \overline{P}]$ and $\theta = \sigma^2 + (Q_{LB} - D)^2$. Next, we study the behavior of $\Pi_{LB}$. For a given price $P_{LB}$, the first order optimality condition, $\dfrac{\partial \Pi_{LB}}{\partial Q_{LB}} = 0$, results an optimal quantity $Q^*_{LB}$ . This quantity is also a unique maximizer of $\Pi_{LB}$, since the second order derivative can easily be shown negative i.e., $\dfrac{\partial^2 \Pi_{LB}}{\partial Q^2_{LB}} = -\dfrac{\sigma^2\,(\mathcal{C} + P_{LB})}{2\,\theta^{3/2}} \leq 0$. Likewise in previous analysis, for a given quantity $Q_{LB}$, we study $\dfrac{\partial^2 \Pi_{LB}}{\partial P^2_{LB}}$.

$$\frac{\partial^2 \Pi_{LB}}{\partial P^2_{LB}} = \frac{D'\left(2\left(-D + Q_{LB} + \sqrt{\theta}\right)\theta - \sigma^2\,(\mathcal{C} + P_{LB})\,D'\right)}{2\,\theta^{\frac{3}{2}}} \tag{4.11}$$

Given, $Q_{LB} \geq D$ and $D' < 0$, thus from Equation 4.11, it is evident that $\dfrac{\partial^2\,\Pi_{LB}}{\partial P^2_{LB}} \leq 0$. Considering the current analysis of $\Pi_{LB}$, we can conclude that when one of the decision variables is known (fixed) the $\Pi_{LB}$ is quasi-concave on the other variable. Next, to show $\Pi_{LB}$ is quasi-concave in both $P_{LB}$ and $Q_{LB}$, we study the Hessian matrix $\mathbf{H}$, as follows:

It is already identified that the first principal minors of $\mathbf{H}$ are negative. Now we show that the second principal minor is also non-negative. This will prove the concavity of $\Pi_{LB}$.

Here,

$$\frac{\partial^2 \Pi_{LB}}{\partial P_{LB}\, \partial Q_{LB}} = \frac{(D - Q_{LB})\ \theta + \theta^{\frac{3}{2}} + \sigma^2\ (\mathcal{C} + P_{LB})\ D'}{2\, \theta^{\frac{3}{2}}} \tag{4.12}$$

Thus,

$$|\mathbf{H}| = -\frac{(\,D - Q_{LB} + \sqrt{\theta})^2}{4\,\theta} - \frac{\sigma^2\,(\mathcal{C} + P_{LB})\ D'}{\theta^{3/2}} \tag{4.13}$$

Knowing the inequality, $\sqrt{\theta} \geq |Q_{LB} - D|$, as well as, $Q_{LB} \geq D$, it is concluded from Equation 3.19 that $|\mathbf{H}| \geq \mathbf{0}$ such that: i) $P_{LB} \geq -\frac{\sigma}{4\,D'}$; and ii) $Q_{LB} \geq D$, with an additional condition, $\mathcal{C} < P_{LB}$. The computational procedure again uses FMINCON with these newly established constraints.

## 4.3 The Random Yield Case

In this section, the goal is to determine the optimal pricing and quantity using distribution free approach for a manufacturing firm such that the worst possible revenue is maximized. A decision to produce $Q$ items would result $G(Q)$ good items, where $G(Q)$ is a random variable. A simplification assumption would be to consider the same probability for an item to be good, and it follows binomial distribution. Let us assume the probability for an item being good to be $\rho$, and follows binomial distribution. Then the expected (mean) number of good items would be $\rho\, Q$ with corresponding variance $\rho\,(1-\rho)\,Q$. The revenue function in this case becomes:

$$\begin{aligned}
\Pi(P^*, Q^*) &= \max_{P,Q}\ P E_\xi\left[\min\{G(Q), \mathcal{D}\}\right] - C\,Q \\
&= (\rho P - C)\,Q - P E_\xi\left[G(Q) - \mathcal{D}\right]^+ \\
&= (\rho P - C)\,Q - P \int_{\underline{\xi}}^{\rho\,Q - D} \Phi(\xi)\, d\,\xi
\end{aligned} \tag{4.14}$$

The optimal quantity $Q^*$ is:

$$Q^* = \rho(D + \Phi^{-1}(\varrho)) \tag{4.15}$$

where $\varrho = \dfrac{\rho P - C}{\rho P}$, and revenue function is reduced to:

$$\Pi(P^*) = \max_P \rho\,(\rho P - C)\,(D + \Phi^{-1}(\varrho)) - P \int_{\underline{\xi}}^{\rho^2(D+\Phi^{-1}(\varrho))-D} \Phi(\xi)\,d\,\xi \tag{4.16}$$

The proof of quasi-concavity of $\Pi$ is not presented, although the computational procedure to determine $P^*$ and $Q^*$ remains the same as mentioned in previous cases.

Now a LB estimate on revenue, $\Pi_{LB}$, by using the distribution free approach is proposed:

$$E_\xi\,[G(Q) - \mathcal{D}]^+ \leq \frac{(\sigma^2 + Q\rho(1-\rho) + (Q\rho - D)^2)^{1/2} + (\rho Q - D)}{2} \tag{4.17}$$

Substituting the distribution free estimate of $E_\xi\,[G(Q) - \mathcal{D}]^+$ into Equation 4.11, resulting:

$$\begin{aligned} \Pi_{LB}(P_{LB}, Q_{LB}) &= \rho P_{LB} Q_{LB} - P_{LB} \frac{(\sigma^2 + \rho\, Q_{LB}(1-\rho) + (\rho\, Q_{LB} - D)^2)^{1/2} + (\rho\, Q_{LB} - D)}{2} \\ &- C Q_{LB} \end{aligned} \tag{4.18}$$

Now we study for the quasi-concavity of $\Pi$ established in Equation 4.18:

$$\frac{\partial^2 \Pi_{LB}}{\partial Q_{LB}^2} = \frac{\rho^2\,((\rho-1)\,(4\ D + \rho - 1) - 4\,\sigma^2)\,P_{LB}}{8\,\theta^{3/2}} \tag{4.19}$$

Where $\theta = \sigma^2 + (D - \rho\, Q_{LB})^2 - (\rho - 1)\,\rho\, Q_{LB}$. It is easy to show in Equation 4.19 that $\dfrac{\partial^2 \Pi_{LB}}{\partial Q_{LB}^2} \leq 0$, for $\sigma \geq \frac{1}{2}$. Furthermore, the partial second derivative of $\Pi_{LB}$ with respect to $P_{LB}$ is presented in the following equation.

$$\frac{\partial^2 \Pi_{LB}}{\partial P_{LB}^2} = \frac{D'\,(2\,\theta\,(-D + \rho\, Q_{LB} + \sqrt{\theta}) - P_{LB}\,(\sigma^2 - (\rho-1)\,\rho\, Q_{LB})\,D')}{2\,\theta^{3/2}} \tag{4.20}$$

From Equation 4.20, for $Q_{LB} \geq \frac{D}{\rho}$ and $D$ is IPE, we can easily show that $\frac{\partial^2 \Pi_{LB}}{\partial P_{LB}^2} \leq 0$,

Now, we further study the Hessian matrix $\mathbf{H}$ to investigate the quasi-concavity of $\Pi_{LB}$ in price $P_{LB}$ and quantity $Q_{LB}$. The determinant of $\mathbf{H}$ is determined as follows:

$$\frac{\partial^2 \Pi_{LB}}{\partial P_{LB}\, \partial Q_{LB}} = \frac{1}{4}\rho \left(\frac{2\ D + \rho - 2\,\rho\, Q_{LB} - 1}{\sqrt{\theta}} + \frac{P_{LB}\,(2\,\sigma^2 - (\rho-1)\,(D + \rho\, Q_{LB}))\, D'}{\theta^{3/2}} + 2\right) \tag{4.21}$$

Also

$$\begin{aligned}
|\mathbf{H}| \;=\;& \\
& -\frac{1}{16\,\theta^2} \\
& (\rho^2\,(\theta\,(2\ D + \rho + 2\,\sqrt{\theta} - 1)^2 + (\rho-1)^2\, P_{LB}^2\,(D')^2 - \\
& \quad 4\,\theta\,\rho\, Q_{LB}\,(2\ D + \rho - \rho\, Q_{LB} + 2\,\sqrt{\theta} - 1) - \\
& \quad 2\, P_{LB} \\
& \qquad (-2\,(\rho + 4\,\sqrt{\theta} - 1)\,\sigma^2 + (-2\, D^2 + 6\,\sqrt{\theta}\ D + \sqrt{\theta}\,(\rho-1))\,(\rho-1) + \\
& \qquad\quad 2\,(\rho-1)\,\rho\, Q_{LB}\,(2\ D + \rho - \rho\, Q_{LB} + \sqrt{\theta} - 1))\ D'))
\end{aligned} \tag{4.22}$$

Where from Equation 4.22, we may conclude that a further investigation of $\mathbf{H}$, such that $\mathbf{H} \geq \mathbf{0}$, is a prohibiting task. Therefore, the prove which may guarantee, $\Pi_{LB}$ is quasi-concave in $P_{LB}$ and $Q_{LB}$ is not presented. Previously used computational method are also suggested for this case study, however, these methods unlike previous cases do not guarantee a global optimal solution. The $\Pi_{LB}$ will be optimized subject to a linear constraint $\rho\, Q \geq D$, with two other aforementioned assumptions: (i)$\sigma \geq \frac{1}{2}$; and (ii) linear IPE riskless demand $D$. A build-in function in MATLAB, FMINCON is used to solve the problem.

## 4.4 The Multiple Product Case

Now we consider multiple products pricing problem subject to budget constraint. The problem can also be analyzed with additional constraints such as capacity. The problem with only capacity constraint is a typical airline RM problem with a joint seat allocation and fare pricing control without nested seat inventory control (Weatherford and Bodily (1992)). Furthermore, the cost of acquiring the seat inventory is ignored. In the current analysis, $n$ items are assumed, for each item $i$, the quantity $Q_i$ must be manufactured or purchased with a unit cost $C_i$. $P_i$ would be the price of an item $i$. This is the case of RM with limited amount of budget. The constraint on budget results a competition among items for this scare resource. In the context of manufacturing cost minimization while producing multiple commodities competing for a budget allocation, is entitled as stochastic product mix problem by Johnson and Montgomery (1974).

The problem is written as:

$$\Pi\left(\mathbf{P}^{*}, \mathbf{Q}^{*}\right) = \max_{\mathbf{P},\mathbf{Q}} \sum_{i=1}^{n} E_{\xi_i}\left[P_i \min\{Q_i, \mathcal{D}_i\} - C_i\, Q_i\right] \tag{4.23}$$

Subject to:

$$\sum_{i=1}^{n} C_i\, Q_i \leq B \tag{4.24}$$

Where, $\mathbf{P} = (P_1, P_2, \cdots, P_n)$, and $\mathbf{Q} = (Q_1, Q_2, \cdots, Q_n)$.

This is a constrained non-linear optimization problem with quasi-concave objective function and a single linear constraint, as identified earlier there exist a global optimal solution to this problem. A solution procedure would be to jointly optimize $\mathbf{P}$ and $\mathbf{Q}$ using standard non-linear optimization methods. This study uses FMINCON, the search bounds would be $P_{LB_i} \in [\underline{P}_i, \overline{P}_i]$ and $Q_{LB_i} \in [0, D_{P_i=\underline{P}_i} + \overline{\xi}]$, $\forall\ i = \{1, 2, \cdots, n\}$.

The distribution free approach to aforementioned problem with budget constraint is:

$$\Pi\left(\mathbf{P}^*_{\mathbf{LB}}, \mathbf{Q}^*_{\mathbf{LB}}\right) = \max_{\mathbf{P_{LB}}, \mathbf{Q_{LB}}} \sum_{i=1}^{n} (P_{LB_i} - C_i)\, Q_{LB_i} - P_{LB_i} \frac{\left(\sigma_i^2 + (D_i - Q_{LB_i})^2\right)^{1/2} - (D_i - Q_{LB_i})}{2}$$

Subject to:

$$\sum_{i=1}^{n} C_i\, Q_{LB_i} \leq B \tag{4.25}$$

Where, $\mathbf{P_{LB}} = (P_{LB_1}, P_{LB_2}, \cdots, P_{LB_n})$, and $\mathbf{Q_{LB}} = (Q_{LB_1}, Q_{LB_2}, \cdots, Q_{LB_n})$. The objective function is summation of payoff resulted from each item $i$, $\forall i = \{1, 2, \cdots, n\}$,

According to a theorem stated in Winston (2004), the global optimal control $\mathbf{P}^*_{\mathbf{LB}}$ and $\mathbf{Q}^*_{\mathbf{LB}}$, is found when additional linear constraints, $Q_{LB_i} \geq D_i$, $\forall i = \{1, 2, \cdots, n\}$, are also considered with a constraint identified in Equation 4.25.

The bound on $P_{LB_i}$ would also be modified to $P_{LB_i} \in \left[\max\left(C_i, -\frac{\sigma_i}{4D_i'}\right), \overline{P}_i\right] \forall\, i = \{1, 2, \cdots, n\}$. Again FMINCON is suggested for this constrained non-linear optimization.

## 4.5 Numerical Analysis

In this section, numerical results are presented to demonstrate the performance of the proposed distribution free approach on standard newsvendor problem and its extensions. In the experimentation, a linear riskless demand function, $D = \alpha - \beta\, P_{LB}$ is assumed. The study is reported based on over 100 randomly generated problems. The random problem generation scheme follow uniform distribution such that, $C \sim \mathrm{U}[20, 100]$, $\alpha \sim \mathrm{U}[100, 200]$ and $\beta \sim \mathrm{U}[0.05, 0.3]$. The market randomness, $\xi$ for the additive model is bounded in $[-\psi, \psi]$ with $\psi \sim \mathrm{U}(0, 30]$. The analysis mainly focuses two aspect: i) Performance evaluation of distribution free approach on standard newsvendor problem and its suggested extensions; and ii) The impact study of suggested extensions compared to standard newsvendor problem.

### 4.5.1 Extension to Shortage Cost Penalty

The numerical study is also performed with standard newsvendor problem extended to shortage cost penalty. The same data set tested on the standard newsvendor problem is also considered in here. The new parameter, shortage cost is generated such that $S \sim$ U$[0.1, 0.2] \times$ C. The results with 100 randomly generated instances are summarized in Table 4.1. There is a slightly small trend of over-stocking when the outcomes of distribution free approach are compared with distribution based approach using uniform and normal distributions. Again there is a trend of over-pricing when using distribution free estimates. The deviation is about .85% and 0.80% for uniform and normal distribution respectively. EVAI assuming uniform and normal distribution is about 0.26% and 0.14% respectively.

| | | | Uniform | | | Normal | | |
|---|---|---|---|---|---|---|---|---|
| | $\sigma$ | $\overline{\xi}$ | $\frac{Q^*}{Q^*_{LB}}$ | $\frac{P^*}{P^*_{LB}}$ | $\frac{\Pi(P^*,Q^*)}{\Pi(P^*_{LB},Q^*_{LB})}$ | $\frac{Q^*}{Q^*_{LB}}$ | $\frac{P^*}{P^*_{LB}}$ | $\frac{\Pi(P^*,Q^*)}{\Pi(P^*_{LB},Q^*_{LB})}$ |
| Avg. | 8.68 | 15.03 | 0.9967 | 1.0085 | 1.0026 | 0.9698 | 1.0080 | 1.0014 |
| Min. | 0.68 | 1.18 | 0.8571 | 1.0007 | 1.0000 | 0.8368 | 1.0007 | 1.0000 |
| Max. | 16.94 | 29.34 | 1.0573 | 1.0245 | 1.0342 | 0.9992 | 1.0235 | 1.0129 |
| Std Err. | | 8.34 | 0.0348 | 0.0056 | 0.0051 | 0.0318 | 0.0053 | 0.0023 |

Table 4.1: Extension to shortage cost penalty

### 4.5.2 The Recourse Case

In the numerical experimentation continued using standard newsvendor problem data, the recourse cost, $\mathcal{C}$ is generated by following $\mathcal{C} \sim$ U$[1.2, 1.4] \times$ C. The findings of numerical study are summarized in Table 4.2. Similar trends of over-stocking and under-pricing are observed as noticed earlier in the case of the standard newsvendor problem extended to shortage penalty cost. While the EVAI improvement is about 0.26% and 0.14% for uniform and normal distributions respectively which is similar to the shortage penalty cost case.

| | | | Uniform | | | Normal | | |
|---|---|---|---|---|---|---|---|---|
| | $\sigma$ | $\overline{\xi}$ | $\frac{Q^*}{Q^*_{LB}}$ | $\frac{P^*}{P^*_{LB}}$ | $\frac{\Pi(P^*,Q^*)}{\Pi(P^*_{LB},Q^*_{LB})}$ | $\frac{Q^*}{Q^*_{LB}}$ | $\frac{P^*}{P^*_{LB}}$ | $\frac{\Pi(P^*,Q^*)}{\Pi(P^*_{LB},Q^*_{LB})}$ |
| Avg. | 8.68 | 15.03 | 0.9914 | 1.0081 | 1.0029 | 0.9650 | 1.0077 | 1.0017 |
| Min. | 0.68 | 1.18 | 0.8512 | 1.0007 | 1.0000 | 0.8316 | 1.0007 | 1.0000 |
| Max. | 16.94 | 29.34 | 1.0550 | 1.0234 | 1.0395 | 0.9987 | 1.0221 | 1.0145 |
| Std Err. | | 8.34 | 0.0348 | 0.0053 | 0.0060 | 0.0327 | 0.0050 | 0.0026 |

Table 4.2: The recourse case

### 4.5.3 The Random Yield Case

In the numerical experimentation along with the parameters of standard newsvendor problem, the probability of an item being defective, $\rho$ is generated such that $\rho \sim$ U$[0.5, 0.9]$. The results of the numerical study are summarized in Table 4.3 show trends of over-stocking and under-pricing. The EVAI improvement when demand randomness follows uniform and normal distributions are 0.42% and 0.22% respectively.

| | | | Uniform | | | Normal | | |
|---|---|---|---|---|---|---|---|---|
| | $\sigma$ | $\overline{\xi}$ | $\frac{Q^*}{Q^*_{LB}}$ | $\frac{P^*}{P^*_{LB}}$ | $\frac{\Pi(P^*,Q^*)}{\Pi(P^*_{LB},Q^*_{LB})}$ | $\frac{Q^*}{Q^*_{LB}}$ | $\frac{P^*}{P^*_{LB}}$ | $\frac{\Pi(P^*,Q^*)}{\Pi(P^*_{LB},Q^*_{LB})}$ |
| Avg. | 8.68 | 15.03 | 0.9947 | 1.0080 | 1.0042 | 0.9673 | 1.0076 | 1.0022 |
| Min. | 0.68 | 1.18 | 0.8845 | 0.9972 | 1.0000 | 0.8593 | 0.9970 | 1.0001 |
| Max. | 16.94 | 29.34 | 1.0552 | 1.0231 | 1.0446 | 1.0049 | 1.0236 | 1.0108 |
| Std Err. | | 8.34 | 0.0346 | 0.0061 | 0.0074 | 0.0274 | 0.0060 | 0.0027 |

Table 4.3: The random yield case

### 4.5.4 The Multiple Product Case

The numerical study considers two products with the same parameter generation scheme for each product as reported in the Section 4.5 and used for single product case in the standard newsvendor problem. An additional parameter, the budget $B$, is generated following a uniform distribution such that $B \sim \text{U}[10000, 15000]$.

| | $\frac{Q_1^*}{Q_{LB_1}^*}$ | $\frac{P_1^*}{P_{LB_1}^*}$ | $\frac{Q_2^*}{Q_{LB_2}^*}$ | $\frac{P_2^*}{P_{LB_2}^*}$ | $\frac{\Pi(\mathbf{P}^*,\mathbf{Q}^*)}{\Pi(\mathbf{P}_{\mathbf{LB}}^*,\mathbf{Q}_{\mathbf{LB}}^*)}$ |
|---|---|---|---|---|---|
| Avg. | 0.9895 | 1.0088 | 0.9910 | 1.0095 | 1.0051 |
| Min. | 0.8555 | 0.9926 | 0.7546 | 0.9772 | 1.0001 |
| Max. | 1.0601 | 1.0385 | 1.0975 | 1.0276 | 1.1475 |
| Std Err. | 0.0400 | 0.0078 | 0.0441 | 0.0074 | 0.0158 |

Table 4.4: Uniform Distribution two product case

| | $\frac{Q_1^*}{Q_{LB_1}^*}$ | $\frac{P_1^*}{P_{LB_1}^*}$ | $\frac{Q_2^*}{Q_{LB_2}^*}$ | $\frac{P_2^*}{P_{LB_2}^*}$ | $\frac{\Pi(\mathbf{P}^*,\mathbf{Q}^*)}{\Pi(\mathbf{P}_{\mathbf{LB}}^*,\mathbf{Q}_{\mathbf{LB}}^*)}$ |
|---|---|---|---|---|---|
| Avg. | 0.9694 | 1.0064 | 0.9685 | 1.0070 | 1.0020 |
| Min. | 0.8433 | 0.9914 | 0.7461 | 0.9892 | 1.0000 |
| Max. | 1.0249 | 1.0237 | 1.0714 | 1.0224 | 1.0197 |
| Std Err. | 0.0398 | 0.0065 | 0.0427 | 0.0062 | 0.0029 |

Table 4.5: Normal distribution two product case

| | Distribution Free | Uniform | Normal |
|---|---|---|---|
| Avg. | 0.7675 | 0.7623 | 0.7466 |
| Min. | 0.2961 | 0.2992 | 0.2907 |
| Max. | 1.0000 | 1.0000 | 1.0000 |
| Std Err. | 0.1936 | 0.1967 | 0.1962 |

Table 4.6: Budget utilization

In Tables 4.4-4.5, the computational experience with budget constraint is reported while assuming uniform and normal distribution respectively. When information about the distribution is not known, there is a trend of over-stocking and under-pricing. EVAI in the case of uniform distribution is about 0.51% and for the case of normal distribution this value is about 0.20%. In Table 4.6, the budget utilization ratio is reported. The trend of over-stocking while using distribution free approach results in highest budget utilization when compared with uniform and normal distribution, however this increase is not substantial.

## 4.6 A Statistical Comparative Study

A comparative study is presented using statistical tools. The purpose of the study is to calibrate the difference between standard newsvendor problem and its extensions. The analysis uses paired comparison to analyze the impact of various extension studied on standard newsvendor problem with respect to payoff, quantity and pricing. The paired comparisons are identified in Table 4.7. The proposed statistical analysis is based on t-test. The results are briefly discussed as follows: In Table 4.8, the outcome of quantity comparison is presented. The analysis considers a distribution free approach, random demand following uniform and normal distribution. As it is revealed in the table that all quantity comparisons are significant at 5% level. The mean reported in the table represents percent difference of the comparison. A negative (positive) sign in the mean stands for percentage of over-stocking (under-stocking) for the extended problems compared to standard newsvendor problem. The numerical experimentation provides an statistical evidence, there is significant over-stocking in the case of random yield. Also there is substantial amount of under-stocking while the random demand follows uniform distribution. In each case study, the trends in pricing and quantity allocation along with revenue variation for distribution free and normal distribution are closely comparable. A similar analysis with pricing is reported in Table 4.9, again negative (positive) sign in the mean difference

| Paired Difference | Label |
|---|---|
| Standard Newsvendor problem - Extension to shortage cost penalty | A |
| Standard Newsvendor problem - Extension to shortage and holding cost | B |
| Standard Newsvendor problem - Extension to recourse case | C |
| Standard Newsvendor problem - Extension to random yield case | D |

Table 4.7: Comparison labels

| | | Paired Differences | | | | t | df | Sig. |
|---|---|---|---|---|---|---|---|---|
| | | Mean | Std. | 95% CI | | | | (2-tailed) |
| | Comparison | | Error Mean | Lower | Upper | | | |
| Distribution Free | A | -0.169 | 0.013 | -0.194 | -0.144 | -13.437 | 99 | 0 |
| | B | 1.490 | 0.086 | 1.319 | 1.660 | 17.315 | 99 | 0 |
| | C | -1.373 | 0.105 | -1.581 | -1.166 | -13.117 | 99 | 0 |
| | D | -37.69 | 1.92 | -41.51 | -33.87 | -19.58 | 99 | 0 |
| Uniform Distribution | A | 22.867 | 2.513 | 17.880 | 27.854 | 9.098 | 99 | 0 |
| | B | 23.524 | 2.496 | 18.572 | 28.476 | 9.426 | 99 | 0 |
| | C | 22.370 | 2.524 | 17.362 | 27.379 | 8.862 | 99 | 0 |
| | D | -57.76 | 2.68 | -63.08 | -52.43 | -21.52 | 99 | 0 |
| Normal Distribution | A | -0.107 | 0.011 | -0.128 | -0.086 | -9.921 | 99 | 0 |
| | B | 0.814 | 0.050 | 0.714 | 0.914 | 16.142 | 99 | 0 |
| | C | -0.810 | 0.085 | -0.979 | -0.642 | -9.533 | 99 | 0 |
| | D | -37.44 | 1.95 | -41.31 | -33.57 | -19.20 | 99 | 0 |

Table 4.8: Quantity comparison

describes over-pricing (under-pricing). All comparisons reported in the table are found significant. For the case of random yield, there is a substantial over-pricing of over 2.75% in both distribution free and distribution based approaches. A statistical study with the revenue is reported in Table 4.10. Compared to standard newsvendor problem, in all extensions, the revenue decreases significantly. The most substantial decrease is observed in the case of random yield which is more than 7%.

| | Comparison | Paired Differences Mean | Std. Error Mean | 95% CI Lower | 95% CI Upper | t | df | Sig. (2-tailed) |
|---|---|---|---|---|---|---|---|---|
| Distribution Free | A | -0.014 | 0.002 | -0.018 | -0.011 | -8.896 | 99 | 0 |
| | B | 0.107 | 0.008 | 0.092 | 0.123 | 14.199 | 99 | 0 |
| | C | -0.104 | 0.012 | -0.128 | -0.081 | -8.720 | 99 | 0 |
| | D | -2.80 | 0.23 | -3.25 | -2.35 | -12.24 | 99 | 0 |
| Uniform Distribution | A | -0.010 | 0.002 | -0.014 | -0.007 | -6.345 | 99 | 0 |
| | B | 0.069 | 0.008 | 0.054 | 0.085 | 8.900 | 99 | 0 |
| | C | -0.068 | 0.011 | -0.089 | -0.046 | -6.171 | 99 | 0 |
| | D | -2.75 | 0.22 | -3.18 | -2.31 | -12.52 | 99 | 0 |
| Normal Distribution | A | -0.009 | 0.001 | -0.012 | -0.007 | -7.170 | 99 | 0 |
| | B | 0.066 | 0.006 | 0.054 | 0.078 | 10.559 | 99 | 0 |
| | C | -0.065 | 0.009 | -0.083 | -0.047 | -7.000 | 99 | 0 |
| | D | -2.75 | 0.22 | -3.19 | -2.31 | -12.37 | 99 | 0 |

Table 4.9: Pricing comparison

| | | Paired Differences | | | | t | df | Sig. |
|---|---|---|---|---|---|---|---|---|
| | | Mean | Std. | 95% CI | | | | (2-tailed) |
| | Comparison | | Error Mean | Lower | Upper | | | |
| Uniform Distribution | A | 0.017 | 0.003 | 0.011 | 0.024 | 5.476 | 99 | 0 |
| | B | 0.628 | 0.057 | 0.515 | 0.741 | 11.007 | 99 | 0 |
| | C | 0.127 | 0.025 | 0.077 | 0.176 | 5.105 | 99 | 0 |
| | D | 8.06 | 0.67 | 6.74 | 9.39 | 12.05 | 99 | 0 |
| Normal Distribution | A | 0.020 | 0.003 | 0.013 | 0.026 | 6.178 | 99 | 0 |
| | B | 0.489 | 0.042 | 0.405 | 0.573 | 11.541 | 99 | 0 |
| | C | 0.150 | 0.026 | 0.098 | 0.202 | 5.735 | 99 | 0 |
| | D | 7.76 | 0.64 | 6.49 | 9.04 | 12.09 | 99 | 0 |

Table 4.10: Payoff comparison

## 4.7 Conclusions

A distribution free approach is presented for joint determination of pricing and capacity in the context of revenue management. The approach is used to develop lower bound estimate on revenue and is first addressed on standard newsvendor problem. Later other extensions such as: i) shortage cost penalty, ii) shortage and holding cost, iii) recourse cost, and vi) random yield case are studied. Furthermore, the problem when multiple items are competing for a budget allocation is also discussed. For each case, the distribution free approach provides a lower bound quasi-concave in both pricing and capacity. This results a global optimal control on capacity and pricing for most of the case studies presented in this work such that the revenue is maximized. Numerical experiments are performed with the two most commonly observed viz, uniform and normal distributions. Results show that the distribution free approach results near optimal estimates on capacity and pricing. The work presented in this chapter has impact on revenue management practice as it provides an integrated framework to jointly optimize the price and quantity of perishable assets using minimal information about the behavior of customer demand.

Future work might be to further investigate this approach with an application to industries where revenue management is mostly practiced such airlines, hotels, car rentals, cruise liners etc. Another area is to extend a new version of newsvendor pricing problem which is to incorporate the setup cost. A research can also be to assume some more realistic demand behaviors such as resaleable return option which is most commonly observed in retailing industries. Moreover, the distribution free approach can also be used for competitive pricing in the context of multiple firms.

# Chapter 5

# Conclusions and Research Agenda

In this chapter, the summary of this research is presented for each of the three parts. The usefulness of the work is identified in the current RM literature. Finally, a research agenda is presented to enumerate the possibilities of the future research opportunities.

## 5.1 Summary: Fare Pricing Competition

A price-based RM is studied considering the fare pricing competition. The study determines competitive fare prices for the competing airlines in duopoly environment. With the assumption that the seat allocation is predetermined and known to airlines, the game reduces to fare pricing game. We show the uniqueness of Nash equilibrium for the game and determine expressions for fare pricing at Nash equilibrium. The customer demand is assumed to be price sensitive and both the deterministic and stochastic demands are considered. A sensitivity analysis is also reported with the expressions for unique fare prices for competing airline in each fare class under various demand possibilities.

The analysis presented is this part can be extended into several directions. A possible direction of research is the multi-period pricing problem. In this case, each retailer/airline makes a sequence of decisions and the decision of one period effects all of the following periods. A sequential game may be used to analyze this multi-period decision process. Another extension would be to incorporate the overbooking into the fare pricing model.

## 5.2 Summary: Fare Pricing Competition with Seat Inventory Control

In this research, the practice of price-based and quantity-based revenue management are merged to developed an integrated framework that jointly controls the fare pricing and seat inventory control. The integrated framework also assumes market competition. We developed two models based on additive and multiplicative random demand for the joint determination of competitive fare pricing and seat inventory control in a single flight leg. Both the non cooperative and cooperative bargaining games are studied. The cooperative bargaining game is further classified into bargaining with no side payments options and side payment options. For non cooperative game fare prices at Nash equilibrium along with booking limits are determined numerically. In the case cooperative bargaining games, the Nash bargaining solution is computed. A numerical study is reported and statistical evidence that fare pricing improves the revenue gain of competing airlines is established. A cooperation with the side payments option further improves revenue over the no side payment option. A statistical design of experiments is also carried out to determine the sensitivity of various modeling parameters over revenue gain, fare pricing and seat inventory control of each of competing airline under aforementioned cooperative and non cooperative games.

Future work may include, the possibility to incorporate the overbooking; this work does not consider the dynamic version of the fare price competition in either a single leg or a flight network setting. We would like to manifest the potential benefits of dynamically updating fare prices in today's competitive environment. Neuro-Dynamic programming is a good tool to study the dynamic version of the fare pricing competition in a single flight leg as well as in flight network settings. This research only studies the horizontal competition. A suggested extension can be to consider the combination of horizontal and vertical competition in a stochastic network. The proposed research would be an extension to Netessine and Shumsky's (2005) work.

## 5.3 Summary: Distribution Free Approach for Pricing in Revenue Management

A distribution free approach is used to develop lower bound estimate on revenue in a monopolistic situation. The use of the approach also enables the joint control of pricing and capacity that maximizes the revenue against the worst possible distribution. The most commonly observed problem in revenue management, standard newsvendor problem is considered. Later an extension to standard newsvendor problem which also includes holding and shortage cost is studied. Both the additive and multiplicative approaches for the random demand are assumed. The lower bound estimate on revenue is shown to be quasi-concave for standard newsvendor problem and its extended study while assuming both the additive and multiplicative modeling approach. In a numerical study, we also show that the revenue generated using distribution free approach is near optimal.

The approach is also tested on several other extensions while only using additive modeling approach. The extensions include: standard newsvendor problem with shortage cost penalty; recourse cost case, where there is a second purchasing opportunity; random yield case, in which non conforming products are considered; and multiple product case. In the most case studies, it is shown that the lower bound estimate on revenue is quasi-concave. In numerical experimentation, it is also discover the lower bound estimate of revenue is quite comparable with the optimal revenue estimate while assuming standard distributions for the random demand. Furthermore, in a statistical paired comparison study, a statistical evidence is established that the outcomes of standard newsvendor problem are significantly different from its aforementioned extensions.

Several extensions could be of interest. A suggested future study may be to incorporate the resaleable return and investigate the problem of joint control of pricing and capacity using distribution free approach. This problem is commonly observed in retail revenue management. The distribution free approach for pricing in revenue management may also consider balking.

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