Simultaneous LQ Control of a Set of LTI Systems using Constrained Generalized Sampled-Data Hold Functions

Javad Lavaei and Amir G. Aghdam

Department of Electrical and Computer Engineering, Concordia University
Montréal, QC Canada H3G 1M8
{j_lavaei, aghdam}@ece.concordia.ca

Abstract

In this paper, sampled-data control of a set of continuous-time LTI systems is considered. It is assumed that a predefined guaranteed continuous-time quadratic cost function, which is, in fact, the sum of the performance indices for all systems, is given. The main objective here is to design a decentralized periodic output feedback controller with a prespecified form, e.g., polynomial, piecewise constant, exponential, etc., which minimizes the above mentioned guaranteed cost function. This problem is first formulated as a set of matrix inequalities, and then by using a well-known technique, it is reformulated as a LMI problem. The set of linear matrix inequalities obtained provides necessary and sufficient conditions for the existence of a decentralized optimal simultaneous stabilizing controller with the prespecified form (rather than a general form). Moreover, an algorithm is presented to solve the resultant LMI problem. Finally, the efficiency of the proposed method is demonstrated in two numerical examples.

Key words: Simultaneous stabilization, H_2 optimal control, Generalized sampled-data hold function, Decentralized, LMI

1 Introduction

There has been a considerable amount of interest in the past several years towards control of continuous-time systems by means of periodic feedback, or so-called generalized sampled-data hold functions (GSHF). The idea of using GSHF instead of a simple zero-order hold (ZOH) in control systems was first introduced in Chammas and Leondes (1978). Kabamba investigated several applications and properties of GSHF in control systems (Kabamba, 1987; Kabamba and Yang, 1991). He showed that many of the advantages of state feedback controllers, without the requirement of using state estimation procedures, can be obtained by using a GSHF. A comprehensive frequency-domain analysis was presented by Feuer and Goodwin to examine robustness, sensitivity, and intersample effect of GSHF (Feuer and Goodwin, 1994).

Simultaneous stabilization of a set of systems, on the other hand, is of special interest in the control literature (Fonte, Zasadzinski, and Bernier-Kazantsev, 2001), and has applications in the following problems:

- A system which is desired to be stabilized by a fixed controller in different modes of operations, e.g., failure mode.
- A nonlinear plant which is linearized at several equilibria.
- A system which is desired to be stabilized in presence of uncertainties in its parameters.

Despite numerous efforts made to solve the simultaneous stabilization problem, it still remains an open problem. In the special case, when there are only two plants, the problem is completely solved in Youla, Bongiorno, and Lu (1974); Vidyasagar and Viswanadham (1982), and for the case of three and four plants, some necessary and sufficient relations in the form of polynomial are presented in Jia and Ackermann (2001). However, no necessary and sufficient condition has been obtained for simultaneous stabilization of more than four plants, so far. Moreover, it is proved in Blondel and Gevers (1993) that if the number of plants is more than two, then the problem is rationally undecidable. It is also shown in Toker and Ozbay (1995) that the problem is NP-hard. These results clearly demonstrate complexity level of the problem. Since there does not exist any LTI simultane-

[1] This work has been supported by the Natural Sciences and Engineering Research Council of Canada under grant RGPIN-262127-03.

ous stabilizing controller in many cases, a time-varying controller is considered in Miller and Rossi (2001). It is shown that for any set of stabilizable and observable plants, there exists a time-varying controller consisting of a sampler, a ZOH, and a time-varying discrete-time compensator which not only stabilizes all of the plants, but also acts as a near-optimal controller for each plant. This result points to the usefulness of sampling in simultaneous stabilization problem. Nevertheless, fast sampling requirement and large control gain are the drawbacks of this approach.

Stabilizing a set of plants simultaneously by means of a periodic controller is also investigated in the literature (Kabamba and Yang, 1991; Tarn and Yang, 1988). A method is proposed in Kabamba and Yang (1991) to not only minimize a guaranteed cost function corresponding to all of the systems, but also accomplish desired pole placement. The drawback, however, is that the problem is formulated as a two-boundary point differential equation whose analytical solution is cumbersome, in general. Some algorithms are proposed to solve the resultant differential equation numerically, in the particular case of only one LTI system, which is no longer a simultaneous stabilization problem (Hyslop, Schattler, and Tarn, 1992; Werner, 1996). Design of a high-performance simultaneous stabilizer in the form of a piecewise constant GSHF is investigated in Cao and Lam (2001).

On the other hand, it is not realistic in many practical problems to assume that all of the outputs of a system are available to construct any particular input of the system. In other words, it is often desired to have some form of decentralization. Problems of this kind appear, for example, in electric power systems, communication networks, large space structures, robotic systems, economic systems and traffic networks, to name only a few.

This paper deals with the problem of simultaneous stabilization of a set of systems by means of a decentralized periodic controller. It is assumed that a discrete-time decentralized compensator is given for a set of detectable and stabilizable LTI systems. This compensator is employed to simplify the simultaneous stabilizer design problem. In certain cases, however, the problem may not be solvable without using a proper compensator (e.g., in presence of unstable fixed modes (Davison and Chang, 1990)). The objective here is to design a GSHF which satisfies the following constraints:

i) The GSHF along with the discrete-time compensator simultaneously stabilize the plants.
ii) It has the desired decentralized structure.
iii) It has a prespecified form such as polynomial, piecewise constant, etc.
iv) It minimizes a predefined guaranteed cost function, which is the sum of the performance indices of all plants.

It is to be noted that condition (iii) given above is motivated by the following practical issues:

- In many problems involving robustness, noise rejection, simplicity of implementation, elimination of fixed modes, etc., it is desired to design GSHFs with a specific form, e.g. piecewise constant, exponential, etc. (Wang, 1982; Kabamba, 1987).
- Design of a high performance simultaneously stabilizing *piecewise constant* GSHF with no compensator is studied in Cao and Lam (2001) for the centralized case, which can be simply extended to the decentralized case. However, the present paper solves the problem in the general case by considering any arbitrary form for the GSHF, such as exponential, and by including a compensator in the control configuration as well.
- In the case of a sufficiently small sampling period, the optimal simultaneous stabilizer (whose exact solution, as pointed out earlier, involves complicated computations), can be approximated by a polynomial (e.g., the truncated Taylor series).

This paper is organized as follows. The simultaneous stabilizing periodic feedback problem is formulated in Section 2. A necessary and sufficient condition for the existence of a stabilizing GSHF and compensator with the desired structure is obtained as a set of matrix inequalities in Section 3. The problem is then converted to a linear matrix inequality (LMI) problem by using a well-known technique, and an algorithm is presented to solve it. The effectiveness of the proposed method is demonstrated in two numerical examples in Section 4. Finally, some concluding remarks are given in Section 5.

2 Problem formulation

Consider a set of η continuous-time detectable and stabilizable LTI systems $\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_\eta$ with the following state-space representations:

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t) \tag{1a}$$
$$y_i(t) = C_i x_i(t) \tag{1b}$$

where $x_i \in \Re^{n_i}$, $u_i \in \Re^m$ and $y_i \in \Re^l$, $i \in \bar{\eta} := \{1, 2, ..., \eta\}$, are the state, the input and the output of $\mathcal{S}_i$, respectively. Assume that the discrete-time compensator K_c^i, $i \in \bar{\eta}$, with the following representation is given:

$$\begin{aligned} z_i[\kappa+1] &= E z_i[\kappa] + F y_i[\kappa] \\ \phi_i[\kappa] &= G z_i[\kappa] + H y_i[\kappa] \end{aligned} \tag{2}$$

and assume also that $z_i[0] = 0$. It is to be noted that the discrete argument corresponding to the samples of any signal is enclosed in brackets (e.g., $y_i[\kappa] = y_i(\kappa h)$). K_c^i can be either decentralized with block-diagonal transfer function matrix or centralized. Suppose now that the system $\mathcal{S}_i$, $i \in \bar{\eta}$ is desired to be controlled by the compensator K_c^i and the hold controller K_h^i represented by:

$$u_i(t) = f(t)\phi_i[\kappa], \quad \kappa h \leq t < (\kappa+1)h, \quad \kappa = 0, 1, 2, ...$$

where h is the sampling period, and $f(t) = f(t+h)$, $t \geq 0$. Note that $f(t)$ is a sampled-data hold function, which is desired to be described by the following set of basis functions:

$$\mathbf{f} := \{f_1(t), f_2(t), ..., f_k(t)\}$$

where $f_i(t) \in \Re^{m \times l_i}$, $i = 1, 2, ..., k$. Thus, $f(t)$ can be written as a linear combination of the basis functions in $\mathbf{f}$ as follows:

$$f(t) = f_1(t)\alpha_1 + f_2(t)\alpha_2 + \cdots + f_k(t)\alpha_k \tag{3}$$

where some of the entries of the variable matrices $\alpha_i \in \Re^{l_i \times l}$, $i = 1, 2, ..., k$, are set equal to zero and the other entries are free variables so that the structure of $f(t)$ complies with the desired control constraint, which is determined by a given information flow matrix (Davison and Chang, 1990). This is illustrated later in Example 2. Furthermore, the set of basis functions $\mathbf{f}$ is obtained according to the desirable form of GSHF (e.g, exponential, polynomial, etc.). This will be demonstrated in Examples 1 and 2. Note that the motivation for considering a special form for $f(t)$ is discussed in the introduction.

For any $i \in \{1, 2, ..., k\}$, put all of the indices of the zeroed entries of α_i in the set $\mathbf{E_i}$. Assume now the expected value of $x_i(0)x_i(0)^T$, which is referred to as the covariance matrix of the initial state $x_i(0)$, is known and denoted by X_0^i for any $i \in \bar{\eta}$. The objective is to obtain the constrained matrices $\alpha_1, ..., \alpha_k$ such that the following performance index is minimized:

$$J = E\left\{ \sum_{i=1}^{\eta} \int_0^\infty \left(x_i(t)^T Q_i x_i(t) + u_i(t)^T R_i u_i(t)\right) dt \right\} \tag{4}$$

where $R_i \in \Re^{m \times m}$ and $Q_i \in \Re^{n_i \times n_i}$ are symmetric positive definite and symmetric positive semi-definite matrices, respectively, and $E\{\cdot\}$ denotes the expectation operator. Note that by minimizing the cost function given above, the stability of the system $\mathcal{S}_i$ under the discrete-time compensator K_c^i and the hold controller K_h^i, for any $i \in \bar{\eta}$, is achieved because the cost function becomes infinity otherwise. Note also that since (4) is a continuous-time performance index, it takes the intersample ripple effect into account.

The equation (3) can be written as $f(t) = g(t)\alpha$, where:

$$g(t) := [f_1(t) \;\; f_2(t) \;\; ... \;\; f_k(t)], \;\; \alpha := \left[\alpha_1^T \;\; \alpha_2^T \;\; ... \;\; \alpha_k^T\right]^T \tag{5}$$

Define a new set $\mathbf{E}$ based on the sets $\mathbf{E_1}, ..., \mathbf{E_k}$, such that any of the entries of α whose index belongs to $\mathbf{E}$ is equal to zero. On the other hand, it is known that:

$$x_i(t) = e^{(t-\kappa h)A_i} x_i(\kappa h) + \int_{\kappa h}^{t} e^{(t-\tau)A_i} B_i u_i(\tau) d\tau$$

for any $\kappa h \leq t \leq (\kappa+1)h$, $\kappa \geq 0$. Let the following matrices be defined for any $i \in \bar{\eta}$:

$$M_i(t) := e^{tA_i}, \;\; \bar{M}_i(t) := \int_0^t e^{(t-\tau)A_i} B_i g(\tau) d\tau$$

Therefore:

$$x_i(t) = M_i(t-\kappa h)x_i[\kappa] + \bar{M}_i(t-\kappa h)\alpha\phi_i[k] \tag{6}$$

for any $\kappa h \leq t \leq (\kappa+1)h$. It can be easily concluded from (1b), (2), and (6) by substituting $t = (\kappa+1)h$, that $\mathbf{x}_i[\kappa+1] = \tilde{M}_i(h, \alpha)\mathbf{x}_i[\kappa]$ for any $\kappa \geq 0$, where $\mathbf{x}_i[\kappa] = \left[\, x_i[\kappa]^T \;\; z_i[\kappa]^T \,\right]^T$, and:

$$\tilde{M}_i(h, \alpha) := \begin{bmatrix} M_i(h) + \bar{M}_i(h)\alpha H C_i & \bar{M}_i(h)\alpha G \\ F C_i & E \end{bmatrix} \tag{7}$$

It is straightforward to show that:

$$\mathbf{x}_i[\kappa] = \left(\tilde{M}_i(h, \alpha)\right)^\kappa \mathbf{x}_i[0], \quad \kappa = 0, 1, 2, ...$$

3 Optimal Structurally Constrained GSHF

It is desired now to find out when the structurally constrained GSHF $f(t)$ exists such that the system $\mathcal{S}_i$ is stable under the compensator K_c^i and the hold controller K_h^i, for any $i \in \bar{\eta}$.

Lemma 1 *There exists a GSHF $f(t)$ with the desired form (given by the equation (3)) such that the system $\mathcal{S}_i$ is stable under the compensator K_c^i and the hold controller K_h^i for any $i \in \bar{\eta}$, if and only if there exists an output feedback with the constant gain α, with the properties that:*

1. *Each entry of α whose index belongs to the set $\mathbf{E}$ is equal to zero.*
2. *It simultaneously stabilizes all of the η systems $\bar{\mathcal{S}}_1, \bar{\mathcal{S}}_2, ..., \bar{\mathcal{S}}_\eta$, where the system $\bar{\mathcal{S}}_i$, $i \in \bar{\eta}$, is represented by:*

$$\bar{x}_i[\kappa+1] = \begin{bmatrix} M_i(h) & 0 \\ F C_i & E \end{bmatrix} \bar{x}_i[\kappa] + \begin{bmatrix} \bar{M}_i(h) \\ 0 \end{bmatrix} \bar{u}_i[\kappa]$$
$$\bar{y}_i[\kappa] = \begin{bmatrix} H C_i & G \end{bmatrix} \bar{x}_i[\kappa]$$

(note that each of the two 0's in the above equation represents a zero matrix with proper dimension).

Proof: The proof follows from the fact that the system $\mathcal{S}_i,\ i \in \bar{\eta}$, is stable under K_c^i and K_h^i if and only if all of the eigenvalues of the matrix $\tilde{M}_i(h,\alpha)$ given in (7) are located inside the unit circle in the complex plane. ■

Remark 1 *Lemma 1 presents a necessary and sufficient condition for the existence of a structurally constrained GSHF $f(t)$ with a desired form, which simultaneously stabilizes all of the systems $\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_\eta$ along with a given discrete-time compensator. The condition obtained is usually referred to as "simultaneous stabilization of a set of LTI systems via structured static output feedback", which has been investigated intensively in the literature. For instance, one can exploit the LMI algorithm proposed in Cao and Lam (2001) to solve the simultaneous stabilization problem given in Lemma 1 in order to obtain a stabilizing matrix α denoted by $\breve{\alpha}$ (which is later used as the initial point in the main algorithm), or conclude the non-existence of such GSHF, otherwise.*

Define now the following matrices for any $i \in \bar{\eta}$:

$$
\begin{aligned}
P_0^i &:= \int_0^h \left(M_i(t)^T Q_i M_i(t)\right) dt \\
P_1^i &:= \int_0^h \left(M_i(t)^T Q_i \bar{M}_i(t)\right) dt \\
P_2^i &:= \int_0^h \left(\bar{M}_i(t)^T Q_i \bar{M}_i(t) + g(t)^T R_i g(t)\right) dt \\
q_0^i(\alpha) &:= P_0^i + P_1^i \alpha H C_i + (P_1^i \alpha H C_i)^T \\
&\quad + (\alpha H C_i)^T P_2^i (\alpha H C_i) \\
q_1^i(\alpha) &:= P_1^i \alpha G + (\alpha H C)^T P_2^i \alpha G \\
N_i(\alpha) &:= \begin{bmatrix} q_0^i(\alpha) & q_1^i(\alpha) \\ q_1^i(\alpha)^T & G^T \alpha^T P_2^i \alpha G \end{bmatrix}
\end{aligned}
$$

Theorem 1 *The optimal GSHF $f(t)$ can be obtained by minimizing J given below:*

$$
J = trace\left(\sum_{i=1}^{\eta} K_i \begin{bmatrix} X_0^i & 0 \\ 0 & 0 \end{bmatrix}\right) \tag{8}
$$

where $K_1, K_2, ..., K_\eta$ satisfy the following inequalities:

$$
\begin{bmatrix} -K_i + N_i(\alpha) & \tilde{M}_i^T(h,\alpha) K_i \\ K_i \tilde{M}_i(h,\alpha) & -K_i \end{bmatrix} < 0, \quad i = 1, 2, ..., \eta \tag{9}
$$

Proof: The proof is in line with that given in Lavaei and Aghdam (2006), and is omitted here. ■

Lemma 2 *The matrix P_2^i is positive definite if and only if there does not exist a constant nonzero vector x such that $g(t)x = 0$ for all $t \in [0,h]$.*

Proof: The proof is given in Lavaei and Aghdam (2006). ■

Lemma 2 presents a necessary and sufficient condition for the positive definiteness of the matrix P_2^i, which "almost always" holds in practice. It is assumed in the remainder of the paper that the matrix P_2^i is positive definite, as this assumption is required for the development of the main result.

Theorem 2 *The matrix inequality (9) is equivalent to the following matrix inequality:*

$$
\begin{bmatrix} \Phi_1^i & (\Phi_2^i)^T & (\Phi_4^i)^T \\ \Phi_2^i & \Phi_3^i & (\Phi_5^i)^T \\ \Phi_4^i & \Phi_5^i & -I \end{bmatrix} < 0 \tag{10}
$$

where

$$
\begin{aligned}
\Phi_1^i &:= -K_i + \begin{bmatrix} P_0^i + P_1^i \alpha H C_i + \left(P_1^i \alpha H C_i\right)^T & P_1^i \alpha G \\ (P_1^i \alpha G)^T & 0 \end{bmatrix}, \\
\Phi_2^i &:= K_i \begin{bmatrix} M_i(h) & 0 \\ F C_i & E \end{bmatrix}, \\
\Phi_3^i &:= -K_i - K_i \begin{bmatrix} \bar{M}_i(h)(P_2^i)^{-1}\bar{M}_i(h)^T & 0 \\ 0 & 0 \end{bmatrix} K_i, \\
\Phi_4^i &:= (P_2^i)^{\frac{1}{2}} \alpha \begin{bmatrix} H C_i & G \end{bmatrix}, \\
\Phi_5^i &:= \begin{bmatrix} (P_2^i)^{-\frac{1}{2}} \bar{M}_i(h)^T & 0 \end{bmatrix} K_i
\end{aligned}
$$

Proof: One can write the inequality (9) as follows:

$$
\begin{bmatrix} \Phi_1^i & (\Phi_2^i)^T \\ \Phi_2^i & \Phi_3^i \end{bmatrix} - \begin{bmatrix} (\Phi_4^i)^T \\ (\Phi_5^i)^T \end{bmatrix} (-I) \begin{bmatrix} \Phi_4^i & \Phi_5^i \end{bmatrix} < 0
$$

The matrix inequality (10) yields by applying the Schur complement formula to the above inequality. ■

It can be easily verified that in the absence of the block entry Φ_3^i, the matrix given in the left side of (10) is in the form of LMI. Moreover, this block entry cannot be converted to the LMI form due to the negative quadratic term inside it. Thus, the technique introduced in Cao, Sun, and Lam (1999) will now be used to remedy this drawback. Consider the arbitrary positive definite matrices $\Gamma_1, \Gamma_2, ..., \Gamma_\eta$ with the same dimensions as $K_1, K_2, ..., K_\eta$, respectively. Since P_2^i is assumed to be positive definite, one can write $(K_i - \Gamma_i)\Omega_i(K_i - \Gamma_i) \geq 0,\ i \in \bar{\eta}$, where:

$$\Omega_i := \begin{bmatrix} \bar{M}_i(h)(P_2^i)^{-1}\bar{M}_i(h)^T & 0 \\ 0 & 0 \end{bmatrix}$$

Therefore:

$$-K_i\Omega_i K_i \leq \Pi_i, \quad i \in \bar{\eta} \tag{11}$$

where $\Pi_i := \Gamma_i\Omega_i\Gamma_i - K_i\Omega_i\Gamma_i - \Gamma_i\Omega_i K_i,\ i \in \bar{\eta}$.

Theorem 3 *There exist positive definite matrices $K_1, K_2, ..., K_\eta$ satisfying the matrix inequality (10) if and only if there exist positive definite matrices $K_1, K_2, ..., K_\eta$ and $\Gamma_1, \Gamma_2, ..., \Gamma_\eta$ satisfying the following matrix inequalities:*

$$\begin{bmatrix} \Phi_1^i & (\Phi_2^i)^T & (\Phi_4^i)^T \\ \Phi_2^i & -K_i + \Pi_i & (\Phi_5^i)^T \\ \Phi_4^i & \Phi_5^i & -I \end{bmatrix} < 0, \quad i \in \bar{\eta} \tag{12}$$

Proof: If there exist positive definite matrices $K_1, K_2, ..., K_\eta$ and $\Gamma_1, \Gamma_2, ..., \Gamma_\eta$ satisfying (12), then according to (11), the matrices $K_1, K_2, ..., K_\eta$ satisfy (10) as well. On the other hand, suppose that there exist positive definite matrices $K_1, K_2, ..., K_\eta$ satisfying the matrix inequality (10). Choosing $\Gamma_i = K_i$ for $i = 1, 2, ..., \eta$, one can easily verify that $\Phi_3^i = -K_i + \Pi_i$. Hence, the inequality (10) is equivalent to the inequality (12) in this case. ■

It is to be noted that the matrix inequality (12) is LMI for the variables $K_i,\ i \in \bar{\eta}$, and α, if the matrices Γ_i, $i \in \bar{\eta}$, are set to be fixed. The following algorithm is proposed based on Theorems 1, 2 and 3, to compute the coefficients $\alpha_1, \alpha_2, ..., \alpha_\eta$ in order to obtain the desired GSHF $f(t)$.

Algorithm 1:

Step 1) Set $\alpha = \breve{\alpha}$ (where $\breve{\alpha}$ is defined in Remark 1) and solve the (linear matrix) inequality (9) in order to obtain $K_1, K_2, ..., K_\eta$.
Step 2) Set $\Gamma_i = K_i$ for all $i \in \bar{\eta}$, where the matrices $K_i,\ i \in \bar{\eta}$, are obtained in Step 1.
Step 3) Minimize J given by (8) for $K_1, K_2, ..., K_\eta$ and α subject to
· The LMI constraint (12)
· $K_i > 0$ for all $i \in \bar{\eta}$
· The constraint that each entry of α whose index belongs to the set $\mathbf{E}$ must be zero.
Step 4) If $\sum_{i=1}^{\eta} \|K_i - \Gamma_i\| < \delta$, where δ is a predetermined error margin, go to Step 6.
Step 5) Set $\Gamma_i = K_i$ for $i = 1, 2, ..., \eta$, where the matrices $K_i,\ i \in \bar{\eta}$, are obtained by solving the optimization problem in Step 3. Go to Step 3.
Step 6) The value obtained for α is sufficiently close to the optimal value, and substituting the resultant matrices $K_i,\ i \in \bar{\eta}$, into (8) gives the minimum value of J. Note that the coefficients $\alpha_1, \alpha_2, ..., \alpha_k$ can be obtained from (5).

Remark 2 *It can be easily verified that the value of J decreases each time that the optimization problem of Step 3 is solved, which indicates that the algorithm is monotone decreasing. On the other hand, since the inequality (11) will be converted to the equality if $\Gamma_i = K_i$, Algorithm 1 should ideally stop when $\sum_{i=1}^{\eta} \|K_i - \Gamma_i\| = 0$ in order to obtain the exact result. However, since it is desirable that the algorithm be halted in a finite time, Step 4 is added. It is to be noted that δ determines (indirectly) the closeness of the performance index obtained to its minimum value.*

4 Numerical examples

Example 1: This example can be found in Howitt et al. (1993), and represents the ship-steering system with two distinct modes. Consider two systems with the following parameters:

$$A_1 = \begin{bmatrix} -0.298 & -0.279 & 0 \\ -4.370 & -0.773 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.116 \\ -0.773 \\ 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -0.428 & -0.339 & 0 \\ -2.939 & -1.011 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.150 \\ -1.011 \\ 0 \end{bmatrix}$$

and $C_1 = C_2 = I$. Assume that the initial state of each of these systems is a random variable whose covariance matrix is equal to the identity matrix, and that $h = 0.1sec$. Assume also that it is desired to find a GSHF which minimizes the performance index J given by (4) with $R_i = Q_i = I,\ i = 1, 2$, while it has the following structure:

$$f(t) = \begin{bmatrix} * + * \sin(t) & * & * + * \, e^{-t} \end{bmatrix} \tag{13}$$

where the symbol "*" represents constant values which are to be found. Note that no compensator is considered in this example. The following basis functions and coefficient matrices can therefore be defined for (13):

$$f_1(t) = \sin(t), \quad f_2(t) = 1, \quad f_3(t) = e^{-t}$$
$$\alpha_1 = \begin{bmatrix} * & 0 & 0 \end{bmatrix}, \ \alpha_2 = \begin{bmatrix} * & * & * \end{bmatrix}, \ \alpha_3 = \begin{bmatrix} 0 & 0 & * \end{bmatrix}$$

It is to be noted that the "*" elements used above imply that these entries of the vectors α_1, α_2 and α_3 are the ones that are not set equal to zero. The optimal GSHF obtained from Algorithm 1 (by using the initial point $\breve{\alpha}$ given in Lavaei and Aghdam (2006)) will be:

$$\left[-3.925 + 2.805\, \sin(t) \quad 2.117 \quad -0.188 + 1.480\; e^{-t} \right]$$

The corresponding performance index is 31.581.

Example 2: Consider a two-input two-output system $\mathcal{S}$ consisting of two single-input single-output (SISO) agents with the following state-space matrices:

$$A = \text{diag}([0.5 \quad -2.5 \quad -5.5])$$
$$B = \begin{bmatrix} 0 & -2 & -4 \\ -2 & 2 & 0 \end{bmatrix}^T, \quad C = \begin{bmatrix} -2 & 0 & 2 \\ 0 & -2 & 2 \end{bmatrix}$$

It is desired to design a high-performance decentralized controller with the diagonal information flow structure for this system. It can be easily concluded from Davison and Chang (1990) that $\lambda = 0.5$ is a decentralized fixed mode (DFM) of the system. Thus, there is no LTI controller to stabilize the system. As a result, the available methods to design a continuous-time LTI controller (e.g., see Cao et al. (1999)) are incapable of handling this problem. Choose now $h = 1sec$, and denote the discrete-time equivalent model of the system $\mathcal{S}$ with $\mathcal{S}_d$. If the algorithm presented in Cao and Lam (2001) is exploited to design a discrete-time *static* stabilizing controller for the system $\mathcal{S}_d$, it will fail. This signifies that in order to design a decentralized controller for the system $\mathcal{S}$, two different types of controllers can be used: a dynamic discrete-time controller or a periodic controller. These two possibilities are explained in the following:

i) Let a deadbeat dynamic stabilizing controller K_c for the system $\mathcal{S}_d$ be designed by using the method given in Davison and Chang (1990). Assume that $Q = R = I$, and that the initial state of the system $\mathcal{S}$ is a random variable with the identity covariance matrix. In this case, the corresponding performance index will be equal to 83439.49, which is inadmissibly large. To improve the performance, a hold controller K_h is desired to be added to the control system. Assume that the hold function $f(t)$ is desirable to have the following form:

$$\text{diag}([* + *\, cos(850t) \qquad * + *\;\, cos(850t)])$$

Using Algorithm 1 with several iterations results in the hold function $f(t) = \text{diag}([1.003 - 0.071cos(850t)\ ,\ 0.958 + 0.754cos(850t)])$, and the corresponding performance index turns out to be 81517.97. This indicates an improvement of about 2.36% by using the hold controller K_h. However, this enhancement is not noticeable.

ii) It is desired to find out whether there exists a hold controller K_h to stabilize the system $\mathcal{S}$ by itself (i.e., without any compensator K_c). Consider the following basis functions for the hold function $f(t)$:

$$f_i(t) = u_e\left(t - \frac{i-1}{2}\right) - u_e\left(t - \frac{i}{2}\right), \quad i = 1, 2$$

where $u_e(\cdot)$ denotes the unit-step function. It is to be noted that this GSHF is equivalent to a piecewise constant function with two different levels. Applying the result of Cao and Lam (2001) to Lemma 1 leads to the controller K_h with the hold function:

$$\text{diag}([-1.4f_1(t) - 0.185f_2(t) \quad 0.5f_1(t) + f_2(t)]) \tag{14}$$

The resulting performance index is equal to 2121.18. Hence, Algorithm 1 can now be utilized to adjust the coefficients of this hold function properly. The optimal $f(t)$ obtained will be equal to:

$$\text{diag}([-2.71f_1(t){+}1.08f_2(t) \quad 0.97f_1(t){-}0.30f_2(t)]) \tag{15}$$

and the performance index of the closed-loop system will be 301.73. This implies that a high-performance stabilizing controller is designed for the ill-controllable system $\mathcal{S}$. The outputs of the first and the second agents of $\mathcal{S}$ under the hold functions (14) and (15) are illustrated in Figures 1(a) and 1(b) for $x(0) = [0.5 \quad 0.5 \quad 0.5]^T$. In addition, the inputs of the first and the second agents of $\mathcal{S}$ are depicted in Figures 2(a) and 2(b). The value of the cost function J is plotted for the first 150 iterations in Figure 3.

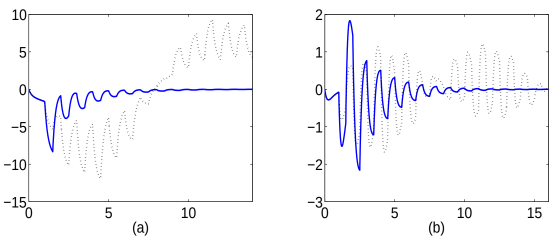

Fig. 1. The outputs of the first agent and the second agent are depicted in (a) and (b), respectively, under the GSHFs (14) (dotted curves), and (15) (solid curves).

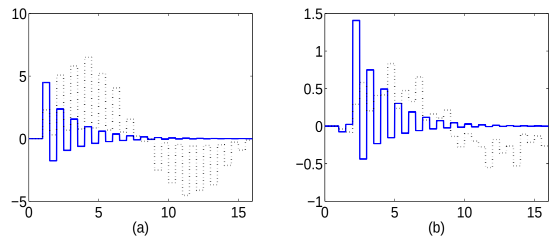

Fig. 2. The inputs of the first agent and the second agent are depicted in (a) and (b), respectively, under the GSHFs (14) (dotted curves), and (15) (solid curves).

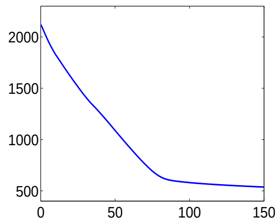

Fig. 3. The value of J for the first 150 iterations.

5 Conclusions

In this paper, a method is proposed to design a decentralized periodic output feedback with a prescribed form, e.g. polynomial, piecewise constant, sinusoidal, etc., to simultaneously stabilize a set of continuous-time LTI systems and minimize a predefined guaranteed continuous-time quadratic performance index, which is, in fact, the sum of the performance indices of all of the systems. The design procedure is accomplished in three phases: First, the problem is formulated as a set of matrix inequalities. Next, it is converted to a set of linear matrix inequalities, which represent necessary and sufficient conditions for the existence of such a structurally constrained controller with the prespecified form. An algorithm is then presented to solve the resultant LMI problem. Simulation results demonstrate the effectiveness of the proposed method.

References

Blondel, V., & Gevers, M. (1993). Simultaneous stabilizability of three linear systems is rationally undecidable. *Mathematics of Control, Signals, and Systems.* 6(2), 135–145.

Cao, Y. Y., & Lam, J. (2001). A computational method for simultaneous LQ optimal control design via piecewise constant output feedback. *IEEE Transactions on Systems, Man, and Cybernetics.* 31(5), 836–842.

Cao, Y. Y., Sun, Y. X., & Lam, J. (1999). Simultaneous stabilization via static output feedback and state feedback. *IEEE Transactions on Automatic Control.* 44(6), 1277–1282.

Chammas, A. B., & Leondes, C. T. (1978). On the design of linear time-invariant systems by periodic output feedback: Part I, discrete-time pole placement. *International Journal of Control.* 27, 885–894.

Davison, E. J., & Chang, T. N. (1990). Decentralized stabilization and pole assignment for general proper systems. *IEEE Transactions on Automatic Control.* 35(6), 652–664.

Feuer, A., & Goodwin, G. C. (1994). Generalized sample hold functions-frequency domain analysis of robustness, sensitivity, and intersample difficulties. *IEEE Transactions on Automatic Control.* 39(5), 1042–1047.

Fonte, C., Zasadzinski, M., Bernier-Kazantsev, C., & Darouach, M. (2001). On the simultaneous stabilization of three or more plants. *IEEE Transactions on Automatic Control.* 46(7), 1101–1107.

Howitt, G. D., & Luus, R. (1993). Control of a collection of linear systems by linear state feedback control. *International Journal of Control.* 58, 79–96.

Hyslop, G. L., Schattler, H. , & Tarn, T. J. (1992). Descent algorithms for optimal periodic output feedback control. *IEEE Transactions on Automatic Control.* 37(12), 1893–1904.

Jia, Y., & Ackermann, J. (2001). Condition and algorithm for simultaneous stabilization of linear plants. *Automatica.* 37(9), 1425–1434.

Kabamba, P. T. (1987). Control of linear systems using generalized sampled-data hold functions. *IEEE Transactions on Automatic Control.* 32(9), 772–783.

Kabamba, P. T., & Yang, C. (1991). Simultaneous controller design for linear time-invariant systems. *IEEE Transactions on Automatic Control.* 36(1), 106–111.

Miller, D. E., & Rossi, M. (2001). Simultaneous stabilization with near optimal LQR performance. *IEEE Transactions on Automatic Control.* 46(10), 1543–1555.

Tarn, T. J., & Yang, T. (1988). Simultaneous stabilization of infinite-dimensional systems with periodic output feedback. *Linear Circuit Systems and Signals Processing: Theory and Application* (pp. 409–424).

Toker, O., & Özbay, H. (1995). On the NP-hardness of solving bilinear matrix inequalities and simultaneous stabilization with static output feedback, *Proceedings of the 1995 American Control Conference.* Seattle, Washington (pp. 2525–2526).

Vidyasagar, M., & Viswanadham, N. (1982). Algebraic design techniques for reliable stabilization. *IEEE Transactions on Automatic Control.* 27(5), 1085–1095.

Wang, S. H. (1982). Stabilization of decentralized control systems via time-varying controllers. *IEEE Transactions on Automatic Control.* 27(3), 741–744.

Werner, H. (1996). An iterative algorithm for suboptimal periodic output feedback control. *UKACC International Conference on Control* (pp. 814–818).

Lavaei, J., & Aghdam, A. G. (2006). High-Performance Simultaneous Stabilizing Periodic Feedback Control with a Constrained Structure. *Proceedings of the 2006 American Control Conference.* Minneapolis, Minnesota.

Youla, D., Bongiorno, J., & Lu, C. (1974). Single-loop feedback stabilization of linear multivariable dynamical plants. *Automatica.* 10(2), 159–173.