A duality-based convex optimization approach to L_2-gain control of piecewise affine slab differential inclusions ⋆

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Abstract

This paper introduces the dual parameter set of a piecewise affine (PWA) system. This is a key concept to enable a convex formulation of PWA controller synthesis for PWA slab differential inclusions using a new convex relaxation. Another important contribution of the paper is to present PWA L_2-gain analysis and synthesis results for PWA systems whose output is also a PWA function of the state (as opposed to a piecewise-linear function). Unlike other results existing in the literature, the sufficient LMI conditions in this paper are valid for synthesis, even when the PWA systems include sliding modes. A numerical example with sliding modes illustrates the new approach.

Key words: Piecewise affine systems; Dual parameter set, Controller synthesis; Lyapunov functions.

1 Introduction

PWA models can describe a large class of nonlinear systems, including linear systems with certain memoryless nonlinearities, such as saturation and dead-zone. In addition to being a broad and complex class of models, PWA systems are locally linear, which enables one to formulate *sufficient* conditions for the stability and performance analysis of PWA systems as convex optimization problems subject to linear matrix inequalities (LMI) (Boyd et al., 1994). This has been the trend of research in the work on stability analysis of PWA systems based on Lyapunov functions and LMIs (Johansson and Rantzer, 1998; Hassibi and Boyd, 1998; Decarlo et al., 2000; Gonçalves et al., 2003; Rodrigues, 2004). The analysis of L_2-gain performance of PWA systems is also formulated as a set of LMIs in Hassibi and Boyd (1998); Johansson (2003) and Rantzer and Johansson (2000).

In addition to the work on analysis, controller synthesis for L_2-gain performance of PWA systems has also attracted growing attention (Feng, 2002; Feng et al., 2002; Feng, 2004) and, more recently, controller synthesis based on input to state stability has also been considered in Heemels et al. (2007). In his work on L_2-gain controller synthesis for uncertain PWA systems, Feng (2002) formulates the synthesis problem as a set of LMIs based on a piecewise quadratic (PWQ) Lyapunov function provided that the structure of the PWA controller was constrained. Feng et al. (2002) proposed a method to design PWL controllers for PWL systems based on a PWQ Lyapunov function to limit the L_2-gain of the system. The method was later extended to uncertain PWL systems in Feng (2004). However, the approaches in Feng et al. (2002) and Feng (2004) do not use any S-procedure in the design process, which means that each closed-loop subsystem of the PWL system has to be stable and this makes the proposed methods conservative. They also ignore attractive sliding modes. There is therefore no guarantee for the closed-loop system to be stable in general.

A very important subclass of PWA systems is the class of PWA *slab systems* (Rodrigues and Boyd, 2005), for which the partition of the state space is a function of a scalar variable. The synthesis of PWL controllers for stability and performance of PWA slab systems is formulated in Hassibi and Boyd (1998) as a set of LMIs. However, for PWA controllers, it is said in Hassibi and Boyd (1998) that "*It doesn't seem that the condition for stabilizability using this type of input command can be cast as an LMI*". Rodrigues and Boyd (2005) showed that by considering an affine term in the controller, the synthesis problem for PWA slab systems can be formulated as a set of LMIs parametrized by a vector. Three different

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algorithms for controller synthesis have been proposed in Rodrigues and Boyd (2005) and the bisection method has been used to find the controller that maximizes the decay rate of the trajectories.

To the best of our knowledge, no convex optimization problem has been proposed for PWA controller design for stability and performance without limiting the structure of the controller. To fill this gap in the literature, this paper formulates PWA controller synthesis for PWA slab systems as a set of LMIs. This result is based on a new key concept: the *dual parameter set* for PWA slab differential inclusions that is introduced in this paper. Considering PWA slab differential inclusions (as opposed to equations) enables the design for stability and performance of nonlinear systems that can be *included* by a PWA envelope.

The structure of the paper is as follows. Some mathematical preliminaries and PWA slab differential inclusions are introduced in sections 2 and 3, respectively. Performance analysis is presented in section 4. Section 5 addresses L_2-gain control design. Finally, a numerical example is shown in section 6 and conclusions are drawn in section 7.

2 Mathematical preliminaries

Consider the nonlinear system

$$\begin{cases} \dot{x} = f(x) + g(x)w \\ y = h(x) \end{cases} \tag{1}$$

where $f(x)$ and $g(x)$ are defined *almost everywhere* and bounded for bounded $\|x\|$.

The L_2 gain from w to y is defined as

$$\sup_{0<\|w\|_2<\infty} \frac{\|y\|_2}{\|w\|_2} \tag{2}$$

where the L_2 norm of a signal z is defined as

$$\|z\|_2 = \left[\int_0^\infty z^{\mathrm{T}}(\tau)z(\tau)d\tau\right]^{\frac{1}{2}} \tag{3}$$

and the supremum is taken over all nonzero trajectories assuming $x_0 = 0$.

The notion of storage function originates in the early work on dissipativity theory started by Willems (1972). The following sufficient condition for finite L_2 gain uses this notion.

Definition 1 *The nonlinear system (1) has finite L_2-gain less than $\gamma > 0$ if there exists a locally bounded function $V : \mathbb{R}^n \to \mathbb{R}$, called* storage function*, such that $V(x) \geq 0$ for all $x \in \mathbb{R}^n$, $V(0) = 0$ and*

$$\forall t \geq 0, V(x(t)) \leq V(x_0) + \int_0^t W(\tau)d\tau \tag{4}$$

where $W(\tau) = -\|y(\tau)\|^2 + \gamma^2\|w(\tau)\|^2$ is called the supply rate.

3 Polytopic PWA slab differential inclusions

Polytopic PWA slab differential inclusions are a generalization of polytopic linear differential inclusions in Boyd et al. (1994). A polytopic PWA slab differential inclusion is described by

$$\begin{aligned} \dot{x}(t) &= A(x,t)x + a(x,t) + B_u(x,t)u + B_w(x,t)w \\ y(t) &= C(x,t)x + c(x,t) + D_u(x,t)u + D_w(x,t)w \end{aligned} \tag{5}$$

where $x(t) \in \mathbb{R}^n$ denotes the state, $u(t) \in \mathbb{R}^{n_u}$ is the control input, $w(t) \in \mathbb{R}^{n_w}$ is the exogenous input and $y(t) \in \mathbb{R}^{n_y}$ is the output. The initial state is $x(0) = x_0$. It is assumed that system (5) satisfies

$$\begin{aligned} \dot{x} &\in \mathrm{Conv}\{A_{i\kappa}x + a_{i\kappa} + B_{u_{i\kappa}}u + B_{w_{i\kappa}}w, \kappa = 1, 2\} \\ y &\in \mathrm{Conv}\{C_{i\kappa}x + c_{i\kappa} + D_{u_{i\kappa}}u + D_{w_{i\kappa}}w, \kappa = 1, 2\} \end{aligned} \tag{6}$$

for $x \in \mathcal{R}_i$ where Conv stands for the convex hull of a set and $\mathcal{R}_i$, $i = 1, \ldots, N$ are N slab regions partitioning the cross product of a slab subset of the state space $\mathcal{X} \subset \mathbb{R}^n$ and the space of the exogenous input $\mathcal{W}$, defined as

$$\mathcal{R}_i = \{(x, w) \mid \sigma_i < C_{\mathcal{R}}x + D_{\mathcal{R}}w < \sigma_{i+1}\}, \tag{7}$$

where $C_{\mathcal{R}} \in \mathbb{R}^{1\times n}$, $D_{\mathcal{R}} \in \mathbb{R}^{1\times n_w}$ and σ_i for $i = 1, \ldots, N+1$ are scalars such that

$$\sigma_1 < \sigma_2 < \ldots < \sigma_{N+1} \tag{8}$$

It is assumed that $a_{i\kappa} = 0$ and $c_{i\kappa} = 0$ for $i \in \mathcal{I}(0, 0)$ and $\kappa = 1, 2$ where

$$\mathcal{I}(x, w) = \{i \mid (x, w) \in \overline{\mathcal{R}}_i\} \tag{9}$$

and $\overline{\mathcal{R}}_i$ denotes the closure of $\mathcal{R}_i$. Note that if $(x, w) \in \mathcal{R}_i$, then $\mathcal{I}(x, w) = \{i\}$. Each slab region can be described by the following degenerate ellipsoid

$$\mathcal{R}_i = \{(x, w) \mid |L_ix + l_i + M_iw| < 1\} \tag{10}$$

where $L_i = 2C_{\mathcal{R}}/(\sigma_{i+1} - \sigma_i)$, $l_i = -(\sigma_{i+1} + \sigma_i)/(\sigma_{i+1} - \sigma_i)$ and $M_i = 2D_{\mathcal{R}}/(\sigma_{i+1} - \sigma_i)$.

We consider the following definition of solutions for PWA slab differential inclusions

Definition 2 *An absolutely continuous function $x(t)$ is defined to be a solution of (5) if $x(t) \in \mathcal{X}, \forall t \geq 0$ and it satisfies*

$$\dot{x}(t) \in \mathcal{F}(x, u, w) \tag{11}$$

for almost every $t \geq 0$ *where*

$$\mathcal{F}(x,u,w) \triangleq \mathrm{Conv}\left\{ f_{i\kappa} \,\middle|\, \begin{smallmatrix} f_{i\kappa}=A_{i\kappa}x+a_{i\kappa}+B_{u_{i\kappa}}u+B_{w_{i\kappa}}w \\ \text{for all } i\in\mathcal{I}(x,w), \kappa=1,2 \end{smallmatrix} \right\} \tag{12}$$

Remark 3 *Johansson (2003) defines a PWA differential inclusion as*

$$\begin{cases} \dot{x} = A_i(t)x(t) + a_i(t) + B_i(t)u(t) \\ y(t) = C_i(t)x(t) + c_i(t) + D_i(t)u(t) \end{cases}, \quad \text{for } x \in \overline{\mathcal{R}}_i \tag{13}$$

where

$$\begin{bmatrix} A_i(t) & a_i(t) & B_i(t) \\ C_i(t) & c_i(t) & D_i(t) \end{bmatrix} = \sum_{\kappa=1}^{\mathcal{K}_i} \alpha_\kappa(t) \begin{bmatrix} A_{i\kappa} & a_{i\kappa} & B_{i\kappa} \\ C_{i\kappa} & c_{i\kappa} & D_{i\kappa} \end{bmatrix} \tag{14}$$

with $\alpha_\kappa(t) \geq 0$ and $\sum_{k=1}^{\mathcal{K}_i} \alpha_\kappa(t) = 1$. A solution for the inclusion (13) is defined in Johansson (2003) as an absolutely continuous function $x(t)$ such that for almost all $t \geq 0$, it satisfies

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} \in \mathrm{Conv}\left\{ \begin{bmatrix} A_{i\kappa} & a_{i\kappa} & B_{i\kappa} \\ C_{i\kappa} & c_{i\kappa} & D_{i\kappa} \end{bmatrix} \begin{bmatrix} x(t) \\ 1 \\ u(t) \end{bmatrix} \,\middle|\, k = 1, \ldots, \mathcal{K}_i \right\} \tag{15}$$

for $x(t) \in \overline{\mathcal{R}}_i$.

In this paper, we consider the state equation and the output equation separately as it is seen in (6). Therefore (6) introduces a larger class of inclusions. In particular, the definition of solutions for PWA differential inclusions considered in this paper accomodates attractive sliding modes for which $x(t)$ stays at the boundary of two or more regions for a nonzero time interval.

The next section addresses performance analysis of PWA differential inclusions.

4 Analysis

In this section, performance analysis of PWA differential inclusions is considered. The concept of the *parameter set* of a PWA differential inclusion is also introduced. This concept will be used to derive sufficient conditions for the analysis.

Consider the following PWA differential inclusion

$$\begin{aligned} \dot{x} &\in \mathrm{Conv}\{A_{i\kappa}x + a_{i\kappa} + B_{w_{i\kappa}}w, \kappa = 1,2\},\ (x,w) \in \mathcal{R}_i \\ y &\in \mathrm{Conv}\{C_{i\kappa}x + c_{i\kappa} + D_{w_{i\kappa}}w, \kappa = 1,2\} \\ \mathcal{R}_i &= \{(x,w) |\ \|L_i x + l_i + M_i w\| < 1\} \end{aligned} \tag{16}$$

It is assumed that $a_{i\kappa} = 0,\ c_{i\kappa} = 0$ for $i \in \mathcal{I}(0,0)$ and $\kappa = 1,2$. The *parameter set* of (16) is defined in the following.

Definition 4 *The* parameter set *of the differential inclusion (16) is defined as*

$$\Phi = \left\{ \begin{bmatrix} A_{i\kappa_1} & a_{i\kappa_1} & B_{w_{i\kappa_1}} \\ L_i & l_i & M_i \\ C_{i\kappa_2} & c_{i\kappa_2} & D_{w_{i\kappa_2}} \end{bmatrix} \,\middle|\, i=1,\ldots,N,\ \kappa_1=1,2,\ \kappa_2=1,2 \right\} \tag{17}$$

The following proposition describes sufficient conditions for the differential inclusion (16) to have a finite L_2-gain from w to y.

Proposition 5 *For a given $\gamma > 0$, the PWA slab differential inclusion (16) has a finite L_2-gain less than $\sqrt{2}\gamma$ from w to y if there exists $P = P^T \in \mathbb{R}^{n\times n}$ such that $P > 0$, for $i \in \mathcal{I}(0,0)$ verifies*

$$\begin{bmatrix} A_{i\kappa_1}^T P + PA_{i\kappa_1} + C_{i\kappa_2}^T C_{i\kappa_2} & * \\ B_{w_{i\kappa_1}}^T P + D_{w_{i\kappa_2}}^T C_{i\kappa_2} & -\gamma^2 I + D_{w_{i\kappa_2}}^T D_{w_{i\kappa_2}} \end{bmatrix} < 0 \tag{18}$$

and $\lambda_{i\kappa_1\kappa_2} < 0,\ i \notin \mathcal{I}(0,0)$ verify (19) for $\kappa_1 = 1,2,\ \kappa_2 = 1,2$. ■

Proof Sketch: Let $V(x) = x^{\mathrm{T}}Px$. It can be shown that inequalities (18) and (19) imply

$$\nabla V(x)^{\mathrm{T}} f_1 + \|y_1\|^2 - \gamma^2\|w\|^2 < 0 \tag{20}$$
$$\nabla V(x)^{\mathrm{T}} f_2 + \|y_1\|^2 - \gamma^2\|w\|^2 < 0 \tag{21}$$
$$\nabla V(x)^{\mathrm{T}} f_1 + \|y_2\|^2 - \gamma^2\|w\|^2 < 0 \tag{22}$$
$$\nabla V(x)^{\mathrm{T}} f_2 + \|y_2\|^2 - \gamma^2\|w\|^2 < 0 \tag{23}$$

where

$$f_1 = A_{i1}x + a_{i1} + B_{w_{i1}}w \tag{24}$$
$$f_2 = A_{i2}x + a_{i2} + B_{w_{i2}}w \tag{25}$$
$$y_1 = C_{i1}x + c_{i1} + D_{w_{i1}}w \tag{26}$$
$$y_2 = C_{i2}x + c_{i2} + D_{w_{i2}}w \tag{27}$$

It can then be concluded that the PWA differential inclusion (16) is dissipative with the storage function $2V(x)$ and the supply rate $W(x,w) = 2\gamma^2\|w\|^2 - \|y\|^2$ and the L_2-gain of (16) from w to y is less than $\sqrt{2}\gamma$. □

Remark 6 *Note that for $i \in \mathcal{I}(0,0)$, we have $(0,0) \in \mathcal{R}_i$. It then follows from (10) that for $i \in \mathcal{I}(0,0)$, $|l_i| < 1$. The element $(2,2)$ of the matrix in inequality (19) is therefore nonnegative. This makes inequality (19) infeasible for $i \in \mathcal{I}(0,0)$ and this is why it cannot be used for $i \in \mathcal{I}(0,0)$.*

The *dual parameter set* is defined in the following. Note that, in this paper, we consider polytopic PWA *slab* differential inclusions. Therefore $l_i = l_i^{\mathrm{T}}$ is a scalar.

$$\begin{bmatrix} (A_{i\kappa_1}^{\mathrm{T}}P+PA_{i\kappa_1}+C_{i\kappa_2}^{T}C_{i\kappa_2}+\lambda_{i\kappa_1\kappa_2}L_i^TL_i) & * & * \\ (a_{i\kappa_1}^{T}P+c_{i\kappa_2}^{T}C_{i\kappa_2}+\lambda_{i\kappa_1\kappa_2}l_iL_i) & (\lambda_{i\kappa_1\kappa_2}(l_i^2-1)+c_{i\kappa_2}^{T}c_{i\kappa_2}) & * \\ (B_{w_{i\kappa_1}}^{\mathrm{T}}P+D_{w_{i\kappa_2}}^{\mathrm{T}}C_{i\kappa_2}+\lambda_{i\kappa_1\kappa_2}M_i^TL_i) & (D_{w_{i\kappa_2}}^{\mathrm{T}}c_{i\kappa_2}+\lambda_{i\kappa_1\kappa_2}l_iM_i^{\mathrm{T}}) & (-\gamma^2I+D_{w_{i\kappa_2}}^{\mathrm{T}}D_{w_{i\kappa_2}}+\lambda_{i\kappa_1\kappa_2}M_i^{\mathrm{T}}M_i) \end{bmatrix} < 0 \tag{19}$$

Definition 7 *The* dual parameter set *of (16) is defined as*

$$\Phi^T = \left\{ \begin{bmatrix} A_{i\kappa_1}^T & L_i^T & C_{i\kappa_2}^T \\ a_{i\kappa_1}^T & l_i & c_{i\kappa_2}^T \\ B_{w_{i\kappa_1}}^T & M_i^T & D_{w_{i\kappa_2}}^T \end{bmatrix} \middle| \, i=1,\ldots,N,\ \kappa_1=1,2,\ \kappa_2=1,2 \right\} \tag{28}$$

The importance of the dual parameter set is that if we write the LMIs in Proposition 5 for Φ^{T}, the resulting LMIs are performance conditions equivalent to those of Proposition 5. This is shown in the following proposition.

Proposition 8 *For a given $\gamma > 0$, the PWA slab differential inclusion (16) has a finite L_2-gain less than $\sqrt{2}\gamma$ from w to y if there exists $Q = Q^T \in \mathbb{R}^{n\times n}$ such that $Q > 0$, for $i \in \mathcal{I}(0,0)$, verifies*

$$\begin{bmatrix} A_{i\kappa_1}Q + QA_{i\kappa_1}^T + B_{w_{i\kappa_1}}B_{w_{i\kappa_1}}^T & * \\ C_{i\kappa_2}Q + D_{w_{i\kappa_2}}B_{w_{i\kappa_1}}^T & -\gamma^2 I + D_{w_{i\kappa_2}}D_{w_{i\kappa_2}}^T \end{bmatrix} < 0 \tag{29}$$

and $\mu_{i\kappa_1\kappa_2} < 0$, $i \notin \mathcal{I}(0,0)$ verifies (30) for $\kappa_1 = 1,2$ and $\kappa_2 = 1,2$. ■

Proof Sketch: It can be shown that the conditions of Proposition 8 and that of Proposition 5 are equivalent under the change of variables $Q = \gamma^2 P^{-1}$ and $\mu_{i\kappa_1\kappa_2} = \lambda_{i\kappa_1\kappa_2}^{-1}$. □

Proposition 8 is an important result that enables the formulation of the L_2-gain synthesis as a convex optimization problem in the next section.

5 State-feedback controller synthesis

In this section, PWA controller synthesis for PWA slab differential inclusions will be formulated as a convex optimization program.

The objective is to design a PWA control signal of the form

$$u = K_i x + k_i \text{ for } x \in \mathcal{R}_i \tag{31}$$

to limit the L_2-gain from w to y of the following differential inclusion

$$\begin{aligned} &\dot{x} \in \mathrm{Conv}\{A_{i\kappa}x + a_{i\kappa} + B_{u_{i\kappa}}u + B_{w_{i\kappa}}w,\ \kappa = 1,2\}, \\ &y \in \mathrm{Conv}\{C_{i\kappa}x + c_{i\kappa} + D_{u_{i\kappa}}u + D_{w_{i\kappa}}w\}, \\ &\text{for } (x,w) \in \mathcal{R}_i = \{(x,w) |\ \|L_i x + l_i + M_i w\| < 1\} \end{aligned} \tag{32}$$

It follows from Proposition 8 that there exists a PWA control signal such that the L_2-gain from w to y for the PWA slab differential inclusion (32) in $\mathcal{X}$ is less than $\sqrt{(2)}\gamma$ if there exists $Q = Q^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ such that for all $i \in \mathcal{I}(0,0)$,

$$Q > 0, \tag{33}$$

$$\begin{bmatrix} \begin{pmatrix} A_{i\kappa_1}Q + B_{u_{i\kappa_1}}Y_i \\ +QA_{i\kappa_1}^{\mathrm{T}} + Y_i^{\mathrm{T}}B_{u_{i\kappa_1}}^{\mathrm{T}} \\ +B_{w_{i\kappa_1}}B_{w_{i\kappa_1}}^{\mathrm{T}} \end{pmatrix} & * \\ \begin{pmatrix} C_{i\kappa_2}Q + D_{u_{i\kappa_2}}Y_i \\ +D_{w_{i\kappa_2}}B_{w_{i\kappa_1}}^{\mathrm{T}} \end{pmatrix} & -\gamma^2 I + D_{w_{i\kappa_2}}D_{w_{i\kappa_2}}^{\mathrm{T}} \end{bmatrix} < 0 \tag{34}$$

and for all $i \notin \mathcal{I}(0,0)$, there exist $\mu_i < 0$, such that the matrix inequality (35) is satisfied for $\kappa_1 = 1,2$ and $\kappa_2 = 1,2$, where

$$Y_i = K_i Q \tag{36}$$
$$Z_i = \mu_i k_i \tag{37}$$
$$W_i = \mu_i k_i k_i^{\mathrm{T}} \tag{38}$$

The main problem in designing a PWA controller using constraints (33)–(38) is the equality constraint (38) because it prevents the problem to be formulated as a convex optimization. In the following, we propose a method to overcome this difficulty.

Convex relaxation: To formulate PWA controller synthesis as a convex program, considering $\mu_i < 0$ and (38), we have

$$W_i \le 0 \tag{39}$$

Therefore the term $B_{u_{i\kappa}}W_iB_{u_{i\kappa}}^{\mathrm{T}}$ in (35) is always negative semi-definite and can be omitted to make the problem convex. This idea leads to the following proposition.

Proposition 9 *For a given $\gamma > 0$, there exists a PWA controller of the form (31) such that the PWA slab differential inclusion (32) has a finite L_2-gain less than $\sqrt{2}\gamma$ from w to y if there exist $Q = Q^T \in \mathbb{R}^{n\times n}$ and $\mu_i \in \mathbb{R}$, such that $Q > 0$ verifies (34) for all $i \in \mathcal{I}(0,0)$ and $\mu_i < 0$ verifies (35) with $W_i = 0$ for all $i \notin \mathcal{I}(0,0)$ and for $\kappa_1 = 1,2,\ \kappa_2 = 1,2$.* ■

The PWA controller gains for Proposition 9 can be computed as

$$K_i = Y_i Q^{-1} \tag{40}$$

$$k_i = \begin{cases} 0 & \text{if } i \in \mathcal{I}(0) \\ \frac{1}{\mu_i}Z_i & \text{otherwise} \end{cases} \tag{41}$$

$$\begin{bmatrix} (A_{i\kappa_1}Q+QA_{i\kappa_1}^{\mathrm{T}}+B_{w_{i\kappa_1}}B_{w_{i\kappa_1}}^{\mathrm{T}}+\mu_{i\kappa_1\kappa_2}a_{i\kappa_1}a_{i\kappa_1}^{\mathrm{T}}) & * & * \\ (L_iQ+M_iB_{w_{i\kappa_1}}^{\mathrm{T}}+\mu_{i\kappa_1\kappa_2}l_ia_{i\kappa_1}^{\mathrm{T}}) & (\mu_{i\kappa_1\kappa_2}(l_i^2-1)+M_iM_i^{\mathrm{T}}) & * \\ (C_{i\kappa_2}Q+D_{w_{i\kappa_2}}B_{w_{i\kappa_1}}^{\mathrm{T}}+\mu_{i\kappa_1\kappa_2}c_{i\kappa_2}a_{i\kappa_1}^{\mathrm{T}}) & (D_{w_{i\kappa_2}}M_i^{\mathrm{T}}+\mu_{i\kappa_1\kappa_2}l_ic_{i\kappa_2}) & (-\gamma^2I+D_{w_{i\kappa_2}}D_{w_{i\kappa_2}}^{\mathrm{T}}+\mu_{i\kappa_1\kappa_2}c_{i\kappa_2}c_{i\kappa_2}^{\mathrm{T}}) \end{bmatrix} < 0 \qquad (30)$$

$$\begin{bmatrix} \begin{pmatrix} A_{i\kappa_1}Q+B_{u_{i\kappa_1}}Y_i+QA_{i\kappa_1}^{\mathrm{T}}+Y_i^{\mathrm{T}}B_{u_{i\kappa_1}}^{\mathrm{T}} \\ +B_{w_{i\kappa_1}}B_{w_{i\kappa_1}}^{\mathrm{T}}+\mu_ia_{i\kappa_1}a_{i\kappa_1}^{\mathrm{T}} \\ +a_{i\kappa_1}Z_i^{\mathrm{T}}B_{u_{i\kappa_1}}^{\mathrm{T}}+B_{u_{i\kappa_1}}Z_ia_{i\kappa_1}^{\mathrm{T}}+B_{u_{i\kappa_1}}W_iB_{u_{i\kappa_1}}^{\mathrm{T}} \end{pmatrix} & * & * \\ L_iQ+M_iB_{w_{i\kappa_1}}^{\mathrm{T}}+\mu_il_ia_{i\kappa}^{\mathrm{T}}+l_iZ_i^{\mathrm{T}}B_{u_{i\kappa}}^{\mathrm{T}} & \mu_i(l_i^2-1)+M_iM_i^{\mathrm{T}} & * \\ \begin{pmatrix} C_{i\kappa_2}Q+D_{u_{i\kappa_2}}Y_i \\ +D_{w_{i\kappa_2}}B_{w_{i\kappa_1}}^{\mathrm{T}}+\mu_ic_{i\kappa_2}a_{i\kappa_1}^{\mathrm{T}} \\ +c_{i\kappa_2}Z_i^{\mathrm{T}}B_{u_{i\kappa_1}}^{\mathrm{T}}+D_{u_{i\kappa_2}}Z_ia_{i\kappa_1}^{\mathrm{T}} \end{pmatrix} & \begin{pmatrix} D_{w_{i\kappa_2}}M_i^{\mathrm{T}} \\ +\mu_il_ic_{i\kappa_2} \\ +l_iD_{u_{i\kappa_2}}Z_i \end{pmatrix} & \begin{pmatrix} -\gamma^2I+D_{w_{i\kappa_2}}D_{w_{i\kappa_2}}^{\mathrm{T}} \\ +\mu_ic_{i\kappa_2}c_{i\kappa_2}^{\mathrm{T}}+c_{i\kappa_2}Z_i^{\mathrm{T}}D_{u_{i\kappa_2}}^{\mathrm{T}} \\ +D_{u_{i\kappa_2}}Z_ic_{i\kappa_2}^{\mathrm{T}} \end{pmatrix} \end{bmatrix} < 0 \qquad (35)$$

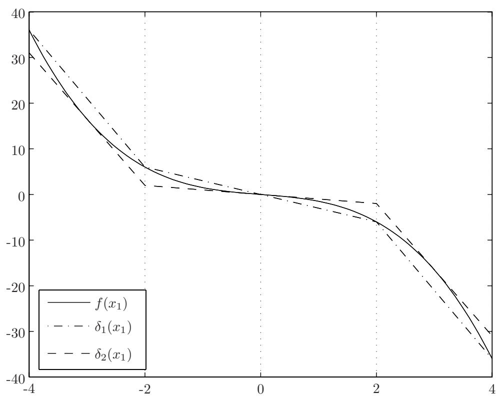

Fig. 1. The computed bounding envelope for a scalar function

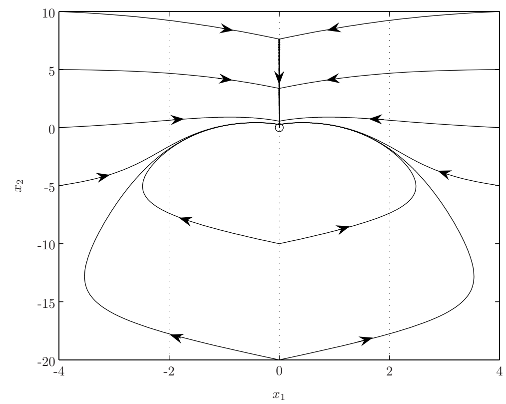

Fig. 2. Trajectories of the open loop nonlinear system

6 Numerical example

In this section, we consider a nonlinear system with a discontinuous vector field. The model is described by the following state equations.

$$\begin{aligned} \dot{x}_1 &= f(x_1) - 2\,\mathrm{sgn}(x_1)x_2 + w \\ \dot{x}_2 &= 2|x_1| - 2x_2 + u \\ y &= x_1 + x_2 - w \end{aligned} \qquad (42)$$

where $f(x_1) = -x_1 - 0.5x_1^3$. Using the proposed method in Samadi and Rodrigues (2007), a bounding envelope is computed for the nonlinear function $f(x_1)$ which is shown in Fig. 1. Trajectories of the open loop system are shown in Fig. 2. Notice that there is a sliding mode at $x_1 = 0$.

The objective is to design a PWA controller u to limit the L_2-gain from the disturbance w to the output y. Substituting $f(x_1)$ by its PWA bounds in (42), one gets a PWA differential inclusion with

$$\mathcal{R}_1 = (\,-4\ -2\,), \qquad \mathcal{R}_2 = (\,-2\ 0\,),$$
$$\mathcal{R}_3 = (\,0\ 2\,), \qquad \mathcal{R}_4 = (\,2\ 4\,)$$

Using the PWA approximation, the nonlinear system (42) can be described by the following differential inclusion

$$\begin{aligned} &\dot{x} \in \mathrm{Conv}\{A_{i\kappa}x + a_{i\kappa} + B_uu + B_ww\},\ x \in \mathcal{R}_i \\ &y = Cx + D_ww + D_uu \end{aligned} \qquad (43)$$

where $i = 1, \ldots, 4$, $\kappa = 1, 2$ and

$$B_w = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right],\ C = \left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right],\ D_u = 0,\ D_w = 0 \qquad (44)$$

Using Proposition 8, one can solve a set of LMIs using SeDuMi (Strum, 2001) and Yalmip (Löfberg, 2004) to compute $\gamma = 1.2$ for the open loop system. The following PWA controller can then be calculated using Proposition 9 to achieve $\gamma = 1$ for the closed loop system.

$$\begin{aligned} K_1 &= [\,-22.4985\ -83.6140\,],\ k_1 = 0.7609 \\ K_2 &= [\,-31.6286\ -117.2289\,],\ k_2 = 0 \\ K_3 &= [\,-25.6282\ -96.8814\,],\ k_3 = 0 \\ K_4 &= [\,-35.2165\ -114.3215\,],\ k_4 = 0.0489 \end{aligned} \qquad (45)$$

Fig. 3 shows the trajectories of the nonlinear system (42) in closed loop connection with the PWA controller. Note

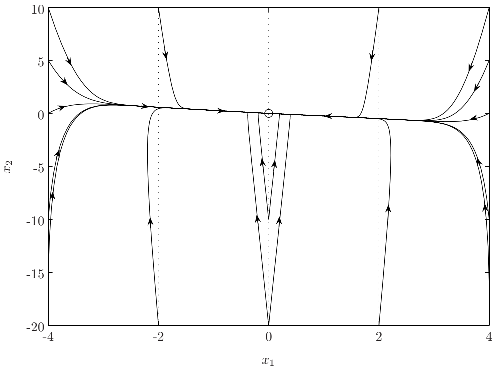

Fig. 3. Trajectories of the closed loop nonlinear system

that the PWA controller designed for the differential inclusion is guaranteed to limit the L_2 gain of the nonlinear system.

7 Conclusions

In this paper, an interesting duality relation was revealed in the LMIs describing sufficient conditions for finite L_2 gain of PWA slab differential inclusions. The definition of the regions of a PWA slab system was extended, the L_2-gain controller design was formulated as a set of LMIs and the synthesis was extended to PWA slab systems with an output that is also a PWA function of the state. The new method presented in the paper enables performance analysis, as well as controller synthesis, for a large class of nonlinear systems as a solution of convex optimization problems.

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