Effects of Lesson Sequencing on Preservice Teachers' Mathematical Knowledge of Place-Value

Diana A. Royea

A Thesis

In

The Department

Of

Education

Presented in Partial Fulfillment of the Requirements

for the degree of Master of Arts (Child Study) at

Concordia University

Montreal, Quebec, Canada

September 2012

CONCORDIAN UNIVERISTY

School of Graduate Studies

This is to certify that the thesis prepared

By: Diana Royea

Entitled: The Effects of Lesson Sequencing on Preservice Teachers' Place-Value Knowledge

and submitted in partial fulfillment of the requirements for the degree of

Master of Arts (Child Study)

complies with the regulation of the University and meets the accepted standards with respect to originality and quality.

Signed by the final Examining Committee:

______________________________________Examiner

Dr. Anna Sierpinska

______________________________________Examiner

Dr. Sandra Martin-Chang

_____________________________________Supervisor

Dr. Helena Osana

Approved by ____________________________________

Richard Schmid
Chair of the Department

_________ 2012 ___________________________________

Brian Lewis
Dean of Faculty

ABSTRACT

The Effects of Lesson Sequencing on Preservice Teachers' Place-Value Knowledge

Diana Royea

Elementary students' mathematical achievement is a focal point of mathematics education research. Place-value is a foundational topic in the elementary mathematics curriculum. In order to teach place-value in a manner that is in line with mathematics reform practices, teachers must possess strong conceptual, procedural, and specialized content knowledge (SCK) of place-value. At the same time, preservice teachers tend possess mathematical knowledge that is conceptually and procedurally weak. This study used a pretest-posttest design to investigate the effects of lesson sequencing on preservice teachers' conceptual, procedural, SCK, and transfer knowledge of place-value. Preservice teachers were assigned to one of three conditions: Concepts-first, Procedures-First, or Iterative. All of the participants were exposed to the same eight lessons, four conceptual and four procedural. The differences between the conditions was the order the lessons were received in. The results were analyzed quantitatively and where there were significant effects, those results were further analyzed from a qualitative perspective. Quantitative results indicated that there was a significant time × group interaction for conceptual knowledge. The Iterative condition significantly outperformed the Concepts-first and the Procedures-first conditions. While there was no main effect of condition on procedural knowledge, SCK, and transfer, there was a main effect of time for all three of these knowledge types. Furthermore, qualitative analyses revealed that the pathway of conceptual knowledge acquisition was affected by lesson sequencing. Finally, limitations, future research, and practical implications of this study are discussed.

Acknowledgments

First and foremost, I would like to express my sincerest gratitude to my supervisor, Dr. Helena Osana. Probably without realizing it, you have had profound, positive influence on my life. When I found myself looking for direction, you opened my eyes to possibilities I never would have known to exist and provided me with opportunities that have opened many doors.

I would also like to thank my committee members. Dr. Sandra Martin-Chang, you have served as role model for me from my first semester as a graduate student. You have helped me develop confidence in myself and I will always appreciate your upfront and insightful advice. Dr. Anna Sierpinska, thank you for providing your unique perspective and helping me appreciate the value of qualitative research.

In addition, I would like to thank Vanessa Rayner, Allyson Cooperman, and Kathy Pzednowek for their advice, insight, and friendship. Last, but not least, I would to extend my undying gratitude to a few people who are closest to me. Thanks mom, for helping in every way that you could. Nicolas Maibroda, thank you for always being available and for your continuous encouragement. Alexandra Bremner, thank you for encouraging me to open myself up to new experiences and places. Kevin Branco, thank you for helping me make the initial decision to pursue an MA. Most importantly, thank you Dylan Bockus for your constant support, always believing in my abilities and not letting me chalk my achievements up to good luck, and helping me so much in so many ways especially as the thesis process was coming to an end.

Table of Contents

List of Figures

Figure

List of Tables

List of Appendices

Appendix

Statement of the Problem

Learning Mathematics with Understanding

Concerns with elementary students' mathematical achievement have been a driving force of mathematics education research (Gould, 2005; Mack, 1995; Saxe, Gearhart, & Nasir, 2001). These concerns are further mirrored in current mathematical reform principles (Ministère de l'Éducation, Loisirs et Sport, 2001; National Council of Teachers of Mathematics, 2009). Fostering the type of mathematical learning advocated by the latest reform requires elementary school teachers to possess sound mathematical knowledge that is both conceptually and procedurally rich (Ball, 1996; Ball, Hill, & Bass, 2005; Hill, Rowan, & Ball, 2005), a form of mathematics that is unfamiliar to them (Comiti & Ball, 1996; Frykolm, 1999). Preservice teachers' mathematical knowledge has also been undergoing significant scrutiny, however. A plethora of research has identified a variety of concerns relating to the quality of preservice teachers' mathematical understanding and the types of knowledge that are required to adequately perform within their profession (Ball, Lubienski, & Mewborn, 2001; Hill, Ball, & Schilling, 2008; Kahan, Cooper, & Bethea, 2003; Osana, Lacroix, Tucker, & Desrosiers, 2006; Stacey, Helme, Steinle, Baturo, Irwin, & Bana, 2001; Rowland, Huckstep, & Thwaites, 2005; Tirosh, 2000; Tsamir & Tirosh, 2008).

The general consensus is that preservice teachers are lacking in conceptual and procedural mathematical knowledge (Ball, 1990; Morris, Hiebert, & Spitzer, 2009; Rizvi & Lawson, 2007). Conceptual knowledge of mathematics is the knowledge of concepts and principles underlying mathematical procedures and the interrelationships between these concepts. Conversely, procedural mathematics is often the rote knowledge of mathematical procedures and algorithms. Possessing strong conceptual knowledge has both short and long term benefits for students and preservice teachers, including flexible thinking, transfer, and improved access to and increases in mathematical knowledge (Hiebert & Wearne, 1992). For teachers, possessing conceptual knowledge, procedural knowledge, and an understanding of the conceptual basis underlying procedures is vital to engaging in a reform-oriented practice (Ball, Thames, & Phelps, 2008; Frykholm, 1999; Steele, 2001).

Mathematical Knowledge for Teaching and Teacher Training

Ball, Thames, and Phelps (2008) developed an influential and informative model of teachers' mathematical knowledge. According to this model, mathematical knowledge for teaching (MKT) is separated into two major categories: subject matter knowledge and pedagogical content knowledge. For the purpose of this study, two sub-domains of subject matter knowledge will be addressed. Those sub-domains are common content knowledge (CCK) and specialized content knowledge (SCK).

CCK includes the basic skills that typical mathematically literate adults have (Ball, 1990; Hill & Ball, 2004). That is, CCK is the mathematical knowledge required to perform basic calculations and successfully navigate through everyday mathematics problems in order to arrive at correct answers (Ball, et al., 2008). Teachers must possess CCK because they need to know the material they are teaching and be able to differentiate between correct and incorrect student solutions. While this type of knowledge is essential for teachers, it is not specific to teaching *per se* (Ball, 1990; Ball et al., 2008).

On the other hand, SCK is a type of knowledge that is uniquely required by mathematics teachers (Ball et al., 2005). It includes mathematical reasoning and unpacking in a manner that is distinctively required by the work of teachers. Tasks that require SCK include teaching mathematics with meaning and analyzing and understanding students’ solutions and errors (Ball et al., 2008). Evidently, SCK requires mathematical understanding that goes well beyond the knowledge that is taught to students. In particular, teachers require conceptual knowledge of mathematics, procedural knowledge of mathematics, and an understanding of how mathematics concepts and procedures are related in order to adequately develop SCK (Ball et al., 2008; Ball & Bass, 2000; Rizvi & Lawson, 2007).

Even though it is often assumed that preservice teachers are mathematically proficient by the time they enter a teacher education program, preservice teachers’ mathematical knowledge during and after their teacher training remains disconnected and conceptually weak (Ball, Lubienski, & Mewborn, 2001; Khoury & Zazkis, 1994; Newton, 2008). That is, the mathematical procedures that they know are not supported by conceptual understanding. Given the positive correlation between teachers’ mathematical knowledge and student achievement (Hill et al., 2008; Hill, Rowan, & Ball, 2005), these findings are disconcerting and justify further research regarding how to improve preservice teachers’ conceptual and procedural knowledge of mathematics, including how and why mathematical procedures work, during their teacher training.

Knowledge Acquisition: Concepts and Procedures

How to develop SCK in preservice teachers is not clear. At the same time, developing SCK is dependent, at least in part, on developing sufficient conceptual and procedural knowledge of mathematics. In children, the process by which conceptual knowledge and procedural fluency are acquired remains somewhat of a debate (Byrnes &

Wasik, 1991; Gelman & Williams, 1998; Rittle-Johnson & Koedinger, 2002; Siegler, 1991; Siegler & Crowley, 1994; Sophian, 1997). In an attempt to improve elementary students' conceptual understanding and procedural knowledge of mathematics, prior research studies have examined the impact of an iterative sequencing of conceptual and procedural lessons (Rittle-Johnson & Koedinger, 2002). Rittle-Johnson and Koedinger (2009) found that iterating between conceptual and procedural lessons resulted in comparable gains in conceptual mathematical knowledge, increased gains in procedural knowledge, and increased transfer compared to presenting all the conceptual lessons before all procedural lessons. The effect of lesson sequencing on preservice teachers' conceptual and procedural mathematical knowledge has not yet been studied, however.

The purpose of this study is to determine the effects of three different lesson sequences, Concepts-First, Procedures-First, and Iterating between concepts and procedures, on preservice teachers' conceptual knowledge, procedural knowledge, and SCK of numeration, place-value, and multi-digit arithmetic. An additional purpose is to examine the nature of preservice teachers knowledge of place-value and how this knowledge changes as a result of different lesson sequences. The results of this study will have several practical implications. Not only will the results contribute to the literature on the acquisition of mathematical knowledge, but will also provide practical guidelines on how to foster preservice teachers' mathematical knowledge and SCK during their teacher training. Increasing preservice teachers' SCK is especially important because for most preservice teachers, their teacher training will be their first exposure to the type of rich, interconnected knowledge of mathematics they will be expected to use in their future reform-oriented practices (Frykholm, 1999)

Literature Review

The following review begins with the definitions of conceptual mathematical knowledge, procedural mathematical knowledge, and specialized content knowledge (Ball, Thames, & Phelps, 2008). Then, the mathematical knowledge that preservice teachers actually possess and the type of mathematical knowledge that they need to be effective mathematics teachers is discussed. One way to measure effective teaching is by looking at student outcomes. As such, the effect of teachers' knowledge on student achievement is briefly followed by a review of the literature on a mathematical topic that poses particular difficulties for children: place-value. While the nature of students' place-value knowledge and means of remediating this knowledge is well documented, little is known about preservice teachers' understanding of place-value and how to improve their conceptual knowledge, procedural knowledge, and specialized content knowledge of place-value concepts and relevant procedures. Finally, the relationships between conceptual and procedural knowledge of mathematics is explored and related to designing and implementing mathematical lessons to foster conceptual and procedural mathematical knowledge acquisition.

Conceptual Knowledge, Procedural Knowledge, and Specialized Content Knowledge

Conceptual knowledge. Rittle-Johnson and Siegler (1998) broadly define conceptual knowledge as the "explicit or implicit understanding of the principles that govern a domain and the interrelations between pieces of knowledge in a domain" (p. 77). More specifically, concepts are the governing principles or ideas of a given mathematical domain. Conceptual knowledge is characterized by an understanding of these principles and the relationships between these principles (Rittle-Johnson & Alibali, 1999); the relationships together form a connected network of knowledge (Rittle-Johnson & Siegler, 1998). Prior knowledge is linked to or tied to new knowledge through these networks to create meaningful learning through the process of assimilation (Byrnes & Wasik, 1991; Rittle-Johnson & Siegler, 1998). Conceptual knowledge is freed from any particular context and lies within a more abstract level of thinking and reflection (Tchoshanov, 2011). For the purpose of this study, conceptual knowledge is defined as knowledge of the ideas underlying mathematical procedures that cannot be learned through memorization but rather through reflection on the relationships between various pieces of mathematical knowledge.

Procedural knowledge. Unlike conceptual knowledge, procedural knowledge is context specific (Hiebert & Lefevre, 1986). Rittle-Johnson and Siegler (1998) defined procedural knowledge as the “action sequences for solving problems” (p. 77). That is, procedural knowledge is the knowledge of prescribed steps required to solve a mathematical problem. This type of knowledge is composed of two distinct parts. The first part consists of formal mathematical language and symbols, whereas the second part consists of algorithms and rules for completing mathematical tasks such as the standard procedures taught in school (Hiebert, 1992). Therefore, procedural knowledge necessitates the ability to remember and correctly apply mathematical syntax and symbols, in addition to the steps required to solve a problem (Brynes & Wasik, 1986; Fuson & Briars, 1990).

Knowledge of syntax and symbols is considered procedural in nature because this type of knowledge often only demonstrates an understanding of the surface features of any given mathematical problem (Rittle-Johnson & Siegler, 1998). While procedural knowledge is often perceived as the ability to follow the steps required to solve a problem using standard mathematical notation, it can also be non-symbolic. Non-symbolic procedural knowledge means that the steps to solve a problem could involve using concrete objects, or manipulatives, in a consistent way to solve the same type of problem by following the same sequence of steps (Hiebert & Lefevre, 1986). For the purpose of this study, procedural knowledge is operationalized as the knowledge of rules, algorithms, procedures, and formal mathematical language and symbols.

Specialized Content Knowledge. Specialized content knowledge (SCK) is a form of knowledge possessed and applied by teachers of mathematics. It plays a crucial role in teaching for understanding and involves the ability to perform many teaching tasks. Some of these tasks include: (a) being able to use multiple mathematical representations for the same concepts, (b) understanding the relationships between different representations, (c) understanding different situations of division (such as partitive and measurement division), (d) using appropriate mathematical language, (e) selecting tasks to elicit specific mathematical concepts or remediate misunderstandings, and (f) understanding the conceptual basis of mathematical procedures (Ball, Thames, & Phelps, 2008). Evidently, SCK requires both conceptual and procedural knowledge of mathematics. At the same time, SCK further demands an understanding of how conceptual and procedural knowledge are related to one another (Osana & Royea, 2011).

Preservice Teachers' Mathematical Knowledge: What do They Know?

Preservice teachers often struggle with school mathematics. A substantial body of literature documents their difficulties. Concepts related to multiplicative structures, for example, are especially difficult for many preservice teachers. In general, preservice teachers' mathematical reasoning is constrained by additive reasoning making it difficult for them to reason multiplicatively at all (Sowder et al., 1998). Simon and Blume (1994) found that preservice teachers experience difficulties when trying to understand the multiplicative relationship between the sides of similar rectangles. Additionally, many preservice teachers do not have a solid conceptual understanding of multi-digit multiplication or division and as a consequence, they are prone to making systematic procedural errors (Graeber, Tirosh, & Glover, 1989). In terms of division, more often than not, preservice teachers are limited to understanding division in terms of the partitive situation only and struggle to solve and create problems that reflect the measurement situation of division (Simon, 1993). In any context, preservice teachers generally perform poorly when the quotient of a division problem should be less than one (Tirosh & Graeber, 1990). These issues are, at least in part, due to the fact that many preservice teachers have difficulty understanding division procedures and the multiplicative nature of division (Zazkis & Campbell, 1996).

The concepts and procedures related to rational numbers, especially fractions, is another elementary school mathematics topic with which preservice teachers often struggle (Khoury & Zazkis, 1994). Newton (2008) performed an extensive analysis of preservice teachers' conceptual and procedural knowledge of all four mathematical operations with fractions. Overall, the results indicated that preservice teachers maintain many of the same conceptual misunderstandings as children. Furthermore, the preservice teachers in her study also made several procedural errors directly related to their lack of conceptual understanding (Newton, 2008; Toluk-Uçar, 2009). Links between preservice teachers' fraction concepts and procedures are known to be weak (Osana & Royea, 2011) and given preservice teachers' difficulties with division in general (Simon, 1993; Tirosh & Graeber, 1990), it is not surprising that division with fractions is considered especially challenging for this particular population (Flores, Turner, & Bachman, 2005; Kribs-Zaleta, 2006).

Preservice Teachers' Mathematical Knowledge for Teaching: What Should They Know?

Over the years, researchers have struggled to precisely define the specific types of knowledge required to teach mathematics well (Ball, Thames, & Phelps, 2008; Carpenter, Fennema, Peterson, & Carey, 1988; Graeber, 1999; Silverman & Thompson, 2008). Shulman's (1986) seminal work on teacher knowledge has provided the framework for defining the mathematical knowledge teachers need to teach mathematics effectively as recommended by current reform principles (e.g., NCTM, 2009). Shulman contributed to the understanding of teachers' knowledge by marrying content knowledge and pedagogical knowledge to create the notion of pedagogical content knowledge. Content knowledge, also commonly referred to as subject matter knowledge (Ball, 1990; Ball et al., 2008), is the understanding of basic concepts and procedures of any teachable domain. This type of knowledge is not specific only to teaching (Shulman, 1986). Pedagogical knowledge, on the other hand, was specifically conceptualized for the work of teachers. This knowledge encompasses institutionally related knowledge such as the knowledge of curriculum, knowing what needs to be taught at which grade level, and the order in which certain topics should be presented to students (Shulman, 1986). Many subsequent researchers have adopted (Carpenter, Fennema, Peterson, & Carey, 1988;

Rizvi & Lawson, 2007) or extended this influential framework to better understand and study teachers' knowledge (Ball, Hill, & Bass, 2005; Ball et al., 2008).

Common content knowledge and specialized content knowledge. Shulman's (1986) discussion of pedagogical content knowledge changed the way several educational researchers have conceptualized and studied mathematics teacher education. Ball, Thames, and Phelps (2008) have constructed an elaboration of Shulman's original framework that includes an expanded definition of the original content knowledge and pedagogical content knowledge constructs (Ball, Hill, & Bass, 2005). After sorting through the morass of knowledge and skills that elementary teachers are purported to need in their practice, Ball et al. further divided subject matter knowledge and pedagogical content knowledge into more detailed categories specifically suited to mathematics instruction. Under their expanded framework, pedagogical content knowledge and subject matter knowledge consist of three sub-domains, each as illustrated in Figure 1. Pedagogical content knowledge consists of a combination of knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum. By the same token, subject matter knowledge is further subdivided into common content knowledge, specialized content knowledge (SCK), and horizon content knowledge (Ball et al., 2008). Because of the focus of the present study, I will focus the remainder of this review on specialized content knowledge, henceforth referred to as SCK, and more specifically, conceptual mathematical knowledge as a component of SCK.

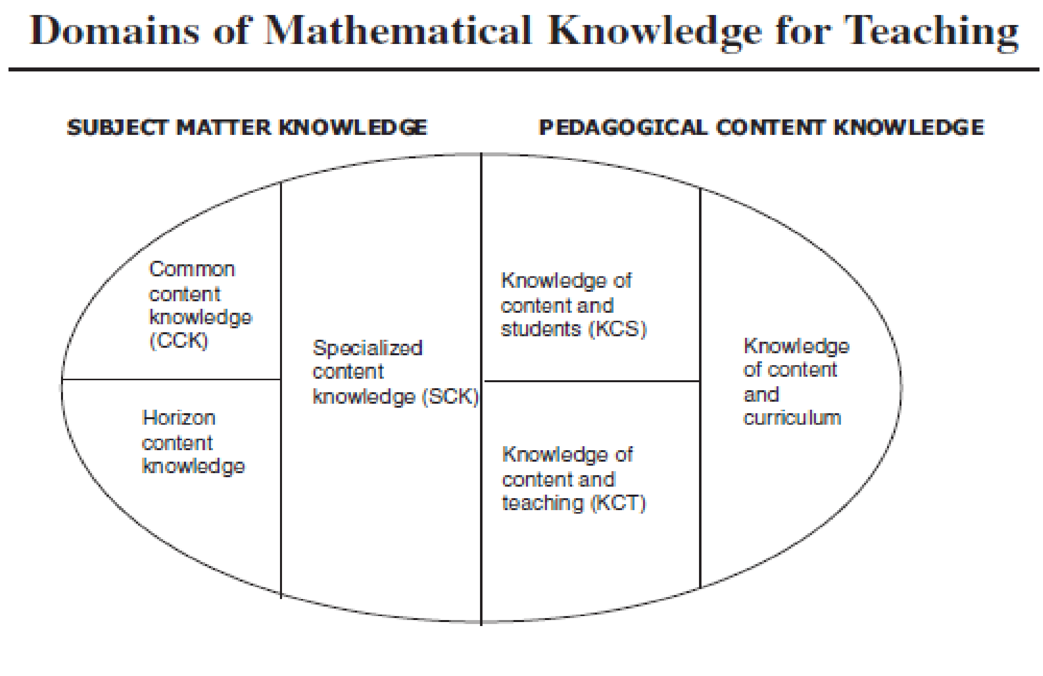


Figure 1. Pictorial representation of mathematical knowledge for teaching as presented by Ball et al. (2008).

Cognitive knowledge types. Ball, Thames and Phelps (2008) are by no means the only researchers to emphasize the role of conceptual knowledge in their framework for mathematical knowledge for teaching. In examining the effects of teachers' mathematical content knowledge on student achievement, Tchoshanov (2011) mapped out three cognitive types of knowledge that teachers have. While these typologies are not explicitly integrated as part of Ball, Thames and Phelps' (2008) mathematical knowledge for teaching framework, there are many parallels between the two. According to Tchoshanov, teachers who possess Type I cognitive knowledge demonstrate the ability to recall basic mathematical facts and correctly apply mathematical rules. This is similar to at least part to the construct of common content knowledge of Ball et al. (2008) in that

Type I cognitive knowledge is procedural in nature and tied to specific mathematical contexts (Tchoshanov, 2011).

Type II and Type III cognitive knowledge parallel SCK (Ball et al., 2008; Tchoshanov, 2011). Teachers who possess Type II cognitive knowledge demonstrate the same skills and understanding as teachers with Type I cognitive knowledge, but are further able to understand the underlying concepts and procedures (Tchoshanov, 2011). Type II cognitive knowledge is not context bound in the same way that Type I cognitive knowledge is. More specifically, teachers with Type II knowledge have a concrete understanding of mathematical concepts that enables them to select multiple representations, transfer knowledge to novel contexts, and solve non-routine problems with relative ease (Tchoshanov, 2011). By this definition, Type II cognitive knowledge is conceptually rich and ties in with the Ball et al. concept of SCK. Finally, Type III cognitive knowledge, according to Tchoshnov (2011), characterizes mathematical knowledge that is removed from any specific mathematical context. It is characterized as encompassing the knowledge of both Type I and Type II cognitive knowledge and the ability to further extend mathematical thinking to making general mathematical statements, designing mathematical tools and models, and proving theorems. Similar to Type II cognitive knowledge, Type III cognitive knowledge requires a rich, conceptual understanding of mathematics and fits into the category of SCK (Ball et al., 2008).

Teachers' Mathematical Knowledge and Student Achievement

The existing research on teachers' mathematical knowledge and student achievement further implicate the important role of teachers' mathematical knowledge in effective teaching. Tchoshnov, Lesser, and Salazar (2008) identified a positive correlation between student achievement and teachers' conceptual knowledge. This correlation was further examined by Tchoshnov (2011), who explored different degrees of teachers' conceptual knowledge (Type I cognitive knowledge, Type II cognitive knowledge, and Type III cognitive knowledge) and how conceptual knowledge translated into classroom teaching practices and student achievement. The results from three sub-studies on these knowledge types indicated that teachers who possessed stronger conceptual knowledge, Type II and Type III, were consequently more conceptual in their teaching and emphasized the relationships between concepts and procedures rather than emphasizing procedural rules alone (Tchoshnov, 2011). At the same time, there was no significant difference in student achievement scores in the classrooms of teachers who possessed Type II cognitive knowledge compared to the classrooms of teachers who possessed Type III cognitive knowledge. More importantly, students of teachers who exhibited both Type II and Type III cognitive knowledge scored significantly higher on measures of mathematical achievement than students of teachers exhibited Type I cognitive knowledge (Tchoshnov, 2011).

Focusing specifically on teachers' SCK, Hill, Rowan, and Ball (2005) similarly found a positive relationship between teachers' SCK and gains in student achievement over a one year period. Moreover, in a correlational study that combined five case studies, a significant positive relationship was found between teachers' level of mathematical knowledge for teaching and students' mathematical achievement (Hill et al., 2008). However the constructs are conceptualized, teachers' conceptual knowledge, procedural knowledge, and SCK appear to be positively related to quality of instruction and student outcomes (Hill, Rowan, & Ball, 2005; Hill et al., 2008; Tchoshnov, 2011).

Therefore, the argument that teachers require strong mathematical knowledge to help students perform in line with reform standards is further supported.

Numeration and Place-Value

Concepts and procedures related to place-value are topics that elementary school teachers will repeatedly encounter during their teaching career. Simply put, place-value refers to the value of digits in relation to the position they appear in a number. For example, the in the number 135, the one does not represent a single unit. Rather, it represents one group of one-hundred units, of 10 groups of 10 units. Similarly the three represents three tens, not three. The position of the digit dictates the digit's value. Enumeration and grouping quantities form the basis of any place-value system. These concepts are further built upon when children learn basic arithmetic. In essence, these mathematical concepts are some of the first that children encounter both informally before starting school (Baroody, Lai, & Mix, 2006; Wynn, 1990) and formally when they do start school (Baroody, 1990; Canobi, Reeves, & Pattison; 2003). Because of the way other mathematical topics build on place-value notions, place-value is arguably one of the most important concepts for elementary students to master (McClain, 2003). Furthermore, children's understanding of place-value and related place-value concepts can serve to help or hinder future mathematical achievement (Fuson, 1990; Fuson & Briars, 1990).

Unfortunately, traditional education produces dire effects on children's understanding of place-value (Hiebert & Wearne, 1992) because teachers approach the topics as a series of rules and procedures for writing numbers and performing operations with little emphasis on concepts and the relationships between those concepts and place-value procedures (Fuson et al., 1997b). Teachers who lack strong conceptual and procedural mathematical knowledge perpetuate the cycle of weak mathematical knowledge. The students of those teachers will themselves acquire mathematical knowledge that is conceptually and procedurally weak. These students often grow up disliking and avoiding mathematics (Jackson & Leffingwell, 1999).

Children's knowledge of place-value. A review of the literature on children's understanding of place-value suggests that there are three general requirements for acquiring meaningful learning of place-value concepts and procedures (Fuson, 1990; Fuson & Briars, 1990). The first essential place-value requirement addresses counting, regrouping, and written notation (Baroody, 1990; Hiebert & Wearne, 1992). Children must understand a variety of mathematical concepts to fulfill this requirement. Not only do children need to learn the number words in the correct sequence and how to write and read numbers, they also need to learn the relationship between number words and the value they represent in relation to the way they are written (Fuson, 1990). In general, children must have a sound understanding of the concepts related to counting and grouping sets of objects including using written notation (Hiebert & Wearne, 1992).

The second requirement of learning place-value with meaning pertains more specifically to the construction of the concept of unit. Conceptually understanding the construction of multiunit structures is a crucial component of the development of conceptual knowledge of place-value (Fuson, 1990; Miura, Okamoto, Kim, Steere & Fayol, 1993). As opposed to seeing numbers as collections of individual units, conceptual understanding of place-value requires children to be able to conceptualize individual and composite units in tandem (Fuson, 1990). For example, given the number 100, conceptual understanding of the multiunit structure requires that children can flexibly conceptualize the number as one unit of a hundred or as 10 units of ten or as 100 single units. The position of each digit dictates the quantity of units or composite units that could be used to represent the number. Groups of units need to be understood and treated as composite units as opposed to "concatenated numbers" (Fuson, 1990; Hiebert & Wearne, 1996). In addition to having conceptual understanding of numeration concepts and notation (Baroody, 1990; Hiebert & Wearne, 1992) and multiunit structures or composite units (Fuson, 1990), children must further conceptually understand the third requirement: how to regroup composite units in such a manner as to preserve quantity (Fuson & Briars, 1990; McClain, 2003).

Elementary school children's difficulties with place-value. Mathematical competence in place-value relies on children developing relationships between their procedural and conceptual knowledge (Bisanz & Lefevre, 1992; Hiebert & Wearne, 1996). It has been well documented that elementary school children have difficulties with place-value (Baroody,1990; Fuson 1986; Fuson 1988; Fuson, 1990, Fuson and Briars, 1990). Moreover, the difficulties associated with place-value are often the result of inadequate or disconnected conceptual knowledge in part because of the mathematics instruction they receive in school (Fuson & Briars, 1990). The basis of many children's difficulties with place-value is embedded in language and counting words (Saxton & Towse, 1998). This is especially true for English speakers (Miura, Okamoto, Kim, Steere, & Fayol, 1993). English counting words do not support the construction of meaningful grouping and place-value. In English decade number words especially, the relationship between the tens and ones is not made obvious by the actual number names (Fuson et al., 1997a).

Miura, Okamoto, Kim, Steere and Fayol (1993) explored the way children from different countries cognitively represented multi-digit numbers. After examining French, Japanese, Korean, Swedish, and American children's conceptions of place-value, it was suggested that the language spoken affects children's cognitive representations of numbers and consequently their conceptual understanding of place-value. English speaking children are more likely to develop the concatenated (Fuson, 1990) understanding of multi-digit numbers, whereas children who speak languages that demonstrated the relationships between the value of the position of the digit and the value of the digits in a multi-digit number ("ten and two" as opposed to "twelve" in English) are less likely to develop this same misconception (Miura et al., 1993).

The most prominent misconception children hold about place-value stems from their conception of multi-digit numbers (Fuson, 1990). Many children perceive multi-digit numbers as unitary (Bowers, Cobb, & McClain, 1999). That is, rather than seeing the number, say 23, as composed of two units or groups of ten and three single units, their understanding of the quantity is limited to the 23 single units interpretation. Fuson (1990) refers to this conceptual misunderstanding as the "concatenated single-digit number" misconception. This particular conceptual misunderstanding is believed to be the catalyst for other place-value misconceptions and associated procedural errors (Kouba et al., 1988).

Counting and multi-digit addition and subtraction are often taught as a sequence of steps (Fuson & Briars, 1990). And as a result, children's difficulties with multi-digit arithmetic are apparent on both a procedural and conceptual level. For instance, procedural fluency does not necessarily indicate that a child possesses adequate conceptual knowledge (Cauley, 1988; Fuson et al., 1997b). Cauley (1988) found that elementary school children with high levels of procedural knowledge of multi-digit subtraction had incomplete conceptual knowledge. In fact, many children are capable of correctly carrying out arithmetic procedures with little or no conceptual understanding of the process. Several other studies have found that among students who followed the appropriate procedures to add or subtract two multi-digit numbers, many arrived at the correct answer but did not understand the crucial aspects of the procedure they followed and were incapable of explaining the value of and reasons for regrouping (Fuson & Briars, 1990; Labinowicz, 1985; Olivier, Murray, and Human, 1990; Resnick & Omanson, 1987). Furthermore, lack of conceptual understanding and reliance on procedures may account for student misconceptions such as always subtracting the smaller number from the bigger number, despite the order in which the numbers are presented (Fuson & Briars, 1990).

Remediating children's knowledge of place-value. Given the importance of place-value and the weak nature of children's place-value knowledge, it is important for teachers to understand and teach place-value in a meaningful and connected manner (McClain, 2003; Yackel, Underwood, & Elias, 2007). Understanding the source of student difficulties and developing activities and lessons that highlight relevant concepts to children can help remediate those difficulties (Ball, Thames, & Phelp, 2008). At the same time, those particular teaching activities require teachers to possess a substantial amount of conceptual knowledge and specialized content knowledge (Ball, 1990; Ball, Hill, & Bass, 2005; Ball et al., 2008).

Even though children have difficulty with place-value concepts, the existing literature shows promise in terms of remediating their performance. The most successful instructional interventions on place-value with children focus on building and strengthening children's conceptual understanding of critical place-value concepts (Fuson, 1990; Fuson & Briars, 1990; McClain, 2003). The main feature of any conceptually based instruction is the notion of creating links and connections between mathematical ideas, whether those ideas are conceptual or procedural in nature (Lesh, Post, & Behr, 1987). With elementary school children, manipulatives are often used to aid in the development of conceptual understanding of mathematical topics and procedures (Canobi, Reeve, & Pattison, 2003; Fuson & Briars, 1990; Osana & Pitsolantis, 2011).

Fuson (1990) observed that children's knowledge of place-value is negatively affected by the traditional manner in which it is taught in the school system. After an examination of traditional elementary school mathematics textbooks, it was determined that place-value is traditionally approached as a series of rote procedures applied without meaning. As a consequence, Fuson designed a conceptually based instructional intervention on place-value concepts mirroring the topics taught within a traditional curriculum while emphasizing the links between the various concepts and procedures. Following the intervention, first-grade students who completed the conceptually based instruction performed significantly better on measures of conceptual knowledge of multi-digit addition and subtraction than first-grade students who followed the traditional place-value lessons as prescribed by a textbook (Fuson, 1990).

In another study, Fuson and Briars (1990) used conceptually based place-value instruction to remediate second graders' place-value understanding. The results demonstrated that students who were average achievers in multi-digit addition and subtraction performed significantly better following the intervention and demonstrated meaningful regrouping for both operations (Fuson & Briars, 1990). Similarly, Hiebert and Wearne (1992) implemented a conceptually based instructional unit on and multi-digit addition and subtraction focusing on the use of manipulatives to make place-value concepts and connections between concepts and procedures explicit. Compared to conventional textbook instruction, first graders who engaged in the conceptual instruction demonstrated significantly higher levels of conceptual understanding of multi-digit addition and subtraction procedures and were more flexible in their selection of problem solving strategies (Hiebert & Wearne, 1992).

Preservice teachers' understanding of place-value. Several important points about place-value and preservice teachers' knowledge have been established in this review thus far. First, place-value is an important topic in the elementary mathematics curriculum (Yackel, Underwood, & Elias, 2007). Secondly, elementary school students often experience difficulties related to their lack of conceptual understanding of place-value concepts (Hiebert & Wearne, 1996, Carpenter & Moser, 1984). Thirdly, conceptually based instruction has been shown to help elementary school children develop conceptual understanding of place-value and procedural fluency (Fuson & Briars, 1990). Finally, it is well known that preservice teachers' knowledge of school mathematics tends to be conceptual weak (Ball, 1990; Newton, 2008) and procedurally rule bound (Zazkis & Campbell, 1996), frequently leading them to hold similar misconceptions and commit the same errors as elementary school students (Harel & Behr, 1995). At the same time, there is a near dearth of research on preservice teachers' understanding and learning of place-value.

McClain (2003) explored whether or not the results obtained from studies conducted on elementary students' understanding of place-value could be used as a guide to developing preservice teachers' understanding of place-value. By designing and implementing conceptually based lessons on place-value using a base-8 context, McClain found that the learning trajectory of preservice teachers was, in fact, similar to that of elementary students. Furthermore, using conceptual lessons in base-8, as opposed to base-10, served to help preservice teachers develop their conceptual understanding of the positional place-value system without interference from prior knowledge and known procedures (McClain, 2003). Yackel, Underwood, and Elias (2007) also used a base-8 place-value system to help develop preservice teachers' conceptual understanding of place-value and to create learning experiences that parallel those of elementary students who have little prior knowledge of place-value when they start formal schooling. Preservice teachers were thus afforded the opportunity to reconceptualise their view of mathematics and what it means to learn and teach mathematics with understanding (Yackel, et al., 2007). It is important to note here that even though these interventions appear to be effective, none of these studies examine the nature of the students' developing knowledge of place-value.

Conceptual and Procedural Knowledge in Mathematics

The relationship between concepts and procedures. Conceptual knowledge and procedural knowledge do not exist independently of one another. While they are considered distinct types of knowledge (Rittle-Johnson & Sielger, 1998), conceptual knowledge and procedural knowledge are interactive. At the same time, the development and exact relationship between conceptual and procedural knowledge is not completely understood. In several different mathematical domains, past research has demonstrated that children who are stronger than their peers in conceptual knowledge tend to also be stronger in procedural knowledge (Baroody & Gannon, 1984; Byrnes & Wasik, 1991; Cowan, Dowker, Christakis & Bailey, 1996; Cowan & Renton, 1996; Dixon & Moore, 1996). Regardless of the multitude of studies that demonstrate the influence of conceptual knowledge on procedural knowledge, it is strongly believed that the relationship between these two types of knowledge is in fact, bidirectional in nature (Rittle-Johnson & Koedinger, 2009; Rittle-Johnson & Siegler, 1998; Rittle-Johnson, Siegler, & Alibali, 2001).

***Conceptual knowledge affecting procedural knowledge*.** In general, two important ways that conceptual knowledge impacts procedural knowledge is in terms of selecting and generating procedures to solve mathematical problems (Rittle-Johnson & Siegler, 1998). Possessing conceptual understanding of a mathematical topic helps children identify the essential elements of a given problem and facilitates the selection of an appropriate known procedure to solve it (Rittle-Johnson & Alibali, 1999). Mestre (2002) found that undergraduate students who scored low on a measure of conceptual understanding relied on memory to select procedures that were used to solve problems that were similar according to superficial characteristics. Thus, they were less successful at choosing an appropriate procedure compared to their peers who scored higher on conceptual understanding. Conceptual knowledge constrains procedural selection by helping children recognize when a given procedure is inappropriate (Rittle-Johnson & Alibali, 1999). Gains in conceptual knowledge might help people recognize the application of incorrect procedures by highlighting inconsistencies with the procedure being used and their conceptual understanding of the mathematical situation (Byrnes & Wasik, 1991; Rittle-Johnson & Siegler, 1998).

Moreover, adequate conceptual knowledge has a positive impact on procedural knowledge in terms of procedural generation. In fact, several theories on knowledge acquisition postulate that the generation of procedures in children is based on the conceptual understanding of the mathematics embedded within the problem being approached (Gelman & Williams, 1997; Halford, 1993). Not only does conceptual understanding constrain procedural selection, it similarly constrains procedural discovery and the adaptation of existing procedures to novel mathematical tasks (Gelman & Gallistel, 1978; Gelman & Meck, 1986; Siegler & Crowley, 1994). Therefore, understanding the underlying concepts and the interrelations between concepts and procedures facilitates the appropriate generation of procedures to solve a mathematical problem while reducing the likelihood of over-generalizing the procedure to an inappropriate context.

In addition to influencing the selection of known procedures (Rittle-Johnson & Siegler, 1998) and the generation and adaptation of procedures (Siegler & Crowley, 1994), conceptual understanding also plays a role in future procedural gain (Hiebert &

Wearne, 1996). Hiebert and Wearne (1996) found that children's conceptual knowledge can also be used to predict future fluency in procedural skill. That is, children who scored higher in conceptual knowledge in early elementary school tended to acquire higher procedural knowledge than children who originally scored lower in conceptual knowledge (Hiebert & Wearne, 1996). Other studies have found similar results (Fuson, 1990; Fuson & Briars, 1990; McClain, 2003).

Procedural knowledge affecting conceptual knowledge. It has been established that conceptual knowledge impacts procedural knowledge, yet the relationship is not unidirectional. Procedural knowledge may positively impact children's acquisition of conceptual knowledge. When children are procedurally fluent, the acquisition of conceptual knowledge is sometimes facilitated (Rittle-Johnson & Alibali, 1999). When solving mathematical problems using well-known, automatic procedures, working memory is freed up, which may promote children's metacognition about conceptual aspects of the procedure. Such reflection may lead to an exploration of why the procedure works and thereby further develop conceptual understanding (Rittle-Johnson & Siegler, 1998). At the same time, the same correct procedural knowledge may further help children expand on their conceptual knowledge by helping them focus more specifically on accurate and related concepts (Rittle-Johnson & Koedinger, 2009).

The acquisition of concepts and procedures. The fact that conceptual and procedural knowledge are related to each other is unquestionable (Byrnes & Wasik, 1991; Fuson, 1990; Rittle-Johnson & Alibali, 1999). At the same time, the exact mechanism by which conceptual and procedural knowledge is acquired is still unclear. Over the years, researchers have studied children across several mathematical domains from enumeration (Baroody, 1990; Gelman & Gallistel, 1978) to fractions (Mack, 1990; Yoshida & Sawano, 2002) in an attempt to understand how mathematical knowledge is acquired. Despite the inconsistencies found within the literature on this topic, three important theories of the acquisition of mathematical concepts and procedures have emerged. Traditionally, two of these theories dominated. The knowledge acquisition debate centered on whether or not mathematical concepts were acquired before procedures, the Concepts-first perspective, or if mathematical procedures were acquired before mathematical concepts, the Procedures-first perspective (Rittle-Johnson & Siegler, 1998). More recently, the third theory, the Iterative perspective, has come to the forefront of the debate (Rittle-Johnson & Koedinger, 2002). The Iterative perspective contends that conceptual and procedural knowledge are not acquired sequentially with one type preceding the other. Instead, mathematical knowledge is acquired in an iterative manner, with gains in one type of knowledge leading to gains in the other, thereby fostering further gains in the first type of knowledge.

Concepts-first perspective. Many researchers have postulated that mathematical knowledge develops in a sequential manner with mathematical concepts being learned before procedures (Geary, 1994; Halford, 1993; Wynn, 1992). If this is, in fact, how mathematical knowledge is acquired, it should have an important influence on how mathematics is taught in the classroom. That is, if concepts are developed before procedures, children should be taught mathematical concepts before traditional algorithms in an attempt to teach mathematics with meaning and in a fashion that corresponds to children's development. In fact, before starting formal education, very young children demonstrate conceptual understanding of enumeration (Gelman &

Gallistel, 1978), certain characteristics of small quantities (Feigenson, 2005; Wood & Spelke, 2005), and even simple arithmetic (Geary, 1994; Wynn, 1992). In some domains, it is widely accepted that conceptual knowledge does precede procedural knowledge. For example, Baroody (1992), Wynn (1990), and Fuson (1988) all supported the claim that children develop a sophisticated conceptual understanding of counting principles well before school-age. Moreover, several studies have found that not only do young children possess these rich conceptual understandings, but at the same time, they do not yet possess procedural fluency, suggesting that conceptual knowledge, at least in certain domains, precedes procedural knowledge (Byrnes & Wasik, 1991; Gelman & Meck, 1983).

Procedures-first perspective. Despite the evidence that children acquire several mathematical concepts before entering school, many school-aged children perform mathematical procedures with very little conceptual understanding (Olivier, Murray, and Human, 1990; Resnick & Omanson, 1987). If conceptual knowledge precedes procedural knowledge, children should not be able to perform mathematical procedures successfully in the absence of knowledge of the related concepts. Nonetheless, children and preservice teachers have demonstrated some procedural fluency and knowledge of fractions despite their impoverished conceptual knowledge of this topic (Mack, 1990; Newton, 2008; Yoshida & Sawano, 2002). Similarly, children have demonstrated the ability to perform multi-digit arithmetic without conceptual understanding of the procedures they are following (Fuson & Briars, 1990; Labinowicz, 1985).

Resolving the paradox. Some researchers have explained the Concepts-First or Procedures-First paradox using the privileged domains hypothesis (Geary, 1995; Gelman

& Meck, 1992). The privileged domains hypothesis stipulates that while in general, procedures are acquired before concepts, some mathematical concepts are easier for children to learn conceptually because they are of potential evolutionary importance and thereby remain evolutionarily privileged (Rittle-Johnson & Siegler, 1998). Therefore, in the privileged domains, concepts exceptionally develop before procedures.

Another attempt to resolve the Concepts-First or Procedures-First paradigm is provided by the frequency of exposure hypothesis (Baroody & Gannon, 1984; Fuson, 1988). Similar to the privileged domains hypothesis, the frequency of exposure hypothesis simply purports that while procedures are generally acquired before concepts, the frequent opportunities provided by children's environments to observe certain activities, such as counting, allow children to imitate and acquire certain conceptual knowledge and skills considered privileged by the former hypothesis (Briars & Siegler, 1984).

Iterating between concepts and procedures. While researchers have been examining children's acquisition of conceptual and procedural mathematical knowledge, it has become apparent that there is no clear cut evidence favouring either the Concept-first perspective or the Procedures-first perspective universally across all mathematical domains (Gelman & Gallistel, 1978; Hiebert & Lefevre, 1986; Siegler, 1991; Wynn, 1990). The third and more recent perspective on the acquisition of mathematical knowledge, the Iterative perspective, takes into account many of the aspects of the Concept-first perspective and the Procedures-first perspective (Rittle-Johnson, Siegler, & Alibali, 2001). The Iterative perspective argues that there is a complex, bidirectional relationship between conceptual and procedural knowledge (Rittle-Johnson & Kodinger, 2009). This relationship is more interactive than either of the other two theories presumes. Regardless of whether or not the initial piece of knowledge is conceptual or procedural in nature, the Iterative perspective argues that increases in one type of knowledge foster increases in the other type of knowledge, thereby further fostering gains in the first type (Rittle-Johnson & Kodinger, 2009; Rittle-Johnson, Siegler, & Alibali, 2001).

Established relationships between conceptual and procedural knowledge, such as the positive relationship between conceptual and procedural knowledge (Cauley, 1988) and the predictive relationship of conceptual knowledge on procedural knowledge (Hiebert & Wearne, 1996), lend credence to the Iterative perspective. Moreover, other studies on the acquisition and nature of children's mathematical knowledge have further suggested that gains in procedural knowledge support gains in conceptual knowledge (Brynes & Wasik, 1991; Hiebert & Wearne, 1996). As previously discussed in this review, conceptual instruction has also been shown to improve procedural fluency (Rittle-Johnson & Siegler, 1998). Further research on children's development of mathematical equivalence that explicitly examined the effect of lesson type provide further support for the bidirectional relationship between conceptual and procedural knowledge (Rittle-Johnson & Alibali, 1999). Rittle-Johnson and Alibali (1999) examined the impact of conceptual and procedural lessons on children's knowledge of equivalence with addition. The researchers reported that children in the fourth and fifth grades who received conceptual instruction demonstrated increased conceptual understanding as well as greater correct procedural generation and transfer, while students who received the procedural instruction also demonstrated increased conceptual understanding, but limited transfer (Rittle-Johnson & Alibali, 1999).

Several other studies directly examined the impact of iterating between conceptual and procedural lessons on children's mathematical knowledge (Rittle-Johnson & Koedinger; 2002; Rittle-Johnson, Siegler, & Alibali, 2001). Rittle-Johnson and Koedinger (2009) assumed that if knowledge develops in an iterative fashion, then iterating between conceptual and procedural lessons should improve learning. Rittle-Johnson and Koedinger (2009) compared the effects of an Iterative lesson sequencing to a Concepts-First lesson sequencing on sixth grade students' knowledge of decimal numeration. The results indicated that while children in both conditions had comparable gains in conceptual knowledge, those in the Iterative condition also gained more procedural knowledge and demonstrated the ability to transfer procedures to novel situations. Perhaps more importantly, pretest knowledge of concepts predicted posttest knowledge of procedures and vice versa, thereby supporting the Iterative perspective of the acquisition of mathematical knowledge (Rittle-Johnson & Koedinger, 2002; Rittle-Johnson & Koedinger, 2009). Rittle-Johnson and Koedinger's (2009) results regarding students' abilities to transfer their knowledge to novel situations also suggest that the Iterative sequencing of lessons resulted in mathematical knowledge that was more connected and conceptually rich, which is required for the adequate development of SCK for teachers (Ball et a., 2008). Iterating between conceptual and procedural lessons may highlight the links between concepts and procedures, thereby creating a tightly woven network of knowledge (Rittle-Johsnon & Koedinger, 2009; Rittle-Johnson & Siegler, 1998; Rittle-Johnson, Siegler, & Alibali, 2001).

The Iterative Perspective and Mathematical Knowledge Gains

While there is evidence to support the Iterative perspective (Rittle-Johnson & Koedinger, 2009; Rittle-Johnson, Siegler, & Alibali, 2001), it is important to explore why iterating between concepts and procedures leads to more gains in knowledge than presenting concepts before procedures. There are three main reasons related to concepts and procedures that may explain why iterating between them may lead to greater gains in mathematics learning. The first explanation is related to the cognitive load associated with presenting all the concepts first. Presenting all the relevant concepts of a mathematical domain in the absence of any procedures takes a toll on working memory (Shrager & Siegler, 1998). Teaching procedures in tandem with the associated concepts, or closely together, may lighten the cognitive load, thereby freeing up the capacity to further reflect on the concepts being used (Rittle-Johnson & Koedinger, 2002; Rittle-Johnson & Siegler, 1998).

In addition to freeing up cognitive space and potentially enabling metacognitive activities, iterating between concepts and procedures may help students better understand how concepts and procedures are related to one another (Rittle-Johnson & Koedinger, 2002). Exposure to a concept followed by an appropriate procedure that capitalizes on that concept may encourage students to integrate both the concept and procedure into a meaningful network of usable knowledge (Resnick & Omanson, 1987). That is, iterating between concepts and procedures may highlight the relevance of concepts on procedures and procedures on concepts, fostering the integration of the two (Rittle-Johnson & Koedinger, 2002).

Finally, another reason that iterating between concepts and procedures may lead to more gains in conceptual knowledge than presenting all the concepts first is related to procedural generalization. Procedural generalization is when known procedures are generalized or applied to situations different from the situation in which they were learned. Iterating between lesson types may encourage appropriate generalizations (Rittle-Johnson & Koedinger, 2002; Rittle-Johnson & Siegler, 1998). In fact, iterating between varied but related tasks is believed to support appropriate generalizations (Anderson, 1993) and discourage overgeneralizations as well (Rittle-Johnson & Albali, 1999).

Present Study

The objective of this study was to examine the effect of lesson sequencing on the acquisition and nature of preservice teachers' mathematical knowledge. More specifically, this study examined the relative effects of an Iterative lesson sequencing, a Concept-first lesson sequencing, and a Procedures-first lesson sequencing on preservice teachers' conceptual knowledge, procedural knowledge, and specialized content knowledge of place-value concepts. This study also further examined the nature of the preservice teacher's place-value knowledge and how this knowledge changed as a result of different lesson sequences.

A group of preservice teachers received a computer-based instructional intervention on place-value concepts and procedures. Those concepts and procedures involved naming numbers, counting, grouping, and multi-digit addition and subtraction. All of the participants completed the same eight lessons on place-value (four conceptual lessons and four procedural lessons), but the order of the lessons differed according to the conditions to which the participants were assigned. The eight lessons were administered over four instructional sessions of two lessons per session. All concepts and procedures taught during the lessons were in a base-7 context. I assumed that using base-7 prevented the participants from using any concepts or procedures they may already have known.

About a week before and a week after the intervention, the participants completed a paper-and-pencil test, called the Numbers and Operations Test (NOT), that measured their base-7 conceptual and procedural knowledge, their base-10 conceptual and procedural knowledge, as well as their SCK of place-value concepts. In addition, the test contained items designed to assess the participants' ability to transfer their procedural knowledge to novel problems in base-5.

The three research questions addressed in the present study are the following:

(1) Will the Iterative sequencing of place-value lessons result in greater increases in conceptual scores, procedural scores, and SCK scores compared to the Concepts-First sequencing?

(2) Will the Iterative sequencing of place-value lessons result in greater increases in conceptual scores, procedural scores, and SCK scores compared to the Procedures-First sequencing?

(3) How will the nature of preservice teachers' knowledge change as a result of lesson sequencing?

Method

Participants

The participants were 33 undergraduate students enrolled in an elementary teacher training program at an urban, English language university in Canada. All of the participants were registered in the first of three required teaching mathematics methods courses. The participants were recruited to participate in this project during the first teaching mathematics method course and because the instructional intervention was directly related to the course curriculum, participation in all instructional sessions, including the pretest and the posttest, was a required part of the course. The intervention took place over the fall semester of 2011 and none of the participants had yet completed any of the three required teaching mathematics methods courses. All 33 participants (Concepts-first, n = 11; Procedures-first, n = 10; Iterative, n = 12) were included in the qualitative analysis, whereas only 29 of the participants (Concepts-first, n = 10; Procedures-first, n = 9; Iterative, n = 10) were included in the quantitative analysis because any participant who missed one or more of the four evaluations was dropped from the quantitative analysis.

The majority of the participants were female (n = 30). The average age of all of the participants was 26 years old with ages ranging from 19 to 43 years. All of the participants, except for one, reported having some teaching experience in the form of teaching internships, private tutoring, or classroom teaching. All of the participants started the program in the fall semester. Twenty-two (22) of the participants had started the program in 2010, eight in 2009, and three in 2008.

Design

This study used a three condition pretest-posttest experimental design. The three conditions were Concepts-first, Procedures-first, and Iterative. All of the participants were randomly assigned to one of the three conditions. Several types of place value knowledge were evaluated throughout this intervention. To evaluate the participants' conceptual base-7 knowledge, procedural base-7 knowledge, and SCK of place-value and the effects of lesson sequencing on this knowledge, the participants were evaluated at four time points throughout the instructional intervention (see Figure 2). The first and last assessment points are called here pretest-T1 and posttest-T4, respectively. Furthermore, the participants' conceptual base-10 knowledge, procedural base-10 knowledge, and procedural transfer in base-5 were evaluated at two time points, namely pretest-T1 and posttest-T4 (see Figure 3).

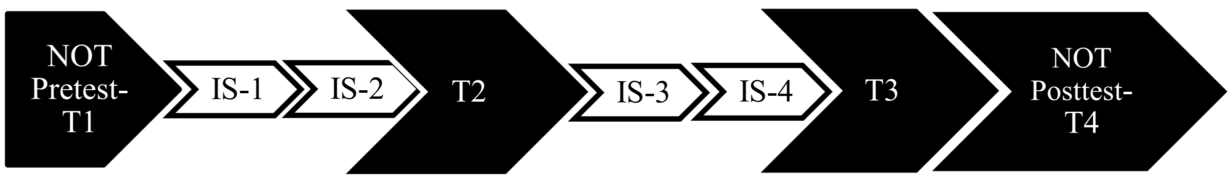


Figure 2. Schematic of assessment time points of conceptual base-7 knowledge, procedural base-7 knowledge, and SCK. The black shapes represent when the assessments took place. The white shapes represent the instructional sessions.

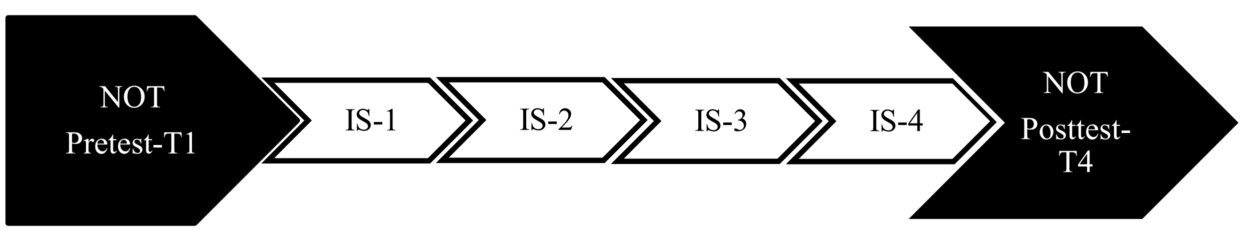


Figure 3. Schematic of assessment time points of conceptual base-10 knowledge, procedural base-10 knowledge, and transfer. The black shapes represent when the assessments took place. The white shapes represent the instructional sessions.

The study was designed so that the pretest and the postttest would take place during lecture time and all of the instructional sessions would take place during the designated lab of the mathematics method course. Lab sessions were scheduled every week for four weeks for this purpose. Two lessons were to be completed during each of the four instructional sessions. Following the administration of the Numbers and Operations Test (NOT), the participants completed the instructional intervention, which consisted of a series of lessons on counting, grouping, and place-value concepts in base-7. Some of the lessons focused on the concepts central to these topics and other lessons focused on related procedures. A week after the instructional intervention, the participants completed the NOT again as the postttest.

There were four instructional sessions and two lessons per instructional session. The Concepts-first condition, Procedures-first condition, and Iterative condition received exactly the same lessons. The difference between the three conditions was the sequencing of the lessons (see Table 1). That is, the order in which the lessons were presented varied depending on the condition. The Concepts-first condition received all of the conceptual lessons before the procedural lessons. The Procedures-first condition received all of the procedural lessons before the conceptual lessons. Consequently, the Iterative condition received one conceptual lesson followed by the related procedural lesson before receiving the subsequent conceptual lesson; the remaining lessons iterated between conceptual and procedural in this fashion.

Table 1

Presentation of Lessons by Condition

	Concepts-first		Procedures-first		Iterative	
Instructional Session 1	C1	C2	P1	P2	C1	P1
Instructional Session 2	C3	C4	P3	P4	C2	P2
Instructional Session 3	P1	P2	C1	C2	C3	P3
Instructional Session 4	P3	P4	C3	C4	C4	P4

Notes. C = conceptual lesson; P = procedural lesson.

After completion of the second instructional session, at which time the Concepts-first condition had received all of the conceptual lessons, the Procedures-first condition had received all of the procedural lessons, and the Iterative condition had received half of the conceptual lessons and half of the procedural lessons (see Table 1), all of the participants' conceptual knowledge, procedural knowledge, and SCK of place-value was evaluated again during the lab session (see T2 in Figure 2). A week later, the participants completed instructional session 3. The fourth and last instructional session was a week after instructional lesson 3. After the fourth session, all of the participants had received all of the conceptual and procedural lessons (see Table 1), their conceptual base-7 knowledge, procedural base-7 knowledge, and SCK was assessed again (see T3 in Figure 2). Finally, at posttest-T4, the participants' conceptual base-7 knowledge, procedural base-7 knowledge, SCK, conceptual base-10 knowledge, procedural base-10 knowledge of place-value, and transfer were evaluated for the final time using the NOT.

To summarize, there were six outcome measures, three measures were used at all four time points. These measures were: (1) conceptual place-value knowledge in base-7, (2) procedural place-value knowledge in base-7, and (3) SCK of place-value. Three additional measures were assessed twice during the instruction intervention. These measures were: (1) conceptual place-value knowledge in base-10, (2) procedural place-value knowledge in base-10, and (3) transfer to base-5 (see Table 2).

Table 2

Assessment of Outcome Measures across all Four Time points

	Outcome Measures					
	CK7	PK7	SCK	CK10	PK10	T
Pretest-T1	X	X	X	X	X	X
T2	X	X	X	--	--	--
T3	X	X	X	--	--	--
Posttest-T4	X	X	X	X	X	X

Notes. CK7= conceptual base-7 knowledge; PK7 = procedural base-7 knowledge; CK10 = conceptual base-10 knowledge; PK10 = procedural base-10 knowledge; T = transfer. An X indicates that the measure was assessed whereas a "--" indicates that the measure was not assessed. T2 = time point 2; T3 = time point 3.

Measures and Instruments

Demographics. As part of the pretest, participants were asked to fill out a demographic survey. This paper-and-pencil survey collected information such as gender, age, semester and year when the participants entered the teacher preparation program, and a summary of the participants' teaching experience (see Appendix A).

Numbers and Operations Test. The Numbers and Operations Test (NOT) is a paper-and-pencil measure that was administered at pretest-T1 and the posttest-T4. It was designed specifically for this study to evaluate the participants' conceptual and procedural knowledge of counting, grouping, multi-digit addition, and multi-digit subtraction in the context of a base-7 place-value system. The NOT also assessed SCK, multi-digit addition and multi-digit subtraction in base-10, as well as transfer of procedural knowledge. While the test was designed to evaluate base-7 and base-10 place-value knowledge, some general information (see Figure 4) about these two place-value systems was provided at the beginning of the NOT at both pretest-T1 and posttest-T4. This was done to prevent the participants, who had probably never explicitly been exposed to a numeration system other than base-10, from being overwhelmed by the test and to prevent a floor effect for the base-7 items. The NOT is presented in Appendix B.

General Information

In our everyday lives, the counting and numeration system that we use is called "**base-10.**" At the same time, other counting and numeration systems exist. Throughout this handout, you will be asked to complete a variety of tasks in both our regular way of counting and in different bases. Please be careful to make sure you are using the correct way of counting as indicated in the instructions. When a number is in base-10, it will be written in the way you are used to seeing.

For example, the following numbers are in base-10: 1, 16, 25, fifty, one-hundred-two.

When a number is in a different base, this will be indicated by subscripts (little numbers written lower and to the right) or written in words.

For example, the following numbers are in base-7:

1_7	16_7	25_7
↑	↑	↑
subscript	subscript	subscript

The following numbers are also in base-7: five-zero base-7, one-zero-two base-7.

Figure 4. Introductory note on the NOT.

The NOT consists of several subscales: a conceptual base-7 subscale, a procedural base-7 subscale, a conceptual base-10 subscale, a procedural base-10 subscale, an SCK subscale, and a transfer subscale consisting of two items assessing procedural knowledge in base-5. Including the two transfer items, there is a total of 38 items. Of these items, 6 assess conceptual knowledge, 28 assess procedural knowledge, 2 assess SCK, and two assess procedural transfer knowledge (see Table 3 for a breakdown of the items by type).

All test items except for the SCK, transfer, and base-10 items are analogous to the types of questions that were used during the instructional intervention and are thus considered familiar tasks. The SCK items require the analysis of hypothetical students' work on multi-digit addition and subtraction problems in base-10 and are considered novel items.

Table 3

Numbers and Operations Test Item Distribution

	Number of Questions			
	Base-7	Base-10	Base-5*	Total
Conceptual Addition	2	1	0	3
Conceptual Subtraction	2	1	0	3
Procedural Addition	5	5	1	11
Procedural Subtraction	3	6	1	10
Procedural Enumeration	9	0	0	9
SCK Addition	0	1	0	1
SCK Subtraction	0	1	0	1
Total	21	15	2	38

Note. *Indicates transfer items.

Conceptual base-7 subscale. There are 4 familiar conceptual items on the NOT that focus on assessing the concepts behind addition and subtraction. To assess concepts of addition, two of the conceptual items display two groups of items (e.g., stars) placed in columns in a table. Participants were asked to join the two quantities in base-7. Similarly, to assess concepts of subtraction, the other two conceptual items also display two groups of items placed in by columns in a table. This time, the participants are asked to subtract the second quantity from the first in base-7. Please refer to Figure 5 to see a sample addition item and Figure 6 for a sample subtraction item.

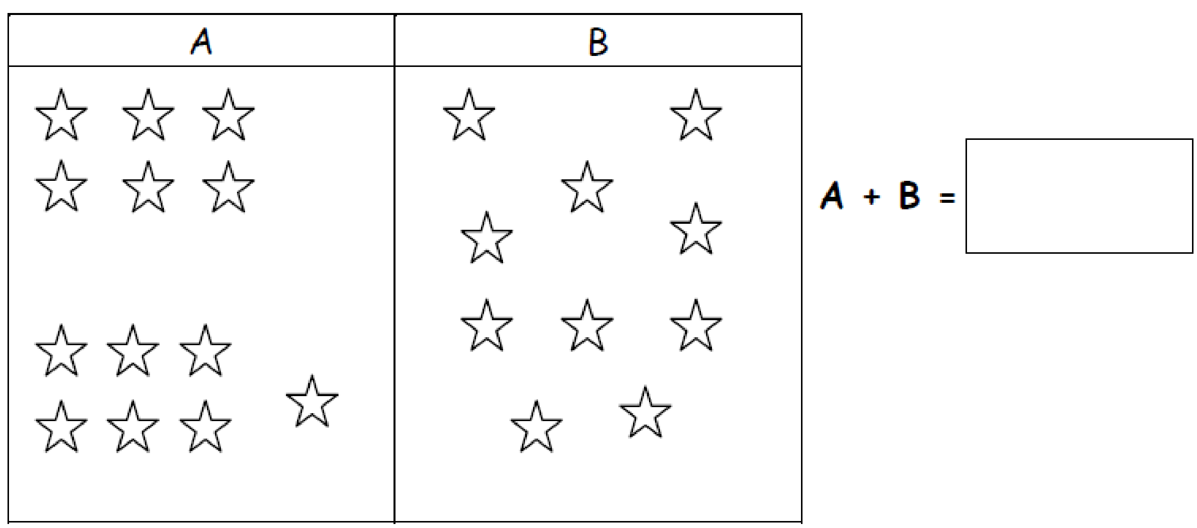


Figure 5. Sample conceptual base-7 addition item. Participants were expected to join the quantities in the two columns. The correct answer is 32_7.

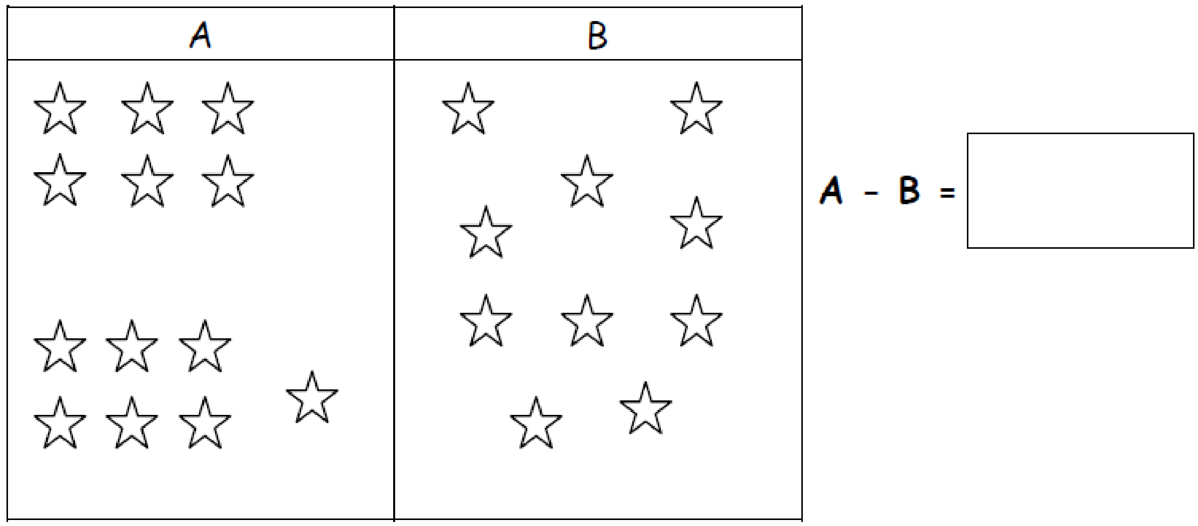


Figure 6. Sample conceptual base-7 subtraction item. Participants were expected to subtract the quantity in the column labelled B from the quantity in the column labelled A. The correct answer is 3_7.

Procedural base-7 subscale. The procedural base-7 subscale contains 17 familiar items. The familiar procedural items assess certain procedural aspects of multi-digit addition, multi-digit subtraction, and enumerating in base-7. For the procedural addition and subtraction items, participants were given double-digit addition problems and subtraction problems and were asked to compute the answers. The addition and subtraction items were presented both vertically (see Figure 7) and horizontally (see Figure 8). Four items were vertical and 4 were horizontal.

$$\begin{array}{r} 14_7 \\ +13_7 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 100_7 \\ -\ 50_7 \\ \hline \end{array}$$

Figure 7. Sample vertically presented addition and subtraction procedural base-7 items from the Numbers and Operations Test.

$60_7 + 116_7 =$ ________ $\qquad 42_7 - 15_7 =$ __________

Figure 8. Sample horizontally presented addition and subtraction procedural base-7 items from the Numbers and Operations Test.

The remainder of the familiar procedural base-7 items required the participants to use procedures to determine missing values. A base-7 number chart (see Figure 9) was provided that paralleled a typical number chart (see Figure 10) often used in early elementary school to teach children various place-value and counting concepts. The participants were asked to fill in missing values on the base-7 number chart and to respond to five questions about numbers that could be answered by recognizing the patterns in the given base-7 number chart. These questions are presented in Figure 11.

0	**1_7**	**2_7**	**3_7**	**4_7**	**5_7**	**6_7**
	11_7	**12_7**	**13_7**	**14_7**	**15_7**	**16_7**
20_7	**21_7**		**23_7**	**24_7**	**25_7**	**26_7**
30_7	**31_7**	**32_7**	**33_7**	**34_7**	**35_7**	**36_7**
	41_7	**42_7**	**43_7**	**44_7**		**46_7**
50_7	**51_7**	**52_7**	**53_7**	**54_7**	**55_7**	**56_7**
60_7	**61_7**	**62_7**	**63_7**	**64_7**	**65_7**	**66_7**

Figure 9. Base-7 number chart with missing values from the Numbers and Operations Test. Participants were asked to fill in the blanks with the appropriate missing value so that the pattern was preserved. The correct answers from left to right are 10_7, 22_7, 40_7, and 45_7.

0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
10	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**
20	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**
30	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**
40	**41**	**42**	**43**	**44**	**45**	**46**	**47**	**48**	**49**
50	**51**	**52**	**53**	**54**	**55**	**56**	**57**	**58**	**59**
60	**61**	**62**	**63**	**64**	**65**	**66**	**67**	**68**	**69**
70	**71**	**72**	**73**	**74**	**75**	**76**	**77**	**78**	**79**
80	**81**	**82**	**83**	**84**	**85**	**86**	**87**	**88**	**89**
90	**91**	**92**	**93**	**94**	**95**	**96**	**97**	**98**	**99**

Figure 10. Typical number chart. This type of chart is used in elementary school to help teach children place-value patterns and counting up and down by numbers other than 1.

Instructions: Use the chart above to fill in the blanks with the appropriate base-7 number.

The next number after 26_7 is _________.

The number before 50_7 is _________.

_________ is seven numbers more than 43_7.

_________ is three numbers less 55_7.

12_7 is _________ numbers more than 2_7.

Figure 11. Sample base-7 procedural knowledge questions using the number chart. The correct answers in order are 30_7, 46_7, 52_7, 52_7, and 7.

Conceptual base-10 subscale. The conceptual base-10 subscale consists of 2 items: one addition and one subtraction. These two items are analogous to the base-7 conceptual items. That is, two columns were presented each containing a certain number of objects (e.g., stars). The participants were then asked to join the objects or to subtract the second column from the first. See Figure 12 for a sample base-10 conceptual item.

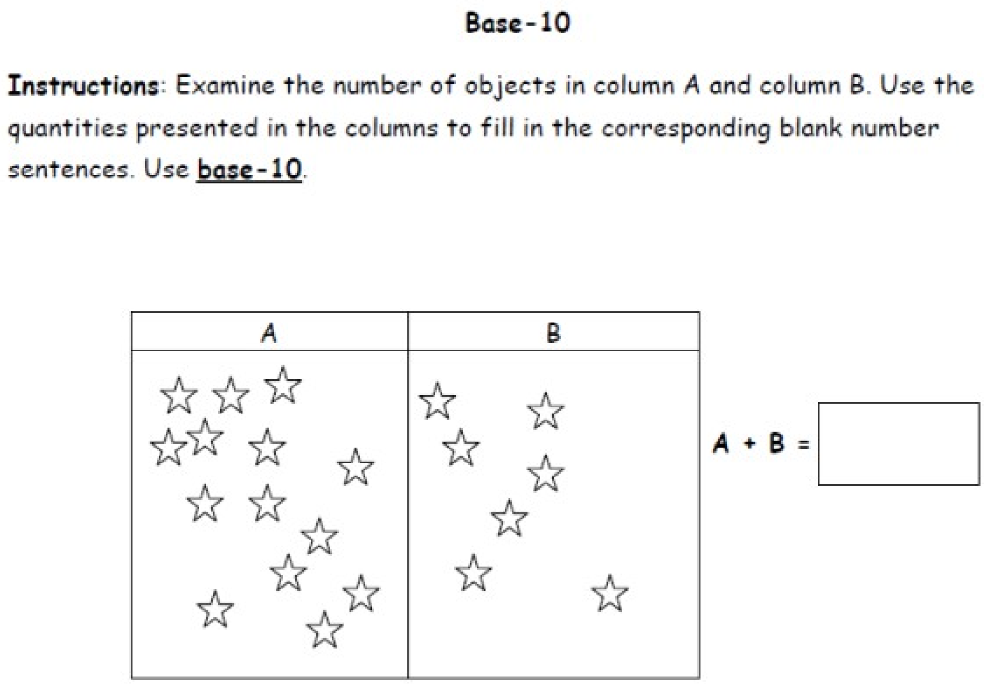


Figure 12. Sample base-10 conceptual addition item. The correct answer is 21.

Procedural base-10 subscale. The procedural base-10 subscale contains 11 items. These procedural items assess the execution of the addition and subtraction algorithms in base-10. Participants were given multi-digit addition and subtraction problems and were asked to compute the answers. The addition and subtraction items were presented both vertically (see Figure 13) and horizontally (see Figure 14). There were 8 horizontal and 3 vertical problems.

Base-10

Instructions: Perform the following calculations. Do the computations in **base-10**.

$$\begin{array}{r} 96\,477 \\ +\quad 623 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 1\;\;983 \\ -\quad 26 \\ \hline \end{array}$$

Figure 13. Sample procedural base-10 addition and subtraction items presented vertically.

Instructions: Perform the appropriate calculations to answer the following problems. Use **base-10**.

172 + 88 = __________ 1 003 - 17 = __________

Figure 14. Sample procedural base-10 addition and subtraction items presented horizontally.

SCK subscale. Two additional transfer tasks designed to assess the participants' ability to interpret elementary students' work constitute the SCK subscale of the NOT. They require the participants to combine their conceptual knowledge and procedural knowledge of the traditional base-10 numeration system to perform an error analysis of two examples of hypothetical students' work. One of the SCK items requires the participants to determine whether a student's solution to a vertically arranged multi-digit addition problem is correct and to further explain the student's answer using relevant concepts and procedures. The other SCK transfer task is exactly the same as the first, except that the hypothetical elementary student solved a multi-digit subtraction problem.

In both SCK items, the student solutions contain typical errors produced by elementary school students. For example, in the addition SCK task (see Figure 15), the student neglected to properly regroup composite units of ten and wrote “12” in the ones position. In the second SCK task (see Figure 16), the student had to subtract nine ones from four ones. The standard procedure is to decompose a group of one hundred into tens and then to decompose a group of ten into ones. Rather than regrouping a group of one hundred into 10 groups of ten, the student erroneously decomposed a group of one hundred and regrouped it as 10 ones.

Base-10

Instructions: Look at the solutions below produced by elementary school students. In each case, indicate if the student got the right answer. If the student solved the problem correctly, explain the steps used. If the student solved the problem incorrectly, describe the mistake(s) made by the student.

Student A's work:

$$\begin{array}{r} 27 \\ +\ 5 \\ \hline 212 \end{array}$$

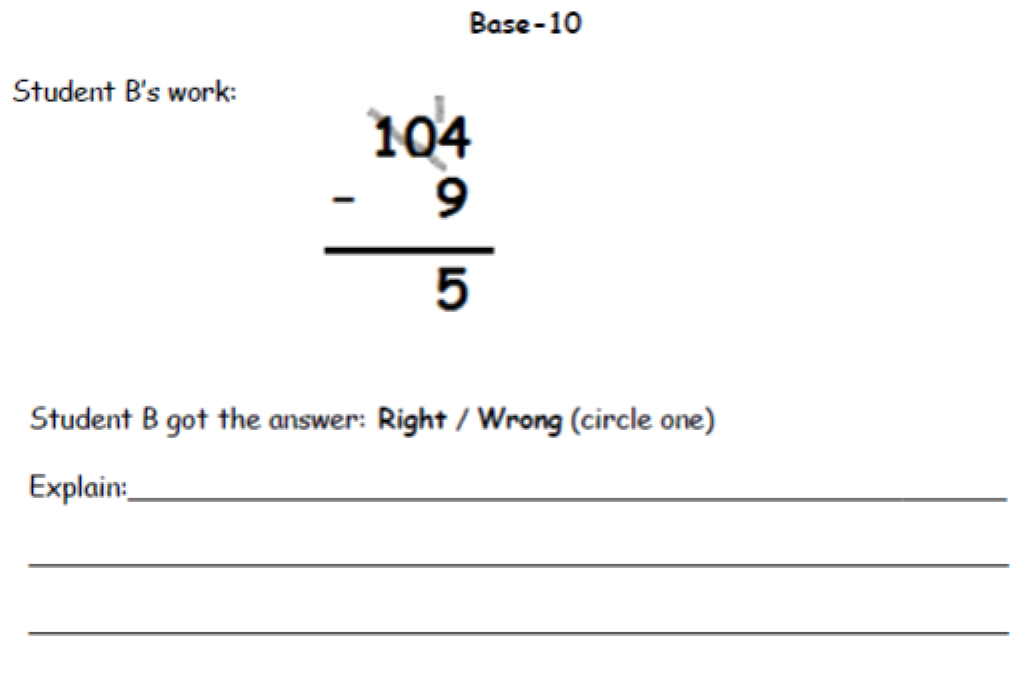
Student A got the answer: **Right / Wrong** (circle one)

Explain:____

____

____

Figure 15. SCK addition item from the NOT. The correct answer is 32, but the student did not group the 12 ones in the one column into one group of 10 and two ones.



Base-10

Student B's work:

$$\begin{array}{r} 104 \\ -\quad 9 \\ \hline 5 \end{array}$$

Student B got the answer: **Right / Wrong** (circle one)

Explain:

Figure 16. SCK subtraction item from the NOT. The correct answer is 95, but rather than regrouping a group of one hundred into 10 groups of ten, the hypothetical student erroneously decomposed a group of one hundred and regrouped it as 10 ones.

Transfer subscale. There are two procedural transfer items. Participants were given with numbers presented symbolically in base-5 and were asked perform one vertical multi-digit addition problem in base-5 and a vertical multi-digit subtraction problem in base-5 (see Figure 17).

Base-5

Instructions: Perform the appropriate operations in the following problems. Do all the computations in base-5.

$$\begin{array}{r} 13_5 \\ +\ 13_5 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 44_5 \\ -\ 32_5 \\ \hline \end{array}$$

Figure 17. Transfer items from the Numbers and Operations Test. The correct answers from left to right are 31_5 and 12_5.

Assessment at T2. After instructional session 2, the participants' conceptual base-7 knowledge, procedural base-7 knowledge, and SCK were assessed. The T2 assessment consisted of one familiar conceptual item, a familiar procedural item, and one SCK item. The familiar items mirrored what was covered in the instructional sessions up to this time point. More specifically, the participants were asked to name a number in base-7 (procedural subscale) and then explain why that number was named that way (conceptual subscale); (see Figure 18). The SCK item at T2 was a multi-digit addition item. The participants were expected to recognize that the hypothetical student produced an incorrect answer and to identify that the student did not regroup a group of ten ones into a group of ten and subtracted in the hundreds column rather than added (Figure 19). See Appendix C for the assessment at T2.

1. In words, name the following number and explain why that is how the number is named.

   614_7 = ______________________________

   Explanation:

Figure 18. Conceptual and procedural item at T2. The number should be named six-one-four base-7. It is named this way because there are four ones (7^0), one group of seven (7^1), and six groups of 49 (7^2). The place-value positions are not ones, tens, and hundreds like in base-10.

Examine the following student solution. Determine if the student is correct. Explain how the student arrived at the answer. If the student is wrong, explain where the student went wrong.

Student's work:

$$\begin{array}{r} 907 \\ +\ 315 \\ \hline 612 \end{array}$$

The student is correct/incorrect.
Explanation:

Figure 19. SCK item at T2. The participants were expected to recognize the hypothetical student's answer as incorrect and to identify that the student did not regroup appropriately in the ones column and subtracted in the hundreds column rather than adding.

Assessment at T3. At T3, the participants had completed all of the instructional sessions. Once again, their conceptual base-7 knowledge, procedural base-7 knowledge, and SCK of place-value was assessed. The conceptual and procedural items were familiar as they replicated the items in the instructional sessions. At T3, procedural base-7 knowledge was assessed by having the participants compute a vertical multi-digit subtraction item (see Figure 20). For the conceptual subscale, the participants were required to use a block model, introduced during the instructional session, to calculate the answer to a horizontal subtraction problem (see Figure 21). Finally, for the SCK subscale, the participants had to again analyze a hypothetical student's solution. This time the student solved a vertical multi-digit subtraction problem in base-10. The participants had to recognize that the student produced an incorrect answer because he subtracted the smaller number from the larger number in the ones column rather than subtracting the subtrahend from the minuend. See Appendix D for the T3 assessment.

Instructions: Solve the following problem.

$$\begin{array}{r} 123_7 \\ -\ 65_7 \\ \hline \end{array}$$

Figure 20. Procedural item at T3. The correct answer is 25_7.

Instructions: Using the block model you learned in the computer lessons. Calculate the following.

$421_7 - 354_7$

Figure 21. Conceptual item at T3. The participants should have represented the quantities using the block model and then cancelled out the blocks in the minuend from the subtrahend. The correct answer is 34_7.

Instructional Intervention

All of the conceptual and procedural lessons were presented using an online survey tool called Survey Monkey. Every conceptual lesson had a corresponding procedural lesson. The topics of the lessons are presented in Table 4. Each lesson was formatted as a survey that presented the lesson to the participants and then asked them to answer a variety of fill-in-the-blank and multiple-choice questions. These questions were used for instructional purposes only and were not used as data in this study. The survey tool collected and stored the responses in a secure online database. In addition to storing the participants' responses, the survey tool also recorded how long it took each participant to complete each lesson.

Table 4

Conceptual and Procedural Lesson Topics

	Conceptual	Procedural
Lesson 1	Counting by one	Naming numbers
	Counting by seven	
Lesson 2	Enumerating collections of objects	Using the base-7 number chart
	Grouping by seven	Base-7 number chart patterns
	Representing quantities with base-7 blocks	
Lesson 3	Adding using base-7 blocks	Adding using the traditional algorithm
Lesson 4	Subtracting using base-7 blocks	Subtracting using the traditional algorithm

Conceptual lessons. All four of the conceptual lessons were designed with careful consideration of how to explain key place-value concepts involved in counting, grouping, addition, and subtraction while avoiding any strategies or procedures for completing the tasks. One way this was facilitated was by using a different base than the participants were used to (McClain, 2003; Yackel, Underwood & Elias, 2007). The use of a different base reduced the likelihood that participants would use already known procedures during the lessons. Thus, all of the lessons on place-value concepts were in base-7. That is, groupings were conducted in groups of seven rather than in 10. In a base-7 numeration system, the digits 0 through 6 are used. Furthermore, every time there are seven at any given place-value position, those seven are regrouped to make a "one" (i.e., one group of seven) in the next denomination.

In addition to using a different base, language, notation, and terminology were considered carefully to help participants build place-value concepts and meaningful relationships. To avoid confusion with base-10 numbers, all of the numbers throughout all of the lessons were presented with the subscript "7" to indicate that the number is in base-7. Furthermore, when naming numbers, only the traditional base-10 number names for the digits 0 through six were used. For multi-digit numbers in base-7, the participants were instructed to name those numbers as a sequence of the individual digits followed by "base-7." For example, the number 146_7 would be called "one-four-six base-7" and not "one-hundred-forty-six," as prescribed by base-10 number names.

Conceptual Lesson 1. The first conceptual lesson focused on counting objects in base-7. The participants were presented with a quantity using pictures (i.e., stars) of a certain number of objects. While counting activities often involve both procedural and conceptual knowledge, coordinating number words with quantities is predominately conceptual (Steffe & Cobb, 1988). In the case of this intervention, the amount of procedural knowledge implicated in the enumeration tasks was minimized because the counting was in a different base and different number names were used.

Rather than explicitly being told that the lesson involves counting, participants were presented with a quantity and told what the quantity is called in base-7. The lesson starts with one object being presented and the lesson states that it is called “one base-7,” (see Figure 22). The subsequent screens present one more object than the previous screen, and each time the participant is shown how to represent the quantity using words (see Figure 23). When there were seven objects presented on the screen, the objects were grouped together (see Figure 24). Objects were counted in this manner using pictures for the quantities 1_7 through 130_7 (see Figure 25).

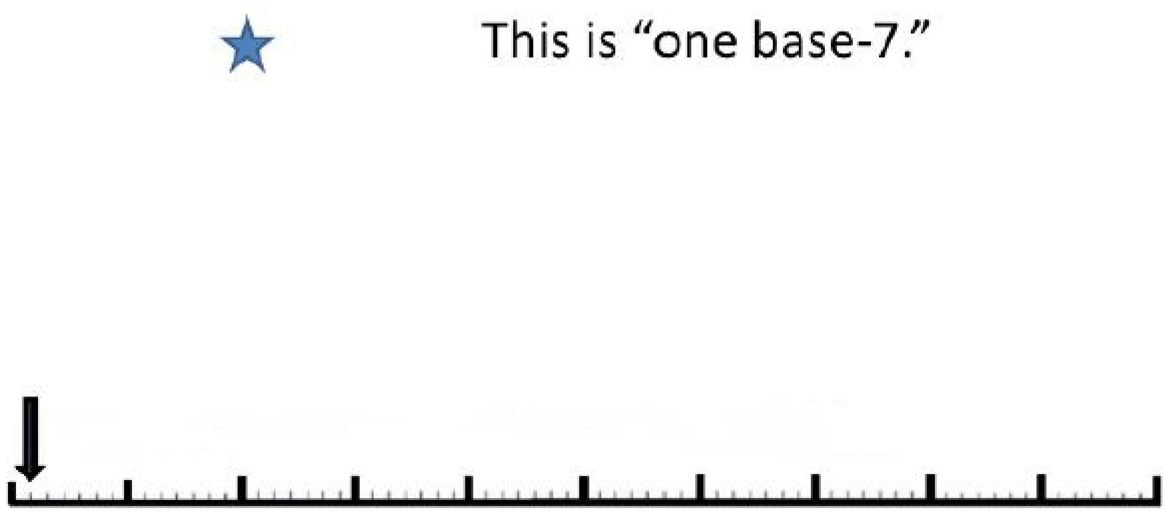


Figure 22. Screen shot of Conceptual Lesson 1.

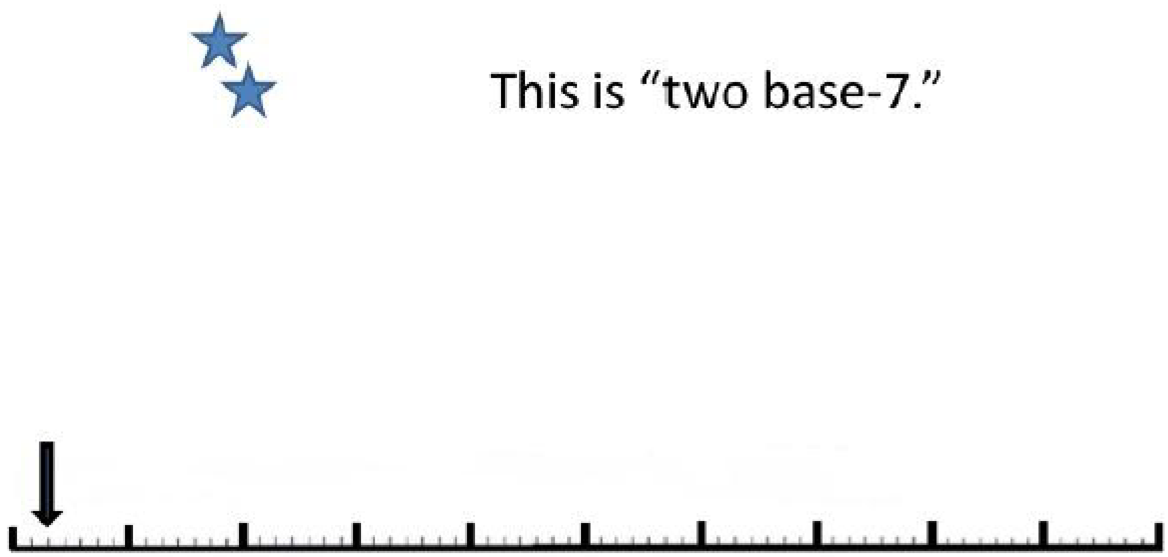


Figure 23. Screen shot of Conceptual Lesson 1.

Figure 24. Screen shot of making a group of seven objects from Conceptual Lesson 1.

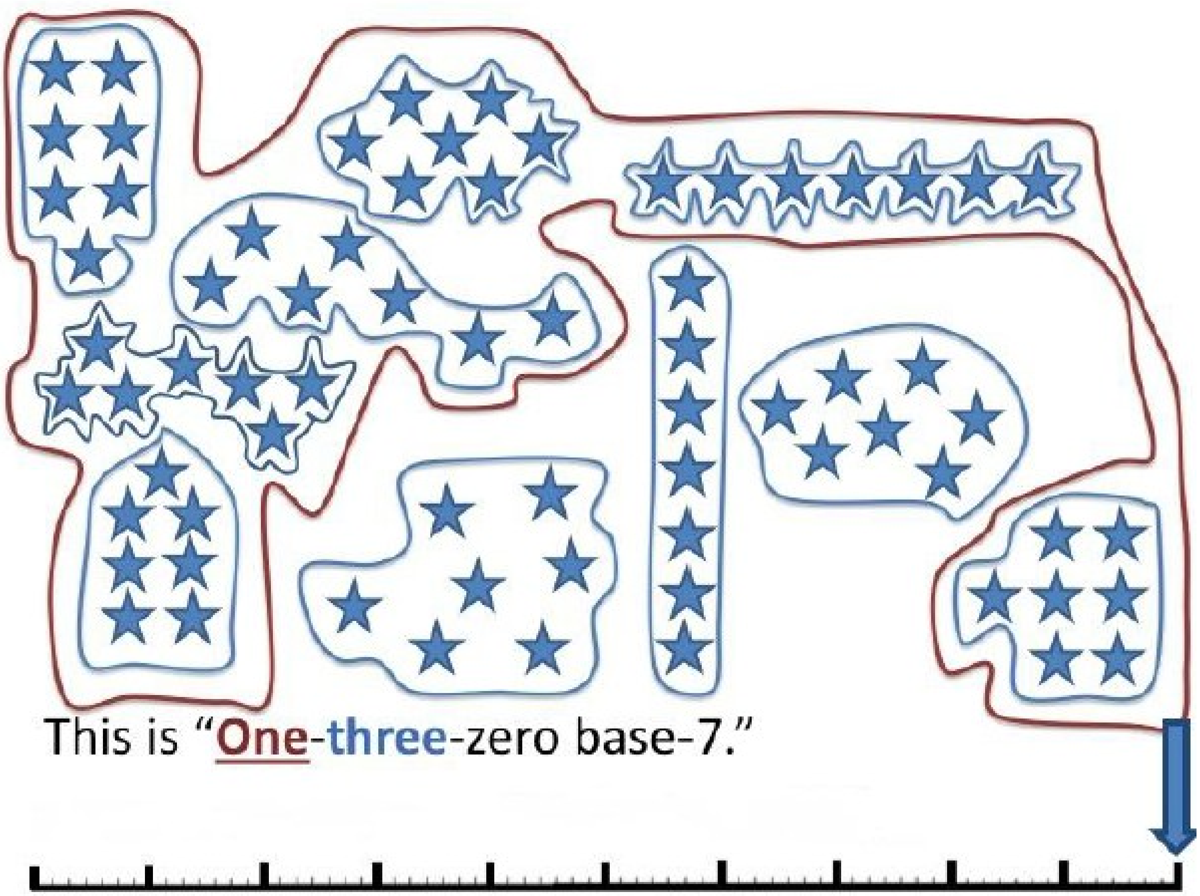


Figure 25. Screen shot of making groups of seven from Conceptual Lesson 1. Here the count went up to one-three-zero base-7.

Later in the lesson, quantities of objects were counted by sevens using a similar pictorial strategy (see Figure 26). At the bottom of each counting slide, a number line was presented that provides a visual representation of the relationship between the quantities that were being counted and their relationship to the previous and next quantities. Using different representations is believed to help solidify the construction of concepts (Rittle-Johnson & Siegler, 1997). See Appendix E for all of the screen shots of Conceptual Lesson 1.

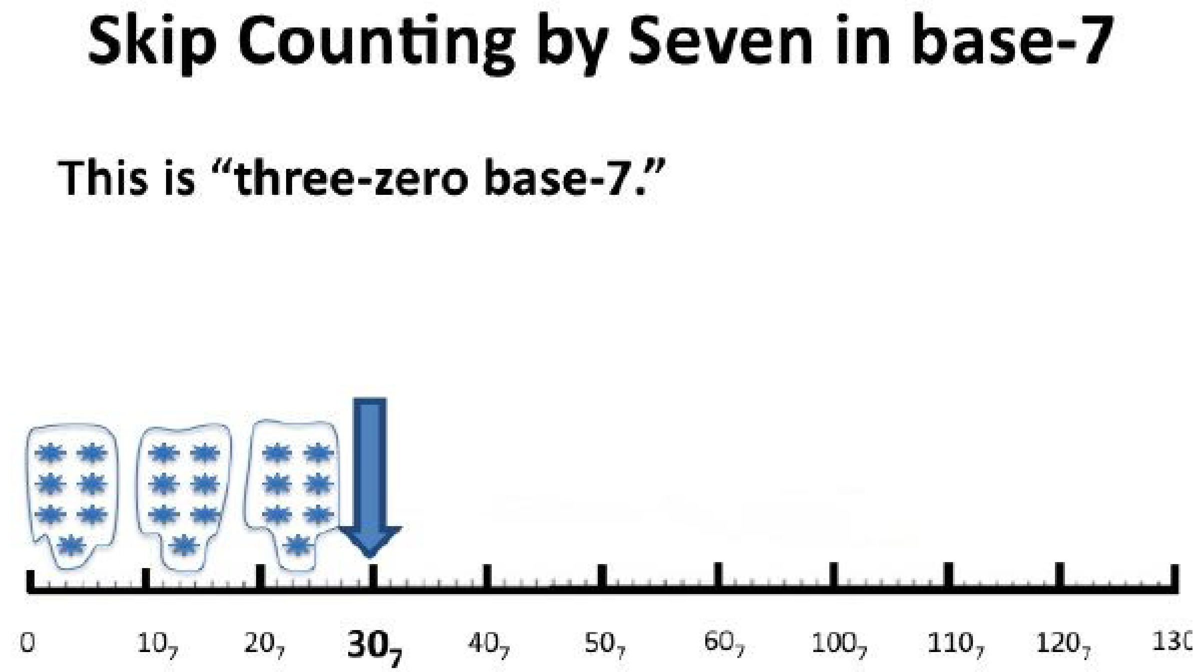


Figure 26. Screen shot of objects with a number line from Conceptual Lesson 1.

Following from Conceptual Lesson 1, there were 10 lesson questions. Eight of the questions were fill-in-the-blanks and two were multiple-choice. For the fill in the blank questions, a set of objects was displayed and the participant was required to write the number name in base-7 that corresponded to the quantity represented by the set (see Figure 27). For the multiple choice questions, a base-7 number in symbolic form was presented and the participants was expected to select which one of four pictures of sets corresponded to the number (see Figure 28). See Appendix F for the end of Conceptual Lesson 1 questions.

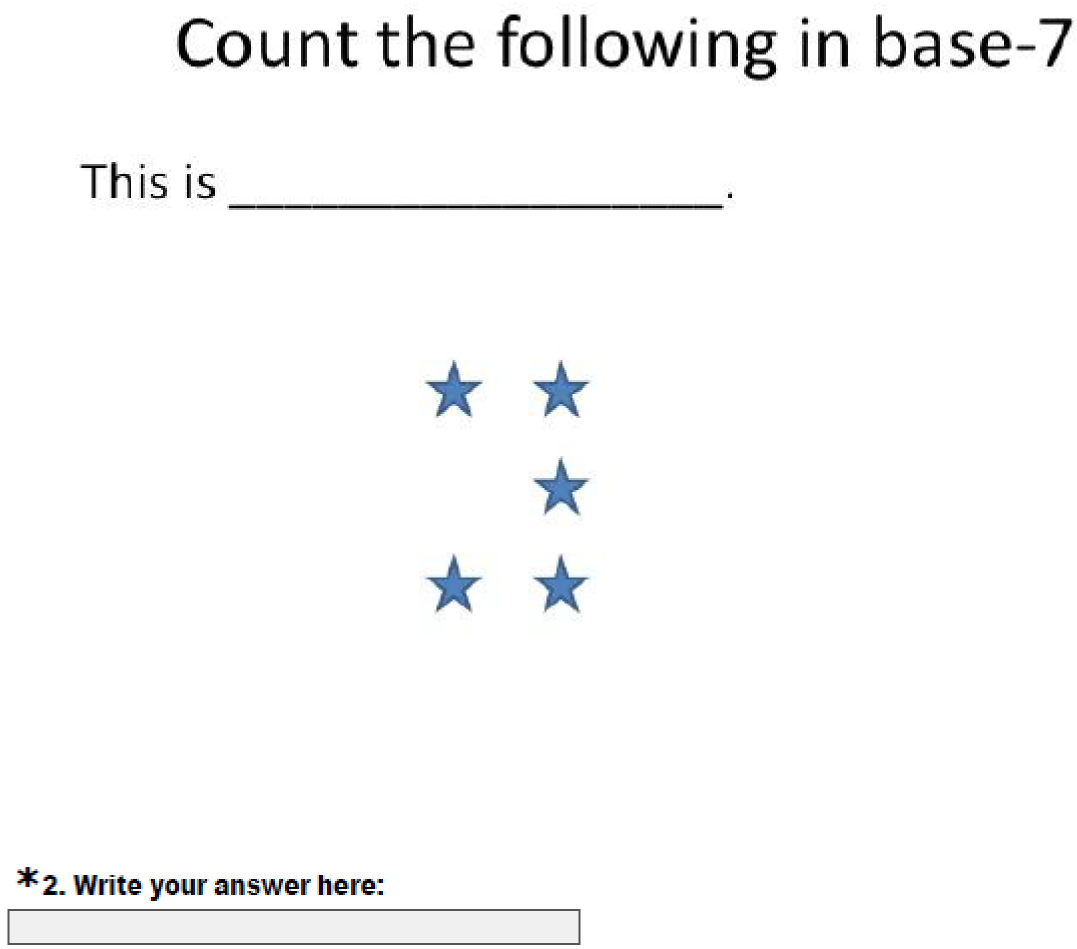


Figure 27. Screen shot of Conceptual Lesson 1 fill-in-the-blank question.

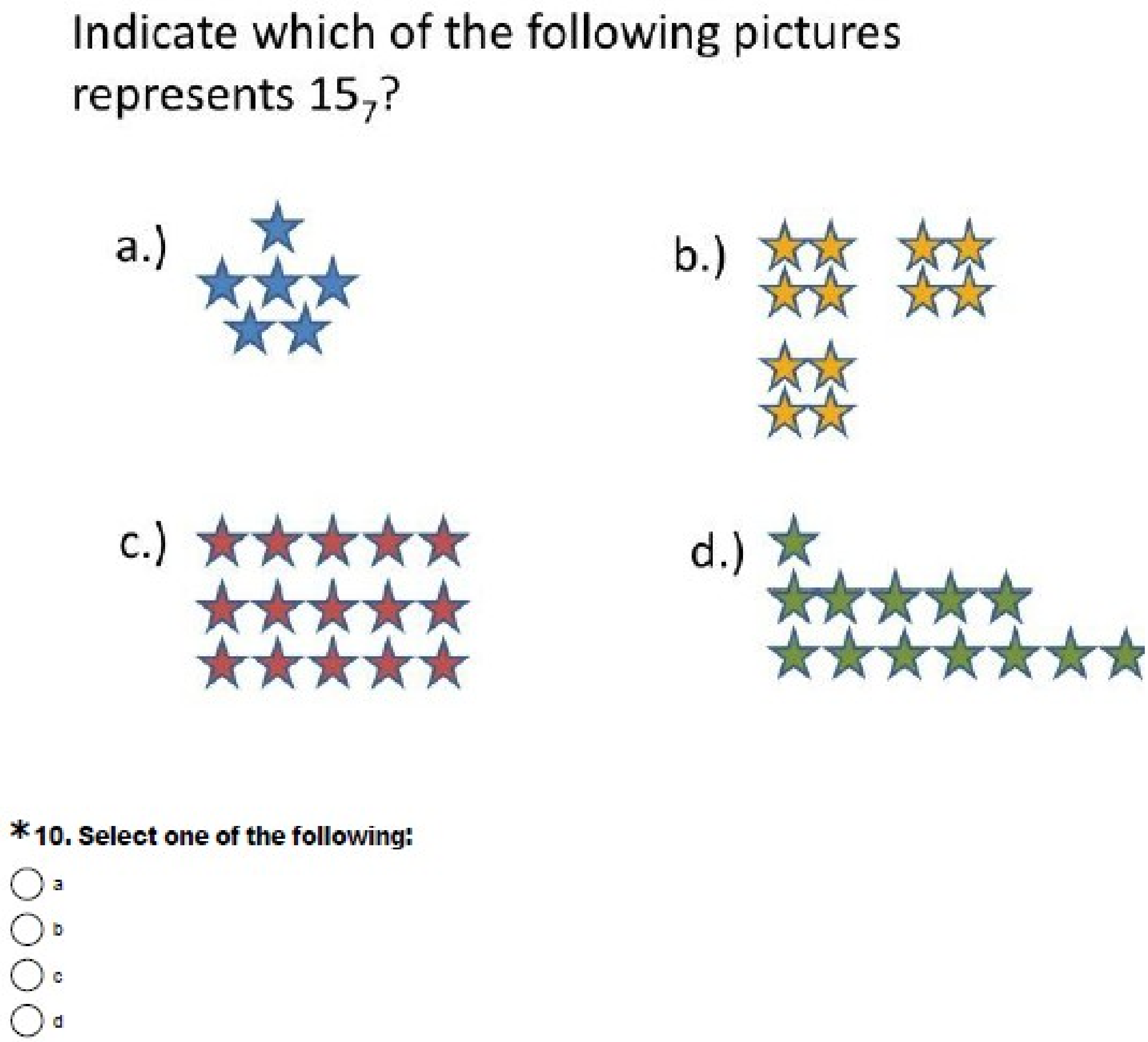


Figure 28. Screen shot of Conceptual Lesson 1 multiple choice question.

Conceptual Lesson 2. Unlike the first conceptual lesson, the second conceptual lesson provided two representations of a quantity in base-7. The first used words and the second used digits (see Figure 29). This lesson focused on counting and grouping sets of objects and in particular, emphasized grouping single objects by seven, called here “groups” (see Figure 30). Mathematically, one single object is represented by 7^0 and one group of units is represented by 7^1. When seven groups are formed, it is called here a “big group” and is mathematically equivalent to 7^2. Making these groupings explicit was meant to foster the participants’ conceptual understanding of composite units and the role of composite units within a positional place-value system.

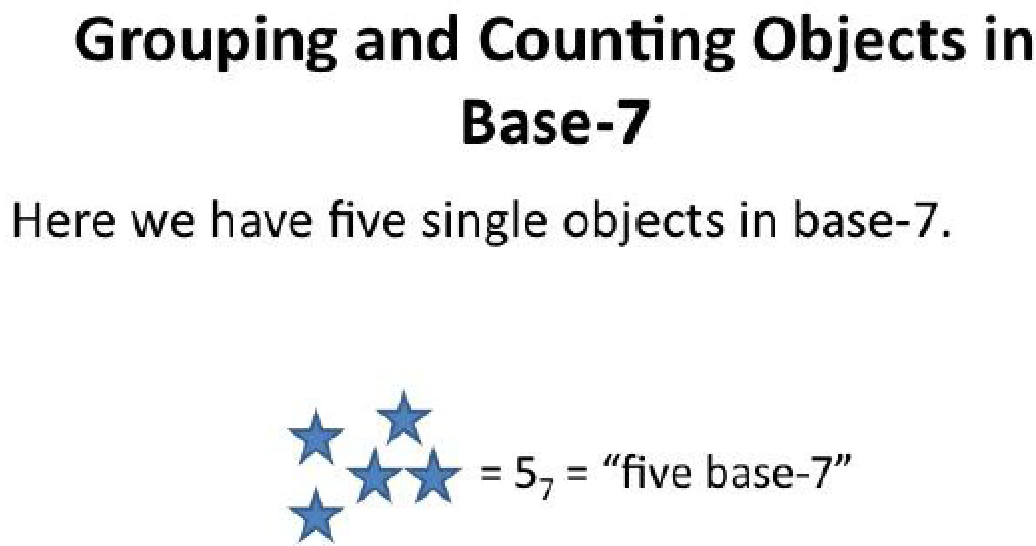


Figure 29. Screen shot of representations of quantities in base-7 from Conceptual Lesson 2.

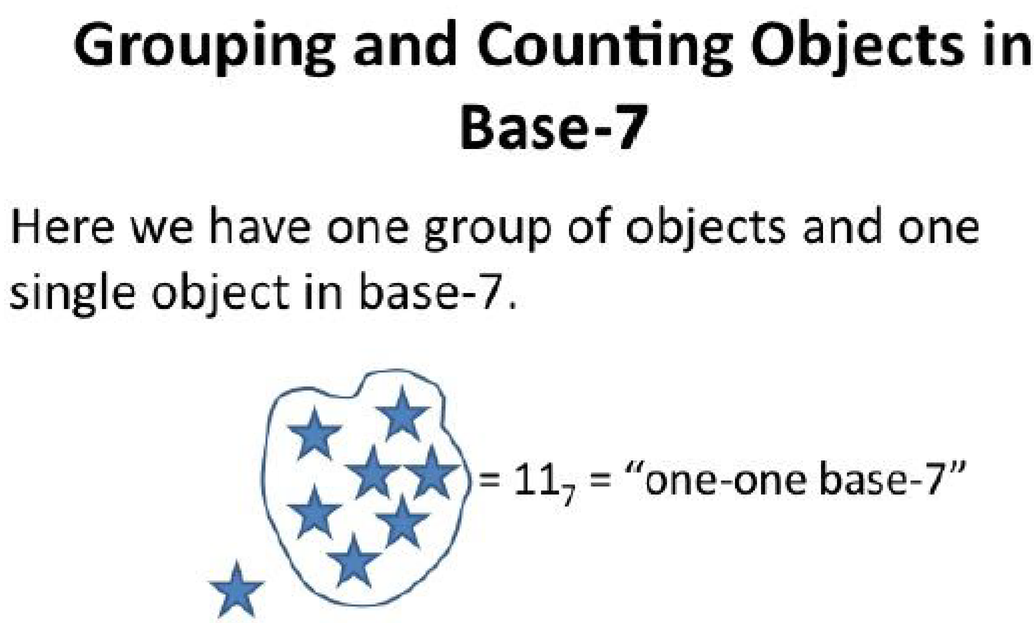


Figure 30. Screen shot of grouping objects by seven from Conceptual Lesson 2.

In this lesson, an alternative representation for counting in base-7 was also introduced (see Appendix G for the complete Conceptual Lesson 2). Hiebert and Wearne (1992) used base-10 blocks with children to build their conceptual understanding of place-value and multi-digit addition and subtraction. To simulate this instructional method, pictures of base-7 blocks were used in this lesson. Representing units and composite units with a proportional model may help foster conceptual understanding of place-value. Coordinating composite units back and forth, such as, in this case, from singles to sevens, requires conceptual knowledge (Cobb, Yackel, & Wood, 1992).

In this lesson, a picture of one small square represented one unit. One rectangle, composed of seven small squares, represented one "group." Finally, a large square, equivalent to seven rectangles (or seven groups), was said to represent one "big group" (see Figure 31). In more explicit terminology, the small square represents a unit (7^0), the rectangle is a composite unit that represents seven units (7^1), and the big square is a composite unit of seven rectangles, or 49 single units (7^2). An additional purpose for providing pictures of base-7 blocks as an alternate representation was to facilitate the use of combining sevens and ones as a conceptually based addition strategy, which appeared in Conceptual Lesson 3. Fuson (1990) and Fuson and Briars (1990) argued that providing exercises that allow the construction and trading of multiunit structures, as well as linking those structure to word names, builds conceptual understanding of place-value.

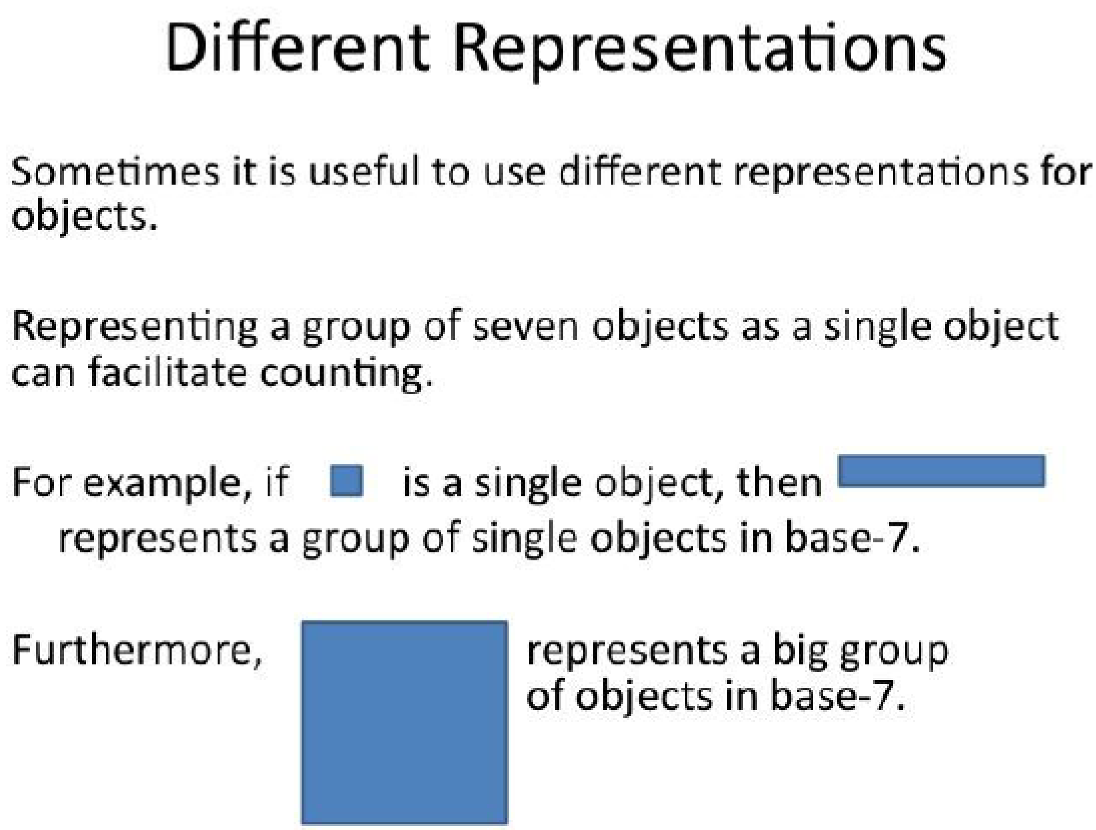
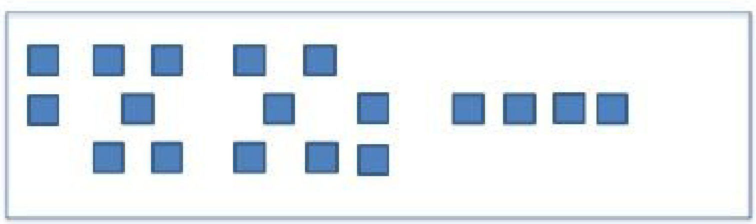


Figure 31. Screen shot of the block model introduced in Conceptual Lesson 2.

Following Conceptual Lesson 2 were 12 lesson questions. Eleven of the questions were fill-in-the-blank and one was a multiple choice question. The fill-in-the-blank questions represented a variety of different tasks, such as naming the number of units, number of groups, and number of big groups for a given set of objects (see Figure 32). Participants were also asked to use symbols to represent how many were in a quantity drawn with base-7 blocks (see Figure 33). The multiple choice questions required the participants to select the picture that accurately represented a given quantity of objects represented by pictures of base-7 blocks. See Appendix H for the end of Conceptual Lesson 2 questions.

Examine the objects in the box and answer the following questions.



*2. How many groups of objects can you make in base-7?

*3. How many big groups of objects can you make in base-7?

*4. In words, how many objects are there in base-7?

Figure 32. Screen shot of naming the number of units question from Conceptual Lesson 2.

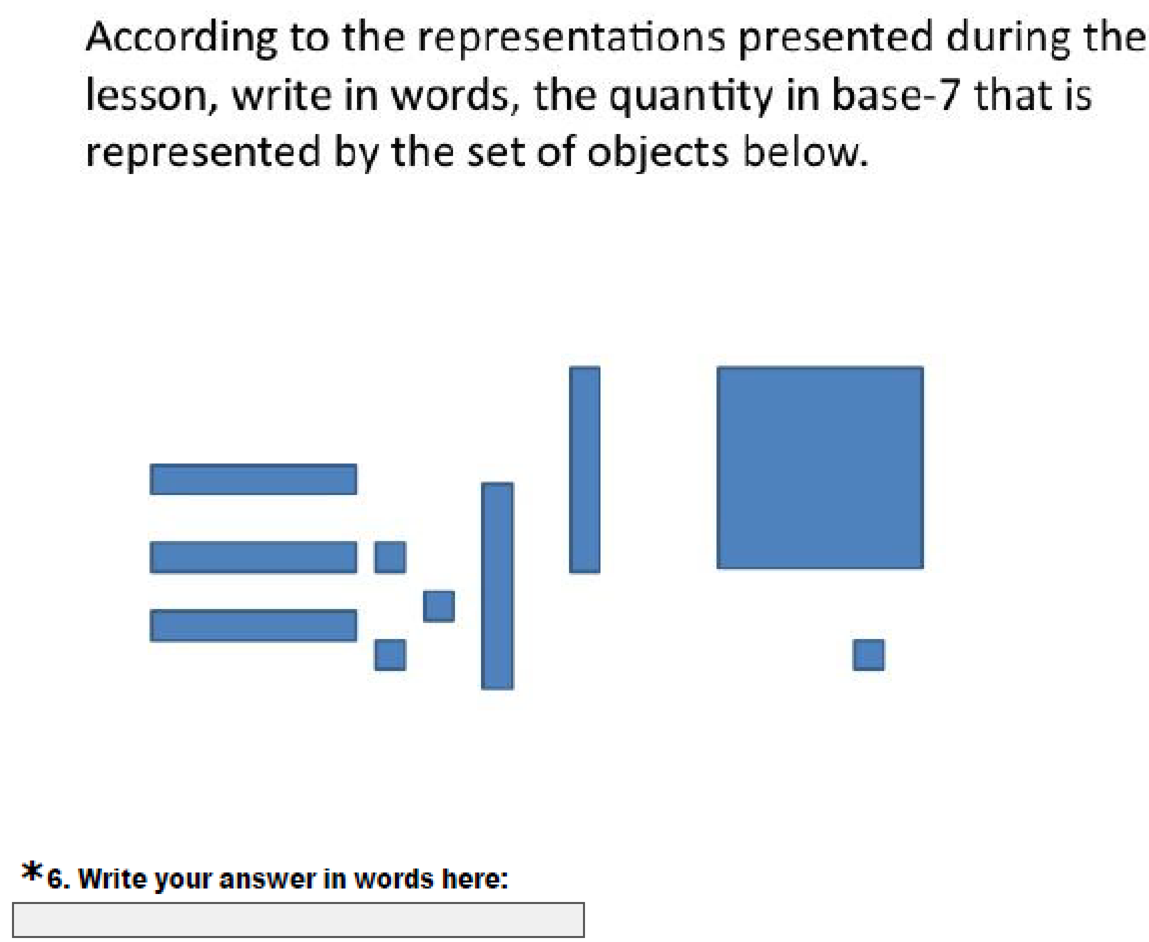


Figure 33. Screen shot of question about objects represented with pictures of base-7 blocks from Conceptual Lesson 2.

Conceptual Lesson 3. The third conceptual lesson focused on combining two sets of objects that were represented with base-7 blocks. Rather than perform the steps associated with the traditional addition algorithm, Conceptual Lesson 3 demonstrated (a) how single units can be grouped into composite units (see Figure 34), and (b) how to combine the single units and composite units from two different sets to arrive at their sum (see Figure 35). Throughout the third conceptual lesson, examples demonstrated how combining sevens and ones can be used to add two sets of objects (see Appendix I for Conceptual Lesson 3). Adding quantities in this way could help the participants develop a conceptual understanding of grouping in the context of addition because this is similar to the invented strategy of combining tens and ones that children use. Meaningful engagement with multi-digit addition and subtraction can lead to conceptual understanding of place-value (Fuson & Briars, 1990). In fact, many studies use multi-digit arithmetic to assess and remediate inaccurate place-value notions (Fuson et al., 1997; Hiebert & Wearne, 1992).

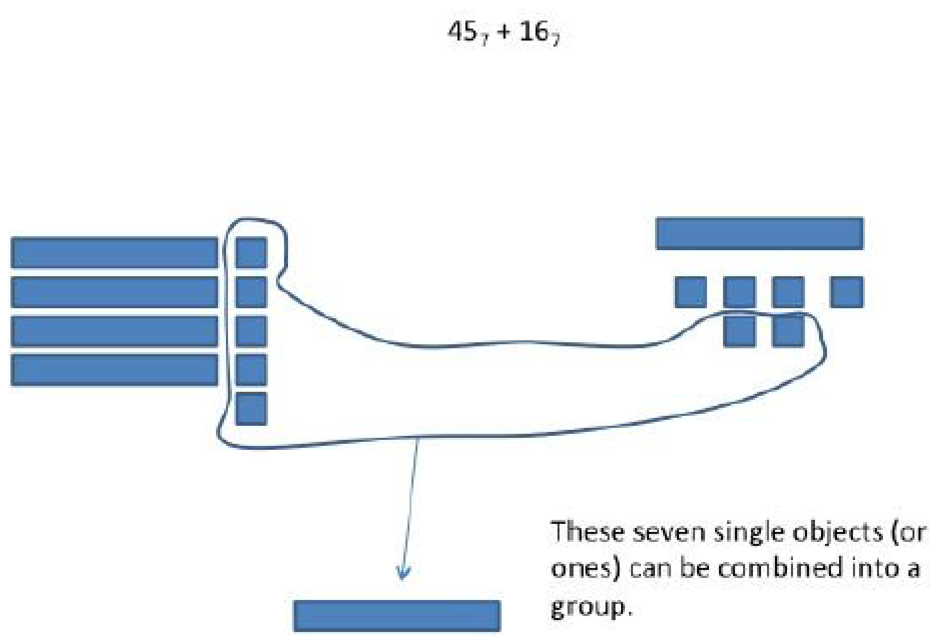


Figure 34. Screen shot demonstrating how units can be combined into composite units from Conceptual Lesson 3.

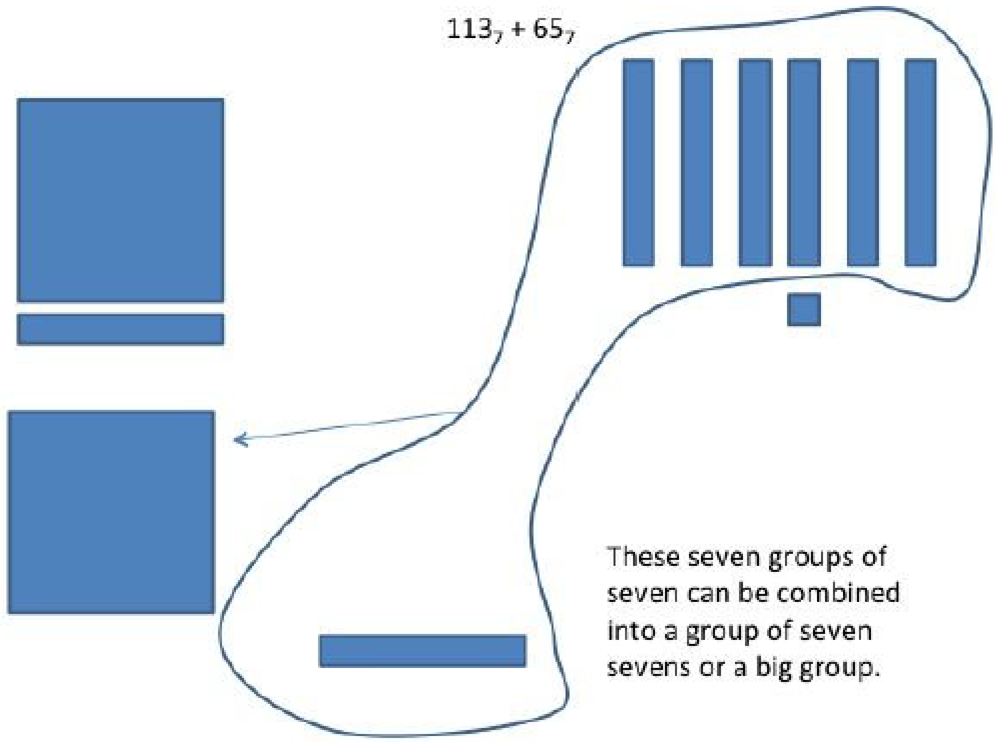


Figure 35. Screen shot demonstrating how to regroup to form composite units from Conceptual Lesson 3.

At the end of Conceptual Lesson 3, there were 10 lesson questions. Three of the questions were multiple choice questions and 7 were fill-in-the-blank. For the multiple choice questions, each item consisted of a number sentence and four possible base-7 block representations. Participants were required to choose the representation that demonstrated the quantities in the number sentence and the resulting sum. Conversely, each fill-in-the-blank question consisted of items that had two sets of objects represented in base-7 blocks and for each item, the participants were required to write the corresponding number sentence in written symbols. See Appendix J for the end of Conceptual Lesson 3 questions.

Conceptual Lesson 4. The fourth and final conceptual lesson was exactly like the third conceptual lesson with one difference. Rather than demonstrating how to add two sets of objects using the "combining sevens and ones" strategy, the final conceptual lesson showed how to subtract one set from another using the same strategy (i.e., subtracting ones and subtracting sevens separately). See Appendix K for Conceptual Lesson 4. At the end of the lesson, there were also 10 lesson questions that were structurally similar to those in Conceptual Lesson 3, but used subtraction (see Appendix L).

Procedural lessons. All four procedural lessons were designed to reflect procedures that were related to the concepts in the corresponding conceptual lessons (see Table 4). As such, all the procedural lessons were also in base-7 and focused on strategies required to count, create and decompose composite units, and add and subtract two multi-digit numbers. Written symbols were used for all representation of quantities in the procedural lessons. During the procedural lessons, the participants did not receive any explanations about why they were learning certain procedures or why these procedures work. Instead, everything was presented as facts and sequential steps. Furthermore, language that is traditionally used when these procedures are taught in elementary school was incorporated into the lessons. That is, in the procedural lessons, the terms used when composite units were created were "carrying" and "borrowing."

Procedural Lesson 1. This lesson focused on naming numbers in base-7 with no relationship to the quantities those number names represent. To accomplish this, the participants were exposed to the symbolic form of a base-7 number and were told what the number name was (see Figure 36). All of the same language and naming conventions were used for all numbers from 1_7 to 130_7. See Appendix M for Procedural Lesson 1.

Naming Numbers in Base-7

This is how you name single digit numbers in base-7:

Digit	Name
0	Zero
1_7	One base-7
2_7	Two base-7
3_7	Three base-7
4_7	Four base-7
5_7	Five base-7
6_7	Six base-7

Figure 36. Screen shot of number names from Procedural Lesson 1.

Following the number naming lesson (i.e., Procedural Lesson 1) were 10 end-of-lesson questions. Three of these questions were true or false questions. For these, participants were given a number followed by a statement pertaining to that number's name. The participant was required to decide if the statement was true or false (see Figure 37). Three additional questions were multiple-choice questions. The multiple-choice questions presented a number in base-7 and four possible names for the number (see Figure 38). The participant had to select the correct number name. The last four lesson questions were fill-in-the-blank questions. For these final questions, the participants were presented with a base-7 number and were required to write out the number's name in words (see Figure 39). See Appendix N for the end of Procedural Lesson 1 questions.

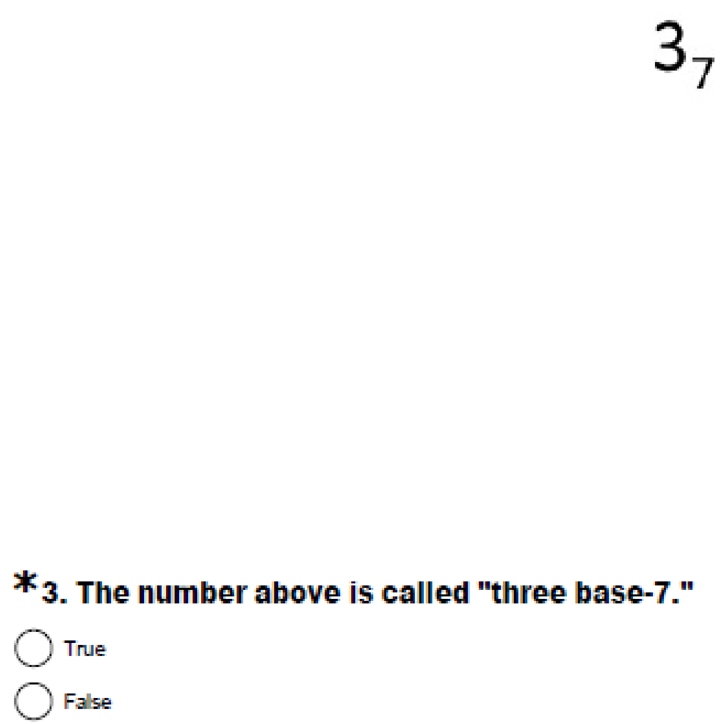


Figure 37. Screen shot of a true or false question from Procedural Lesson 1.

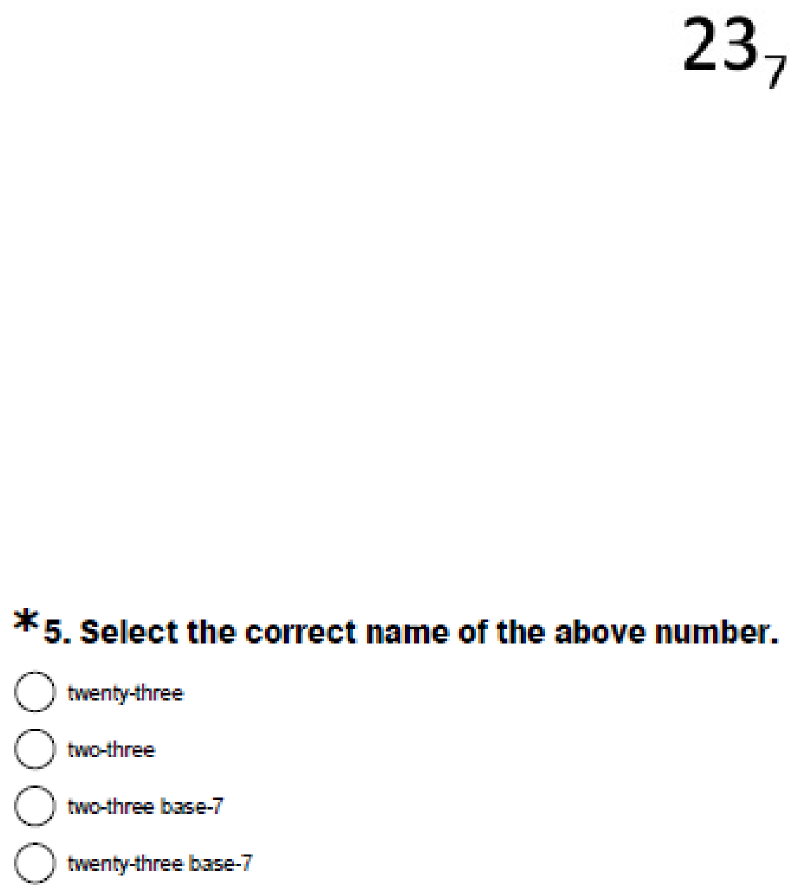


Figure 38. Screen shot of multiple choice question from Procedural Lesson 1.

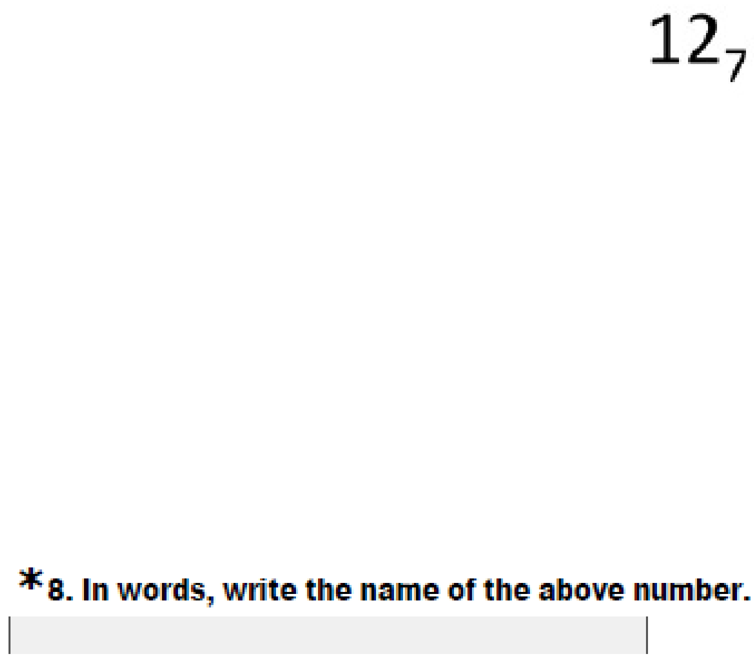


Figure 39. Screen shot of fill-in-the-blank question from Procedural Lesson 1.

Procedural Lesson 2. The second procedural lesson taught the participants about the number chart tool that is often used in a procedural manner in elementary school. In the context of the intervention, it was a base-7 number chart. As part of the instruction, numerical patterns were made explicit and taught as rules. For example, one "rule" was to look at the number directly below a given number to find a number representing seven more (see Figure 40). See Appendix O for Procedural Lesson 2.

Following the lesson on the base-7 number chart and patterns, there were 10 lesson questions. Five of these questions required participants to fill in values that were missing in the number chart while preserving the counting patterns (see Figure 41). The other five questions were fill-in-blank questions about various patterns in the base-7 number chart. Some questions were, “What number is seven more than 4_7?” and “What number is three more than 5_7?” See Appendix P for end of Procedural Lesson 2 questions.

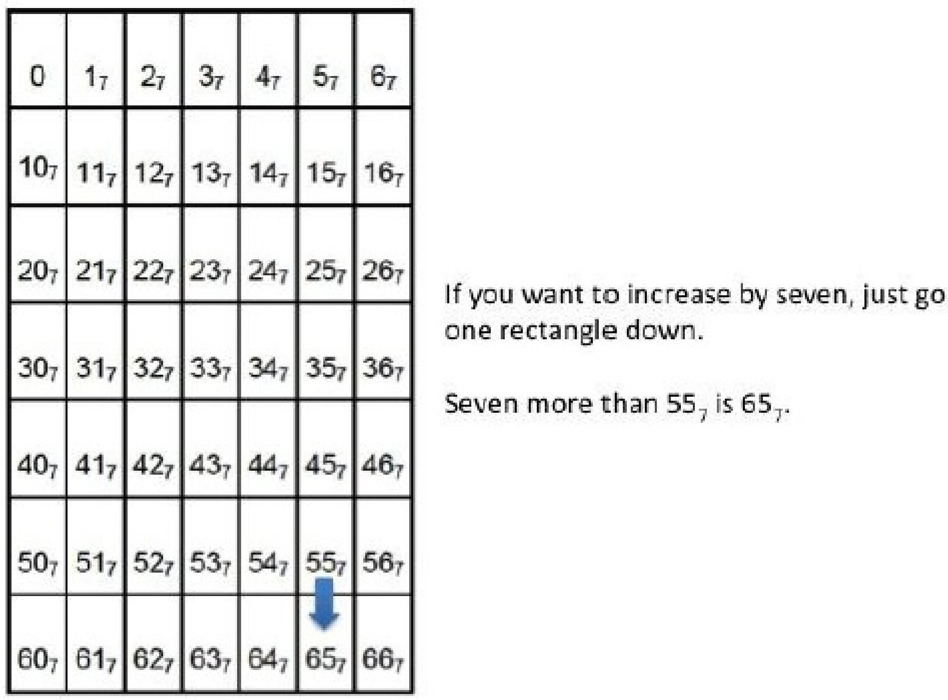


Figure 40. Screen shot of sample "rule" from Procedural Lesson 2.

0	A	2_7	3_7	4_7	5_7	6_7
10_7	11_7	12_7	13_7	14_7	15_7	16_7
20_7	21_7	22_7	23_7	B	25_7	C
D	31_7	32_7	33_7	34_7	35_7	36_7
40_7	41_7	42_7	43_7	44_7	45_7	46_7
50_7	51_7	E	53_7	54_7	55_7	56_7
60_7	61_7	62_7	63_7	64_7	65_7	66_7

***2. Fill in the missing values in a way that preserves the pattern.**

A is:

Figure 41. Screen shot of fill-in-the-blank question from Procedural Lesson 2.

Procedural Lesson 3. The third procedural lesson focused explicitly on the procedure for adding two base-7 multi-digit numbers. This procedure parallels the algorithm taught in school and was taught as a series of steps without any conceptual explanations (see Figure 42). All of the problems were presented vertically because this presentation is believed to reinforce the learning of rote steps for children (Fuson, 1990). Refer to Appendix Q for the entire Procedural Lesson 3.

Following the lesson, there were 10 end-of-lesson questions. Five of the questions were true-or-false questions and five of the questions were fill-in-the-blank. For the true-or-false questions, the participants were shown two base-7 multi-digit numbers presented vertically with a corresponding sum. Participants were required to indicate whether or not the given sum was the correct answer. For the fill-in-the-blank questions, the participants were given two multi-digit numbers arranged vertically and were asked to calculate the sum using the procedure taught during the lesson. All of the end of Procedural Lesson 3 questions are presented in Appendix R.

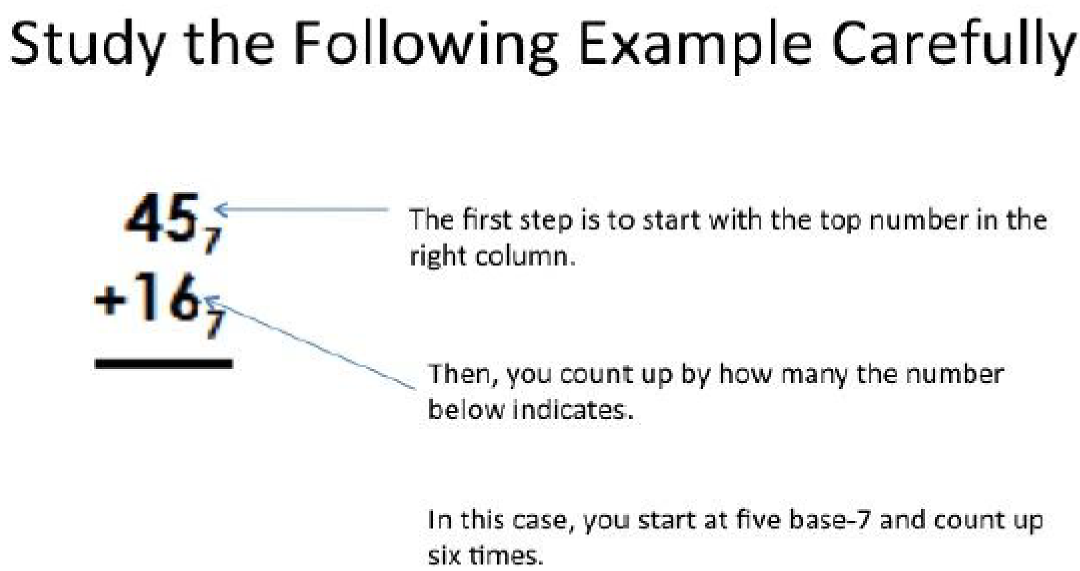


Figure 42. Screen shot of addition algorithm from procedural lesson 3.

Procedural Lesson 4. Procedural Lesson 4, including the end-of-lesson questions, was exactly the same as Procedural Lesson 3 using subtraction instead of addition (see Appendix S for Procedural Lesson 4 and Appendix T for Procedural Lesson 4 questions).

Procedure

During class time on the first day of class, all of the participants who were registered for the first teaching mathematics methods course completed the NOT at pretest-T1. The participants were given the test booklet and instructed to write their student identification numbers on the first page and not to talk to other students while they were completing the test. I explained to the students that number of correct answers they got would not affect their course grade and to try their best even if they were not sure about how to complete some of the questions. The time the students started was noted. They were instructed to turn the test over for me to collect when they were finished. I marked the time on the tests as I collected them. On average, it took 42 minutes to complete the NOT at pretest-T1.

All the participants were scheduled to arrive two days later at the two computer labs that were reserved for the intervention. Prior to arriving, the participants were randomly assigned to one of the three conditions, and were assigned to a computer work station by student identification number. In the computer labs, the participants were seated in groups according to condition.

In the lab sessions of the course following the pretest-T1, all participants completed a total of eight lessons over four instructional sessions. Four lessons were completed during the first instructional session and two lessons were completed in the remaining two weekly instructional sessions. All lessons were delivered on the computer and the participants worked their way through the lessons at individual computer stations. All of the participants met for the instructional sessions at the same time in one of two computer labs books specifically for this purpose. Two computer labs were required because no single computer lab at the university had enough computers to accommodate the number of participants in the study. All of the participants in the Iterative condition were in one computer lab, while all of the Concepts-first and Procedures-first participants were in a different computer lab across the hallway from the first computer lab. To avoid contaminating the conditions, all of the Concepts-first participants were seated on one side of the computer lab and all of the Procedures-first participants were seated on the other side on the computer lab.

Using the FirstClass e-mail software prior to the instructional sessions, e-mails containing the appropriate links to the intervention for each participant's assigned condition were composed, but not sent. For every participant, two e-mails, one per lesson, were composed for each instructional session. Once all of the participants had arrived at the lab rooms and were seated at their assigned computers, they were instructed to open their FirstClass e-mail accounts. They were informed that they would shortly receive a link and were asked to follow the link to the survey containing the first lesson. They were also informed that a second e-mail containing a second link would be sent 25 minutes later.

The participants were supposed to complete the first instructional session, which consisted of two lessons on base-7 place-value, during the lab time of the mathematics methods course. While the participants were gathered in the computer lab to complete the first instructional session, however, the university was evacuated for a fire drill. As a consequence, instructional session 1 and instructional session 2 were completed the following week during the lab time when the second instructional session was scheduled. That is, the participants completed the first four lessons in one lab session that was two weeks after pretest-T1.

The procedure for instructional session 2 was exactly the same as for instructional session 1. When they were finished the first survey, they were instructed to start the second survey. They completed the lessons in the order in which they were sent to their e-mail accounts. They were also be asked to work individually and not to communicate with other participants. Once the lessons were completed, the participants were excused. The same process was repeated for all three instructional sessions. The week following instructional session 4, the same procedure used for pretest-T1 was followed for postttest-T4.

Scoring

All conceptual items on the NOT were coded as either correct or incorrect. Correct answers were awarded 1 point and incorrect answers were awarded 0 points. Any items that were left blank were treated as incorrect and received a score of 0. All of the procedural items, except for the missing value base-7 chart items, were coded in the same manner as the conceptual items. The missing value base-chart item was subdivided into four sub-items,each worth a maximum of 1 point. Therefore, that particular item was worth a maximum of 4 points.

The SCK items were scored differently. Each SCK item was worth a total of five points. One point was awarded for correctly identifying whether or not the student's work was mathematically accurate. An additional point was awarded for accurately identifying the error made by the student, and up to three points were awarded depending on the nature of the participant’s explanation of the student’s work. For a conceptual only or procedural only explanation, the participant received 1 point. Procedural only explanations describe the student's steps without any reference to the justifications for the steps or why the student should have taken different steps. Similarly, a conceptual only explanation describes where the student went wrong from a conceptual perspective (e.g., did not properly regroup) but does not explain the steps the student took. For disjointed conceptual and procedural explanation, the participant received 2 points. An explanation was considered a disjointed conceptual and procedural explanation when the explanation incorporated both concepts and procedures but failed to make a connection between the concepts and the procedures. A linked conceptual and procedural explanation incorporated the relevant concepts and procedures and made the relationship between the concepts and procedures explicit. A linked conceptual and procedural explanation, received 3 points.

For the addition SCK item (see Figure 15), an ideal explanation of the student’s work for five points would include identifying that the child's response was incorrect, describing that the child added the ones to get 12 and made a procedural error by writing “12” in the ones positions. Similarly, for the subtraction SCK item (see Figure 16), an explanation of the student’s work for five points would include discussing that the student was incorrect because 9 could not be subtracted from 4 because you cannot take more away from the amount that exists, and the student decomposed a group of hundreds, and incorrectly regrouped the group of hundreds into ten ones rather than to ten tens in a manner that was conceptually and procedurally linked.

The conceptual items, procedural items, and SCK items at T2 and T3 were coded in the same way as the pretest-T1 and posttest-T4 items.

Data Analysis

Quantitative analysis. After all the items were coded, a percentage for each of the six outcome measures (conceptual base-7 knowledge, procedural base-7 knowledge, conceptual base-10 knowledge, procedural base-10 knowledge, SCK, and transfer) was calculated at each applicable time point. Depending on the measure, these percentages were calculated for pretest-T1, T2, T3, and posttest-T4 or pretest-T1 and posttest-T4 (see Table 2). After the data were coded and the percentages were calculated, mean scores were compared using either a repeated measures 3 x 2 ANOVA with lesson sequence (Iterative, Concepts-first, and Procedures-first) as the between condition factor and time (pretest-T1, posttest-T4) as the within condition factor or a repeated measures 3 x 4 ANOVA with lesson sequence (Iterative, Concepts-first, and Procedures-first) as the between condition factor and time (pretest-T1, T2, T3, posttest-T4) as the within condition factor. Separate analysis were conducted for each of the six dependent measures.

Qualitative coding and analysis. Following the quantitative analysis, any significant effects were further analyzed from a qualitative perspective. Rubrics for all of the time points emerged from the data using a grounded theory approach. All of the participants' responses were examined, and then re-examined, as interesting features were observed in the codes. The examination of the commonalities and differences with and across time points were conducted using codes and observed patterns in the data.

Results

Quantitative Analysis

The data from this study were analyzed both quantitatively and qualitatively. This section specifically addresses the quantitative results of the conceptual knowledge scores in base-7 and in base-10, the procedural knowledge scores in base-7 and in base-10, the SCK scores, and the transfer scores.

Descriptive statistics. The mean scores and standard deviations for conceptual knowledge of base-7 place-value, procedural knowledge of base-7 place-value, and SCK are presented in Table 5. Mean knowledge scores are presented as a function of lesson sequencing (Concepts-first, Procedures-first, Iterative) and time (pretest-T1, T2, T3, posttest-T4). Similarly, the mean scores and standard deviations for conceptual and procedural knowledge of base-10 place-value at pretest-T1 and posttest-T4 are presented in Table 6. Mean knowledge scores are once again presented as a function of lesson sequencing (Concepts-first, Procedures-first, Iterative) and time (pretest-T1, posttest-T4).

Table 5

Mean Knowledge Scores of Base-7 Place-Value and Standard Deviations as a Function of Condition and Time

	Time Points			
	Pretest-T1	T2	T3	Posttest-T4
Conceptual Knowledge				
Concepts-first	0.40 (0.27)	0.20 (0.42)	0.60 (0.51)	0.83 (0.31)
Procedures-first	0.50 (0.31)	0.11 (0.33)	0.78 (0.44)	0.94 (0.17)
Iterative	0.53 (0.32)	1.00 (0.00)	1.00 (0.00)	0.93 (0.17)
Procedural Knowledge				
Concepts-first	0.60 (0.19)	0.89 (0.33)	1.00 (0.00)	0.95 (0.06)
Procedures-first	0.63 (0.23)	0.89 (0.33)	0.89 (0.33)	0.94 (0.07)
Iterative	0.65 (0.23)	1.00 (0.00)	0.80 (0.42)	0.94 (0.10)
SCK				
Concepts-first	0.49 (0.25)	0.59 (0.25)	0.52 (0.24)	0.53 (0.20)
Procedures-first	0.43 (0.24)	0.54 (0.23)	0.54 (0.33)	0.70 (0.17)
Iterative	0.44 (0.07)	0.67 (0.31)	0.53 (0.22)	0.74 (0.13)

Note. Standard deviation scores are reported in parenthesis after the mean knowledge score.

Table 6

Mean Knowledge Scores of Base-10 Place-value and Standard Deviations as a Function of Condition and Time

	Time Points	
Condition	Pretest-T1	Posttest-T4
Conceptual Knowledge		
Concepts-first	0.86 (0.23)	0.95 (0.15)
Procedures-first	0.90 (0.21)	1.00 (0.00)
Iterative	0.95 (0.15)	0.95 (0.16)
Procedural Knowledge		
Concepts-first	0.75 (0.15)	0.80 (0.15)
Procedures-first	0.77 (0.15)	0.87 (0.13)
Iterative	0.84 (0.18)	0.82 (0.17)

Note. Standard deviation scores are reported in parenthesis after the mean knowledge score.

Lesson sequencing and conceptual knowledge of place-value. Separate analyses were conducted for base-7 place-value knowledge and base-10 place-value knowledge. The dependent base-7 measures were the conceptual base-7 subscale of the NOT at pretest-T1, the conceptual item at T2, the conceptual item at T3; and the conceptual base-7 scale of the NOT at posttest-T4. The base-10 conceptual subscale of the NOT was the dependent measure for the base-10 analysis.

Conceptual Base-7 knowledge of place-value. The mean conceptual knowledge scores for the base-7 measures for each of the three conditions and at each of the four time points are graphed in Figure 43.

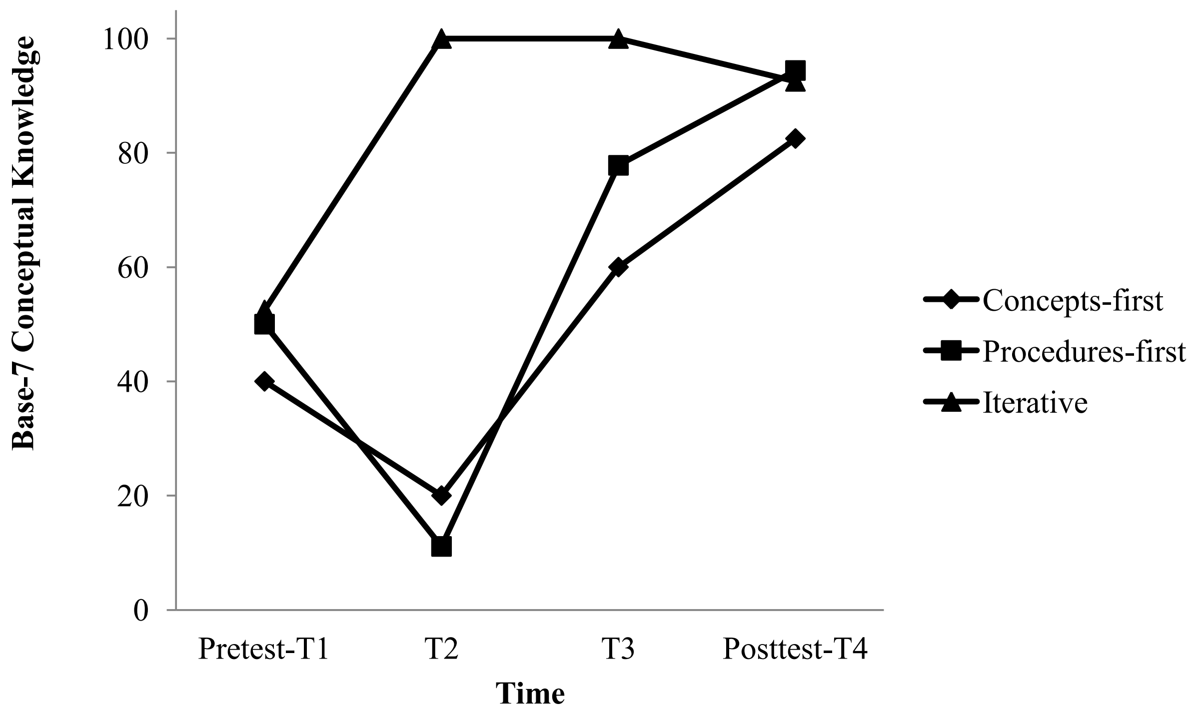


Figure 43. Mean conceptual knowledge percent scores. All three conditions are graphed as a function of both lesson sequencing and time across all four time points.

To test the effects of lesson sequencing on conceptual knowledge of base-7 place-value, I conducted a repeated measures 4 × 3 ANOVA with time (pretest-T1, T2, T3, posttest-T4) as the within group factor and lesson sequencing (Concepts-first, Procedures-first, Iterative) as the between group factor. Results revealed that there was a significant main effect of time, $F(3, 78) = 16.8$, $p < 0.001$. In other words, regardless of condition, there was a significant difference found between the mean conceptual knowledge scores at the four time points. Post-hoc Bonferroni comparisons revealed that between pretest-T1 ($M = 0.474$, $SD = 0.294$) and T3 ($M = 0.793$, $SD = 0.412$, $p = 0.018$), there was a significant increase in conceptual knowledge scores regardless of condition. In addition, there was a significant increase in mean conceptual knowledge scores between pretest-T1 ($M = 0.474$, $SD = 0.294$) and T4 ($M = 0.897$, $SD = 0.227$, $p < 0.001$); and T2 ($M = 0.448$, $SD = 0.506$) and T3 ($M = 0.793$, $SD = 0.412$, $p = 0.004$), and, finally, between T2 ($M = 0.448$, $SD = 0.506$) and T4 ($M = 0.897$, $SD = 0.227$, $p < 0.001$). Without taking into consideration lesson sequencing, it appears that overall, the participants increased their conceptual understanding of place-value from pretest-T1 to posttest-T4.

The same 4 x 3 ANOVA also revealed a significant main effect of group, $F(2, 26) = 11.8$, $p < 0.001$. That is, without taking into consideration the effect of time, there was a significant difference in mean conceptual knowledge scores between conditions. More specifically, post-hoc Bonferroni comparisons indicated that the Iterative condition's mean conceptual knowledge score ($M = 0.863$, $SD = 0.123$) was significantly higher than the Concepts-first condition's mean conceptual knowledge score ($M = 0.506$, $SD = 0.380$, $p < 0.001$). Moreover, the Iterative condition's conceptual knowledge score ($M = 0.863$, $SD = 0.123$) was also significantly higher than the Procedures-first condition's conceptual knowledge score ($M = 0.583$, $SD = 0.312$, $p = 0.005$). At the same time, there was no significant difference between the mean conceptual knowledge scores of the Concepts-first condition ($M = 0.506$, $SD = 0.380$) and the Procedures-first condition ($M = 0.583$, $SD = 0.312$, $p > 0.05$).

Finally, the same 4 x 3 ANOVA indicated that there was a significant time × condition interaction, $F(6, 78) = 5.27$, $p < 0.001$. This indicates that the effect of time on conceptual knowledge was moderated by lesson sequencing, or condition. Simple effects analysis indicated that there were differences between the condition means at T2 only, $F(2, 26) = 24.46$, $p < 0.001$. Post-hoc analysis revealed that the Iterative condition ($M = 1.00$, $SD = 0.00$) outperfomed the Concepts-first condition ($M = 0.20$, $SD = 0.422$, $p < 0.001$) and also the Procedures-first condition ($M = 0.11$, $SD = 0.333$, $p < 0.001$). At the same time, no significant difference was found at T2 between the Concepts-first and the Procedures-first conditions ($p = 0.999$).

Conceptual Base-10 knowledge of place-value. The mean base-10 conceptual knowledge scores for each of the three conditions at pretest-T1 and posttest-T4 are graphed in Figure 44. To examine the effects of base-7 lesson sequencing on conceptual knowledge of base-10 place-value, I conducted a repeated measures 2 × 3 ANOVA with time (pretest-T1, posttest-T4) as the within condition factor and lesson sequencing (Concepts-first, Procedures-first, Iterative) as the between conditions factor. The results revealed that there was no significant effect of time, $F(1, 28) = 1.939$, $p = 0.175$ or condition, $F(2, 28) = 0.463$, $p = 0.634$. Furthermore, there was no significant time × condition interaction, $F(2, 28) = 0.479$, $p = 0.624$.

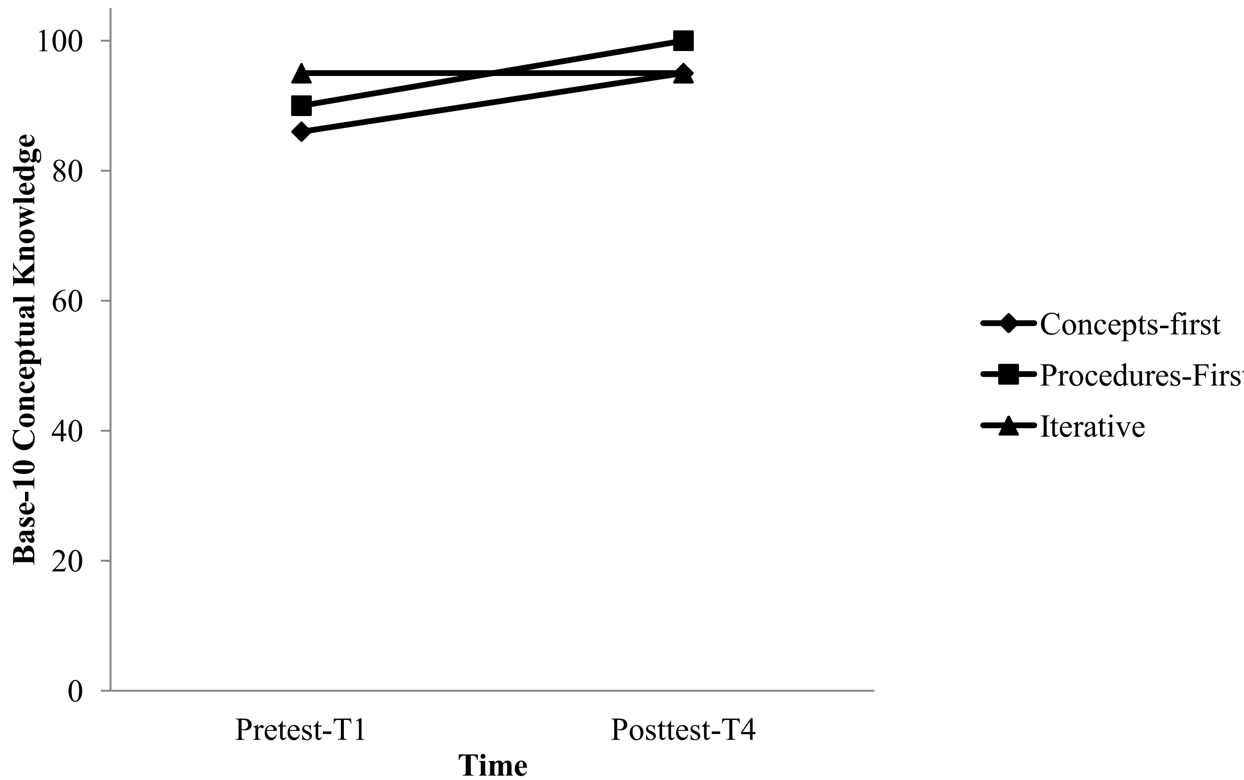


Figure 44. Mean base-10 conceptual knowledge percent scores. All three conditions are graphed as a function of both lesson sequencing and time.

Lesson sequencing on procedural knowledge of place-value. Separate analysis were conducted for base-7 place-value knowledge and base-10 place-value knowledge. The procedural base-7 dependent measure was the procedural base-7 subscale of the NOT at pretest-T1 and posttest-T4, the procedural item at T2, and the procedural item at T3. Similarly, the procedural base-10 knowledge subscale from the NOT at pretest-T1 and posttest-T4 was used as the dependent measure for procedural base-10 place-value knowledge.

Procedural base-7 knowledge of place-value. The mean scores of base-7 procedural knowledge for the Concepts-first, Procedures-first, and Iterative conditions at each of the four time points are graphed in Figure 45.

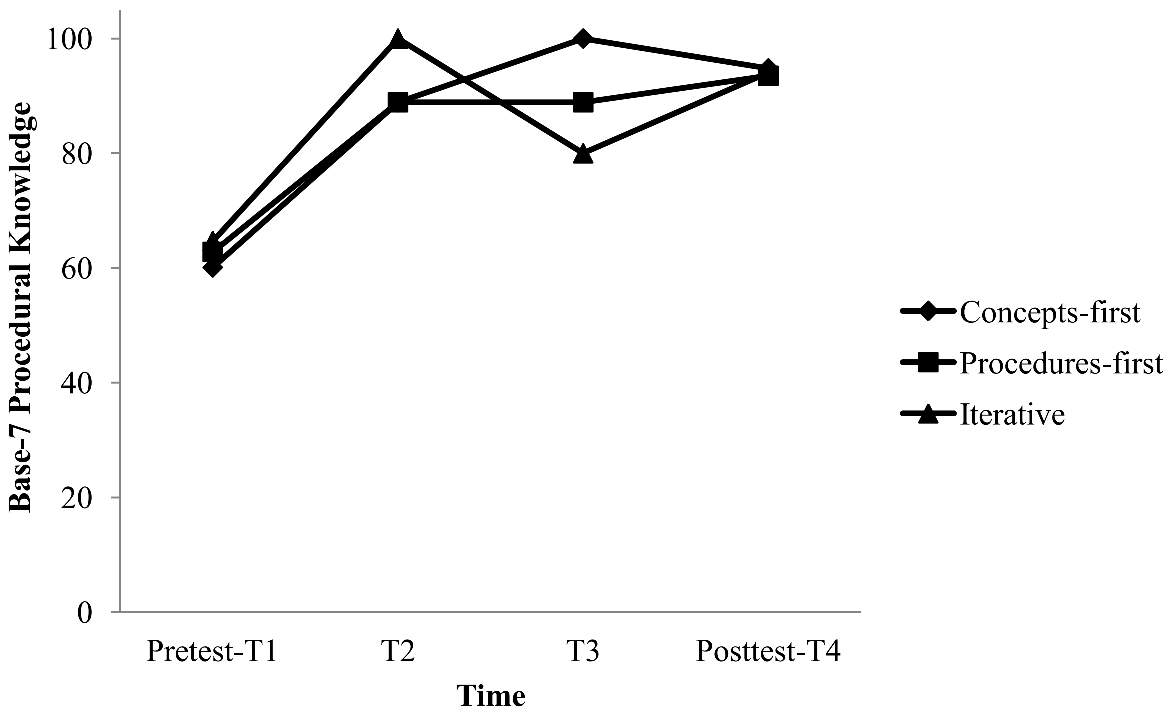


Figure 45. Mean base-7 procedural knowledge percent scores. All three conditions are graphed as a function of both lesson sequencing and time.

To test the effects of lesson sequencing on procedural base-7 knowledge, I conducted a repeated measures 4 × 3 ANOVA with time (pretest-T1, T2, T3, posttest-T4) as the within condition factor and lesson sequencing (Concepts-first, Procedures-first, Iterative) as the between conditions factor. The results revealed that there was a significant main effect of time, $F(3,75) = 11.7$, $p < 0.001$. Post-hoc Bonferroni comparisons revealed that there was a significant increase in procedural knowledge scores from T1 ($M = 0.626$, $SD = 0.209$) to T2 ($M = 0.928$, $SD = 0.262$, $p < 0.001$). Similarly, at T3 ($M = 0.893$, $SD = 0.315$), the mean procedural knowledge score was significantly higher than the mean procedural knowledge score at T1 ($M = 0.626$, $SD = 0.209$, $p = 0.005$). Finally, the mean score at T1 ($M = 0.626$, $SD = 0.225$) was significantly lower than the mean score at T4 ($M = 0.941$, $SD = 0.075$, $p < 0.001$). This means that regardless of lesson sequencing, the participants' procedural knowledge improved significantly from the pretest-T1 to T2, after which their performance remained constant. There was no significant main effects condition, $F(2, 75) = 0.084$, $p = 0.920$, nor was there a significant time × condition interaction, $F(6, 75) = 0.845$, $p = 0.539$.

Procedural base-10 knowledge of place-value. The mean base-10 procedural knowledge scores for each of the three conditions and at pretest-T1 and posttest-T4 are graphed in Figure 46. To examine the effects of base-7 lesson sequencing on procedural base-10 place-value knowledge, I conducted a repeated measures 2 × 3 ANOVA with time (pretest-T1, posttest-T4) as the within condition factor and lesson sequencing (Concepts-first, Procedures-first, Iterative) as the between condition factor. The results revealed that there was no significant effect of time, $F(1, 28) = 2.343$, $p = 0.137$, or condition, $F(1, 28) = 0.608$, $p = 0.551$. Furthermore, there was no significant time × condition interaction, $F(2, 28) = 1.294$, $p = 0.290$.

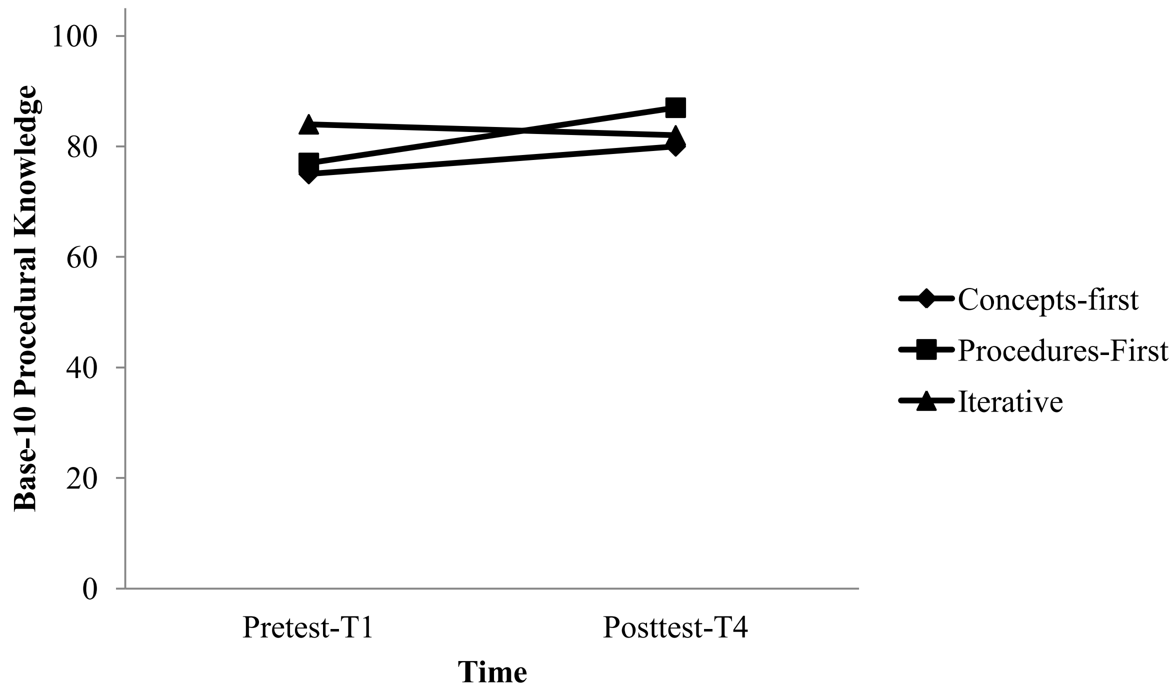


Figure 46. Mean base-10 procedural knowledge scores. All three conditions are graphed as a function of both lesson sequencing and time.

Lesson sequencing and SCK of place-value. The mean SCK scores for each of the three conditions at each of the four time points are graphed in Figure 47. To test the effects of lesson sequencing on SCK, I conducted a repeated measures 4 × 3 ANOVA with time (pretest-T1, T2, T3, posttest-T4) as the within conditions factor and lesson sequencing (Concepts-first, Procedures-first, Iterative) as the between conditions factor. Similar to procedural base-7 knowledge, the results indicated that there was a significant main effect of time, $F(3, 75) = 5.10$, $p = 0.003$, on SCK scores. Post-hoc Bonferroni comparisons revealed that there was a significant increase in SCK scores from pretest-T1 ($M = 0.454$, $SD = 0.193$) to posttest-T4 ($M = 0.661$, $SD = 0.126$, $p < 0.001$). No other pairwise comparisons were significant. This means that regardless of lesson sequencing, the participants' SCK scores improved significantly at posttest-T4 compared to their initial scores at pretest-T1. There was no significant main effect of condition, $F(2, 25) = 0.455$, $p = 0.639$, nor was there a significant time × condition interaction for SCK, $F(6, 75) = 0.965$, $p = 0.455$.

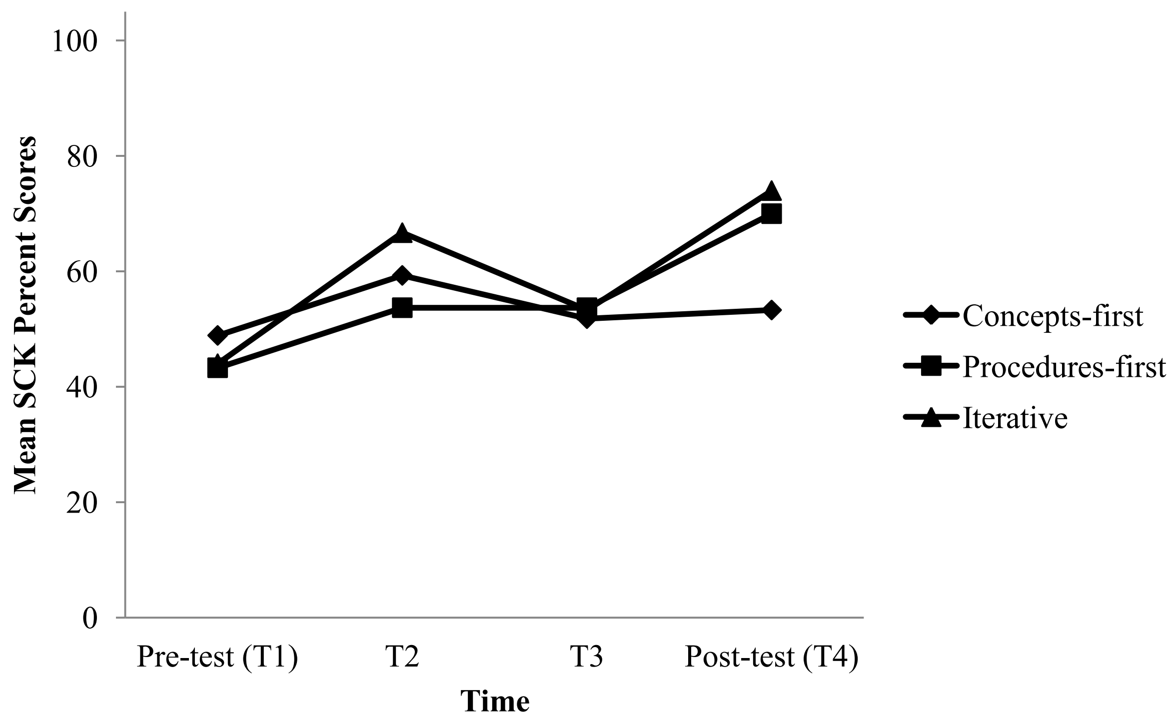


Figure 47. Mean SCK scores. All three conditions are graphed as a function of both lesson sequencing and time across all four time points.

Knowledge transfer to base-5. The mean scores and standard deviations for the transfer measure are presented in Table 7. Mean knowledge scores are presented as a function of lesson sequencing (Concepts-first, Procedures-first, Iterative) and time (pretest-T1, posttest-T4). Furthermore, the mean transfer scores for each of the three conditions and at both time points are graphed in Figure 48.

Table 7

Means and Standard Deviations for Base-5 Transfer Scores as a Function of Condition and Time

	Time Points	
Condition	Pretest-T1	Posttest-T4
Concepts-first	0.27 (0.47)	0.73 (0.47)
Procedures-first	0.40 (0.52)	0.70 (0.48)
Iterative	0.50 (0.53)	0.80 (0.42)

Note. Standard deviation scores are reported in parenthesis after the transfer scores.

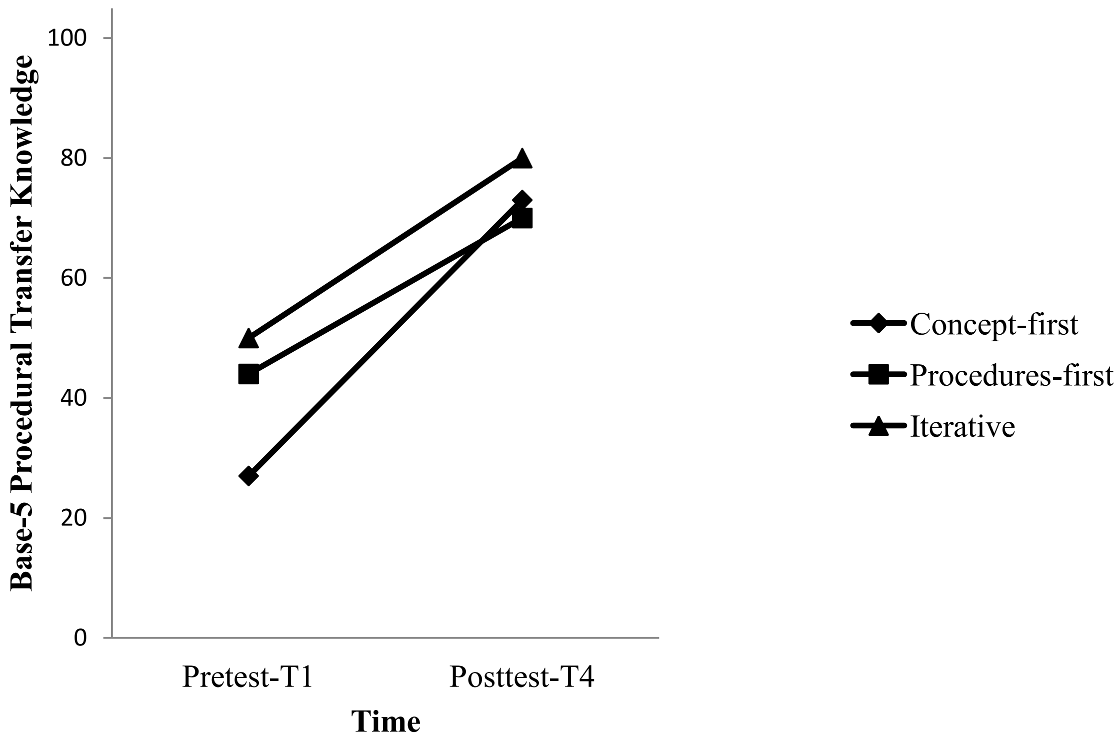


Figure 48. Mean procedural transfer scores. All three conditions are graphed as a function of both lesson sequencing and time from pretest-T1 to posttest-T4.

To evaluate the effects of lesson sequencing on the transfer task, I performed a 2 × 3 repeated measures ANOVA with time (pretest-T1, posttest-T4) as the within condition factor and lesson sequencing (Concepts-first, Procedures-first, Iterative) as the between condition factor. The results indicated that there was a significant main effect of time, $F(1, 28) = 8.28$, $p = 0.008$. That is, without taking into consideration lesson sequencing, there was a significant increase in mean transfer scores from pretest-T1 to posttest-T4. At the same time, there was no main effect of condition, $F(1, 28) = 0.521$, $p = 0.600$, nor was there a significant time × condition interaction, $F(2, 28) = 0.184$, $p = 0.833$.

Qualitative Analysis

In the interest of examining how the participants' knowledge of place-value changed, further qualitative analyses of conceptual base-7 place-value knowledge, procedural base-7 place-value knowledge, SCK, and transfer were performed. The qualitative analysis for each of the three knowledge types varied depending on the types of significant effects revealed by the quantitative analysis. I only looked at differences between conditions where there was a significant time × condition interaction. Subsequently, when there was a significant main effect of time only, I grouped all of the participants together and examined the changes that occurred over the specific time points where there were significant effects. Finally, when there were no significant effects, no qualitative analyses were performed.

Conceptual base-7 knowledge of place-value. For conceptual base-7 place-value knowledge, the quantitative results of this study revealed a significant time × condition interaction, with significant differences between the conditions, at T2. On the other hand, at pretest-T1, T3 and posttest-T4, there were no significant differences between the conditions. For this reason, all three of the conditions were collapsed and the participants' responses were qualitatively analyzed as a single group for these time points. At T2, the Concepts-first and Procedures-first conditions were collapsed and analyzed as one group because there was no significant difference between these two conditions. The collapsed group will be called here the "Combined" condition. The Iterative condition was not collapsed with the other conditions because there was a significant difference between the Iterative condition and the other two conditions at T2. The qualitative differences between the Iterative condition and the Combined condition will be reported in the following section. In particular, the changes in responses from pretest-T1 to T2, and the changes from T2 to T3, on the participants' conceptual base-7 knowledge are described below. Only these particular differences were examined because it was between these points where there were differences between the performances of the conditions. I was interested in investigating how lesson sequencing possibly impacted the acquisition of conceptual knowledge.

Conceptual addition and subtraction at pretest-T1. At pretest-T1, the participants completed several conceptual tasks that involved both adding and subtracting with pictures of objects rather than numbers. For the qualitative analysis, one conceptual addition item and one conceptual subtraction item were selected for further examination. Please refer to Figure 49 for the conceptual addition item and Figure 50 for the conceptual subtraction item that were selected for the qualitative analysis.

Base-7

Instructions: Examine the number of objects in column A and column B. Use the quantities presented in the columns to fill in the blanks in the corresponding number sentences. Use **base-7**.

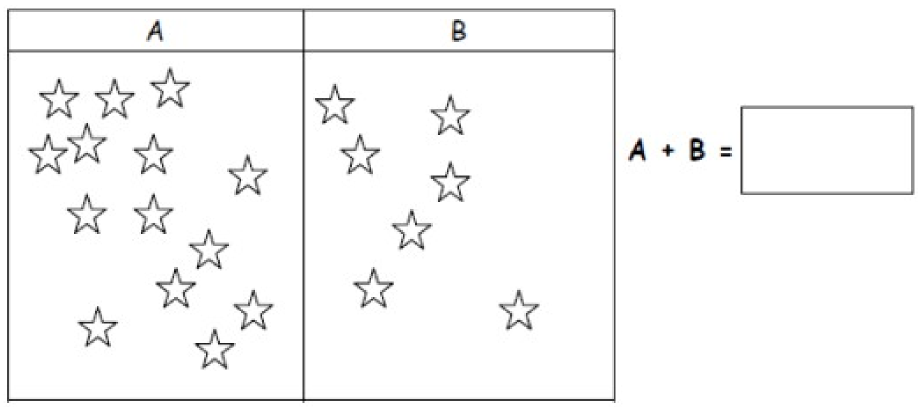


Figure 49. Pretest-T1 and posttest-T4 conceptual addition item. The correct answer should be 30_7.

Instructions: Examine the number of objects in column A and column B. Use the quantities presented in the columns to fill in the blanks in the corresponding number sentences. Use **base-7**.

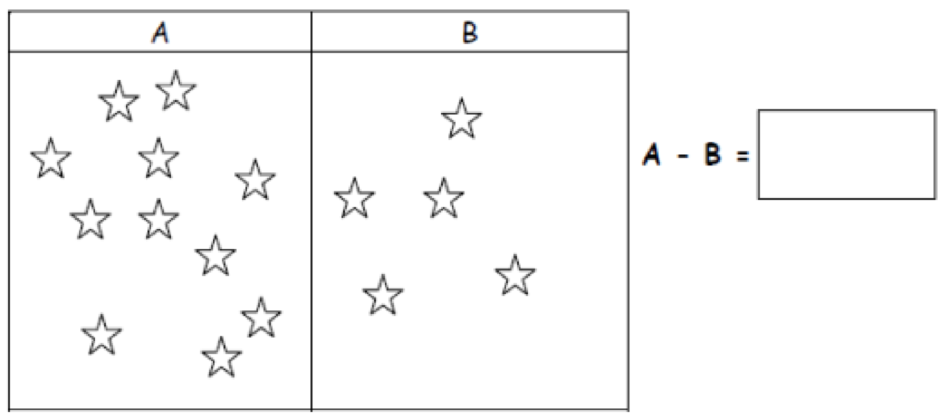


Figure 50. Pretest-T1 and posttest-T4 conceptual subtraction item. The correct answer is 6_7.

Conceptual addition. For the addition item, 10 of participants (30%) got the correct answer whereas seven of the participants (21%) got the correct answer for the subtraction item. Three of the participants who obtained the correct answer showed evidence of counting and grouping in base-7, whereas the other seven did not show any work.

Upon further examination of the incorrect addition items, it became apparent that the participants' base-10 knowledge may have interfered with their ability to complete the addition task in base-7 (see Table 8 for a breakdown of the incorrect responses). Of all of the participants who did not get the correct answer, 12 participants (53% of the incorrect answers) added the objects by counting them in base-10, but using base-7 notation. In these cases, participants counted 21 objects in base-10 and recorded the answer as "21_7" instead of the correct answer, "30_7" (see Figure 51 for an example of this type of error). At the same time, 5 participants counted or operated incorrectly in base-7. For example, one of the participants correctly grouped the objects into three groups of seven, but then recorded "3_7" as the answer rather than "30_7" (see Figure 52). Finally, the last 6 (26% of the incorrect responses) of the participants who did not get the correct answer simply did not respond to the question.

Table 8

Types of Incorrect Base-7 Conceptual Addition Responses

	Incorrect Responses	
Type of Response	*n*	%
Counted in base-10 and incorrectly used base-7 notation	12	53
Counted and operated incorrectly in base-7	5	21
Did not respond	6	26

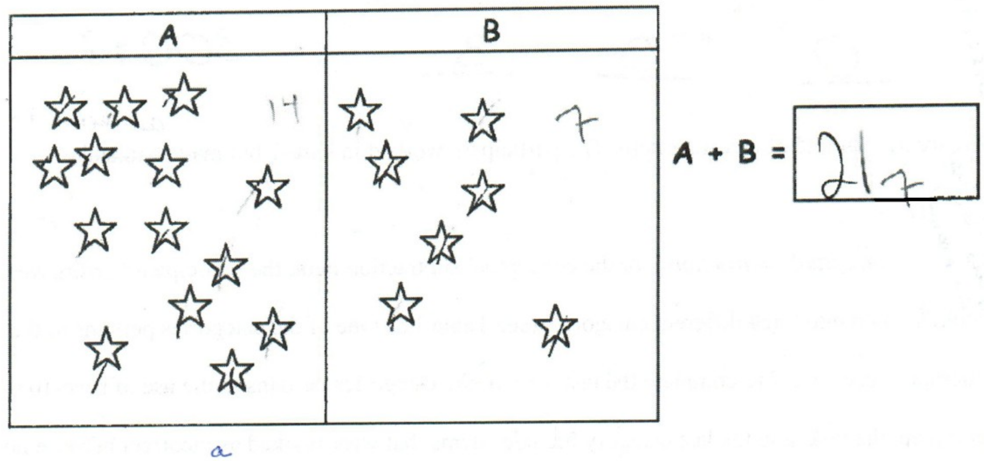


Figure 51. Sample of a participant's incorrect response. The participants worked in base-10 but used base-7 notation.

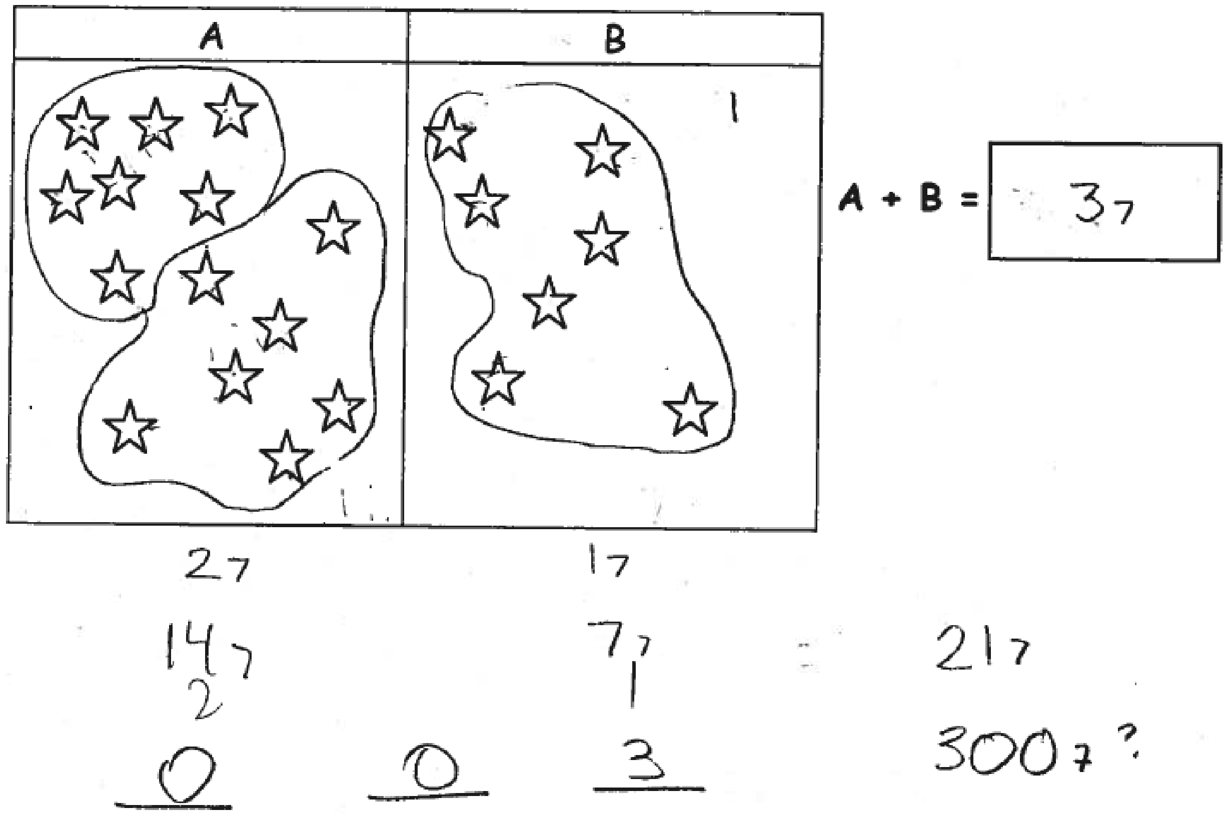


Figure 52. Sample of incorrect item. The participant worked in base-7, but made an error in notation.

Conceptual subtraction. For the conceptual subtraction item, the participants' errors were broken down into three different categories (see Table 9). One of the categories pertains to the attempt to use base-7 to complete the task, one of the categories pertains to the use of base-10 to complete the task, and the last category includes items that were marked as incorrect because no answer was given. Thirteen (13) of the participants who did not get the correct answer used base-7 while answering the question, seven used base-10, and six did not answer the question. Within the errors using base-7 category, several different types of errors were made. For example, some participants accurately grouped and counted the objects in base-7 but failed to use the correct base-7 notation and recorded the answer as "6" rather than "6_7" (see Figure 53 for an example of incorrect base-7 notation). This would be considered a procedural error because the mistake was purely notational. Similarly, one participant crossed out five of the stars from the first quantity to subtract 5_7 from 14_7, but then recorded the answer as "10_7" when it should have been "$6_{7.}$" In this particular case (see Figure 54), the participant demonstrated that she understood what the operation was, attempted to use base-7 (her answer was not in base-10), used base-7 notation, but did not record the correct quantity. One participant attempted to use base-7 but performed the wrong operation. A few other participants failed to keep the quantities separate. For example see Figure 55 that shows a participant's work where she appears to understand the operation, attempts to use base-7, but for some reason fails to keeps the quantities separated. Finally, for several of the participants, the work was difficult to decipher and the answer produced was incorrect (see Figure 56).

Table 9

Types of Incorrect Base-7 Conceptual Subtraction Responses

	Incorrect Responses	
	n	%
Base-7 errors	13	50
Used base-10	7	27
Did not answer	6	23

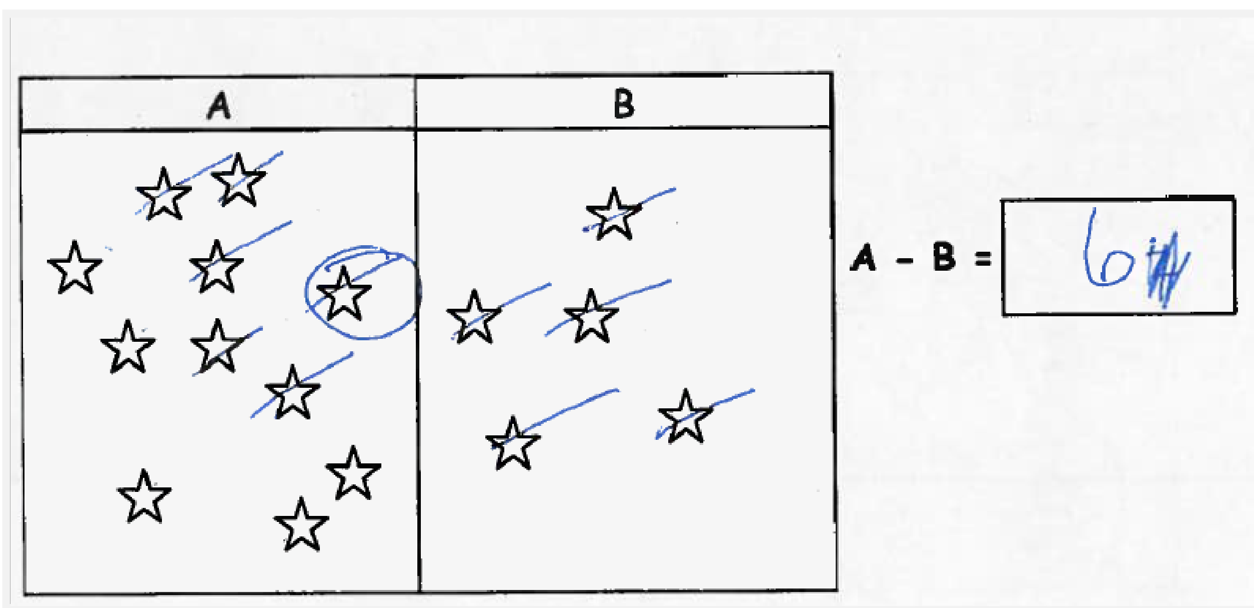


Figure 53. Example of a participant making a base-7 notational error.

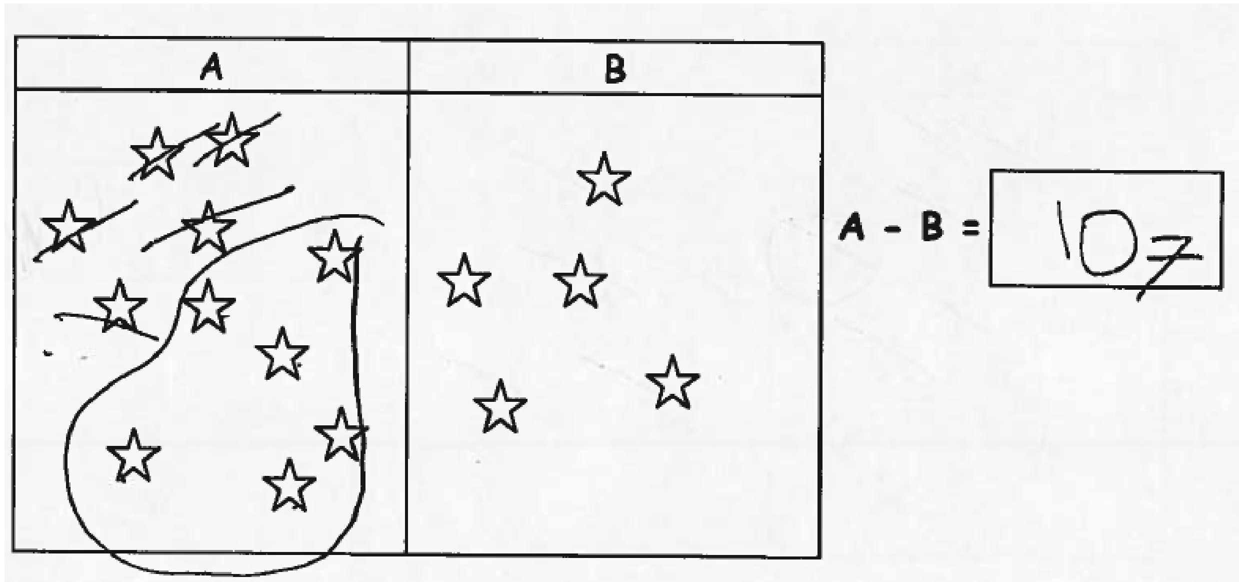


Figure 54. Example of a participant attempting to operate in base-7.

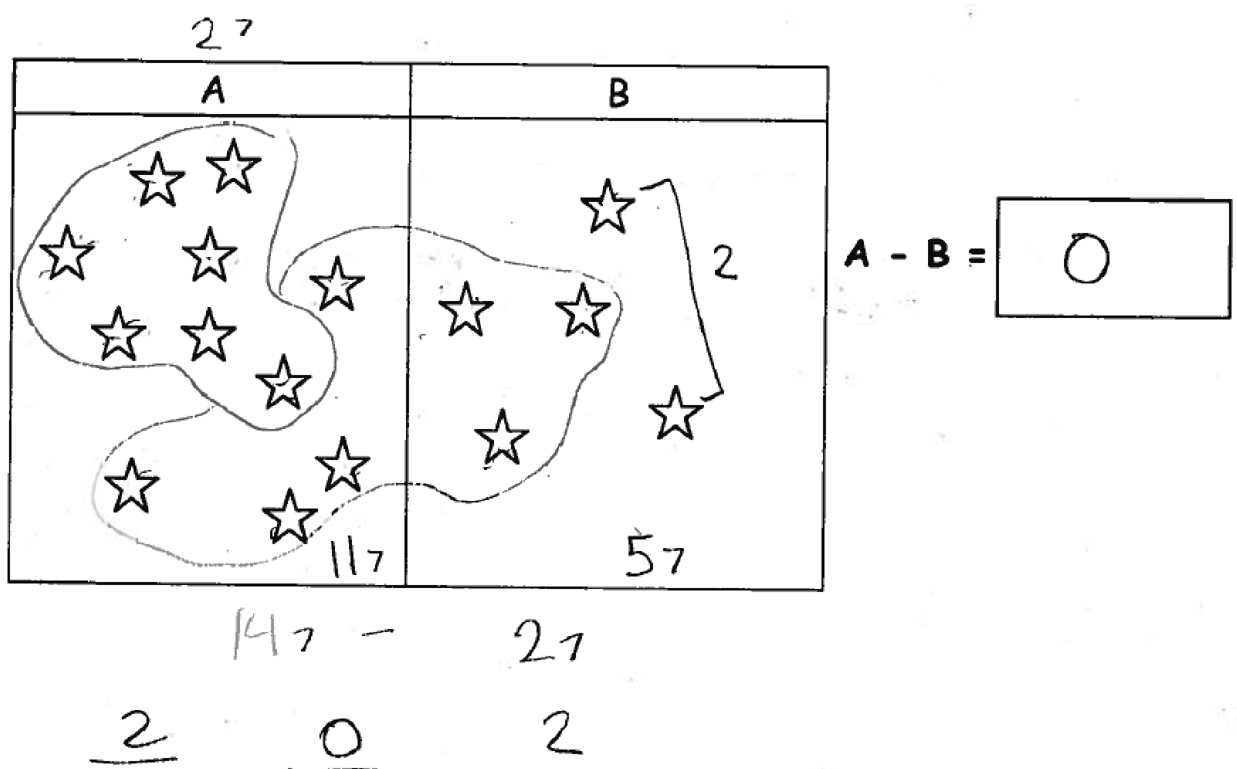


*Figure 55.*Sample of a participant's incorrect response.

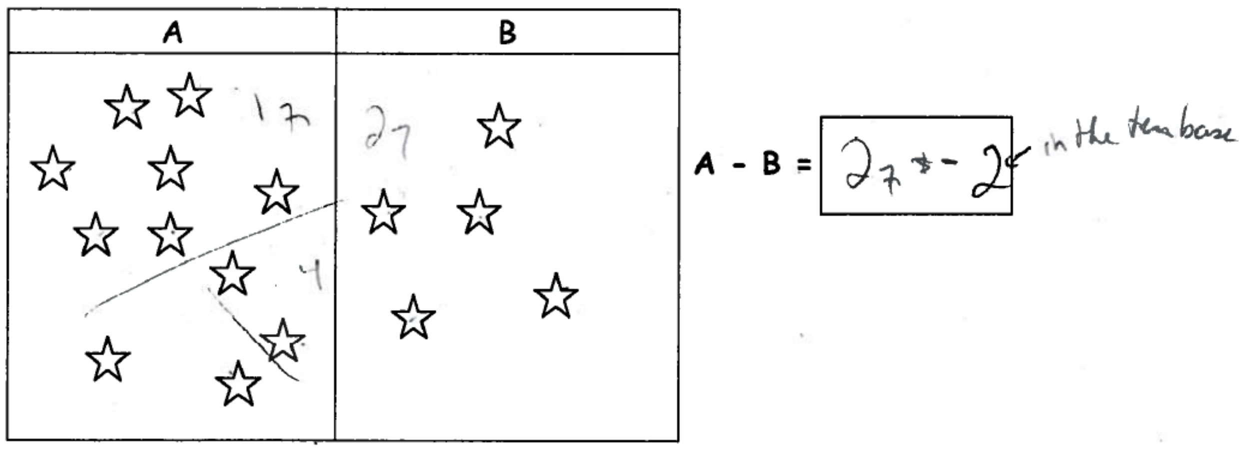


*Figure 56. S*ample of an incorrect response.

Two different types of errors using base-10 were made on the subtraction item. The most common mistake involved using base-10 to subtract the second quantity from the first quantity and then incorrectly trying to convert this difference into base-7. One participant used base-10 and correctly converted to base-7, but performed the wrong operation and added rather than subtracted.

Iterative condition compared to Combined condition at T2. I qualitatively analyzed the differences between the Iterative condition and Combined condition. At this particular time point, the participants were asked to provide an explanation regarding why base-7 numbers are named the way they are. The Iterative condition significantly outperformed the Combined condition. At T2, 11 out of the 12 participants (92%) in the Iterative condition provided an adequate conceptual explanation compared to only 2 of the participants in the Combined condition. That is to say, 92% of the participants in the Iterative condition provided an accurate explanation that was conceptual in nature compared to only 11% of the Combined condition. An example of this kind of explanation provided by a participant in the Iterative condition was "614_7 is called six-one-four base-7 because there are four single objects, one group of seven single objects, and six groups of seven single objects. We cannot say things like six hundred because the six doesn't represent the hundreds." Another correct conceptual explanation was "because there are six groups of 49 things, one group of seven things, and four things."

Incorrect responses were divided into three categories (see Table 10). Those three categories are summarized as follows: the explanation was procedural rather than conceptual, the explanation was in base-10 rather than base-7, or the explanation was incorrect. In the Iterative condition, only one participant made an error on this item, and it was because she did not complete the item. She responded that she "had no idea" why base-7 numbers were named the way they were. For the Combined condition, most of participants who got the item wrong used a procedural explanation. Forty-seven percent of the participants who got the incorrect answer, or 8 participants, explained that 614_7 was named "six-one-four base-7 because that is how it is done," or "because you name the single numbers and then write base-7 after." Another 7 participants in the Combined condition provided a base-10 explanation such as "because there are four ones, one ten, and 6 hundreds." Finally 2 participants in the same condition provided responses that were categorized as incorrect. These responses were either irrelevant, such as "no clue," or incomplete. An example of an incomplete answer is "there are four ones."

Table 10

Incorrect Responses to Conceptual Knowledge Task at T2

	Incorrect Responses			
	Procedural Explanation	Base-10 Explanation	Incorrect Explanation	Total
	n	*n*	*n*	*n*
Iterative	0 (0%)	0 (0%)	1 (100%)	1 (100%)
Combined	8 (47%)	7 (41%)	2 (12%)	17 (100%)

Iterative condition compared to Combined condition from pretest-T1 to T2. While the tasks at pretest-T1 and T2 were both designed to assess the participants' conceptual understanding of base-7 place value, the tasks were not the same. Regardless of this fact, certain differences emerged when comparing how all three of the conditions collapsed into one group performed at pretest-T1 to how the Iterative condition and Combined condition performed at T2. More specifically, the role of negative transfer of base-10 knowledge seemed to differ as a result of time and condition.

At pretest-T1, 13 of the participants (50% of the incorrect answers) who did not get the correct answer on the addition item used base-10 rather than base-7. Similarly, 7 participants who did not get the correct answer on the subtraction item (21% of the incorrect subtraction responses) also used base-10 instead of base-7. This is comparable to the Combined condition at T2. Seven participants in the Combined condition (41% if the incorrect responses) used a conceptual base-10 explanation rather than a conceptual base-7 explanation when describing the naming of base-7 number. At the same time, nobody in the Iterative condition exhibited this kind of negative base-10 transfer at T2 (see Table 11).

Table 11

Negative Base-10 Transfer to Conceptual Knowledge at Pretest-T1 and T2

	Incorrect Responses: Base-10 Negative Transfer	
	n	%
Pretest-T1		
All conditions - Addition	12	52
All conditions - Subtraction	7	27
T2		
Iterative	0	0
Combined	7	41

Conceptual addition and subtraction at T3. At T3, the participants were asked to subtract two quantities using the "block model" that they learned during the instruction (refer to Figure 57). On this particular item, 24 out of the 31 participants (77%) got the correct answer. Correct responses on this item are divided into three categories: correctly used block model only to arrive at correct answer (19 participants), correctly used block model to arrive at correct answer and verified with the algorithm in base-7 (4 participants), and correctly used the block model and incorrectly verified using the base-7 algorithm (1 participant).

Instructions: Using the block model you learned in the computer lessons, calculate the following.

3. $421_7 - 354_7$

Figure 57. Base-7 conceptual knowledge subtraction task at T3. Participants were expected to use the block model presented during the instructional intervention to regroup and trade appropriately and arrive at the difference 34_7.

At T3, the pattern of incorrect responses paralleled that of correct responses. Of all of the incorrect answers, three of the participants incorrectly used the block model. In other words, they attempted to use the block model, but incorrectly represented one or both quantities, incorrectly regrouped, or incorrectly counted the resulting quantity. One of the participants who did not get the correct answer used both the block model and the algorithm. More specifically, that participant incorrectly used the block model, but correctly computed using the algorithm. Finally, three participants correctly computed the answer using the algorithm only. Even though some of the participants arrived at the correct answer using the algorithm, the answer was coded incorrect because they used a procedure, and not their knowledge of the concepts, to arrive at the correct answer.

Iterative condition compared to Combined condition from T2 to T3. As with the conceptual items at pretest-T1 and T2, the conceptual task at T3 was designed to assess the participants' conceptual understanding of base-7 place value. Once again, the tasks at T2 and T3 were not the same, but analyzing the participants' responses at T2 and T3 from a qualitative perspective nevertheless revealed commonalities and differences in the participants' responses. More specifically, the main difference between T2 and T3 is regarding the interference of procedural knowledge on the participants' performance on the conceptual item.

At T2, the Iterative condition significantly outperformed the Combined condition on conceptual knowledge. Furthermore, the qualitative analysis at T2 revealed that almost half of the Combined condition's errors (47%; see Table 11) were attributed to the inappropriate use of a procedural explanation. This type of error did not occur with any of the Iterative condition's participants. At T3, there was no difference between any of the conditions. Nonetheless, the inappropriate application of procedures at T3 was substantially less for all three of the conditions grouped together than it was for the Combined condition at T2. Only three participants altogether inappropriately applied procedures at T3 compared to nine participants in the Combined condition at T2. While there were participants who used the algorithm to verify the answer they obtained using a more conceptual method, fewer participants did not get the answer correct. Furthermore, fewer participants committed themselves to using a procedure only rather than applying their conceptual knowledge.

Procedural base-7 knowledge. For procedural base-7 knowledge, there was a significant difference in base-7 procedural knowledge at the four time points regardless of condition. More specifically, performance pretest-T1 differed significantly from all the other time points. There was no significant main effect of condition nor was there a significant time × condition interaction. Therefore, the differences in the participants' responses at various time points, collapsed across all of the conditions into a single group, were further examined from a qualitative perspective. At pretest-T1, I qualitatively analyzed one procedural addition item and one procedural subtraction item from the NOT. The procedural task at T2 consisted of naming a base-7 number using words. I was interested in examining the nature of the increase in participants' procedural knowledge from pretest-T1 to T2. Because procedural knowledge gains stabilized from T2 to posttest-T4, the only other difference I examined qualitatively was between pretest-T1 and posttest-T4.

Procedural gains from pretest-T1 to T2. As with conceptual knowledge, the participants had more success with procedural addition than procedural subtraction at pretest-T1. At pretest-T1, 10 participants computed the correct answer for the addition item compared to seven participants who computed the correct answer for the subtraction item. For the addition item, all of the participants who computed the correct answer did so by operating and counting in base-7. On the other hand, 18 of the incorrect answers resulted from participants computing in base-10 but using base-7 notation. The remaining five participants did not answer the question.

The results for the subtraction item were similar to results for addition. For all seven of the participants who obtained the correct answer, all of them computed and counted in base-7. The incorrect responses were categorized into four distinct categories (see Table 12). Twenty-one participants computed in base-10 but used base-7 notation.

Similar to this type of error, two of the participants computed in base-10 and performed the wrong operation. That is, they added the two quantities rather than subtracting one from the other. Two other participants computed in base-7, but made a mistake counting or with base-7 notation. Finally, the last participants did not respond to the question.

Table 12

Incorrect Procedural Addition at Pretest-T1 and Posttest-T4

	Incorrect Responses				
	Computed in Base-10 but Used Base-7 Notation	Computed in Base-10 and Wrong Operation	Computed in Base-7 and Made Error	Did Not Answer	Total
	n	*n*	*n*	*n*	*n*
Pretest-T1	21 (81%)	2 (8%)	2 (8%)	1 (4%)	26 (101%)
Posttest-T4	0 (0%)	0 (0%)	2 (67%)	1 (33%)	3 (100%)

Note. The sum of the percentages at pretest-T1 equals 101% because of rounding.

There was a significant increase in base-7 procedural knowledge scores from pretest-T1 (26% accuracy when addition and subtraction scores are averaged together) to T2 (90% accuracy). For the three participants who did not get the correct response at T2, all of the participants used base-10 notation and wrote "base-7" at the end. For example, rather than correctly naming the number "six-one-four base-7," they named the number "six hundred and four base-7."

Procedural gains from pretest-T1 to posttest-T4. There was a substantial increase in base-7 procedural knowledge place-value scores from pretest-T1 to posttest-T4. At posttest-T4, 30 of the participants correctly calculated the response to the addition item compared to 10 participants at pretest-T1. Upon further analysis of the correct responses, it appears that 26 of the participants showed evidence of counting and computing in base-7, whereas 8 participants did not show any work but arrived at the correct answer nonetheless. Presumably, these participants did the work mentally. In contrast, at pretest-T1, all of the correct responses were accompanied by the appropriate written calculations.

As for the three incorrect addition responses at posttest-T4, two participants attempted to compute in base-7, but made errors either regrouping or counting, and one participant did not answer the question. None of the incorrect responses resulted from computing in base-10 and incorrectly using base-7 notation, such as was the case for 21 participants at pretest-T1 (see Table 12).

For the procedural base-7 subtraction at posttest-T4, there was a significant increase in accuracy from 21% at pretest-T1 to 76% at postttest-T4. Of the correct answers at postttest-T4, 22 of the responses were accompanied by the appropriate written work and there was no written work for three of the responses. In contrast, at pretest-T1, all of the correct responses were accompanied by the written work. Of the incorrect responses, three are attributed to using base-10 to compute and incorrectly using base-7 notation, compared to 21 of the incorrect responses at pretest-T1. Furthermore, three of the incorrect answers resulted from using base-7 to compute but making a mistake counting or with notation (compared to two at pretest-T1). Finally, two of the incorrect responses were considered incorrect because the participants did not write down an answer (see Table 13).

Table 13

Incorrect Procedural Subtraction

	Incorrect Responses				
	Computed in Base-10 but Used Base-7 Notation	Computed in Base-10 and Wrong Operation	Computed in Base-7 and Made Error	Did Not Answer	Total
	n	*n*	*n*	*n*	*n*
Pretest-T1	21 (81%)	2 (7%)	2 (7%)	1 (4%)	26 (99%)
Posttest-T4	3 (38%)	0 (0%)	3 (38%)	2 (25%)	8 (101%)

Notes. The percentages for pretest-T1 and posttest-T4 add up to 99% and 101%, respectively because of rounding.

SCK of place-value. The quantitative analysis demonstrated that the only significant effect on the participants' SCK scores was a significant main effect of time indicating that regardless of condition, the participants SCK of place-value improved over time. As such, all three of the conditions were once again collapsed into one group for further qualitative analysis from pretest-T1 to posttest-T4.

At pretest-T1 and posttest-T4, the participants completed two SCK tasks that involved analyzing hypothetical student responses to multi-digit addition and subtraction tasks. For the qualitative analysis, one addition SCK item and one subtraction SCK item were selected for further examination. Please refer to Figure 58 for the SCK addition item and Figure 59 for the SCK subtraction item.

Base-10

Instructions: Look at the solutions below produced by elementary school students. In each case, indicate if the student got the right answer. If the student solved the problem correctly, explain the steps used. If the student solved the problem incorrectly, describe the mistake(s) made by the student.

Student A's work:

$$\begin{array}{r} 27 \\ +\ 5 \\ \hline 212 \end{array}$$

Student A got the answer: **Right / Wrong** (circle one)

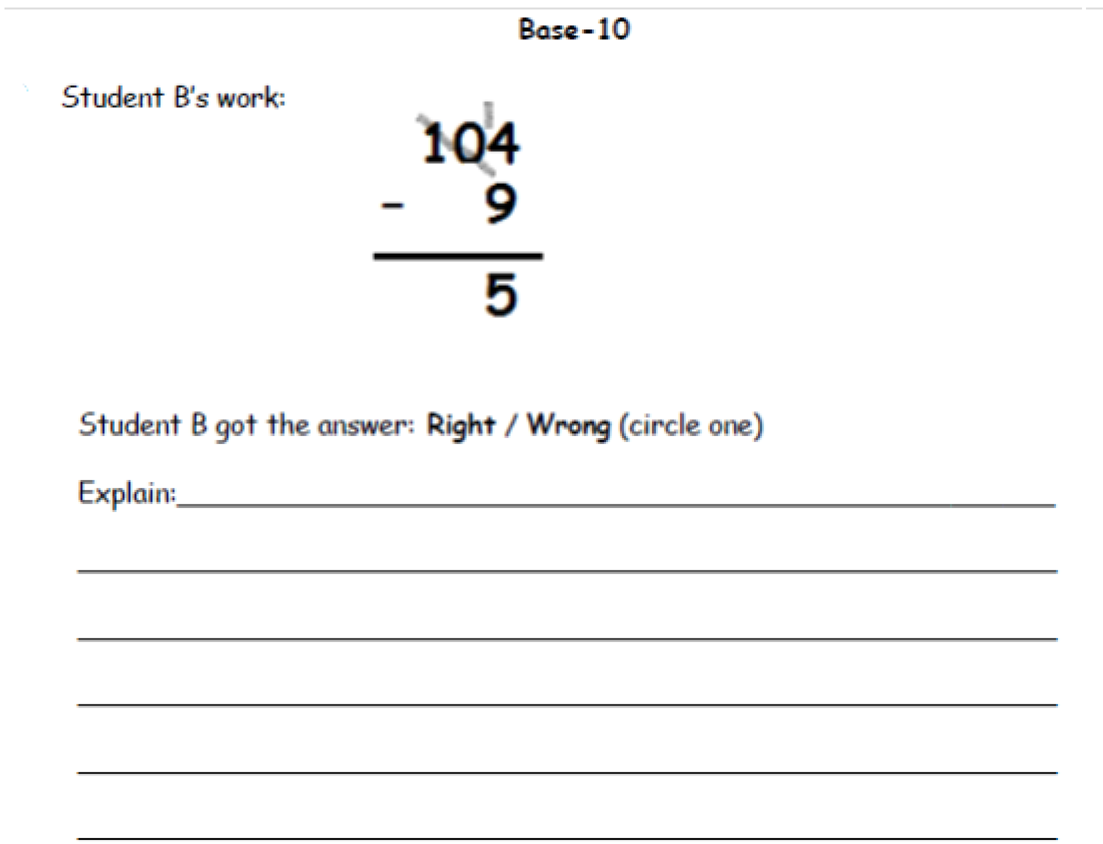
Explain:____

____

____

____

Figure 58. Pretest-T1 and posttest-T4 addition SCK item selected for further qualitative analysis. Participants were expected to recognize that the theoretical student did not get the correct answer and explain that the student wrote the sum of the ones column under the line without regrouping a group of 10 ones into a group of 10.


Base-10

Student B's work:

$$\begin{array}{r} 104 \\ -\quad 9 \\ \hline 5 \end{array}$$

Student B got the answer: **Right / Wrong** (circle one)

Explain:____

Figure 59. Pretest-T1 and posttest-T4 SCK subtraction item selected for further qualitative analysis. Participants were expected to recognize that the theoretical student did not get the correct answer and made an error regrouping one group of one-hundred into 10 ones.

SCK gains from pretest-T1 to posttest-T4. At pretest-T1, 12 participants recognized the student error as "did not regroup a group of 10 ones into one group of 10, called here the "Not Regrouping Error." An alternative correct response that 11 of participants recognized was that the theoretical student "wrote the answer from the ones under the line instead of carrying, or the "Under the Line Error." More specifically, participant explanations were coded as the Not Regrouping Error when the explanation specified that the hypothetical student had not regrouped a group of 10 ones. Whereas explanations that stipulated that the hypothetical student wrote the total of the one's column under the line, without any explanation of regrouping or carrying, were coded as the Under the Line Error. The other 10 participants could not adequately recognize either error. At posttest-T4, a higher number of participants recognized the Not Regrouping Error and the same number recognized the sum under the line error. That is, 21 participants recognized the first error and 11 recognized the second error. Furthermore, only one participant could not recognize either error at postttest-T4, compared to 10 participants at pretest-T1. Please see Table 14 for a summary of the SCK addition responses at pretest-T1 and posttest-T4.

Table 14

Summary of SCK Responses at Pretest-T1 and Posttest-T4

	Response Frequency			
	Not Regrouping Error	Under the Line Error	Did not Recognize an Error	Total
	n	*n*	*n*	*n*
Pretest-T1	12 (36%)	11 (33%)	10 (31%)	33 (100%)
Posttest-T4	21 (64%)	11 (33%)	1 (3%)	33 (100%)

In addition to recognizing the student error, the participants were required to provide an explanation of the error. See Table 15 for a summary of the SCK addition explanations at pretest-T1 and posttest-T4. An analysis of their explanations revealed that for the SCK addition task, 15 of the participants provided a procedural only explanation at pretest-T1 compared to 13 at postttest-T4. Moreover, at pretest-T1 only one participant provided an explanation that was considered conceptual and procedural, but disjointed. Thirteen participants provided the same type of explanation at posttest-T4. A conceptual and procedurally linked explanation was provided by 2 of the participants at pretest-T1 and by 5 participants at posttest-T4. At pretest-T1, a greater number of participants did not provide an explanation compared to posttest-T4; that is, eight participants compared to one participant, respectively. At the same time, only one participant provided an incorrect explanation at posttest-T4 compared to seven participants at pretest-T1.

Table 15

Summary of SCK Addition Explanations at Pretest-T1 and Posttest-T4

	SCK Addition Explanations					
	Procedural Only	Concept-ual and Procedural - Disjointed	Concept-ual and Procedural - Linked	Incorrect Explana-tion	Did not Provide Explana-tion	Total
	n	*n*	*n*	*n*	*n*	*n*
Pretest-T1	15 (45%)	1 (3%)	2 (6%)	7 (21%)	8 (25%)	33 (100%)
Posttest-T4	13 (3%)	13 (21%)	5 (41%)	1 (15%)	1 (15%)	33 (100%)

Table 16 summarizes the qualitative results for the SCK subtraction item. For this SCK item, 11 participants recognized the "Regrouping Error," or that the student had regrouped a group of one hundred into 10 ones rather than 10 tens, and nine participants recognized the error as the "Borrowing error," or that the student had borrowed from the wrong column. At posttest-T4, these numbers changed to seven and 15 participants, respectively. In terms not identifying any error, this number of participants who could not identify an error dropped from 11 at pretest-T1 to eight at posttest-T4. In contrast, non-responses increased from two at pretest-T1 to three at posttest-T4.

Table 16

Summary of SCK Subtraction Responses at Pretest-T1 and Posttest-T4

	Response Frequency				
	Regrouping Error	Borrowing Error	Did Not Recognize an Error	Did Not Respond	Total
	n	*n*	*n*	*n*	*n*
Pretest-T1	11 (33%)	9 (28%)	11 (33%)	2 (6%)	33 (100%)
Posttest-T4	7 (22%)	15 (45%)	8 (24%)	3 (9%)	33 (100%)

There were also qualitative differences in the type of explanations provided at pretest-T1 and posttest-T4. The incidence of procedural only explanations decreased from seven participants at pretest-T1 to one participant at posttest-T4. At the same time, the number of participants who produced disjointed and linked conceptual and procedural explanations increased from one and eight at pretest-T1, to 15 and 12 at posttest-T4, respectively. Not only did the nature of the types of explanations change, but the rate of inaccurate explanations also changed. At pretest-T1, nine of participants responded inaccurately compared to only three at posttest-T4. Finally, eight of the participants did not provide an explanation at pretest-T1 compared to two participants at posttest-T4. See Table 17 for a summary of these contrasts.

Table 17

Summary of SCK Subtraction Explanations at Pretest-T1 and Posttest-T4

	SCK Subtraction Explanations					
	Procedural Only	Conceptual and Procedural - Disjointed	Conceptual and Procedural - Linked	Incorrect Explanation	Did not Provide Explanation	Total
	n	*n*	*n*	*n*	*n*	*n*
Pretest-T1	7 (21%)	1 (3%)	8 (24%)	9 (27%)	8 (24%)	33 (99%)
Posttest-T4	1 (3%)	15 (45%)	12 (37%)	3 (9%)	2 (6%)	33 (100%)

Note. The percentages for pretest-T1 add up to 99% because of rounding.

Transfer knowledge. To evaluate transfer knowledge, participants were asked to compute two items in base-5: one vertically presented addition item and one vertically presented subtraction item (see Figure 60). Similar to SCK, there was a significant main effect of time on the transfer measure. All three conditions were grouped together into a single group because there was no significant main effect of condition or interaction. Given that transfer was evaluated only at pretest-T1 and posttest-T4, the significant main effect of time was an increase in transfer knowledge from pretest-T1 to posttest-T4.

These increases were analyzed qualitatively and reported separately by item in the following sections.

Base-7

Instructions: Perform the following calculations. Do all the computations in **base-7**.

13. $$\begin{array}{r} 14_7 \\ +13_7 \\ \hline \end{array}$$

14. $$\begin{array}{r} 100_7 \\ -\ 50_7 \\ \hline \end{array}$$

Figure 60. Pretest-T1 and posttest-T4 base-7 procedural knowledge items selected for qualitative analysis. Participants were expected to compute using the algorithm taught during the instructional intervention to arrive at the correct answers. The correct answer for these addition and subtraction items are 30_7 and 20_7, respectively.

Addition. At pretest-T1, 13 of the participants correctly answered the addition task compared to 23 at posttest-T4 (see Table 18). The participants' errors were divided into four categories at pretest-T1 (see Table 19). Of the participants who did not get the correct answer on the addition transfer task, 12 calculated in base-10 and used base-5 notation. Another participant unsuccessfully tried to convert the base-5 quantities to base-10 quantities. Finally, seven participants did not provide a response.

Table 18

Correct Transfer Task Responses

	Correct	
	Pretest-T1	Posttest-T4
	n	*n*
Addition	13 (39%)	23 (70%)
Subtraction	21 (63%)	24 (73%)

Table 19

Types of Errors on Addition Transfer Task at Pretest-T1 and Posttest-T4

	Incorrect Responses				
	Calculated in Base-10	Error Converting to Base-10	Error Regrouping in Base-5	Error Adding in Base-5	No Response
	n	*n*	*n*	*n*	*n*
Pretest-T1	12 (60%)	1 (5%)	0 (0%)	0 (0%)	7 (35%)
Posttest-T4	1 (10%)	0 (0%)	1 (10%)	7 (70%)	1 (10%)

At posttest-T4, in contrast, not only did the participants made fewer errors on the addition transfer task, but the types of errors that they did make were different. Errors at posttest-T4 were divided up into five different categories (see Table 19). Unlike at pretest-T1, only one of the incorrect responses was attributed to calculating in base-10, no errors were attributed to incorrectly converting to base-10, and only one of the participants did not provide an answer. At the same time, the majority of errors at posttest-T4 were caused by incorrectly using base-5. More specifically, one participant made a regrouping error. Rather than regrouping a group of five ones (5^0) into one group of five (5^1), it was regrouped into a group of 25 (5^2). This error was considered a regrouping error because the participant included the correct number of units in the group but placed that group in the incorrect denomination. Furthermore, seven of the participants incorrectly added in base-5. For example, several participants incorrectly indicated that the sum of 3_5 and 3_5 was 10_5 or 13_5 instead of the correct sum, 11_5. This was considered an adding error rather than a regrouping error because even though the participants regrouped, it appeared that they miscounted and put either too many units or not enough units into the grouping.

Subtraction. At pretest-T1, the participants' performed better on the subtraction items than they did on addition items. This may be attributed to the fact that there subtraction item did not require regrouping. Nonetheless, 63% of the participants were successful on the subtraction transfer task at pretest-T1. This rate improved to 74% at posttest-T4.

At pretest-T1, the errors were placed into four different categories. Six of the participants' answers were considered incorrect they did not write a response. Base-10 interference appeared to have accounted for a substantial portion of the mistakes made at pretest-T1. Three of the participants calculated in base-10 and incorrectly used base-5 notation. Moreover, one participant unsuccessfully attempted to convert the base-5 quantities to base-10 quantities. The remainder of the errors were attributed to adding in base-5 rather than subtracting. That is, two participants used the wrong operation. See Table 20 for a summary of the response types.

Table 20

Types of Errors on Subtraction Transfer Task at Pretest-T1 and Posttest-T4

	Incorrect Responses				
	Calculated in Base-10	Error Converting to Base-10	Wrong Operation in Base-5	No Response	Total
	n	*n*	*n*	*n*	*n*
Pretest-T1	3 (25%)	1 (8%)	2 (17%)	6 (50%)	12 (100%)
Posttest-T4	1 (14%)	0 (0%)	2 (29%)	4 (57%)	7 (100%)

At posttest-T4, similar error types were made but there were fewer errors overall. Fewer participants did not provide a final answer at posttest-T4. Only four participants down from six at pretest-T1, did not answer the subtraction transfer task. The number of participant errors that were attributed to calculating in base-10 and using base-5 notation went down from three at pretest-T1 and to one at posttest-T4. None of the participants made errors converting the base-5 quantities to base-10 quantities at posttest-T4. On the other hand, the number of participants who performed the wrong operation remained the same.

Discussion

Place-value is an important mathematical topic in the elementary mathematics curriculum. It's importance extends throughout elementary school as a topic that provides the basis for many later mathematical topics. At the same time, preservice teachers' mathematical knowledge of place-value and related topics tends to be conceptually and procedurally weak (Morris, Hiebert, & Spitzer, 2009; Rizvi & Lawson, 2007). With the interest of improving preservice teachers' mathematical knowledge of place-value, the present study examined the effects of an Iterative lesson sequencing on preservice teachers' conceptual, procedural, and SCK of place-value compared to both a Concepts-first and Procedures-first lesson sequencing.

This study supports the prediction that lesson sequencing does have different effects on base-7 place value knowledge. At the same time, the nature of the effects were not what was anticipated prior to commencing the study. The Iterative sequencing did not result in a greater final increase in conceptual base-7 knowledge scores, procedural base-7 knowledge scores, or SCK scores compared to the Concepts-first or the Procedures-first condition. All of the participants in all of the conditions significantly improved on all three of these measures over the course of the instructional intervention. This means that on average, the participants acquired conceptual base-7 place-value knowledge, procedural base-7 place-value knowledge, and SCK of place-value. Furthermore, there was no effect of base-7 place-value lesson sequencing on the participants' base-10 place value knowledge. Nonetheless, there was a significant effect of the intervention on transfer knowledge.

At the same time, there was a difference between the conditions regarding how conceptual base-7 knowledge was acquired over the course of the instructional intervention. In other words, the pathway of conceptual knowledge acquisition for the Iterative condition differed from that of the Concepts-first condition and the Procedures-first condition.

The most interesting difference occurred at T2, where there was a significant time × condition interaction. At T2, the Iterative condition performed significantly higher on conceptual knowledge than both the Concept-first and the Procedures-first condition. There was no significant difference between the Concepts-first and the Procedures-first condition. Furthermore, conceptual knowledge gains for the Iterative condition occurred sooner in the intervention and the Iterative condition maintained those gains throughout the course of the study.

One explanation for the findings at this time point might relate to the amount of time elapsed between exposure to the number naming lesson (Conceptual Lesson 2) and the number naming assessment at T2. For the Iterative condition, this was the last of four lessons received before the assessment. The Concepts-first condition had received this lesson in the same instructional session as the Iterative condition (because of the interruption caused by the fire drill during the first instructional session), but it was the second of four lessons presented during that same instructional session. Finally, for the

Procedures-first condition, those participants had not yet received any conceptual lessons.

While the temporal proximity of the conceptual number naming lesson and the conceptual number naming assessment may have been a factor, I do not think it explains the performance of the Concepts-first and Procedures-first conditions. It is not surprising that the Procedures-first condition did not perform as well as the Iterative condition on the conceptual item at T2 because the Procedures-first condition had not been exposed to any of the conceptual lessons yet. At the same time, no difference was found, however, between the Procedures-first condition and the Concepts-first condition, which had completed all of the conceptual lessons at this time point.

Moreover, even though the Iterative condition had the most recent exposure to the number naming lesson prior to the assessment at T2, the concepts taught in Conceptual Lesson 2 were present throughout Conceptual Lesson 3 and Conceptual Lesson 4, lessons only the Concepts-first group had exposure to at T2. Therefore, if anything, the participants in the Concepts-first condition had more exposure to number naming concepts in base-7 than the Iterative condition. Thus, time is not likely an explanation because the Concepts-first condition did not perform at least as well as the Iterative group at T2. Perhaps the Concepts-first condition had forgotten what they had learned, or were unable to apply it.

An alternative explanation for the Iterative condition's performance at T2 might be more directly related to the Iterative sequencing of place-value concepts and procedures. These differences might be attributed to the Iterative condition's quicker assimilation of base-7 place-value concepts and procedures. Receiving the conceptual lessons and the related procedural lessons in close iteration may have helped these participants create links between the related concepts and procedures. Rittle-Johnson and Koedinger (2009) found that an iterative lesson sequencing fostered a better understanding of the relationships between mathematical concepts and procedures. Further, these relationships may reduce cognitive load. Automaticity of procedures may free up working memory, thereby allowing for deeper reflection of related mathematical concepts (Rittle-Johnson & Koedinger, 2002; Rittle-Johnson & Siegler, 1998). Presenting all the relevant concepts of a mathematical domain in the absence of any procedures takes a toll on working memory (Shrager & Siegler, 1998). Furthermore, exposing concepts and procedures that capitalize on those concepts in close temporal proximity may encourage the integration of both the concepts and procedures into a well-connected network of usable, conceptually rich knowledge (Resnick & Omanson, 1987).

Qualitative analyses of the participants' conceptual base-7 knowledge provided further insight on the nature of preservice teachers' knowledge of place-value and the effects of lesson sequencing on this knowledge. At T2, there was a reduction in negative base-10 transfer to conceptual base-7 knowledge. None of the participants in the Iterative condition demonstrated base-10 interference when explaining why base-7 numbers are named the way they are. This was not the case for either the Concepts-first or the Procedures-first condition at T2. Both conditions experienced substantial negative base-10 transfer. Rittle-Johnson and Koedinger (2002) argued that iterating between lesson types may encourage appropriate generalizations and reduce inappropriate generalizations. Inappropriate overgeneralizations would include using base-10 concepts in a base-7 context. In fact, iterating between concepts and procedures may support appropriate generalizations (Anderson, 1993) while simultaneously discouraging overgeneralizations (Rittle-Johnson & Albali, 1999).

In addition to negative base-10 transfer, the Combined condition also showed signs of procedural interference on the conceptual item at T2. That is, close to half of the Concepts-first and Procedures-first condition's errors at T2 were the direct result of using a procedural explanation rather than a conceptual explanation of the base-7 number name. This type of procedural interference did not present itself in the responses of the Iterative condition at T2. Interestingly, these differences in negative transfer and procedural interference disappeared by T3. That is, negative transfer of base-10 to base-7 concepts and the interference of procedures were substantially lower for all of the participants at T3.

There are several possible explanations as to why these differences disappeared at T3. At this particular time point, the instruction was complete. All of the participants benefited from the instructional intervention, but it required only four lessons for the Iterative condition to improve their conceptual knowledge of place-value. From a methodological perspective, it is also possible that the measures were not sensitive enough to detect differences in the quality of the participants' responses. Finally, it is possible that some of the increase in conceptual knowledge scores was attributed to practice effects. Perhaps with conceptual transfer items, a bigger difference between the conditions on conceptual knowledge would have been observed.

Interesting effects on procedural knowledge were also revealed at T2. That is, at T2, the Concepts-first condition, who had not received any of the procedural lessons, performed just as well on procedural knowledge as the Procedures-first and the Iterative condition, who had received four and two procedural lessons, respectively. The conceptual knowledge and procedural knowledge findings at T2 suggest two things. Firstly, it appears that it is possible to learn procedures from exposure to the relevant concepts alone. Secondly, to learn concepts, it appears that both concepts and procedures need to be taught. This second finding was also highlighted by Rittle-Johnson and Alibali (1999) who found that conceptual knowledge may have a greater influence procedural knowledge than procedural knowledge does on conceptual knowledge.

Qualitative analyses of procedural base-7 knowledge revealed some similar findings. More specifically, at pretest-T1 there was substantial negative base-10 transfer to the base-7 procedural items. On the other hand, there was no difference between the conditions. All three conditions were equally affected by this negative transfer. Fortunately, over the course of the instructional intervention, negative base-10 transfer was considerably reduced.

An additional pattern that was revealed in terms of procedural base-7 knowledge pertains to the written work produced by the participants when responding to the procedural items. Compared to at pretest-T1, there were more correct answers at posttest-T4 that did not provide any evidence of the participant's work. This may suggest that these participants were performing the calculations mentally, thereby suggesting improved procedural fluency (Sohn & Carlson, 1998).

For SCK from pretest-T1 to posttest-T4, there was a significant improvement regardless of condition. The participants' ability to recognize student errors in multi-digit addition and subtraction improved over the course of the instruction. Interestingly, the qualitative nature of the explanations of the students' responses also improved. At pretest-

T1 many of the participants could not adequately explain what the student had done wrong. Moreover, when the participants could provide an adequate explanation, it tended to be limited to a reiteration of the procedure the student had followed. In the rare instances when a deeper explanation was provided, the concepts and procedures used were disjointed.

In contrast, at posttest-T4, improvements did not lie only in the participants' accuracy in identifying the students' errors. There was a dramatic change in the nature of the participants' explanations of the students' work. More specifically, most of the participants referred to the relevant concepts and procedures present in the students' solutions. While most of the participants' explanations remained disjointed, there was also a substantial increase in the number of conceptually and procedurally linked explanations. These findings perhaps indicate that conceptual and procedural knowledge, while perhaps by themselves not sufficient, are important components of SCK. The importance of content knowledge on elementary school teachers' ability to perform many tasks required for a reform oriented practice, such as recognizing errors, analyzing student solutions, and selecting appropriate tasks to elicit mathematical concepts and procedures, is well documented (Ball & Bass, 2000; Ball, Lubienski, & Mewborn, 2001; Ball, Thames, & Phelps, 2008). These teaching tasks are also components of SCK.

Similar to procedural base-7 knowledge, procedural transfer knowledge scores increased over the duration of this study, but there were no differences between the three conditions. As with procedural base-7 knowledge, there was considerable negative base-10 transfer to the procedural base-5 items at pretest-T1. This negative transfer was reduced at posttest-T4. The pathway of acquisition of this transfer knowledge may be veiled by the fact that transfer knowledge was only evaluated at pretest-T1 and posttest-T4 and not at any other time points.

In summary, all of the participants learned place-value concepts and procedures over the course of the study. Furthermore, all of the participants also improved their SCK of place-value. While there was not a clear effect of lesson sequencing on the acquisition of place-value knowledge, lesson sequencing appears to have an impact on the pathway of the acquisition of conceptual knowledge. This pathway has important implications specifically for elementary mathematics teacher educators. More specifically, based on these findings, the importance of teaching both concepts and related procedures needs to be stressed. That is, mathematical concepts and procedures should not be taught separately. Instead, presenting mathematical concepts and related procedures in close temporal proximity, perhaps even in tandem, may have positive effects on the acquisition of conceptual and procedural knowledge.

At the end of the intervention, after both concepts and procedures of place-value had been taught, all of the participants improved their conceptual, procedural, and SCK of place-value, but an iterative sequencing of concepts and procedures appeared to be more efficient. The Iterative condition learned more quickly than the other two conditions. Decreasing the amount of time required to teach mathematics topics is highly desirable given the short amount of time and high volume of material that mathematics teacher educators are expected to cover.

At the same time, further weaknesses of the this study need to be addressed. The group sizes were relatively small from a quantitative perspective, as small samples tend to have more variability than large samples. With small group sizes, one participant's performance could have a greater effect on the overall mean. Additional weaknesses pertain to the measures that were used. Even though certain items were designed to assess conceptual knowledge, there was no way to prevent the participants from following procedures, even mentally, to respond to the items. When it was obvious that this was the case, the items were coded as incorrect. It is impossible, however, to prevent the participants from thinking about some of the conceptual items procedurally or creating their own procedures using generalizations from the instructional intervention. Another weakness associated with the measures was that they were not analogous at all of the assessment points, making the results difficult to compare across time from both a quantitative and a qualitative perspective.

The results of this study contribute the literature in several ways. First of all, it is the first study that I am aware of that looks at the effects of lesson sequencing on the preservice teacher population, a population that is notorious for lacking in mathematical knowledge (Khoury & Zazkis, 1994; Newton, 2008; Toluk-Uçar, 2009). Secondly, the results of this study may actually support a more recent perspective. That is, concepts and procedures may not actually develop iteratively, but rather in tandem or simultaneously (Sarama & Clements, 2009). The closer together the concepts and procedures are presented, the more positive the impact will be on conceptual and procedural knowledge. Finally, this study supports the existing literature on the development of SCK (Ball et al., 2008). It appears that conceptual and procedural knowledge are components that must be addressed when developing SCK. At the same time, on their own, or without strong links being created between concepts and procedures, conceptual knowledge and procedural knowledge are probably not sufficient.

References

Anderson (1993). *Rules of the mind*. Hillsdale, NJ: Erlbaum.

Ball, D. L. (1990). The mathematical understanding that prospective teachers bring to teacher education. *The Elementary School Journal*, *90*, 449-466.

Ball, D. L. (1996). Teacher learning and the mathematics reforms: What we think we know and what we need to learn. *Phi Delta Kappa International*, *77*, 500-508.

Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), Multiple perspectives on mathematics teaching and learning (pp. 83–104). Westport, CT: Ablex Publishing.

Ball, D. L., Hill, H. C., & Bass, H. (2005, Fall). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, 14- 46.

Ball, D. L., Lubienski, S. T., & Mewborn, D. S. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching* (pp. 443-456). New York: Macmillan.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*, 389-407.

Baroody, A. J. (1990). How and when should place-value concepts and skills be taught. *Journal for Research in Mathematics Education*, *21*, 281-286.

Baroody, A. J., & Gannon, K. E. (1984). The development of the commutativity principle and economical addition strategies. *Cognition and Instruction*, *1*, 321-339.

Baroody, A. J., Lai, M., & Mix, K. S. (2006). The development of young children's early number and operation sense and its implications for early childhood education. In *B. Spodek & O. N. Saracho (Eds.), Handbook of research on the education of young children* (2nd ed., pp. 135-152). Mahwah, NJ: Lawrence Erlbaum Associates.

Bisanz, J., & LeFevre, J. A. (1992). Understanding elementary mathematics. In J. I. D. Campbell (Ed.), *The nature and origins of mathematical skill* (pp. 113-136). Amsterdam: Elsevier Science.

Bowers, J., Cobb, P., & McClain, K. (1999). The evolution of mathematical practices: A case study. *Cognition and Instruction*, *17*, 25-64.

Briars, D., & Siegler, R. S. (1984). A featural analysis of preschoolers' counting knowledge. *Developmental Psychology*, *20*, 607-618.

Byrnes, J. P., & Wasik, B. A. (1991). Role of conceptual knowledge in mathematical procedural learning. *Developmental Psychology*, *27*, 777-786.

Canobi, K. H., Reeves, R. A., & Pattison, P. E. (2003). Patterns of knowledge in children's addition. *Developmental Psychology*, *39*(3), 521-534.

Carpenter, T. P., Fennema, E., Peterson, P. L., & Carey, D. A. (1988). Teachers' pedagogical content knowledge of students' problem solving in elementary arithmetic. *Journal for Research in Mathematics Education*, *5*, 385-401.

Carpenter, T. P., & Moser, J. M. (1984). The acquisition of addition and subtraction concepts in grades one through three. *Journal for Research in Mathematics Education*, *15*(3), 179-202.

Cauley, K. M. (1988). Construction of logical knowledge: Study of borrowing in subtraction. *Journal of Educational Psychology*, *80*, 202-205.

Cobb, P., Yackel, E., & Wood, T. (1992). Interaction and learning mathematics classroom situations. *Educational Studies in Mathematics*, *23*, 99-122.

Comiti, C., & Ball, D. L. (1996). Preparing teachers to teach mathematics: A comparative perspective. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education*, (Vol. 2 pp. 1123-1153). Dordrecht, The Netherlands: Kluwer.

Cowan, R., Dowker, A., Christakis, A., & Bailey, S. (1996). Even more precisely assessing children's understanding of the order irrelevance principle. *Journal of Experimental Child Psychology*, *62*

Cowan, R., & Renton, M. (1996). Do they know what they are doing? Children's use of economical addition strategies and knowledge of commutativity. *Educational Psychology*, *16*, 409-422.

Dixon, J. A., & Moore, C. F. (1996). The developmental role of intuitive principles in choosing mathematical strategies. *Developmental Psychology*, *32*, 241-253.

Feigenson, L. (2005). A double-dissociation in infants representations of object arrays. *Cognition*, *13*(2), 150-156.

Flores, A., Turner, E. E., & Bachman, R. C. (2005, October). Posing problems to develop conceptual understanding: Two teachers make sense of division of fractions. *Teaching Children Mathematics,* 117-121.

Frykholm, J. A. (1999). The impact of reform: Challenges for mathematics teacher preparation. *Journal of Mathematics Teacher Education*, *2*, 79-105.

Fuson, K. C. (1986). Roles of representation and verbalization in the teaching of multi-digit addition and subtraction. *European Journal of Psychology of Education*, *1*(2), 35-56.

Fuson, K. C. (1988). Teaching adapted to thinking. *Journal for Research in Mathematics Education*, *19*, 263-267.

Fuson, K. C. (1990). Conceptual structures for multiunit numbers: Implications for learning and teaching multidigit addition, subtraction, and place value. *Cognition and Instruction*, *7*, 343-403.

Fuson, K. C., & Briars, D. J. (1990). Base-ten blocks as a first- and second-grade learning/teaching approach for multidigit addition and subtraction and place-value concepts. *Journal for Research in Mathematics Education*, *21*, 180-206.

Fuson, K. C., Wearne, D., Hiebert, J. C., Murray, H. G., Human, P. G., Olivier, A. I., Carpenter, T. P., & Fennema, E. (1997). Children's conceptual structures for multidigit numbers and methods of multidigit addition and subtraction. *Journal for Research in Mathematics Education*, *28*(2), 130-162.

Geary, D. C. (1994). *Children's mathematical development: Research and practical applications*. Washington, DC: American Psychological Association.

Gelman, R., & Gallistel, C. R. (1978). *The child's understanding of number*. Cambridge, MA: Harvard University Press.

Gelman, R., & Meck, E. (1983). Preschoolers' counting: Principles before skill. *Cognition*, *13*, 343-359.

Gelman, R., & Williams, E. M. (1998). Enabling constraints for cognitive development and learning: Domain specificity and epigenesist. In D. Kuhn & R. S. Siegler

(Eds.), *Handbook of child psychology: Cognition, perception, and language* (5th ed.), (Vol. 2, pp. 575-630). New York, NY: Willey.

Gould, P. (2005). Really broken numbers. *Australian Primary Mathematics Classroom*, *10*(3), 4-10.

Graeber, A. O. (1999). Forms of knowing mathematics: What preservice teachers should learn. *Educational Studies in Mathematics*, *38*, 189-208.

Graeber, A. O., Tirosh, D., & Glover, R. (1989). Preservice teachers' misconceptions in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education*, *20*, 95-102.

Halford, G. S. (1993). *Children's understanding: The development of mental models*. Hillsdale, NJ: Erlbaum.

Harel, G., & Behr, M. (1995). Teachers' solutions for multiplicative problems. *Hiroshima Journal of Mathematics Education*, *3*, 31-51.

Hiebert, J., & LeFevre, J. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1-27). Hillsdale, NJ: Erlbaum.

Hiebert, J., & Wearne, D. (1986). Procedures over concepts: The acquisition of decimal number knowledge. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 199-224). Hillsdale, NJ: Erlbaum.

Hiebert, J., & Wearne, D. (1992). Links between teaching and learning place value with understanding in first grade. *Journal for Research in Mathematics Education*, *23*, 98-122.

Hiebert, J., & Wearne, D. (1996). Instruction, understanding, and skill in multidigit addition and subtraction. *Cognition and Instruction*, *14*, 251-284.

Hill, H. C., & Ball, D. L. (2004). Learning mathematics for teaching: Results from California's Mathematics Professional Development Institutes. *Journal of Research in Mathematics Education*, *35*, 330-351.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge" Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, *39*(4), 372-400.

Hill, H. C., Blunk, M. L., Charalambos, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., & Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction*, *26*(4)*,*430-511.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, *42*, 371-406.

Jackson, C. D., & Leffingwell, R. J. (1999). The role of instructors in creating math anxiety in student from kindergarten through college. *The Mathematics Teacher*, *92*(7), 583-586.

Kahan, J. A., Cooper, D. A., & Bethea, K. A. (2003). The role of mathematics teachers' content knowledge in their teaching: A framework for research applied to a study of student teachers. *Journal of Mathematics Teacher Education*, *6*, 223-252.

Khoury, H. A., Zazkis, R. (1994). On fractions and non-standard representations: Pre-service teachers' concepts. *Educational Studies in Mathematics*, *27*, 191-204.

Kouba, V. L., Brown, C. A., Carpenter, T. P., Lindquist, M. M., Silver, E. A., & Swafford, J. O. (1988). Results from the fourth NAEP assessment of mathematics: Number, operations, and word problems. *Arithmetic Teacher*, *35*(8), 14-19.

Kribs-Zaleta, C. (2006). Invented strategies for division of fractions. In S. Alatorre, J. L. Cortina, M. Saiz,, & A. Mendez (Eds.), *Proceedings of the 28th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*: *Vol. 2*. *Rational and whole numbers* (pp. 371-376). Mérida, Mexico: Universidad Pedagógica Nacional.

Labinowicz, E. (1985). *Learning from children: New beginnings for teaching numerical thinking*. Menlo Park, CA: Addison-Wesley Publishing Company

Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 33-40). Hillsdale, NJ: Erlbaum.

Mack, N. K. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education*, *21*, 16-32.

Mack, N. K. (1995). Confounding whole-number and fraction concepts when building on informal knowledge. *Journal for Research in Mathematics Education*, *26*, 422-441.

McClain, K. (2003). Supporting preservice teachers' understanding of place value and multidigit arithmetic, *Mathematica Thinking and Learning*, *5*, 281-306.

Mestre, J. P. (2002). Probing adults' conceptual understanding and transfer of learning via problem posing. *Applied Developmental Psychology*, *23*, 9-50

Ministère de L'Éducation, Loisirs et Sport (2001). *Quebec education program: Preschool and*

elementary education. Quebec*:* Ministère de L'Éducation, Loisirs et Sport.

Miura, I. T., Okamoto, Y., Kim, C. C., Steere, M., & Fayol, M. (1993). First graders' cognitive representation of number and understanding of place value: Cross-national comparisons – France, Japan, Korea, Sweden, and the United States. *Journal of Educational Psychology*, *85*, 24-30.

Morris, A. K., Hiebert, J., & Spitzer, S. M. (2009). Mathematical knowledge for teaching in planning and evaluating instruction: What can preservice teachers learn? *Journal for Research in Mathematics Education*, *40*(5), 491-529.

National Council of Teachers of Mathematics. (2009). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Newton, K. J. (2008). An extensive analysis of preservice elementary teachers' knowledge of fractions. *American Educational Research Journal*, *45*, 1080-1110. Olivier, Murray, & Human (1990).

Olivier, A., Murray, H., & Human, P. (1990). Building on young children's informal arithmetical knowledge. In G. Booker, P. Cobb, & T. N. Mendicuti (Eds.), *Proceedings of the 14th Psychology of Mathematics Education Conference*, *3* (pp. 297-304). Oaxtepec, Mexico.

Osana, H. P., Lacroix, G. L., Tucker, B. J., & Desrosiers, C. (2006.). The role of content knowledge and problem features on preservice teachers' appraisal of elementary mathematics tasks. *Journal of Mathematics Teacher Education*, *9*, 347-380.

Osana, H. P., & Pitsolantis, N. (in press). Addressing the struggle to link form and understanding in fractions instruction. *British Journal of Educational Psychology*.

Osana, H. P., & Royea, D. A. (2011). Obstacles and challenges in preservice teachers' explorations with fractions: A view from a small-scale intervention study. *Journal of Mathematical Behavior*, *30*, 333-352.

Resnick, L. B., & Omanson, S. F. (1987). Learning to understand arithmetic. In R. Glaser (Ed.), *Advances in instructional psychology* (Vol. 3). Hillsdale, NJ: Erlbaum.

Rittle-Johnson, B., & Alibali, M. W. (1999). Conceptual and procedural knowledge: Does one lead to the other? *Journal of Educational Psychology*, *91*(1), 175-189.

Rittle-Johnson, B., & Koedinger, K. R. (2002). Comparing instructional strategies for integrating conceptual and procedural knowledge. In D. S. Mewborn, P. Sztajin, D. Y. White, H. G. Wiegel, R. L. Bryant, & K. Nooney (eds.), *Proceedings of the 24th Annual Meeting for the International Group for the Psychology of Mathematics Education* (pp.969-978). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Rittle-Johnson, B., & Koedinger, K. (2009). Iterating between lessons on concepts and procedures can improve mathematics knowledge. *British Journal of Educational Psychology*, *79*, 483-500.

Rittle-Johnson, B., & Siegler, R. S. (1998).The relation between conceptual and procedural knowledge in learning mathematics: A review. In C. Donlan (Ed.), *The development of mathematical* skills (pp. 75-110). London: Psychology Press.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology*, *93*(2), 346-362.

Rizvi, N. F. & Lawson, M. J. (2007). Prospective teachers' knowledge: Concept of division. *International Educations Journal*, *8*, 377-392.

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education*, *8*, 255-281.

Sarama, J., & Clements, D. H. (2009). "Concrete" computer manipulatives in mathematics education. *Child Development Perspectives*, *3*, 145-150.

Saxe, G. B., Gearhart, M., & Nasir, N. S. (2001). Enhancing students' understanding of mathematics: A study of three contrasting approached to professional support. *Journal of Mathematics Teacher Education*, *4*, 55-79.

Saxton, M., & Towse, J. N. (1998). Linguistic relativity: The case of place value in multi-digit numbers. *Journal of Experimental Child Psychology*, *69*, 66-79.

Shrager, J., & Siegler, R. S. (1998). A model of children's strategy choices and strategy discoveries. *Psychological Science*, *9*, 405-410.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14.

Siegler, R. S. (1991). In young children's counting, procedures precede principles. *Educational Psychology Review*, *3*(2), 127-135.

Siegler, R. S., & Crowley, K. (1994). Constraints on learning in nonprivileged domains. *Cognitive Psychology*, *27*, 194-226.

Silverman, J., & Thompson, J. (2008). Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education*, *11*, 499-511.

Simon, M. A. (1993). Prospective elementary teachers' knowledge of division. *Journal for Research in Mathematics Education*, *24*(3), 233-254.

Simon, M. & Blume, G. W. (1994). Building and understanding multiplicative relationships: A study of prospective elementary teachers. *Journal for Research in Mathematics Education*, *25*, 472-494.

Sohn, M. H., & Carlson, R. A. (1998). Procedural frameworks for simple arithmetic skills. *Journal of Experiemental Psychology: Learning, Memory, and Cognition*, *24*(4), 1052-1067.

Sophian, C. (1997). Beyond Competence: The significance of performance for conceptual development. *Cognitive Development*, *12*, 281-303.

Sowder, J., Armstrong, B., Lamon, S., Simon, M., Sowder, L., & Thompson, A. (1998). Educating teachers to teach multiplicative structures in the middle grades. *Journal of Mathematics Teacher Education*, *1*, 127-155.

Stacey, K., Helme, S., Steinle, V., Baturo, A., Irwin, K., & Bana, J. (2001). Preservice teachers' knowledge of difficulties in decimal numeration. *Journal of Mathematics Teacher Education*, *4*, 204-225.

Steffe, L. P., & Cobbe, P. (1988). *Construction of arithmetical meanings and strategies*. New York: Springer-Verlag.

Steele, D. F. (2001). The interfacing of preservice and inservice experience of reform-based teaching: A longitudinal study. *Journal of Mathematics Teacher Education*, *4*, 139-172.

Tchoshanov, M. A. (2011). Relationship between teacher knowledge of concepts and connections, teaching practice, and student achievement in middle grades mathematics. *Educational Studies in Mathematics*, *76*, 141-164.

Tchoshnov, M., Lesser, L., & Salazar, J. (2008). Teacher knowledge and student achievement: Revealing patterns. *Journal of Mathematical Education Leadership*, *13*, 39-49.

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education*, *31*, 5-25.

Tirosh, D., & Graeber, A. O. (1990). Evoking cognitive conflict to explore preservice teachers' thinking about division. *Journal for Research in Mathematics Education*, *21*, 98-108.

Toluk-Uçar, Z. (2009). Developing pre-service teachers understanding of fractions through problem posing. *Teaching and Teacher Education*, *25*, 166-175.

Tsamir, P., & Tirosh, D. (2008). Combing theories in research in mathematics teacher education. *ZDM Mathematics Education*, *40*, 861-872.

Wood, J. N., & Spelke, E. S. (2005). Infants' enumeration of actions: Numerical discrimination and its signature limits. *Developmental Science*, *8*(2), 172-181.

Wynn, K. (1990). Children's understanding of counting. *Cognition*, *36*, 155-193.

Yackel, E., Underwood, D., & Elias, N. (2007). Mathematical tasks designed to foster a reconceptualized view of early arithmetic. *Journal of Mathematics Teacher Education,10*, 351-367.

Yoshida, H., & Sawano, K. (2002). Overcoming cognitive obstacles in learning fractions: Equal-partitioning and equal-whole. *Japanese Psychological Research*, *44*, 183-195.

Zazkis, R., & Campball, S. (1996). Divisibility and multiplicative structure of natural numbers: Preservice teachers' understanding. *Journal for Research in Mathematics Education*, *27*, 540-563.

Appendix A

Student Number:_____________________________

Concordia University
Department of Education
EDUC 387/4: Teaching Mathematics I

Participant Demographics

Instructions: Please fill in all the information as accurately as possible. Your information will remain confidential and will be used for research purposes only.

1. Circle your gender: **Male** or **Female**

2. Age:_____

3. When did you begin the ECEE Specialization Program?

   Semester: Fall or Winter

   Year:_______

4. Do you have any individual or classroom-based teaching experience including substitute teaching, teaching stages, tutoring, working as a classroom aide, etc?

   Circle: **Yes** or **NO**

5. If you circled yes, please describe in detail your teaching experiences below.

Type of Teaching Experience	Details of Responsibilities/Tasks	Approximate Duration (in months)

Please feel free to ask for another sheet if you need more space.

Thank-You!

Appendix B

Student Number:______________________

Concordia University
Department of Education
EDUC 386/2: Teaching Mathematics I

Instructions: This worksheet is about reading numbers, writing numbers, and arithmetic. Carefully read all of the instructions, examples, and questions. Please answer <u>all</u> of the questions to the best of your ability in the spaces provided in this booklet.

1

General Information

In our everyday lives, the counting and numeration system that we use is called "**base-10**." At the same time, other counting and numeration systems exist. Throughout this handout, you will be asked to complete a variety of tasks in both our regular way of counting and in different bases. Please be careful to make sure you are using the correct way of counting as indicated in the instructions. When a number is in base-10, it will be written in the way you are used to seeing.

For example, the following numbers are in base-10: 1, 16, 25, fifty, one-hundred-two.

When a number is in a different base, this will be indicated by subscripts (little numbers written lower and to the right) or written in words.

For example, the following numbers are in base-7:

1_7	16_7	25_7
subscript	subscript	subscript

The following numbers are also in base-7: five-zero base-7, one-zero-two base-7.

The following questions are about base-10.

Base-10

Instructions: Perform the following calculations. Do the computations in <u>**base-10**</u>.

96 477 + 623	1 983 - 26
90 000 001 - 4 312	403 x 4
9 999 + 1 191	7 492 x 17

The following questions are about <u>base-7</u>.

Base-7

Instructions: Perform the following calculations. Do all the computations in **base-7**.

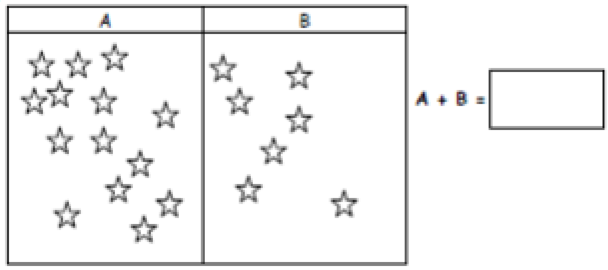
$$\begin{array}{r} 14_7 \\ +13_7 \\ \hline \end{array} \qquad \begin{array}{r} 100_7 \\ -\ 50_7 \\ \hline \end{array}$$

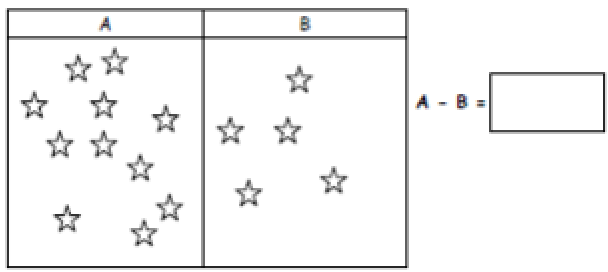
$$\begin{array}{r} 25_7 \\ +\ 25_7 \\ \hline \end{array} \qquad \begin{array}{r} 65_7 \\ -56_7 \\ \hline \end{array}$$

The following questions are about <u>base-10</u>.

Base-10

Instructions: Examine the number of objects in column *A* and column B. Use the quantities presented in the columns to fill in the corresponding blank number sentences. Use **<u>base-10</u>**.

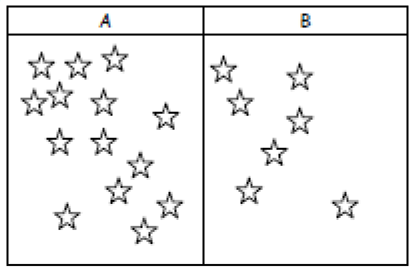




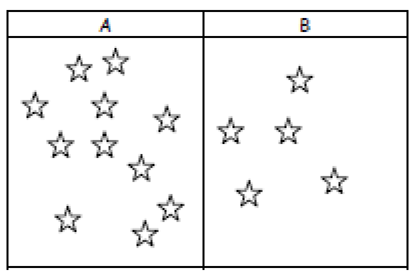
The following questions are about <u>base-7</u>.

Base-7

Instructions: Examine the number of objects in column A and column B. Use the quantities presented in the columns to fill in the blanks in the corresponding number sentences. Use <u>**base-7**</u>.

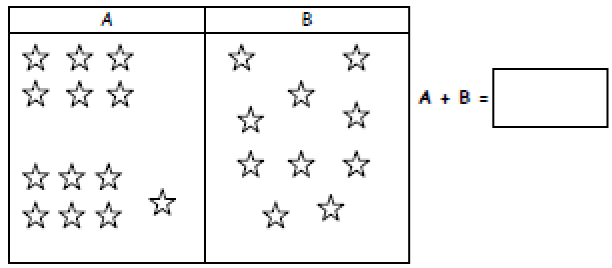


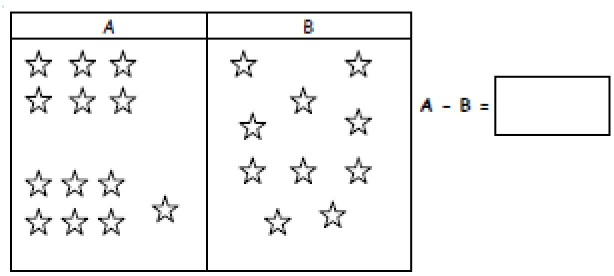
A + B = ☐



A - B = ☐

Base-7





Base-7

Instructions: There are <u>four</u> missing values in the following base-7 number chart. Fill in the missing values so that the pattern is preserved.

0	1_7	2_7	3_7	4_7	5_7	6_7
	11_7	12_7	13_7	14_7	15_7	16_7
20_7	21_7		23_7	24_7	25_7	26_7
30_7	31_7	32_7	33_7	34_7	35_7	36_7
	41_7	42_7	43_7	44_7		46_7
50_7	51_7	52_7	53_7	54_7	55_7	56_7
60_7	61_7	62_7	63_7	64_7	65_7	66_7

Instructions: Use the chart above to fill in the blanks with the appropriate base-7 number.

The next number after 26_7 is _________.

The number before 50_7 is _________.

_________ is seven numbers more than 43_7.

_________ is three numbers less 55_7.

12_7 is _________ numbers more than 2_7.

The following questions are about base-10.

Base-10

Instructions: Perform the appropriate calculations to answer the following problems. Use **base-10**.

172 + 88 = __________

1 003 - 17 = __________

26 X 5 = __________

95 - 88 = __________

126 X 75 = __________

9 X 132 = __________

Base-10

Instructions: Look at the solutions below produced by elementary school students. In each case, indicate if the student got the right answer. If the student solved the problem correctly, explain the steps used. If the student solved the problem incorrectly, describe the mistake(s) made by the student.

Student A's work:

$$\begin{array}{r} 27 \\ +\ 5 \\ \hline 212 \end{array}$$

Student A got the answer: **Right** / **Wrong** (circle one)

Explain:

Base-10

Student B's work:

$$\begin{array}{r} 104 \\ -\quad 9 \\ \hline 5 \end{array}$$

Student B got the answer: **Right** / **Wrong** (circle one)

Explain:

The following questions are about <u>base-7</u>.

Instructions: Perform the appropriate calculations to answer the following problems in **base-7**.

$125_7 + 50_7 =$ ________

$125_7 - 50_7 =$ ________

$60_7 + 116_7 =$ ________

$42_7 - 15_7 =$ ________

The following questions are about base-5.

Base-5

Instructions: Perform the appropriate operations in the following problems. Do all the computations in base-5.

$$\begin{array}{r} 13_5 \\ +\ 13_5 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 44_5 \\ -\ 32_5 \\ \hline \end{array}$$

Appendix C

Session #2 - Questions

Student ID#:____

1. In words, name the following number and explain why that is how the number is named.

   614_7 = ____________________________________

   Explanation:

2. Examine the following student solution. Determine if the student is correct. Explain how the student arrived at the answer. If the student is wrong, explain where the student went wrong.

   Student's work:

   $$\begin{array}{r} 907 \\ +\ 315 \\ \hline 612 \end{array}$$

   The student is correct/incorrect.
   Explanation:

Appendix D

Student ID Number:____________________ Date:______________

Instructions: Solve the following problem. Then, on the lines provided, explain in detail **how** you solved the problem and **why** you solved it that way.

1.

$$\begin{array}{r} 123_7 \\ -\ 65_7 \\ \hline \end{array}$$

Instructions: Examine the following student solution to the problem. Indicate if the student is correct or incorrect. Explain the steps the student used to solve the problem. If the student is incorrect, explain what was done wrong.

2.

$$
\begin{array}{r}
1^{1}73 \\
-\ 84 \\
\hline
191
\end{array}
$$

Instructions: Using the block model you learned in the computer lessons. Calculate the following.

3. $421_7 - 354_7$

Appendix E

*1. Student ID Number:

Introduction

Different numeration systems group numbers in different ways than you are used to.

Understanding how various numeration systems work will help you as a teacher in several ways.

Page 1

You will be able to better understand central concepts about counting and place value that will improve your ability to solve problems with addition, subtraction, multiplication, and division.

As a consequence, you will improve your ability to teach these concepts as well.

Page 2

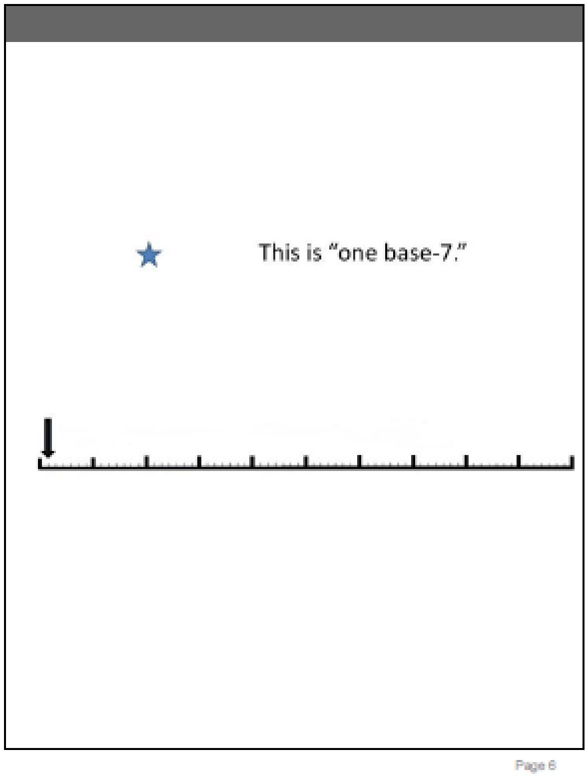
Counting Objects in Base-7

The following pictures demonstrate how to count objects in base-7.

You will see how to name, in base-7, the quantity of objects that are presented.

You will also see how different base-7 quantities relate to each other.

Page 5


This is “one base-7.”
Page 6

Today you will learn about counting objects in a way that is different than you are used to.

You will learn about counting and grouping in base-7.

Instructions

Please read and study the following pictures. The pictures explain the basics of counting and grouping numbers in a different numeration system than you are used to.

Take the time to see the differences and similarities between the pictures.

After you have finished studying the pictures and the explanation, please answer all of the questions that appear at the end of this lesson.

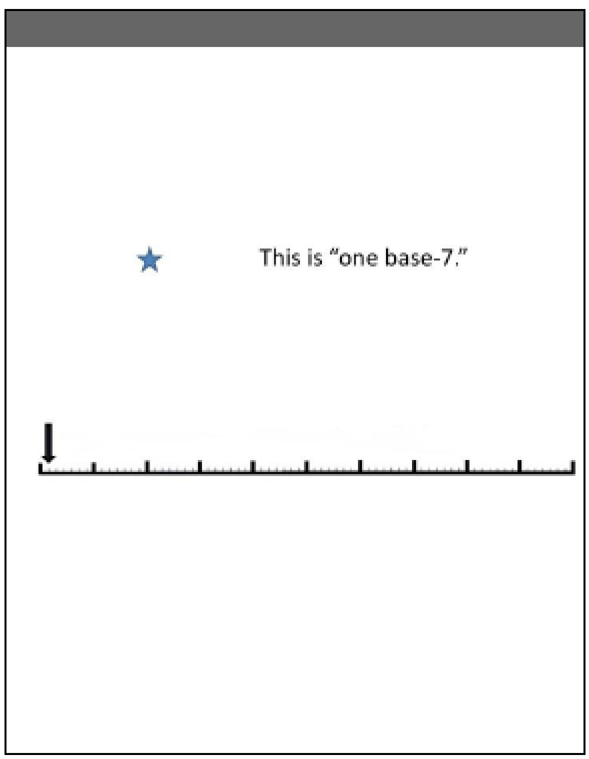
Counting Objects in Base-7

The following pictures demonstrate how to count objects in base-7.

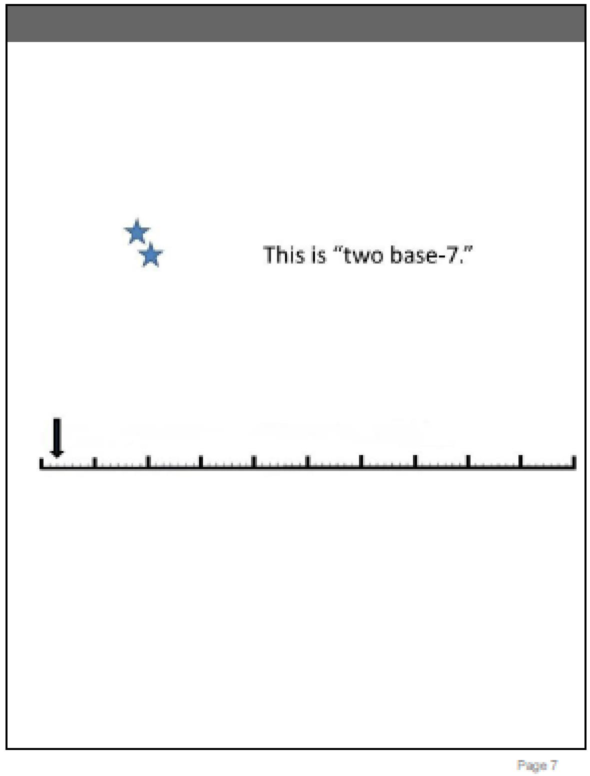
You will see how to name, in base-7, the quantity of objects that are presented.

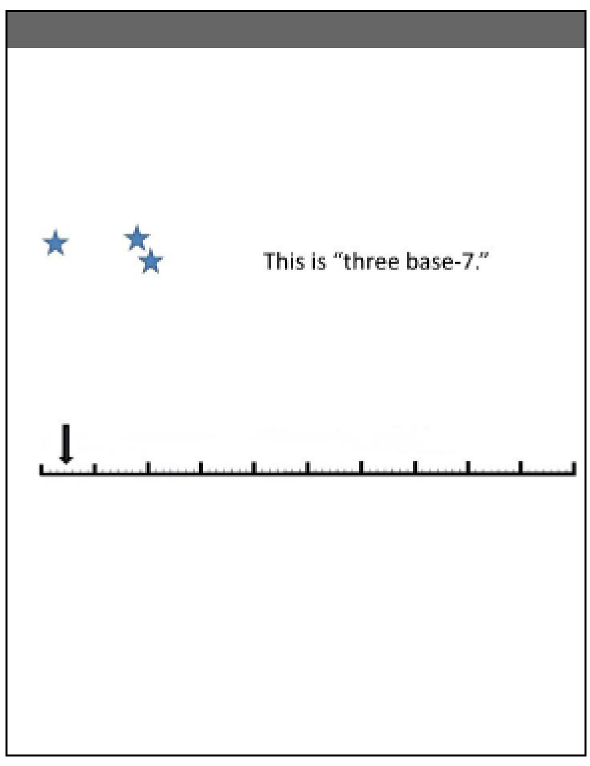
You will also see how different base-7 quantities relate to each other.

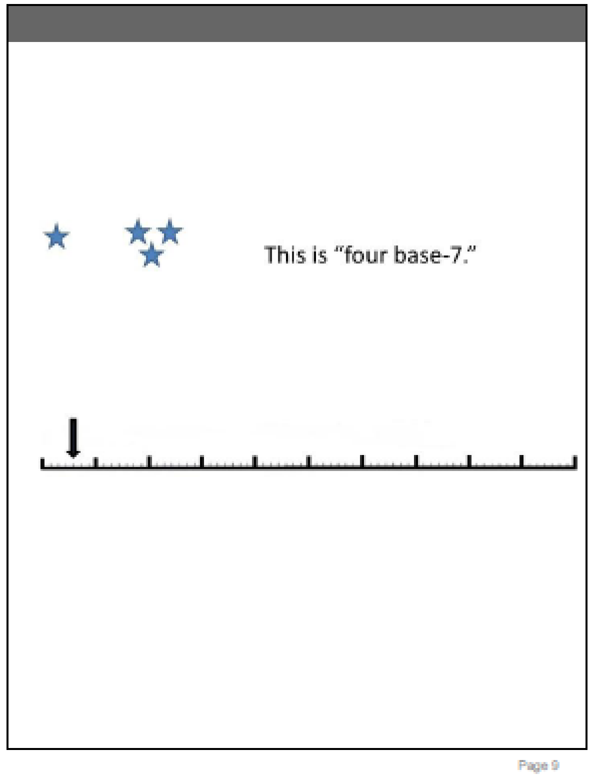
Page 5

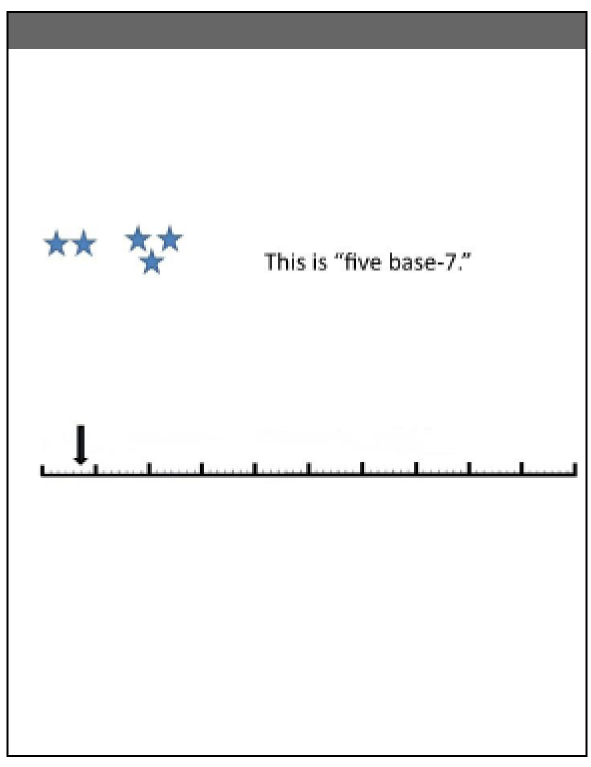


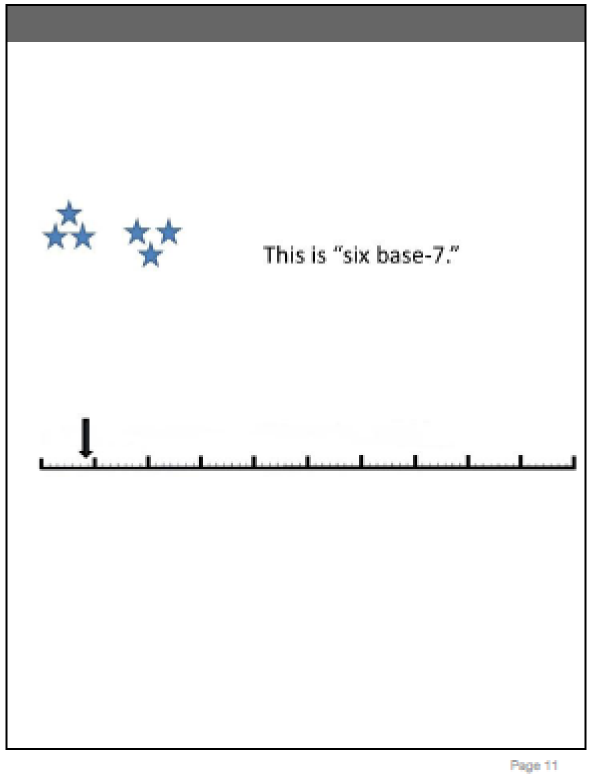
Page 6

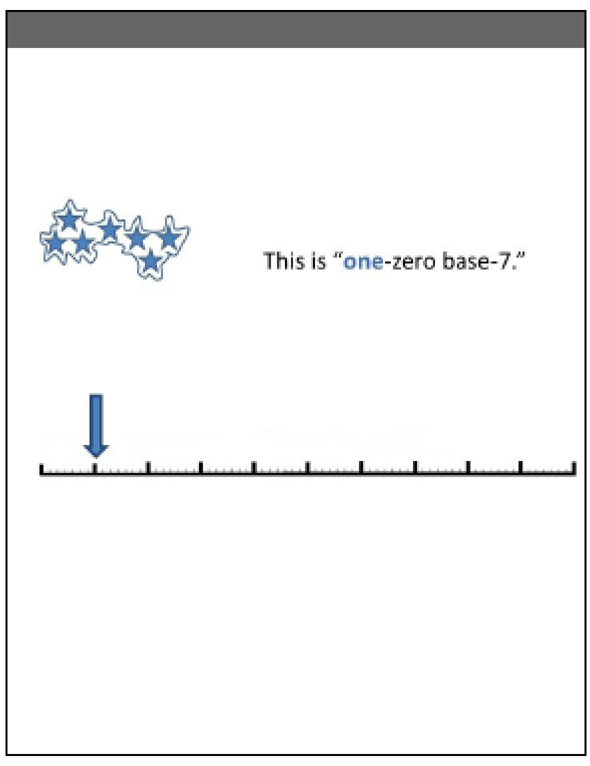

This is "two base-7."
Page 7

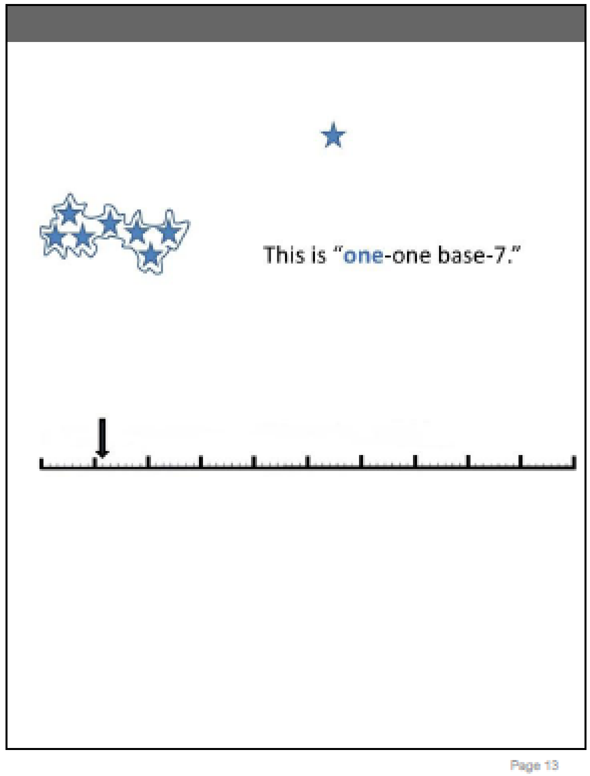

This is "three base-7."
Page 8

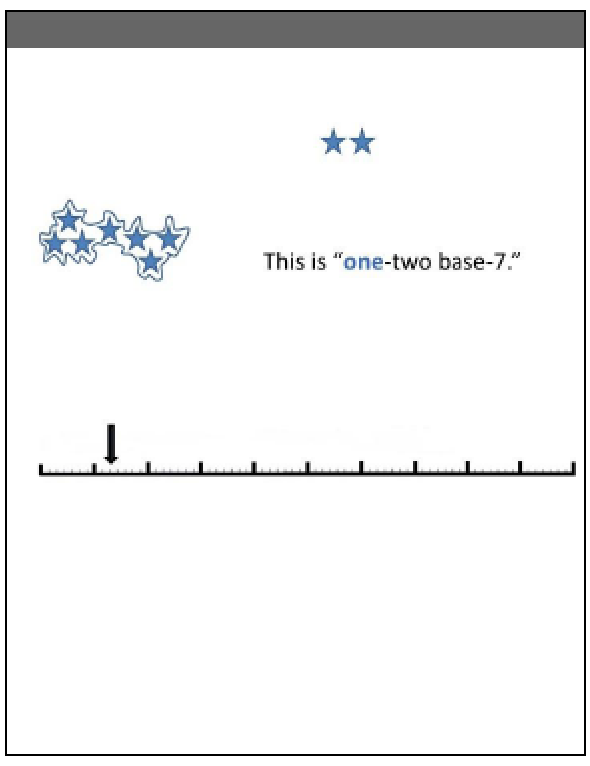

This is "four base-7."
Page 9

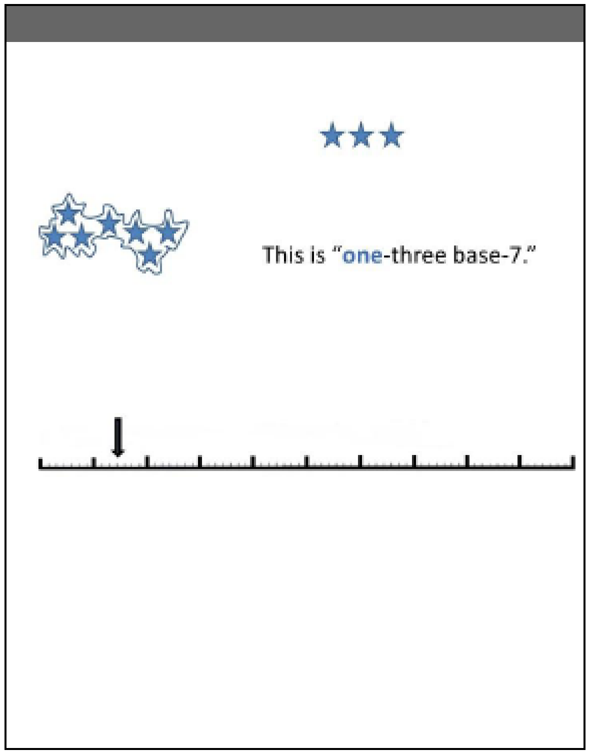

This is "five base-7."
Page 10


This is "six base-7."
Page 11

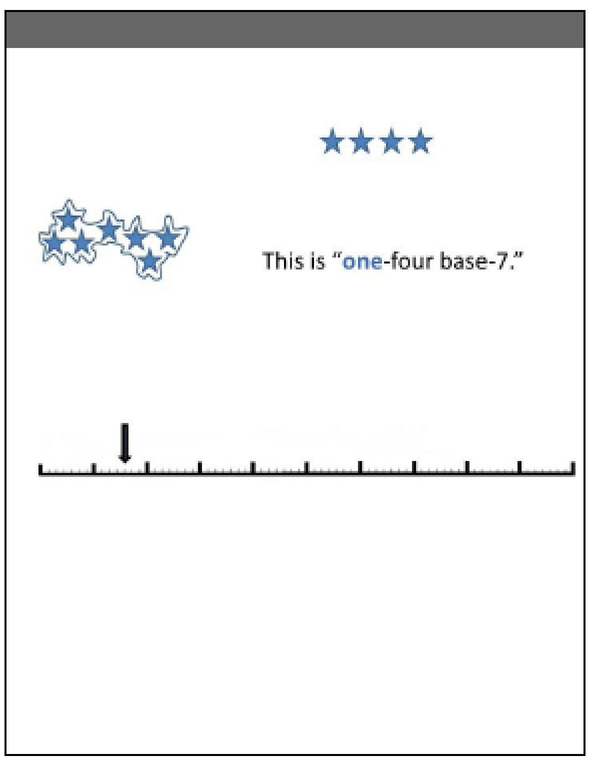

This is "one-zero base-7."
Page 12


This is "one-one base-7."
Page 13

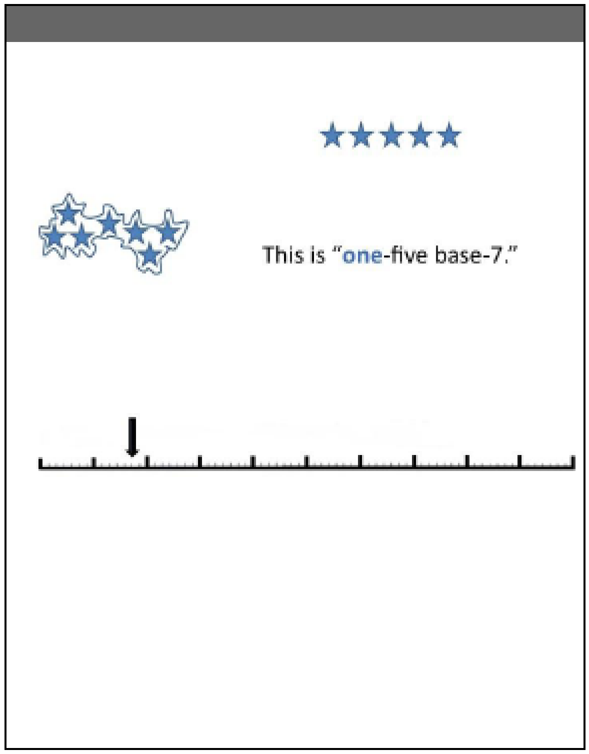

This is "one-two base-7."
Page 14



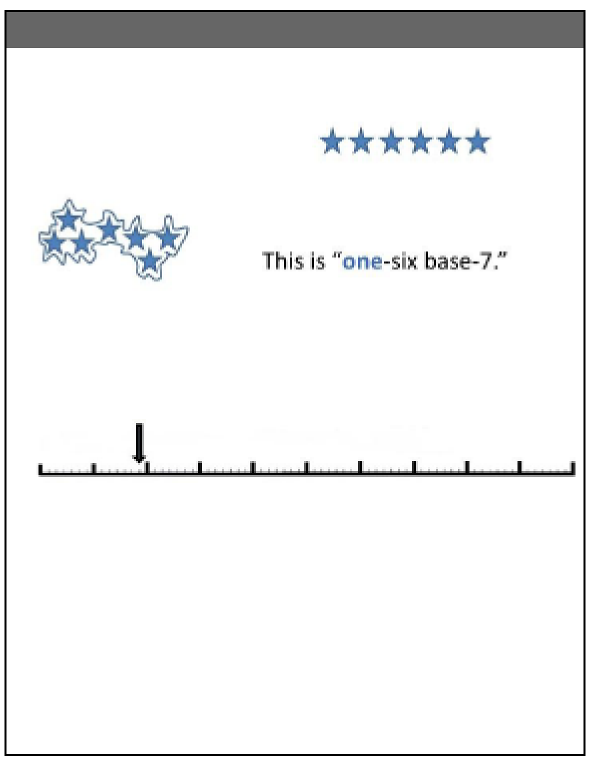
Page 15



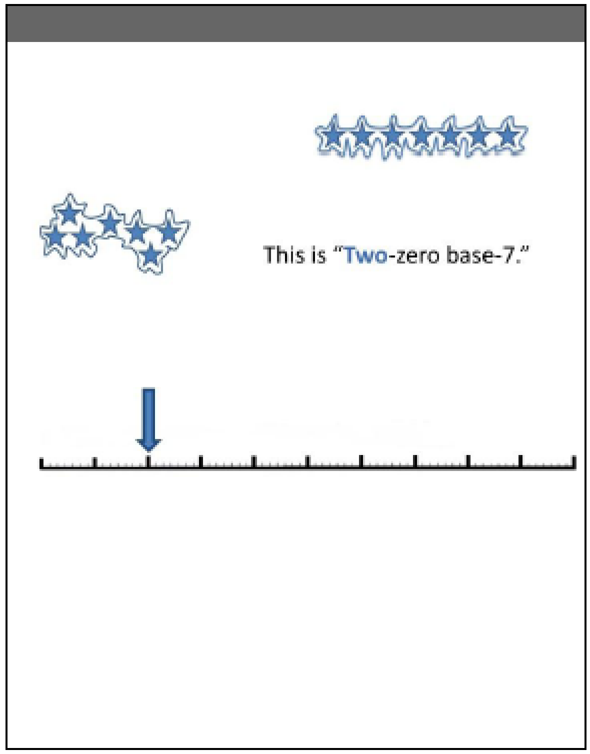
Page 16



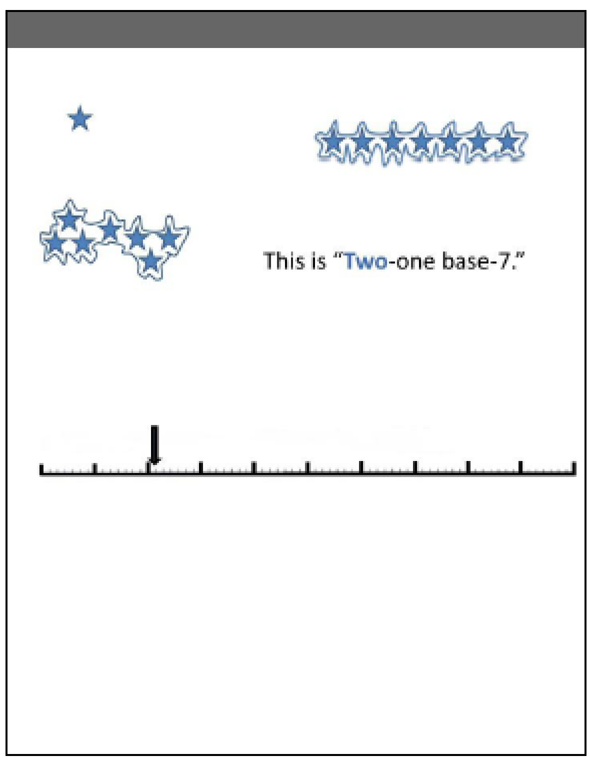
Page 17



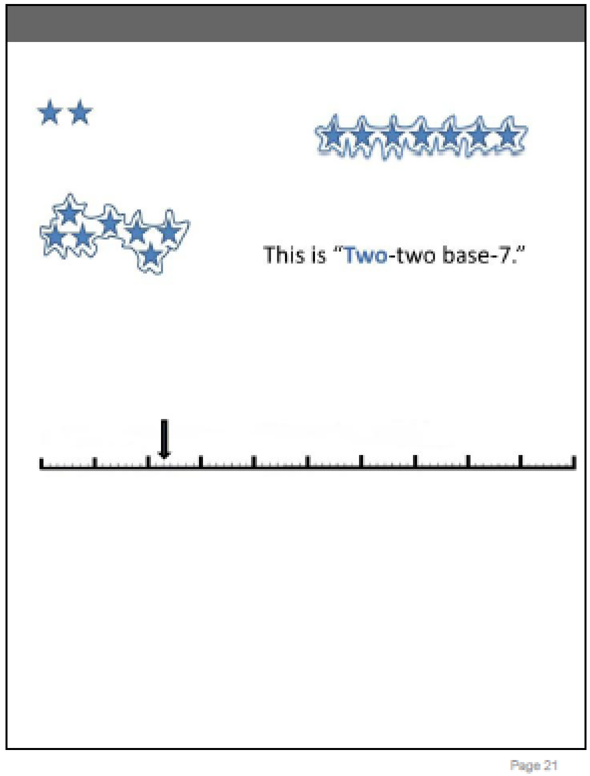
Page 18

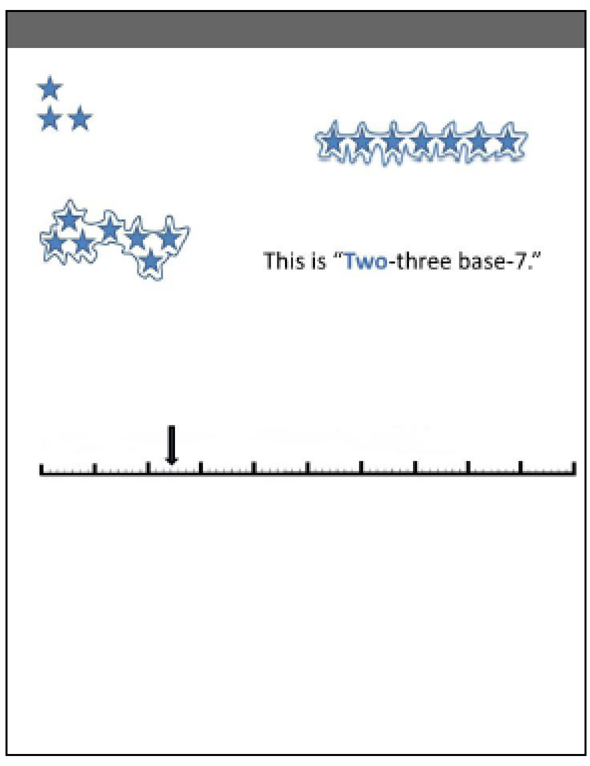


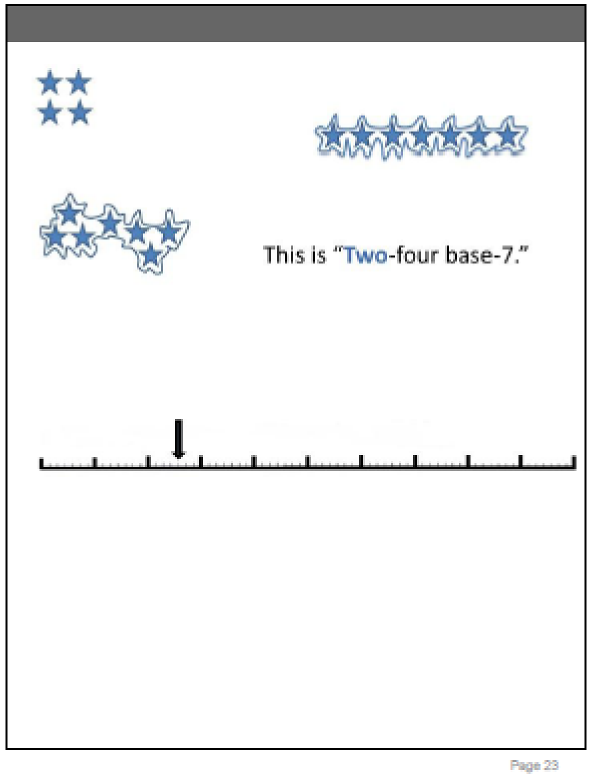
Page 19

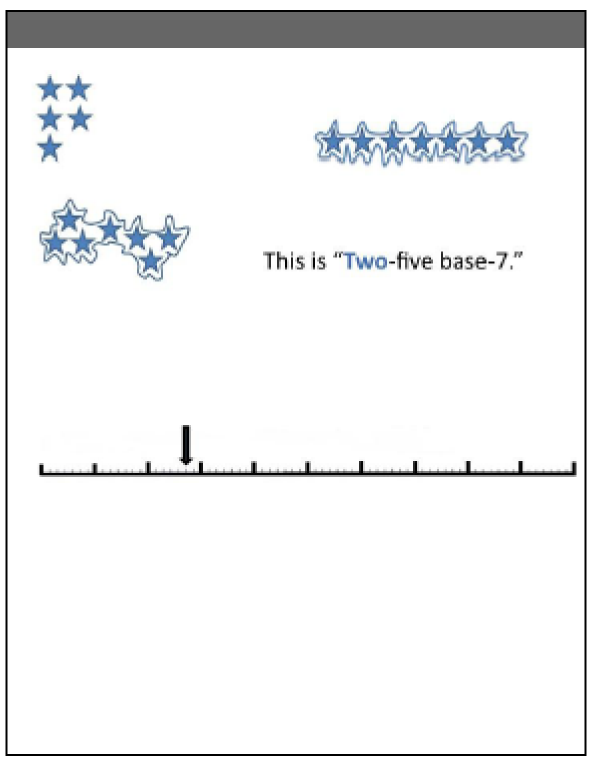


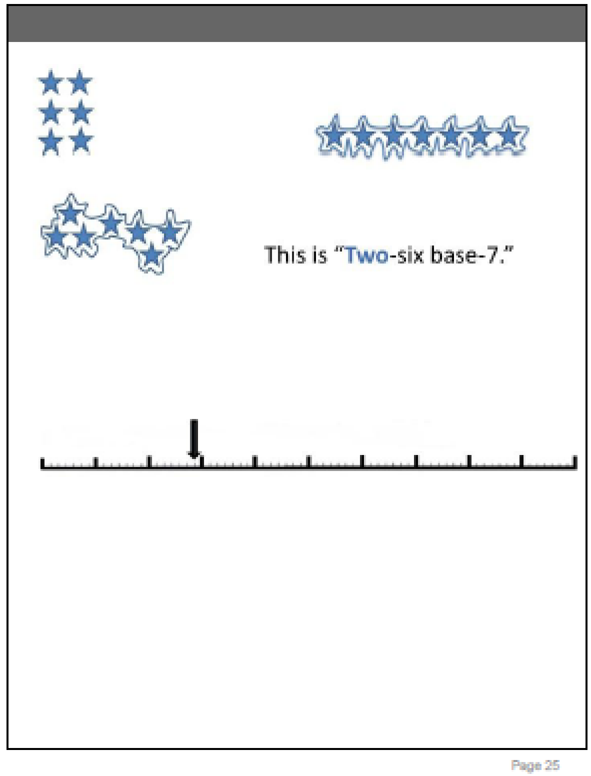
Page 20

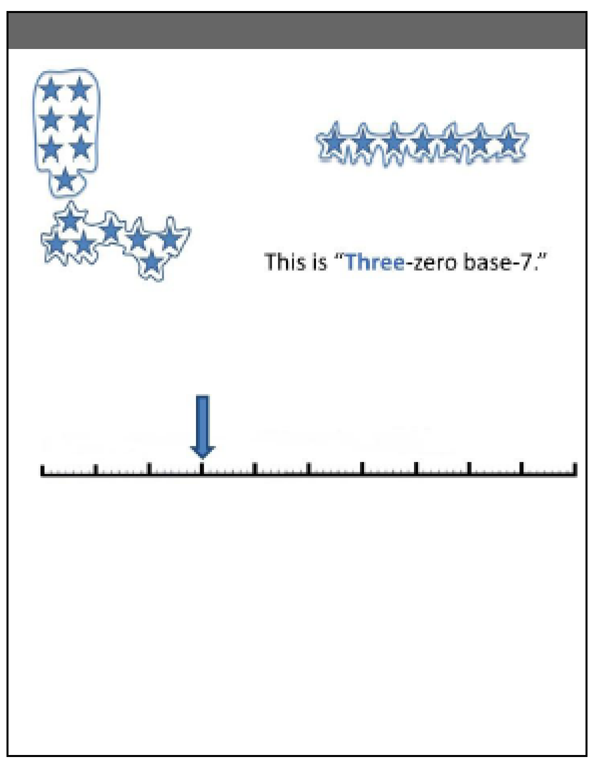

This is "Two-two base-7."
Page 21

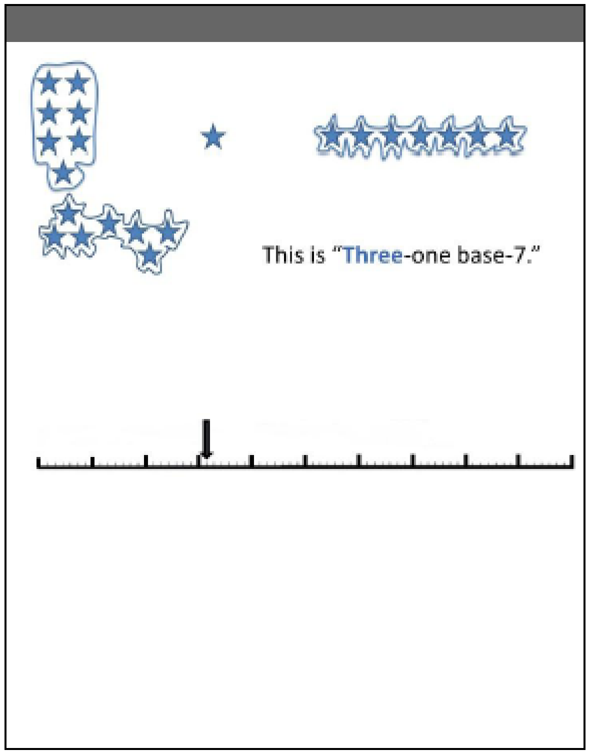

This is "Two-three base-7."
Page 22


This is "Two-four base-7."
Page 23

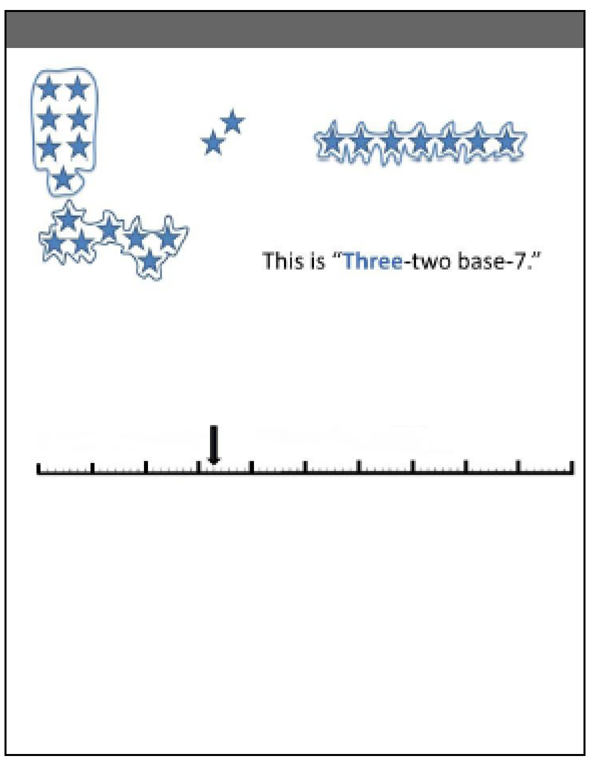

This is "Two-five base-7."
Page 24


This is "Two-six base-7."
Page 25

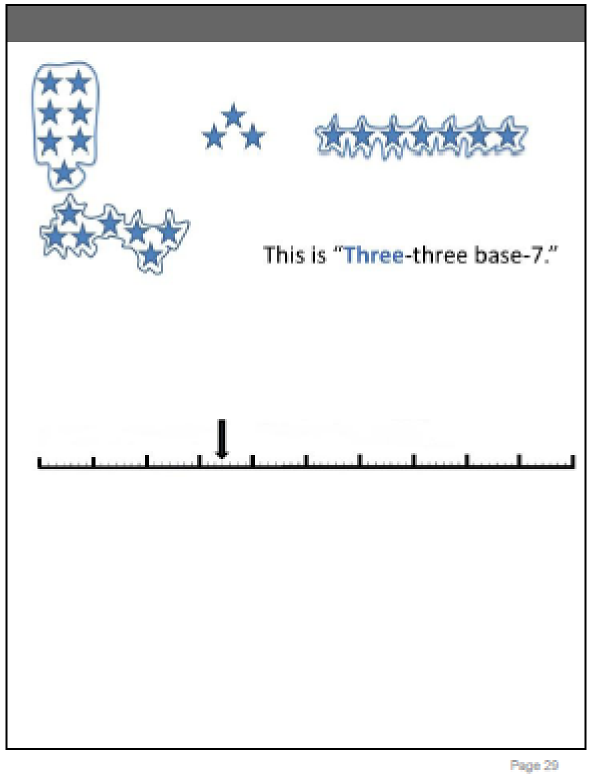

This is "Three-zero base-7."
Page 26

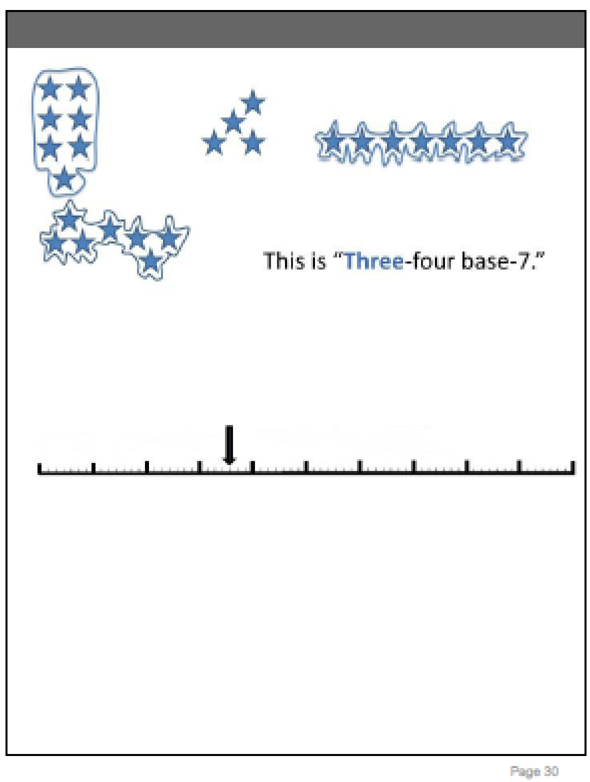


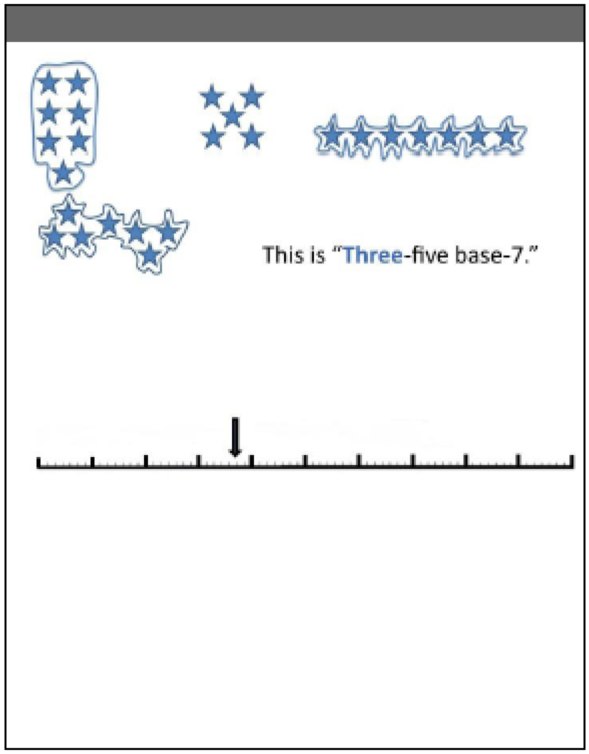
Page 27



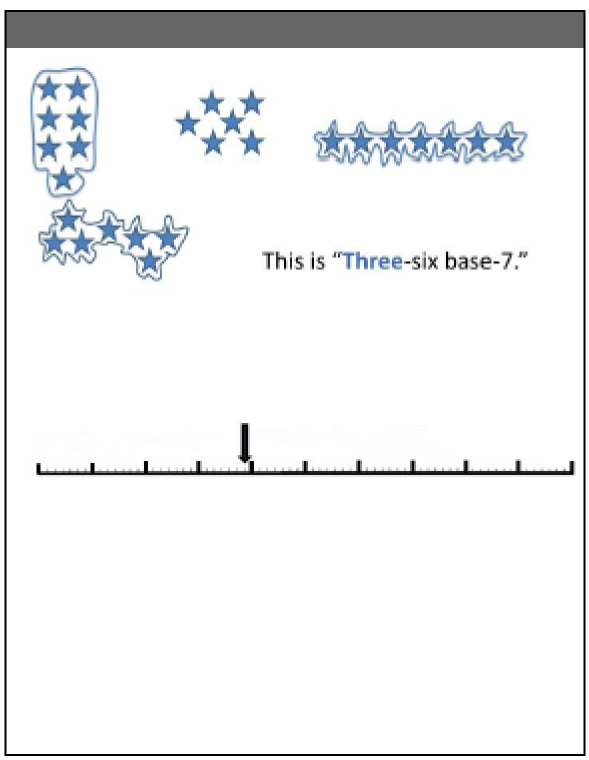
Page 28


This is "Three-three base-7."
Page 29

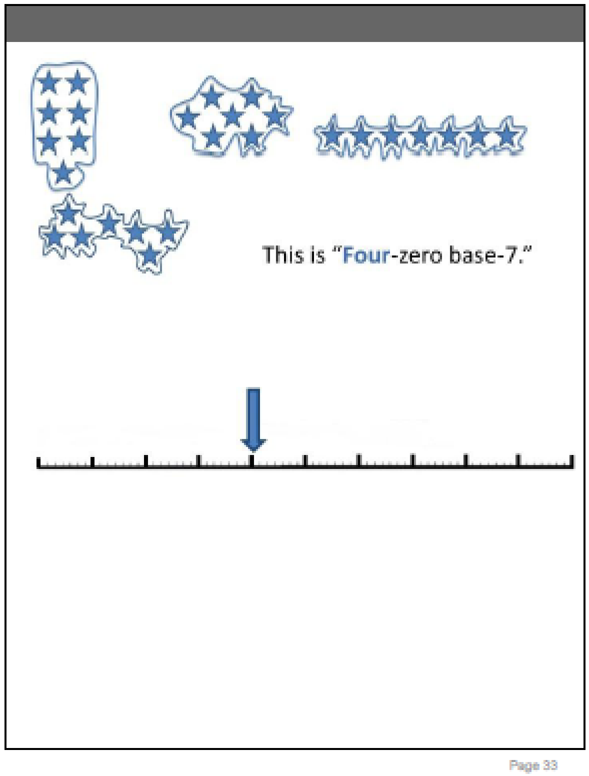

This is "Three-four base-7."
Page 30

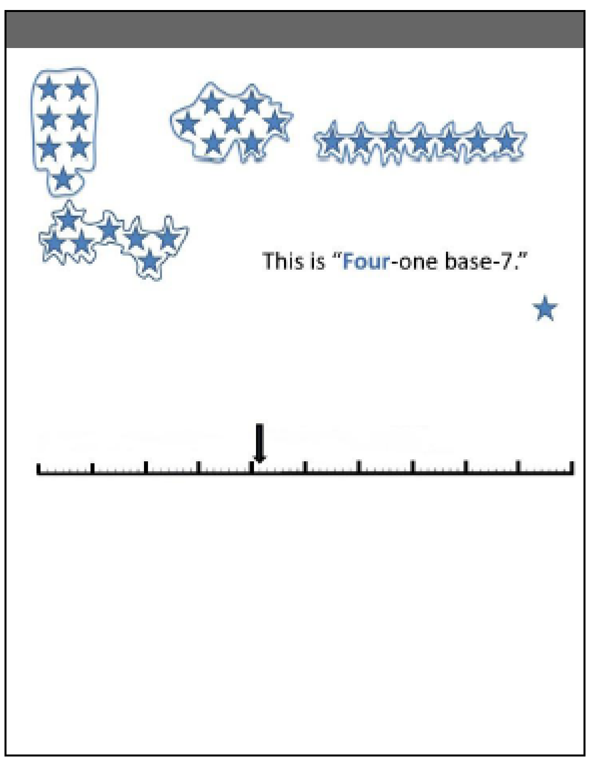


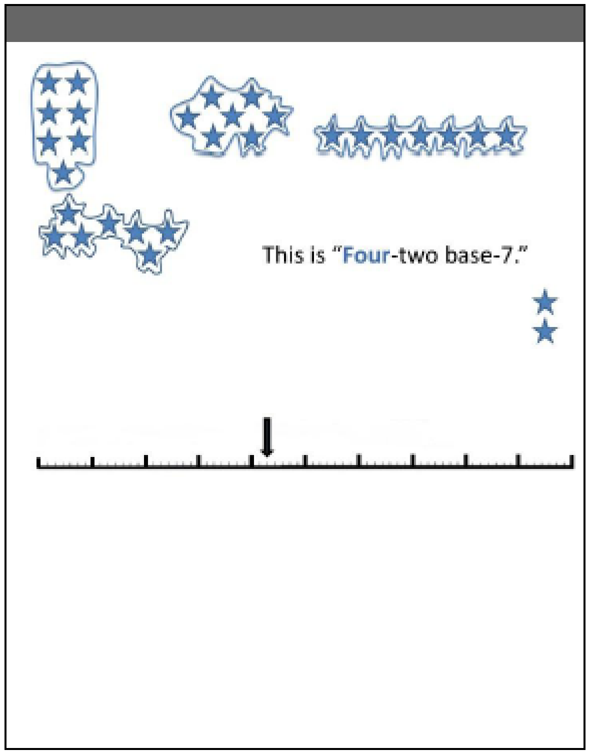
Page 31



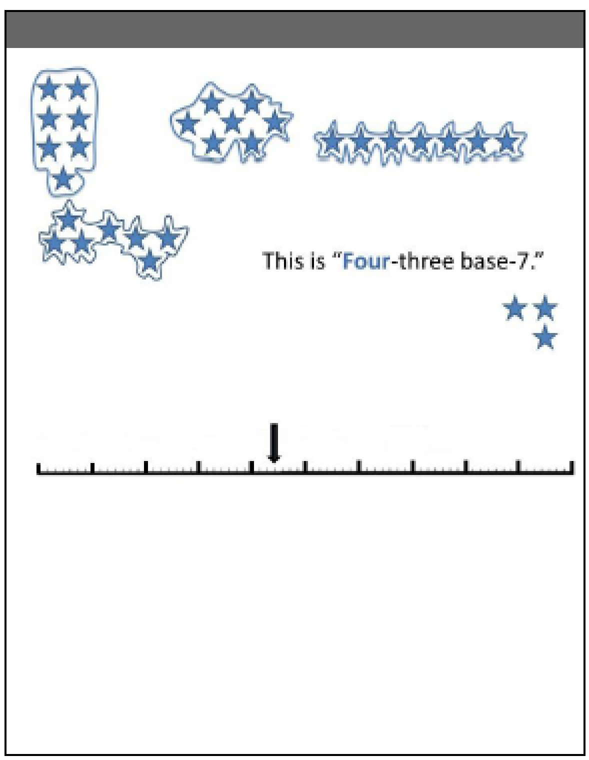
Page 32


This is "Four-zero base-7."
Page 33

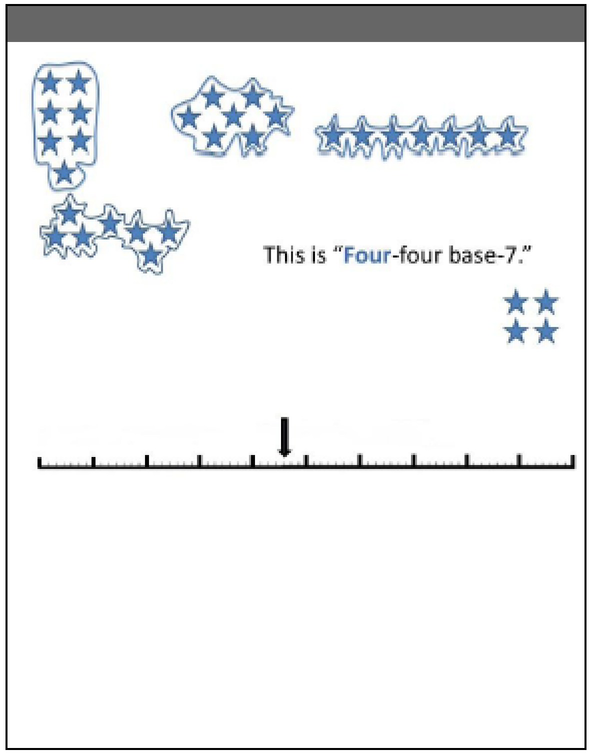

This is "Four-one base-7."
Page 34



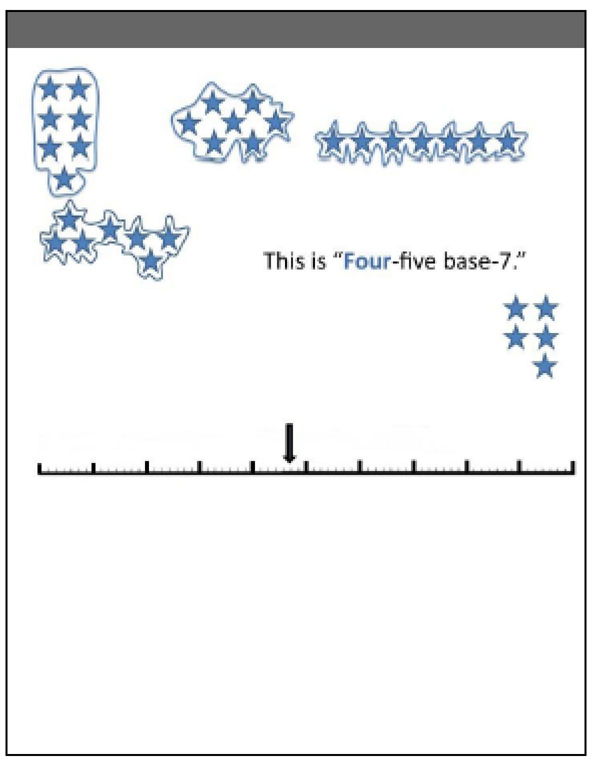
Page 35



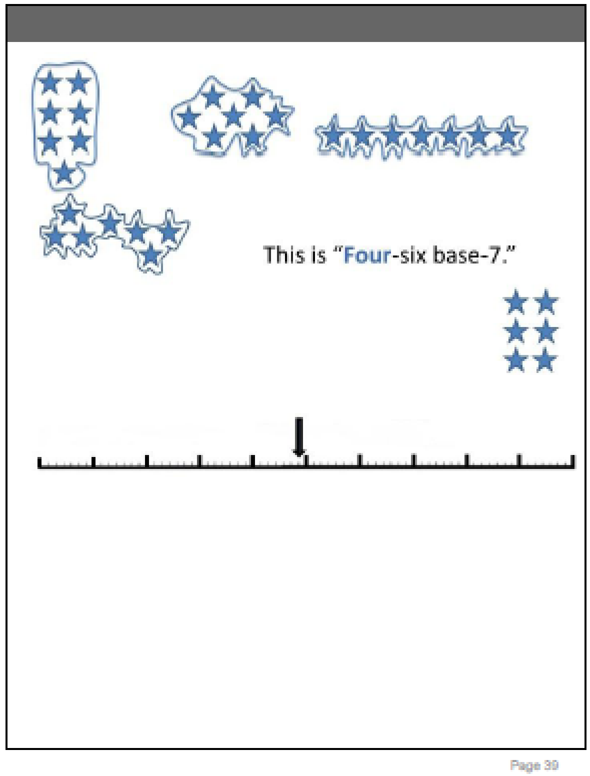
Page 36

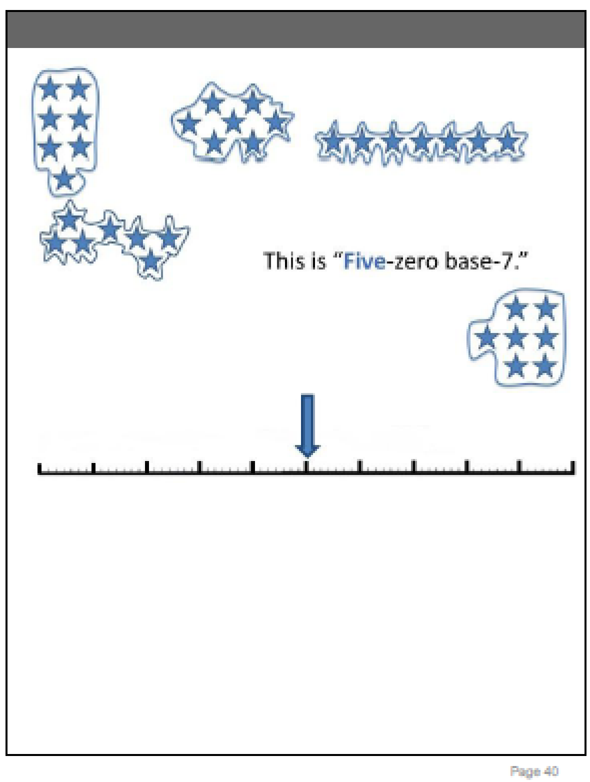


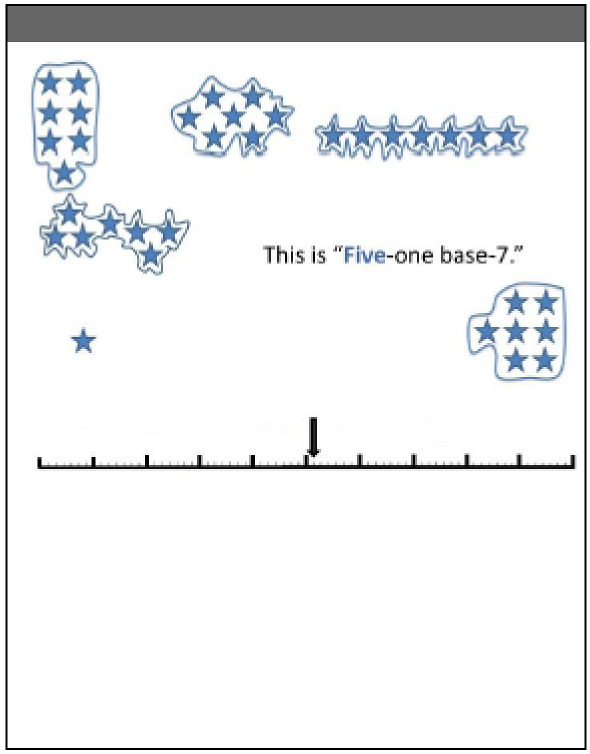
Page 37



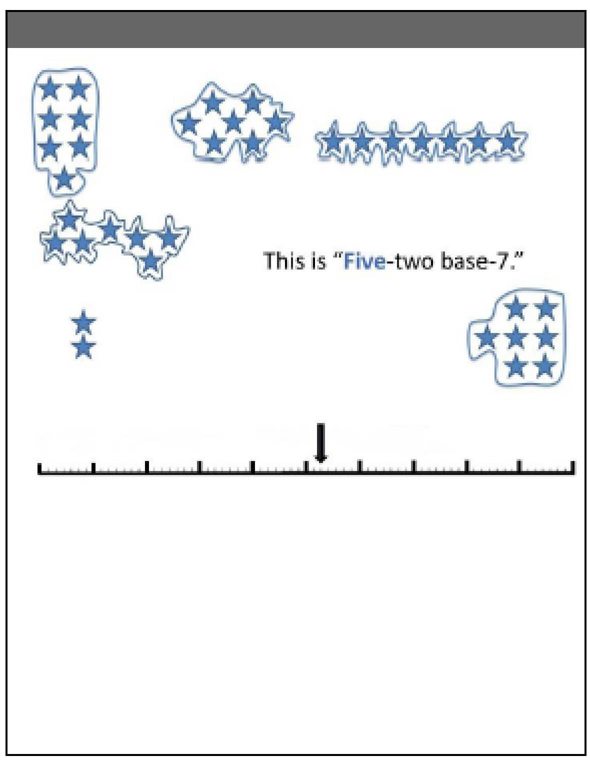
Page 38


This is "Four-six base-7."
Page 39

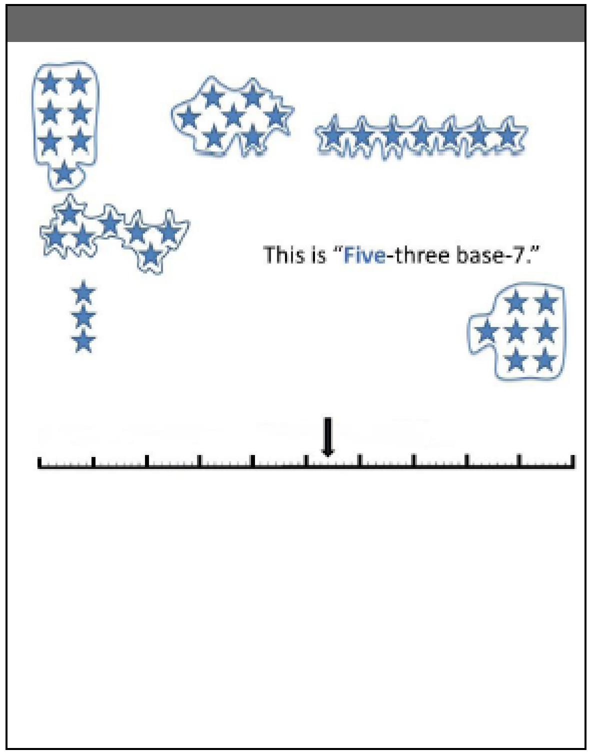

This is "Five-zero base-7."
Page 40



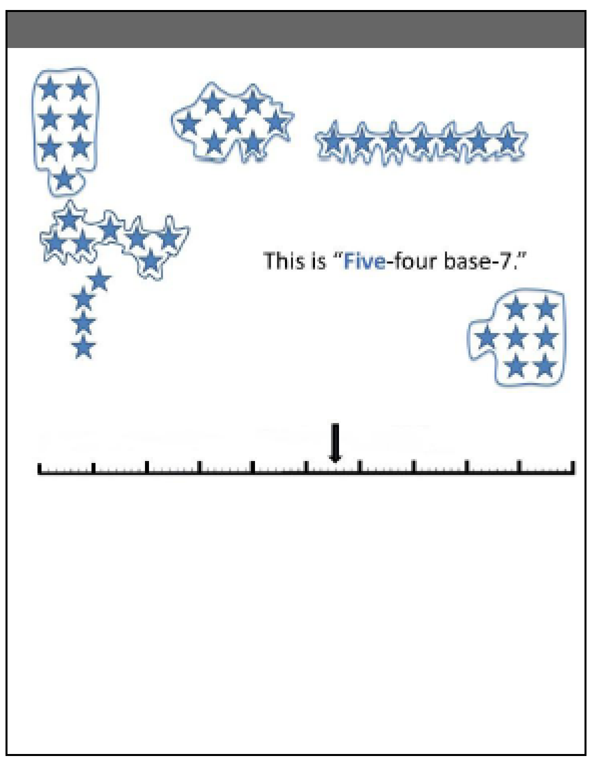
Page 41



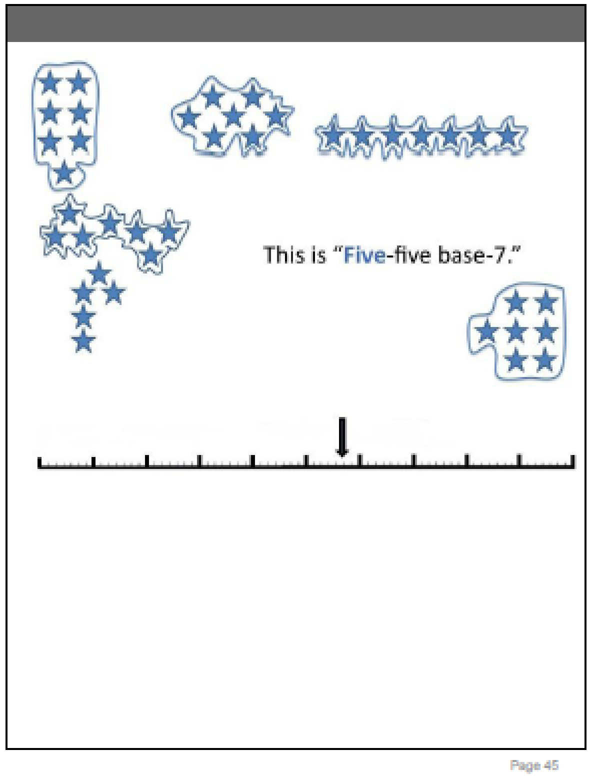
Page 42

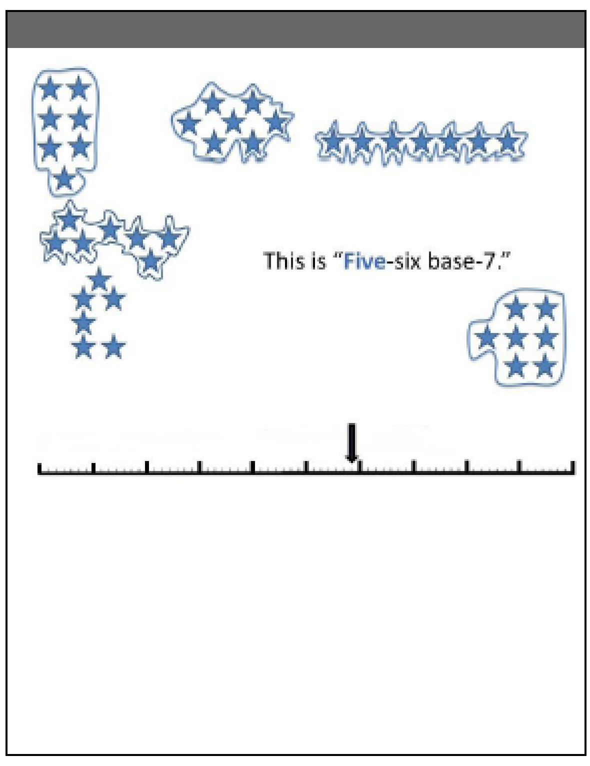


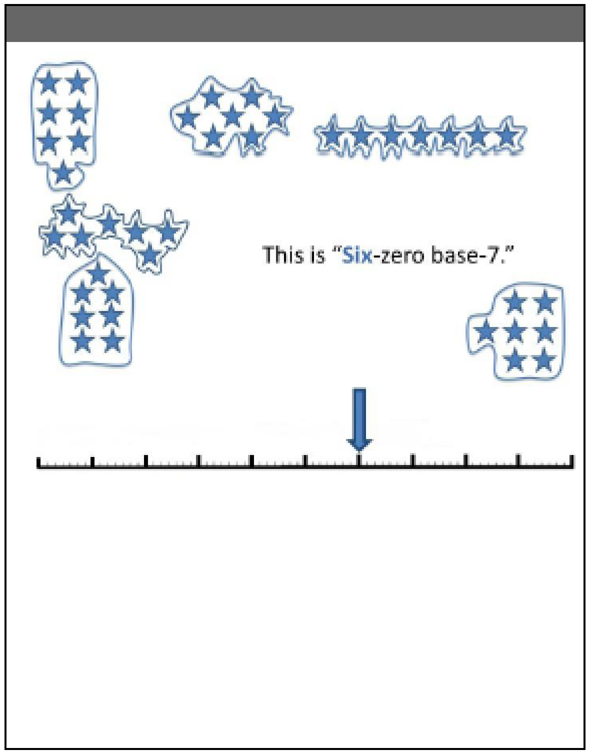
Page 43



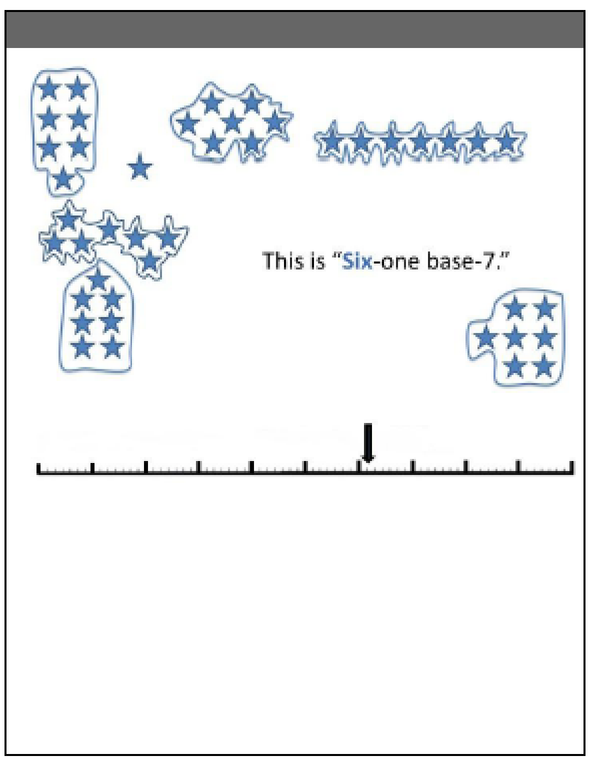
Page 44


This is "Five-five base-7."
Page 45

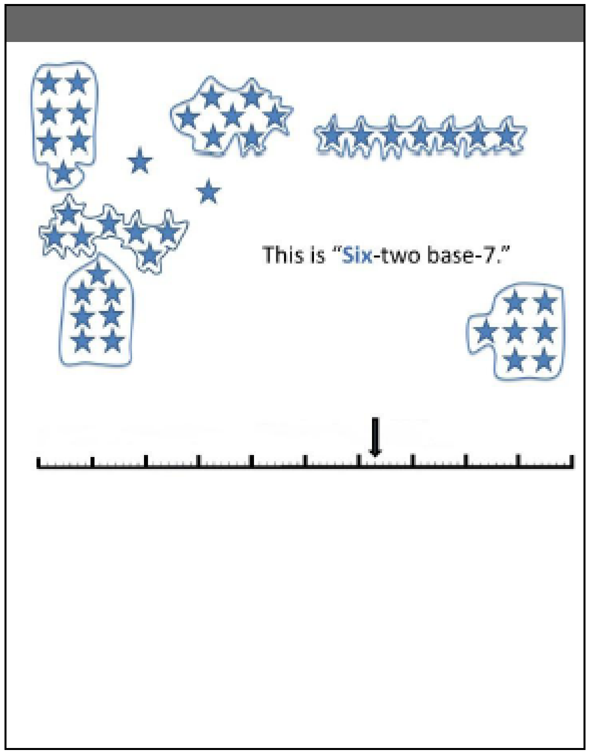

This is "Five-six base-7."
Page 46



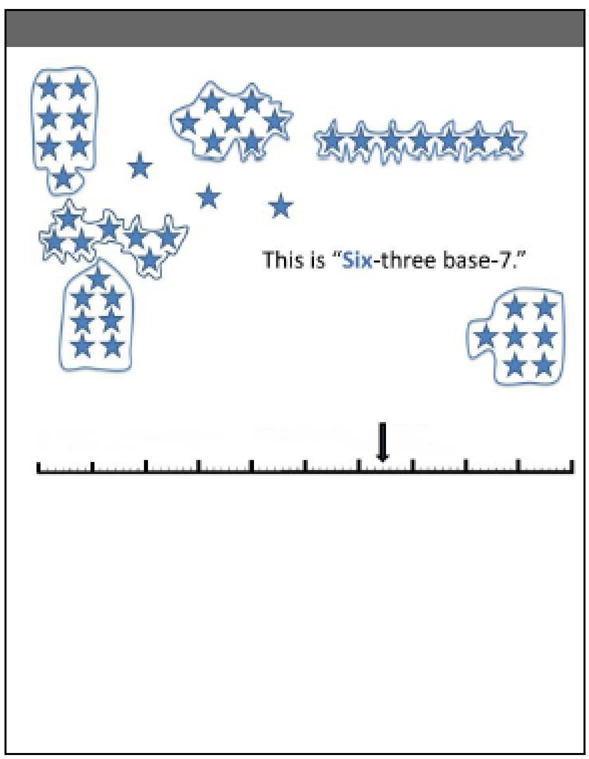
Page 47



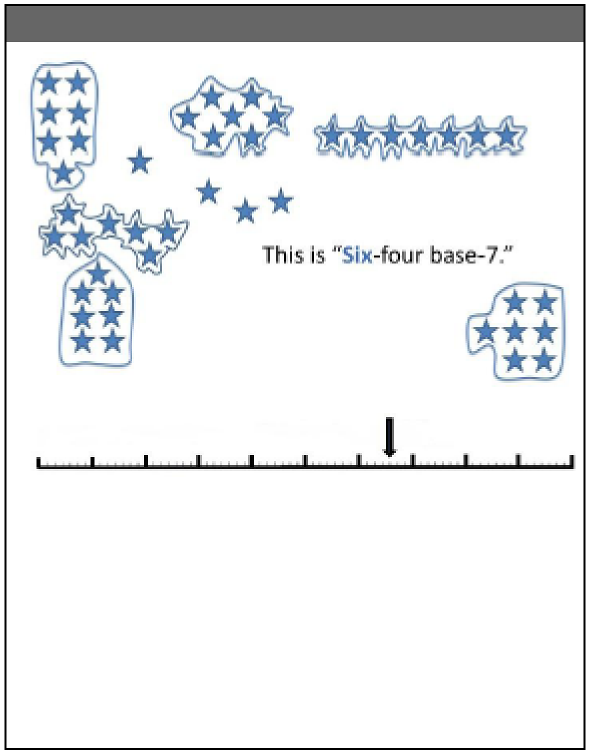
Page 48



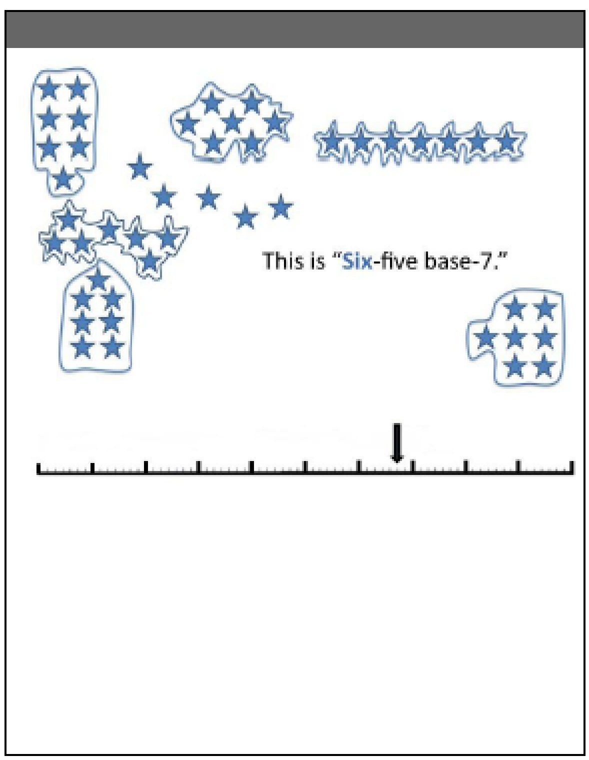
Page 49



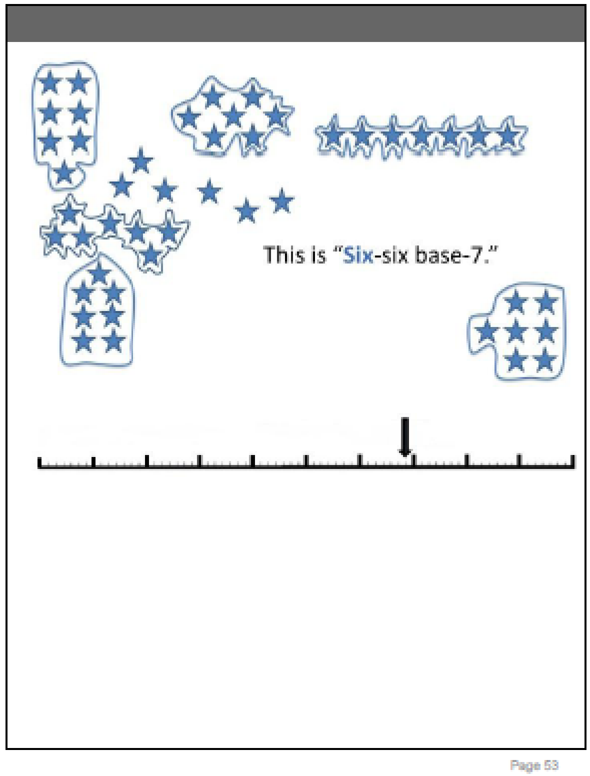
Page 50

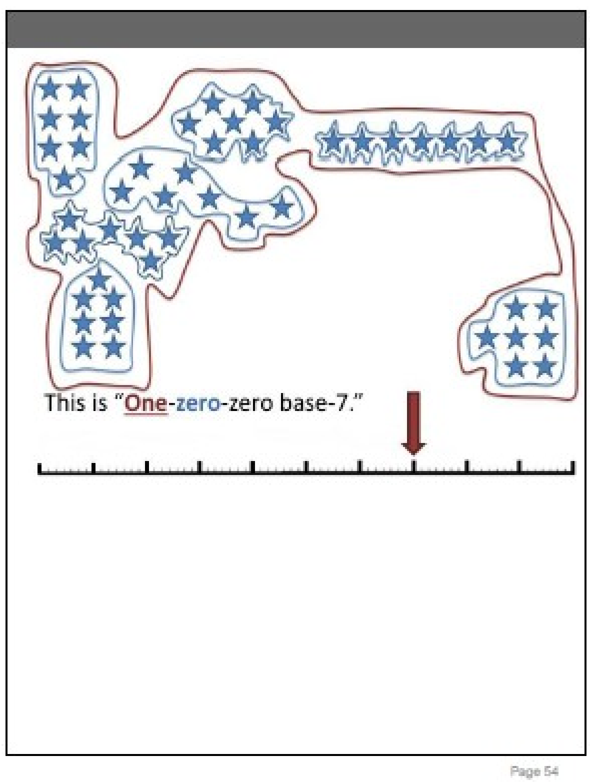


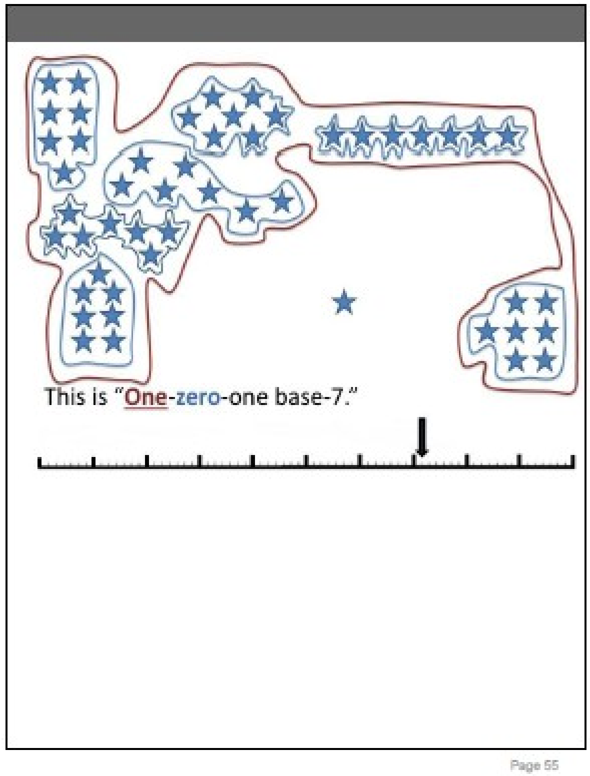
Page 51

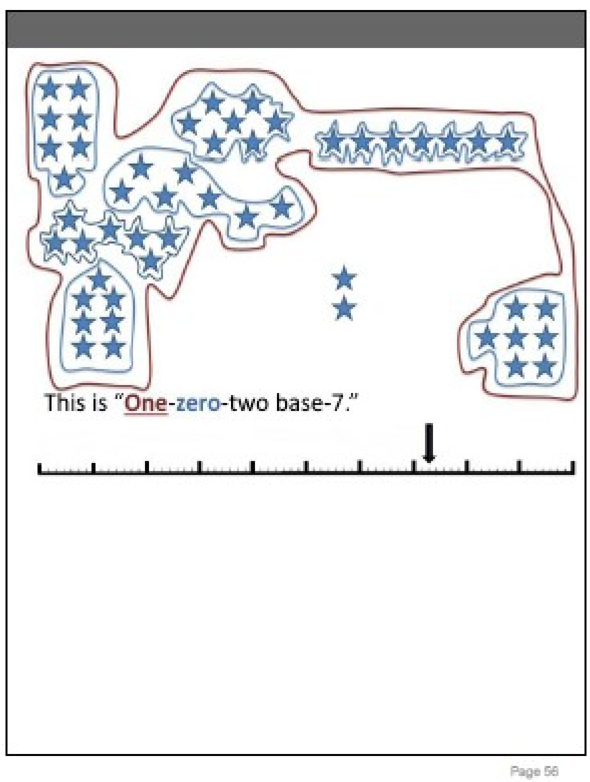


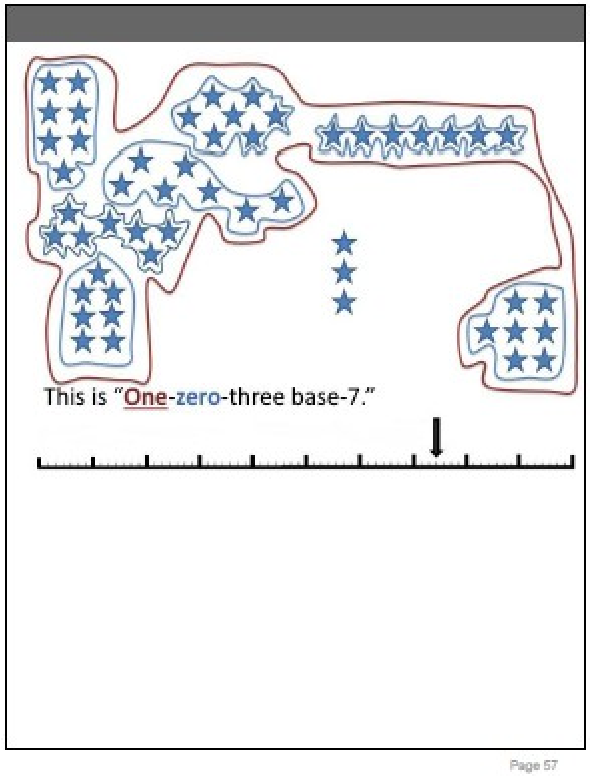
Page 52

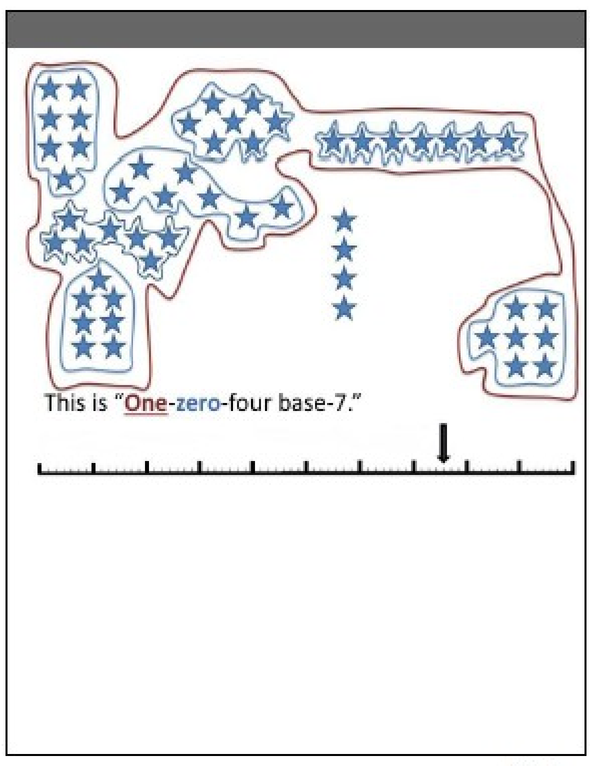

This is "Six-six base-7."
Page 53

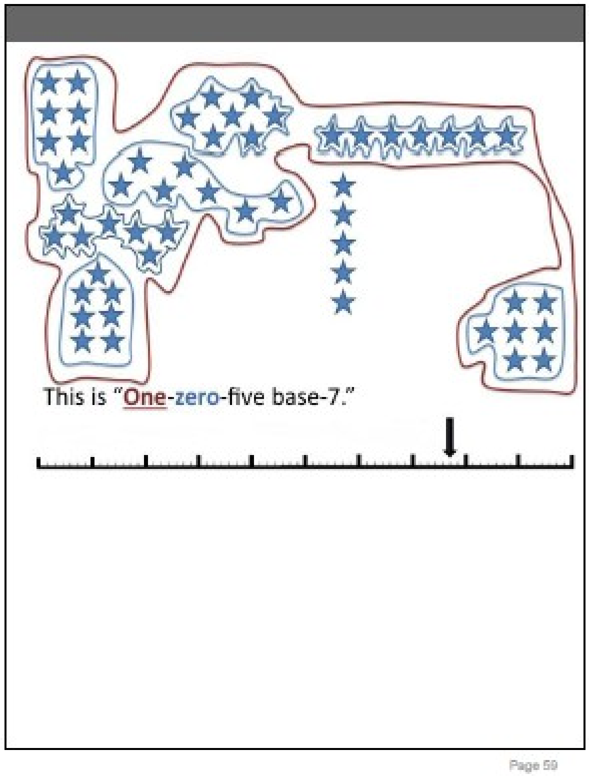

This is "One-zero-zero base-7."
Page 54

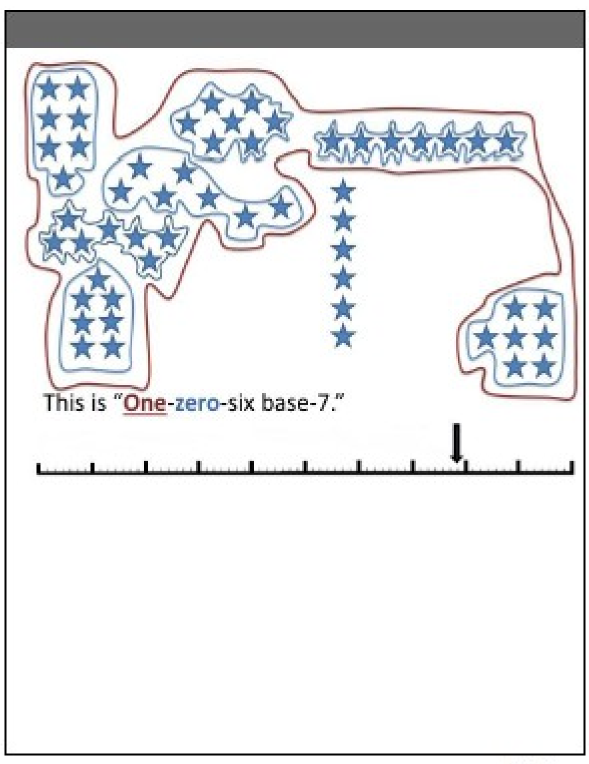

This is "One-zero-one base-7."
Page 55

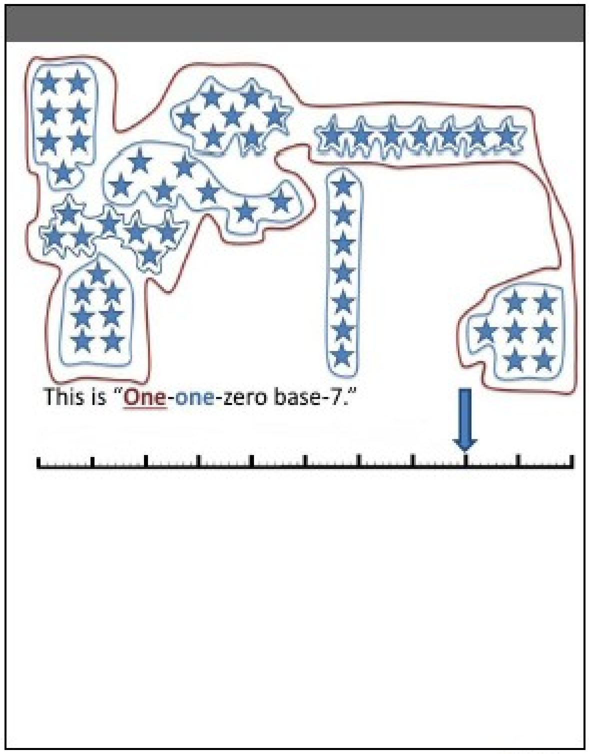

This is "One-zero-two base-7."
Page 56


This is "One-zero-three base-7."
Page 57

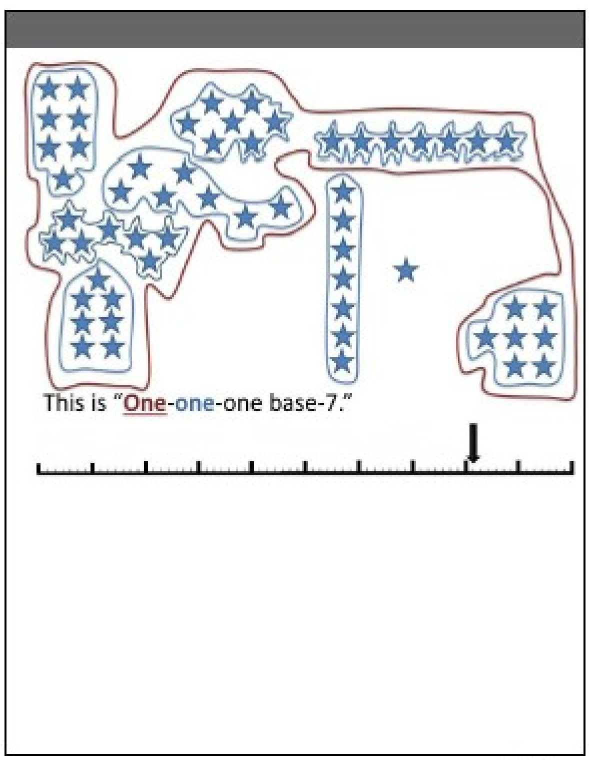

This is "One-zero-four base-7."
Page 58


This is "One-zero-five base-7."
Page 59

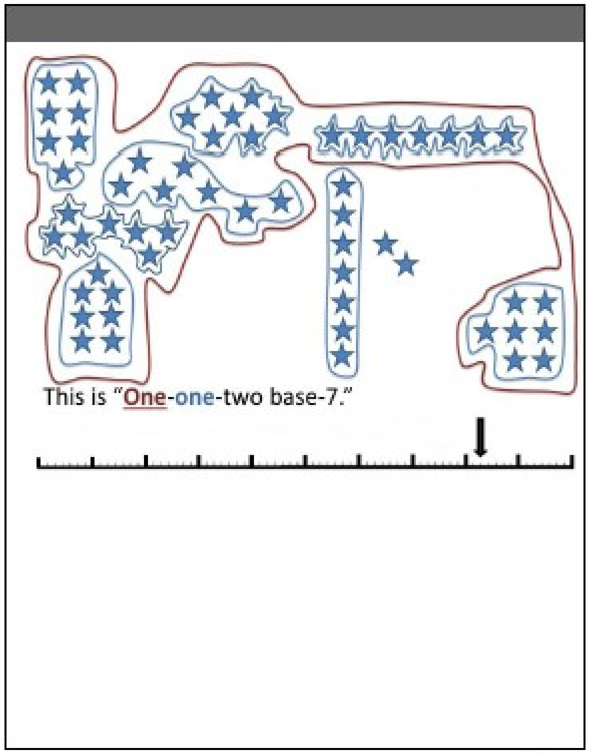

This is "One-zero-six base-7."
Page 60



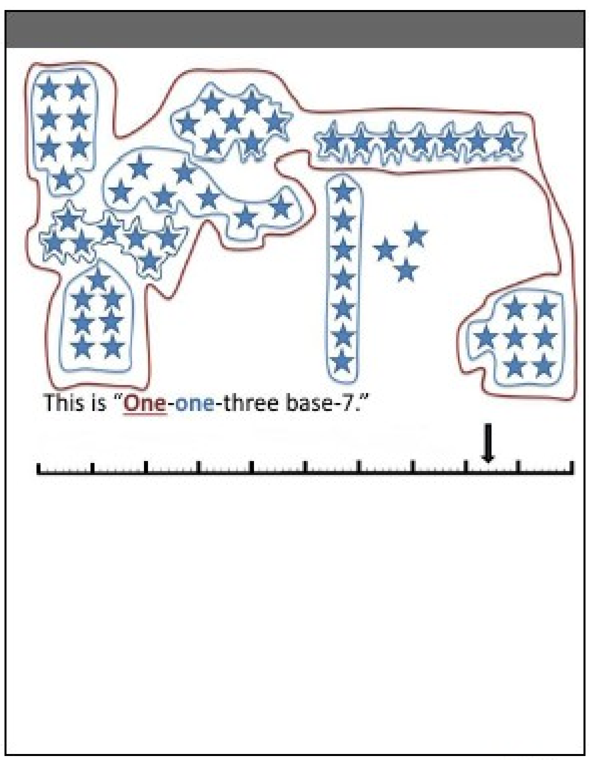
Page 61



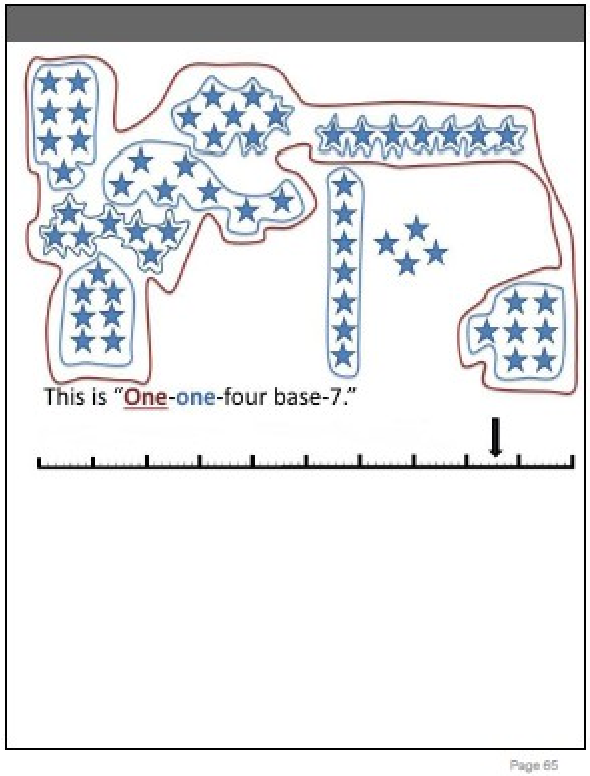
Page 62

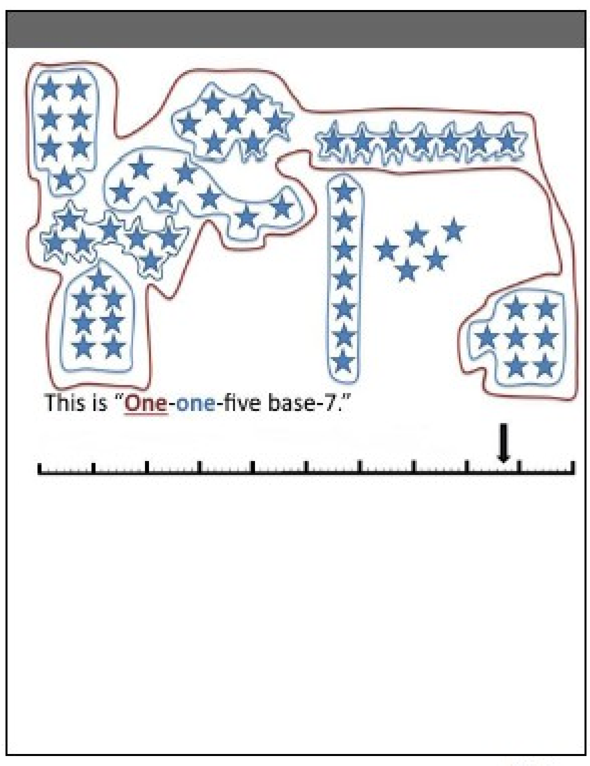


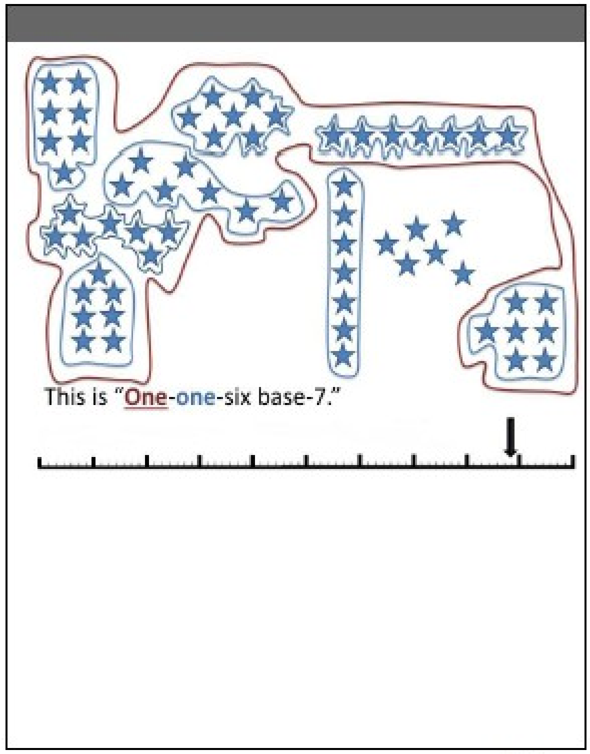
Page 63



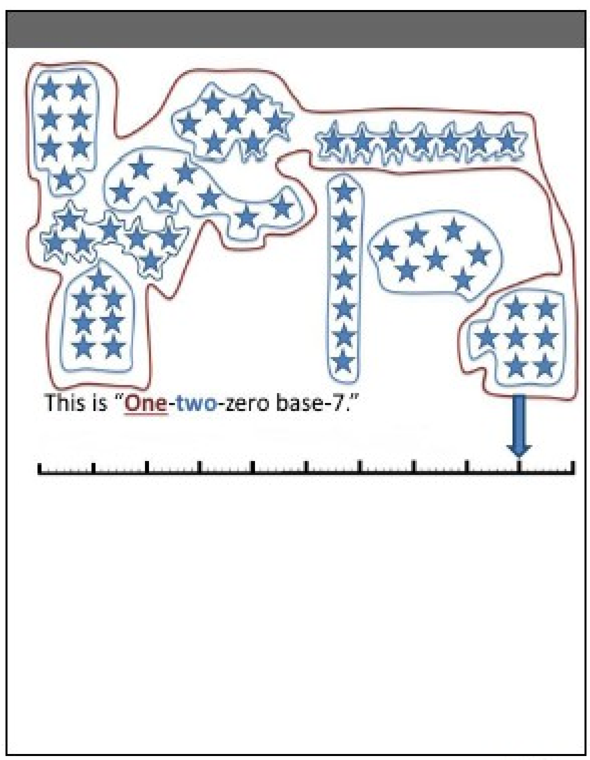
Page 64


This is "One-one-four base-7."
Page 65

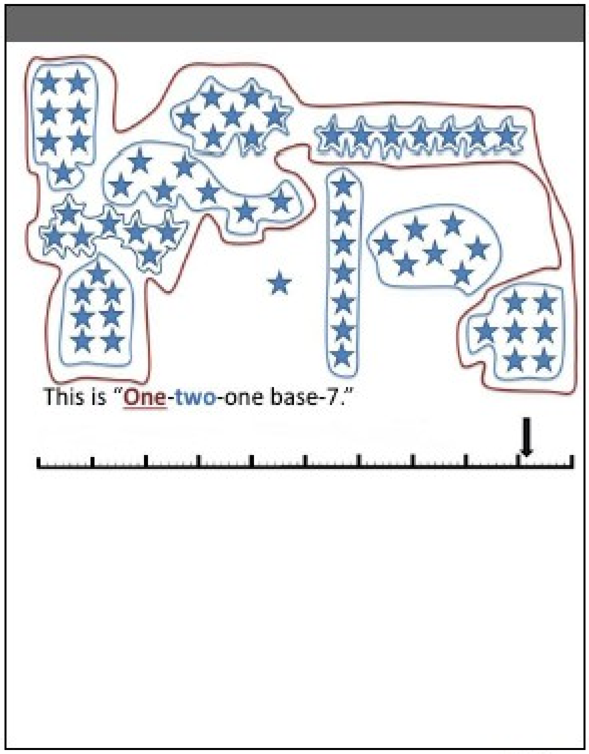

This is "One-one-five base-7."
Page 66



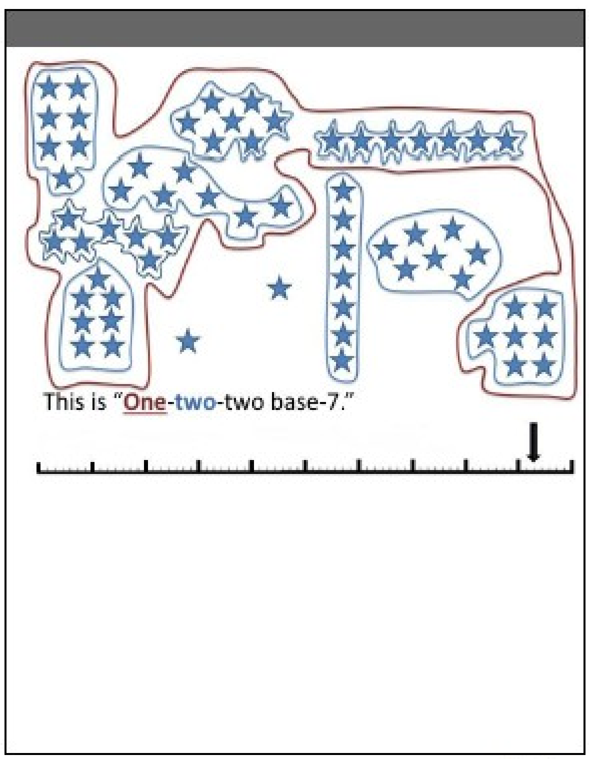
Page 67



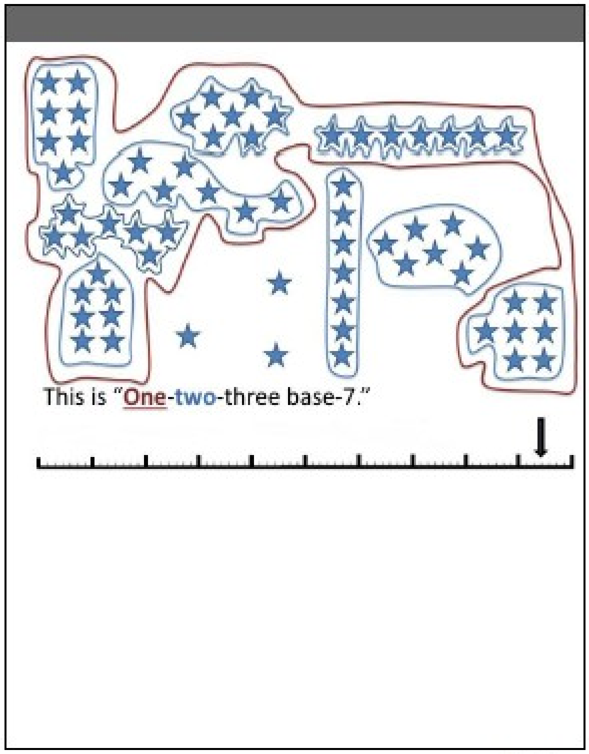
Page 68



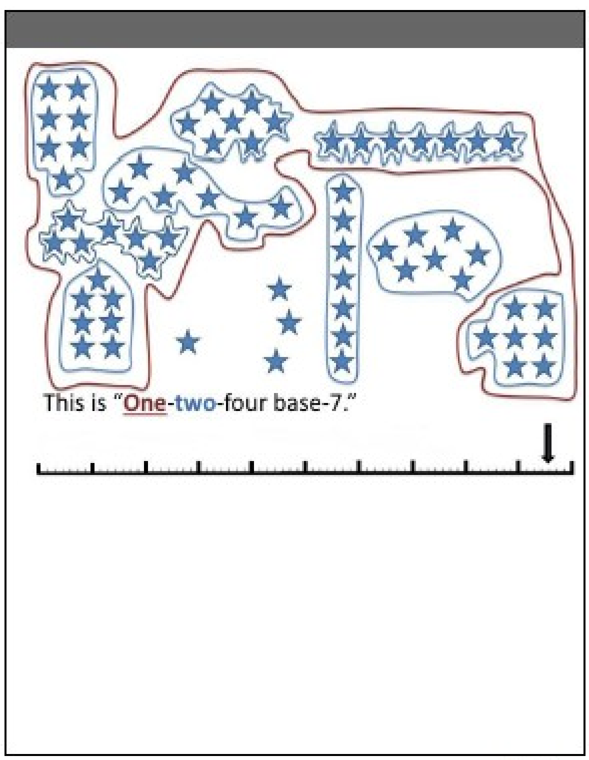
Page 69



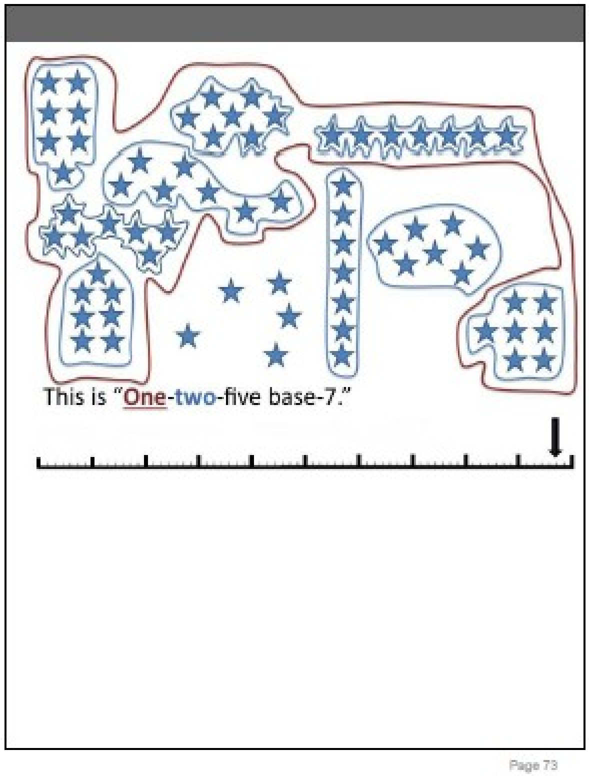
Page 70

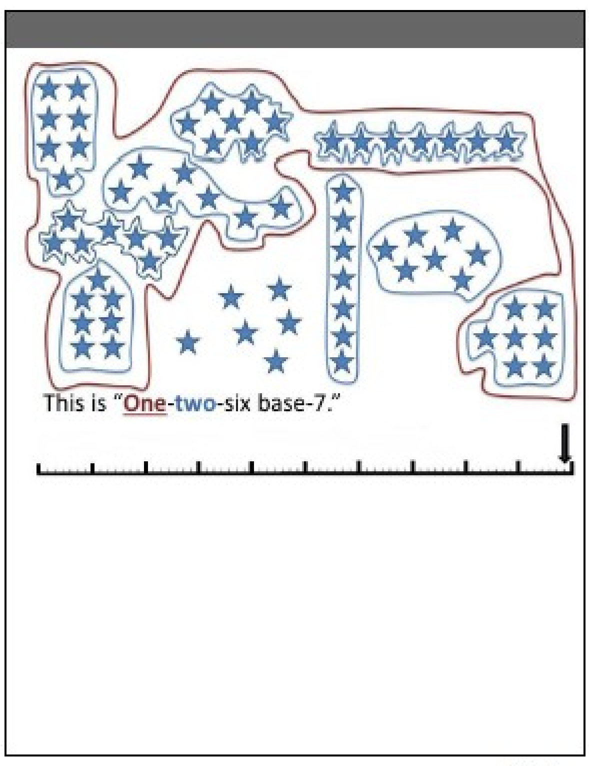


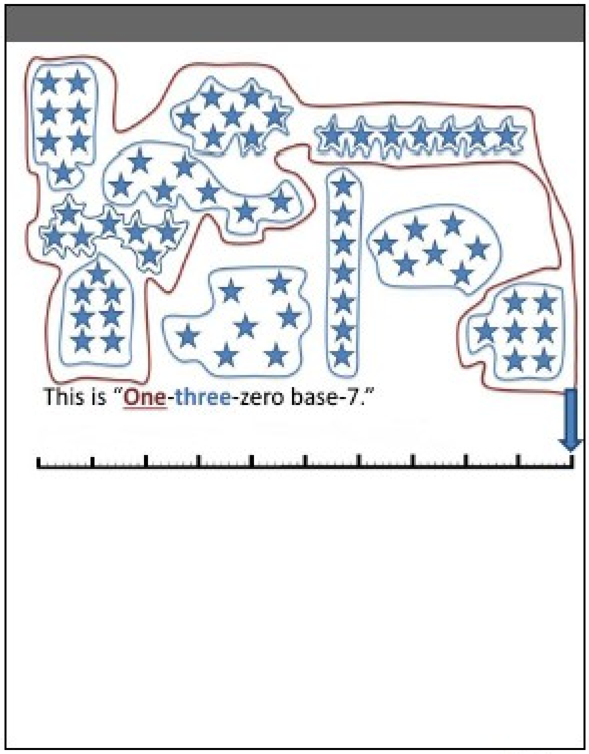
Page 71



Page 72


This is "One-two-five base-7."
Page 73


This is "One-two-six base-7."
Page 74



Page 75

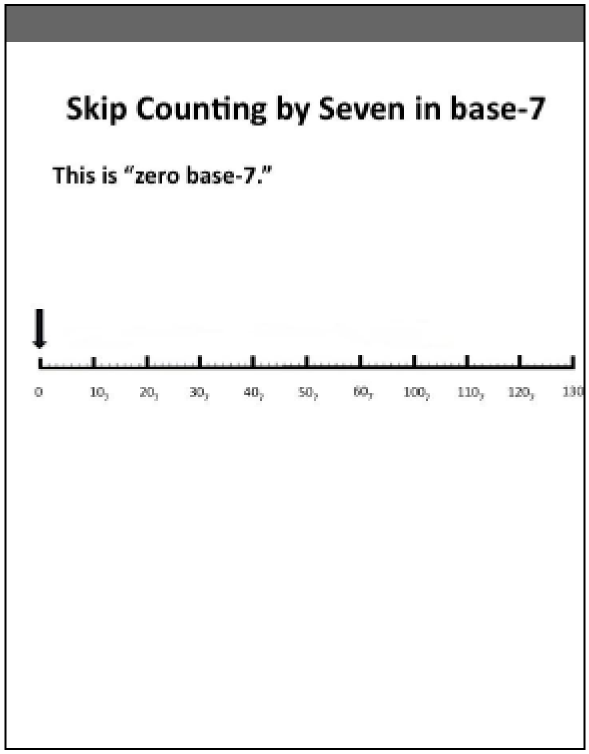
Skip Counting

Now we are going to skip count objects.

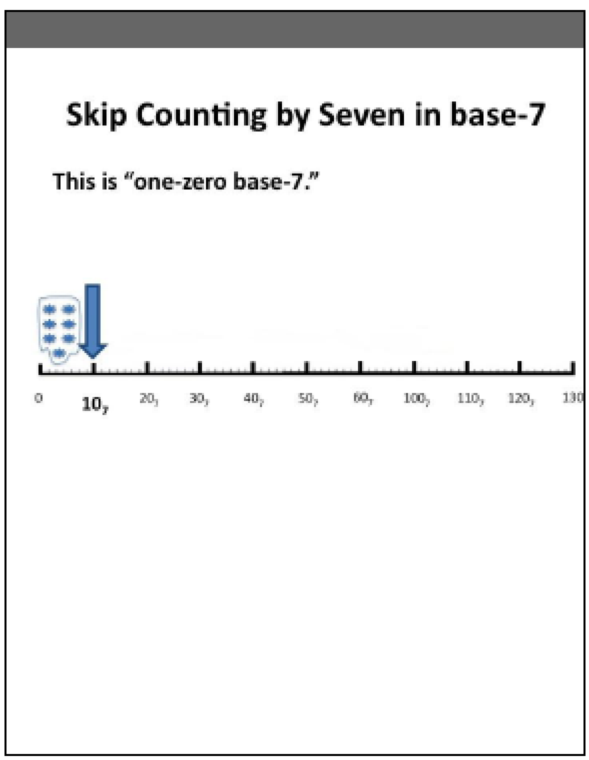
Skip counting means that you do not say every number name as you are counting. Instead you count up by intervals and only say the number names that occur at the given interval.

We are going to skip count by seven in base-7.

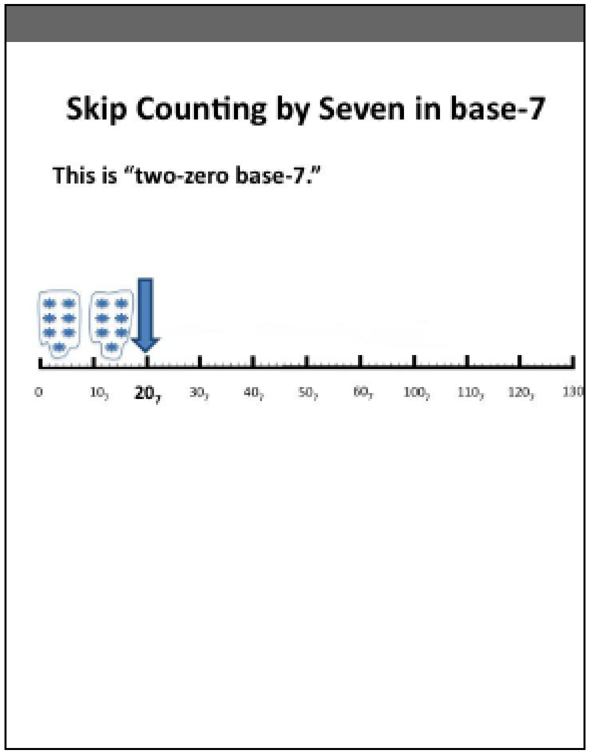
Page 76



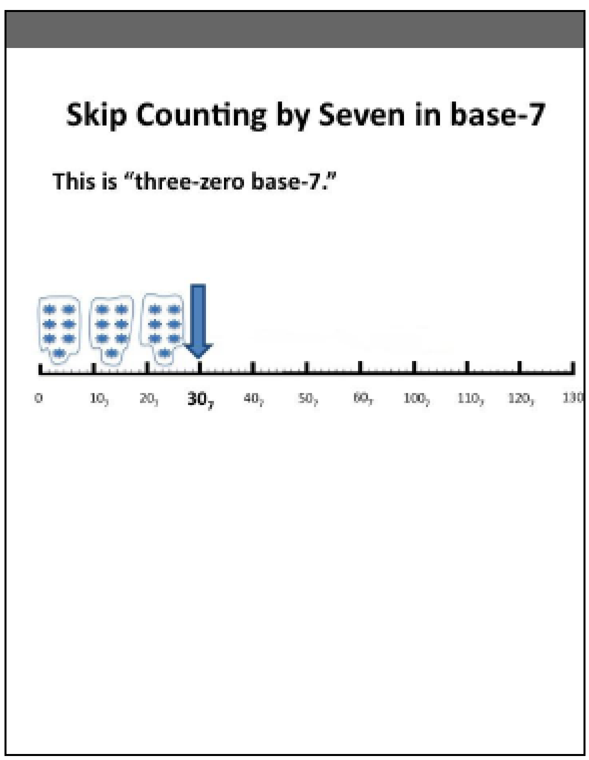
Page 77



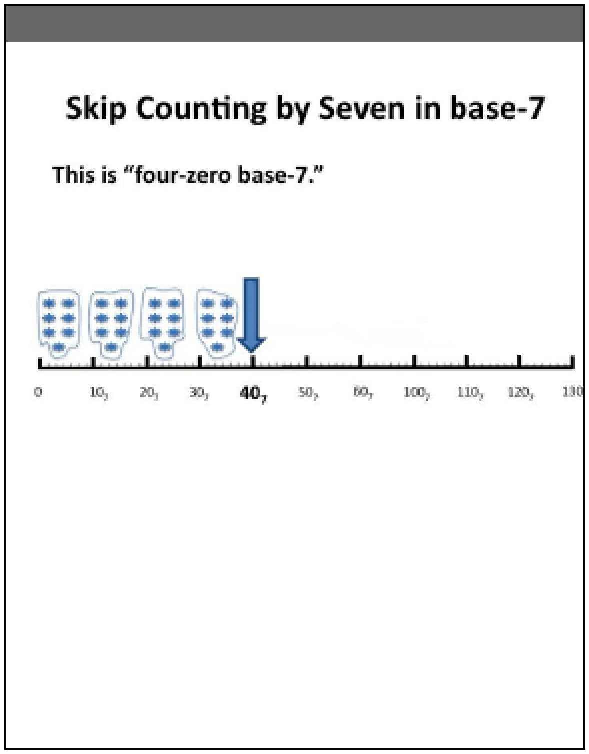
Page 78



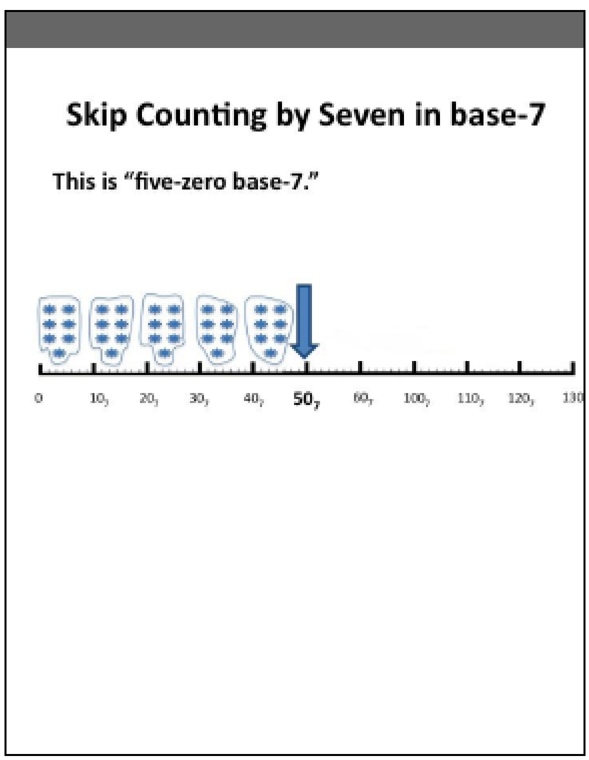
Page 79



Page 80



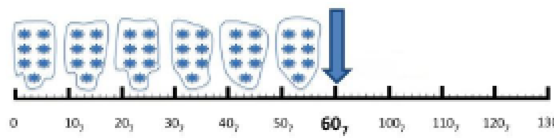
Page 81



Page 82

Skip Counting by Seven in base-7

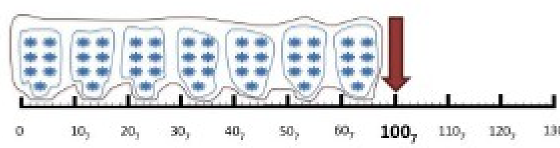
This is "six-zero base-7."



Page 83

Skip Counting by Seven in base-7

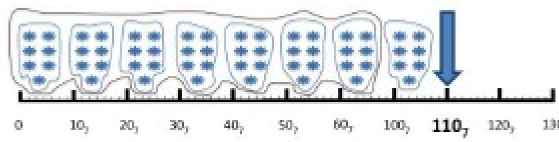
This is "one-zero-zero base-7."



Page 84

Skip Counting by Seven in base-7

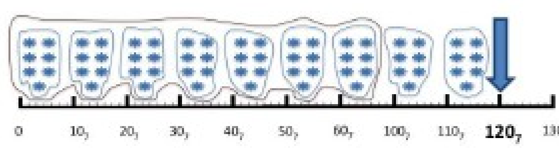
This is “one-one-zero base-7.”



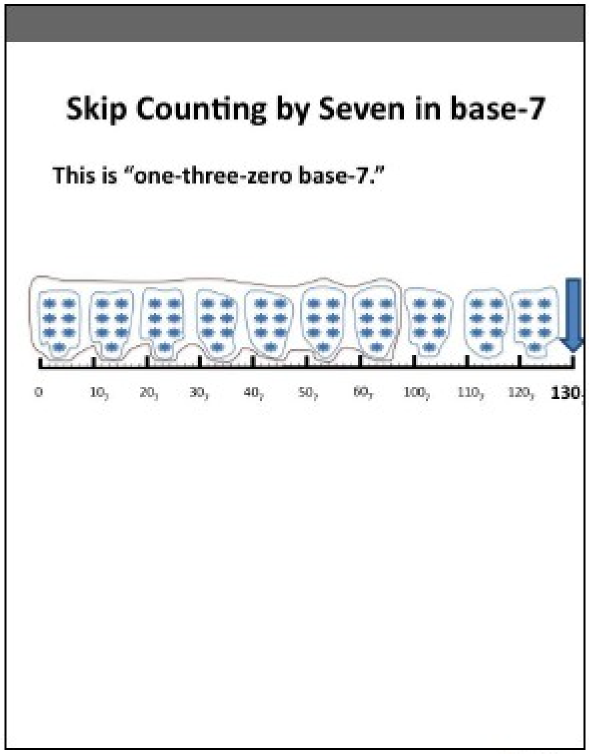
Page 85

Skip Counting by Seven in base-7

This is “one-two-zero base-7.”



Page 86


Skip Counting by Seven in base-7
This is "one-three-zero base-7."
0
10_7
20_7
30_7
40_7
50_7
60_7
100_7
110_7
120_7
130

Page 87

Appendix F

Lesson Questions

This section contains ten questions on counting objects in base-7.

There are:

- Eight fill in the blank questions
- Two multiple choice questions

Page 88

Instructions

Please answers these questions to the best of your ability.

You will not be able to proceed to the next question until you answer the question you are on.

You may refer back to the lesson at any time.

Page 89

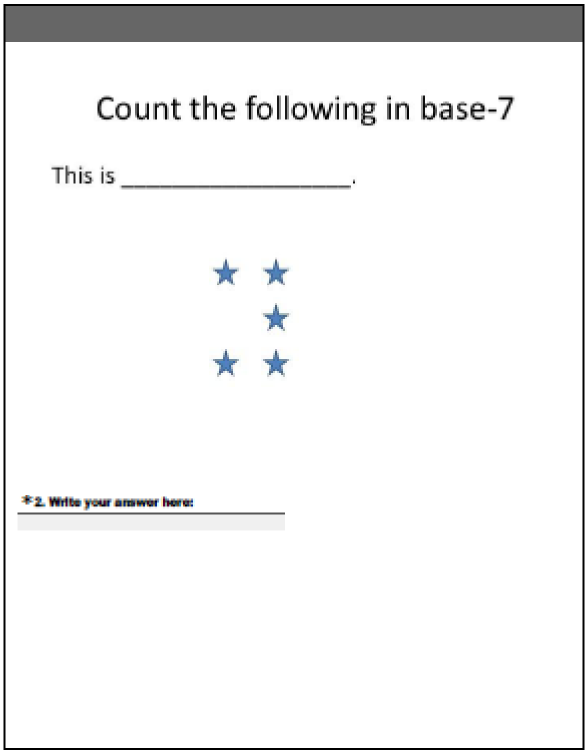
Count the following in base-7.

Make sure you use the same notation that was used during the lesson.

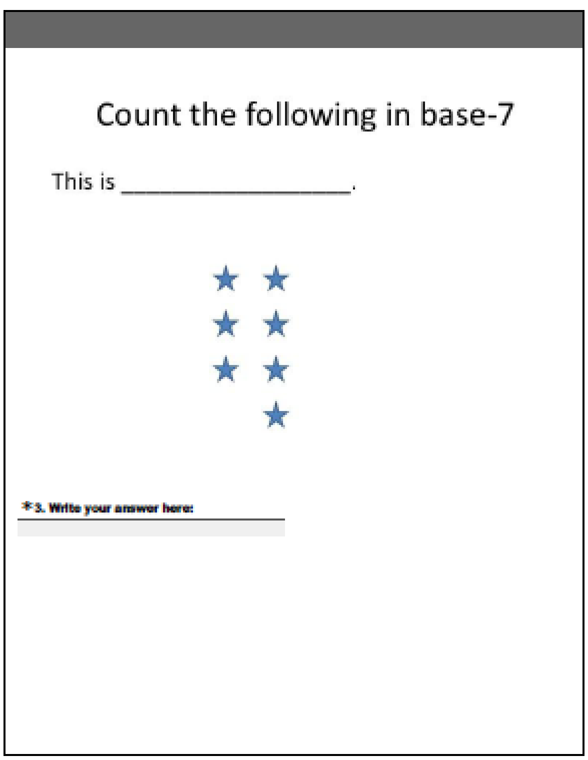
Example:

This is _*two base-7*____.

Page 90



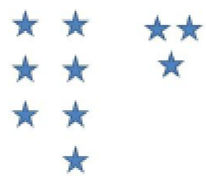
Page 91



Page 92

Count the following in base-7

This is __________________.

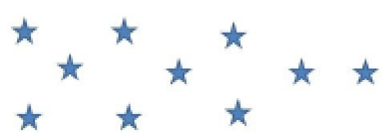


*4. Write your answer here:

Page 93

Count the following in base-7

This is __________________.



*5. Write your answer here:

Page 94

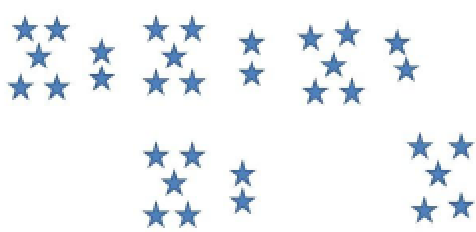
Count the following in base-7

This is ___________________.

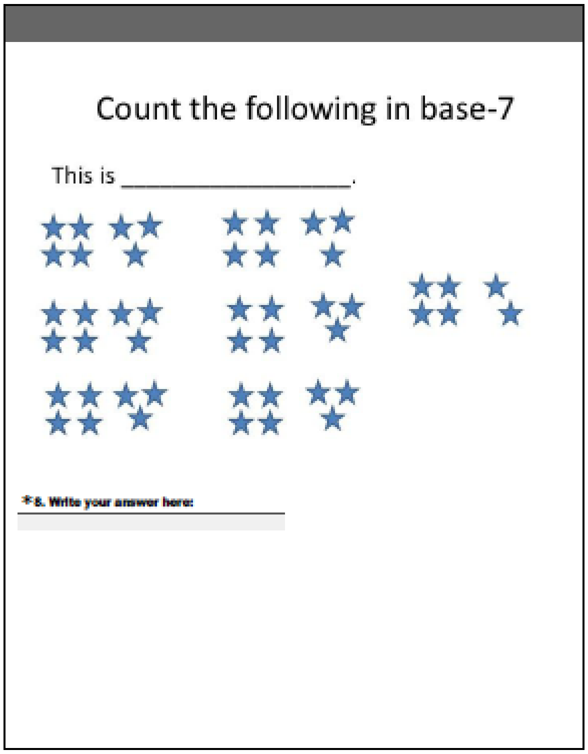
*6. Write your answer here:

Count the following in base-7.

This is ___________________.



*7. Write your answer here:

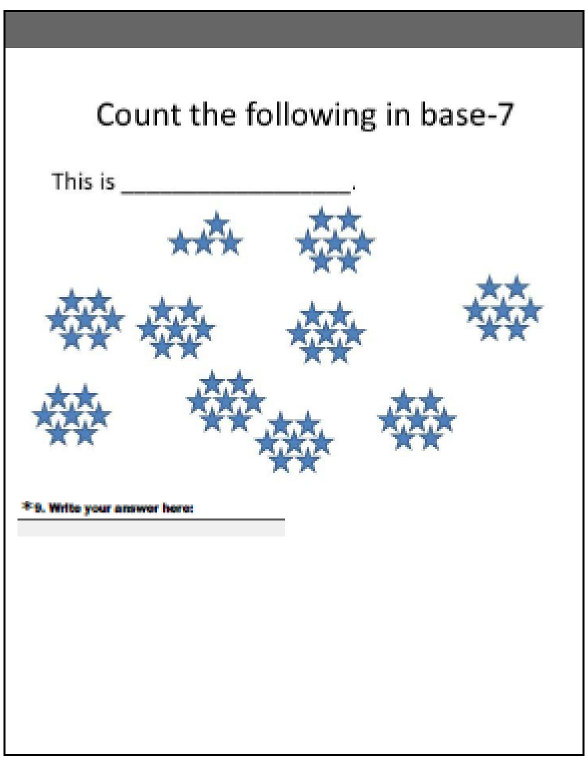


Count the following in base-7

This is ___________________.

*8. Write your answer here:

Page 97

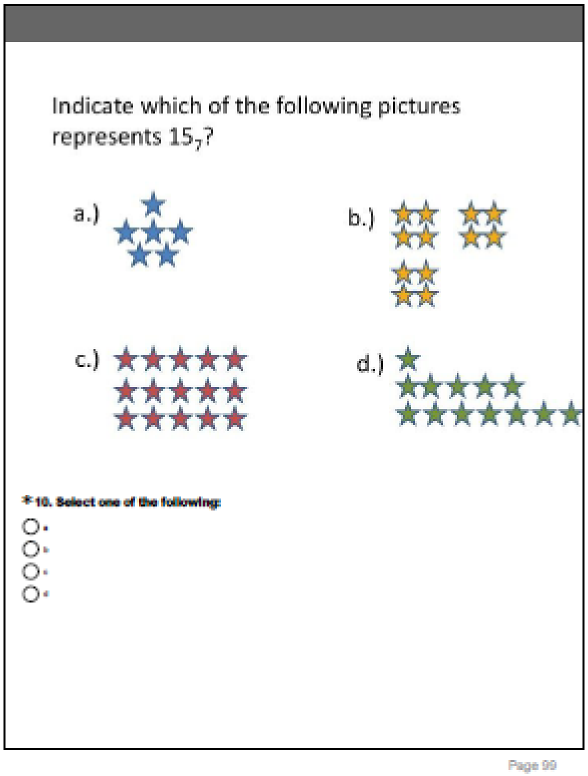


Count the following in base-7

This is ___________________.

*9. Write your answer here:

Page 98



Indicate which of the following pictures represents 15_7?

a.)

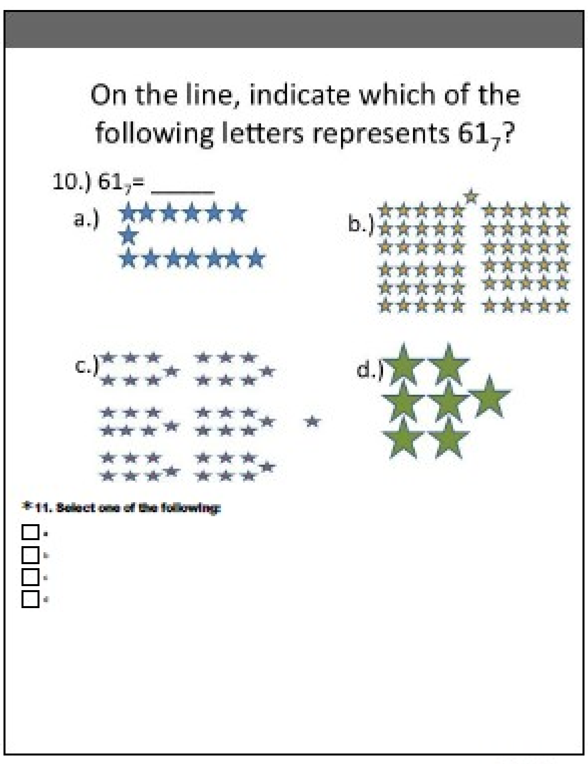
b.)

c.)

d.)

*10. Select one of the following:

Page 99



On the line, indicate which of the following letters represents 61_7?

10.) 61_7= _____

a.)

b.)

c.)

d.)

*11. Select one of the following:

Page 100

Appendix G

*1. Student ID#:

Introduction

You have already learned in a previous lesson how to count objects in base-7.

Today you will learn more about counting and grouping objects in base-7.

Page 1

Instructions

Please read and study the following pictures that explain the basics of grouping numbers in a different numeration system than you are used to.

After you have finished studying the pictures, please answer all of the questions that appear at the end of this lesson.

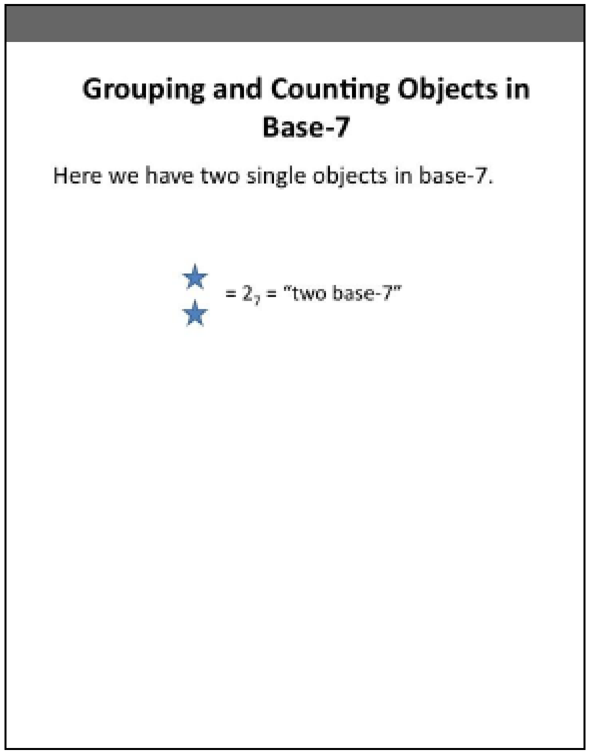
Page 2

Grouping and Counting Objects in Base-7

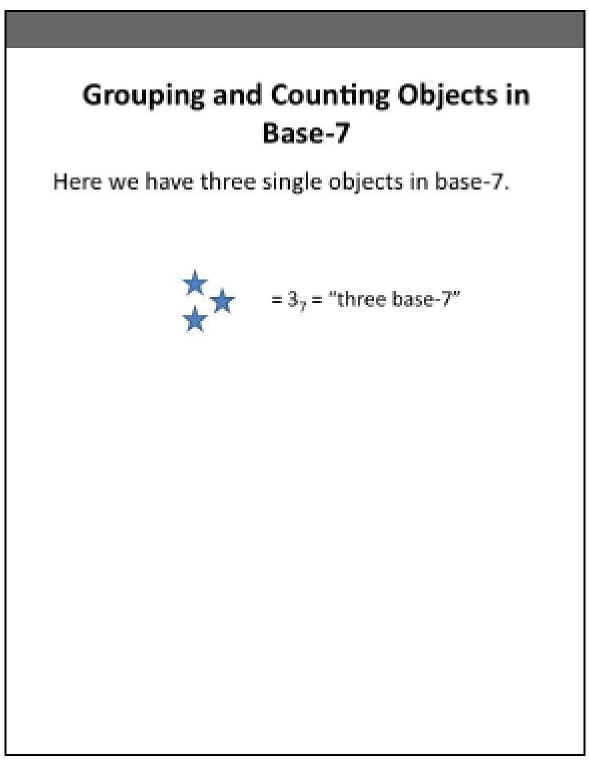
Here we have a single object in base-7.

★ = 1_7 = "one base-7"

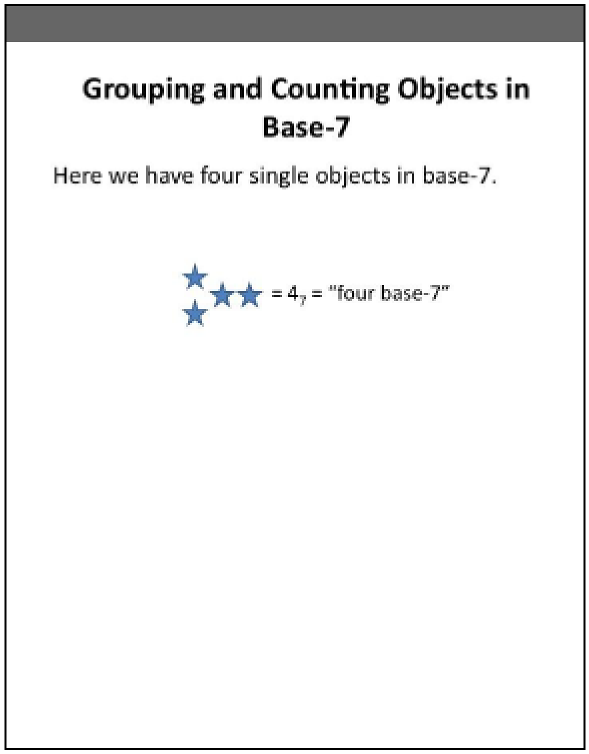
Page 3


Grouping and Counting Objects in Base-7
Here we have two single objects in base-7.
= 2_7 = "two base-7"

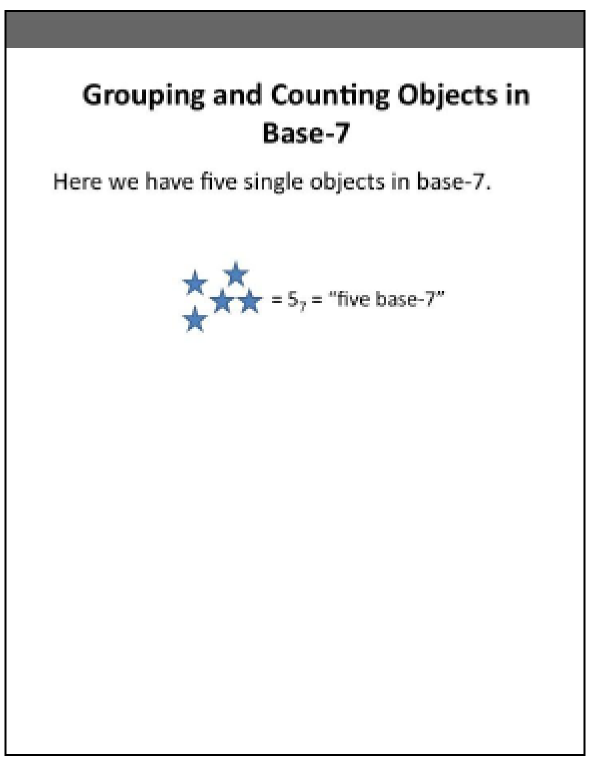
Page 4


Grouping and Counting Objects in Base-7
Here we have three single objects in base-7.
= 3_7 = "three base-7"

Page 5


Grouping and Counting Objects in Base-7
Here we have four single objects in base-7.
= 4_7 = "four base-7"

Page 6


Grouping and Counting Objects in Base-7
Here we have five single objects in base-7.
= 5_7 = "five base-7"

Page 7

Grouping and Counting Objects in Base-7

Here we have six single objects in base-7.

$= 6_7 =$ "six base-7"

Page 8

Grouping and Counting Objects in Base-7

If we add another single object to six single objects in base-7 we can then make a group of objects.

Therefore, here we have one group of objects in base-7.

$= 10_7 =$ "one-zero base-7"

Page 9

Grouping and Counting Objects in Base-7

Here we have one group of objects and one single object in base-7.

$= 11_7$ = "one-one base-7"

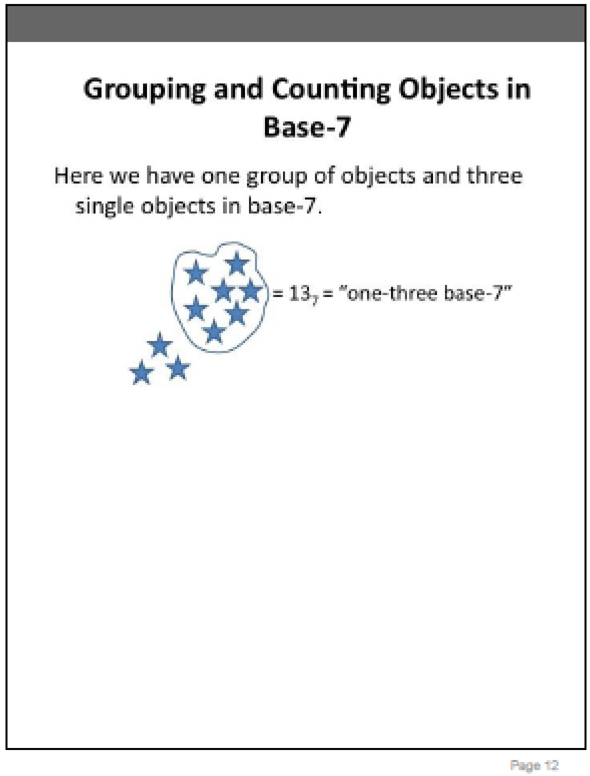
Page 10

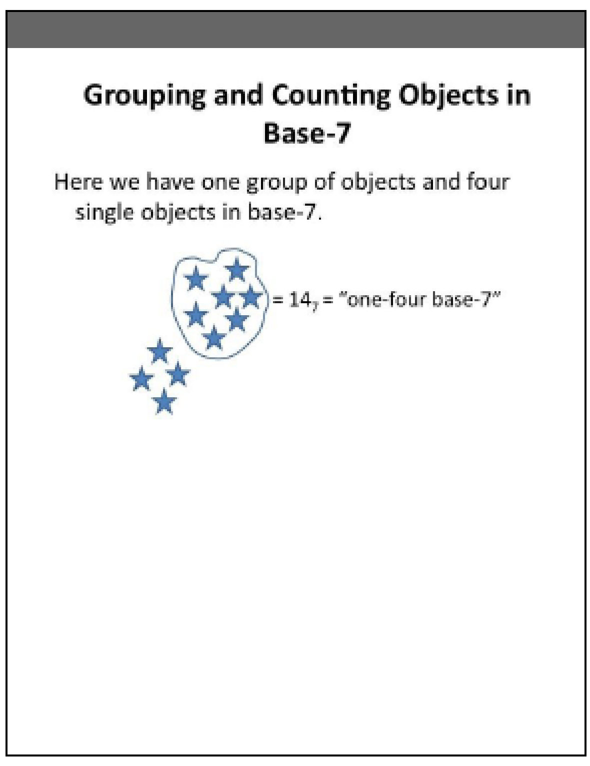
Grouping and Counting Objects in Base-7

Here we have one group of objects and two single objects in base-7.

$= 12_7$ = "one-two base-7"

Page 11


Grouping and Counting Objects in Base-7
Here we have one group of objects and three single objects in base-7.
= 13_7 = "one-three base-7"
Page 12


Grouping and Counting Objects in Base-7
Here we have one group of objects and four single objects in base-7.
= 14_7 = "one-four base-7"
Page 13

Grouping and Counting Objects in Base-7

Here we have one group of objects and five single objects in base-7.

= 15_7 = "one-five base-7"

Page 14

Grouping and Counting Objects in Base-7

Here we have one group of objects and six single objects in base-7.

= 16_7 = "one-six base-7"

Page 15

Grouping and Counting Objects in Base-7

If we add another single object to six single objects, we can then make another group of objects in base-7.

= 20_7 = "two-zero base-7"

Here we have two groups of objects and no single objects in base-7.

Page 16

Grouping and Counting Objects in Base-7

Here we have two groups of objects and one single object in base-7.

= 21_7 = "two-one base-7"

Page 17

Grouping and Counting Objects in Base-7

Here we have two groups of objects and two single objects in base-7.

$= 22_7$ = "two-two base-7"

Page 18

Grouping and Counting Objects in Base-7

Here we have two groups of objects and three single objects in base-7.

$= 23_7$ = "two-three base-7"

Page 19

Grouping and Counting Objects in Base-7

Here we have two groups of objects and four single objects in base-7.

$= 24_7$ = "two-four base-7"

Grouping and Counting Objects in Base-7

Here we have two groups of objects and five single objects in base-7.

$= 25_7$ = "two-five base-7"

Grouping and Counting Objects in Base-7

Here we have two groups of objects and six single objects in base-7.

= 26_7 = "two-six base-7"

Page 22

Grouping and Counting Objects in Base-7

If we add another single object to the six single objects, we can then make another group of objects in base-7.

= 30_7 = "three-zero base-7"

Here we have three groups of objects and no single objects in base-7.

Page 23

Grouping and Counting Objects in Base-7

Here we have three groups and one single object in base-7.

= 31_7 = "three-one base-7"

Page 24

Grouping and Counting Objects in Base-7

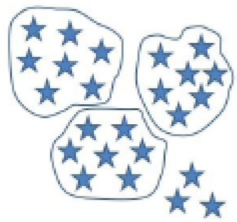
Here we have three groups and two single objects in base-7.

= 32_7 = "three-two base-7"

Page 25

Grouping and Counting Objects in Base-7

Here we have three groups and three single objects in base-7.

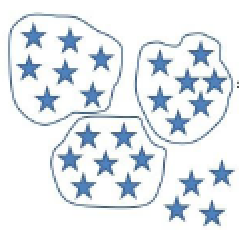


$= 33_7$ = "three-three base-7"

Page 26

Grouping and Counting Objects in Base-7

Here we have three groups and four single objects in base-7.

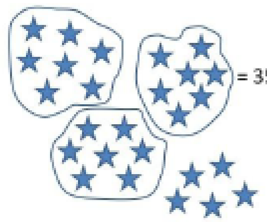


$= 34_7$ = "three-four base-7"

Page 27

Grouping and Counting Objects in Base-7

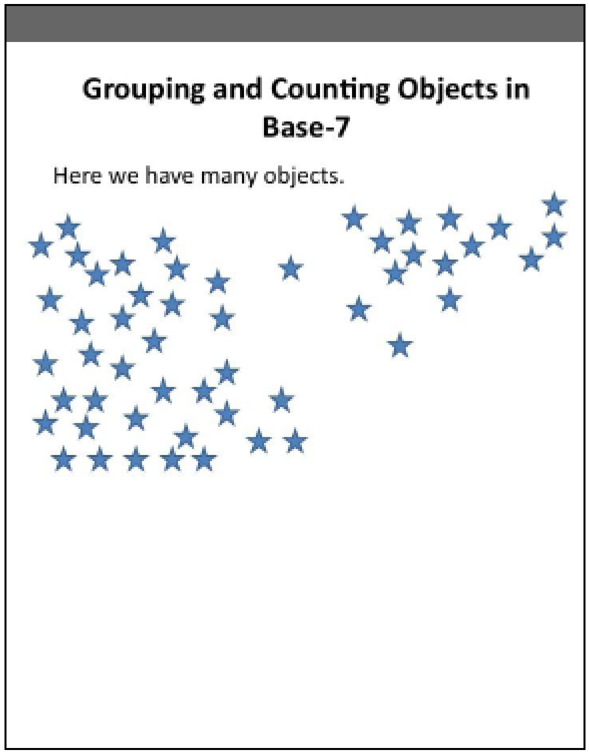
Here we have three groups and five single objects in base-7.



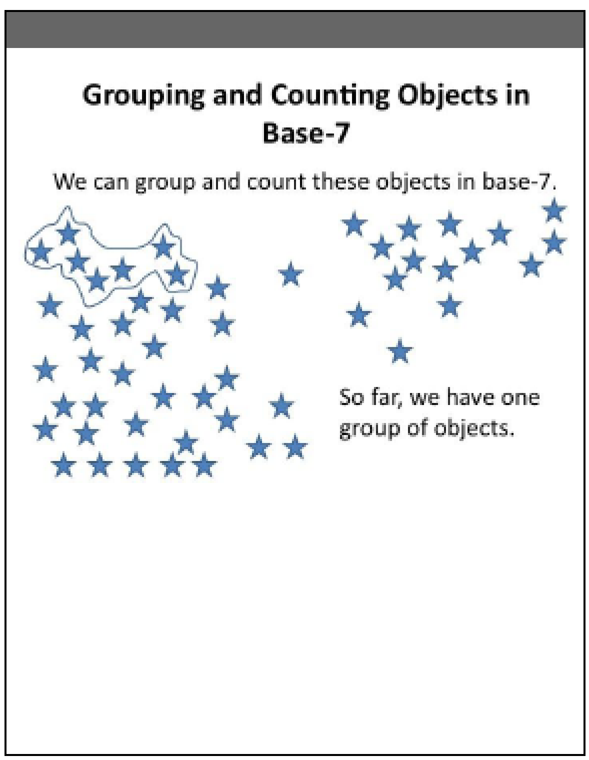
$= 35_7 =$ "three-five base-7"

Objects can be counted in base-7 by grouping them into groups as demonstrated by the previous pictures.

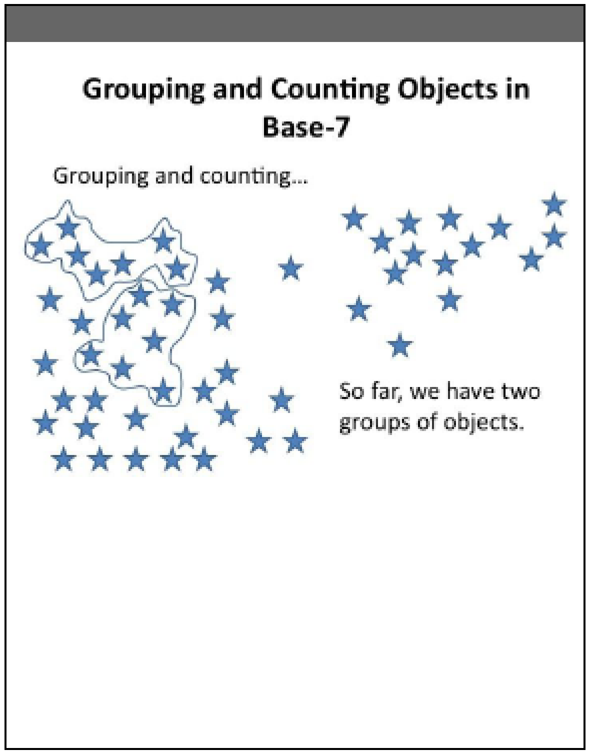
Study how objects are grouped and counted in base-7 in the following pictures.



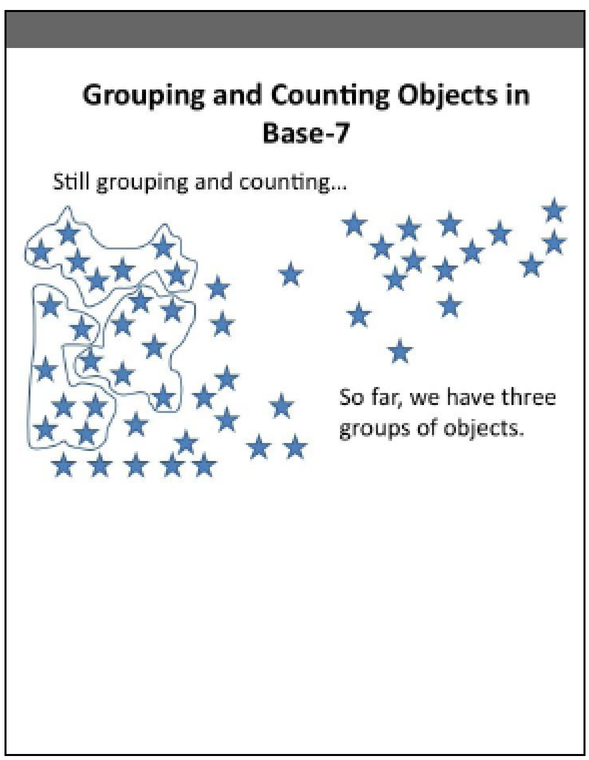
Page 30



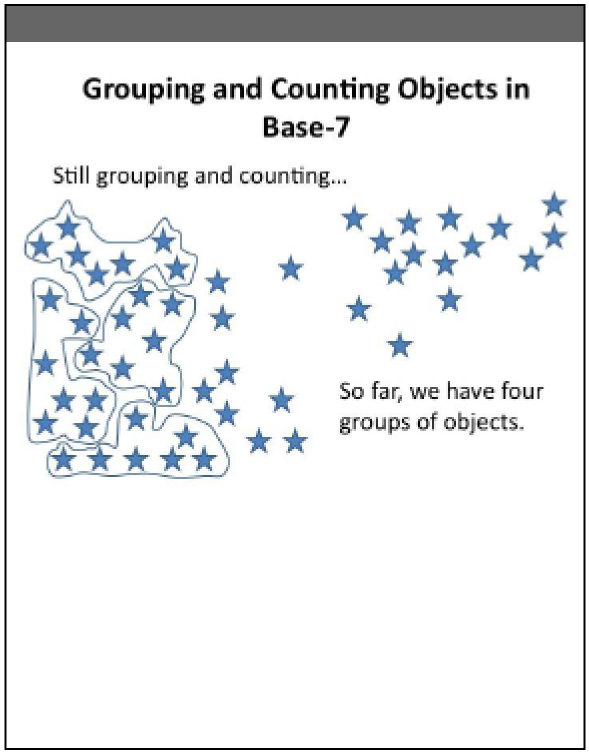
Page 31



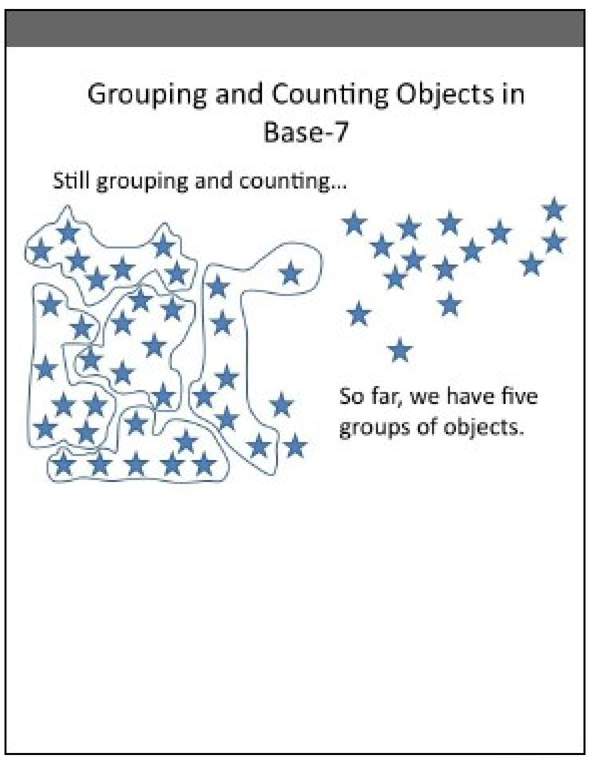
Page 32



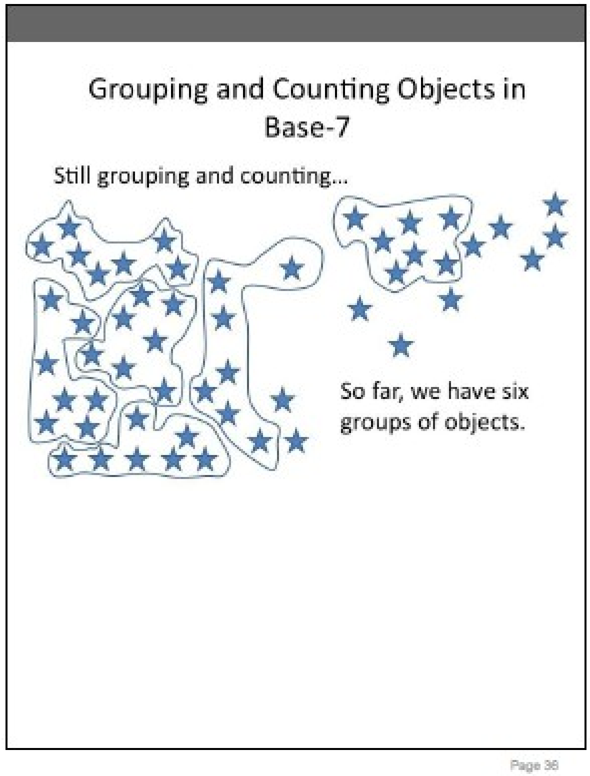
Page 33

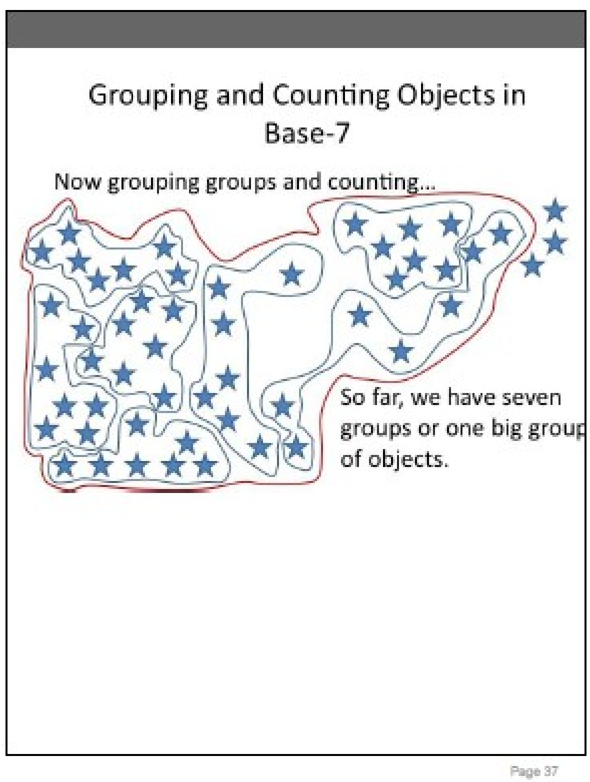


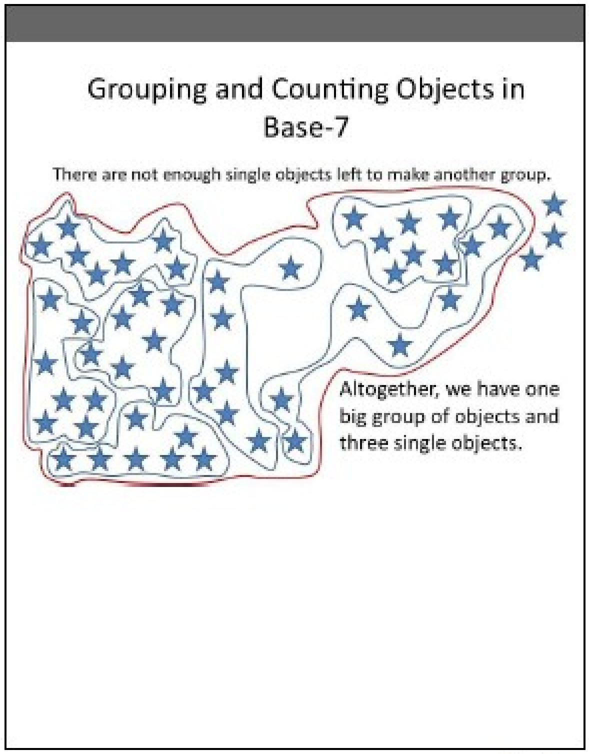
Page 34



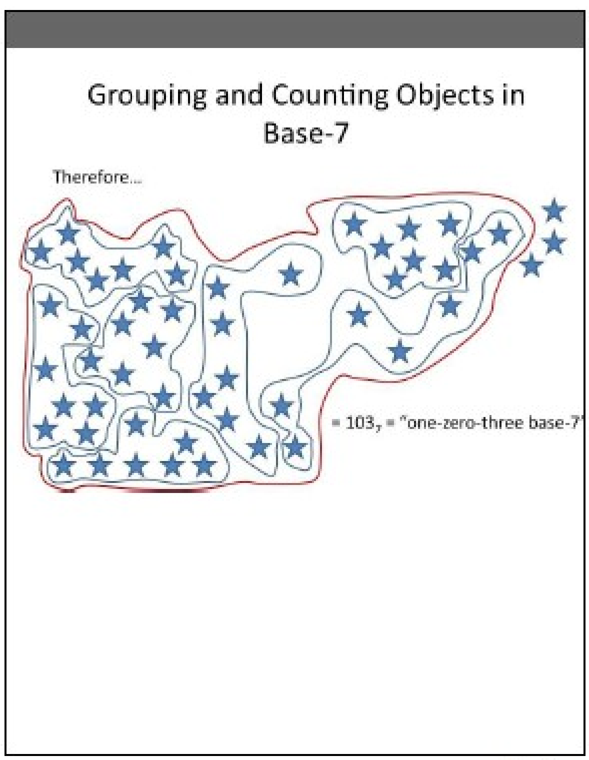
Page 35


Grouping and Counting Objects in Base-7
Still grouping and counting…
So far, we have six groups of objects.
Page 36

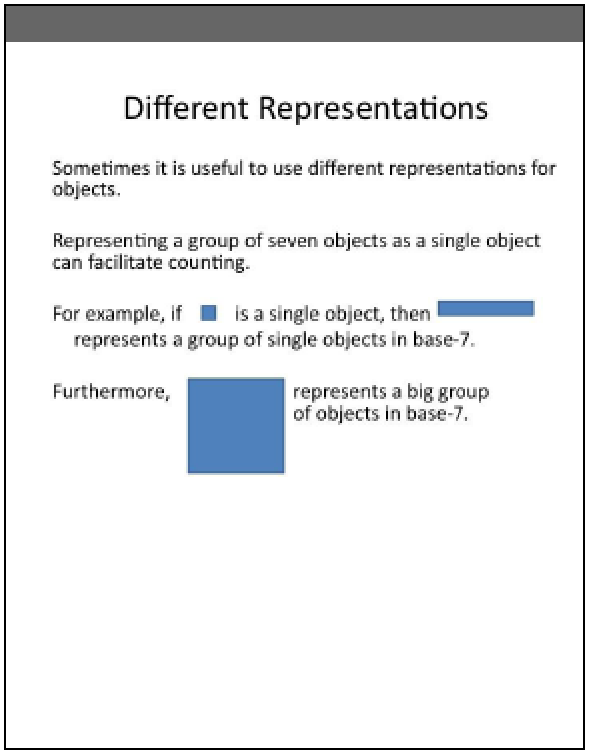

Grouping and Counting Objects in Base-7
Now grouping groups and counting…
So far, we have seven groups or one big group of objects.
Page 37



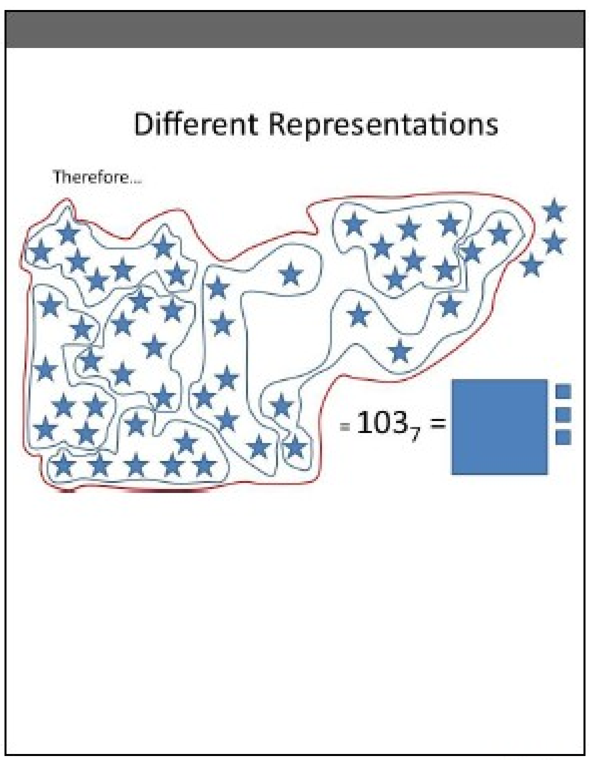
Page 38



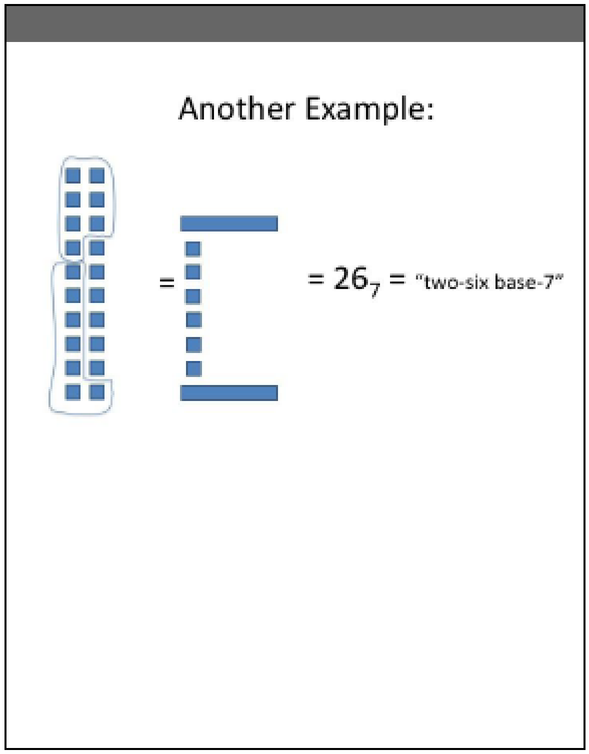
Page 39



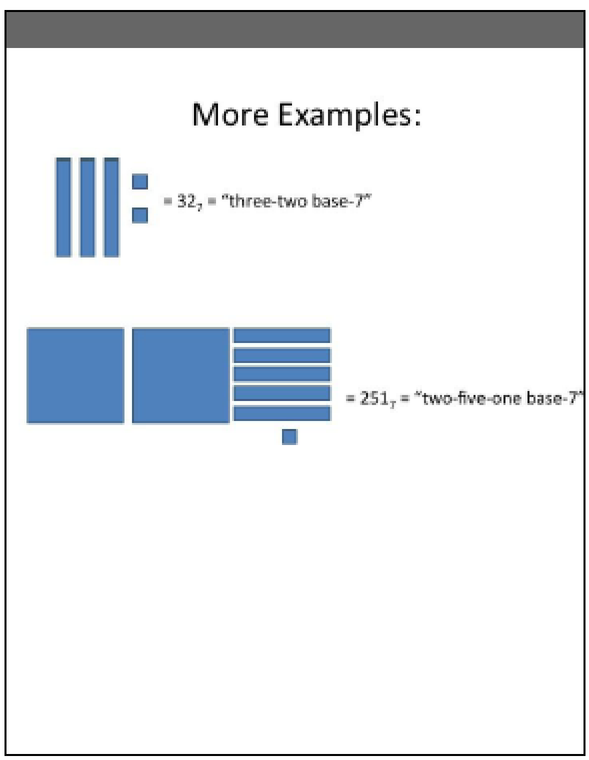
Page 40



Page 41



Page 42



Page 43

Appendix H

Lesson Questions

This section contains 10 questions on grouping and counting objects in base-7.

Page 44

Instructions

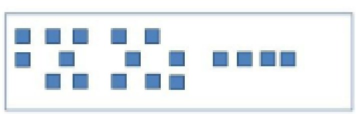
Please answers these questions to the best of your ability.

You will not be able to proceed to the next question until you answer the question you are on.

You may refer back to the lesson at any time.

Page 45

Examine the objects in the box and answer the following questions.



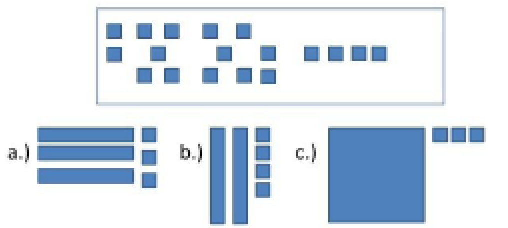
*2. How many groups of objects can you make in base-7?

*3. How many big groups of objects can you make in base-7?

*4. In words, how many objects are there in base-7?

Page 46

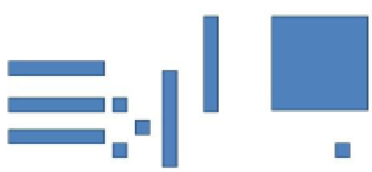
According to the representations presented during the lesson, write the letter that represents the same quantity as the objects in the box below?



*5. Write your answer here:

Page 47

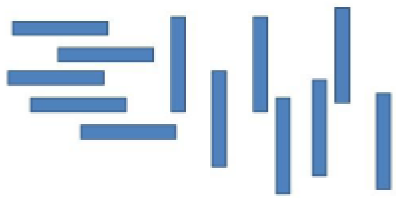
According to the representations presented during the lesson, write in words, the quantity in base-7 that is represented by the set of objects below.



*6. Write your answer in words here:

Page 48

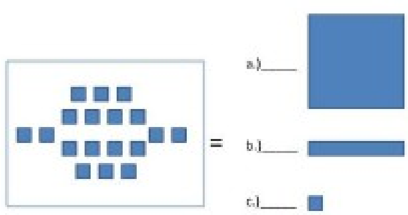
According to the representations presented during the lesson, write, in words, the quantity in base-7 that is represented by the set of objects below.



*7. Write your answer in words here:

Page 49

Indicate how many of each shape is required to create an equivalent representation of the objects in the box.

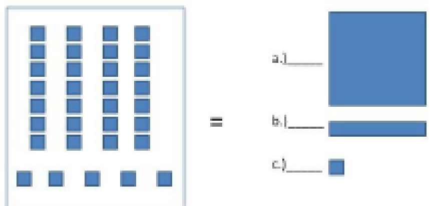


*8. How many of a.), b.) and c.) do you need?

Answer like this: a.)x, b.)xx, c.)xxx

Page 50

Indicate how many of each shape is required to create an equivalent representation of the objects in the box.

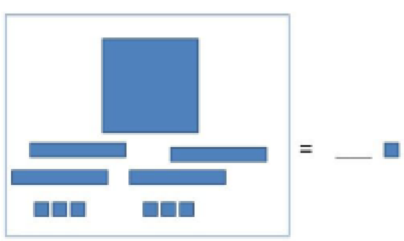


*9. How many of a.), b.) and c.) do you need?

Answer like this: a.)x, b.)xx, c.)xxx

Page 51

Indicate how many of the shape is required to create an equivalent representation of the objects in the box.

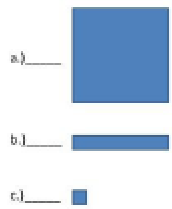


*10. Write your answer here:

Page 52

Indicate how many of each shape is required to create an equivalent representation of the number.

$561_7 =$

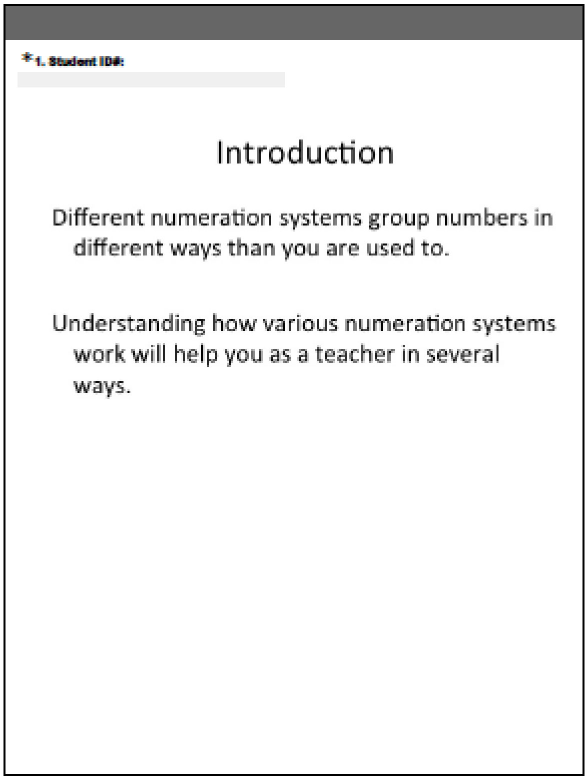


11. How many of a.), b.) and c.) do you need?

Answer like this: a.)x, b.)xx, c.)xxx

Page 53

Appendix I


*1. Student ID#:
Introduction
Different numeration systems group numbers in different ways than you are used to.
Understanding how various numeration systems work will help you as a teacher in several ways.

You will be able to better understand central concepts about counting and place value that will improve your ability to solve problems with addition, subtraction, multiplication, and division.

As a consequence, you will improve your ability to teach these concepts as well.

You have already learned in previous lessons how to count objects and group objects in base-7.

Today you will learn about adding objects in base-7.

Instructions

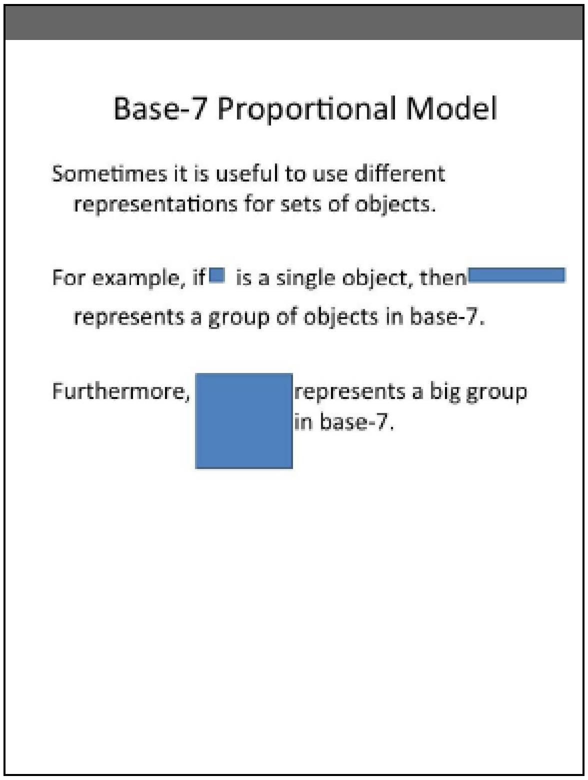
Please read and study the following screens that explain how to add quantities in base-7. The addition strategy that is used may be different than you are used to. The strategy is called combining sevens and ones.

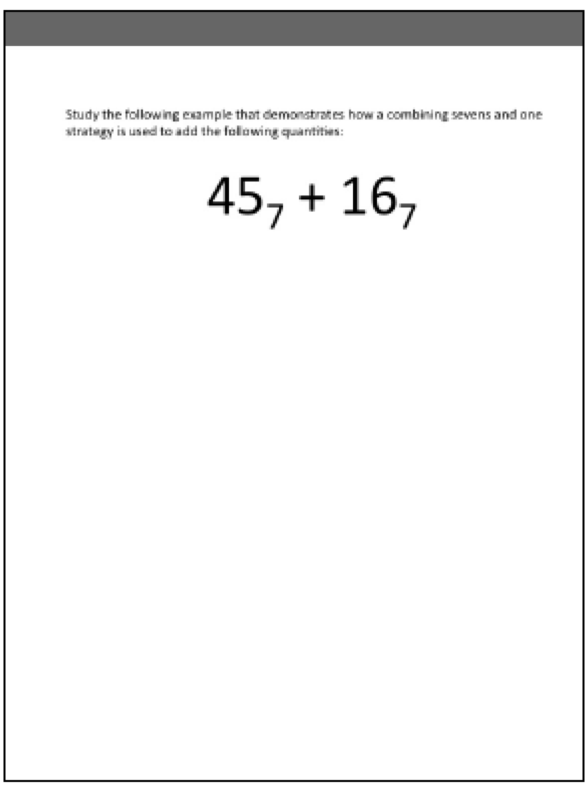
After you have finished studying the screens, please answer all of the questions that appear at the end of this lesson.

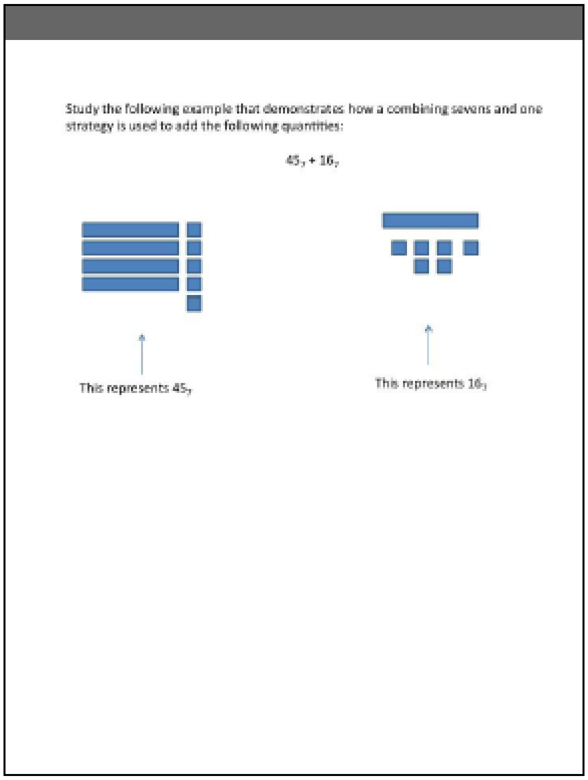
Adding in Base-7 Using a Combing Sevens and Ones Strategy

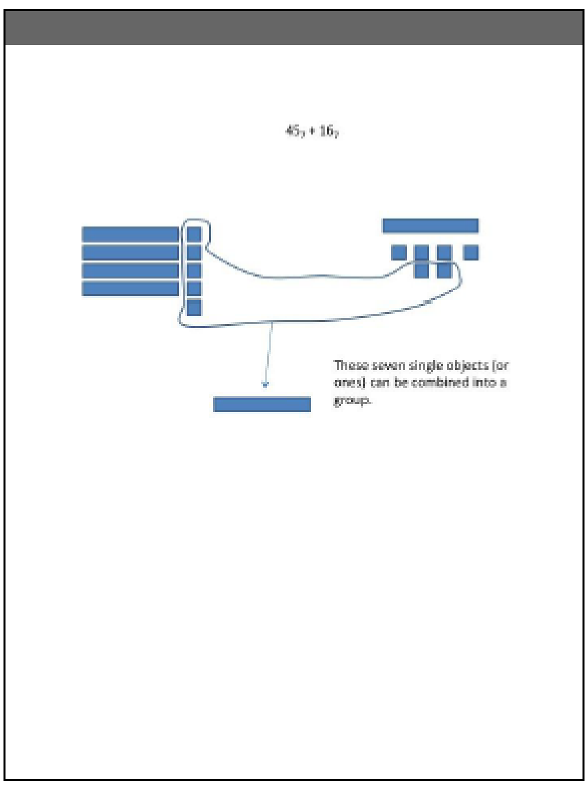
To use this particular addition strategy, you will need to remember the model that you learned about in a previous lesson.

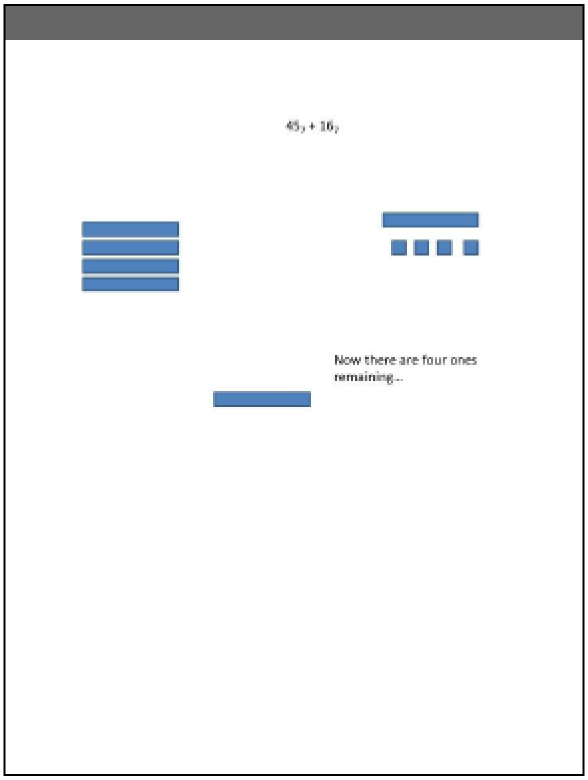
Please familiarize yourself with the model that is presented on the next screen.

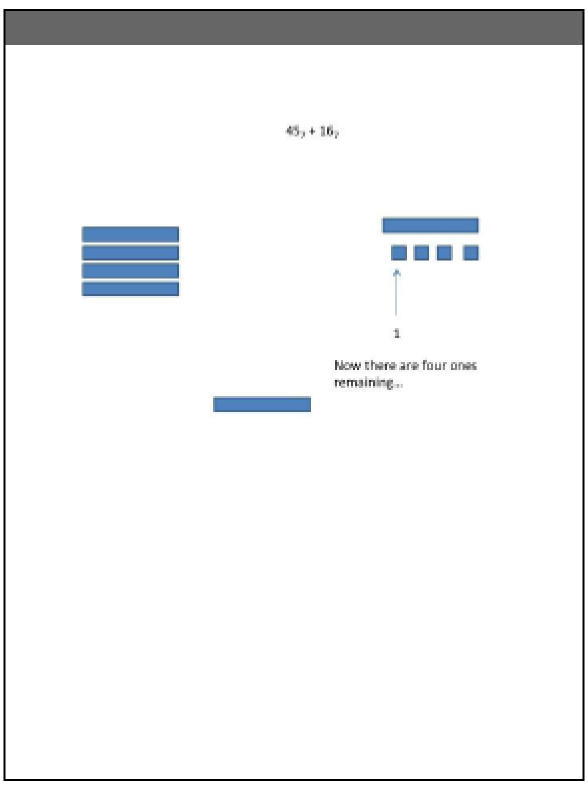

Base-7 Proportional Model
Sometimes it is useful to use different representations for sets of objects.
For example, if is a single object, then
represents a group of objects in base-7.
Furthermore, represents a big group in base-7.

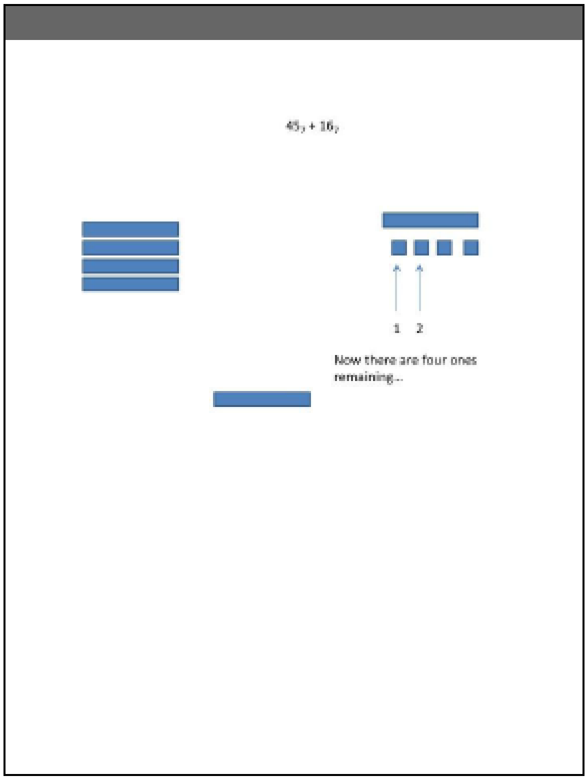

Study the following example that demonstrates how a combining sevens and one strategy is used to add the following quantities:
$45_7 + 16_7$

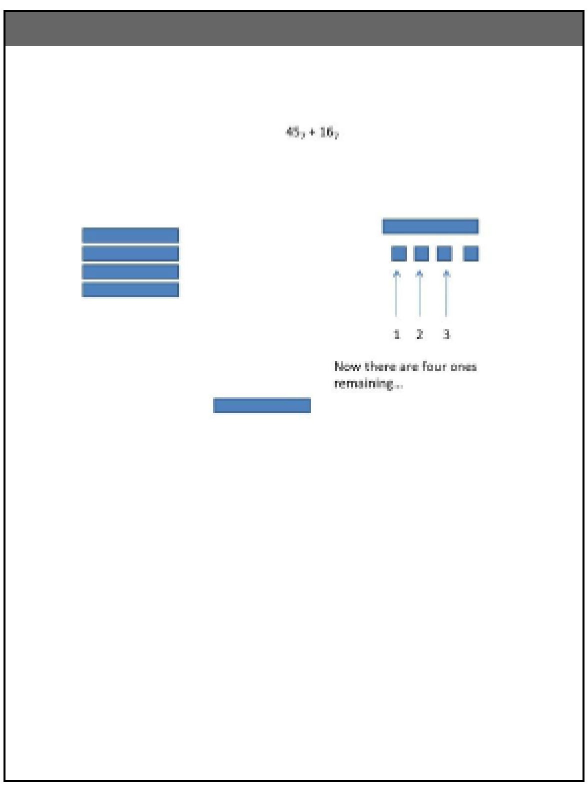

Study the following example that demonstrates how a combining sevens and one strategy is used to add the following quantities:
$45_7 + 16_7$
This represents 45_7
This represents 16_7

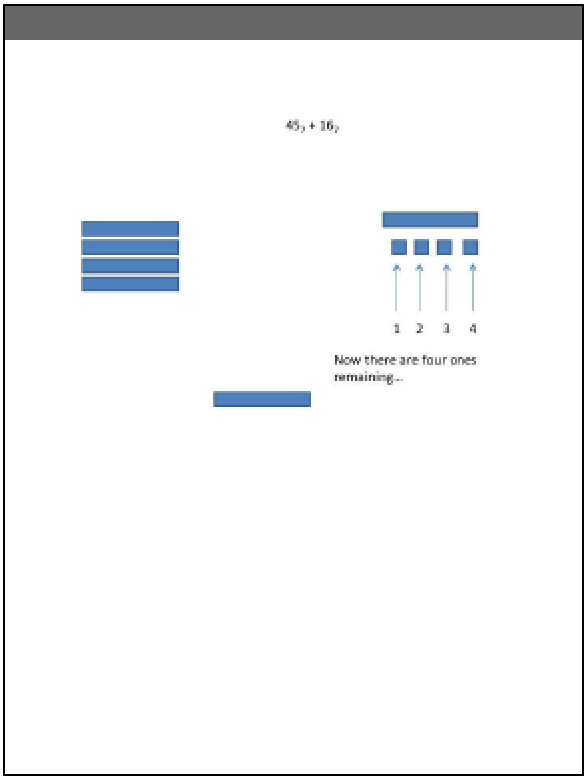

$45_7 + 16_7$
These seven single objects (or ones) can be combined into a group.

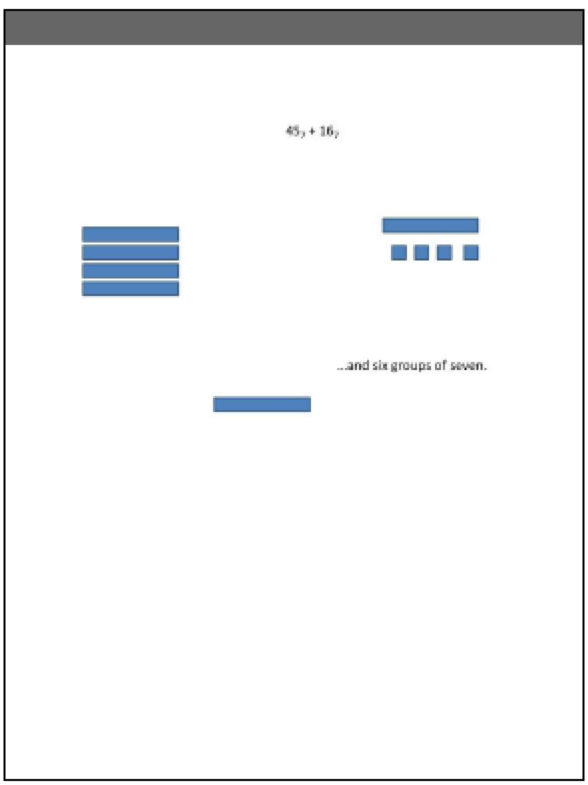

$45_7 + 16_7$
Now there are four ones remaining…

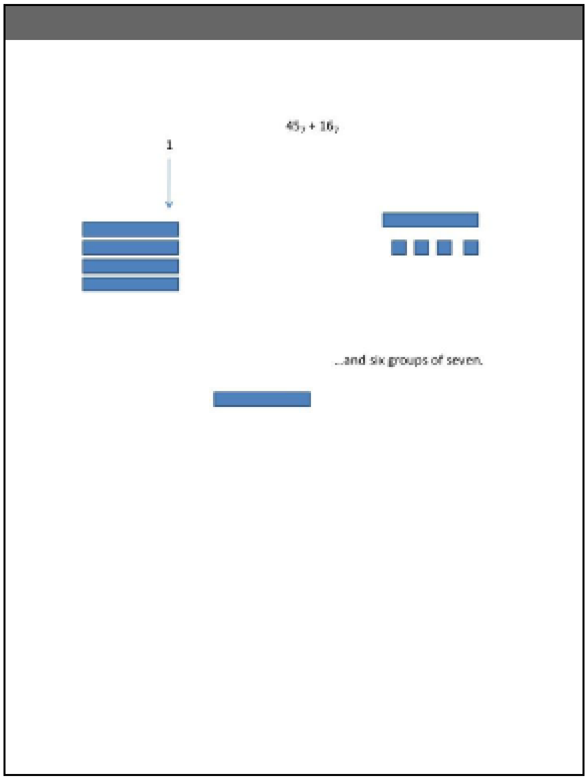

$45_7 + 16_7$
1
Now there are four ones remaining…

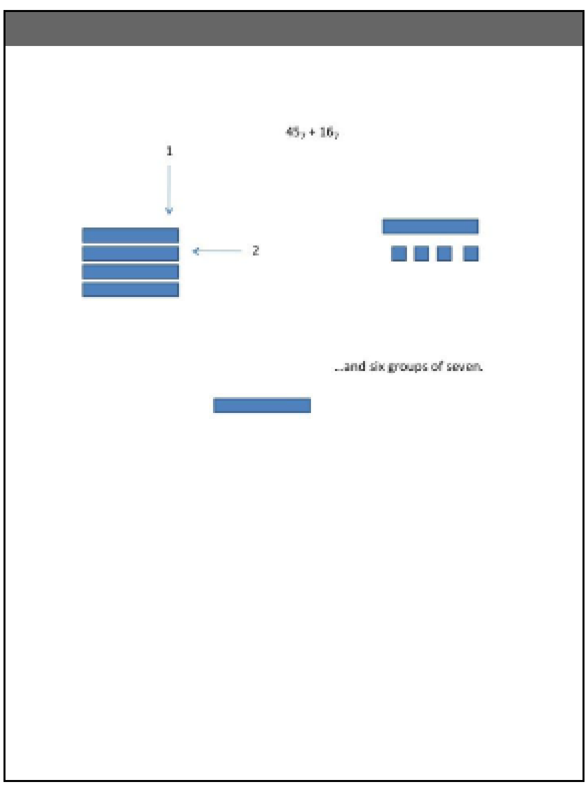

$45_7 + 16_7$
1 2
Now there are four ones remaining…

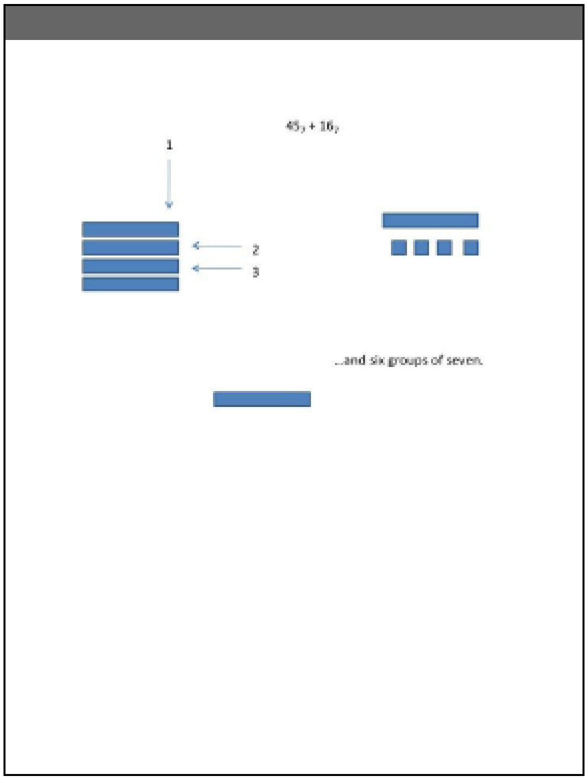

$45_7 + 16_7$
1 2 3
Now there are four ones remaining…

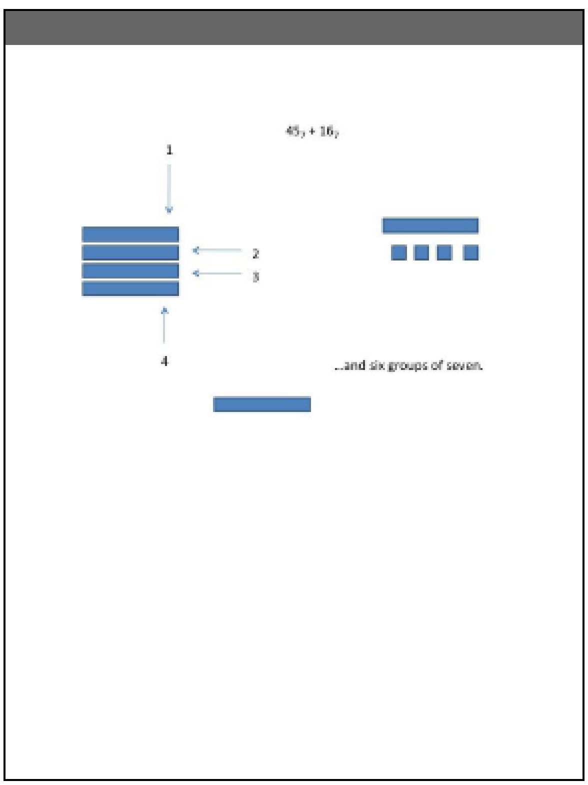

45₇ + 16₇
1 2 3 4
Now there are four ones remaining…

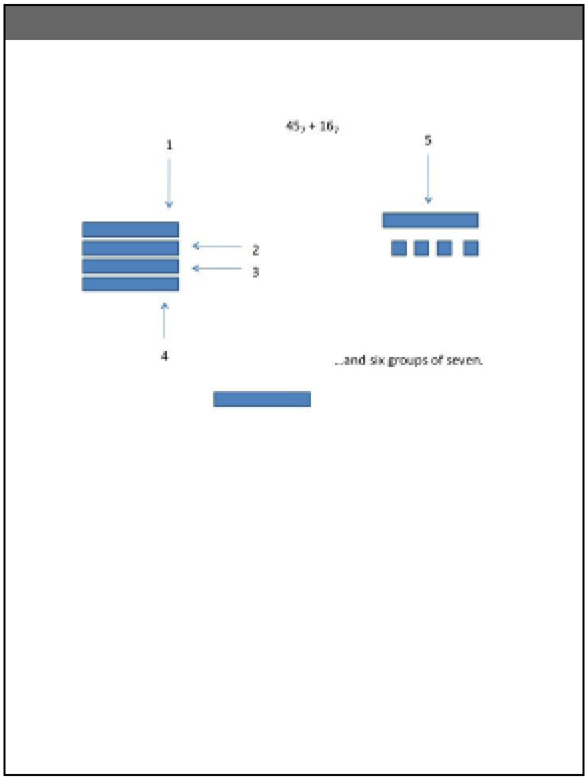

45₇ + 16₇
…and six groups of seven.

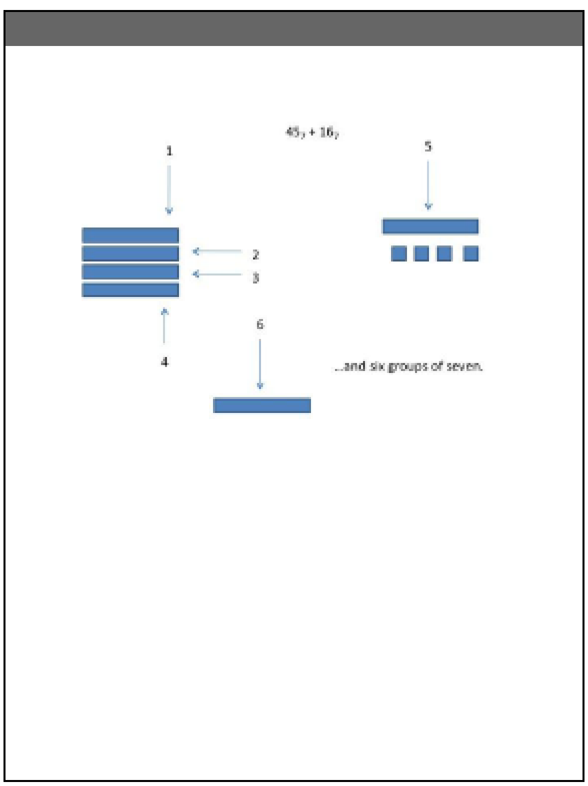

45₇ + 16₇
1
...and six groups of seven.

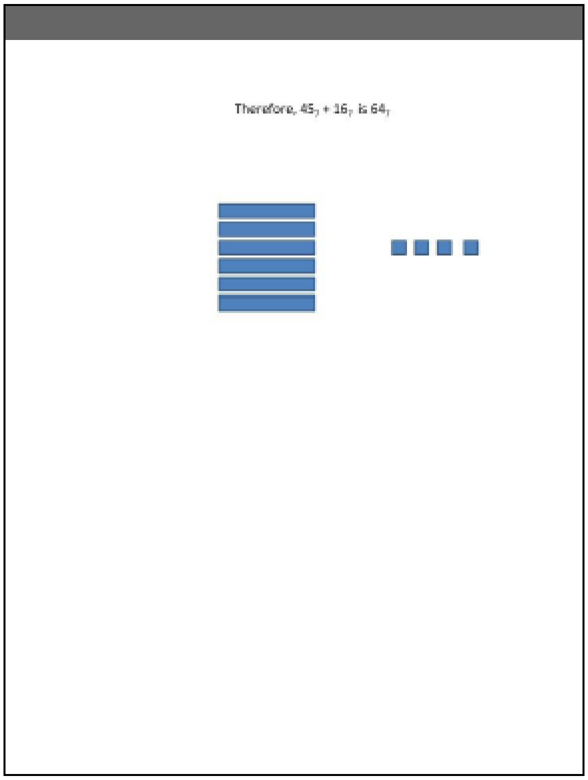

45₇ + 16₇
1
2
...and six groups of seven.

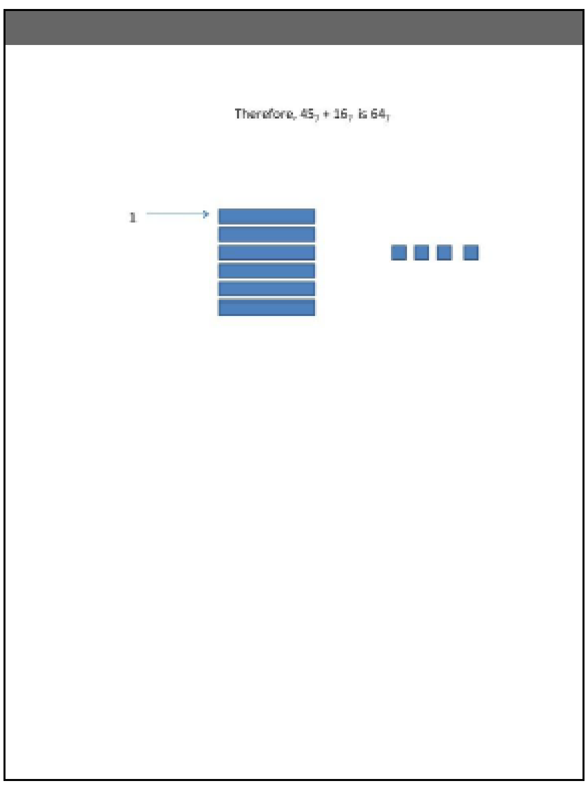

$45_7 + 16_7$
1
2
3
...and six groups of seven.

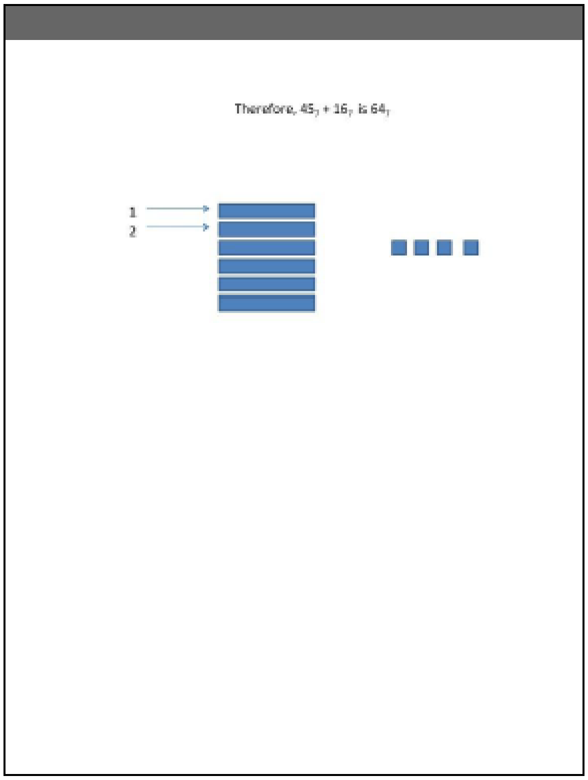

$45_7 + 16_7$
1
2
3
4
...and six groups of seven.

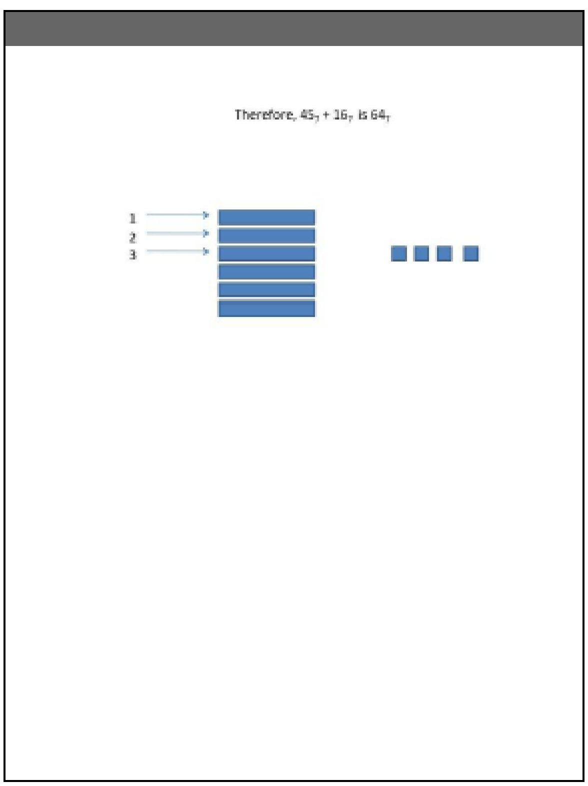

$45_7 + 16_7$
1
5
2
3
4
...and six groups of seven.

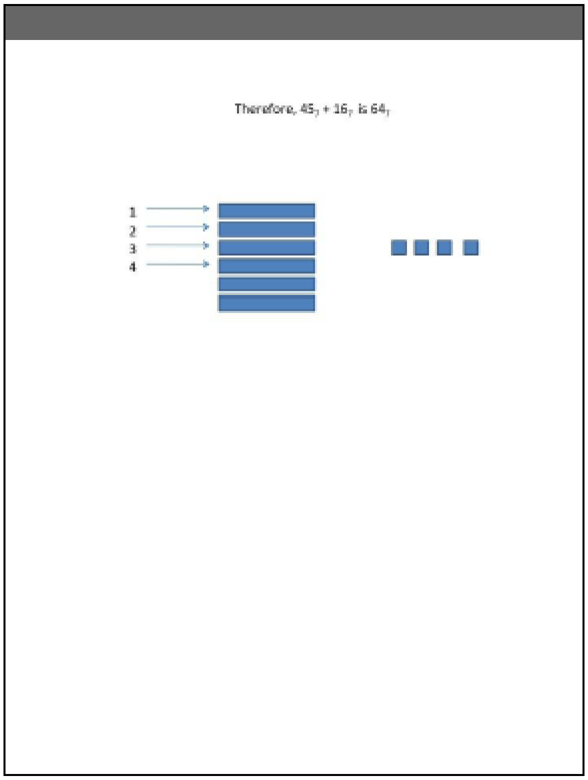

$45_7 + 16_7$
1
5
2
3
6
4
...and six groups of seven.

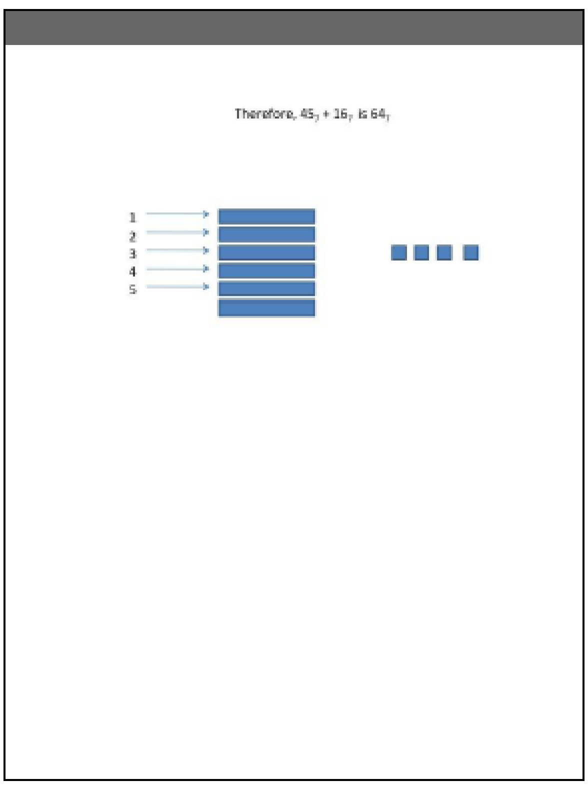

Therefore, 45₇ + 16₇ is 64₇

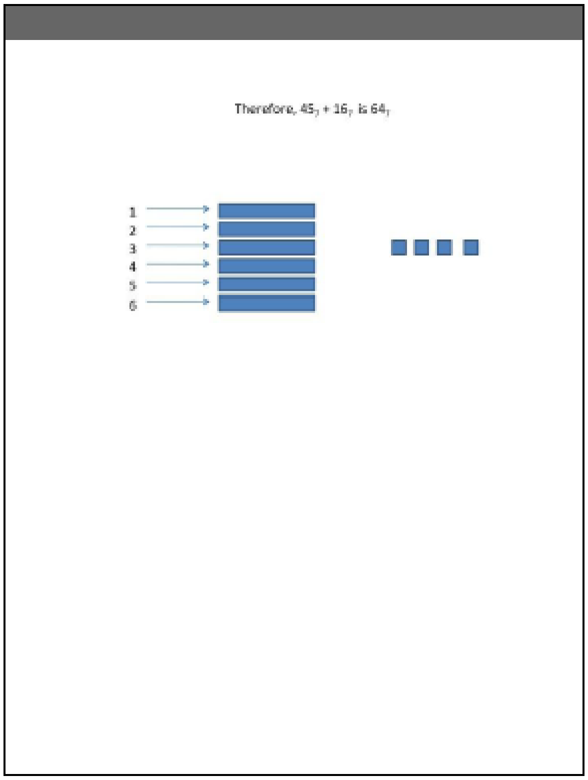

Therefore, 45₇ + 16₇ is 64₇
1

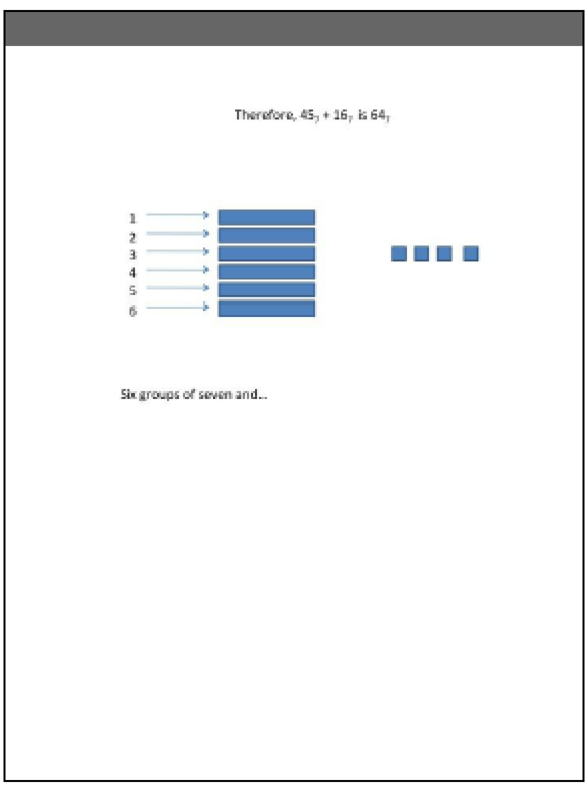

Therefore, $45_7 + 16_7$ is 64_7
1
2

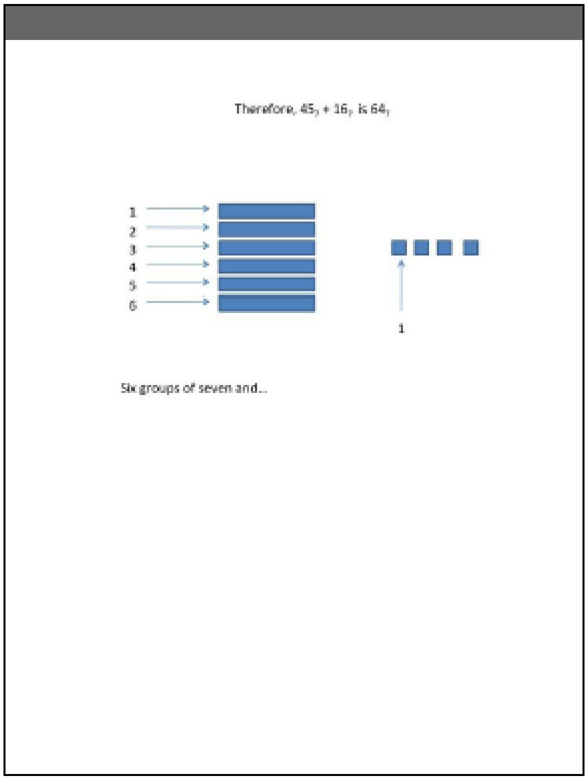

Therefore, $45_7 + 16_7$ is 64_7
1
2
3

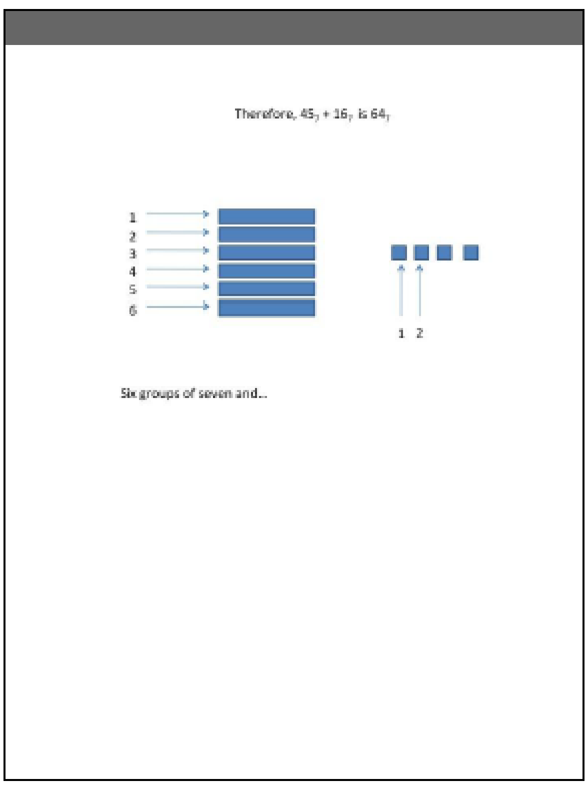

Therefore, $45_7 + 16_7$ is 64_7
1
2
3
4

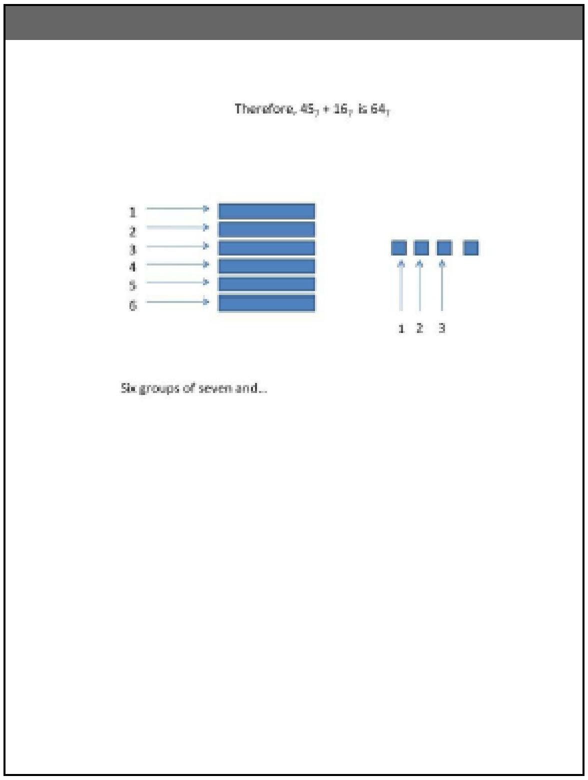

Therefore, $45_7 + 16_7$ is 64_7
1
2
3
4
5

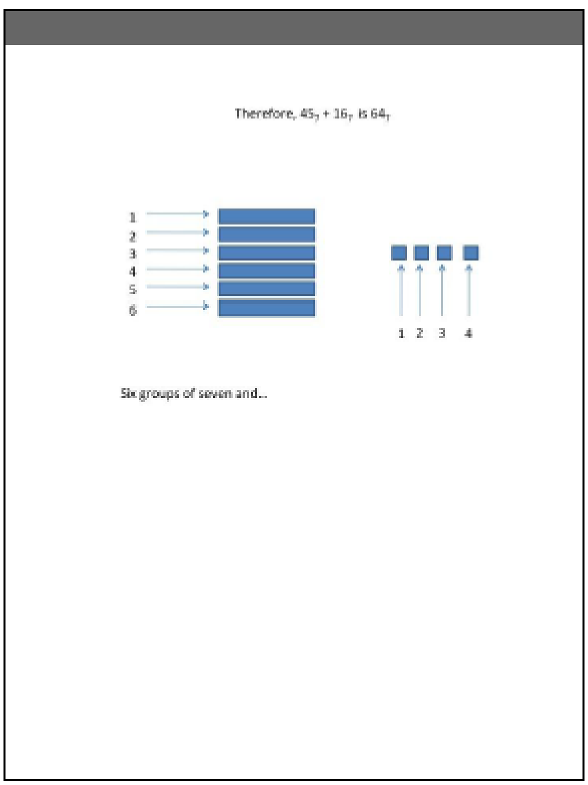

Therefore, 45₇ + 16₇ is 64₇
1
2
3
4
5
6


Therefore, 45₇ + 16₇ is 64₇
1
2
3
4
5
6
Six groups of seven and...

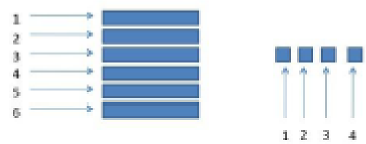

Therefore, 45₇ + 16₇ is 64₇
1
2
3
4
5
6
1
Six groups of seven and...


Therefore, 45₇ + 16₇ is 64₇
1
2
3
4
5
6
1 2
Six groups of seven and...


Therefore, $45_7 + 16_7$ is 64_7
1
2
3
4
5
6
1 2 3
Six groups of seven and…


Therefore, $45_7 + 16_7$ is 64_7
1
2
3
4
5
6
1 2 3 4
Six groups of seven and…

Therefore, $45_7 + 16_7$ is 64_7

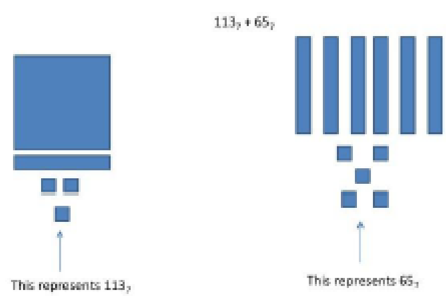


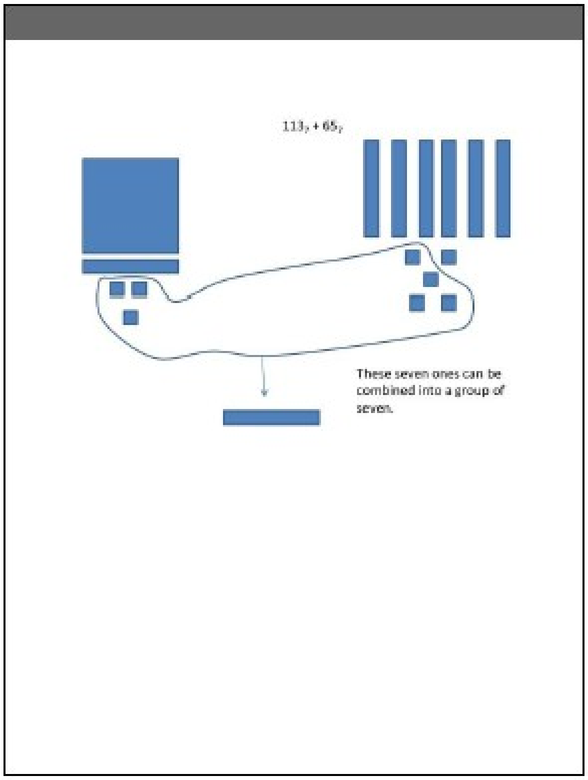
Six groups of seven and four ones or 64_7

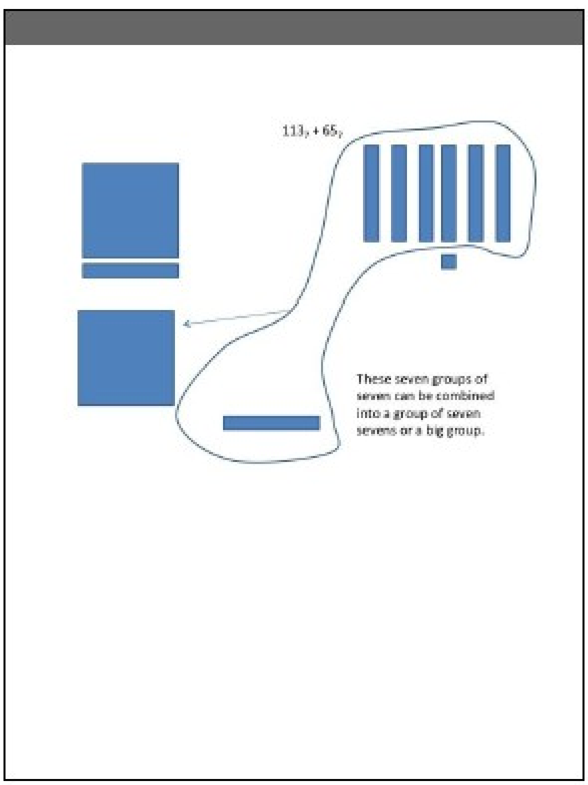
One more Example:

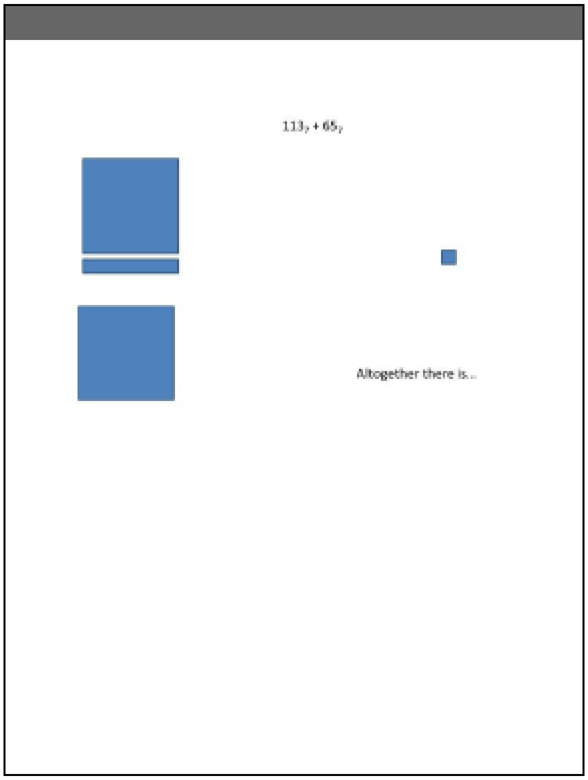
Please study the next example as well. Take your time to understand how the combining sevens and ones strategy works.

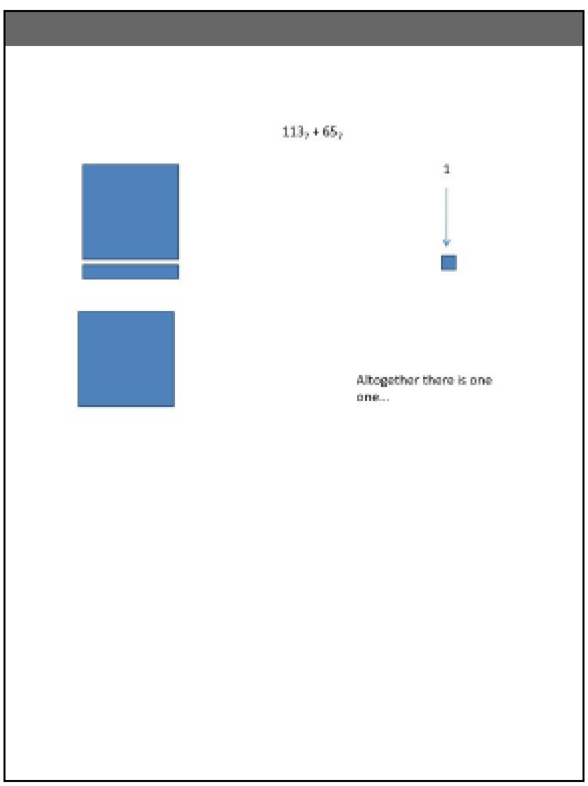
$113_7 + 65_7$

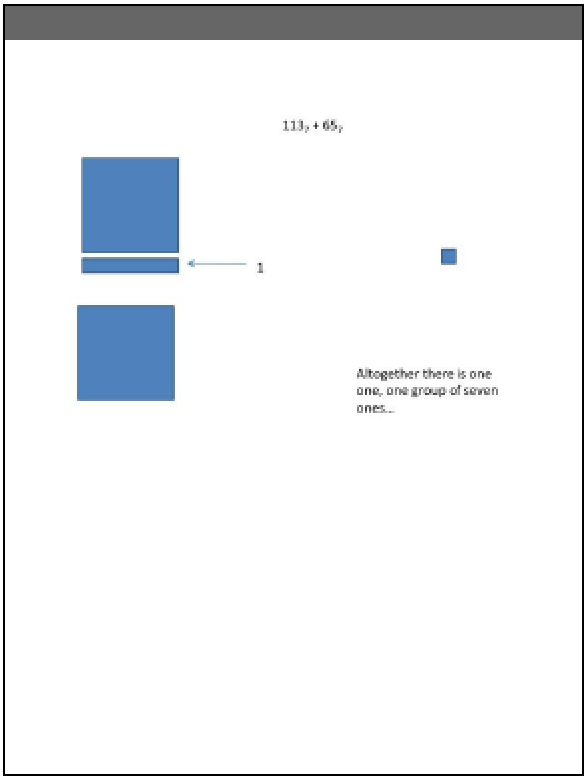


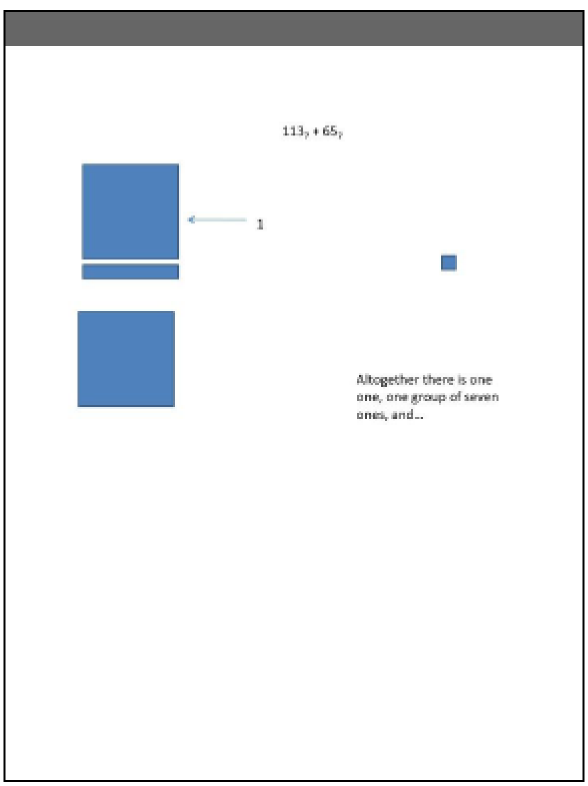

$113_7 + 65_7$
These seven ones can be combined into a group of seven.

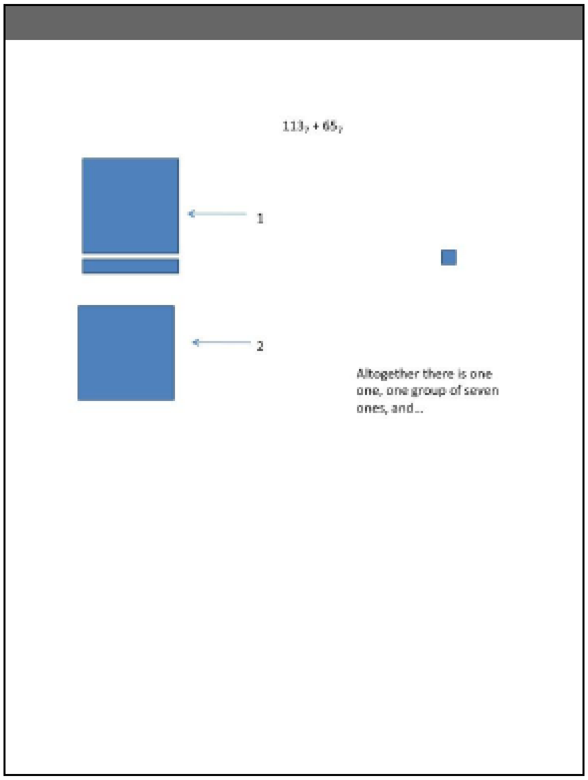

$113_7 + 65_7$
These seven groups of seven can be combined into a group of seven sevens or a big group.

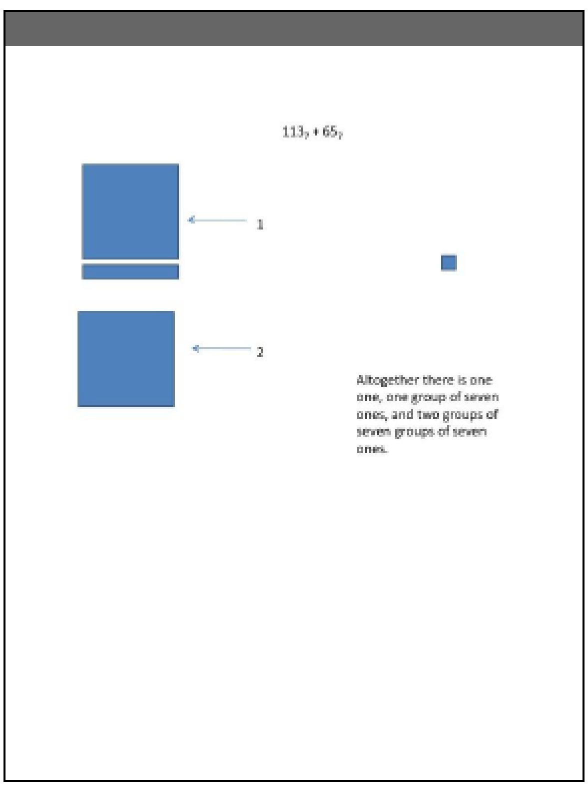

$113_7 + 65_7$
Altogether there is...

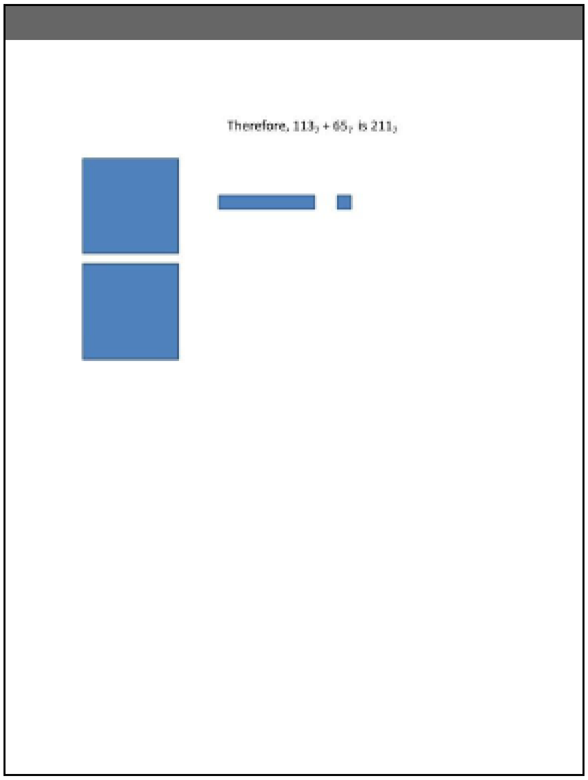

$113_7 + 65_7$
1
Altogether there is one one...


$113_7 + 65_7$
1
Altogether there is one one, one group of seven ones…


$113_7 + 65_7$
1
Altogether there is one one, one group of seven ones, and…


$113_7 + 65_7$
1
2
Altogether there is one one, one group of seven ones, and…


$113_7 + 65_7$
1
2
Altogether there is one one, one group of seven ones, and two groups of seven groups of seven ones.


Therefore, $113_7 + 65_7$ is 211_7

Appendix J

C3 – End of Lesson Questions

ID #:______________________________

1. Compute the following using the algorithm you learned.
   a. $125_7 + 66_7$

   b. $54_7 + 13_7$

c. $341_7 + 233_7$

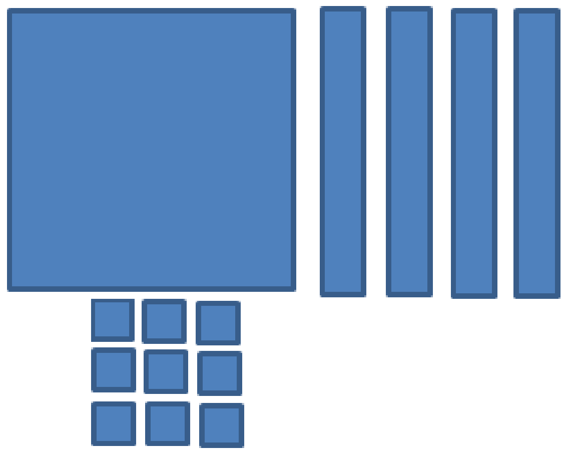
d. $126_7 + 51_7$

e. $43_7 + 14_7$

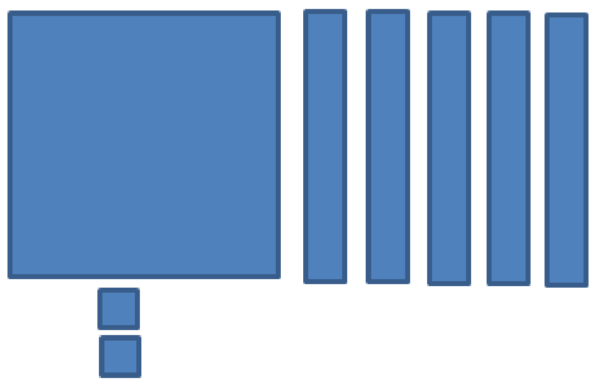
2. Circle the letter that represents the solution to the following problem:

$$124_7 + 25_7.$$

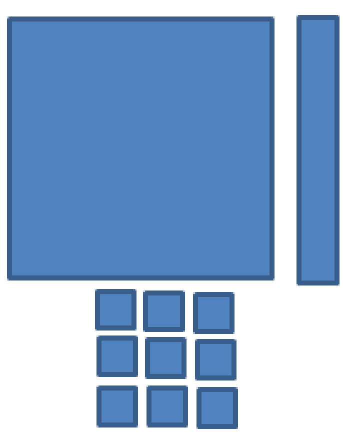
a.



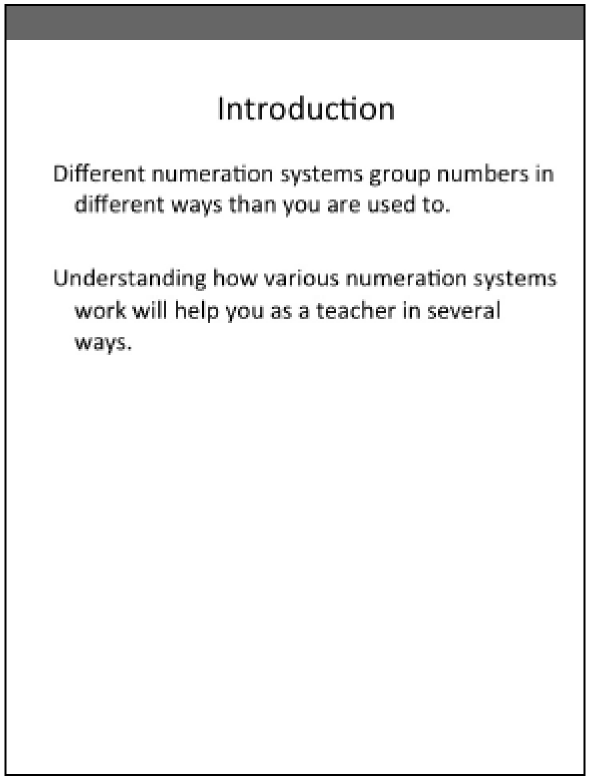
b.



c.



Appendix K


Introduction
Different numeration systems group numbers in different ways than you are used to.
Understanding how various numeration systems work will help you as a teacher in several ways.

You will be able to better understand central concepts about counting and place value that will improve your ability to solve problems with addition, subtraction, multiplication, and division.

As a consequence, you will improve your ability to teach these concepts as well.

You have already learned in previous lessons how to count objects, group objects, and add objects in base-7.

Today you will learn about subtracting objects in base-7.

Instructions

Please read and study the following screens that explain how to subtract quantities in base-7. The strategy that is used may be different than you are used to. The strategy is called combining sevens and ones.

After you have finished studying the screens, please answer all of the questions that appear at the end of this lesson.

Subtracting in Base-7 Using a Combing Sevens and Ones Strategy

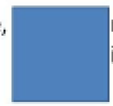
To use this particular subtraction strategy, you will need to remember the model that you learned about in a previous lesson.

Please familiarize yourself with the model that is presented on the next screen.

Base-7 Proportional Model

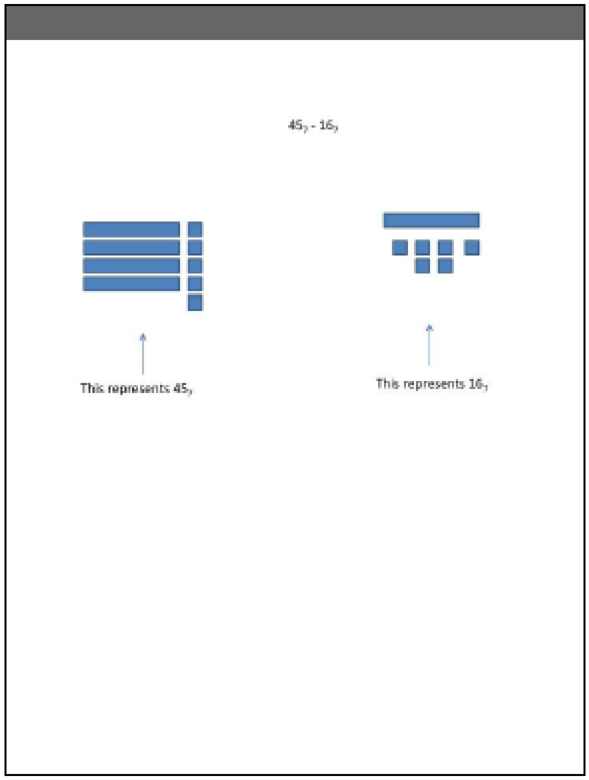
Sometimes it is useful to use different representations for sets of objects.

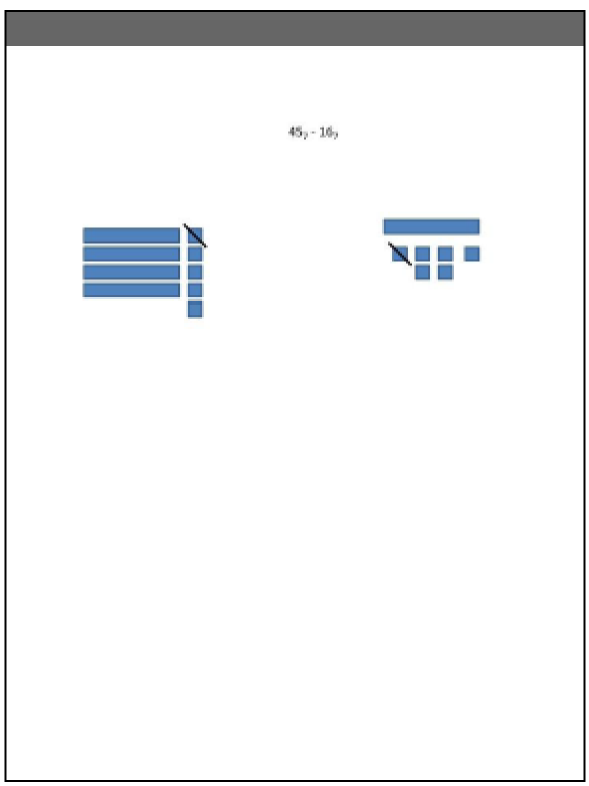
For example, if  is a single object, then  represents a group of objects in base-7.

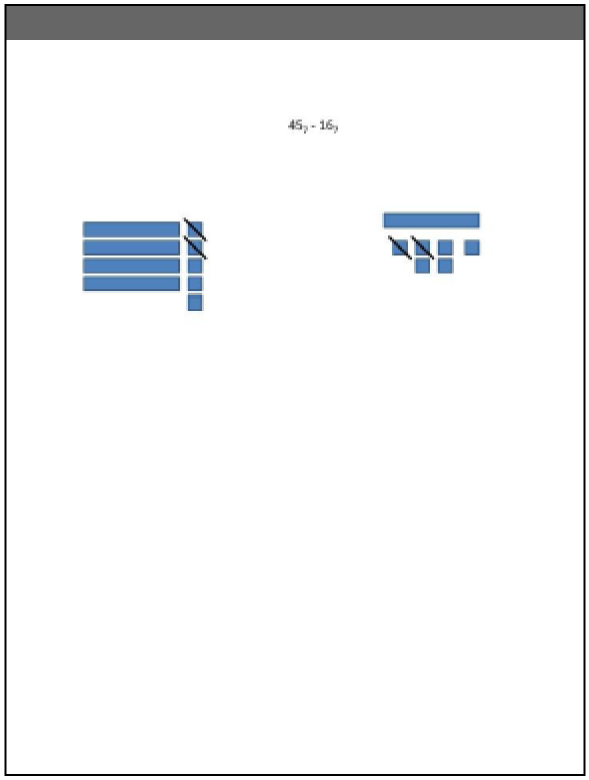
Furthermore,  represents a big group in base-7.

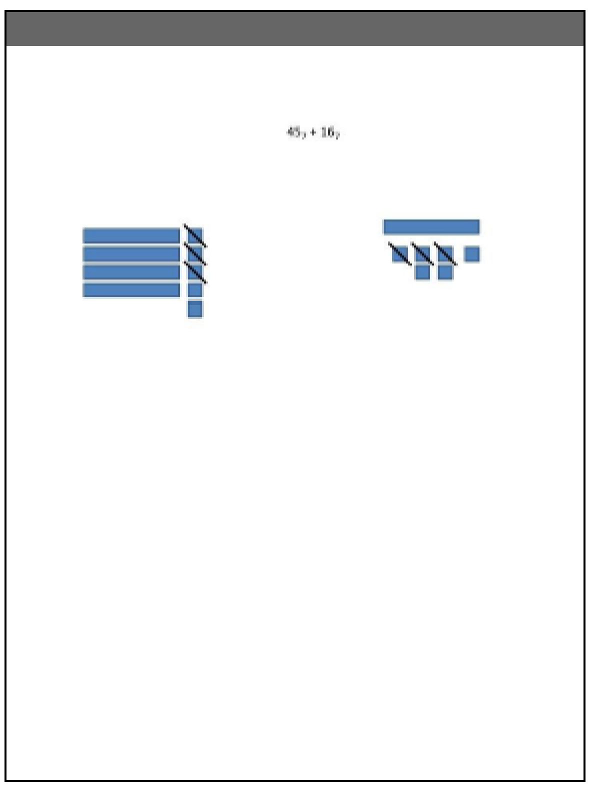
Study the following example that demonstrates how a combining sevens and one strategy is used to subtract the following quantities:

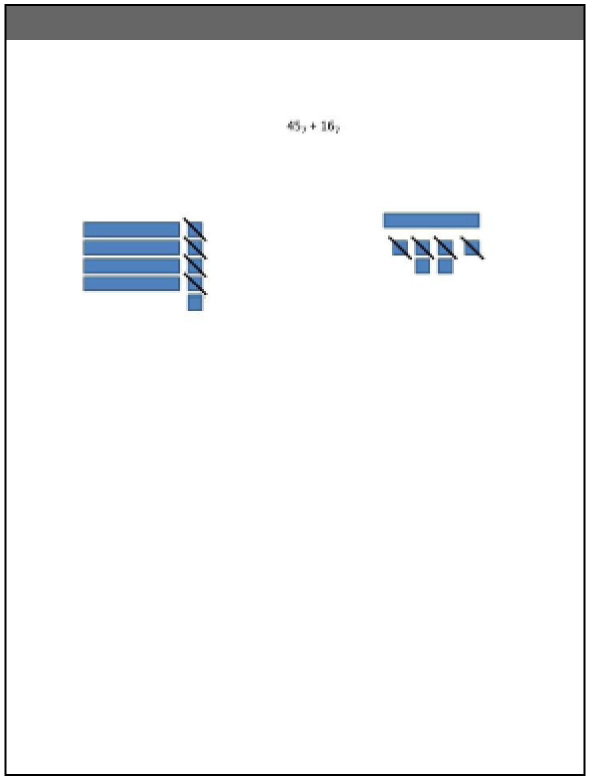
$$45_7 - 16_7$$

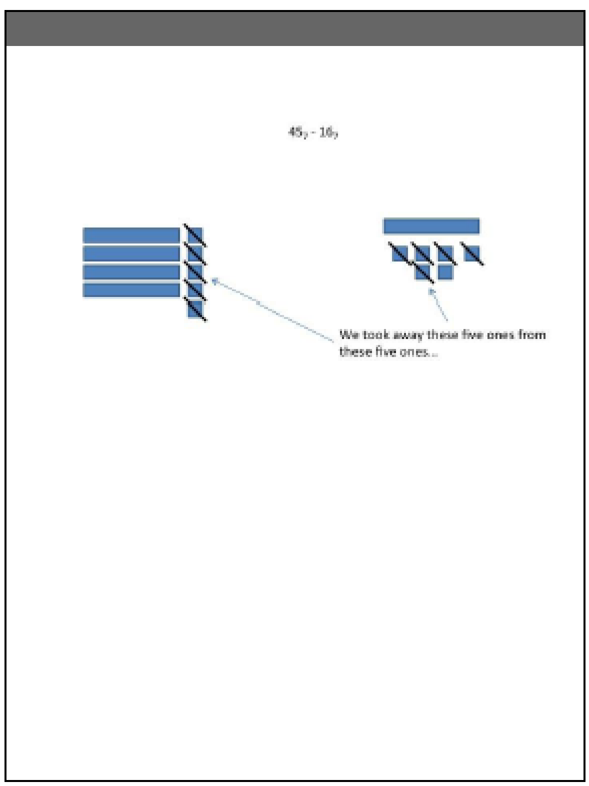

$45_7 - 16_7$
This represents 45_7
This represents 16_7

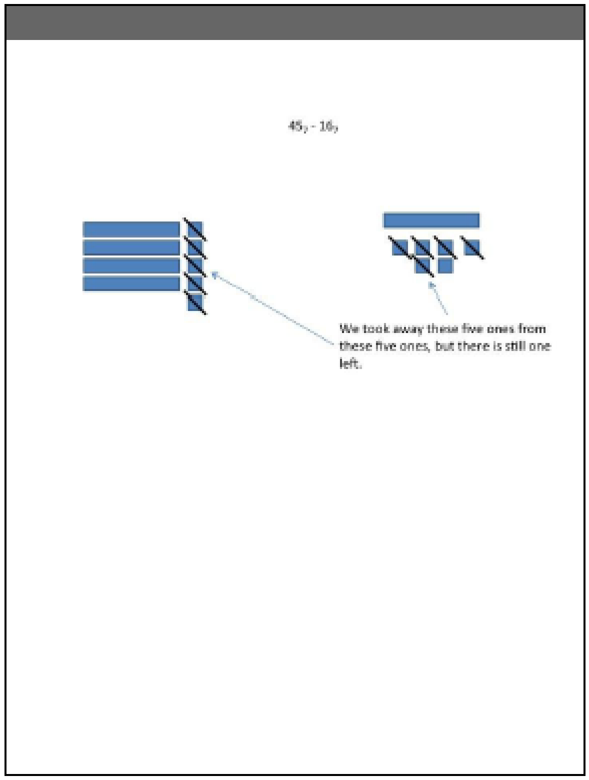

$45_7 - 16_7$

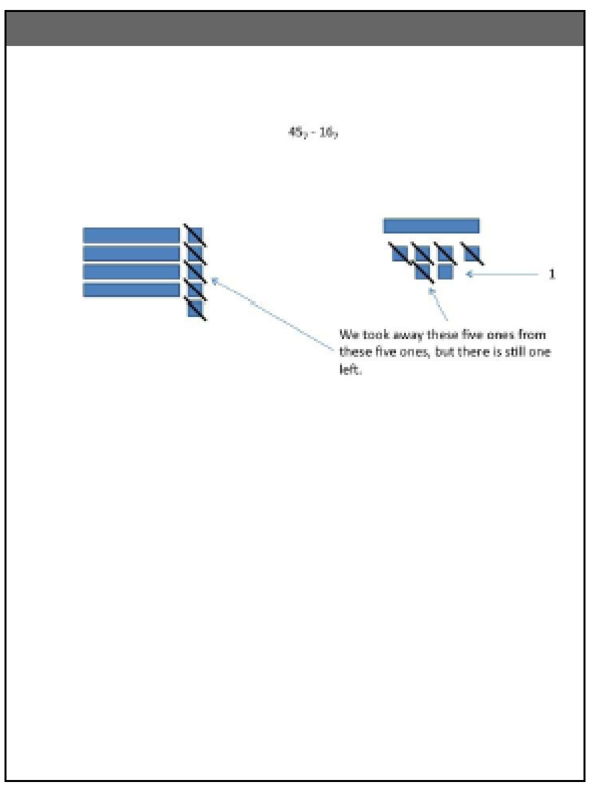

$45_7 - 16_7$

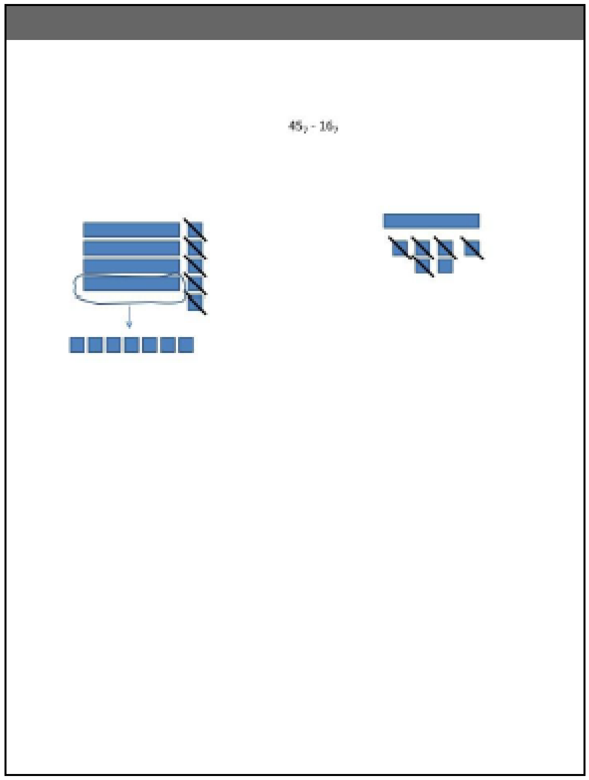

$45_7 + 16_7$

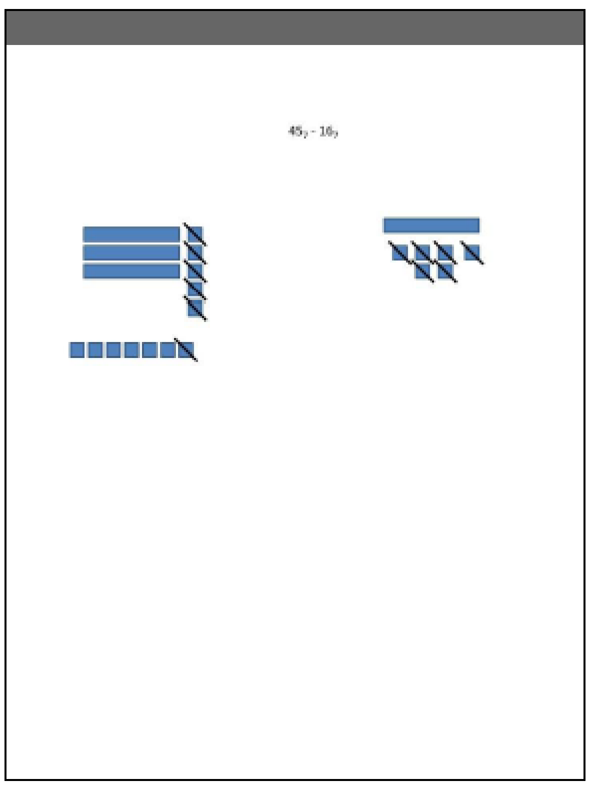

$45_7 + 16_7$

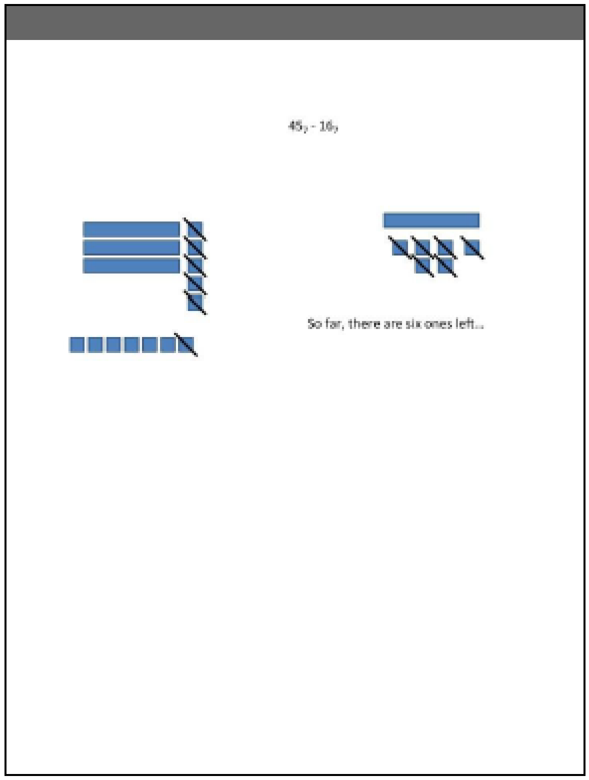

$45_7 - 16_7$
We took away these five ones from these five ones...

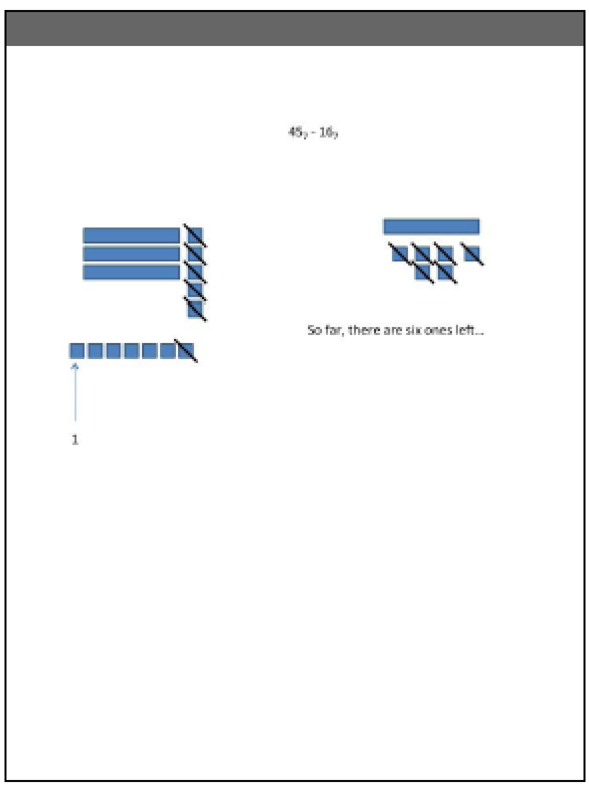

$45_7 - 16_7$
We took away these five ones from these five ones, but there is still one left.

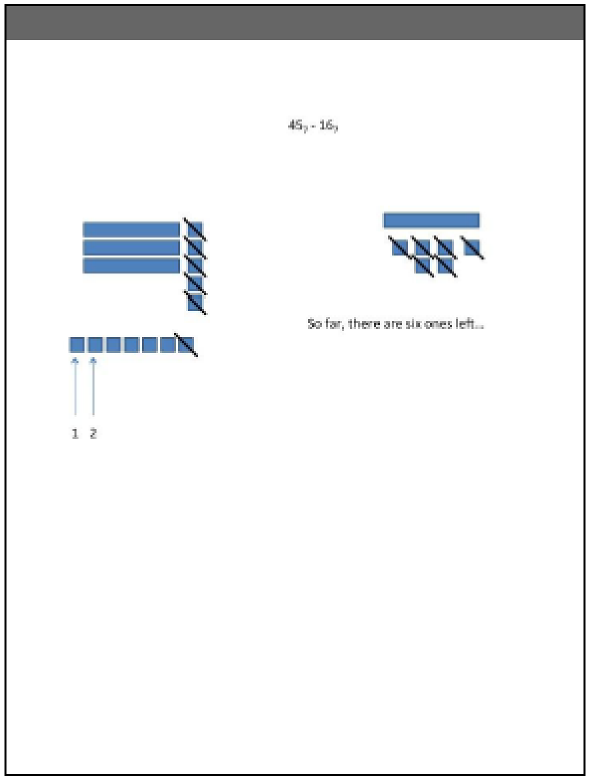

$45_7 - 16_7$
1
We took away these five ones from these five ones, but there is still one left.

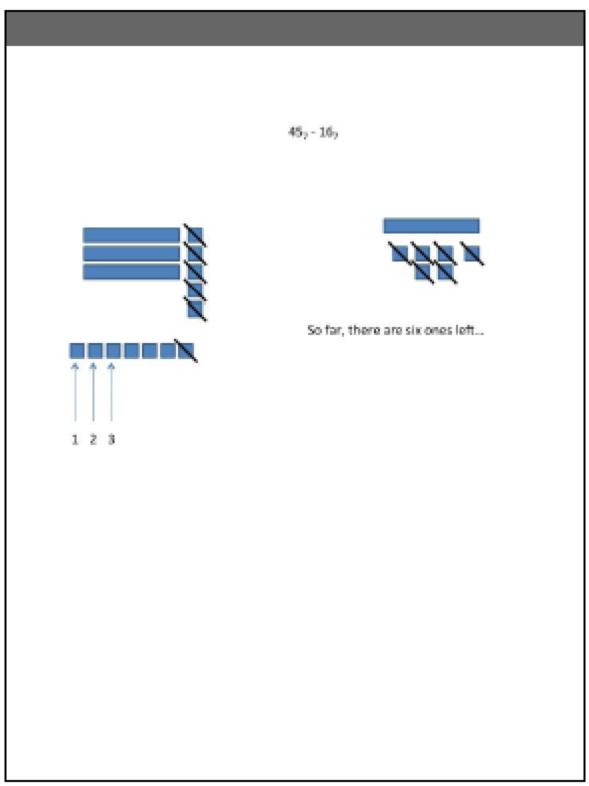

$45_7 - 16_7$

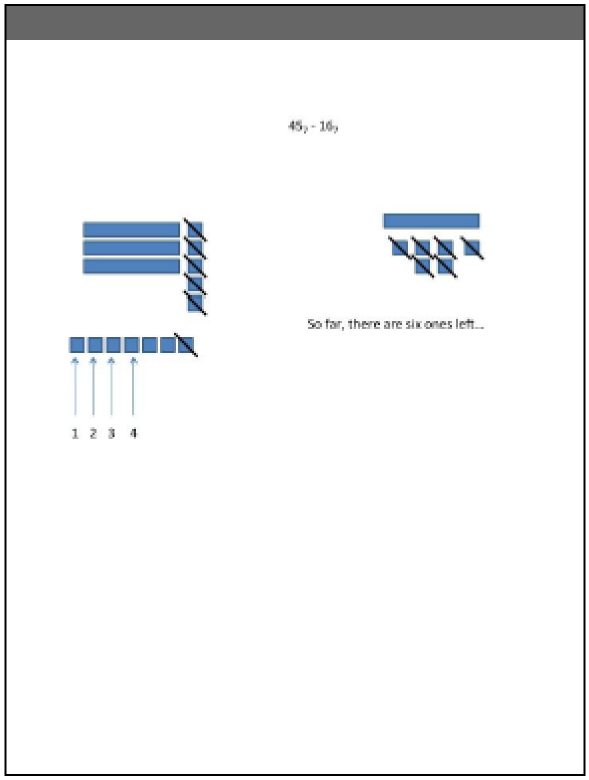

$45_7 - 16_7$

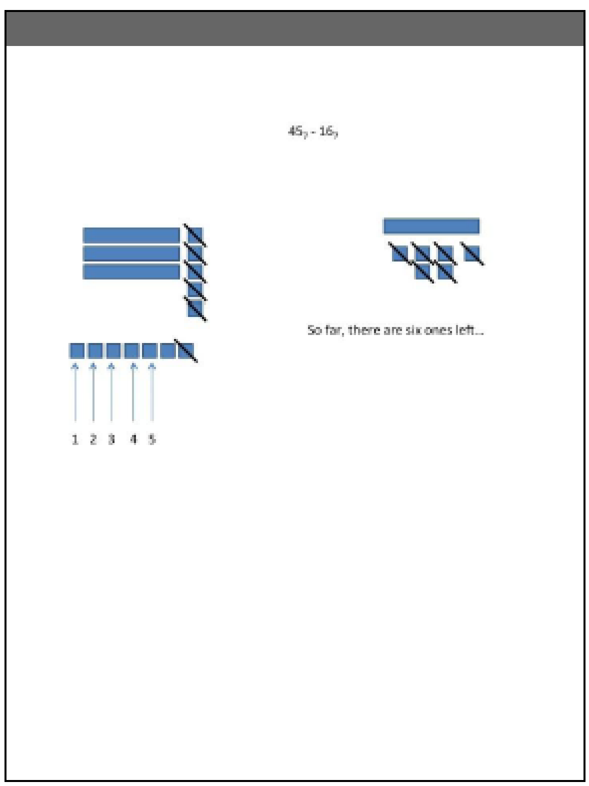

$45_7 - 16_7$
So far, there are six ones left…

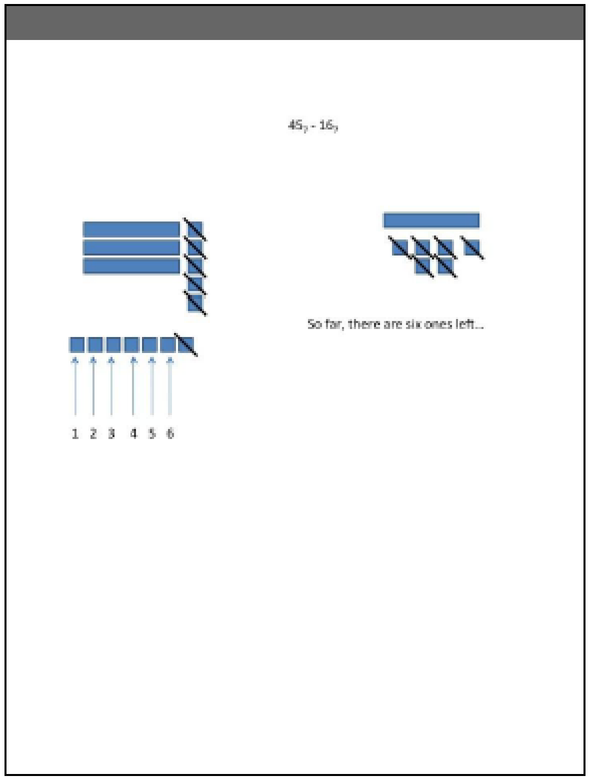

$45_7 - 16_7$
So far, there are six ones left…
1

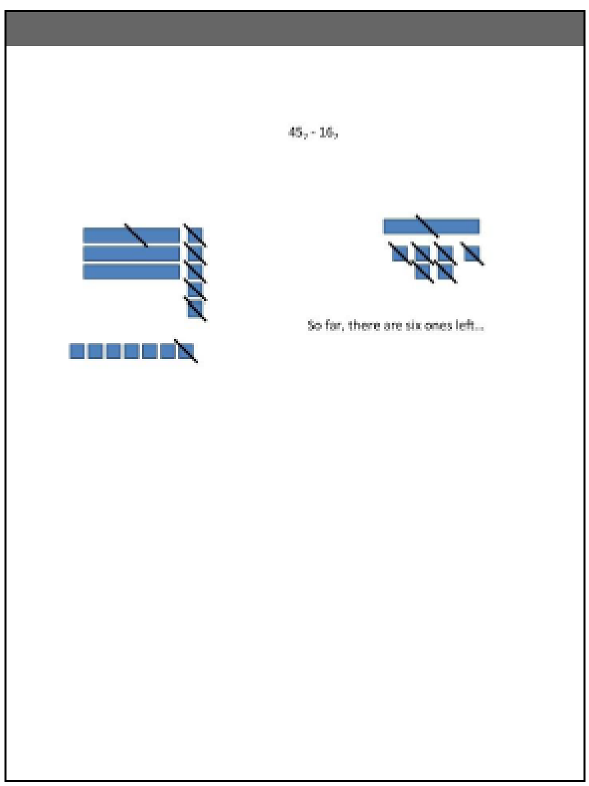

$45_7 - 16_7$
So far, there are six ones left...
1 2

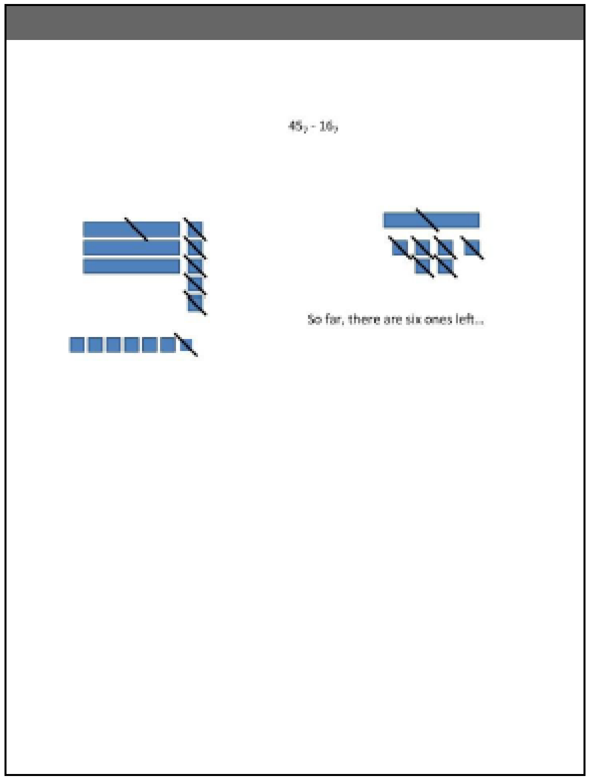

$45_7 - 16_7$
So far, there are six ones left...
1 2 3

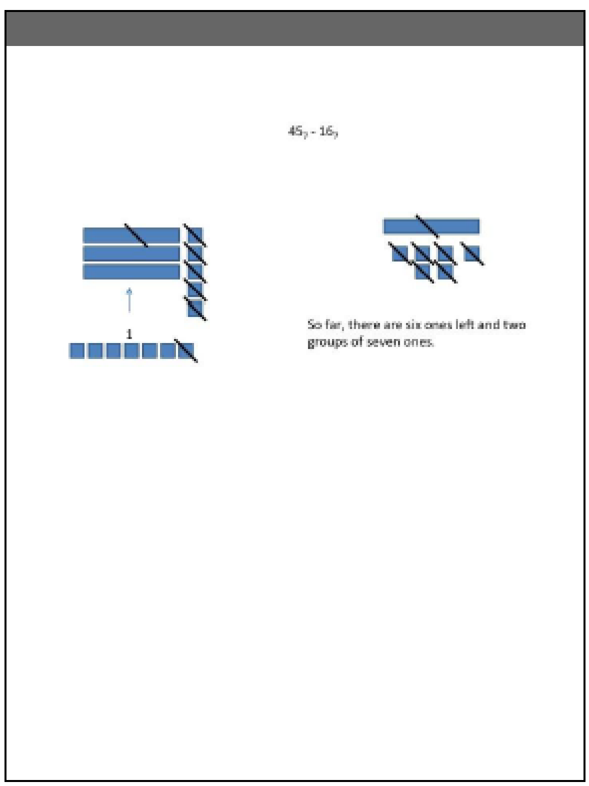

45₇ - 16₇
So far, there are six ones left…
1 2 3 4

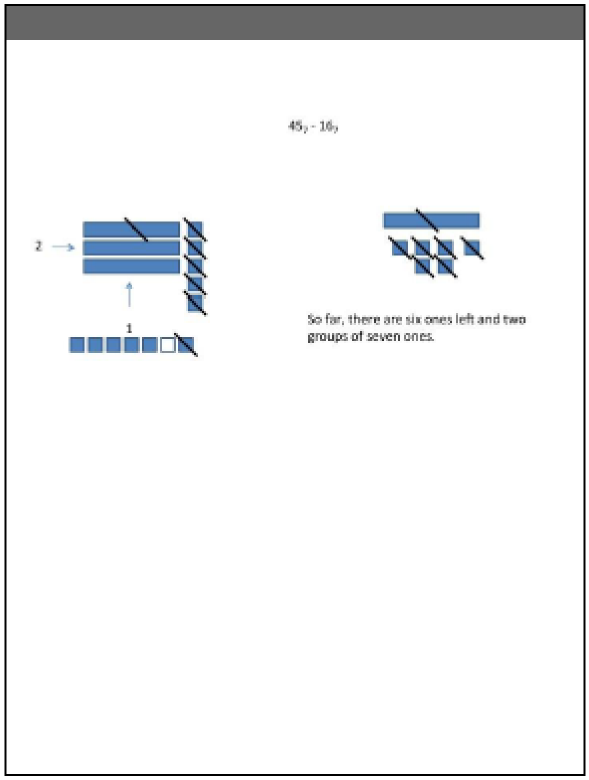

45₇ - 16₇
So far, there are six ones left…
1 2 3 4 5

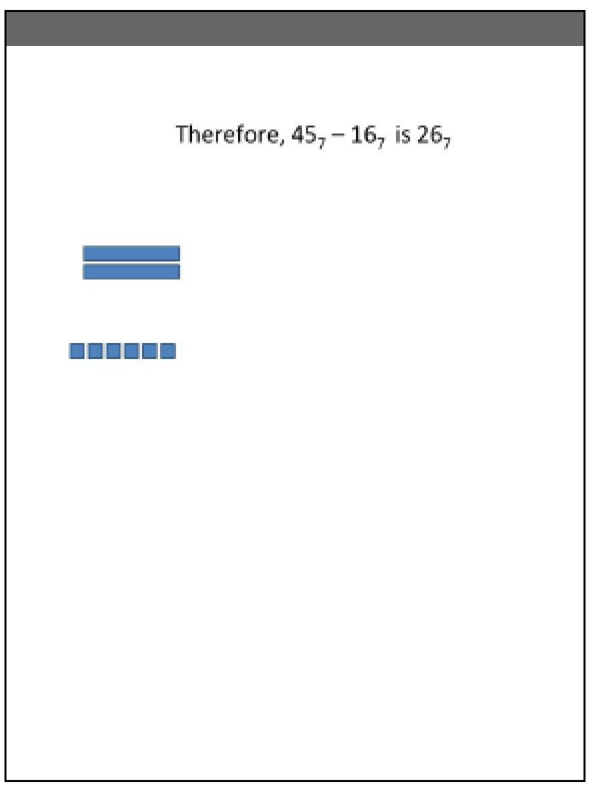

$45_7 - 16_7$
So far, there are six ones left...
1 2 3 4 5 6


$45_7 - 16_7$
So far, there are six ones left...


$45_7 - 16_7$
So far, there are six ones left...

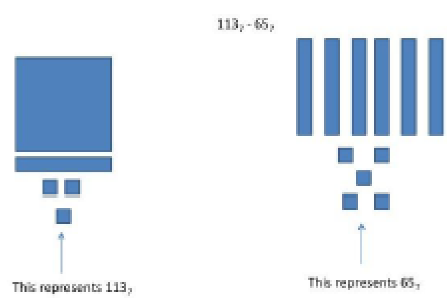

$45_7 - 16_7$
1
So far, there are six ones left and two groups of seven ones.

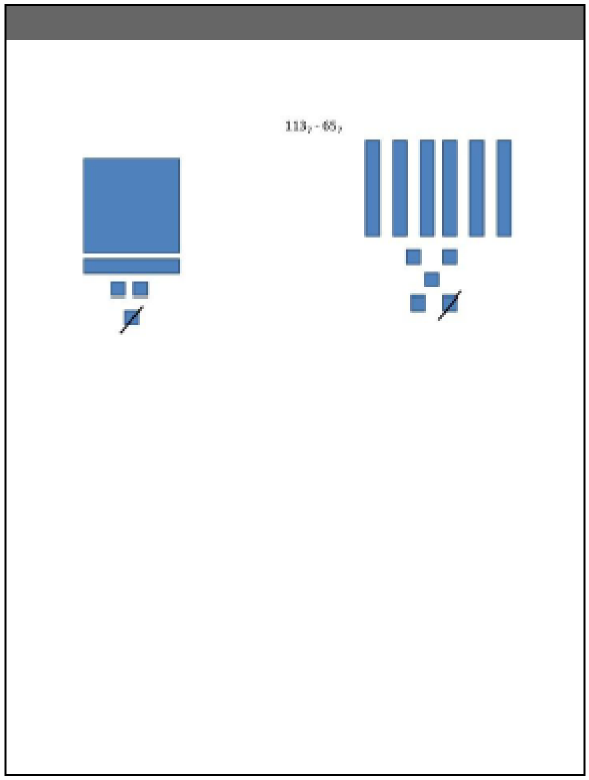

$45_7 - 16_7$
2
1
So far, there are six ones left and two groups of seven ones.

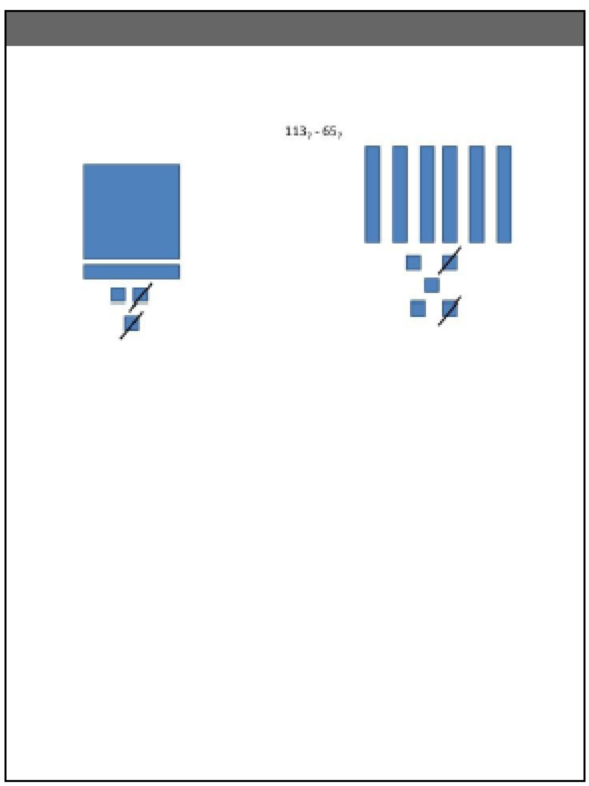

Therefore, $45_7 – 16_7$ is 26_7

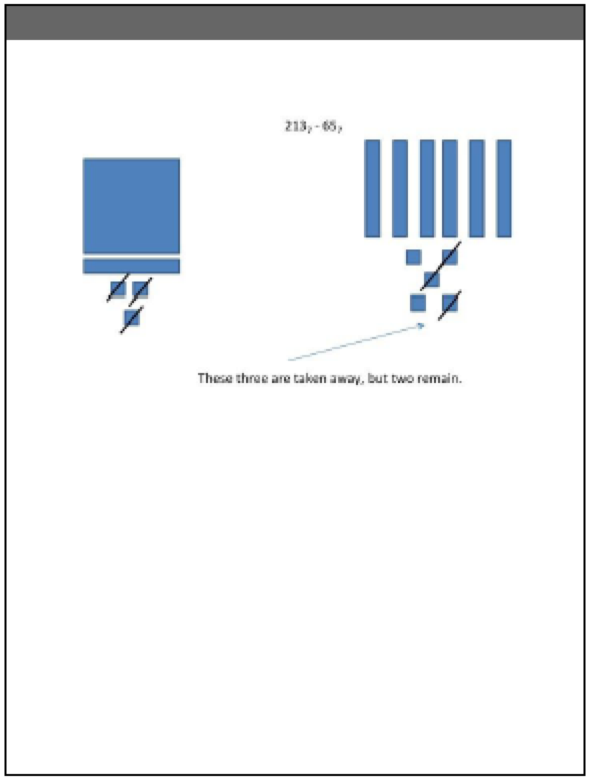
One more Example:

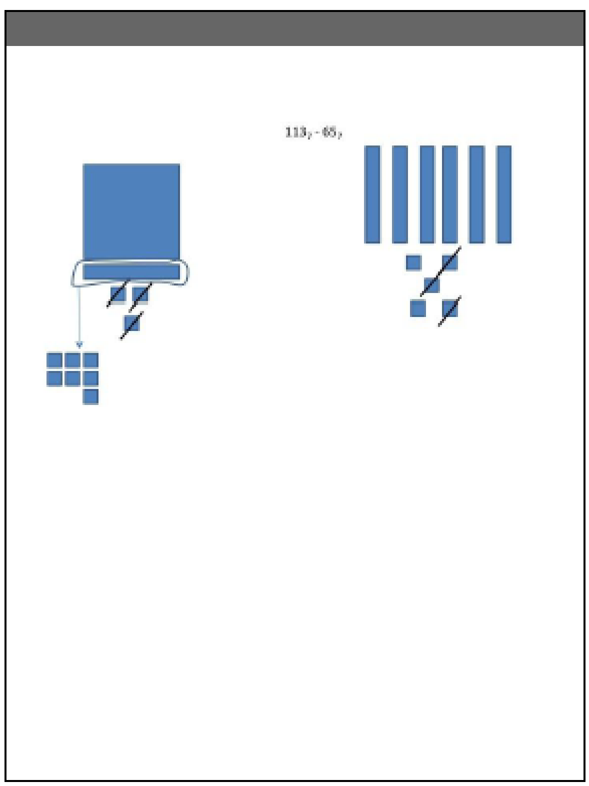
Please study the next example as well. Take your time to understand how the combining sevens and ones strategy works.

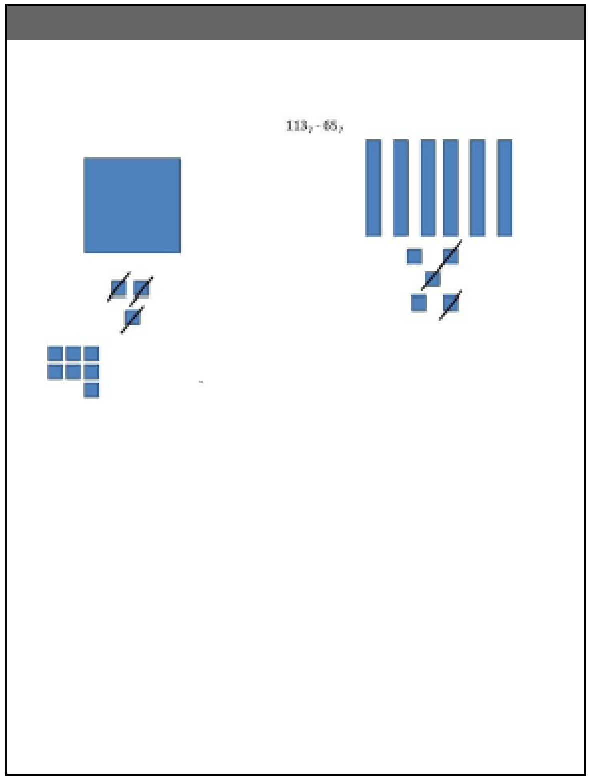


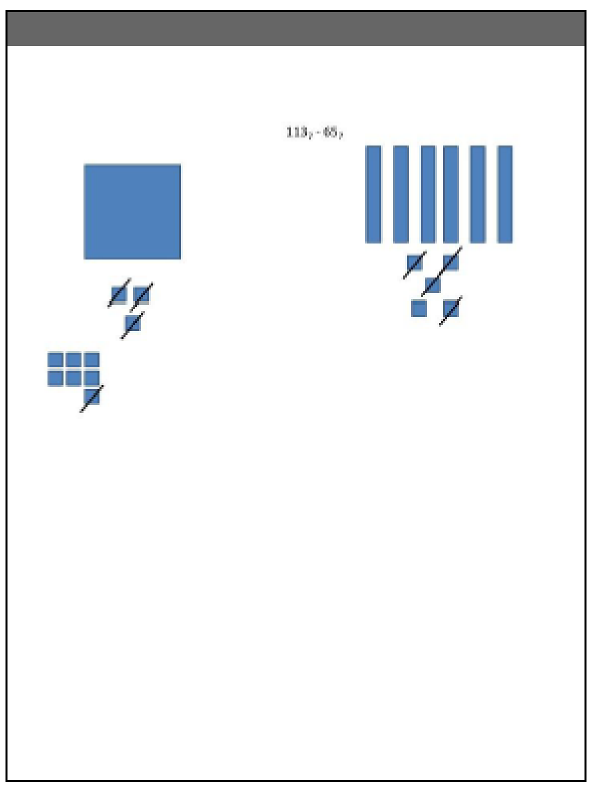

$113_7 - 65_7$

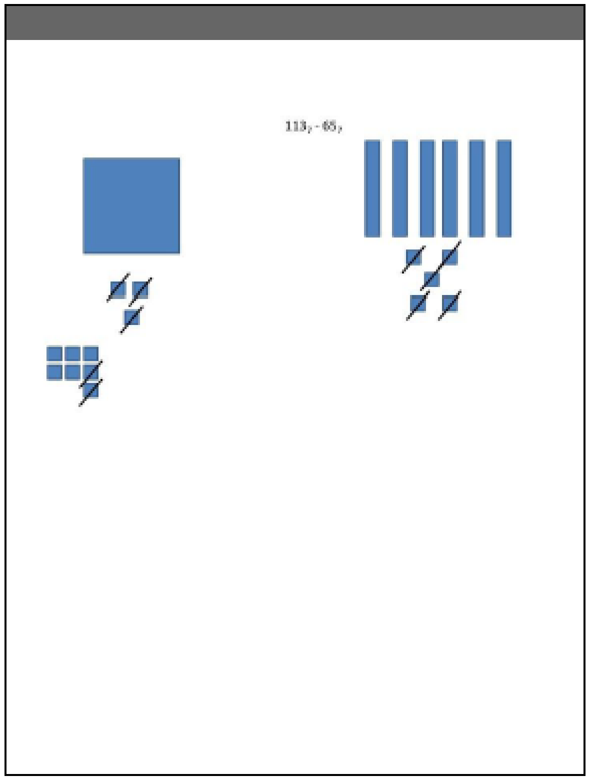

$113_7 - 65_7$

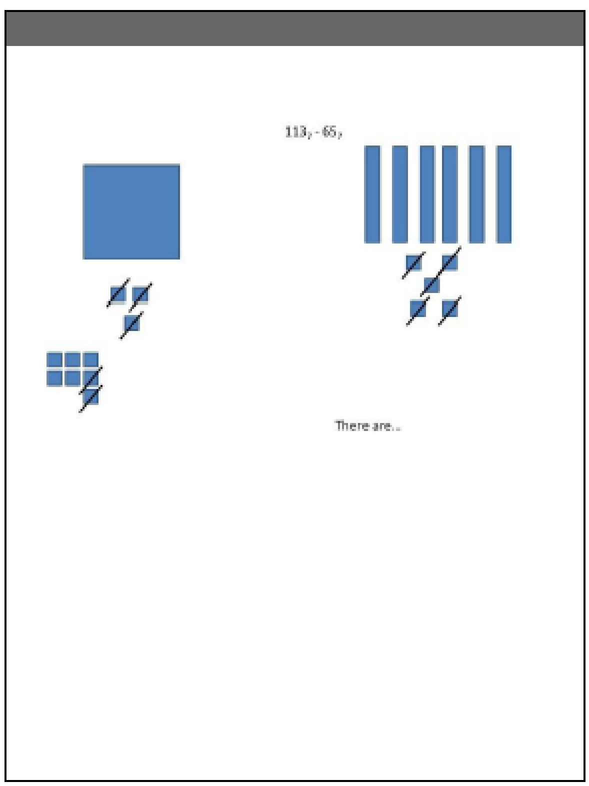

$213_7 - 65_7$
These three are taken away, but two remain.

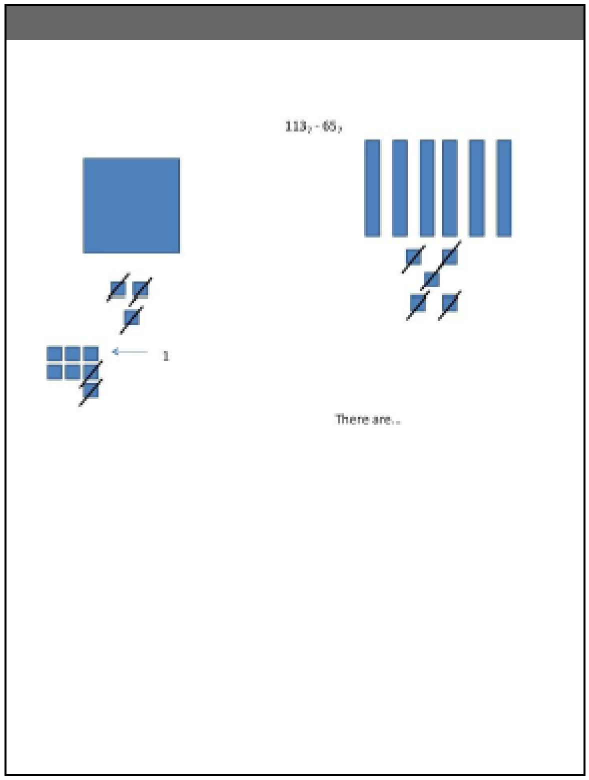

$113_7 - 65_7$

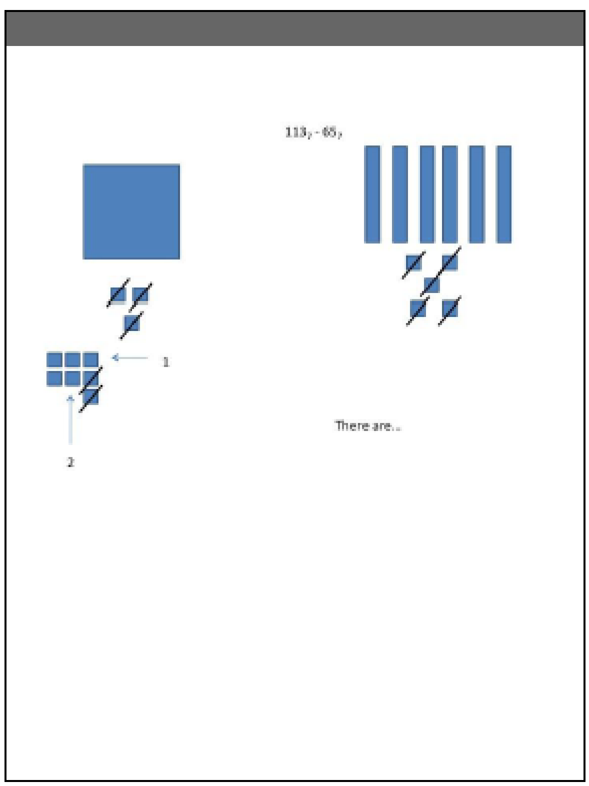

$113_7 - 65_7$

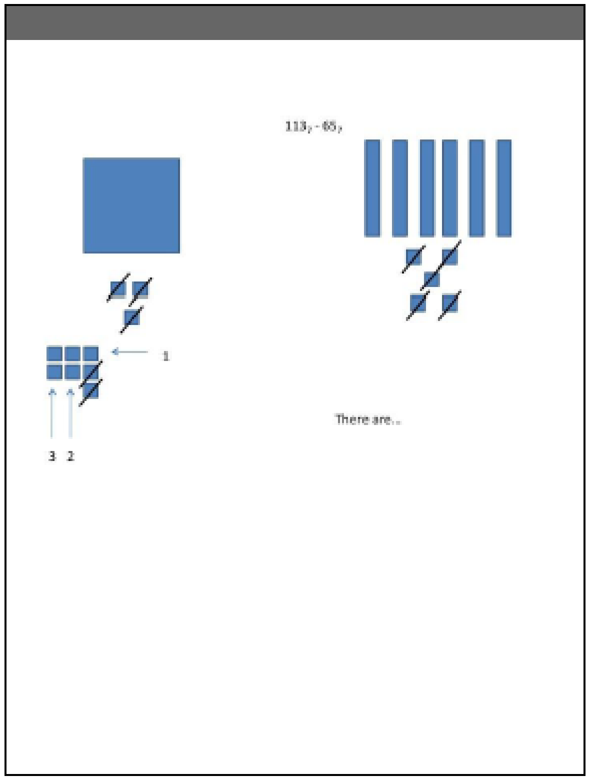

$113_7 - 65_7$

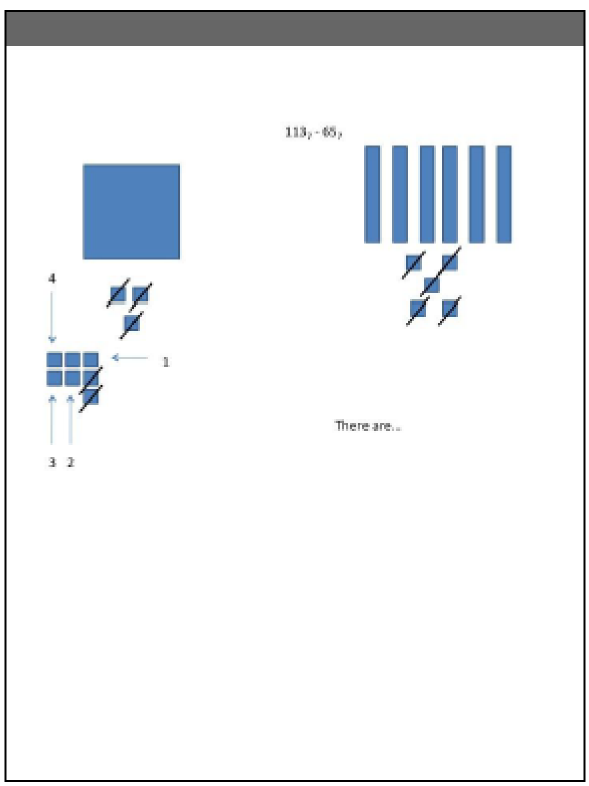

$113_7 - 65_7$

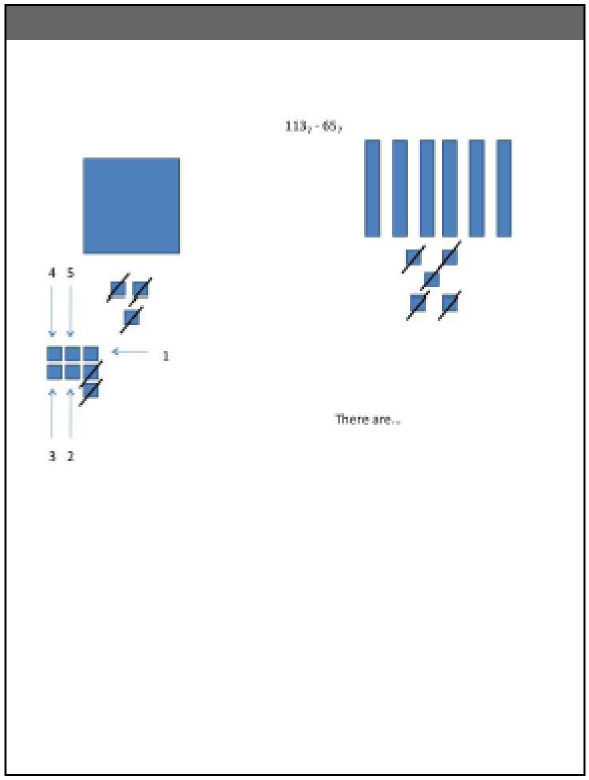

$113_7 - 65_7$
There are..

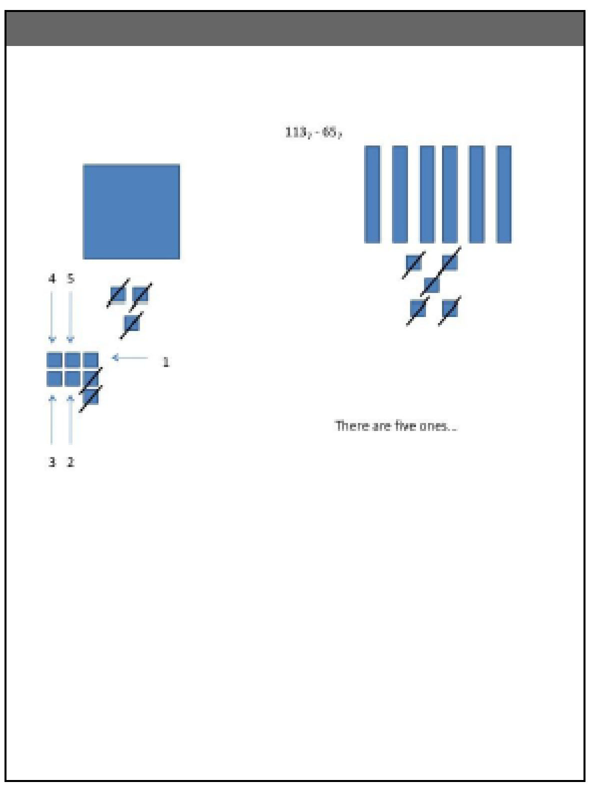

$113_7 - 65_7$
1
There are…

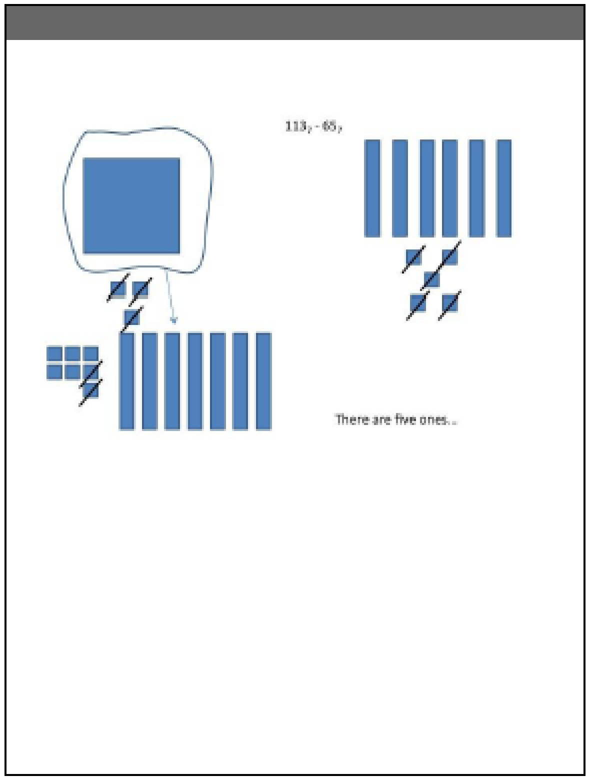

$113_7 - 65_7$
1
2
There are…

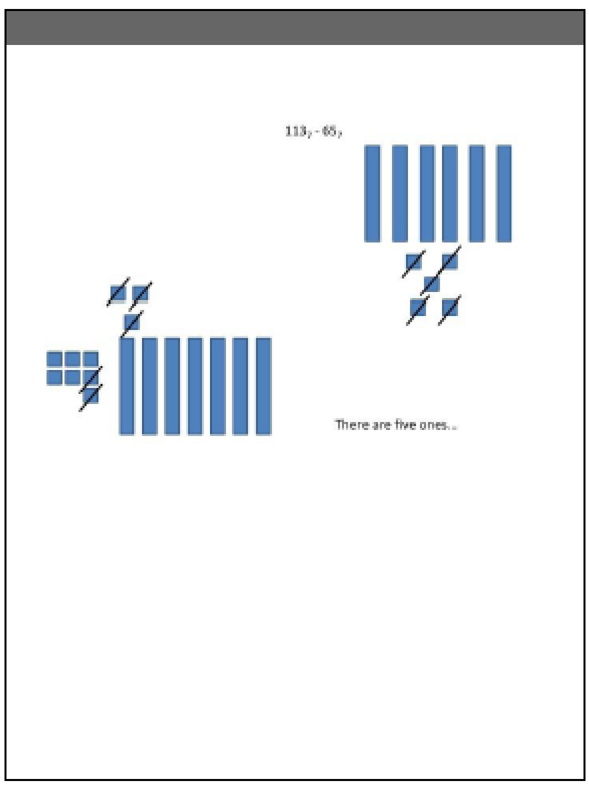

$113_7 - 65_7$
1
3 2
There are…

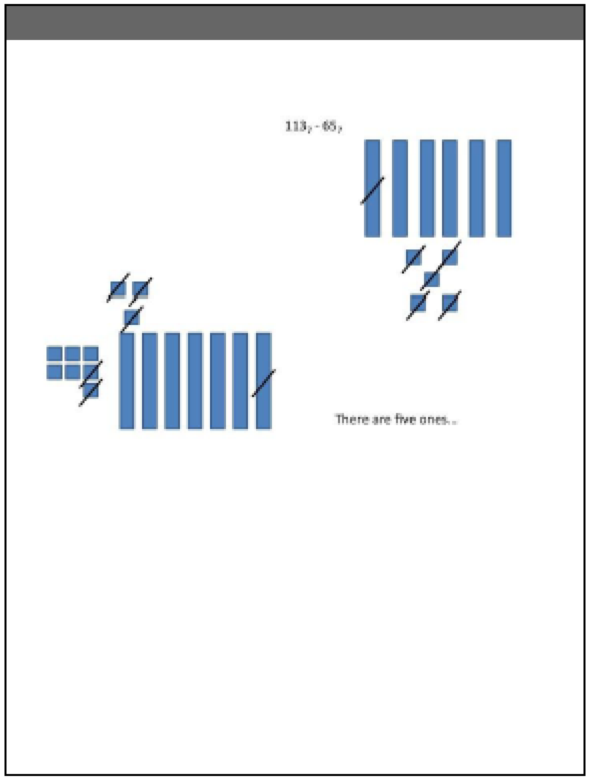

$113_7 - 65_7$
4
1
3 2
There are…

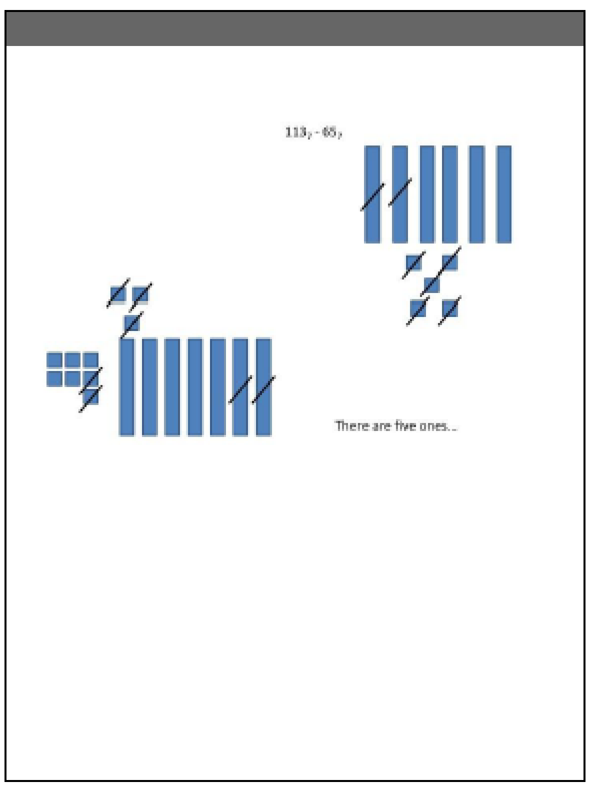

$113_7 - 65_7$
4 5
1
3 2
There are…

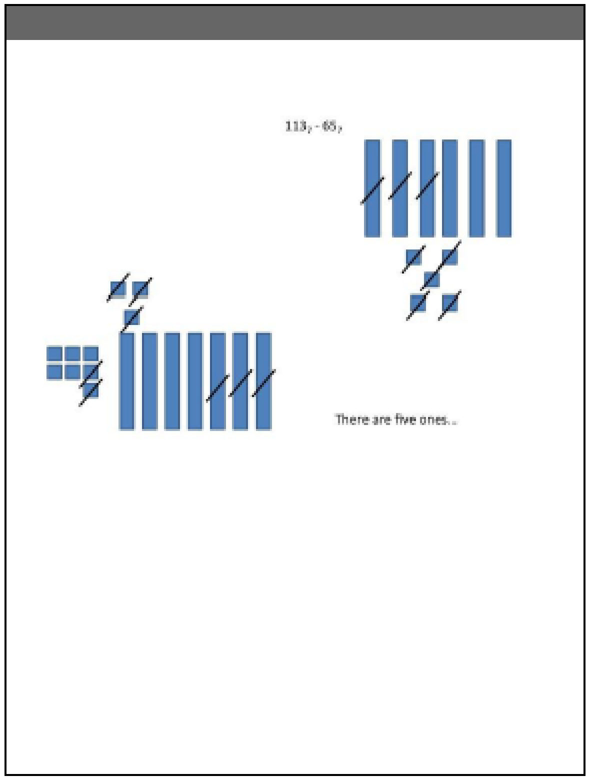

$113_7 - 65_7$
4 5
1
3 2
There are five ones…

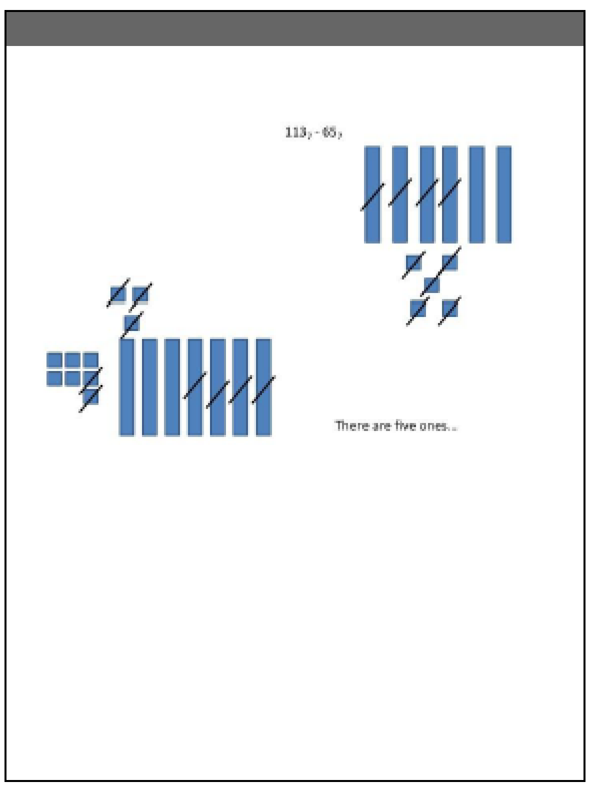

$113_7 - 65_7$
There are five ones…

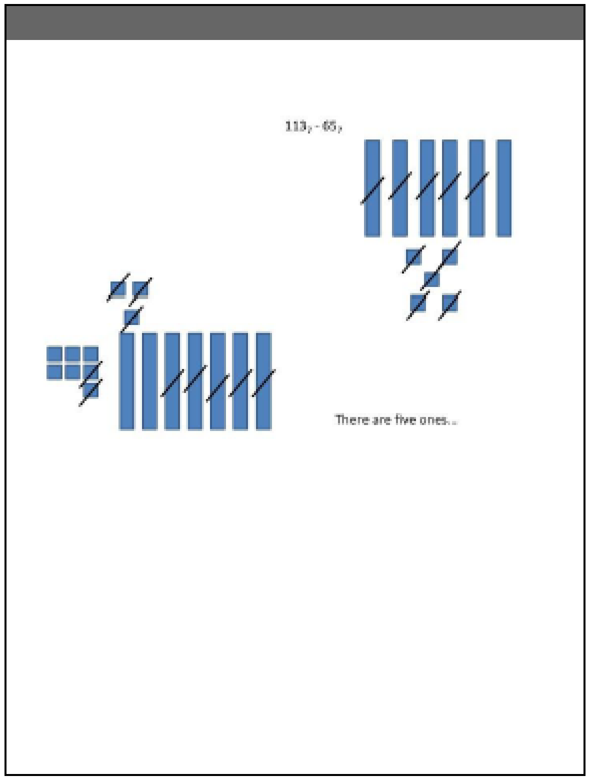

$113_7 - 65_7$
There are five ones…

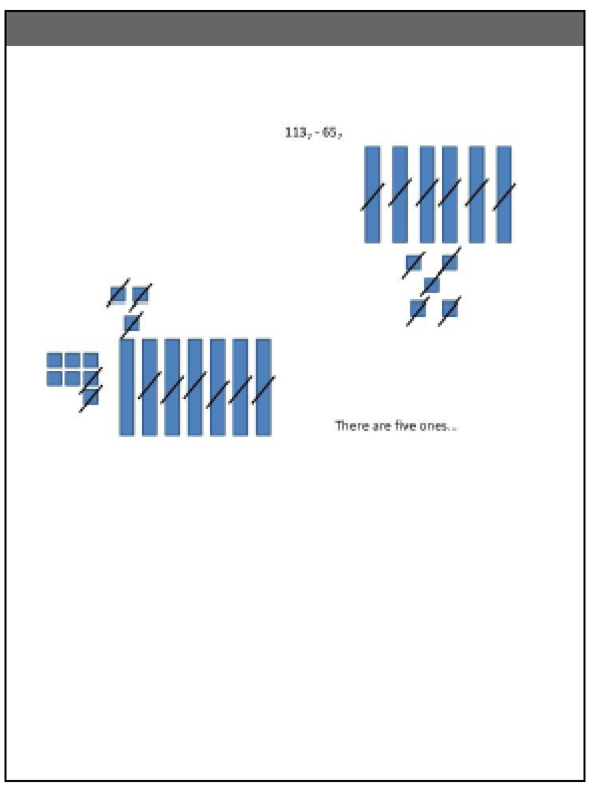

$113_7 - 65_7$
There are five ones…

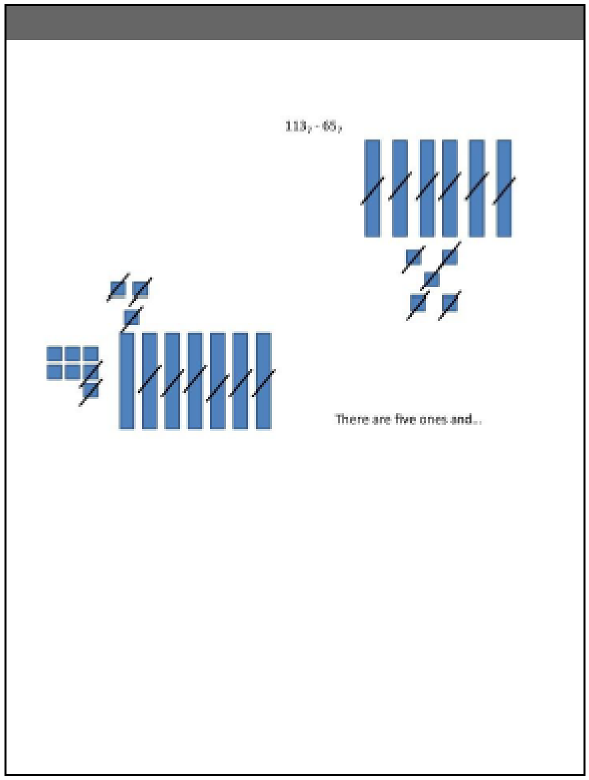

$113_7 - 65_7$
There are five ones…

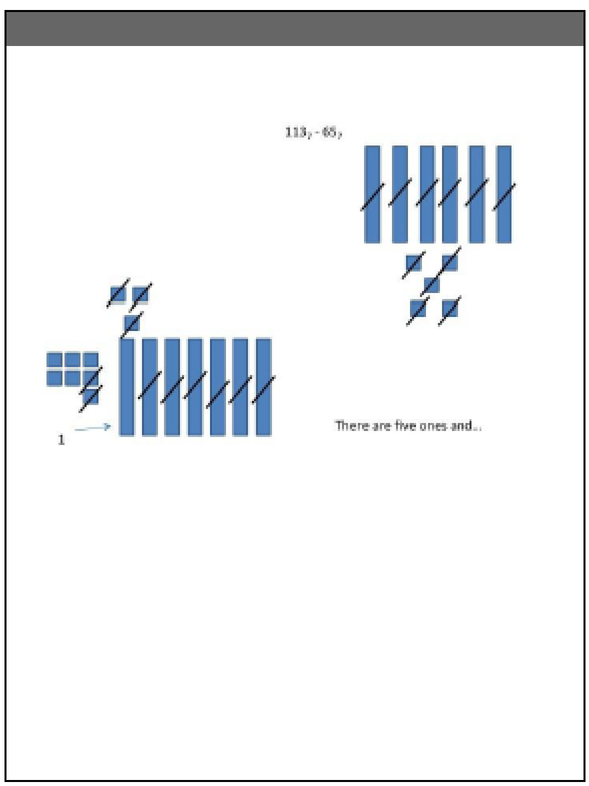

$113_7 - 65_7$
There are five ones…

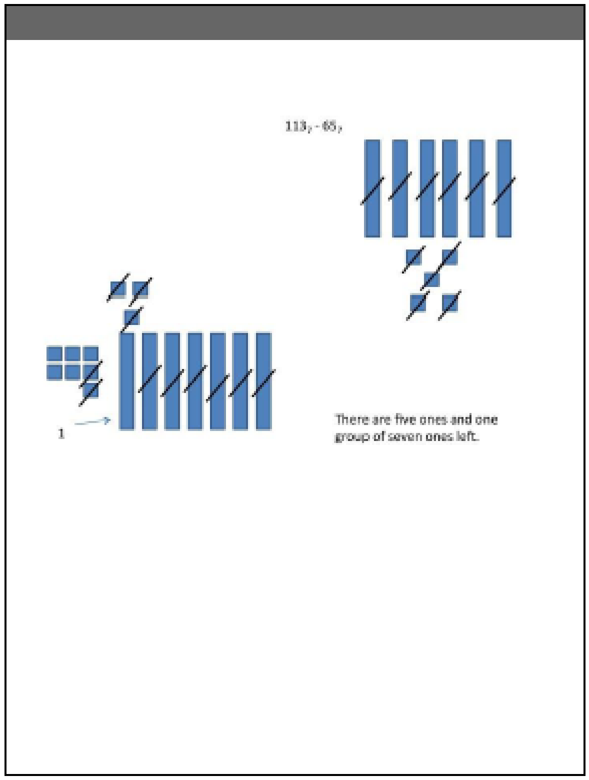

$113_7 - 65_7$
There are five ones…

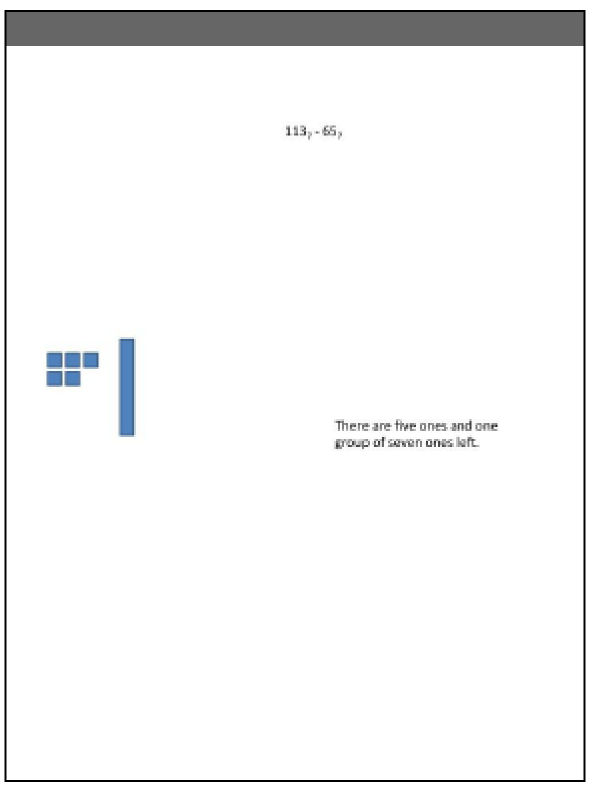

$113_7 - 65_7$
There are five ones...

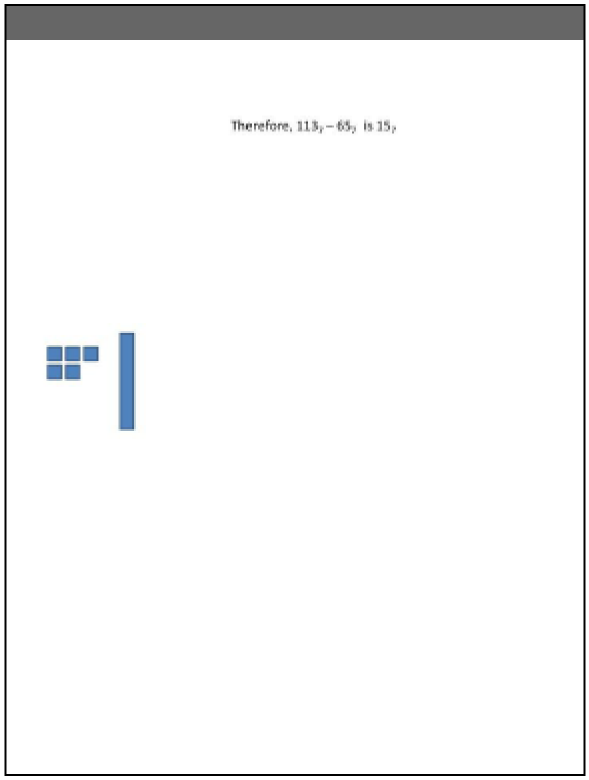

$113_7 - 65_7$
There are five ones...


$113_7 - 65_7$
There are five ones and…


$113_7 - 65_7$
1
There are five ones and…


$113_7 - 65_7$
1
There are five ones and one group of seven ones left.


$113_7 - 65_7$
There are five ones and one group of seven ones left.


Therefore, $113_7 - 65_7$ is 15_7

Appendix L

C4 – End of Lesson Questions

ID #:______________________________

1. Compute the following using the combining sevens and ones method.

   a. $125_7 - 66_7$

   b. $54_7 - 13_7$

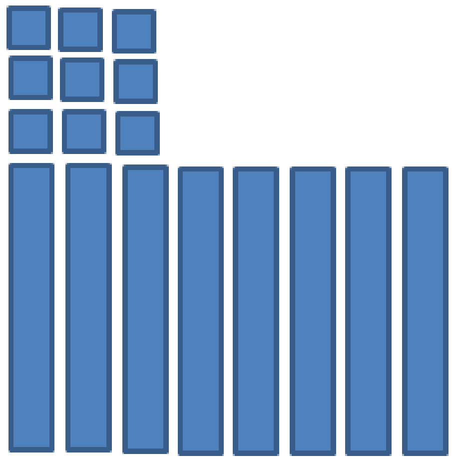
c. $341_7 - 233_7$

d. $126_7 - 51_7$

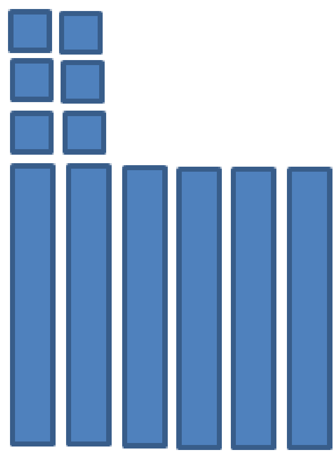
e. $43_7 - 14_7$

2. Circle the letter that represents the solution to the following problem: $124_7 - 25_7$.

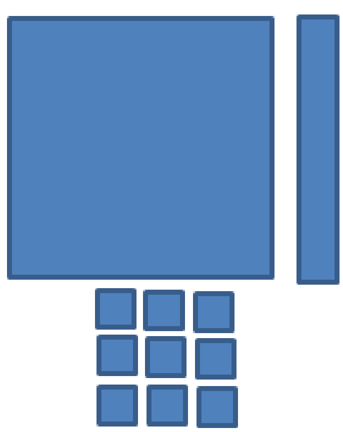
   a.

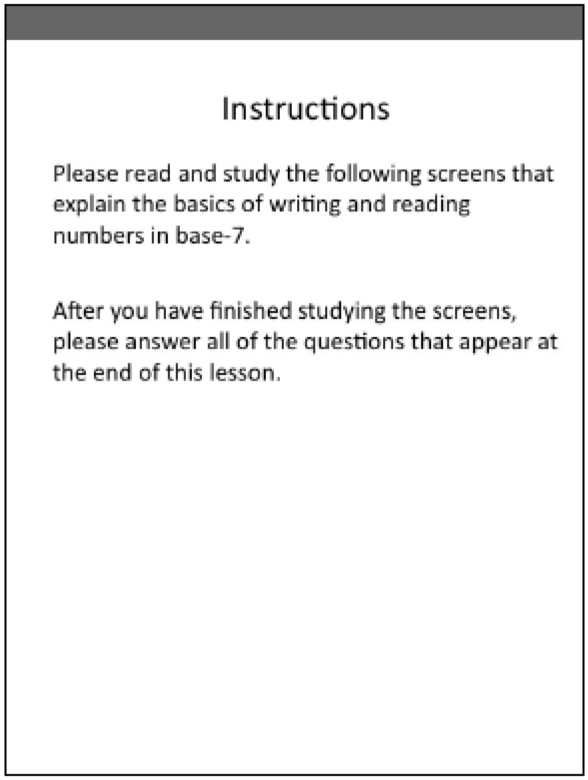
   b.

   c.

Appendix M


Instructions
Please read and study the following screens that explain the basics of writing and reading numbers in base-7.
After you have finished studying the screens, please answer all of the questions that appear at the end of this lesson.

*1. Student ID #:

Introduction

Today you are going to learn about another way to write numbers.

This other way of writing numbers is called base-7.

The purpose of this unit is to show you how to name and write numbers using base-7 notation.

Naming Numbers in Base-7

This is how you name single digit numbers in base-7:

Digit	Name
0	Zero
1_7	One base-7
2_7	Two base-7
3_7	Three base-7
4_7	Four base-7
5_7	Five base-7
6_7	Six base-7

Naming Numbers in Base-7

This is how you name double digit numbers in base-7:

Digit	Name
10_7	One-zero base-7
20_7	Two-zero base-7
30_7	Three-zero base-7
40_7	Four-zero base-7
50_7	Five-zero base-7
60_7	Six-zero base-7

Writing Numbers in Base-7

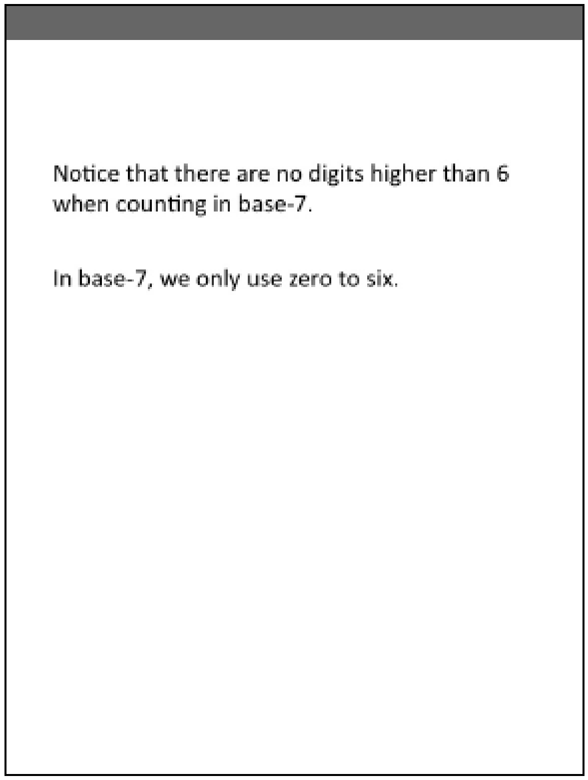
Base-7 is denoted by putting a subscript (or a little number lower and to the right of the number).

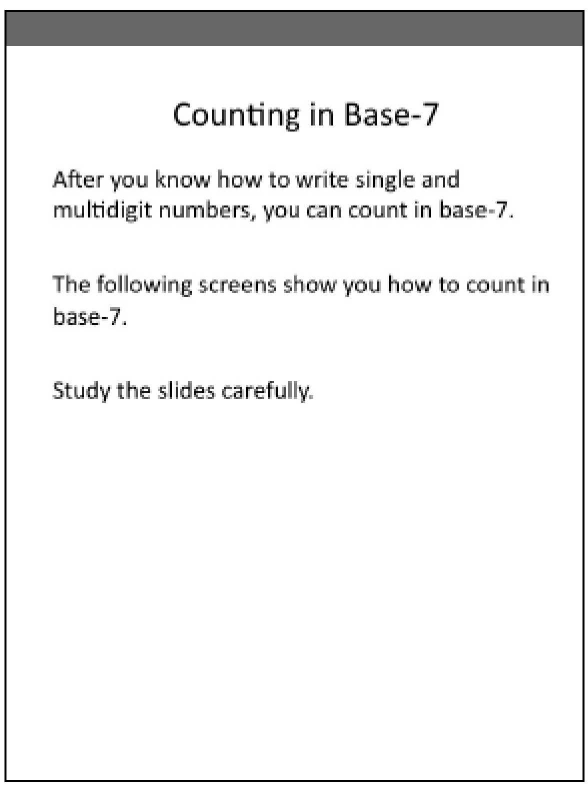
For example:

- 23_7
- 66_7
- 54_7

Subscripts

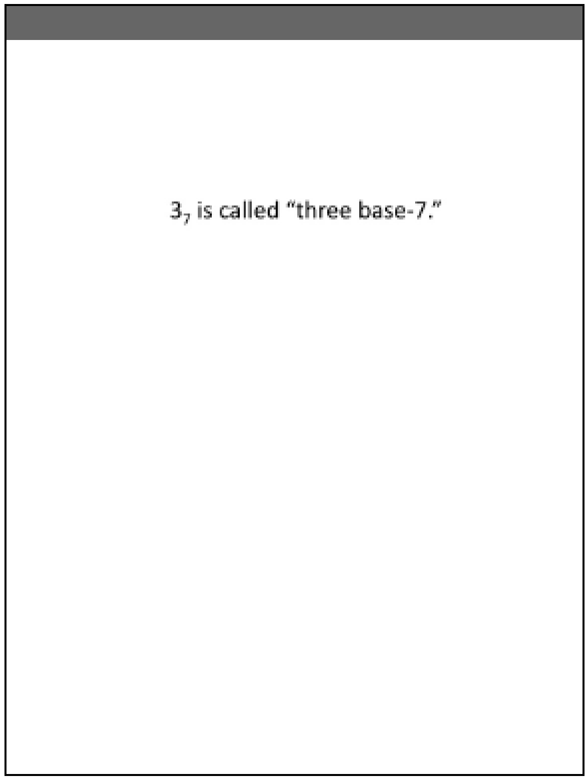
The small 7 ($_7$) following the above two-digit numbers indicates that these numbers are in base-7.

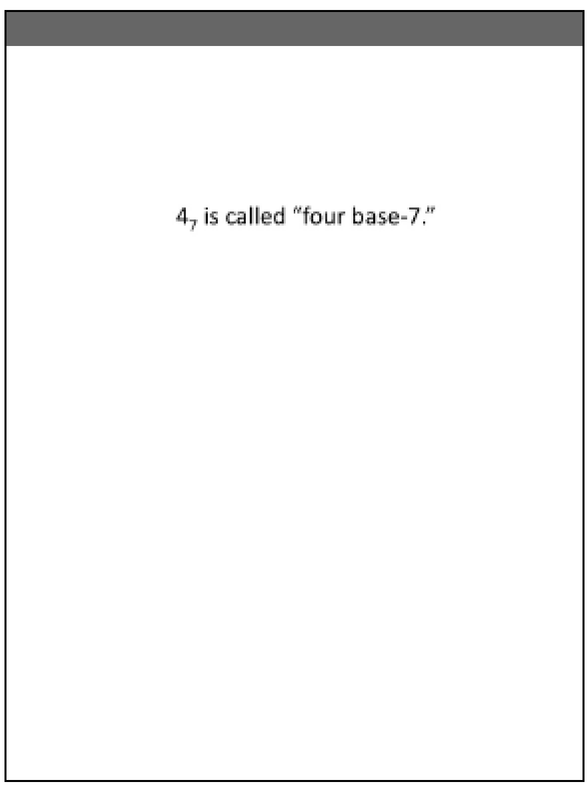

Notice that there are no digits higher than 6 when counting in base-7.
In base-7, we only use zero to six.


Counting in Base-7
After you know how to write single and multidigit numbers, you can count in base-7.
The following screens show you how to count in base-7.
Study the slides carefully.

1_7 is called "one base-7."

2_7 is called "two base-7."


3_7 is called “three base-7.”

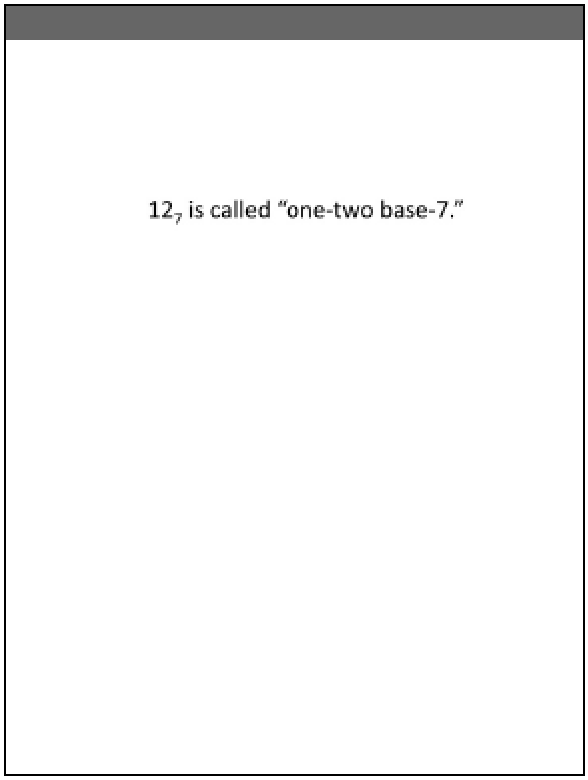

4_7 is called “four base-7.”

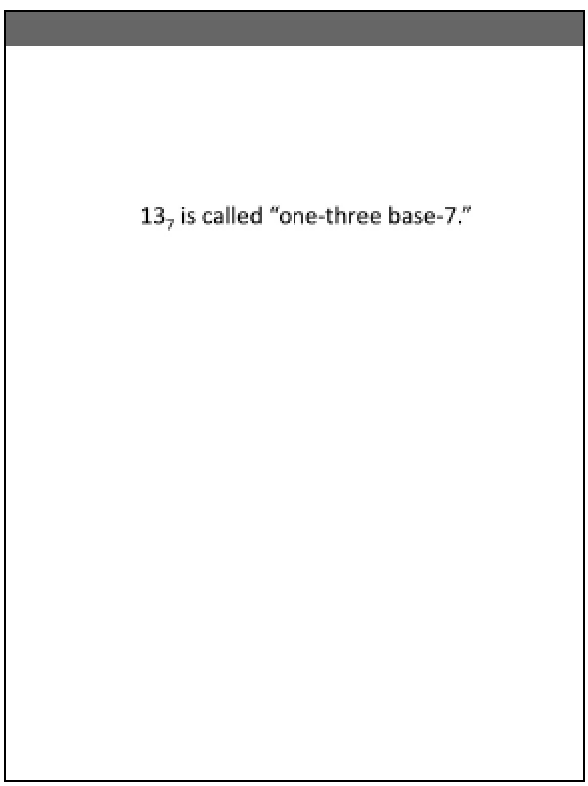
5_7 is called "five base-7."

6_7 is called "six base-7."

10_7 is called “one-zero base-7.”

11_7 is called “one-one base-7.”


12_7 is called "one-two base-7."


13_7 is called "one-three base-7."

14_7 is called "one-four base-7."

15_7 is called "one-five base-7."

16_7 is called "one-six base-7."

20_7 is called "two-zero base-7."

21_7 is called “two-one base-7.”

22_7 is called “two-two base-7.”

23_7 is called "two-three base-7."

24_7 is called "two-four base-7."

25_7 is called “two-five base-7.”

26_7 is called “two-six base-7.”

30_7 is called “three-zero base-7.”

31_7 is called “three-one base-7.”

32_7 is called "three-two base-7."

33_7 is called "three-three base-7."

34_7 is called “three-four base-7.”

35_7 is called “three-five base-7.”

36_7 is called “three-six base-7.”

40_7 is called “four-zero base-7.”

41_7 is called "four-one base-7."

42_7 is called "four-two base-7."

43_7 is called “four-three base-7.”

44_7 is called “four-four base-7.”

45_7 is called "four-five base-7."

46_7 is called "four-six base-7."

50_7 is called "five-zero base-7."

51_7 is called "five-one base-7."

52_7 is called "five-two base-7."

53_7 is called "five-three base-7."

54_7 is called "five-four base-7."

55_7 is called "five-five base-7."

56_7 is called “five-six base-7.”

60_7 is called “six-zero base-7.”

61_7 is called "six-one base-7."

62_7 is called "six-two base-7."

63_7 is called “six-three base-7.”

64_7 is called “six-four base-7.”

65_7 is called “six-five base-7.”

66_7 is called “six-six base-7.”

100_7 is called "one-zero-zero base-7."

101_7 is called "one-zero-one base-7."

102_7 is called "one-zero-two base-7."

103_7 is called "one-zero-three base-7."

104_7 is called “one-zero-four base-7.”

105_7 is called “one-zero-five base-7.”

106_7 is called "one-zero-six base-7."

110_7 is called "one-one-zero base-7."

111_7 is called "one-one-one base-7."

112_7 is called "one-one-two base-7."

113_7 is called "one-one-three base-7."

114_7 is called "one-one-four base-7."

115_7 is called “one-one-five base-7.”

116_7 is called “one-one-six base-7.”

120_7 is called “one-two-zero base-7.”

121_7 is called “one-two-one base-7.”

122_7 is called "one-two-two base-7."

123_7 is called "one-two-three base-7."

124_7 is called “one-two-four base-7.”

125_7 is called “one-two-five base-7.”

126_7 is called "one-two-six base-7."

130_7 is called "one-three-zero base-7."

Naming Multidigit Numbers in Base-7

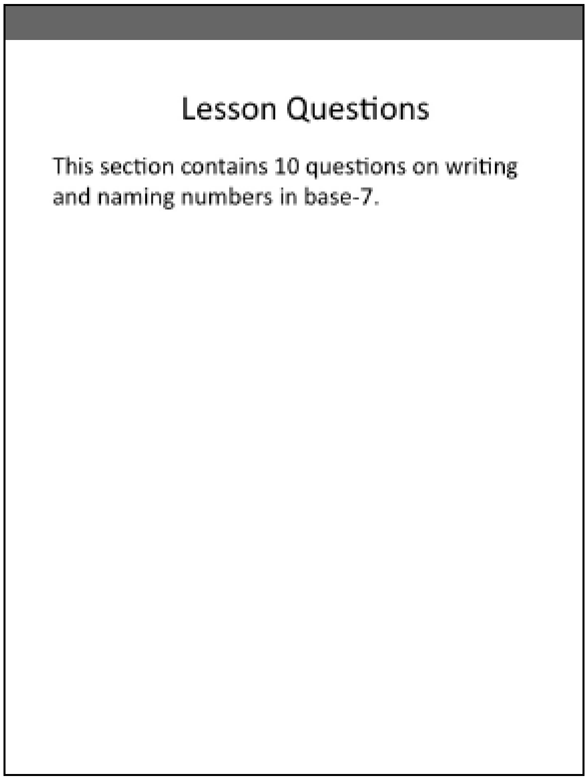
By following the same patterns you have seen in the previous screens, you can name any number in base-7.

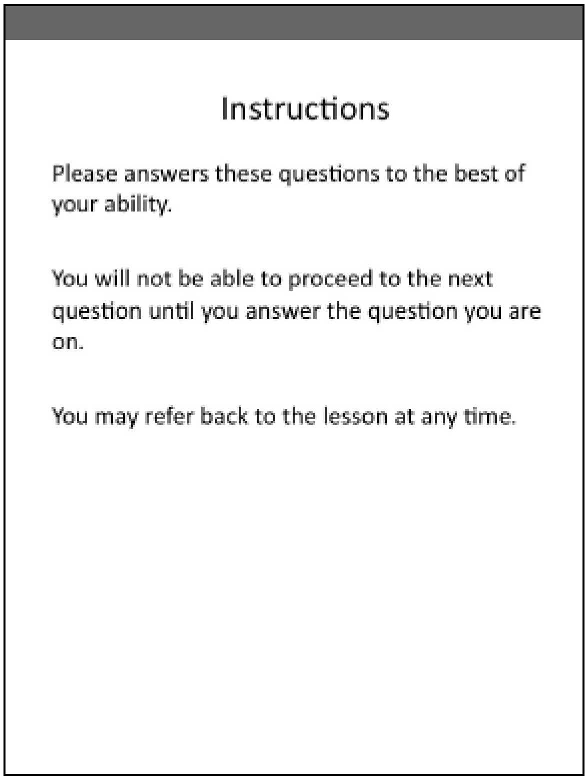
Look at the following table to see the names of some multidigit numbers in base-7.

This is how you name multidigit numbers in base-7:

Digit	Name
135_7	One-three-five base-7
$6\,512_7$	Six-five-one-two base-7
$21\,543_7$	Two-one-five-four-three base-7
425_7	Four-two-five base-7
$146\,534\,000_7$	One-four-six-five-three-four-zero-zero-zero base-7
600_7	Six-zero-zero base-7

Appendix N


Lesson Questions
This section contains 10 questions on writing and naming numbers in base-7.


Instructions
Please answers these questions to the best of your ability.
You will not be able to proceed to the next question until you answer the question you are on.
You may refer back to the lesson at any time.

16_7

*2. The number above is called "sixteen".

- ○ True
- ○ False

3_7

*3. The number above is called "three base-7."

- ○ True
- ○ False

103_7

*4. The number above is called "one-hundred-three base-7".

- ○ True
- ○ False

23_7

*5. Select the correct name of the above number.

- ○ twenty three
- ○ two three
- ○ two three base 7
- ○ twenty three base 7

216_7

*6. Select the correct name of the number above.

- ○ two-one-six base 7
- ○ two hundred and sixteen
- ○ two-one-six
- ○ two hundred and sixteen base 7

15_7

*7. Select the correct name of the above number.

- ○ fifteen
- ○ one-five base 7
- ○ fifteen base 7
- ○ one-five

12_7

*8. In words, write the name of the above number.

501_7

*9. In words, write the name of the above number.

254_7

*10. In words, write the name of the above number.

605_7

*11. In words, write the name of the above number.

Appendix O

*1. Student ID Number

Introduction

You have previously learned about another way to write numbers and count in base-7.

Today you are going to learn more about numbers and counting in base-7.

Instructions

Please read and study the following screens that explain the basics of writing and reading numbers in base-7.

After you have finished studying the screens, please answer all of the questions that appear at the end of this lesson.

Number Charts

Number charts are useful tools elementary school teachers often use to help their students learn many concepts related to numbers such as counting, adding, and subtracting.

The following screen shows you a base-7 number chart.

Study the chart and the instructional slides that follow carefully.

This is a Base-7 Number Chart

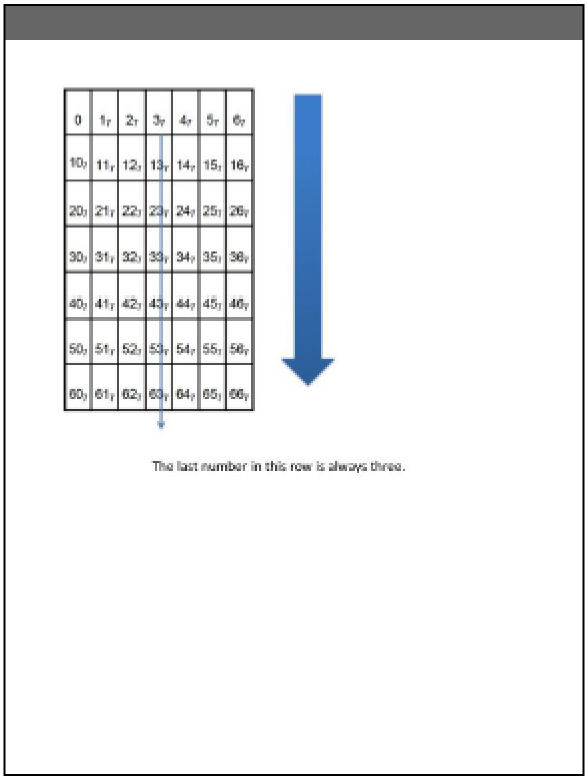
0	1_7	2_7	3_7	4_7	5_7	6_7
10_7	11_7	12_7	13_7	14_7	15_7	16_7
20_7	21_7	22_7	23_7	24_7	25_7	26_7
30_7	31_7	32_7	33_7	34_7	35_7	36_7
40_7	41_7	42_7	43_7	44_7	45_7	46_7
50_7	51_7	52_7	53_7	54_7	55_7	56_7
60_7	61_7	62_7	63_7	64_7	65_7	66_7

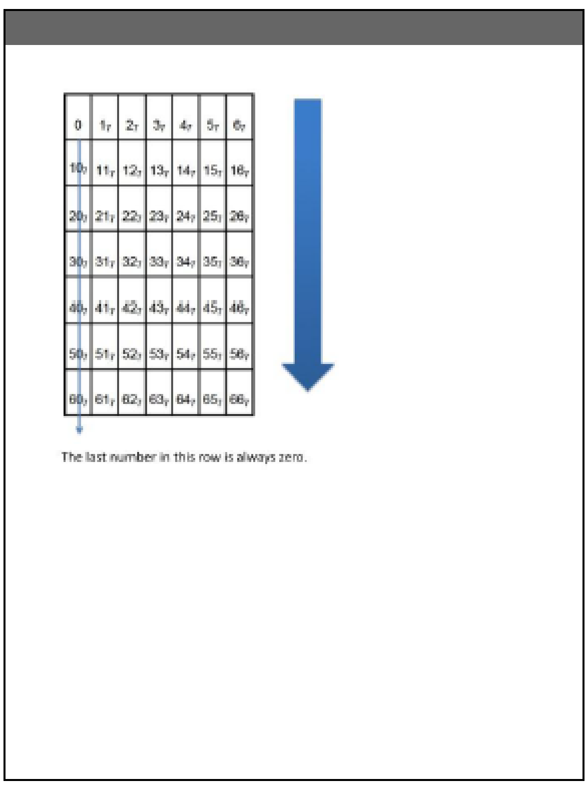
Patterns in the Base-7 Number Chart

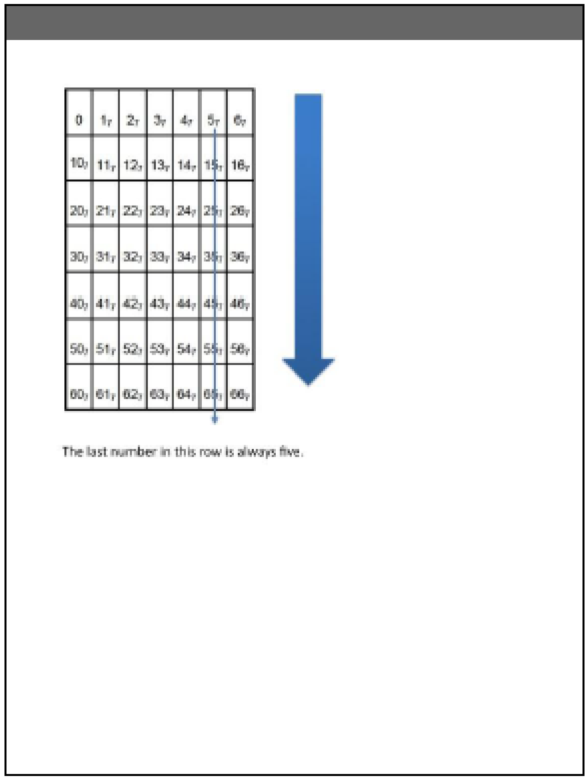
Notice that there are seven columns across the chart.

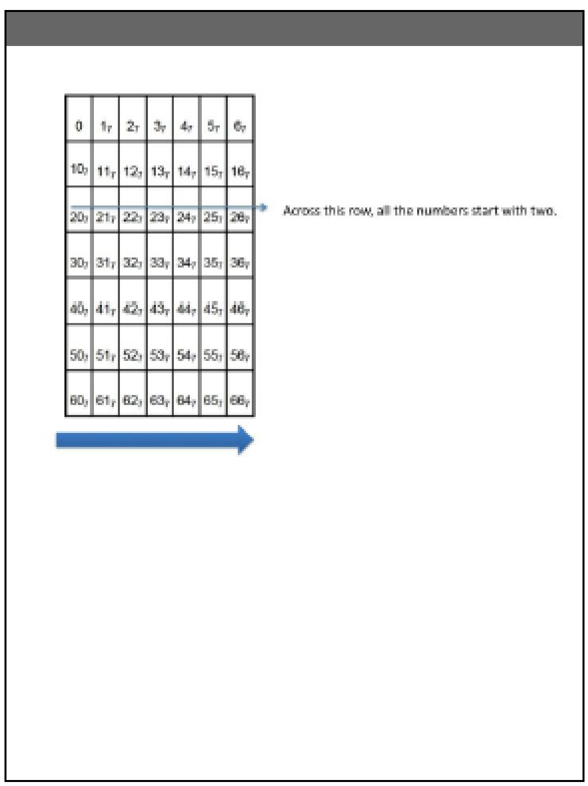
Furthermore, when you go down the columns in the chart, the last number is always the same.

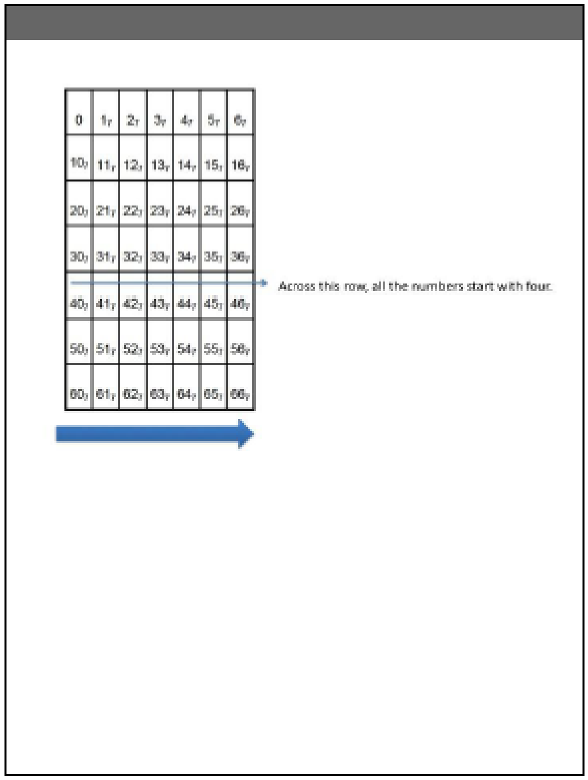
When you go across the rows, the first number always stays the same (except for the first row).

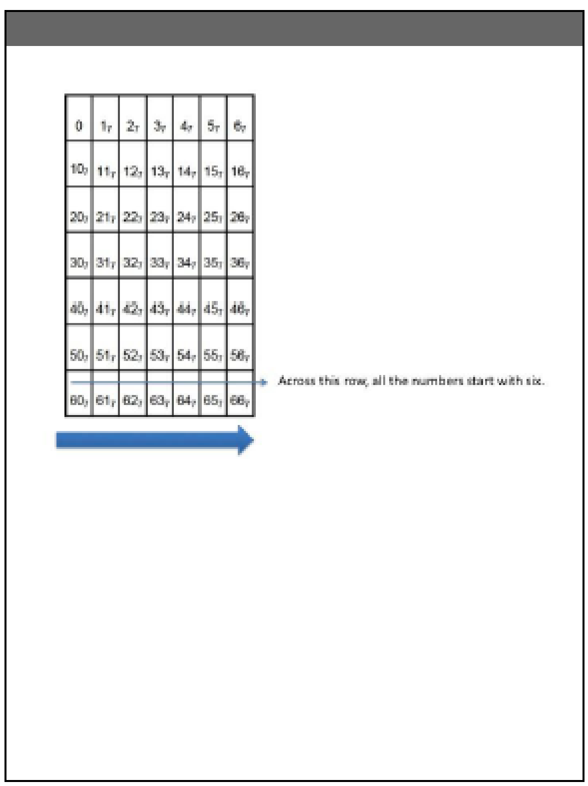

The last number in this row is always three.

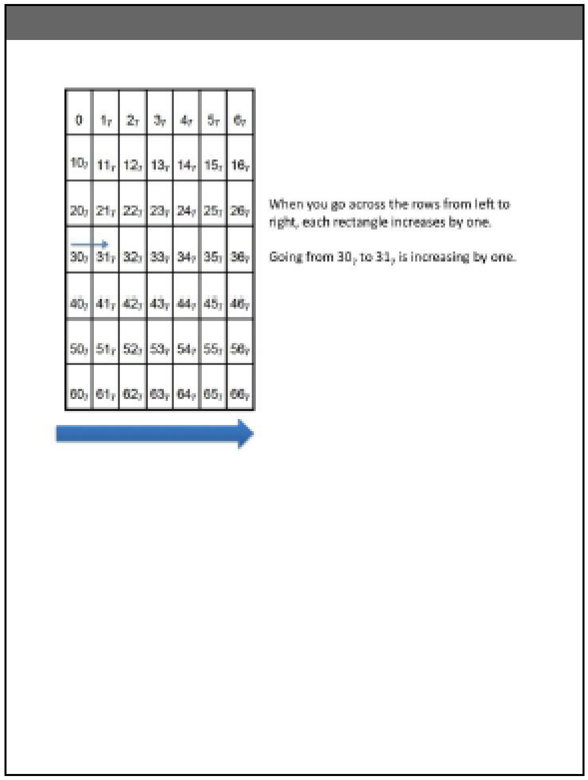

The last number in this row is always zero.

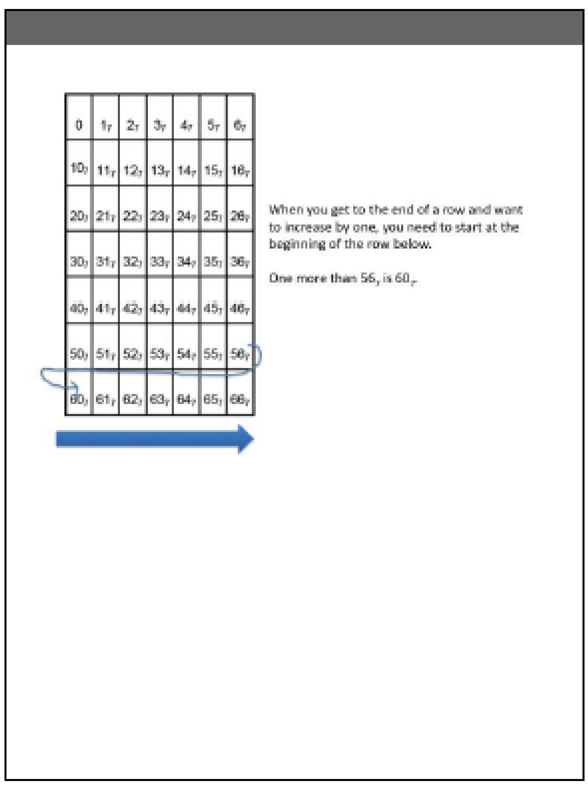

The last number in this row is always five.

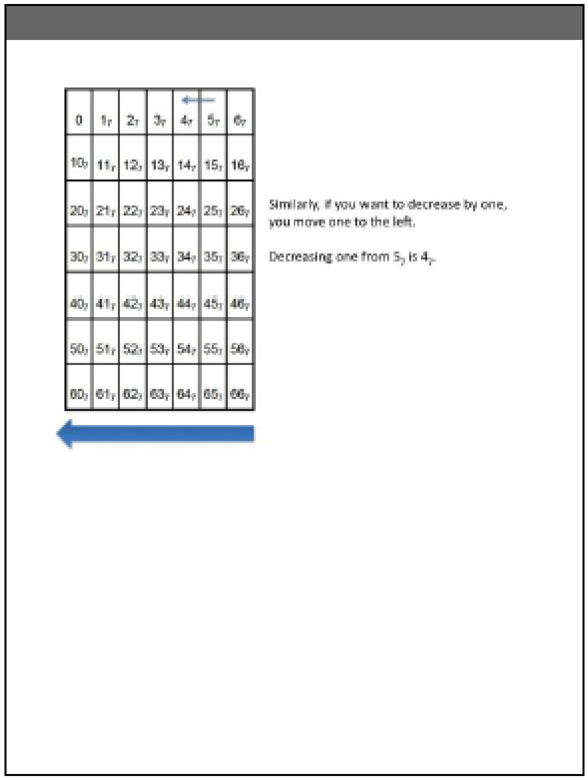

Across this row, all the numbers start with two.

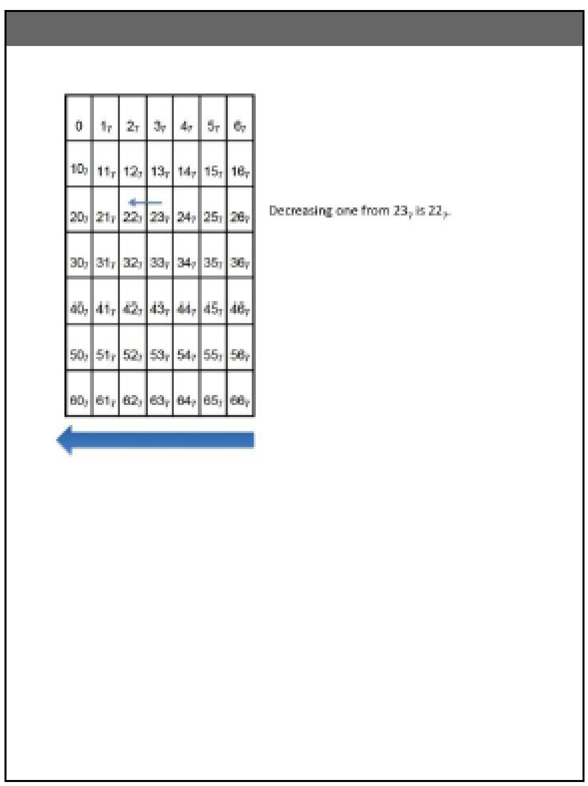

Across this row, all the numbers start with four.

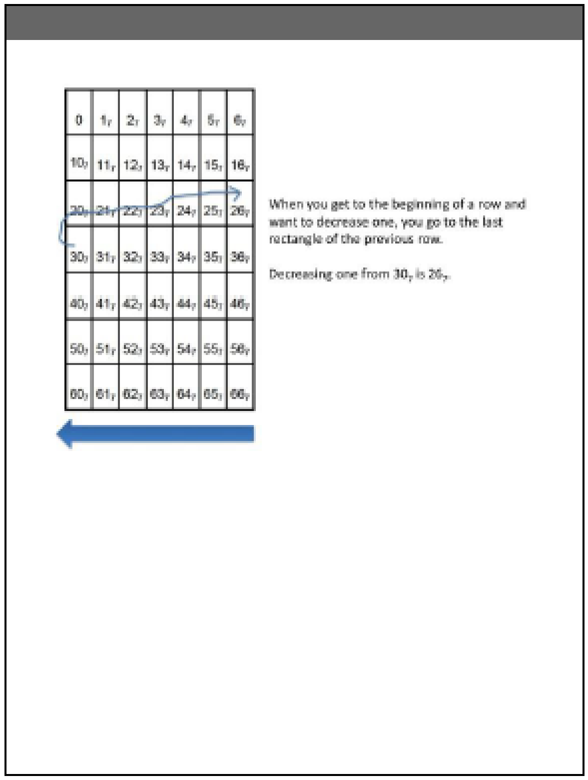

Across this row, all the numbers start with six.

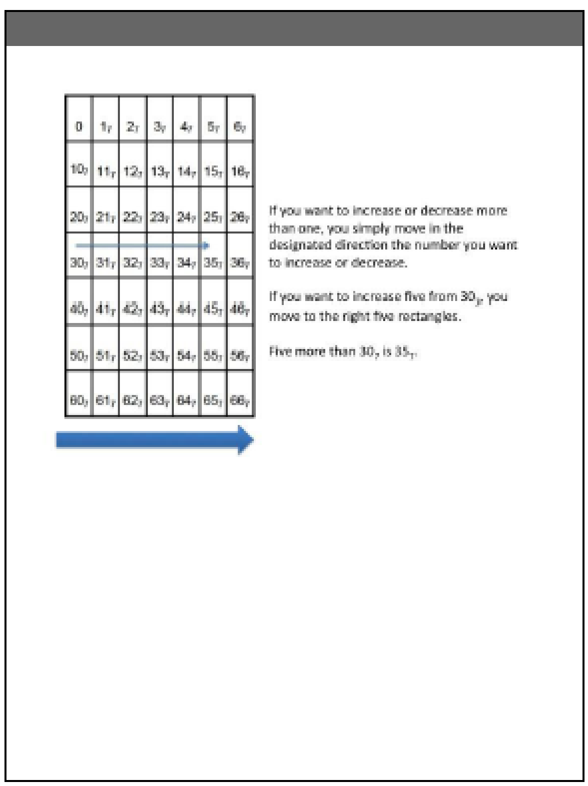

0 1_7 2_7 3_7 4_7 5_7 6_7
10_7 11_7 12_7 13_7 14_7 15_7 16_7
20_7 21_7 22_7 23_7 24_7 25_7 26_7
30_7 31_7 32_7 33_7 34_7 35_7 36_7
40_7 41_7 42_7 43_7 44_7 45_7 46_7
50_7 51_7 52_7 53_7 54_7 55_7 56_7
60_7 61_7 62_7 63_7 64_7 65_7 66_7
When you go across the rows from left to right, each rectangle increases by one.
Going from 30_7 to 31_7 is increasing by one.

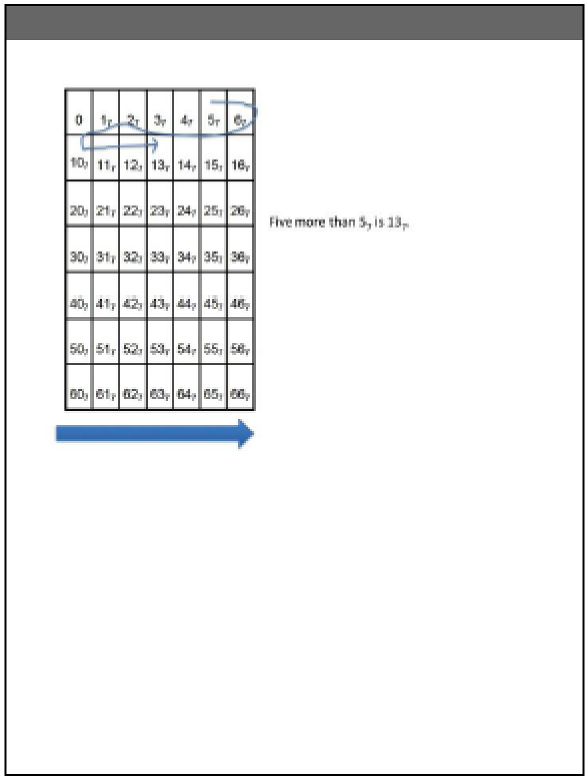

0 1_7 2_7 3_7 4_7 5_7 6_7
10_7 11_7 12_7 13_7 14_7 15_7 16_7
20_7 21_7 22_7 23_7 24_7 25_7 26_7
30_7 31_7 32_7 33_7 34_7 35_7 36_7
40_7 41_7 42_7 43_7 44_7 45_7 46_7
50_7 51_7 52_7 53_7 54_7 55_7 56_7
60_7 61_7 62_7 63_7 64_7 65_7 66_7
When you get to the end of a row and want to increase by one, you need to start at the beginning of the row below.
One more than 56_7 is 60_7.

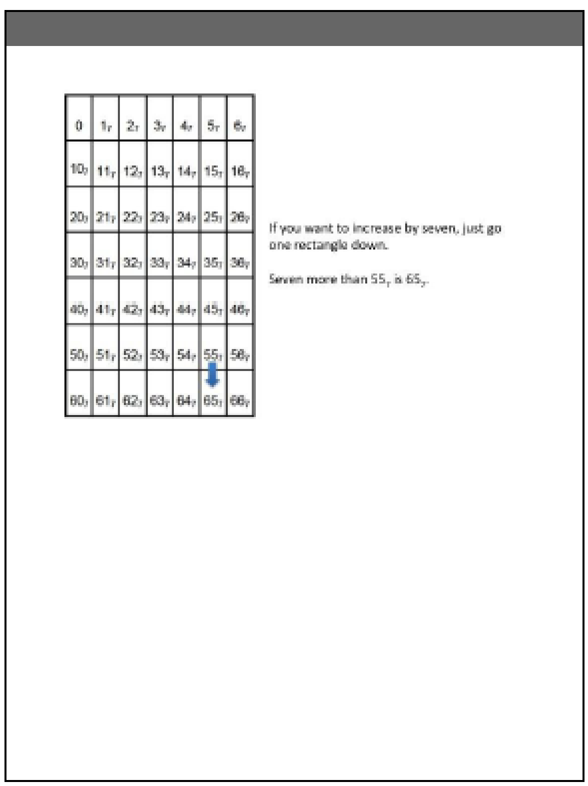

0 1_7 2_7 3_7 4_7 5_7 6_7
10_7 11_7 12_7 13_7 14_7 15_7 16_7
20_7 21_7 22_7 23_7 24_7 25_7 26_7
30_7 31_7 32_7 33_7 34_7 35_7 36_7
40_7 41_7 42_7 43_7 44_7 45_7 46_7
50_7 51_7 52_7 53_7 54_7 55_7 56_7
60_7 61_7 62_7 63_7 64_7 65_7 66_7
Similarly, if you want to decrease by one, you move one to the left.
Decreasing one from 5_7 is 4_7.

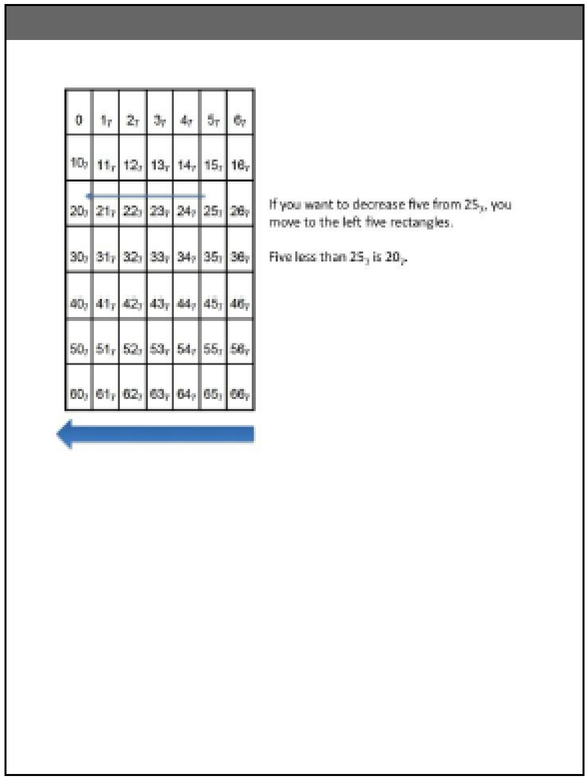

0 1_7 2_7 3_7 4_7 5_7 6_7
10_7 11_7 12_7 13_7 14_7 15_7 16_7
20_7 21_7 22_7 23_7 24_7 25_7 26_7
30_7 31_7 32_7 33_7 34_7 35_7 36_7
40_7 41_7 42_7 43_7 44_7 45_7 46_7
50_7 51_7 52_7 53_7 54_7 55_7 56_7
60_7 61_7 62_7 63_7 64_7 65_7 66_7
Decreasing one from 23_7 is 22_7.

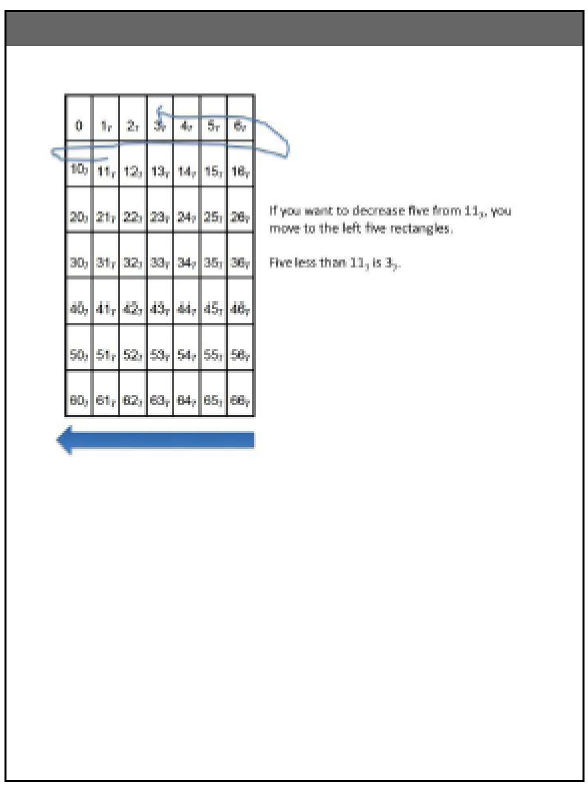

When you get to the beginning of a row and want to decrease one, you go to the last rectangle of the previous row.
Decreasing one from 30_7 is 26_7.

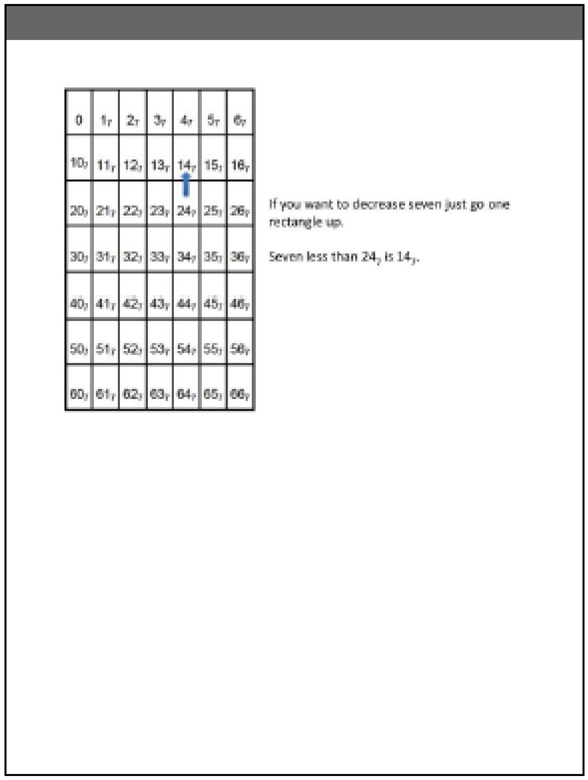

If you want to increase or decrease more than one, you simply move in the designated direction the number you want to increase or decrease.
If you want to increase five from 30_7, you move to the right five rectangles.
Five more than 30_7 is 35_7.


0 | 1_7 | 2_7 | 3_7 | 4_7 | 5_7 | 6_7
10_7 | 11_7 | 12_7 | 13_7 | 14_7 | 15_7 | 16_7
20_7 | 21_7 | 22_7 | 23_7 | 24_7 | 25_7 | 26_7
30_7 | 31_7 | 32_7 | 33_7 | 34_7 | 35_7 | 36_7
40_7 | 41_7 | 42_7 | 43_7 | 44_7 | 45_7 | 46_7
50_7 | 51_7 | 52_7 | 53_7 | 54_7 | 55_7 | 56_7
60_7 | 61_7 | 62_7 | 63_7 | 64_7 | 65_7 | 66_7
Five more than 5_7 is 13_7.


0 | 1_7 | 2_7 | 3_7 | 4_7 | 5_7 | 6_7
10_7 | 11_7 | 12_7 | 13_7 | 14_7 | 15_7 | 16_7
20_7 | 21_7 | 22_7 | 23_7 | 24_7 | 25_7 | 26_7
30_7 | 31_7 | 32_7 | 33_7 | 34_7 | 35_7 | 36_7
40_7 | 41_7 | 42_7 | 43_7 | 44_7 | 45_7 | 46_7
50_7 | 51_7 | 52_7 | 53_7 | 54_7 | 55_7 | 56_7
60_7 | 61_7 | 62_7 | 63_7 | 64_7 | 65_7 | 66_7
If you want to increase by seven, just go one rectangle down.
Seven more than 55_7 is 65_7.

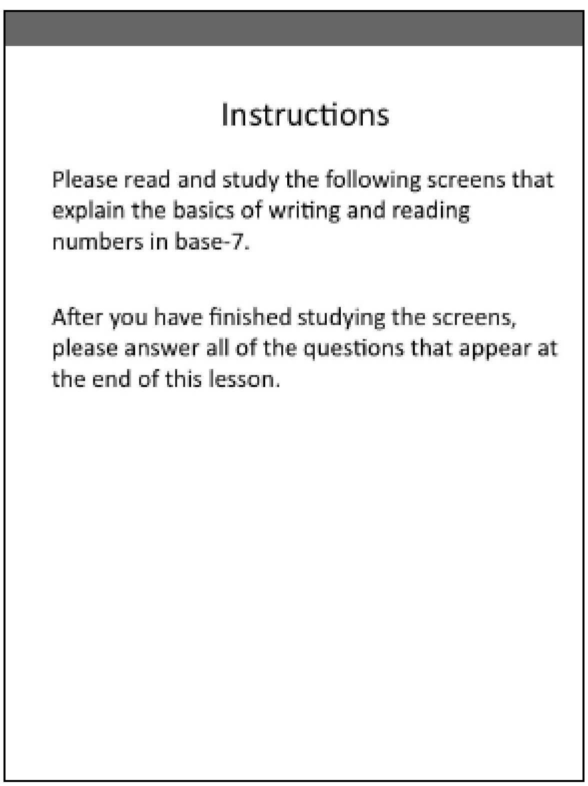

If you want to decrease five from 25_7, you move to the left five rectangles.
Five less than 25_7 is 20_7.


If you want to decrease five from 11_7, you move to the left five rectangles.
Five less than 11_7 is 3_7.


0 1_7 2_7 3_7 4_7 5_7 6_7
10_7 11_7 12_7 13_7 14_7 15_7 16_7
20_7 21_7 22_7 23_7 24_7 25_7 26_7
30_7 31_7 32_7 33_7 34_7 35_7 36_7
40_7 41_7 42_7 43_7 44_7 45_7 46_7
50_7 51_7 52_7 53_7 54_7 55_7 56_7
60_7 61_7 62_7 63_7 64_7 65_7 66_7
If you want to decrease seven just go one rectangle up.
Seven less than 24_7 is 14_7.

Appendix P


*1. Student ID Number
Introduction
You have previously learned about another way to write numbers and count in base-7.
Today you are going to learn more about numbers and counting in base-7.


Instructions
Please read and study the following screens that explain the basics of writing and reading numbers in base-7.
After you have finished studying the screens, please answer all of the questions that appear at the end of this lesson.

Number Charts

Number charts are useful tools elementary school teachers often use to help their students learn many concepts related to numbers such as counting, adding, and subtracting.

The following screen shows you a base-7 number chart.

Study the chart and the instructional slides that follow carefully.

This is a Base-7 Number Chart

0	1_7	2_7	3_7	4_7	5_7	6_7
10_7	11_7	12_7	13_7	14_7	15_7	16_7
20_7	21_7	22_7	23_7	24_7	25_7	26_7
30_7	31_7	32_7	33_7	34_7	35_7	36_7
40_7	41_7	42_7	43_7	44_7	45_7	46_7
50_7	51_7	52_7	53_7	54_7	55_7	56_7
60_7	61_7	62_7	63_7	64_7	65_7	66_7

Patterns in the Base-7 Number Chart

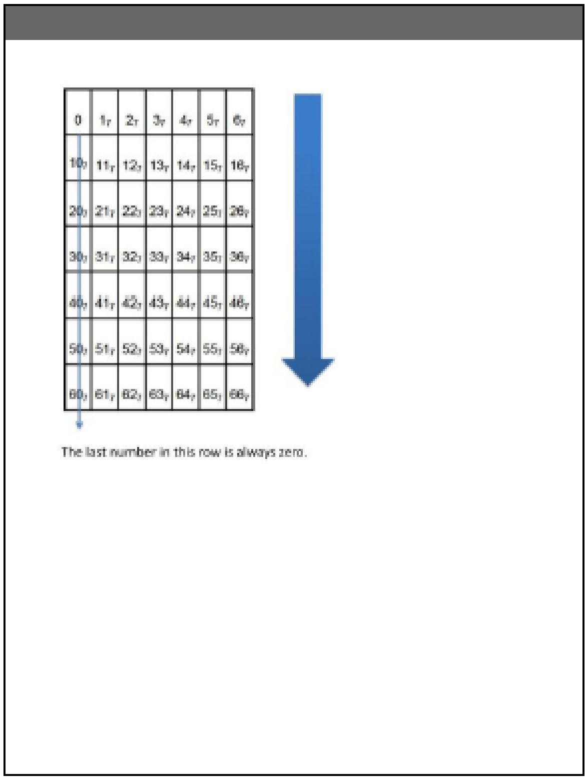
Notice that there are seven columns across the chart.

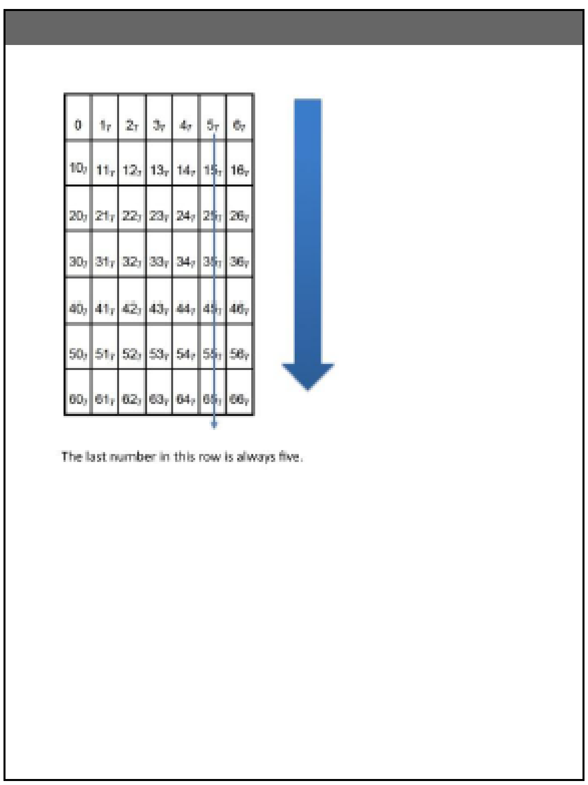
Furthermore, when you go down the columns in the chart, the last number is always the same.

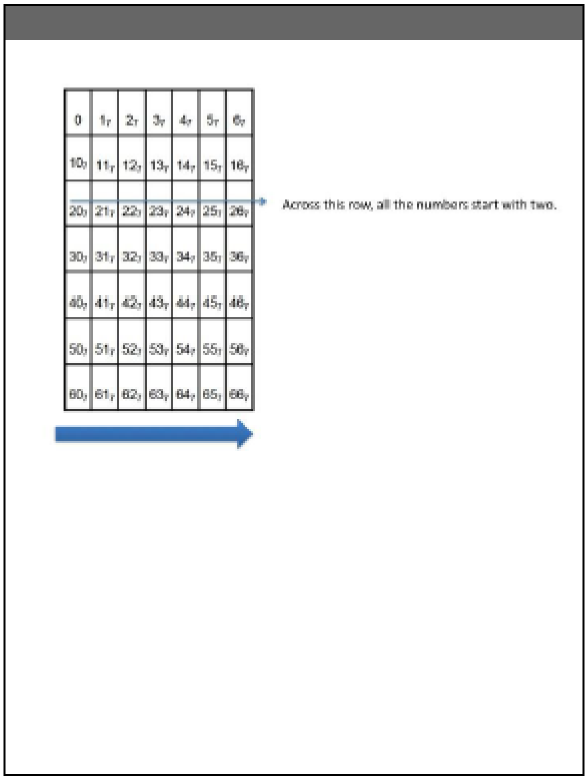
When you go across the rows, the first number always stays the same (except for the first row).

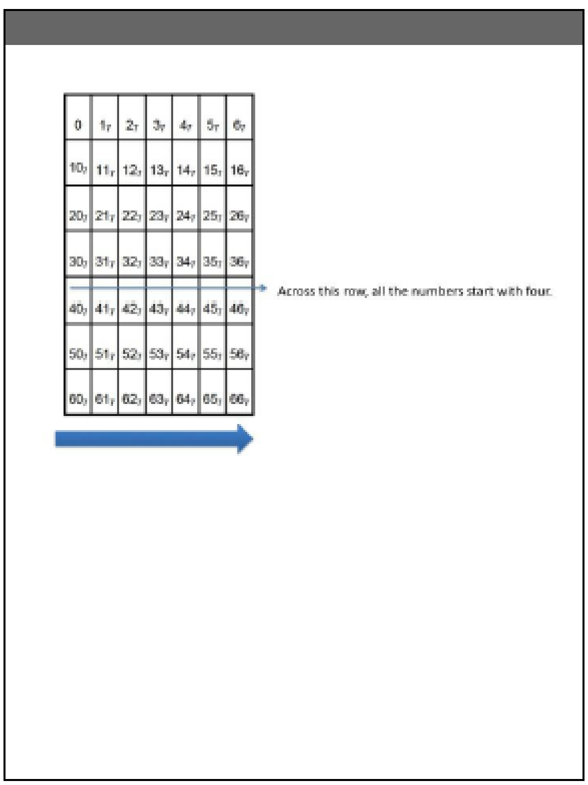
0	1_7	2_7	3_7	4_7	5_7	6_7
10_7	11_7	12_7	13_7	14_7	15_7	16_7
20_7	21_7	22_7	23_7	24_7	25_7	26_7
30_7	31_7	32_7	33_7	34_7	35_7	36_7
40_7	41_7	42_7	43_7	44_7	45_7	46_7
50_7	51_7	52_7	53_7	54_7	55_7	56_7
60_7	61_7	62_7	63_7	64_7	65_7	66_7

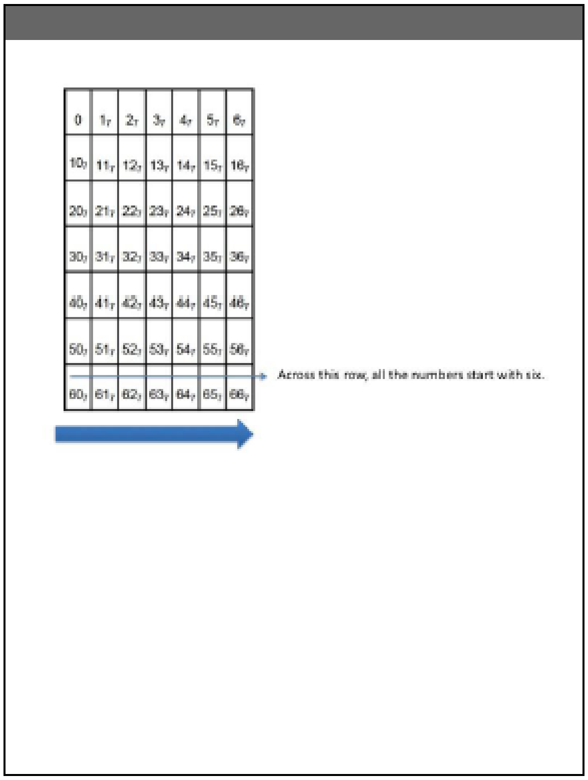
The last number in this row is always three.

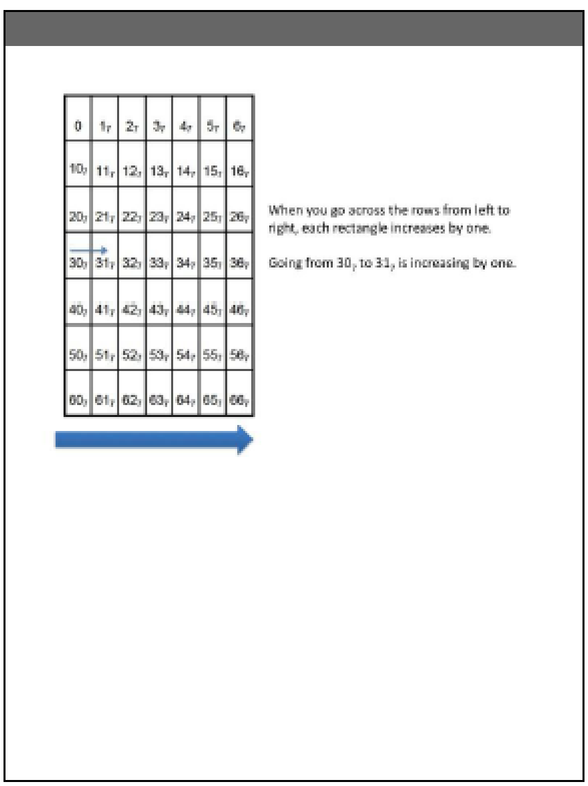

The last number in this row is always zero.

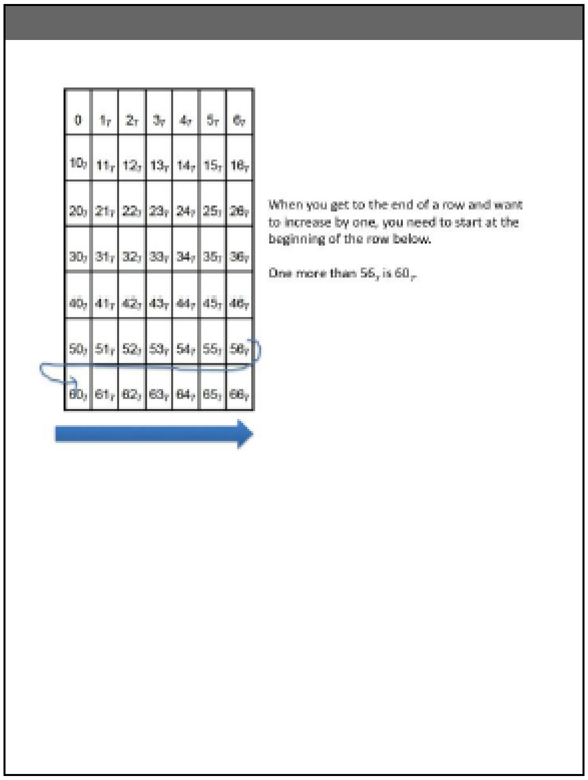

The last number in this row is always five.

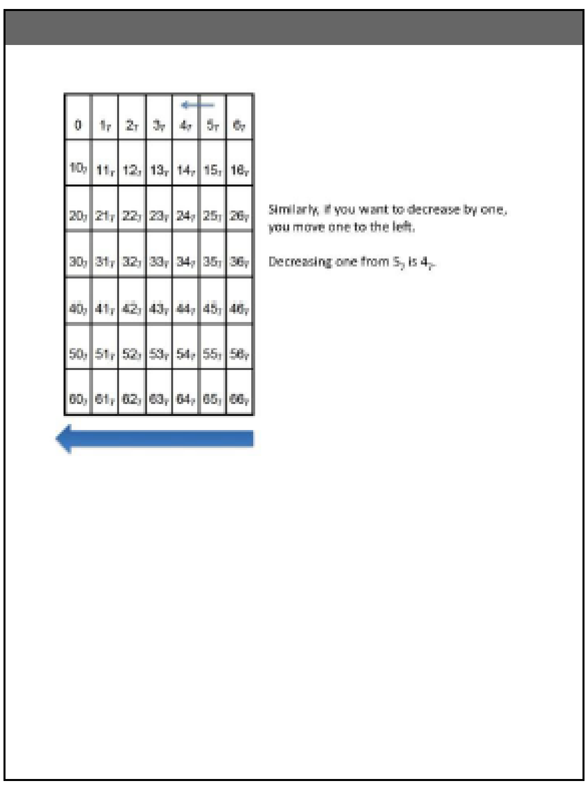

Across this row, all the numbers start with two.

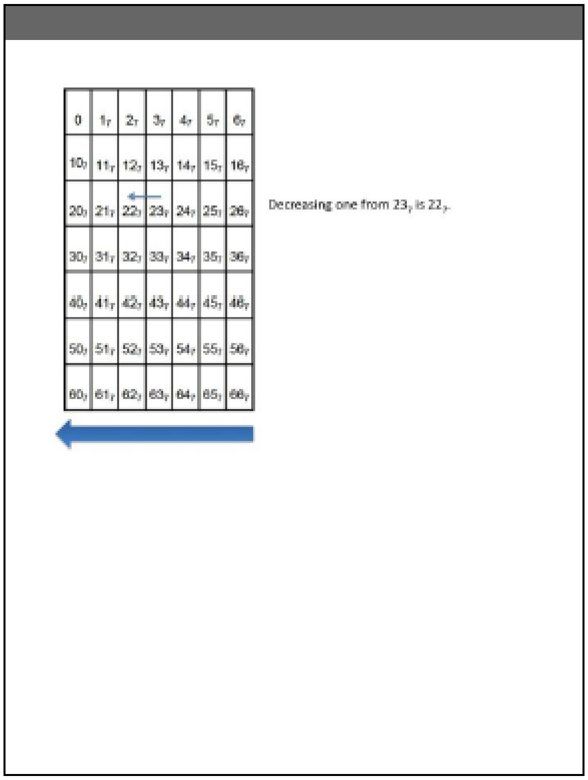

Across this row, all the numbers start with four.

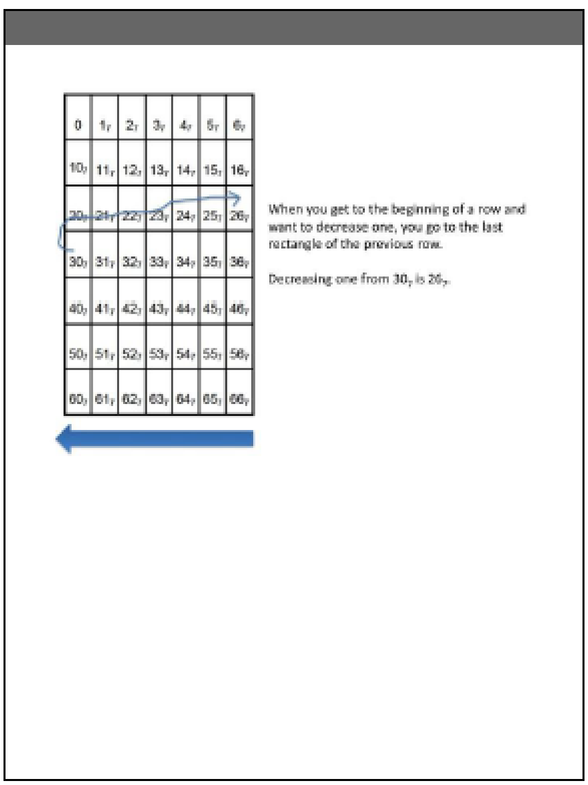

Across this row, all the numbers start with six.


When you go across the rows from left to right, each rectangle increases by one.
Going from 30_7 to 31_7 is increasing by one.


When you get to the end of a row and want to increase by one, you need to start at the beginning of the row below.
One more than 56_7 is 60_7.


Similarly, if you want to decrease by one, you move one to the left.
Decreasing one from 5_7 is 4_7.


0 | 1_7 | 2_7 | 3_7 | 4_7 | 5_7 | 6_7
10_7 | 11_7 | 12_7 | 13_7 | 14_7 | 15_7 | 16_7
20_7 | 21_7 | 22_7 | 23_7 | 24_7 | 25_7 | 26_7
30_7 | 31_7 | 32_7 | 33_7 | 34_7 | 35_7 | 36_7
40_7 | 41_7 | 42_7 | 43_7 | 44_7 | 45_7 | 46_7
50_7 | 51_7 | 52_7 | 53_7 | 54_7 | 55_7 | 56_7
60_7 | 61_7 | 62_7 | 63_7 | 64_7 | 65_7 | 66_7
Decreasing one from 23_7 is 22_7.


0 | 1_7 | 2_7 | 3_7 | 4_7 | 5_7 | 6_7
10_7 | 11_7 | 12_7 | 13_7 | 14_7 | 15_7 | 16_7
20_7 | 21_7 | 22_7 | 23_7 | 24_7 | 25_7 | 26_7
30_7 | 31_7 | 32_7 | 33_7 | 34_7 | 35_7 | 36_7
40_7 | 41_7 | 42_7 | 43_7 | 44_7 | 45_7 | 46_7
50_7 | 51_7 | 52_7 | 53_7 | 54_7 | 55_7 | 56_7
60_7 | 61_7 | 62_7 | 63_7 | 64_7 | 65_7 | 66_7
When you get to the beginning of a row and want to decrease one, you go to the last rectangle of the previous row.
Decreasing one from 30_7 is 26_7.

0	1_7	2_7	3_7	4_7	5_7	6_7
10_7	11_7	12_7	13_7	14_7	15_7	16_7
20_7	21_7	22_7	23_7	24_7	25_7	26_7
30_7	31_7	32_7	33_7	34_7	35_7	36_7
40_7	41_7	42_7	43_7	44_7	45_7	46_7
50_7	51_7	52_7	53_7	54_7	55_7	56_7
60_7	61_7	62_7	63_7	64_7	65_7	66_7

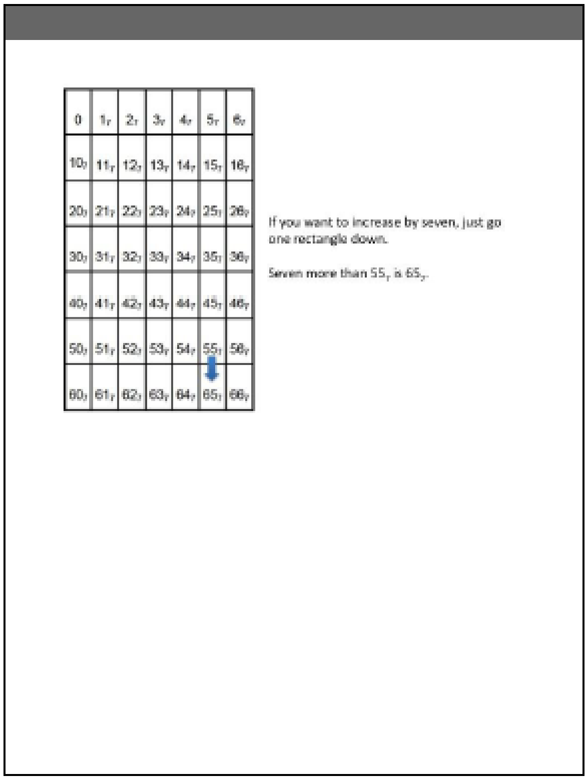
If you want to increase or decrease more than one, you simply move in the designated direction the number you want to increase or decrease.

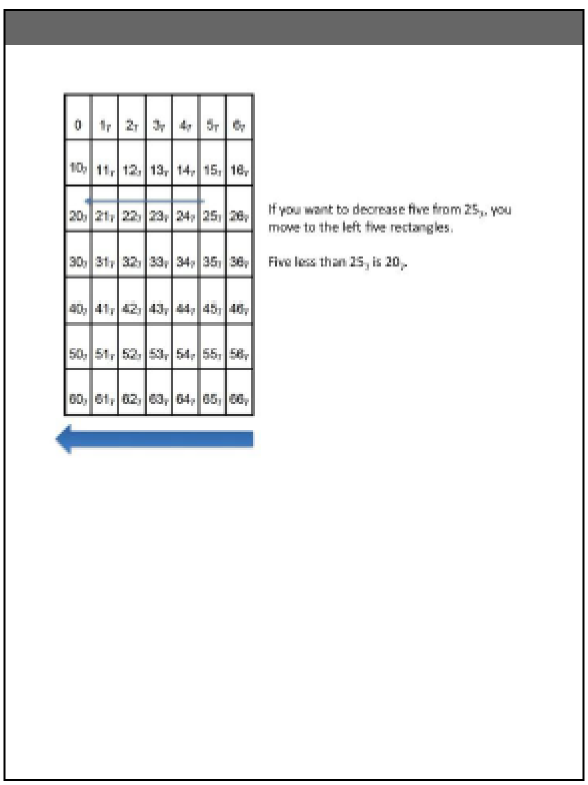
If you want to increase five from 30_7, you move to the right five rectangles.

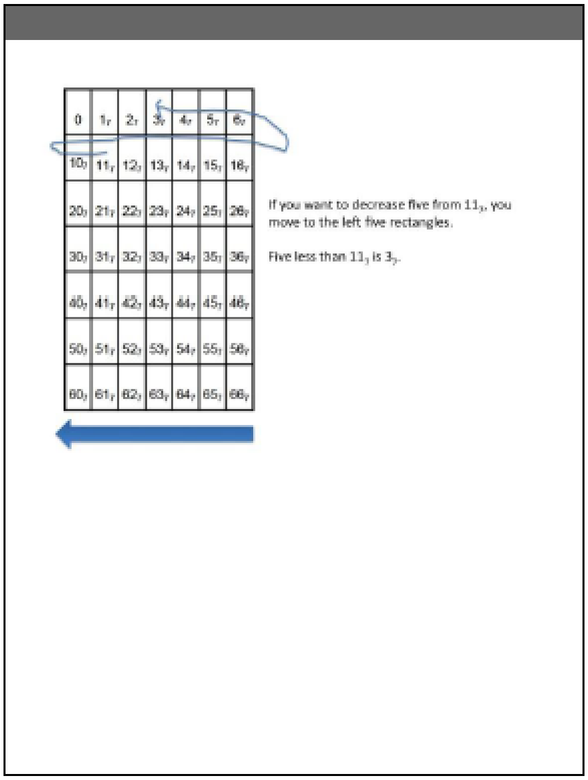
Five more than 30_7 is 35_7.

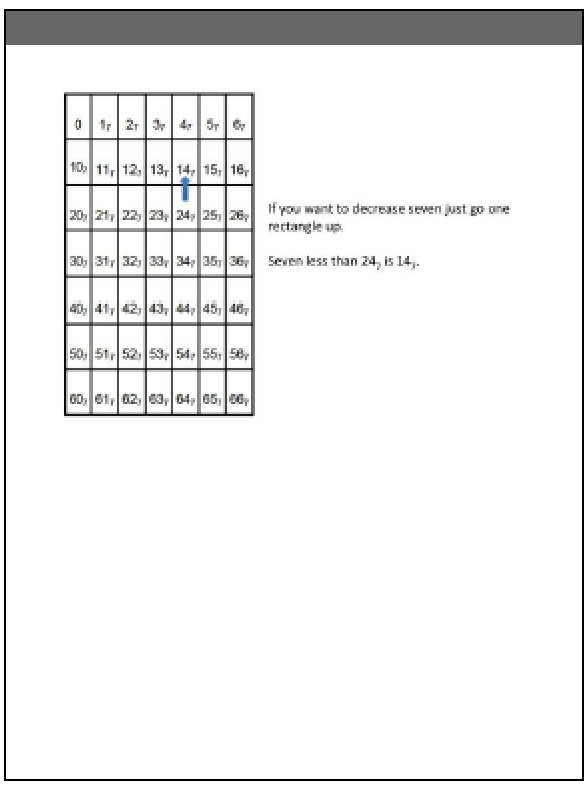
0	1_7	2_7	3_7	4_7	5_7	6_7
10_7	11_7	12_7	13_7	14_7	15_7	16_7
20_7	21_7	22_7	23_7	24_7	25_7	26_7
30_7	31_7	32_7	33_7	34_7	35_7	36_7
40_7	41_7	42_7	43_7	44_7	45_7	46_7
50_7	51_7	52_7	53_7	54_7	55_7	56_7
60_7	61_7	62_7	63_7	64_7	65_7	66_7

Five more than 5_7 is 13_7.

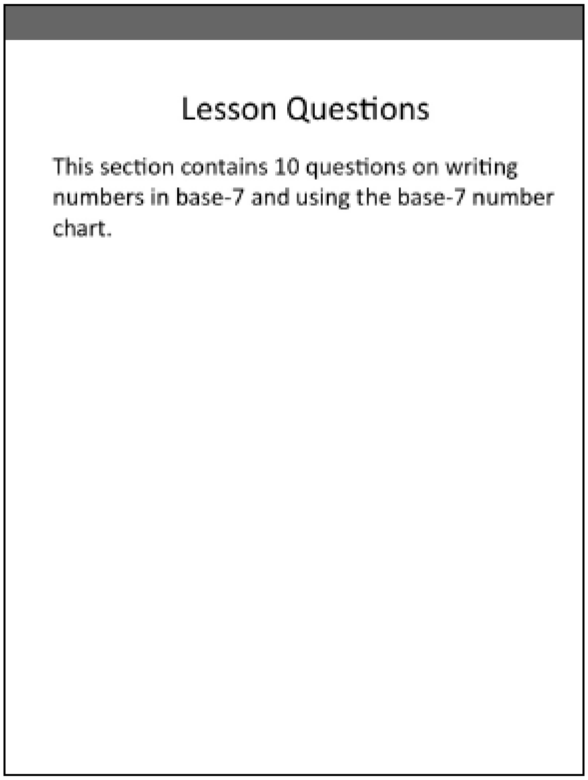

0 1_7 2_7 3_7 4_7 5_7 6_7
10_7 11_7 12_7 13_7 14_7 15_7 16_7
20_7 21_7 22_7 23_7 24_7 25_7 26_7
30_7 31_7 32_7 33_7 34_7 35_7 36_7
40_7 41_7 42_7 43_7 44_7 45_7 46_7
50_7 51_7 52_7 53_7 54_7 55_7 56_7
60_7 61_7 62_7 63_7 64_7 65_7 66_7
If you want to increase by seven, just go one rectangle down.
Seven more than 55_7 is 65_7.


0 1_7 2_7 3_7 4_7 5_7 6_7
10_7 11_7 12_7 13_7 14_7 15_7 16_7
20_7 21_7 22_7 23_7 24_7 25_7 26_7
30_7 31_7 32_7 33_7 34_7 35_7 36_7
40_7 41_7 42_7 43_7 44_7 45_7 46_7
50_7 51_7 52_7 53_7 54_7 55_7 56_7
60_7 61_7 62_7 63_7 64_7 65_7 66_7
If you want to decrease five from 25_7, you move to the left five rectangles.
Five less than 25_7 is 20_7.


If you want to decrease five from 11_7, you move to the left five rectangles.
Five less than 11_7 is 3_7.


If you want to decrease seven just go one rectangle up.
Seven less than 24_7 is 14_7.

Appendix P


Lesson Questions
This section contains 10 questions on writing numbers in base-7 and using the base-7 number chart.

Instructions

Please answers these questions to the best of your ability.

You will not be able to proceed to the next question until you answer the question you are on.

You may refer back to the lesson at any time.

0	A	2_7	3_7	4_7	5_7	6_7
10_7	11_7	12_7	13_7	14_7	15_7	16_7
20_7	21_7	22_7	23_7	B	25_7	C
D	31_7	32_7	33_7	34_7	35_7	36_7
40_7	41_7	42_7	43_7	44_7	45_7	46_7
50_7	51_7	E	53_7	54_7	55_7	56_7
60_7	61_7	62_7	63_7	64_7	65_7	66_7

***2. Fill in the missing values in a way that preserves the pattern.**

A is:

***3. Fill in the missing values in a way that preserves the pattern.**

B is:

4. Fill in the missing values in a way that preserves the pattern.

C is:

*5. Fill in the missing values in a way that preserves the pattern.

D is:

*6. Fill in the missing values in a way that preserves the pattern.

E is:

0	A	2_7	3_7	4_7	5_7	6_7
10_7	11_7	12_7	13_7	14_7	15_7	16_7
20_7	21_7	22_7	23_7	B	25_7	C
D	31_7	32_7	33_7	34_7	35_7	36_7
40_7	41_7	42_7	43_7	44_7	45_7	46_7
50_7	51_7	E	53_7	54_7	55_7	56_7
60_7	61_7	62_7	63_7	64_7	65_7	66_7

*7. In words, write which number is seven more than A.

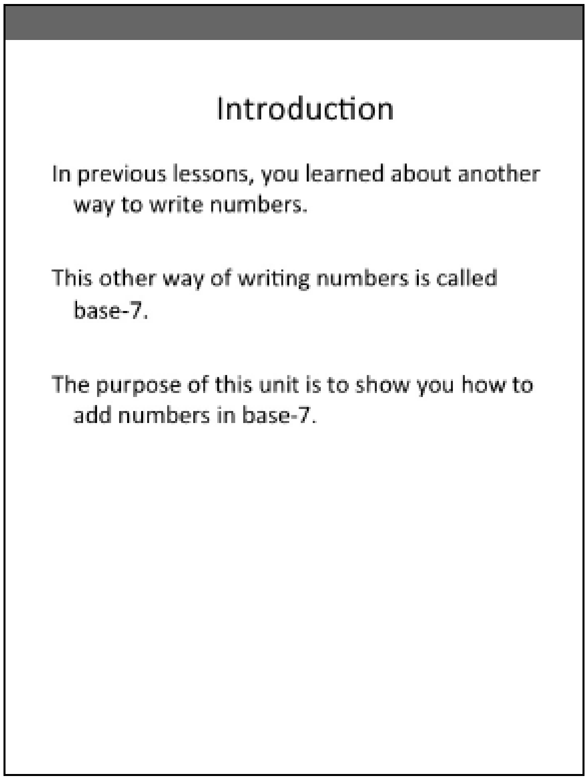
*8. In words, write which number is six more than D.

*9. In words, write which number is seven less than C.

*10. In words, write which number is four less than E.

*11. In words, write which number is four less than B.

Appendix Q


Introduction
In previous lessons, you learned about another way to write numbers.
This other way of writing numbers is called base-7.
The purpose of this unit is to show you how to add numbers in base-7.

Instructions

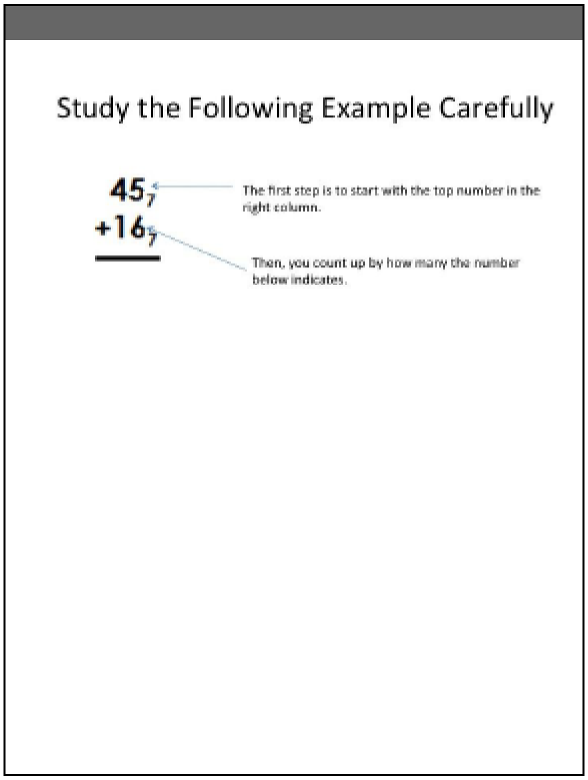
Please read and study the following screens that explain how to add numbers using a traditional algorithm in base-7.

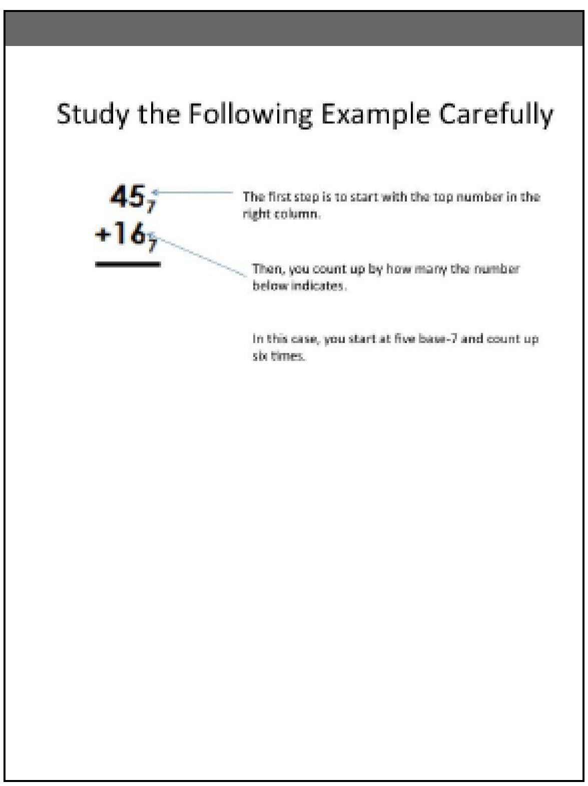
After you have finished studying the screens, please answer all of the questions that appear at the end of this lesson.

Study the Following Example Carefully

$$\begin{array}{r} 45_7 \\ +16_7 \\ \hline \end{array}$$

The first step is to start with the top number in the right column.


Study the Following Example Carefully
45_7
$+16_7$
The first step is to start with the top number in the right column.
Then, you count up by how many the number below indicates.


Study the Following Example Carefully
45_7
$+16_7$
The first step is to start with the top number in the right column.
Then, you count up by how many the number below indicates.
In this case, you start at five base-7 and count up six times.

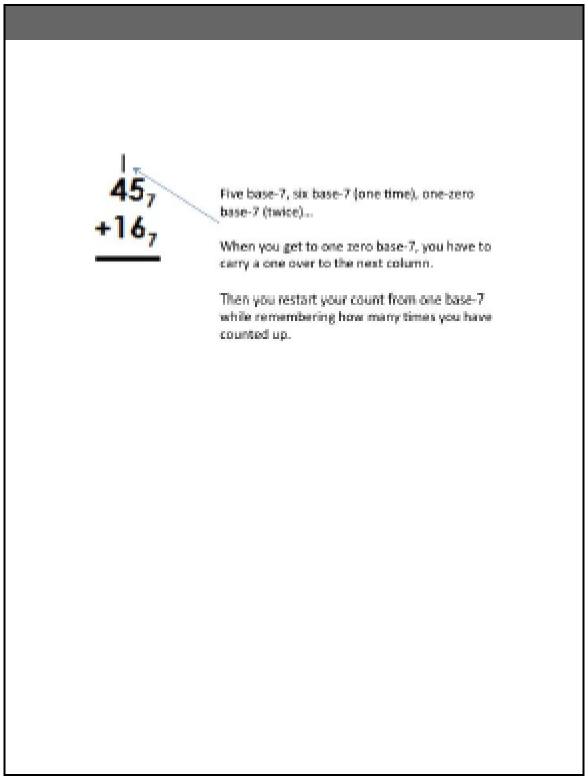
$$\begin{array}{r} 45_7 \\ +16_7 \\ \hline \end{array}$$

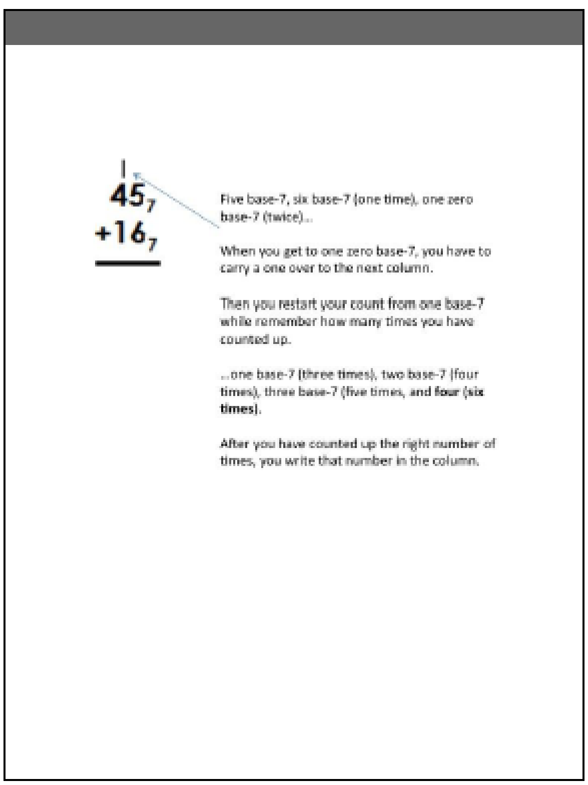
Five base-7, six base-7(one time), one-zero base-7 (twice)...

$$\begin{array}{r} 1 \\ 45_7 \\ +16_7 \\ \hline \end{array}$$

Five base-7, six base-7 (one time), one-zero base-7 (twice)...

When you get to one zero base-7, you have to carry a one over to the next column.


1
45_7
$+16_7$
Five base-7, six base-7 (one time), one-zero base-7 (twice)...
When you get to one zero base-7, you have to carry a one over to the next column.
Then you restart your count from one base-7 while remembering how many times you have counted up.


1
45_7
$+16_7$
Five base-7, six base-7 (one time), one zero base-7 (twice)...
When you get to one zero base-7, you have to carry a one over to the next column.
Then you restart your count from one base-7 while remember how many times you have counted up.
...one base-7 (three times), two base-7 (four times), three base-7 (five times, and **four (six times)**.
After you have counted up the right number of times, you write that number in the column.

$$\begin{array}{r} 1 \\ 45_7 \\ +16_7 \\ \hline 4_7 \end{array}$$

Five base-7, six base-7 (one time), one-zero base-7 (twice)...

When you get to one zero base-7, you have to carry a one over to the next column.

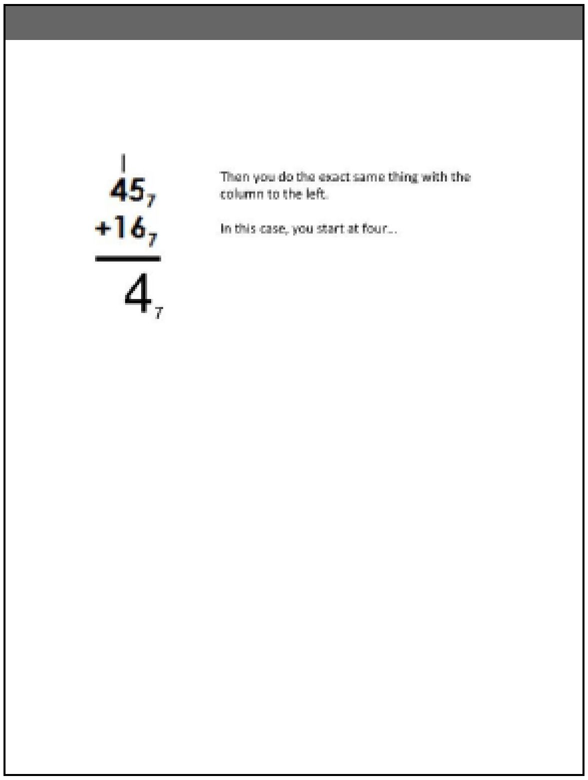
Then you restart your count from one base-7 while remembering how many times you have counted up.

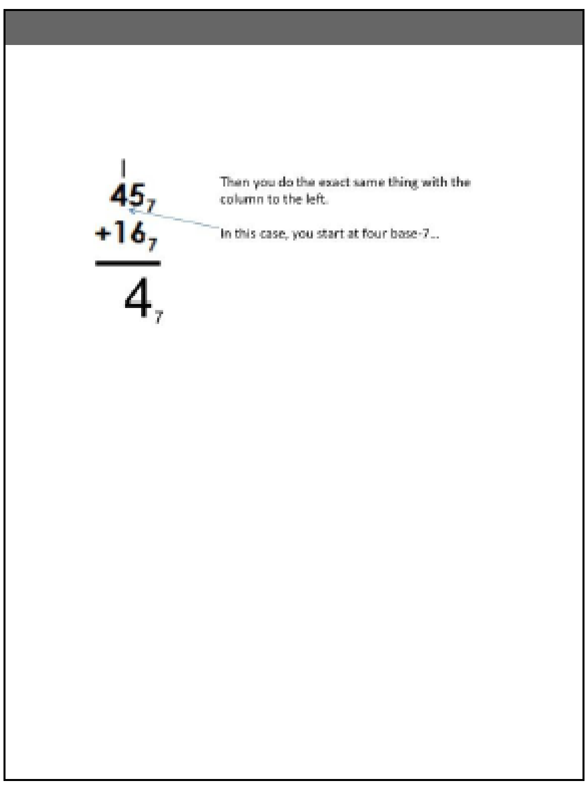
...one base-7 (three times), two base-7 (four times), three base-7 (five times, and **four (six times)**.

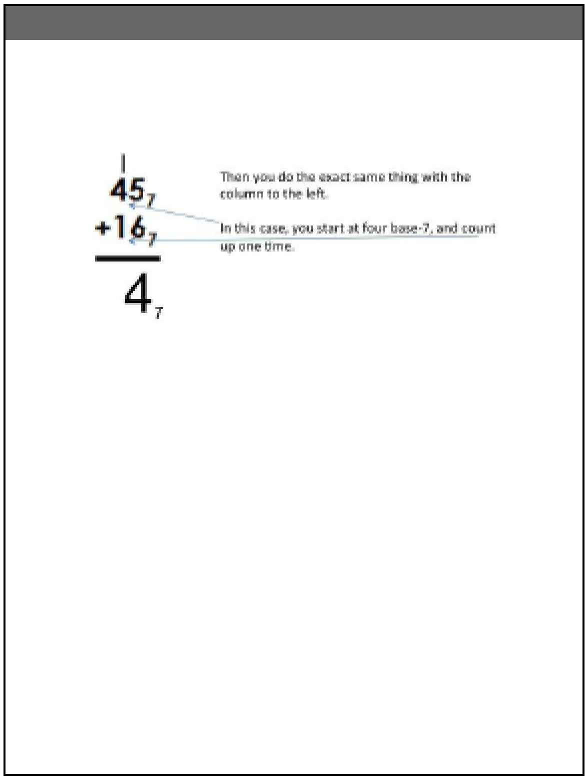
After you have counted up the right number of times, you write that number in the column.

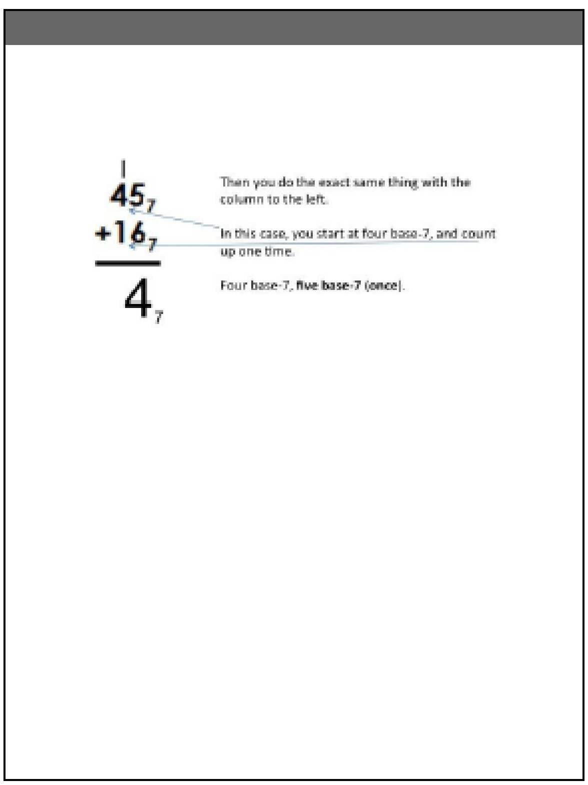
$$\begin{array}{r} 1 \\ 45_7 \\ +16_7 \\ \hline 4_7 \end{array}$$

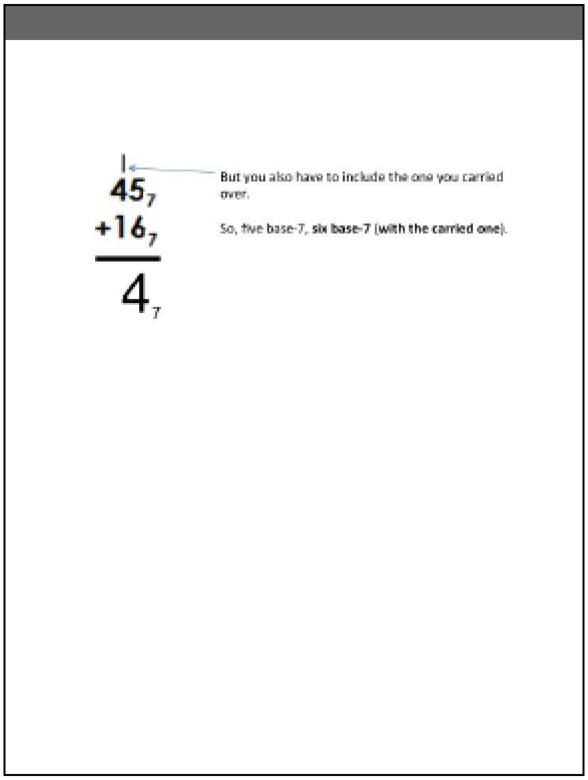
Then you do the exact same thing with the column to the left.

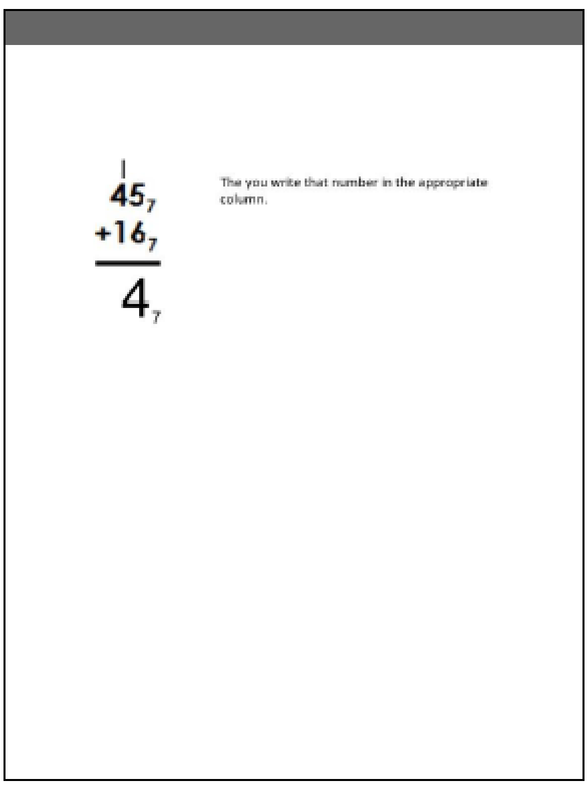

1
45_7
$+16_7$
4_7
Then you do the exact same thing with the column to the left.
In this case, you start at four...

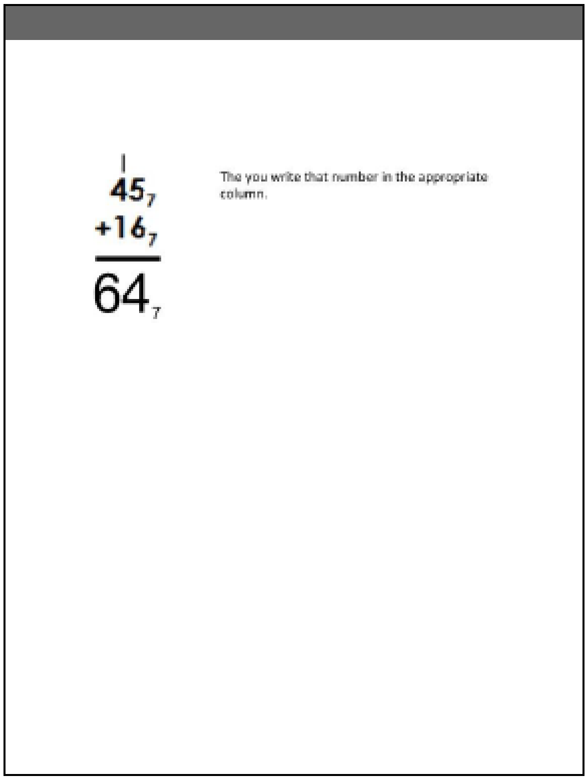

1
45_7
$+16_7$
4_7
Then you do the exact same thing with the column to the left.
In this case, you start at four base-7...

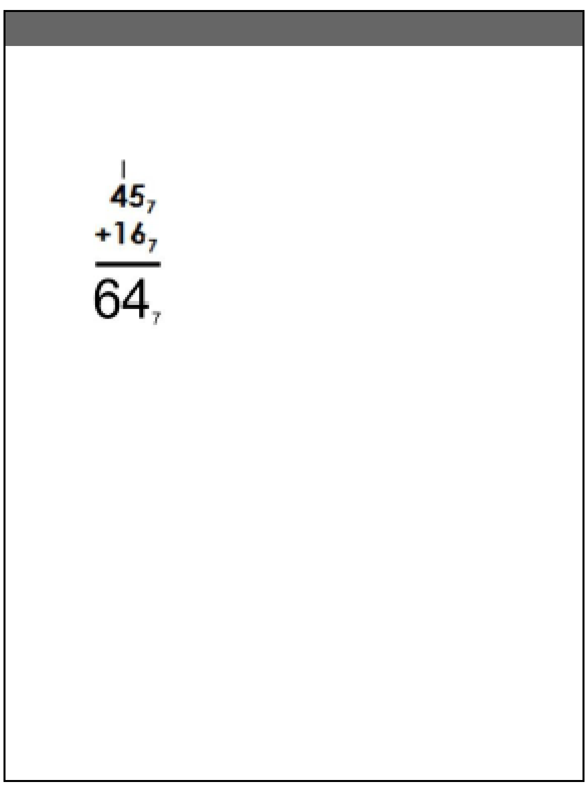

1
45_7
$+16_7$
4_7
Then you do the exact same thing with the column to the left.
In this case, you start at four base-7, and count up one time.


1
45_7
$+16_7$
4_7
Then you do the exact same thing with the column to the left.
In this case, you start at four base-7, and count up one time.
Four base-7, **five base-7 (once)**.


1
45_7
$+16_7$
4_7
But you also have to include the one you carried over.
So, five base-7, six base-7 (with the carried one).


1
45_7
$+16_7$
4_7
The you write that number in the appropriate column.


$\overset{1}{45_7}$
$+16_7$
64_7
The you write that number in the appropriate column.


$\overset{1}{45_7}$
$+16_7$
64_7

One more Example:

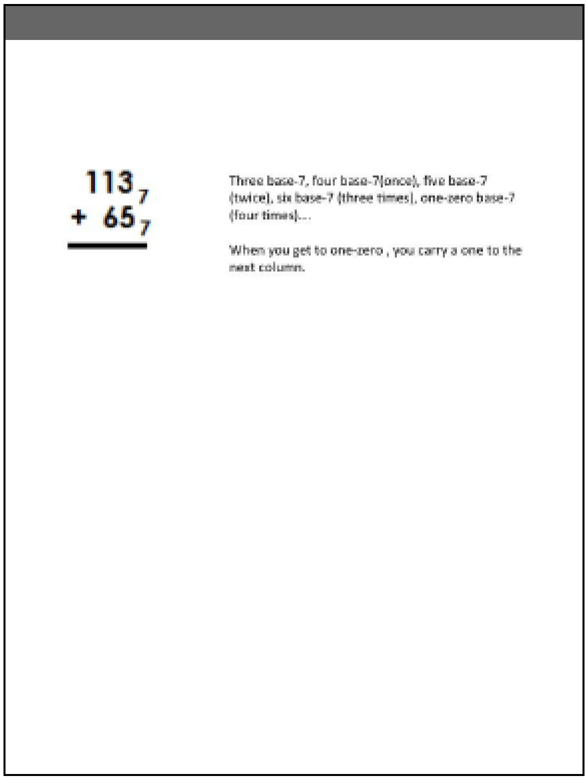
Please study the next example as well. Take your time to understand how the algorithm works.

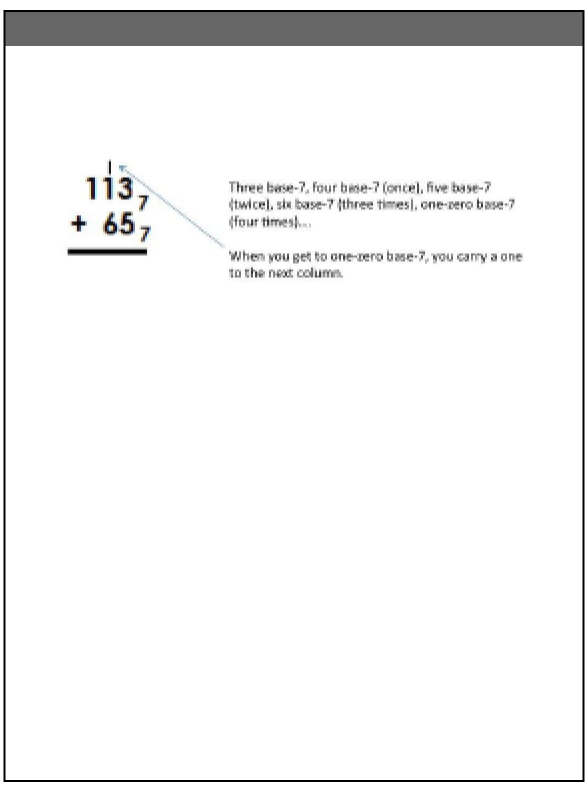
$$\begin{array}{r} 113_7 \\ +\ 65_7 \\ \hline \end{array}$$

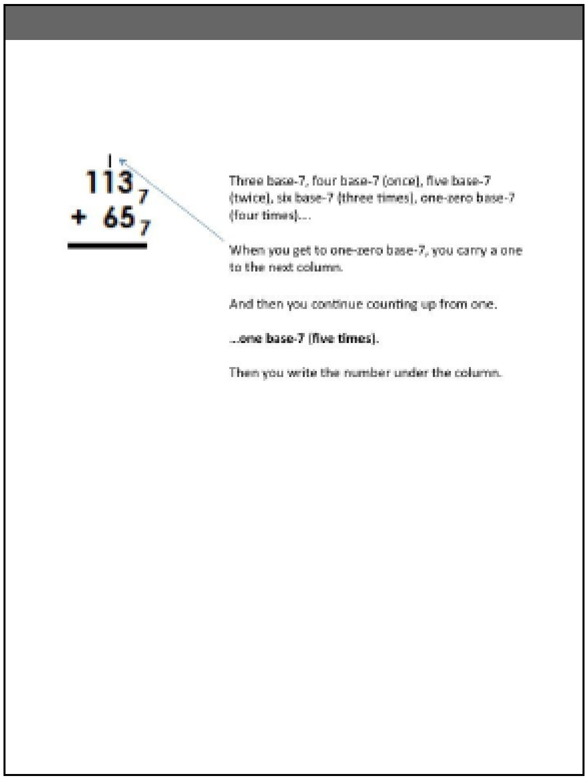
The first step is to start with the top number in the right column.

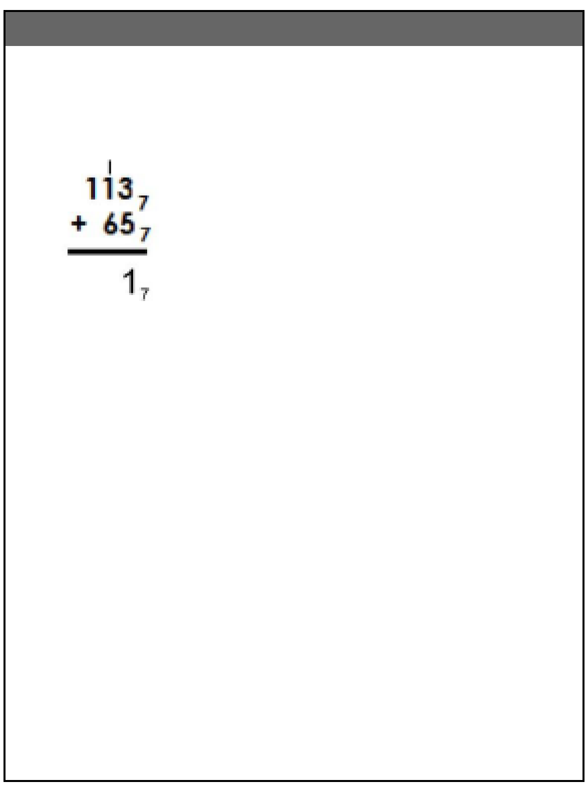
Then, you count up by how many the number below indicates.

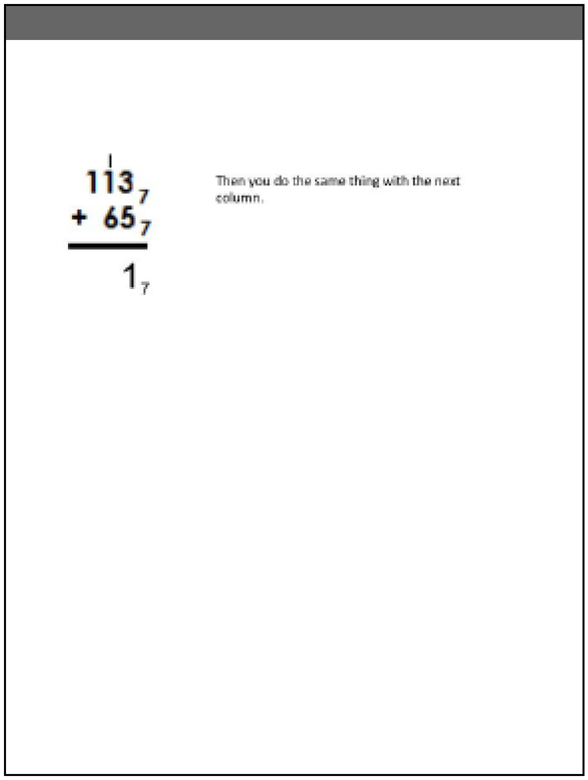
In this case, you start at three base-7 and count up five times.


113₇
+ 65₇
Three base-7, four base-7(once), five base-7 (twice), six base-7 (three times), one-zero base-7 (four times)...
When you get to one-zero , you carry a one to the next column.

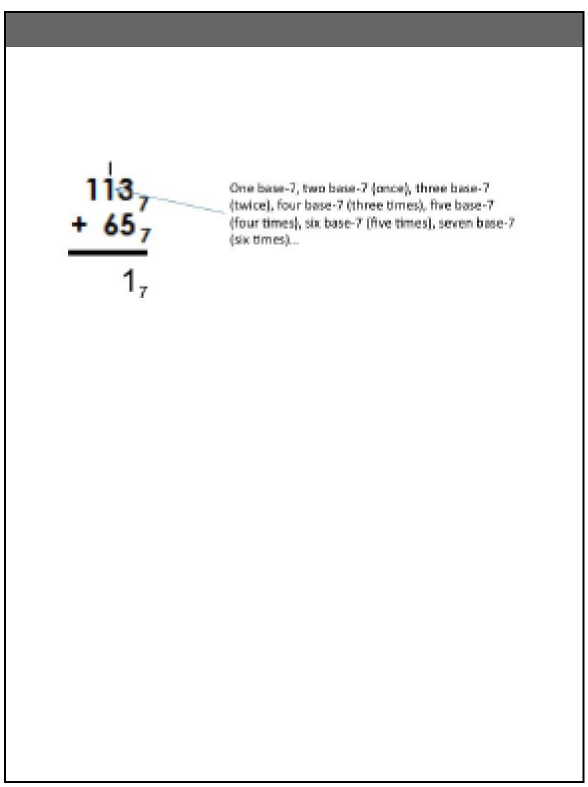

1
113₇
+ 65₇
Three base-7, four base-7 (once), five base-7 (twice), six base-7 (three times), one-zero base-7 (four times)...
When you get to one-zero base-7, you carry a one to the next column.


1
113₇
+ 65₇
Three base-7, four base-7 (once), five base-7 (twice), six base-7 (three times), one-zero base-7 (four times)....
When you get to one-zero base-7, you carry a one to the next column.
And then you continue counting up from one.
...one base-7 (five times).
Then you write the number under the column.


1
113₇
+ 65₇
1₇



1
113₇
+ 65₇
1₇
Then you do the same thing with the next column.



1
113₇
+ 65₇
1₇
One base-7, two base-7 (once), three base-7 (twice), four base-7 (three times), five base-7 (four times), six base-7 (five times), seven base-7 (six times)...

$$\begin{array}{r} 1\,1 \\ 113_7 \\ +\ 65_7 \\ \hline 1_7 \end{array}$$

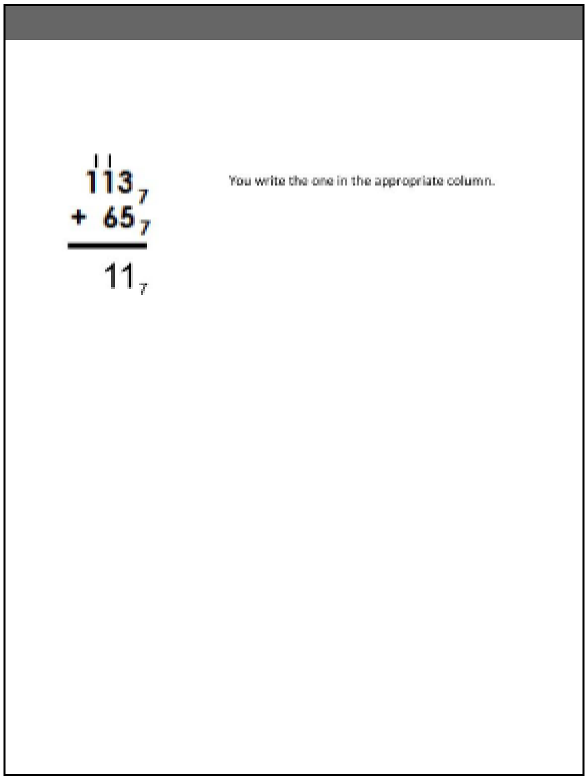
One base-7, two base-7 (once), three base-7 (twice), four base-7 (three times), five base-7 (four times), six base-7 (five times), seven base-7 (six times)...

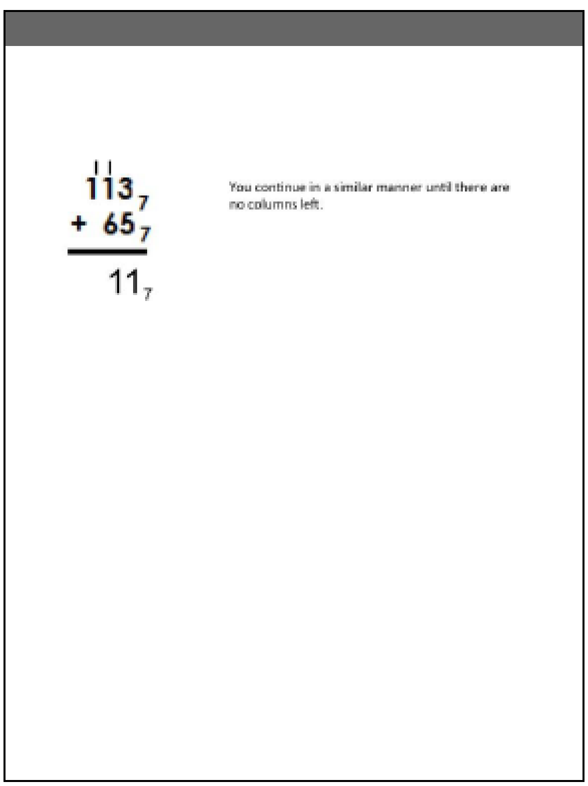
If there was no one carried over, you would write a zero under the column. But because there is a one carried over from the previous column, it must be included.

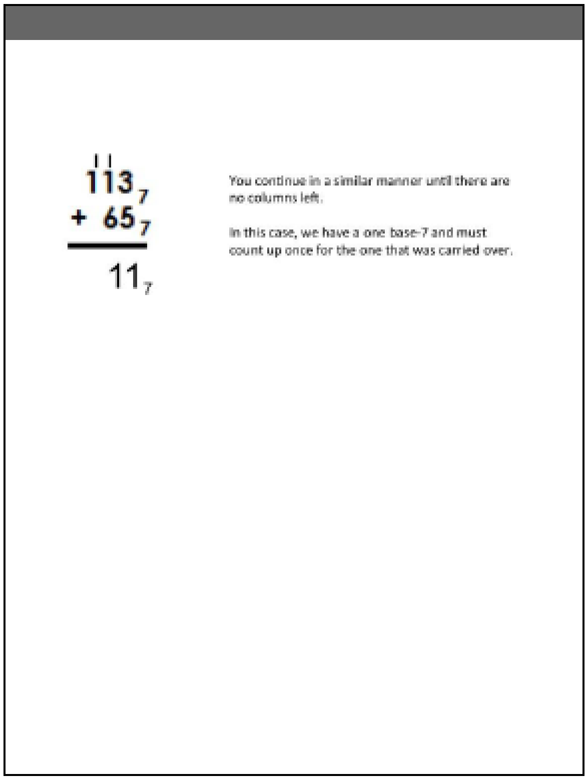
... one base-7 (including the carried one).

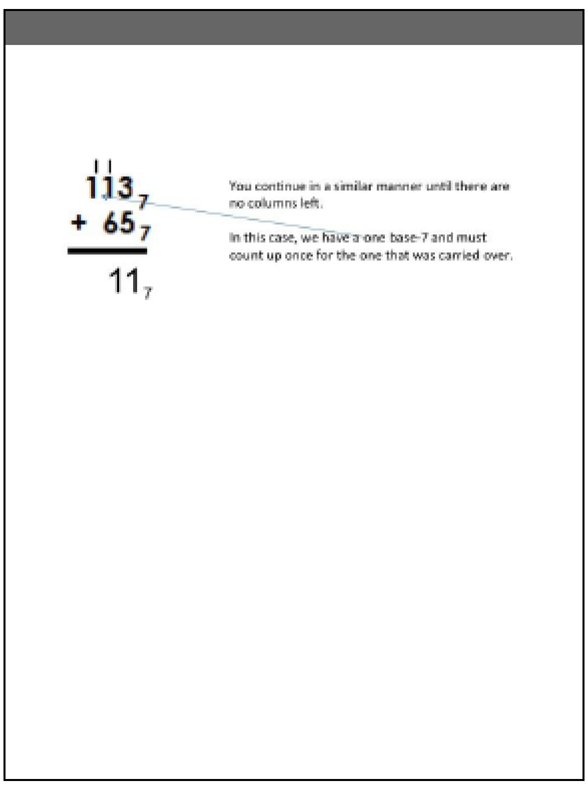
$$\begin{array}{r} 1\,1 \\ 113_7 \\ +\ 65_7 \\ \hline 1_7 \end{array}$$

You write the one in the appropriate column.


1 1
113_7
$+ 65_7$
11_7
You write the one in the appropriate column.


1 1
113_7
$+ 65_7$
11_7
You continue in a similar manner until there are no columns left.


1 1
113₇
+ 65₇
11₇
You continue in a similar manner until there are no columns left.
In this case, we have a one base-7 and must count up once for the one that was carried over.


1 1
113₇
+ 65₇
11₇
You continue in a similar manner until there are no columns left.
In this case, we have a one base-7 and must count up once for the one that was carried over.

$$\begin{array}{r} {\scriptstyle 1\,1} \\ 113_7 \\ +\ 65_7 \\ \hline 11_7 \end{array}$$

You continue in a similar manner until there are no columns left.

In this case, we have a one base-7 and must count up once for the one that was carried over.

One base-7, **two base-7 (carried one)**.

$$\begin{array}{r} {\scriptstyle 1\,1} \\ 113_7 \\ +\ 65_7 \\ \hline 211_7 \end{array}$$

Then you write the number in the appropriate column.

$$\begin{array}{r} \overset{1}{1}\overset{1}{1}3_7 \\ +\ 65_7 \\ \hline 211_7 \end{array}$$

Appendix R

P3 End of Session Questions

Student ID #:___________________________________

Use the algorithm you learned to compute the following

1. $$\begin{array}{r} 341_7 \\ +\ 233_7 \\ \hline \end{array}$$

2. $$\begin{array}{r} 143_7 \\ +\ 65_7 \\ \hline \end{array}$$

3. $$\begin{array}{r} 245_7 \\ +\ 66_7 \\ \hline \end{array}$$

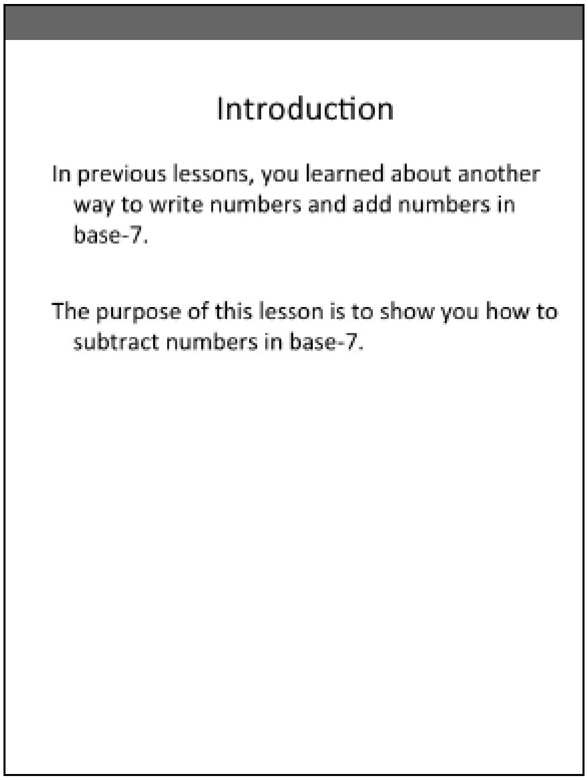
4.

$$\begin{array}{r} 64_7 \\ +\ 23_7 \\ \hline \end{array}$$

5

$$\begin{array}{r} 44_7 \\ +14_7 \\ \hline \end{array}$$

Appendix S


Introduction
In previous lessons, you learned about another way to write numbers and add numbers in base-7.
The purpose of this lesson is to show you how to subtract numbers in base-7.

Instructions

Please read and study the following slides that explain how to subtract numbers using a traditional algorithm in base-7.

After you have finished studying the slides, please answer all of the questions that appear at the end of this lesson.

Study the Following Example Carefully

$$\begin{array}{r} 45_7 \\ \underline{-16_7} \end{array}$$

The first step is to start with the top number in the right column.

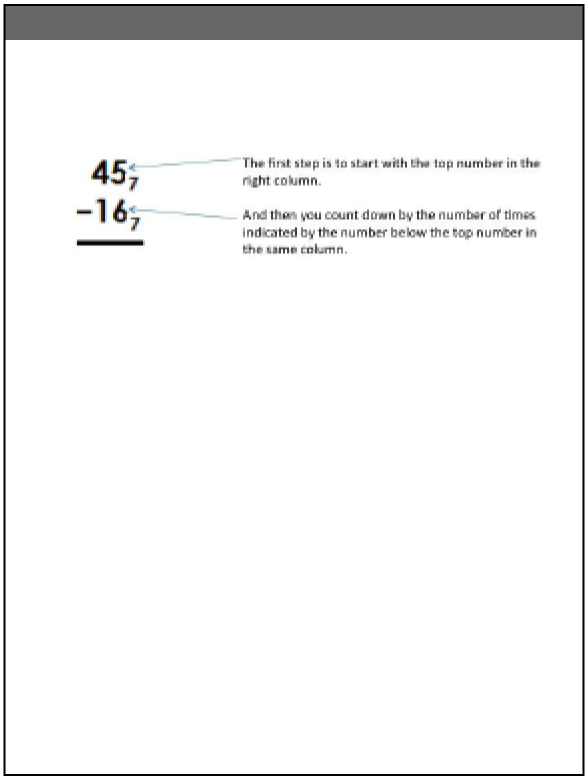
$$\begin{array}{r} 45_7 \\ -16_7 \\ \hline \end{array}$$

The first step is to start with the top number in the right column.

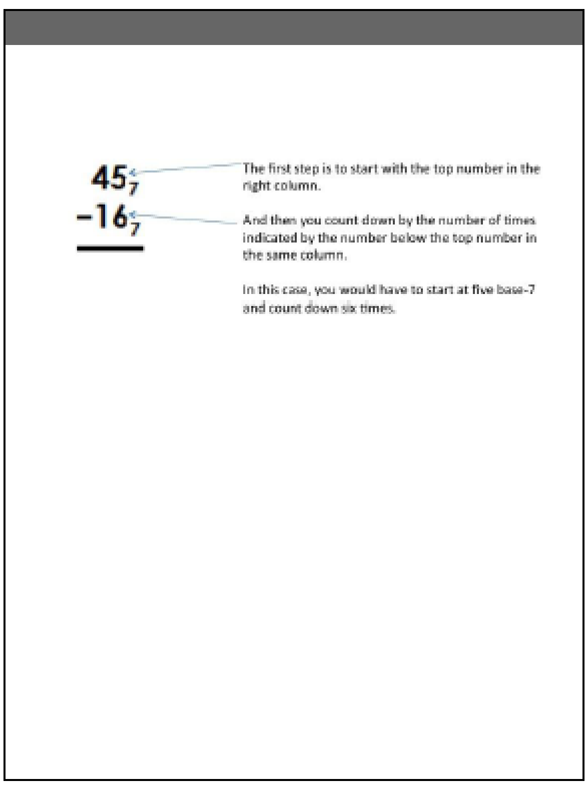
$$\begin{array}{r} 45_7 \\ -16_7 \\ \hline \end{array}$$

The first step is to start with the top number in the right column.

And then you count down by the number of times indicated by the number below the top number in the same column.



45_7
-16_7
The first step is to start with the top number in the right column.
And then you count down by the number of times indicated by the number below the top number in the same column.



45_7
-16_7
The first step is to start with the top number in the right column.
And then you count down by the number of times indicated by the number below the top number in the same column.
In this case, you would have to start at five base-7 and count down six times.

$$\begin{array}{r} 45_7 \\ -16_7 \\ \hline \end{array}$$

The first step is to start with the top number in the right column.

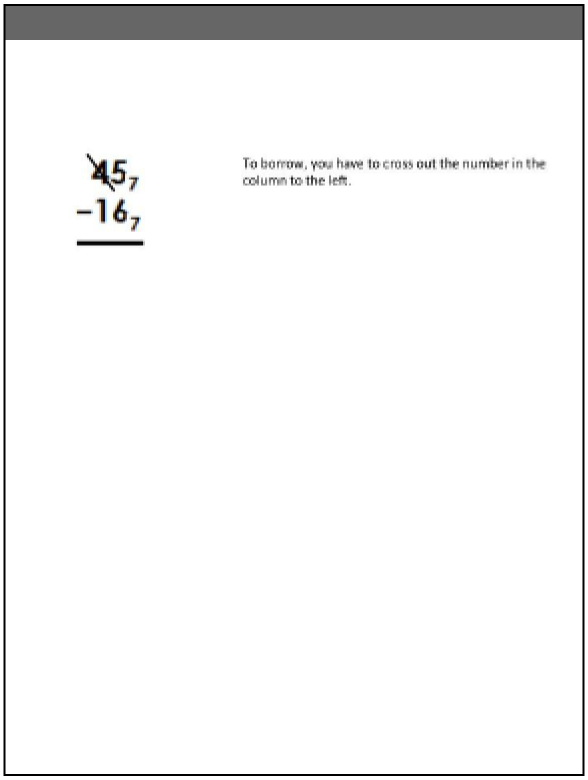
And then you count down by the number of times indicated by the number below the top number in the same column.

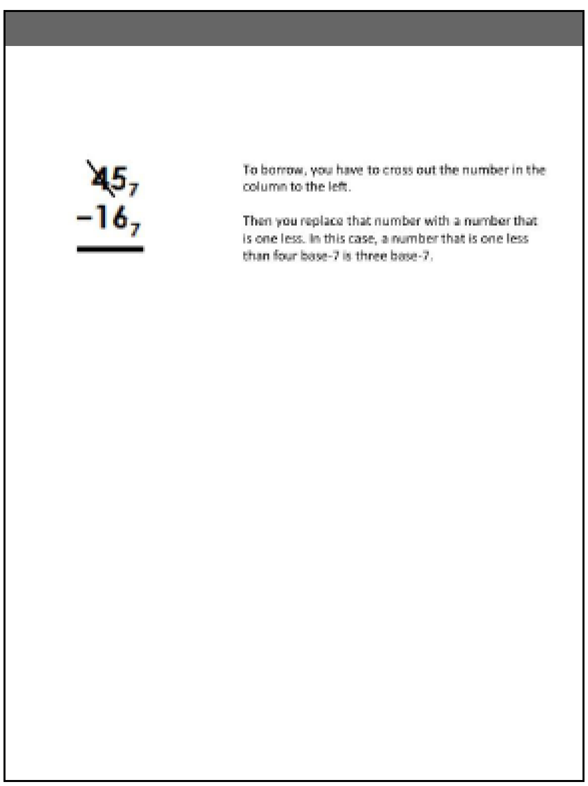
In this case, you would have to start at five base-7 and count down six times.

Because six is more than five base-7, you have to borrow from the next column.

$$\begin{array}{r} 45_7 \\ -16_7 \\ \hline \end{array}$$

To borrow, you have to cross out the number in the column to the left.


$\cancel{4}5_7$
-16_7
To borrow, you have to cross out the number in the column to the left.


$\cancel{4}5_7$
-16_7
To borrow, you have to cross out the number in the column to the left.
Then you replace that number with a number that is one less. In this case, a number that is one less than four base-7 is three base-7.

$$\begin{array}{r} \overset{3}{\cancel{4}}5_7 \\ -16_7 \\ \hline \end{array}$$

To borrow, you have to cross out the number in the column to the left.

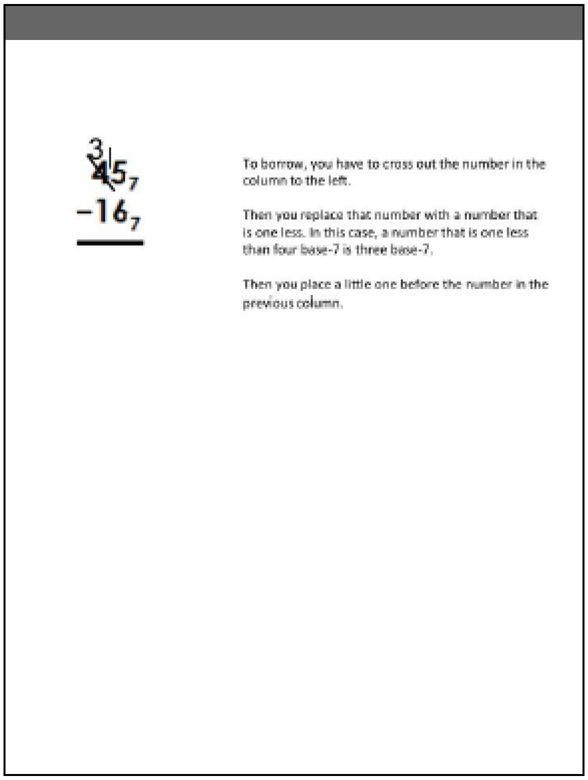
Then you replace that number with a number that is one less. In this case, a number that is one less than four base-7 is three base-7.

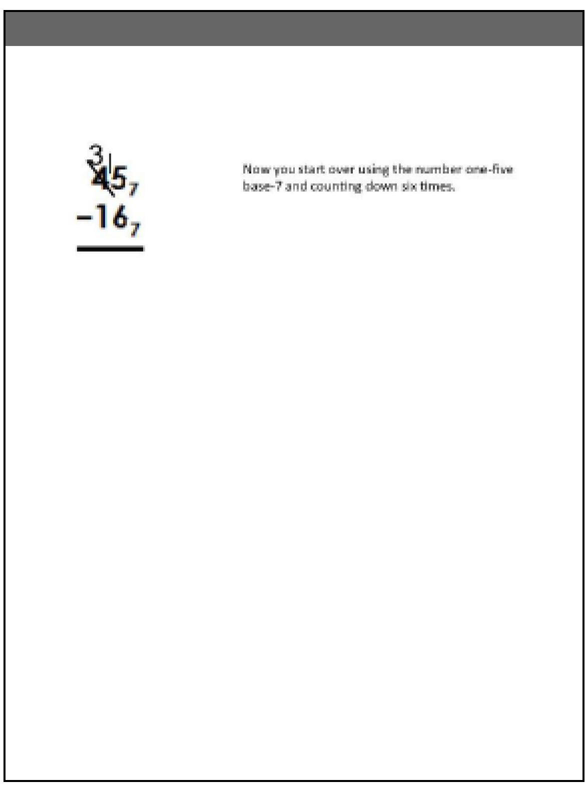
$$\begin{array}{r} \overset{3}{\cancel{4}}5_7 \\ -16_7 \\ \hline \end{array}$$

To borrow, you have to cross out the number in the column to the left.

Then you replace that number with a number that is one less. In this case, a number that is one less than four base-7 is three base-7.

Then you place a little one before the number in the previous column.


3
$\cancel{4}\,{}^{1}5_7$
-16_7
To borrow, you have to cross out the number in the column to the left.
Then you replace that number with a number that is one less. In this case, a number that is one less than four base-7 is three base-7.
Then you place a little one before the number in the previous column.


3
$\cancel{4}\,{}^{1}5_7$
-16_7
Now you start over using the number one-five base-7 and counting down six times.

$$\begin{array}{r} \overset{3}{\cancel{4}}{}^{1}5_7 \\ -16_7 \\ \hline \end{array}$$

Now you start over using the number one-five base-7 and counting down six times.

One-five base-7, one-four base-7 (once), one-three base-7 (twice), one-two base-7 (three times), one-one base-7 (four times), one-zero base-7 (five times), **six base-7** (**six times**).

$$\begin{array}{r} \overset{3}{\cancel{4}}{}^{1}5_7 \\ -16_7 \\ \hline \end{array}$$

Now you start over using the number one-five base-7 and counting down six times.

One-five base-7, one-four base-7 (once), one-three base-7 (twice), one-two base-7 (three times), one-one base-7 (four times), one-zero base-7 (five times), **six base-7** (**six times**).

The you write that number in the column.

$$\begin{array}{r} \overset{3}{\cancel{4}}{}^{1}5_7 \\ -16_7 \\ \hline 6 \end{array}$$

Now you start over using the number one-five base-7 and counting down six times.

One-five base-7, one-four base-7 (once), one-three base-7 (twice), one-two base-7 (three times), one-one base-7 (four times), one-zero base-7 (five times), **six base-7** (**six times**).

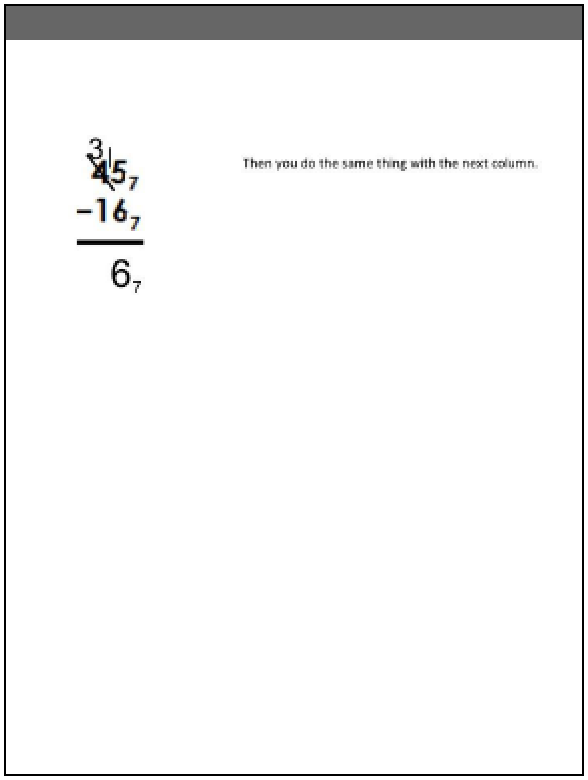
The you write that number in the column.

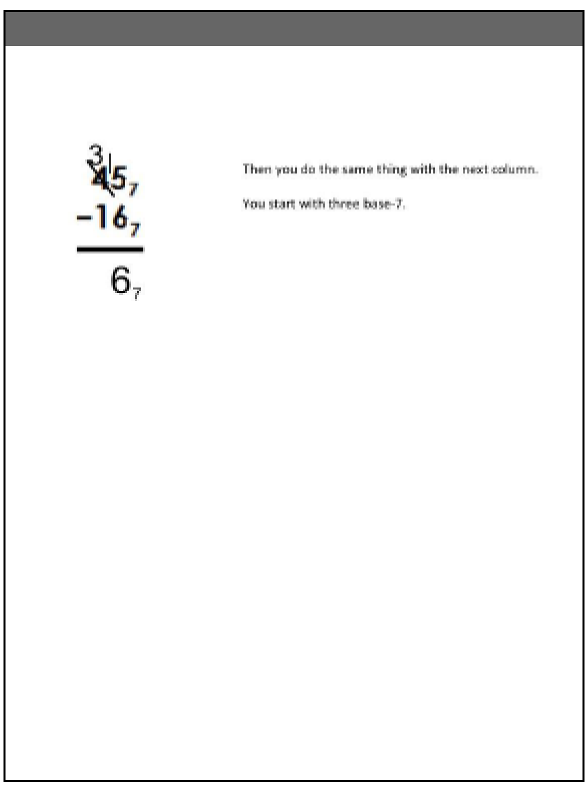
$$\begin{array}{r} \overset{3}{\cancel{4}}{}^{1}5_7 \\ -16_7 \\ \hline 6_7 \end{array}$$

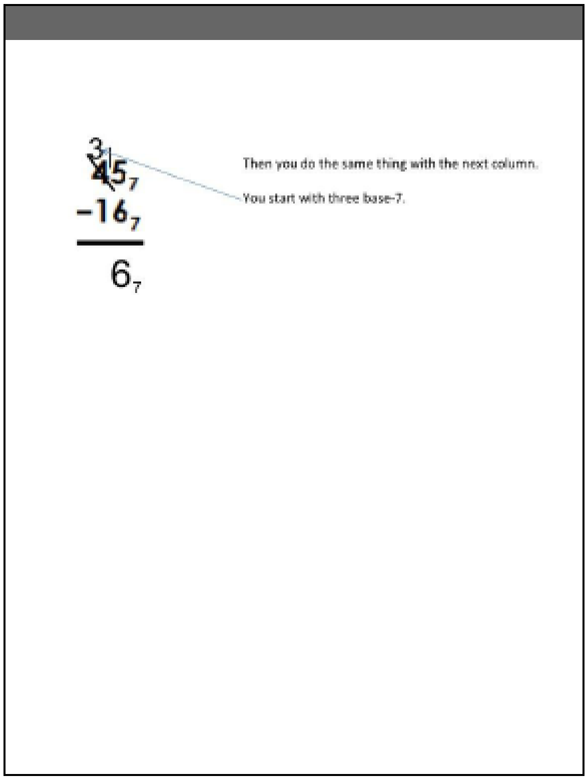
Now you start over using the number one-five base-7 and counting down six times.

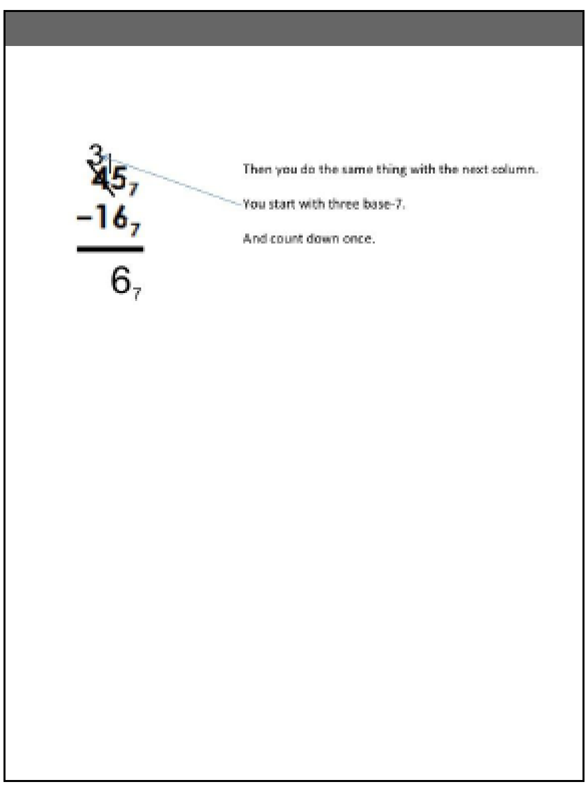
One-five base-7, one-four base-7 (once), one-three base-7 (twice), one-two base-7 (three times), one-one base-7 (four times), one-zero base-7 (five times), **six base-7** (**six times**).

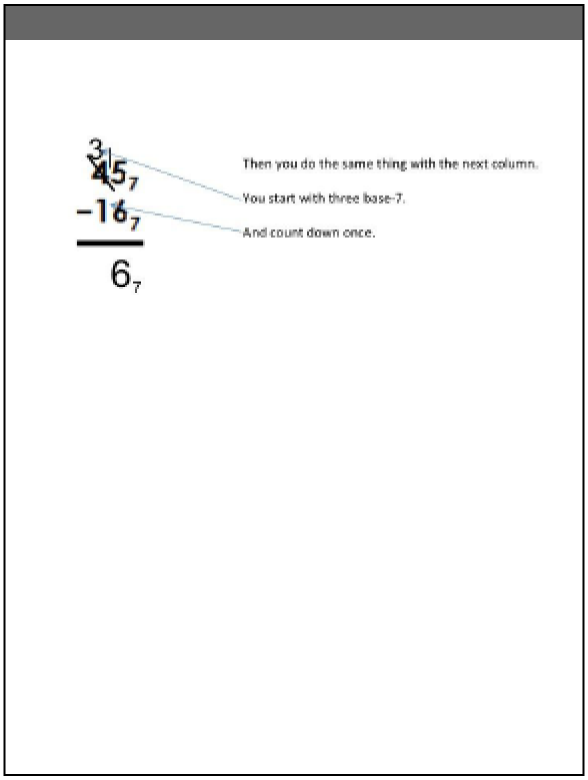
The you write that number in the column.

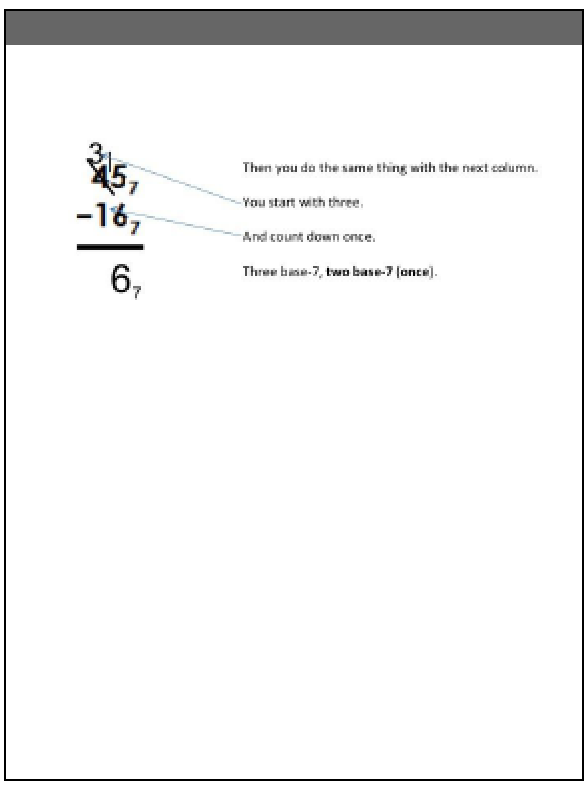

$\overset{3}{\cancel{4}}\,{}^{1}5_7$
-16_7
6_7
Then you do the same thing with the next column.

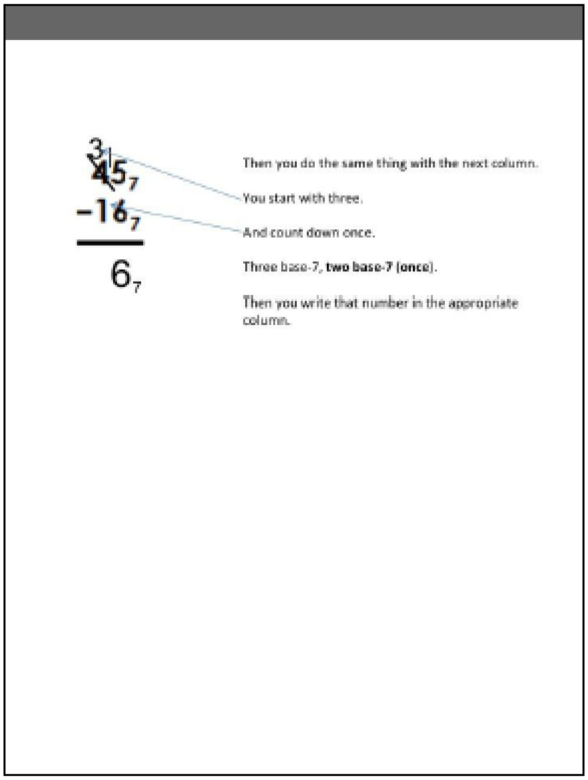

$\overset{3}{\cancel{4}}\,{}^{1}5_7$
-16_7
6_7
Then you do the same thing with the next column.
You start with three base-7.

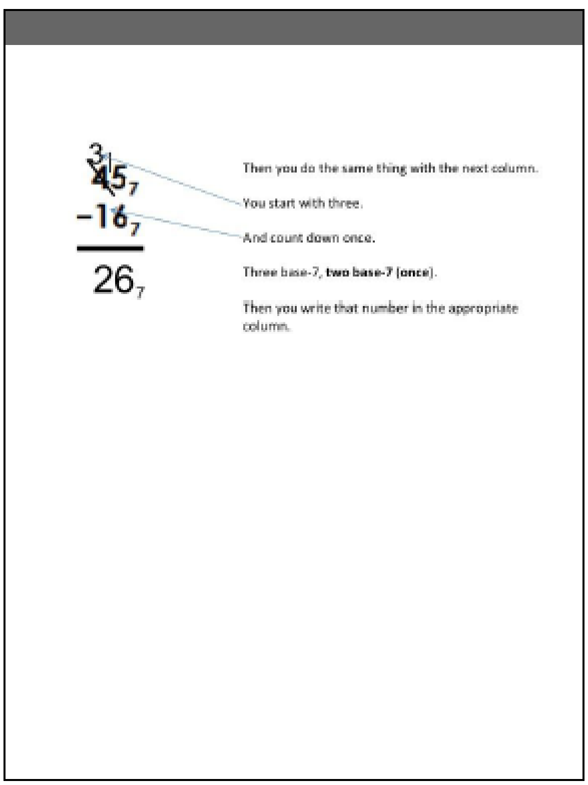

3
45₇
−16₇
6₇
Then you do the same thing with the next column.
You start with three base-7.


3
45₇
−16₇
6₇
Then you do the same thing with the next column.
You start with three base-7.
And count down once.


Then you do the same thing with the next column.
You start with three base-7.
And count down once.


Then you do the same thing with the next column.
You start with three.
And count down once.
Three base-7, two base-7 (once).


3
45₇
−16₇
6₇
Then you do the same thing with the next column.
You start with three.
And count down once.
Three base-7, **two base-7 (once)**.
Then you write that number in the appropriate column.

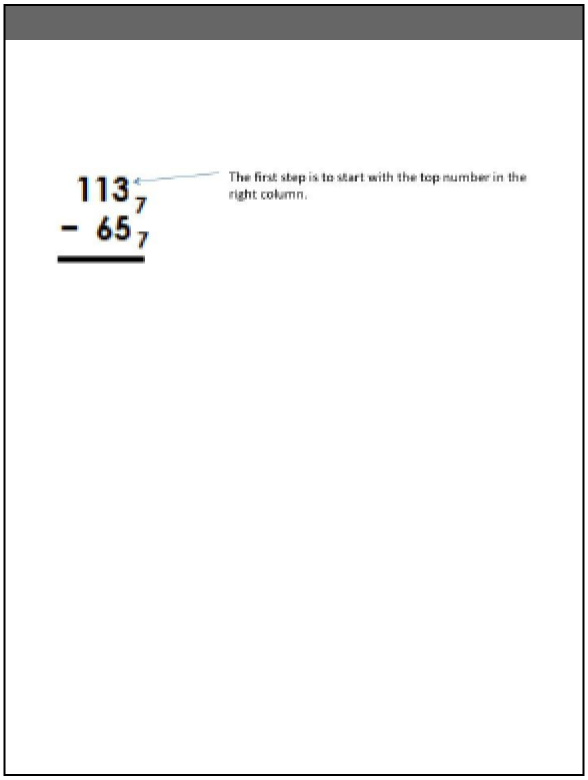

3
45₇
−16₇
26₇
Then you do the same thing with the next column.
You start with three.
And count down once.
Three base-7, **two base-7 (once)**.
Then you write that number in the appropriate column.

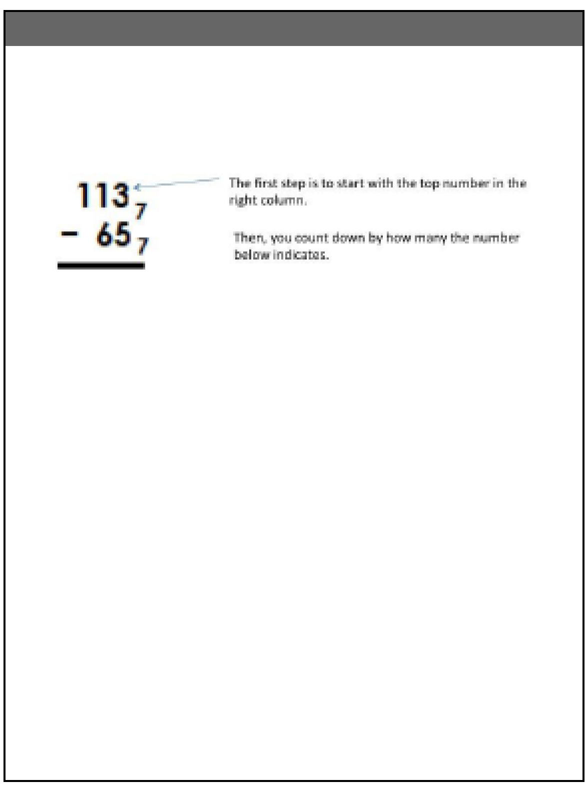
One more Example:

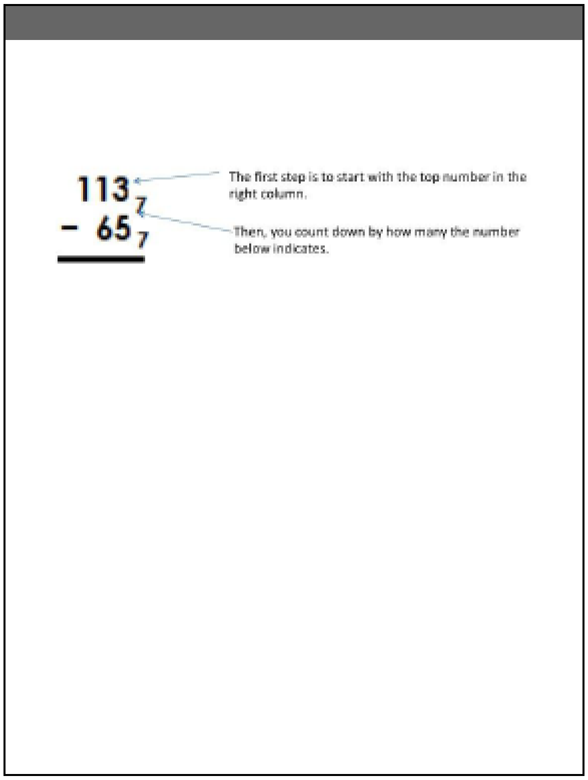
Please study the next example as well. Take your time to understand how the algorithm works.

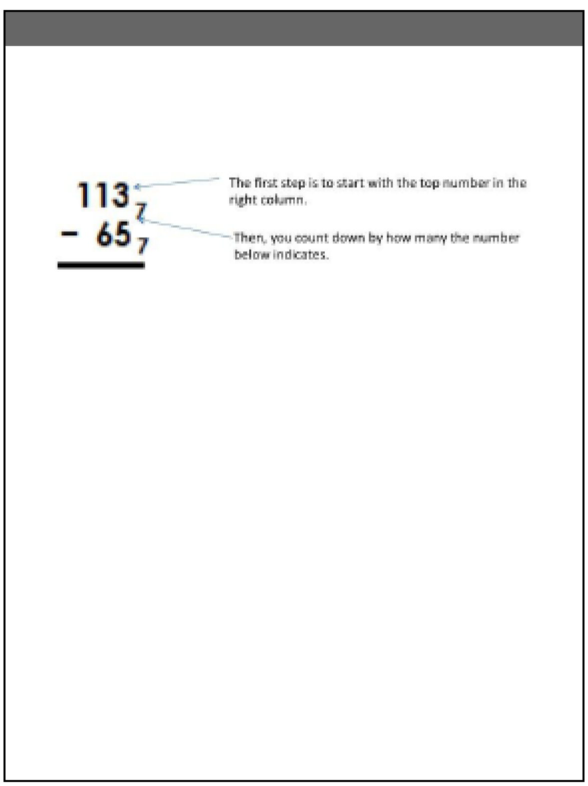
$$\begin{array}{r} 113_7 \\ -\ 65_7 \\ \hline \end{array}$$

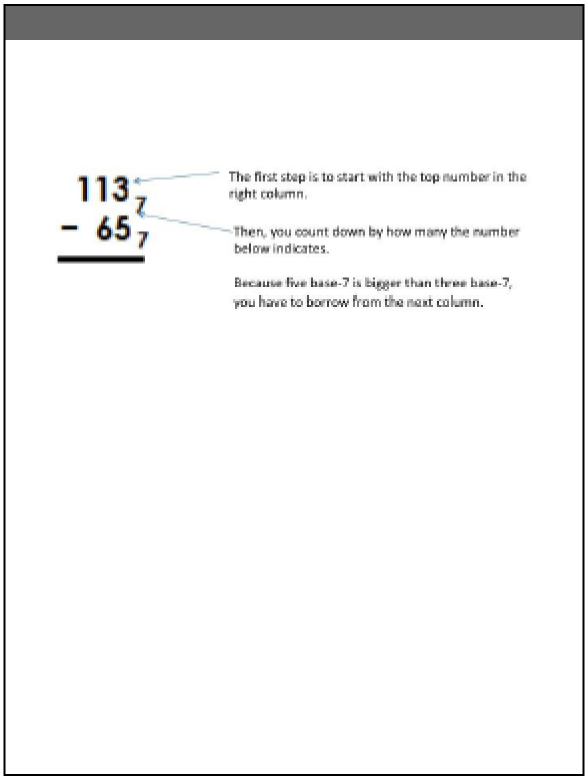
The first step is to start with the top number in the right column.

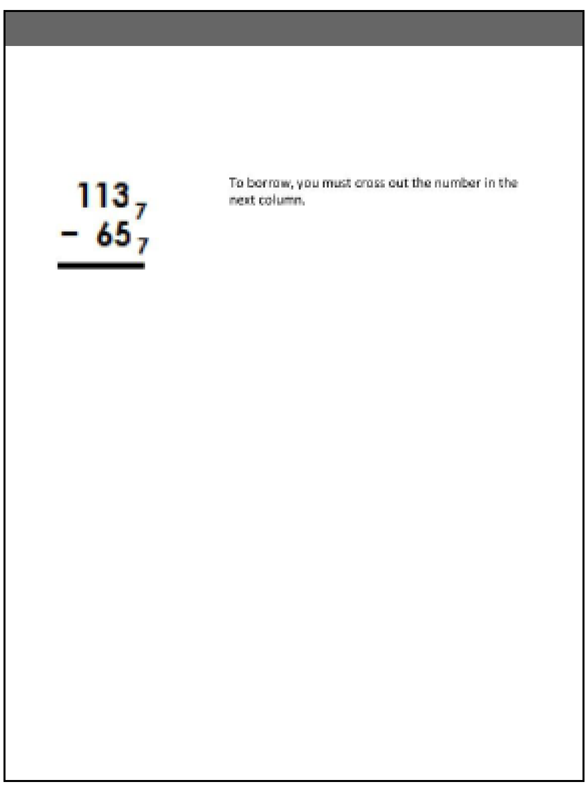

113_7
$-\ 65_7$
The first step is to start with the top number in the right column.

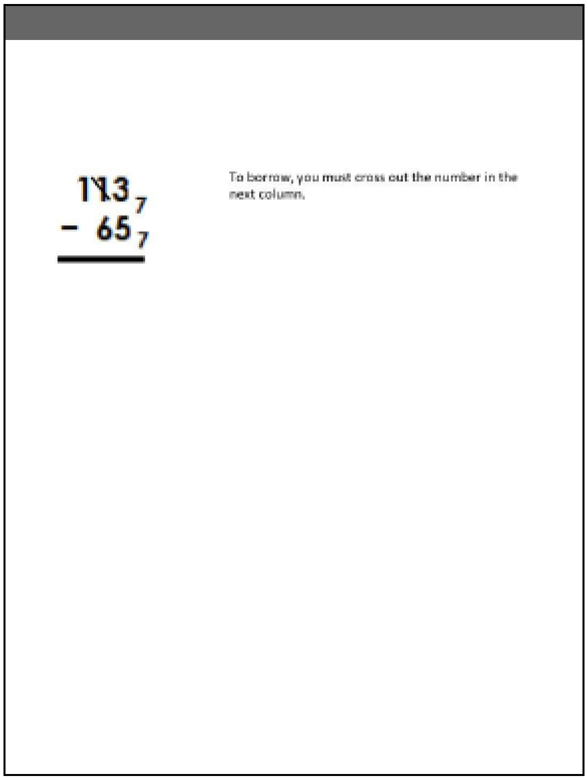

113_7
$-\ 65_7$
The first step is to start with the top number in the right column.
Then, you count down by how many the number below indicates.

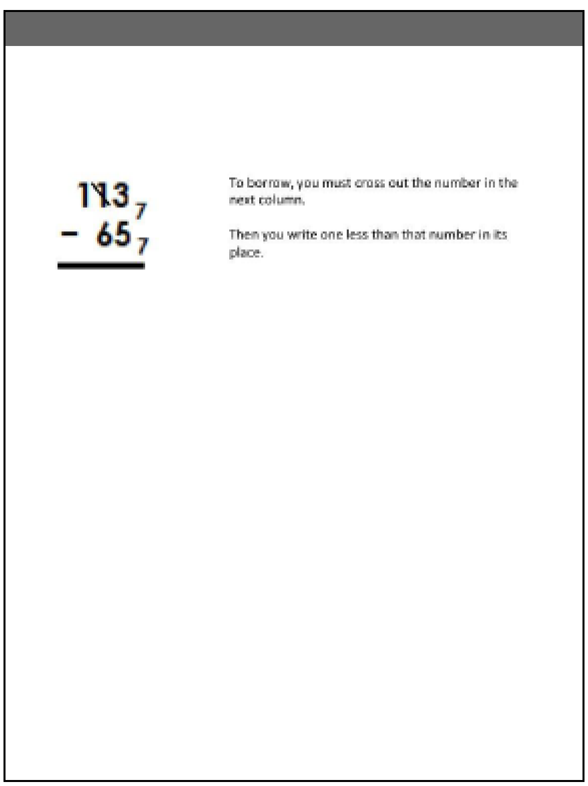

113 7
- 65 7
The first step is to start with the top number in the right column.
Then, you count down by how many the number below indicates.

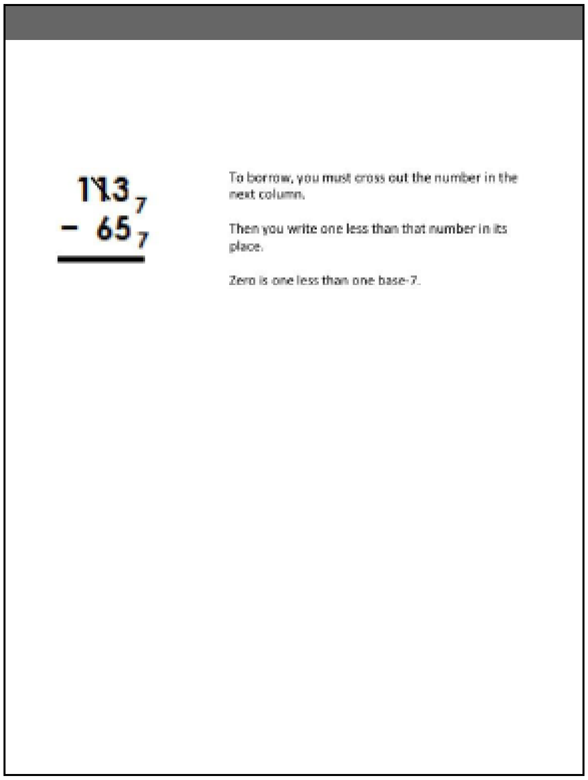

113 7
- 65 7
The first step is to start with the top number in the right column.
Then, you count down by how many the number below indicates.

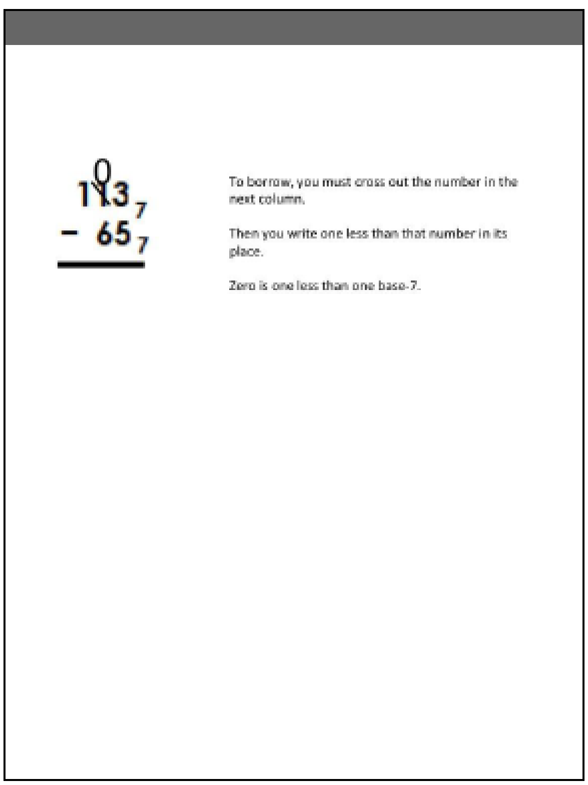

113_7
$- 65_7$
The first step is to start with the top number in the right column.
Then, you count down by how many the number below indicates.
Because five base-7 is bigger than three base-7, you have to borrow from the next column.


113_7
$- 65_7$
To borrow, you must cross out the number in the next column.


113_7
$-\ 65_7$
To borrow, you must cross out the number in the next column.


113_7
$-\ 65_7$
To borrow, you must cross out the number in the next column.
Then you write one less than that number in its place.


$1\cancel{1}3_7$
$-\ 65_7$
To borrow, you must cross out the number in the next column.
Then you write one less than that number in its place.
Zero is one less than one base-7.


0
$1\cancel{1}3_7$
$-\ 65_7$
To borrow, you must cross out the number in the next column.
Then you write one less than that number in its place.
Zero is one less than one base-7.

$$\begin{array}{r} 1\overset{0}{\cancel{1}}3_7 \\ -\ 65_7 \\ \hline \end{array}$$

To borrow, you must cross out the number in the next column.

Then you write one less than that number in its place.

Zero is one less than one base-7.

Then you write a one before the number you are borrowing to.

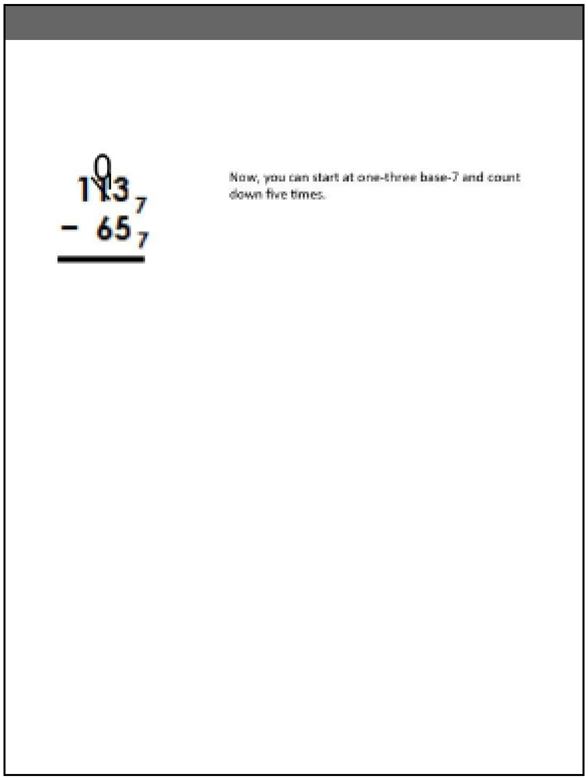
$$\begin{array}{r} 1\overset{0}{\cancel{1}}3_7 \\ -\ 65_7 \\ \hline \end{array}$$

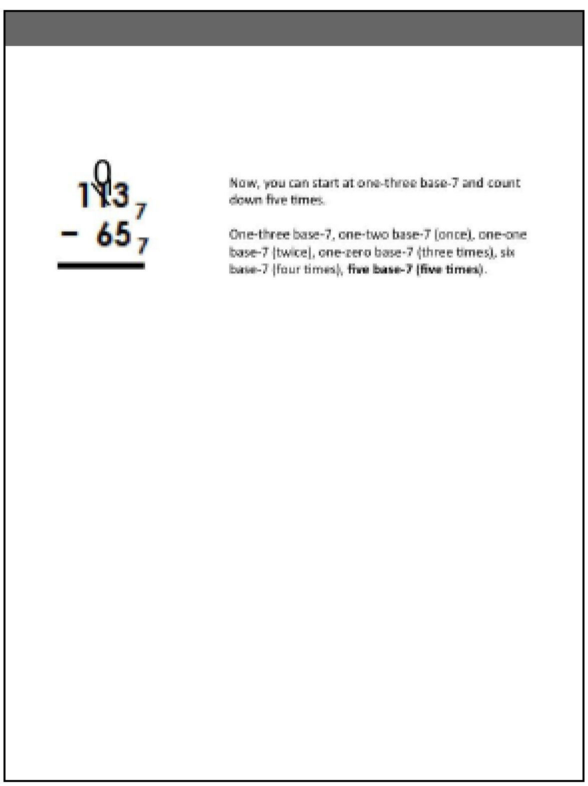
To borrow, you must cross out the number in the next column.

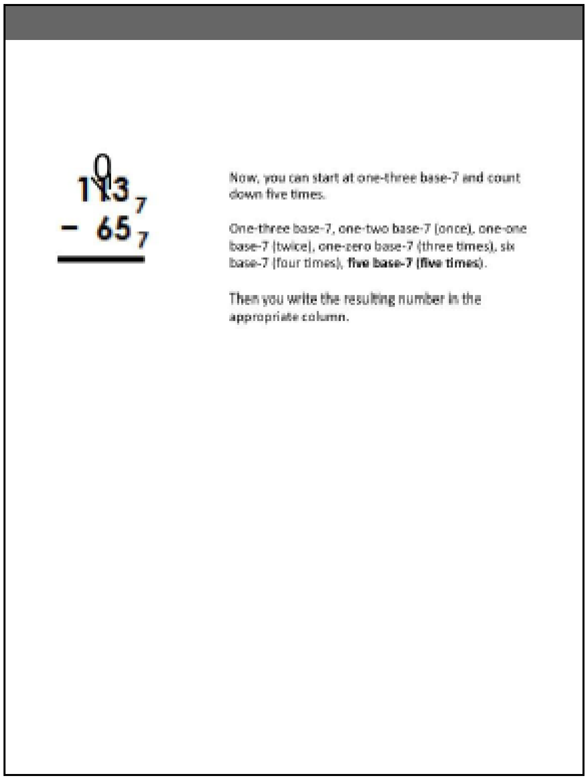
Then you write one less than that number in its place.

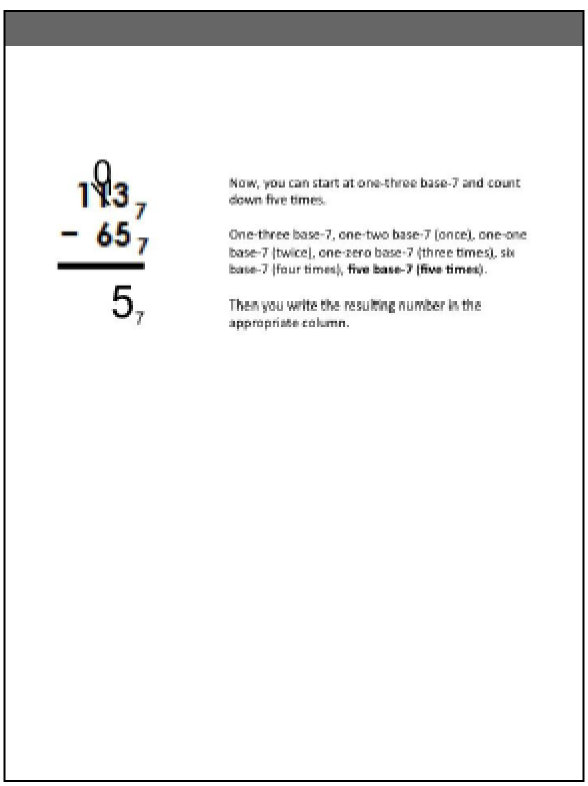
Zero is one less than one base-7.

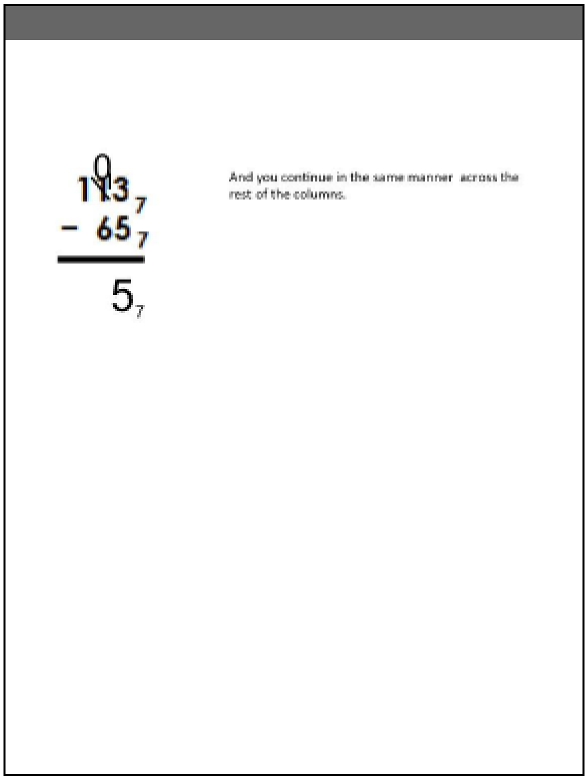
Then you write a one before the number you are borrowing to.

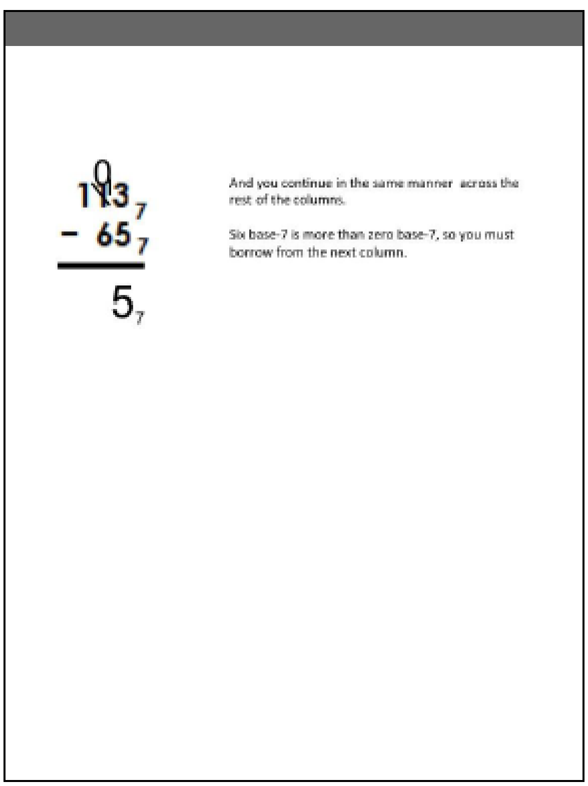

$1\overset{0}{\cancel{1}}3_7$
$-\ 65_7$
Now, you can start at one-three base-7 and count down five times.

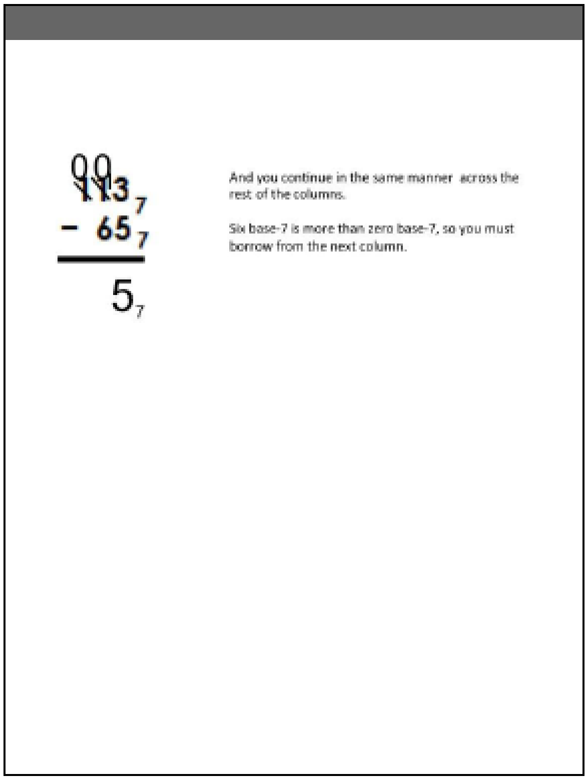

$1\overset{0}{\cancel{1}}3_7$
$-\ 65_7$
Now, you can start at one-three base-7 and count down five times.
One-three base-7, one-two base-7 (once), one-one base-7 (twice), one-zero base-7 (three times), six base-7 (four times), **five base-7 (five times)**.

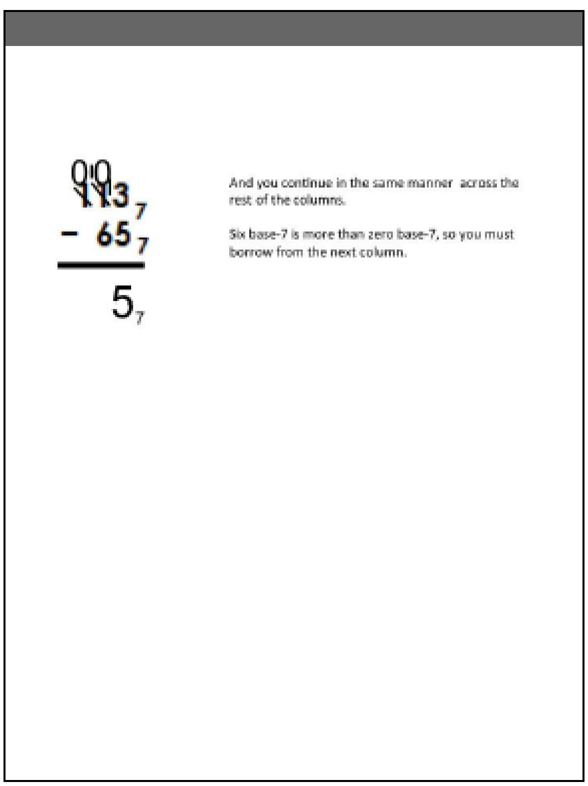

$\overset{0}{1\cancel{1}3_7}$
$-\ 65_7$
Now, you can start at one-three base-7 and count down five times.
One-three base-7, one-two base-7 (once), one-one base-7 (twice), one-zero base-7 (three times), six base-7 (four times), **five base-7 (five times)**.
Then you write the resulting number in the appropriate column.

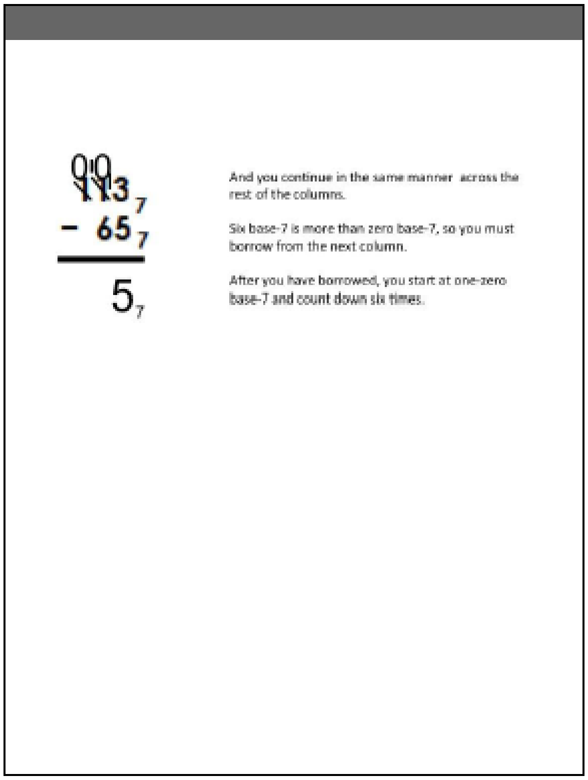

$\overset{0}{1\cancel{1}3_7}$
$-\ 65_7$
5_7
Now, you can start at one-three base-7 and count down five times.
One-three base-7, one-two base-7 (once), one-one base-7 (twice), one-zero base-7 (three times), six base-7 (four times), **five base-7 (five times)**.
Then you write the resulting number in the appropriate column.

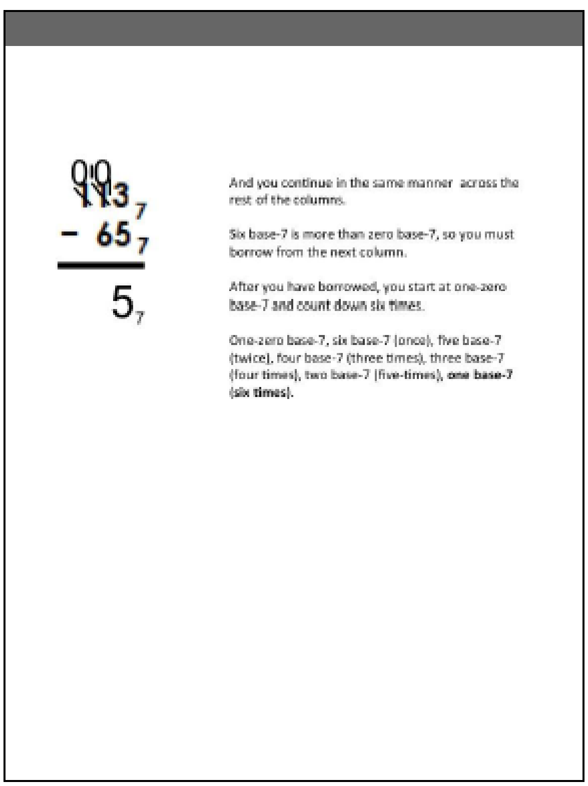

$1\overset{0}{\cancel{1}}3_7$
$-\ 65_7$
5_7
And you continue in the same manner across the rest of the columns.

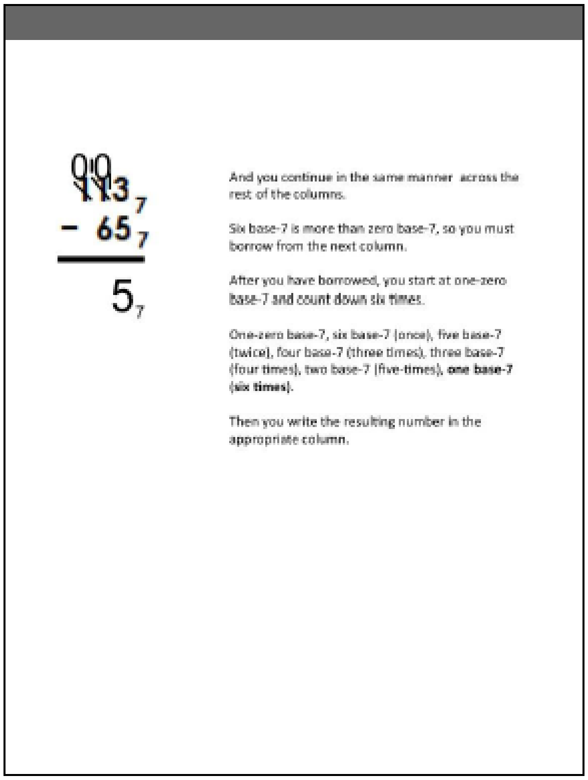

$1\overset{0}{\cancel{1}}3_7$
$-\ 65_7$
5_7
And you continue in the same manner across the rest of the columns.
Six base-7 is more than zero base-7, so you must borrow from the next column.

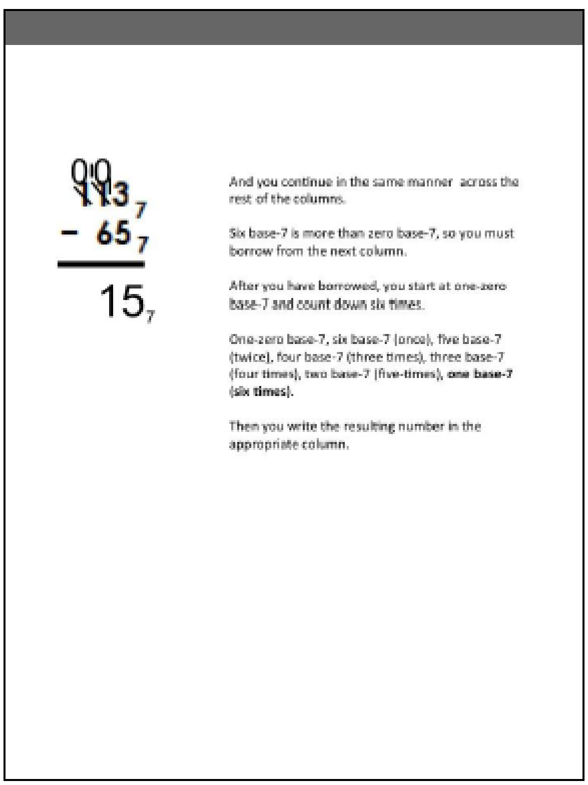

$\cancel{1}\cancel{1}3_7$
$-\ 65_7$
5_7
And you continue in the same manner across the rest of the columns.
Six base-7 is more than zero base-7, so you must borrow from the next column.

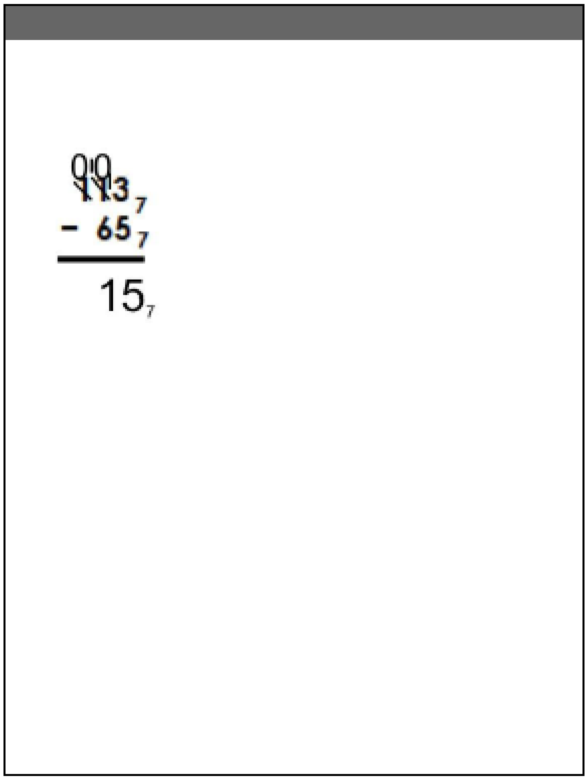

$\cancel{1}\cancel{1}3_7$
$-\ 65_7$
5_7
And you continue in the same manner across the rest of the columns.
Six base-7 is more than zero base-7, so you must borrow from the next column.


0 10
$\cancel{1}\cancel{1}3_7$
$-\ 65_7$
5_7
And you continue in the same manner across the rest of the columns.
Six base-7 is more than zero base-7, so you must borrow from the next column.
After you have borrowed, you start at one-zero base-7 and count down six times.


0 10
$\cancel{1}\cancel{1}3_7$
$-\ 65_7$
5_7
And you continue in the same manner across the rest of the columns.
Six base-7 is more than zero base-7, so you must borrow from the next column.
After you have borrowed, you start at one-zero base-7 and count down six times.
One-zero base-7, six base-7 (once), five base-7 (twice), four base-7 (three times), three base-7 (four times), two base-7 (five-times), **one base-7 (six times).**


And you continue in the same manner across the rest of the columns.
Six base-7 is more than zero base-7, so you must borrow from the next column.
After you have borrowed, you start at one-zero base-7 and count down six times.
One-zero base-7, six base-7 (once), five base-7 (twice), four base-7 (three times), three base-7 (four times), two base-7 (five-times), **one base-7 (six times)**.
Then you write the resulting number in the appropriate column.


And you continue in the same manner across the rest of the columns.
Six base-7 is more than zero base-7, so you must borrow from the next column.
After you have borrowed, you start at one-zero base-7 and count down six times.
One-zero base-7, six base-7 (once), five base-7 (twice), four base-7 (three times), three base-7 (four times), two base-7 (five-times), **one base-7 (six times)**.
Then you write the resulting number in the appropriate column.


113_7
$- 65_7$
15_7

Appendix T

P4 End of Session Questions

Student ID #:___________________________________

Use the algorithm you learned to compute the following

1. $$\begin{array}{r} 341_7 \\ -\ 233_7 \\ \hline \end{array}$$

2. $$\begin{array}{r} 143_7 \\ -\ 65_7 \\ \hline \end{array}$$

3. $$\begin{array}{r} 245_7 \\ -\ 66_7 \\ \hline \end{array}$$

4. $$\begin{array}{r} 64_7 \\ -\ 23_7 \\ \hline \end{array}$$

5 $$\begin{array}{r} 44_7 \\ -14_7 \\ \hline \end{array}$$