RELATING MODULUS AND POINCARÉ INEQUALITIES ON MODIFIED SIERPIŃSKI CARPETS

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ABSTRACT

Relating modulus and Poincaré inequalities on modified Sierpiński carpets

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This thesis investigates the question of whether a doubling metric measure space supports a Poincaré inequality and explains the relationship between the existence of such an inequality and the non-triviality of the respective modulus. It discusses in detail a general class of modified Sierpiński carpets presented by Mackay, Tyson, and Wildrick [15], which are the first examples of spaces that support Poincaré inequalities for a renormalized Lebesgue measure that are also compact subsets of Euclidean space with empty interior. It describes the intricate relationship between the sequence used in the construction of a modified Sierpiński carpet and the validity of Poincaré inequalities on that space.

To my wife Katie whose support made this possible, and to our son Benjamin.

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Introduction

The Poincaré inequality is a fundamental tool in the study of partial differential equations (PDEs) and numerical analysis. It is part of the general theory of Sobolev inequalities which are fundamental to working with PDE's, and is used to prove, for example, energy estimates and results concerning the reaction-diffusion equation. For more information concerning this topic refer to [6], [9], and [14]. The Poincaré inequality provides us with a bound for the variance of a function on a ball based on the integral of its derivative. There are many examples of metric spaces that support a Poincaré inequality and we include a proof for $\mathbb{R}^n$. Some others are Riemannian manifolds that have non-negative Ricci curvature, the Heisenberg group, Carnot groups, Laakso's spaces, and linearly locally contractible manifolds with good volume growth. For additional examples please see [10], [12], [13], [15], and [16].

It is important to discuss why we are concerned with Poincaré inequalities, what they determine about the geometry of the underlying space. Metric spaces with doubling measures that support a p-Poincaré inequality have a first order calculus that is very similar to that of Euclidean spaces. Furthermore, in [2] it is shown that any space which supports a p-Poincaré inequality must be connected. The strongest of the inequalities is the 1-Poincaré and it is a known result that if a metric space supports a 1-Poincaré inequality then the space is quasiconvex [4]. What this means essentially is that any pair of points in the space can be connected by curves that are not "too" long.

It should be acknowledged here that there are many different variations of the

Poincaré inequality. We state three versions in Chapter 2, one for domains in $\mathbb{R}^n$, one for balls in $\mathbb{R}^n$, and a weak Poincaré inequality for metric measure spaces. The weak Poincaré inequality for metric measure spaces (2.4) is the primary definition used in this thesis and is the definition used in [15].

Some interesting spaces on which to study Poincaré inequalities are presented by Mackay, Tyson, and Wildrick in [15]. They are a class of non-self similar modified Sierpiński carpets, some of which support a Poincaré inequality and some do not. The unique aspect of their research is that they present the first examples of compact subsets of Euclidean space that have empty interior, yet support Poincaré inequalities for the renormalized Lebesgue measure. Before discussing the carpets described in [15], we will first review the standard Sierpiński carpet obtained from the unit square in $\mathbb{R}^2$ by iterating the following: divide all current squares into nine sub-squares of equal size, and then remove the central square. Each resulting square should have side length equal to $\frac{1}{3}$ the side length of the original square. The Sierpiński carpet can also be presented as the attractor of an iterated function system as defined in section 1.3; at each stage 8 copies of itself are scaled by a factor of $1/3$ and so it can be shown that the Sierpiński carpet has Hausdorff dimension $\frac{\log 8}{\log 3}$.

The standard Sierpiński carpet is a generalization of the middle-thirds Cantor set to 2 dimensions. The Sierpiński carpet in fact contains the product of the middle-thirds Cantor set and the value $1/2$. Information on the construction of the middle-thirds Cantor set is widely available so it will not be included here. It is a subset of the unit interval $[0, 1]$ possessing interesting properties; it is self-similar, compact, contains no intervals, has Lebesgue measure 0, but is uncountable. By inclusion, the complexity demonstrated by these properties is carried forward into the Sierpiński carpet, and this is what makes this field of study so attractive. There has also been a lot research carried out into the properties of fat Cantor sets which retain many of the properties of the original middle-third Cantor. It is easy to create a fat Cantor set that is still uncountable, contains no intervals, but now has positive Lebesgue measure. They are all topologically the same despite their quantitative differences.

These results are very similar in style to the results presented by Mackay, Tyson, and Wildrick in [15]. Instead of using the ratio of 1/3 as in the construction of the standard Sierpiński carpet, they consider carpets formed by at each stage in the construction taking a scaling ratio that is the reciprocal of *any* odd integer. These ratios then form an infinite sequence and they present results showing that it is the behaviour of the sequence with respect to its ℓ^q norm that determines whether the resulting carpet supports a p-Poincaré inequality. If the given sequence converges fast enough to zero then the resulting carpet is "fat" enough to support a Poincaré inequality, despite still having empty interior. On the other hand, the standard Sierpiński carpet, defined by a constant sequence, does not support such an inequality. As with the fat Cantor sets, the modified Sierpiński carpets are all topologically equivalent to each other, and to the standard Sierpiński carpet, despite their obvious geometric differences.

It is often possible and advantageous to reduce problems concerning Poincaré inequalities to a geometric problem involving the support of a "thick" family of curves. It is the modulus of a curve family that embodies this notion and so to prove the validity of a Poincaré inequality one need only prove the existence of a curve family having non-trivial modulus. This technique is used throughout this thesis and provides some very elegant and concise results, and is also a fundamental tool in the study of quasiconformal mappings. Relating Poincaré inequalities to non-trivial modulus also provides an intuitive understand of when and why Poincaré inequalities might fail. Often the existence of a "tunnel" or "collar" where curves are forced through too narrow a space can destroy the validity of a Poincaré inequality. An example of this sort of problem is given as part of the discussion on modulus in section 2.2.

This is an expository thesis whose goal is to explain the relationship mentioned above between the Poincaré inequality and modulus. Much of the content is based on the results of Mackay, Tyson and Wildrick in [15], and their modified Sierpiński carpets. This thesis also includes results from many other sources on the subject, and where possible, provides any details missing from the original work.

Much of the terminology and background knowledge necessary to understand the results discussed in this thesis are presented in Chapter 1. Most of Chapter 2 is devoted to Poincaré inequalities and their relation to modulus. There is also a section on the similarly defined concept of capacity. The most significant result in Chapter 2 is Proposition 2.4.1, which states that under certain hypothesis, a metric measure space admits a p-Poincaré inequality if and only if its corresponding p-modulus is non-trivial. The proof of this proposition is included, and follows from, results in [12] and [14]. Chapter 3 begins by detailing the construction of the modified Sierpiński carpets presented in [15], followed by a discussion outlining some basic properties of these carpets. The chapter concludes by stating and proving some very interesting results put forth by Mackay, Tyson, and Wildrick concerning how the ℓ^q norm of the generating sequence impacts the validity of the Poincaré inequality on the resulting carpet.

Chapter 1

Preliminaries

1.1 Metric Measure Space Properties

Here we will cover briefly many metric measure space definitions and introduce some potentially new notation. For this paper we let (X, d, μ) be a metric space equipped with a Borel measure μ that is finite and positive on balls. We let $B(x, r)$ denote a ball with radius $r > 0$ and center $x \in X$. More formally this means

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

If we write λB then we are referring to the dilated ball $B(x, \lambda r)$, where $\lambda > 0$.

A measure μ is said to be a **doubling** measure if there is a constant $C > 0$ such that for all balls $B(x, r) \in X$ we have that

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)).$$

The doubling property of a measure is one that will play a significant role in this paper. Most of the measures we consider will support this property and many of the results will rely in some way on the doubling constant C.

We say that a Borel measure μ on (X, d) is **Q-regular** if there exists a constant $C > 0$ and a radius $r_0 > 0$ such that

$$C^{-1} r^Q \leq \mu(B(x, r)) \leq C r^Q \tag{1.1}$$

for any metric ball $B(x,r) \subset X$ with $0 < r < r_0$. We say that μ is **Ahlfors Q-regular** if there is a constant $C > 0$ such that (1.1) holds for all $B(x,r)$ with $0 < r < 2\text{diam}(X)$. We say that μ is **Ahlfors regular** if it is Ahlfors Q-regular for some $Q > 0$.

The following definitions on paths are collected from [19], and despite their formal appearance are quite intuitive and extremely important in the later sections of this paper. A **path** or **curve** in X is a continuous mapping $\gamma : I \to X$ for some interval $I \subset \mathbb{R}$. The curve γ is considered **rectifiable** if its length, defined by

$$\ell(\gamma) := sup \left\{ \sum_{i=1}^{N} d(\gamma(t_i), \gamma(t_{i+1})) : t_1 < t_2 < \cdots < t_N \ with \ t_1, t_2, \ldots, t_N \in I \right\},$$

is finite, and we say γ is **locally rectifiable** if all its closed sub-curves are rectifiable. If there is a curve representing the shortest path possible between two points we call this curve a **geodesic**. We then say a metric space (X,d) is a **geodesic space** if for ever pair of points $x, y \in X$, there exists a geodesic γ joining x to y. We call the metric space X a **length space** if for every x and y with γ a geodesic joining them we have

$$d(x,y) = \ell(\gamma).$$

This is a metric space where for any two points, the length of a geodesic joining them is actually equal to the distance between the two points. A result from Chapter 2 of [18], which will be of use in Chapter 2, states that a geodesic space is actually a length space.

We say that a metric space is **quasiconvex** if there exists a uniform constant $C \geq 1$ such that any distinct pair of points $x, y \in X$ can be connected by a rectifiable curve γ_{xy} with length satisfying

$$\ell(\gamma_{xy}) \leq Cd(x,y).$$

To each curve there is an associated mapping $s : I = [a,b] \to [0, \ell(\gamma)]$ given by $s(t) = \ell(\gamma|_{[a,t]})$. This is a monotone increasing map and so is differentiable almost everywhere, therefore we can set $|\gamma'(t)| := s'(t)$. Using this map we construct the **arc-length parametrization** of γ, denoted γ_0, such that $\gamma_0 : [0, \ell(\gamma)] \to X$, and $|\gamma_0'| = 1$ almost everywhere on $[0, \ell(\gamma)]$

Now if γ is a rectifiable path, and $\rho : X \to [0, \infty]$ is a Borel function, we get that $\rho \circ \gamma$ is a Lebesgue measurable function on I. We can then define

$$\int_\gamma \rho \; ds := \int_0^{\ell(\gamma)} \rho \circ \gamma_0(s) \; ds.$$

A Borel function $\rho : X \to [0, \infty]$ is an **upper gradient** of a function $u : X \to \mathbb{R}$ if

$$|u(x) - u(y)| \leq \int_\gamma \rho \; ds$$

for all rectifiable curves γ joining x and y.

1.1.1 Topological Definitions

The following are some topological definitions that are largely taken from Munkres [17]. We say that a metric space (X, d) is **compact** if every open cover of X has a finite subcover, and a metric space is called **proper** if every closed ball $\{y \in X : d(x, y) \leq r\}$ is compact. There are different notions of compactness, but for this paper the topological definition will suffice. This is a property that will often allow us to take information known locally in a neighbourhood and extend it globally.

For a point $x \in X$ we say X is **locally connected at x** if for every open set V containing x there exists a connected, open set U with $x \in U \subset V$. The space X is **locally connected** if it is locally connected at x for every $x \in X$. A **cut point** is a point of a connected space such that if we were to remove it would cause the resulting space to be disconnected.

1.1.2 Lipschitz Functions

Here we define a class of functions that will be used when we discuss fractals later in this chapter, but also more importantly in Chapter 2 when we prove Theorem 2.4.1.

More on these definitions and Lipschitz functions in general can be found in [8] or [9].

Definition 1.1.1. *A function $u : X \to Y$ such that there exists a constant $C < \infty$ for which*

$$|u(x) - u(y)| \leq C|x - y|$$

for all $x, y \in X$ is called a ***Lipschitz function****. We define the* ***upper*** *and the* ***lower Lipschitz constant*** *at a point $x \in X$ by*

$$Lip\ u(x) = \limsup_{r \to 0} \frac{L(x,u,r)}{r}, \qquad lip\ u(x) = \liminf_{r \to 0} \frac{L(x,u,r)}{r},$$

where

$$L(x,u,r) = \sup\{|u(x) - u(y)| : \ d(x,y) \leq r\} \quad for\ r > 0.$$

We then say Lip(X) is the class of all Lipschitz functions on a domain X, and Lip_0(X) is the collection of all Lipschitz functions with compact support in X.

Before ending this section we will state and prove a claim found also in [8]. This will serve as a valuable tool when proving Theorem 2.4.1, as it will allow us to jump from the standard p-Poincaré inequality to one in which $Lip\ u$ plays the role of the gradient.

Claim 1.1.2 (Lemma 6.7, [8])**.** *If $u \in$ Lip(X), then $Lip\ u$ is an upper gradient for u.*

Proof. We let $\gamma : [a, b] \to X$ be a rectifiable curve parametrized by arc-length that connects x and y. The function $u \circ \gamma$ is Lipschitz continuous and so absolutely continuous, therefore it is differentiable almost everywhere. Then because

$$|(u \circ \gamma)'(t)| \leq lip\ u(\gamma(t)) \leq Lip\ u(\gamma(t))$$

at every point t of differentiability of $u \circ \gamma$, we get the following inequality,

$$|u(x) - u(y)| = \left| \int_a^b \frac{d}{dt} u(\gamma(t))\ dt \right| \leq \int_a^b Lip\ u(\gamma(t))\ dt,$$

which completes the proof. It is of some interest here to note that this argument also holds for lip u, not just Lip u.

□

1.2 Sobolev Spaces

For some parts of this paper we will only be concerned with functions defined on $\mathbb{R}^n$, and their weak derivatives. It can then be convenient to understand and work with Sobolev spaces which we will soon define. First, a weak derivative is a generalization of the concept of the derivative of a function for functions assumed only to be integrable and not necessarily differentiable. For a function $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ if there exists a function $v \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} u\partial_i\phi \; dx = -\int_{\mathbb{R}^n} v\phi \; dx, \qquad \phi \in C_0^\infty(\mathbb{R})^\ltimes, \tag{1.2}$$

then we say v is the **weak ith partial derivative** of u and set $\partial_i u := v$. If for all $i = 1, \ldots, n$ the weak ith partial derivative exists then we say $\nabla u := (\partial_1 u, \partial_2 u, \ldots, \partial_n u)$.

Definition 1.2.1. *The vector space of all locally integrable functions u for which locally integrable weak partial derivatives $\partial_i u$ exist for all $i = 1, \ldots, n$ is denoted by*

$$W^{1,1}_{loc} = W^{1,1}_{loc}(\mathbb{R}^n)$$

*and called the **local Sobolev space**. When we consider u that are globally integrable, with weak derivatives that are also globally integrable, we have the space*

$$W^{1,1} = W^{1,1}(\mathbb{R}^n).$$

If the functions and their weak first derivatives are all locally or globally L^p-integrable for $1 \leq p \leq \infty$ we have the spaces

$$W^{1,p}_{loc} = W^{1,p}_{loc}(\mathbb{R}^n), \qquad W^{1,p} = W^{1,p}(\mathbb{R}^n).$$

The Sobolev space $W^{1,p}$ with the norm

$$||u||_{1,p} = ||u||_p + ||\nabla u||_p$$

is a Banach space for all $1 \leq p \leq \infty$.

1.2.1 Sobolev Embedding Theorems

Here we will outline some important embedding inequalities for Sobolev spaces. A consequence of these theorems is used in Chapter 2 to prove the validity of the Poincaré inequality on $\mathbb{R}^n$.

Theorem 1.2.2. *For a function $u \in W^{1,p}(\mathbb{R}^n)$, we have*

$$||u||_{\frac{np}{n-p}} \leq C(n,p)||\nabla u||_p \qquad if\ 1 \leq p < n;$$

if $p > n$, then u has a continuous representative satisfying

$$|u(x) - u(y)| \leq C(n,p)|x-y|^{1-n/p}||\nabla u||_p$$

for $x, y \in \mathbb{R}^n$.

These inequalities are known as the Sobolev Embedding Theorems and play a central role in Sobolev space theory. They state essentially that $W^{1,p}$ is continuously embedded into L^{p*}, where

$$p* = \frac{np}{n-p},$$

the Sobolev conjugate of p for $1 \leq p < n$. What is also very useful is that this theorem can be extended to Sobolev spaces $W^{1,p}(\Omega)$, where Ω is an open, bounded, Lipschitz domain in $\mathbb{R}^n$. With this in mind we state what is known as the **Rellich-Kondrachov theorem** which states that the embedding is in fact compact.

Theorem 1.2.3. *Let $\Omega \subseteq R^n$ be an open, bounded, Lipschitz domain, and let $1 \leq p < n$. Set*

$$p^* := \frac{np}{n-p}.$$

Then the Sobolev space $W^{1,p}(\Omega)$ is continuously embedded in the space $L^{p^*}(\Omega)$, and is compactly embedded in $L^q(\Omega)$ for every $1 \leq q < p^*$.

A Lipschitz domain is a domain in Euclidean space with a "sufficiently regular" boundary. It is sometimes referred to as a domain with Lipschitz boundary and the most intuitive way to understand the concept is to envision the boundary as locally being the graph of a Lipschitz continuous function. A more rigorous definition can be found in [4].

There is a very useful consequence of this theorem which is a result we will make use of in Chapter 2 when proving Theorem 2.4.1. If the Rellich-Kondrachov theorem holds, we have that any uniformly bounded sequence in $W^{1,p}(\Omega)$ has a convergent subsequence in $L^q(\Omega)$. The definitions and theorems included in this section are sufficient for the purposes of this paper, but the reader is encouraged to see Chapter 3 of [9] for a more detailed discussion of Sobolev spaces.

1.3 Geometric Properties of Fractals

When discussing fractals and in particular their dimension it can be useful to note that fractals are often self similar. This means that they are either exactly or approximately similar to a smaller part of themselves. The self similarity property can actually be used to very concisely define a fractal as an iterated function system (IFS). We will define this concept in a moment, but before we do it is important to note that although we will have some discussion on the standard Sierpiński carpet which *is* self similar, we will primarily focus on modified Sierpiński carpets which, due to their construction, are not necessarily self similar. These carpets therefore require some additional consideration and are slightly more complicated to deal with. The definitions contained in this section are primarily chosen from Chapters 2 and 9 of [7].

We will first introduce the notion of Hausdorff measure and then Hausdorff dimension, both of which are conveniently defined for any set. It is unfortunate though, that the calculations involved can sometimes make it quite difficult to actually determine or estimate Hausdorff dimension.

We begin with some preliminary definitions and introduce some notation that perhaps the reader may not be familiar with. If U is a subset of $\mathbb{R}^n$ then we say the **diameter** of U is defined as

$$\mathrm{diam}(U) = sup\{|x-y| : x, y \in U\}.$$

Furthermore, if $\{U_i\}$ is a countable collection of sets that cover F with diameters less than some value $\delta > 0$, we say that $\{U_i\}$ is a δ**-cover** of F.

Definition 1.3.1. *For a given subset F of $\mathbb{R}^n$, $s \geq 0$, and for any $\delta > 0$, we define*

$$\mathcal{H}^s_\delta(F) = \inf\left\{\sum_{i=1}^{\infty}(\mathrm{diam(U_i)})^{\mathrm{s}} : \{U_i\} \textit{ is a } \delta\textit{-cover of } F\right\}.$$

Here, as δ decreases the number of permissible covers of F also decreases. We therefore get that the infimum increases and so approaches a limit as $\delta \to 0$. This limit exists for any subset F of $\mathbb{R}^n$ and the limiting value is usually either 0 or ∞. For more information on this jump refer to Chapter 2 in [7].

Definition 1.3.2. *The **s-dimensional Hausdorff measure** of F is defined as*

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F), \tag{1.3}$$

*and the **Hausdorff dimension** of F is*

$$\dim_H F = \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}. \tag{1.4}$$

If $s = \dim_H F$, then $\mathcal{H}^s(F)$ can take on any value; zero, infinity, or any value in between. With respect to scaling, Hausdorff dimension behaves as we would expect it to. If we magnify an object by a factor of λ we get that the s-dimensional Hausdorff measure will scale by a factor of λ^s.

Definition 1.3.3. *Let D be closed subset of $\mathbb{R}^n$, with possibly $D = \mathbb{R}^n$. Then a mapping $S : D \to D$ is a **contraction** on D if there exists a number $0 < c < 1$ such* *that*

$$|S(x) - S(y)| \leq c|x - y| \tag{1.5}$$

*for all $x, y \in D$. Note that here all we have is a Lipschitz function where the Lipschitz constant is strictly less than one. A family of contractions $\{S_1, S_2, \ldots, S_m\}$ with $m \geq 2$ is called an **iterated function system** or IFS. Furthermore, if a non-empty compact subset F of D satisfies*

$$F = \bigcup_{i=1}^{m} S_i(F)$$

*then we call F the **attractor** of the IFS. It is important to note that an IFS uniquely determines its attractor.*

We note here that if equality holds in (1.5) then the function S actually transforms sets into geometrically similar sets and so we have the following closely related definition.

Definition 1.3.4. *The transformations $S_1, \ldots, S_m : \mathbb{R}^n \to \mathbb{R}^n$ are considered **similarities**, or **contracting similarities** if*

$$|S_i(x) - S_i(y)| = c_i|x - y|$$

*for $x, y \in \mathbb{R}^n$ and $0 < c_i < 1$. Here c_i is called the **ratio** of S_i, and the attractor of a collection of similarities is referred to as a **self-similar set**. We often refer to a similarity with ratio c_i as a $\mathbf{c_i}$**-similarity** for convenience.*

Similarities play an important role in the relationship between p-moduli and p-Poincaré inequalities. Some standard examples of self-similar sets are the middle-thirds Cantor set, both the standard Sierpiński carpet and the Sierpiński triangle, as well as the von Koch curve. References for these examples are easily found but Falconer describes them all and more in the Introduction to [7].

We will soon introduce a theorem that allows us to easily calculate Hausdorff dimension using very limited information about the scaling ratios used in the construction of the IFS. In order to make a connection between the dimension of a set and the scaling ratios used in the construction of the set we must first ensure that the following condition holds. If we consider the attractor F and its components $S_i(F)$, this condition ensures that the components do not have 'too much' overlap.

Definition 1.3.5. *Assuming the transformations $S_i, \ldots, S_m$ are similarities, we say they satisfy the* ***open set condition*** *if there exists a non-empty open, bounded set V such that*

$$V \supset \bigcup_{i=1}^{m} S_i(V),$$

and this union is disjoint.

So we are now in a position to state the primary theorem of this section. Although we do not prove it here, the theorem and its detailed proof can be found in [7] under Theorem 9.3. As stated before, this is the theorem that allows us to find the dimension of many self similar fractals, and in particular what we will use to calculate the dimension of the self similar Sierpiński carpets defined in the next section.

Theorem 1.3.6. *Suppose that the open set condition holds for the similarities S_i on $\mathbb{R}^n$ with ratios $0 < c_i < 1$ for $1 \leq i \leq m$. If F is the attractor of the IFS $S_1, \ldots, S_m$, i.e.*

$$F = \bigcup_{i=1}^{m} S_i(F),$$

then $\dim_H F = s$, where s satisfies

$$\sum_{i=1}^{m} c_i^s = 1.$$

In fact, we also get that for this value of s, $0 < \mathcal{H}^s(F) < \infty$.

Chapter 2

Poincaré Inequalities, Modulus, and Capacity

2.1 Poincaré Inequalities

We begin this chapter by stating the classical Poincaré inequality for Ω, a subset of $\mathbb{R}^n$ satisfying some specific conditions, followed by a proof outlining the validity of such an inequality on the Sobolev space $W^{1,p}(U)$. We then extend the definition to a Poincaré inequality defined on balls and outline a proof having similar results but this time on the Sobolev space $W^{1,p}(B(x,r))$.

We then move on to define two important notions, modulus and capacity, both of which are shown to be instrumental when proving the validity of Poincaré inequalities. We will also show that on metric measure spaces that are geodesic and proper, modulus and capacity both provide the same numerical value. Using modulus to prove results about Poincaré inequalities is a common technique and some examples follow in this chapter.

The final section of Chapter 2 presents the most important theorem of this thesis, Theorem 2.4.1 which relates the non-triviality of the p-modulus to the validity of the p-Poincaré inequality on doubling metric measure spaces. Following this theorem we present an example with the goal of developing in the reader some form of intuition concerning this relationship. Before proving Theorem 2.4.1 in the last section of the chapter, we state and prove a proposition that relates Poincaré inequalities and modulus on complete, doubling, metric measure spaces.

We must first introduce some notation so that the Poincaré inequality can be stated in a less complicated format. For a metric measure space (X, d, μ) and a subset $E \subset X$ of positive measure, we denote the **mean value of a function** $u : E \to \mathbb{R}$ by,

$$u_E = \fint_E u \, d\mu = \frac{1}{\mu(E)} \int_E u \, d\mu.$$

This definition will be used in a variety of ways throughout this thesis; however the definition should be clear from the context of use.

2.1.1 Poincaré Inequalities on $\mathbb{R}^n$

We begin our analysis of Poincaré inequalities with the more intuitive case of $\mathbb{R}^n$, and discuss the existence of Poincaré inequalities in this setting. We will be proving the validity of the Poincaré inequality on the Sobolev space $W^{1,p}(\Omega)$ defined in Chapter 1, where Ω is a subset of $\mathbb{R}^n$ satisfying specific conditions.

Theorem 2.1.1 (Classical Poincaré Inequality)**.** *Take $1 \leq p \leq \infty$, and consider $\Omega \subset \mathbb{R}^n$ a bounded, connected, open subset with a C^1 boundary $\partial\Omega$. Then there exists a constant C that depends only on Ω, n, and p, such that*

$$||u - u_\Omega||_{L^p(\Omega)} \leq C||\nabla u||_{L^p(\Omega)}$$

for every function $u \in W^{1,p}(\Omega)$.

Proof. The proof of this theorem from [6] is a proof by contradiction, rather than a proof using modulus as seen in the rest of this paper. For the sake of contradiction we begin by assuming that for each integer value $k = 1, 2, \ldots$ we have a function $u_k \in W^{1,p}(\Omega)$ such that

$$||u_k - (u_k)_\Omega||_{L^p(\Omega)} > k||\nabla u_k||_{L^p(\Omega)}. \tag{2.1}$$

We renormalise this function by defining

$$v_k = \frac{u_k - (u_k)_\Omega}{||u_k - (u_k)_\Omega||_{L^p(\Omega)}}$$

for $k \geq 1$, so that $(v_k)|_\Omega = 0$ and $||v_k||_{L^p(\Omega)} = 1$. Therefore by (2.1) we get that

$$||\nabla v_k||_{L^p(\Omega)} < \frac{1}{k} \tag{2.2}$$

and so the functions $\{v_k\}_{k\geq 1}$ are bounded in $W^{1,p}(\Omega)$. Therefore knowing that the Rellich-Kondrachov Theorem 1.2.3 holds, we get as a consequence that there must exist a subsequence $\{v_{k_j}\}_{j\geq 1} \subset \{v_k\}_{k\geq 1}$ and a function $v \in L^p(\Omega)$ such that

$$v_{k_j} \to v \in L^p(\Omega)$$

Taking limits we get that $v_\Omega = 0$ and $||v||_{L^p(\Omega)} = 1$. However, we know that for every $i = 1, 2, \ldots, n$ and $\varphi \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}\int_\Omega v\varphi_{x_i}\, dx &= \lim_{k_j\to\infty} \int_\Omega v_{k_j}\varphi_{x_i}\, dx \\ &= -\lim_{k_j\to\infty} \int_\Omega v_{k_j,x_i}\varphi\, dx \\ &= 0 \qquad \text{by (2.2).}\end{aligned}$$

Therefore, $v \in W^{1,p}(\Omega)$ and $\nabla v = 0$ a.e. Furthermore, because Ω is connected we can conclude v is constant, and since $v_\Omega = 0$ we must have that $v \equiv 0$. This contradicts $||v||_{L^p(\Omega)} = 1$ and we are done. □

We will now consider the case where we restrict the set U to a ball $B = B(x, r)$.

Corollary 2.1.2 (Poincaré inequality for a ball)**.** *Assume $1 \leq p \leq \infty$. Then there exists a constant C, depending only on n and p, such that*

$$||u - u_B||_{L^p(B(x,r))} \leq Cr||\nabla u||_{L^p(B(x,r))} \tag{2.3}$$

for each ball $B = B(x, r) \subset \mathbb{R}^n$ and each function $u \in W^{1,p}(B(x, r))$.

Proof. First, we know that for $\Omega = B_0 = B(0,1)$ equation (2.3) follows from Theorem 2.1.1. If $u \in W^{1,p}(B(x,r))$, define

$$v(y) := u(x+ry), \qquad y \in B(0,1),$$

so that $v \in W^{1,p}(B(0,1))$. We also have that

$$||v - v_{B_0}||_{L^p(B(0,1))} \leq C||\nabla v||_{L^p(B(0,1))}.$$

We write this in terms of u;

$$\left(\int_{B(0,1)} \left| u(x+ry) - \fint_{B(0,1)} u(x+ry)\ dy \right|^p\ dy\right)^{1/p} \leq C\left(\int_{B(0,1)} |r\nabla u(x+ry)|^p\ dy\right)^{1/p},$$

and then by a change of variable we get

$$\left(\int_{B(x,r)} \left| u(t) - \fint_{B(x,r)} u(t)\ dt \right|^p\ dt\right)^{1/p} \leq Cr\left(\int_{B(x,r)} |\nabla u(t)|^p dt\right)^{1/p},$$

which completes the proof. $\square$

2.1.2 Poincaré Inequalities on Metric Measure Spaces

We will now state the Poincaré inequality for metric measure spaces. This is the variation used by Mackay, Tyson and Wildrick in [15], and which we will use throughout Chapter 3. It is what we call a **weak p-Poincaré inequality** as the integral on the right is now taken over the dilated ball $\lambda B = B(x, \lambda r)$ where $\lambda \geq 1$.

Definition 2.1.3. *For $p \geq 1$, a metric measure space (X, d, μ) is said to support a* ***weak p-Poincaré inequality*** *if there exist constants $C, \lambda \geq 1$ such that for any continuous function $u : X \to \mathbb{R}$ with upper gradient $\rho : X \to [0, \infty]$, the following inequality*

$$\fint_B |u - u_B|\ d\mu \leq Cr\left(\fint_{\lambda B} \rho^p\ d\mu\right)^{1/p} \tag{2.4}$$

holds for every ball $B = B(x,r) \subset X$.

This weak Poincaré inequality can be derived from an inequality similar to (2.3) (for metric measure spaces) by an application of Holder's inequality:

$$\int_B |u - u_B| \, d\mu \le \left(\int_B |u - u_B|^p \, d\mu \right)^{1/p} \mu(B(x,r))^{1-1/p} \le Cr \left(\int_B \rho^p \, d\mu \right)^{1/p} \mu(B(x,r))^{1-1/p}.$$

If we take integral averages then

$$\fint_B |u - u_B| \, d\mu \le Cr \left(\fint_B \rho^p \, d\mu \right)^{1/p}.$$

2.2 Moduli of Curve Families

We will now describe one of the fundamental tools used in this paper, that of p-moduli of curve families. Readers should note that all of these definitions can be extended to locally rectifiable curves as well.

Definition 2.2.1. *Given a collection Γ of paths in X, the set of* ***admissible functions*** *of Γ, denoted $A(\Gamma)$, is the set of all Borel functions $\rho : X \to [0, \infty]$ such that for any $\gamma \in \Gamma$ we get*

$$\int_\gamma \rho \, ds \ge 1. \tag{2.5}$$

For $1 \le p \le \infty$, the ***p-modulus*** *of Γ is the value*

$$\mathrm{mod}_p \Gamma = \inf_{\rho \in A(\Gamma)} \int_X \rho^p \, d\mu. \tag{2.6}$$

Here we must note that if $A(\Gamma)$ is empty (for example if Γ contains a constant path), then $\mathrm{mod}_p(\Gamma) = \infty$. Also, we will sometimes be interested in considering admissible functions where ρ is perhaps a Lipschitz function satisfying inequality (2.5) rather than just a Borel function. If $\mathcal{F}(X)$ is a collection of Borel functions then

$$\mathrm{mod}_p(\Gamma; \mathcal{F}(X))$$

refers to the same definition of modulus except we only consider $\rho \in \mathcal{F}(X)$ satisfying (2.5) as admissible functions. For the sake of simplicity, if E and F are disjoint subsets of X, then we define

$$\text{mod}_p(E, F) = \text{mod}_p\Gamma,$$

where Γ is the collection of all curves in X connecting the two sets E and F. Furthermore, we write $\text{mod}_p(x, y)$ when it is clear from the context that we mean $\text{mod}_p(\{x\}, \{y\})$ for $x, y \in X$.

The notion of p-moduli of curve families allows us to think of curve families with non-trivial moduli as 'thick' curve families and so we can prove Poincaré inequalities in a geometric way. Not only can this be a more intuitive a tool for some, but it is also an extremely powerful one as we will see later in this chapter. We will now summarize some basic properties of p-moduli which appear in Proposition 4.1.6 of [14].

Proposition 2.2.2. *Let (X, d, μ) be a metric measure space and let $p \geq 1$. Then*

(i) $\text{mod}_p\emptyset = 0$*;*

(ii) if $\Gamma_1 \subset \Gamma_2$ *then* $\text{mod}_p\Gamma_1 \leq \text{mod}_p\Gamma_2$*;*

(iii) $\text{mod}_p \cup_i \Gamma_i \leq \sum_i \text{mod}_p\Gamma_i$ *for any countable set of collections* Γ_i*;*

(iv) if $p > 1$ *and* $\Gamma_1 \subset \Gamma_2 \subset \Gamma_3 \subset \cdots$*, then* $\text{mod}_p\cup_i\Gamma_i = lim_{i\to\infty}\text{mod}_p\Gamma_i$*;*

(v) a curve family Γ *has p-modulus zero if and only if there is* $\rho \in L^p(X)$ *such that* $\int_\gamma \rho\, ds = \infty$ *for all locally rectifiable curves* $\gamma \in \Gamma$*;*

(vi) if Γ_1, Γ_2 *are curve families with the property that each curve in* Γ_1 *contains a sub-curve in* Γ_2*, then* $\text{mod}_p\Gamma_1 \leq \text{mod}_p\Gamma_2$*;*

Note that properties (i), (ii), and (iii) show that mod_p is an outer measure on the collection of all rectifiable curves. The remaining three properties serve to further develop the reader's intuition and understanding of p-moduli, but will not be proved in this paper.

The following lemma from [14] states that the p-moduli of a family of curves is comparable to the p-moduli of the image of that family under a similarity transformation. Mackay and Tyson use this lemme to prove a proposition similar to Proposition 3.2.6 however because the modified Sierpiński carpets are not self-similar we are not able to make use of it. The utility of such a proposition when computing modulus on self-similar fractals should certainly be noted though.

Lemma 2.2.3. *If S is a c-similarity of a Q-regular metric measure space (X, d, μ) then there is a constant $C > 0$ such that*

$$\frac{1}{C} c^{Q-p} \mathrm{mod}_p \Gamma \leq \mathrm{mod}_p S(\Gamma) \leq C c^{Q-p} \mathrm{mod}_p \Gamma$$

for every curve family Γ and every $p \geq 1$.

2.3 Capacity

Here we will define the notion of p-capacity. It is very similar to the definition of p-moduli; however, the infimum is taken over a different set of functions. We recall that in the definition of p-moduli, equation (2.6), the infimum was taken over all admissible functions defined as those satisfying inequality (2.5). Contrast that to the following definition of capacity:

Definition 2.3.1. *For distinct subsets E and F of a metric measure space (X, d, μ), and for $p \geq 1$, the p-**capacity** of the pair (E,F) is*

$$\mathrm{cap}_p(E, F) = \inf \int_X \rho^p \, d\mu, \tag{2.7}$$

where the infimum is taken over all upper gradients of each real valued Borel function u that satisfies $u|_E \leq 0$ and $u|_F \geq 1$.

It should be noted that the same number is obtained in (2.7) if we say $u|_E = 0, u|_F = 1$ and $0 \leq u \leq 1$. For more information on this please see [11]. As before, we will often write $\mathrm{cap}_p(x, y)$ when it is clear from the context that we mean

$\text{cap}_p(\{x\},\{y\})$ for $x, y \in X$. We also use the notation

$$\text{cap}_p(E, F; \mathcal{F}(X))$$

to refer to capacity where we only consider functions $u \in \mathcal{F}(X)$ when calculating the infimum in (2.7), where $\mathcal{F}(X)$ is any collection of Borel functions. If necessary we include the measure in the notation for modulus and capacity as follows;

$$\text{mod}_p(E, F; \mu) \quad \text{and} \quad \text{cap}_p(E, F; \mu).$$

The following proposition will relate p-capacity to p-moduli if we restrict ourselves to spaces that are geodesic and proper. For the sake of this paper this is extremely useful, allowing us to extend results proved using capacity to results about moduli, which is very helpful in verifying the validity of Poincaré inequalities.

Proposition 2.3.2. *For distinct subsets E and F of a geodesic and proper metric measure space (X, d, μ) with $\mu(X) < \infty$, we have that*

$$\text{mod}_p(E, F) = \text{cap}_p(E, F) = \text{cap}_p(E, F; Lip_0(X)).$$

Proof. First we note that the equality

$$\text{mod}_p(E, F) = \text{cap}_p(E, F)$$

is valid for any sets E and F in a metric space and as seen in Theorem 7.31 of [9], can be verified in the following way. We take u a function on X such that $u|_E \leq 0$ and $u|_F \geq 1$, and note that if ρ is an upper gradient of u we have

$$1 \leq |u(x) - u(y)| \leq \int_\gamma \rho \, ds$$

for all rectifiable curves γ joining $x \in E$ to $y \in F$. This means that any ρ considered when calculating capacity is also an admissible function for Γ, the family of curves joining E and F. Therefore because both definitions are infimums we get that

$$\text{mod}_p(E, F) \leq \text{cap}_p(E, F).$$

Now we must show the opposite inequality. To do so we consider ρ an admissible function for Γ joining E and F, then define

$$u(x) = \inf \int_{\gamma_x} \rho \, ds,$$

where now the infimum is taken over all curves joining the set E and the point $x \in X$. Then we know $u|_E = 0$, $u|_F \geq 1$, and also that ρ is an upper gradient of u. So now we have that any admissible function ρ for Γ is actually the upper gradient of some function u considered in the calculation of capacity. By the same reasoning as before we get that

$$\operatorname{cap}_p(E, F) \leq \operatorname{mod}_p(E, F),$$

and so equality holds.

Recalling that the definition of capacity involves computing the infimum, we note that Lip_0(X) is a subset of all Borel functions, and so the inequality

$$\operatorname{cap}_p(E, F) \leq \operatorname{cap}_p(E, F; \mathrm{Lip}_0(X))$$

holds. Therefore, to finish the proof we need only to show

$$\operatorname{cap}_p(E, F; \mathrm{Lip}_0(X)) \leq \operatorname{mod}_p(E, F),$$

and we will be done. It is possible actually to show that equality holds here, and this follows directly from the following two Lemmas whose proofs will not be included in this paper, but can be found in [12]. □

Lemma 2.3.3. *Let E and F be disjoint subset of a proper metric measure space (X, d, μ) with $\mu(X) < \infty$. Then*

$$\operatorname{cap}_p(E, F; \mathrm{Lip}_0(X)) = \operatorname{cap}_p(E, F; \mathrm{Lip}(X)).$$

Lemma 2.3.4. *Let E and F be disjoint compact subsets of a geodesic and proper metric measure space (X, d, μ) with $\mu(X) < \infty$. Then*

$$\operatorname{cap}_p(E, F; \mathrm{Lip}(X)) = \operatorname{mod}_p(E, F).$$

So we have shown that on geodesic and proper metric measure spaces of finite measure the notions of modulus and capacity coincide. Because of this equality when we want to estimate modulus it is often more convenient to consider capacity instead. We have also shown that even if we only consider Lipschitz function or even only the Lipschitz functions with compact support, the capacity still retains the same value.

2.4 Relating p-Moduli and p-Poincaré Inequalities

As stated in the introduction, it is known that there is a very fundamental relationship between the validity of a p-Poincaré inequality on a metric space and the non-triviality of the p-modulus. Before discussing this dependancy, we will first state a theorem outlining this:

Theorem 2.4.1. *Fix $p \geq 1$. Let (X, d, μ) be a complete, doubling metric measure space. Then (X, d, μ) admits a p-Poincaré inequality if and only if there exist constants $C_1 > 0$ and $C_2 \geq 1$ so that*

$$d(x,y)^{1-p} \leq C_1 \mathrm{mod}_p(\Gamma_{xy}; \mu_{xy}^{C_2}) \tag{2.8}$$

for every pair of distinct points $x, y \in X$.

To understand this relationship between Poincaré inequalities and modulus one can think of trying to find for any two distinct points x and y, a "large" family of curves joining the two, such that the length of the curves is comparable to the distance between the points. If this is possible then the modulus is said to be non-trivial and as a result the spaces supports a Poincaré inequality. In general too narrow of a passage can cause problems for the Poincaré inequality. Narrow passages force the constant to become larger and larger and in extreme cases will cause it to fail. When trying to construct a counterexample, any tunnel that becomes more and more narrow will cause the Poincaré inequality to fail

To further illustrate this concept and also to emphasize why these Sierpiński carpet examples are so interesting, we consider a classic but interesting example in $\mathbb{R}^n$ where the Poincaré inequalities will fail for any $1 \le p \le n$.

Example 2.4.2. *(Bow-tie) Let $n \ge 2$ and denote $x \in \mathbb{R}^n$ as $x = (x_1, x_2, \dots, x_n)$. Let*

$$\begin{aligned} X_+ &= \{x \in \mathbb{R}^n; x_j \ge 0, j = 1, \dots, n\}, \\ X_- &= \{x \in \mathbb{R}^n; x_j \le 0, j = 1, \dots, n\}, \end{aligned} \tag{2.9}$$

equipped with the Euclidean metric and the Lebesgue measure.

Then if $1 \le p \le n$ we get that $X = X_+ \cup X_-$ does not support a p-Poincaré inequality even though on their own both X_+ and X_- each support 1-Poincaré inequalities. The problem with the union is that any curves between the two subspaces must pass through the origin. The proof of these statements as well as more information on this topic in general can be found in Chapter 5 of [2].

To understand Theorem 2.4.1 we need to first examine the notation involved. In this equation, $\mathrm{mod}_p(\Gamma_{xy}; \mu_{xy}^C)$ represents the p-modulus of the family of curves Γ_{xy} joining x and y, and the measure μ_{xy}^C is defined as follows:

$$\mu_{xy}^C(E) = \int_{E \cap B_{xy}^C} \frac{d(x,z)}{\mu(B(x, d(x,z)))} + \frac{d(y,z)}{\mu(B(y, d(y,z)))} d\mu(z), \tag{2.10}$$

with $B_{xy}^C = B(x, Cd(x,y)) \cup B(y, Cd(x,y))$. We say that μ_{xy}^C is the symmetric Riesz kernel of μ at x and y, and note that μ_{xy}^C is absolute continuous with respect to μ. Furthermore, if μ is doubling, we get that $\mu_{xy}^C(X) < \infty$. For more information on this see [9] and [12]. Returning to our interpretation of Theorem 2.4.1, we recall that by definition

$$\mathrm{mod}_p(\Gamma; \nu) := \inf \int \rho^p d\nu$$

for a Borel measure ν on (X, d), and the infimum is taken over all admissible Borel functions ρ, those where

$$\int_\gamma \rho ds \ge 1$$

for any rectifiable curve $\gamma \in \Gamma$. The proof of this theorem is presented at the end of this chapter and more information on this topic can be found in [12]. The primary use of this theorem will be in proving the following:

Proposition 2.4.3. *Let* (X, d, μ) *be a complete, doubling metric measure space. If* (X, d, μ) *supports a p-Poincaré inequality, then* $\mathrm{mod}_p \Gamma > 0$ *for some curve family* Γ.

Proof of Proposition 2.4.3. By Theorem 2.4.1 we know that because (X, d, μ) supports a p-Poincaré inequality we must have that

$$\mathrm{mod}_p(\Gamma_{xy}; \mu^C_{xy}) > 0$$

for any distinct points $x, y \in X$, and $C = C_2$ from Theorem 2.4.1. So we fix two such points x, y, let $r = d(x, y)$, and then fix $\epsilon > 0$ such that $\mathrm{mod}_p(\Gamma_{xy}; \mu^C_{xy}) \geq \epsilon$. Now define

$$A := B(x, \frac{2}{3}r) \setminus B(x, \frac{1}{3}r).$$

Let Γ_A be the family of curves joining $B(x, \frac{2}{3}r)$ to $B(x, \frac{1}{3}r)$. Then we know that if $\sigma \in \Gamma_{xy}$ is a curve joining x and y, there must be a curve $\gamma \in \Gamma_A$ such that γ is a subcurve of σ. Therefore,

$$\int_\gamma \rho \; ds = \int_\sigma \rho \cdot \mathcal{X}_A \; ds$$

This means that if ρ is admissible for Γ_A we know $\rho_0 = \rho \cdot \mathcal{X}_A$ is admissible for Γ_{xy}. So for $\rho \in \Gamma_A$ we get that

$$\begin{aligned} \inf_{\rho \in \Gamma_A} \int_A \rho^p \; d\mu^C_{xy} &= \inf_{\rho \in \Gamma_A} \int_X (\rho \cdot \mathcal{X}_A)^p \;\; d\mu^C_{xy} \\ &\geq \inf_{\rho \in \Gamma_{xy}} \int_X \rho_0^p \; d\mu^C_{xy} \\ &\geq \epsilon. \end{aligned} \tag{2.11}$$

Now we consider the relationship between $\mu^C_{xy}|_A$ and $\mu|_A$. First note that because $C \geq 1$ we have that $A \subset B^C_{xy}$ and so $B^C_{xy} \cap A = A$. In addition, we have that for $z \in A$,

$$d(x, z) \leq \frac{2}{3}r, \qquad d(y, z) \leq \frac{5}{3}r,$$

and

$$\mu(B(x, \frac{1}{3}r)) \leq \mu(x, d(x,z)).$$

Then, we take λ large enough to ensure

$$B(x, \frac{1}{3}r) \subset B(y, \lambda d(y,z)).$$

Now we know (X, d, μ) is doubling so there is a value C_D depending on the doubling constant, such that

$$\begin{aligned} C_D\mu(B(y,z)) &\geq \mu(B(y, \lambda d(y,z))) \\ &\geq \mu(B(x, \frac{1}{3}r)). \end{aligned}$$

Putting this all together we get the following:

$$\begin{aligned} \mu_{xy}^C(E)|_A = \mu_{xy}^C(E \cap A) &\leq \int_{E \cap A} \frac{\frac{2}{3}r}{\mu(B(x, \frac{1}{3}r))} + \frac{\frac{5}{3}r}{C_D\mu(B(x, \frac{1}{3}r))} \, d\mu(z) \\ &\leq \frac{C_D\frac{2}{3}r + \frac{5}{3}r}{C_D\mu(B(x, \frac{1}{3}r))}\mu(E \cap A) \\ &\leq \tilde{C}\mu(E)|_A \end{aligned} \tag{2.12}$$

where here $\tilde{\mathrm{C}}$ is a constant depending on C_D, r, and x all of which are fixed. From this and (2.11) we conclude that there is a $\delta > 0$ such that for any $\rho \in \Gamma_A$

$$\delta \leq \int_A \rho^p \, d\mu.$$

Therefore we get that $\mathrm{mod_p}\Gamma_\mathrm{A} \geq \delta > 0$. □

2.5 Proof of Theorem 2.4.1

We would now like to prove Theorem 2.4.1. The statement follows directly from a result proved by Keith in [12] which is stated next. It is not a direct proof; however the steps taken by Keith in proving Theorem 2.5.1 serve to illustrate the complexity of the relationship between Poincaré inequalities and modulus, by breaking it down into a series of smaller proofs. These proofs rely on the results stated in Propositions 2.5.2, and 2.5.3 which we include without proofs, but readers can find more information on the subject in [12].

Theorem 2.5.1. *Let $p \geq 1$ and let (X, d, μ) be a complete, geodesic, and proper metric measure space with μ doubling, and such that every ball in X has measure in $(0, \infty)$. Then the following are equivalent:*

1. *(X, d, μ) admits a weak p-Poincaré inequality (2.4) for all measurable functions,*

2. *(X, d, μ) admits a weak p-Poincaré inequality (2.4) for all compactly supported Lipschitz functions and their compactly supported Lipschitz upper gradients,*

3. *there exists constants $C, \lambda \geq 1$, such that*

$$\fint_B |u - u_B| \; d\mu \leq C \, diam B \left(\fint_{\lambda B} (Lip \; u)^p \; d\mu \right)^{1/p}, \tag{2.13}$$

   for every $u \in Lip_0(X)$ and for every ball B in X;

4. *there exists a constant $C \geq 1$ such that*

$$d(x, y)^{1-p} \leq C \mathrm{mod_p}(\Gamma_{\mathrm{xy}}; \mu_{\mathrm{xy}}^{\mathrm{C}}) \tag{2.14}$$

   for every pair of distinct points $x, y \in X$.

In order to prove Theorem 2.4.1 we need only to prove that (1) $\iff$ (4), but to do this we will need to prove that (1) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (1). As stated in the introduction to this chapter, the chain of implications here shows us in more detail why this relationship holds. First we will need the following propositions.

Proposition 2.5.2. *Let $p \geq 1$, and let (X, d, μ) be a geodesic metric measure space with μ doubling. Suppose u is a real-valued continuous function, ρ is a real-valued*

Borel function, both on X, satisfying a weak p-Poincaré inequality (2.4) *for some $C, \lambda \geq 1$, and for all balls $B \in X$. Then there exists a value $L \geq 1$ such that*

$$|u(x) - u(y)|^p \leq Ld(x,y)^{p-1} \int_X \rho^p \, d\mu_{xy}^{C_2} \tag{2.15}$$

for all distinct points $x, y \in X$.

Although stated by Keith, Proposition 2.5.2 is largely credited to Heinonen and Koskela's work in [10], and more information is also available in [9].

Proposition 2.5.3. *Let $p \geq 1$, and let (X, d, μ) be a metric measure space with μ doubling. Then if there is a value $L \geq 1$ such that for every pair of functions $u : X \to [0, \infty]$ and $\rho : X \to [0, \infty]$, where u is measurable and ρ is an upper gradient for u, equation* (2.15) *holds for μ almost every $x, y \in X$, then (X, d, μ) admits a p-Poincaré inequality for all measurable functions, with constants depending on L, p, and the doubling constant of μ.*

For more information on the preceding proposition see Theorem 9.5 of [9] and Proposition 11 of [12]. We are now in a position to prove proposition 2.5.1.

Proof of Theorem 2.5.1. Proof that (1) $\Rightarrow$ (3). Here we need only refer to Claim 1.1.2, where we showed that if $u \in \mathrm{Lip(X)}$, then *Lip* u is an upper gradient for u.

□

Proof that (3) $\Rightarrow$ (2). Assume (3) holds, and let $u, \rho \in \mathrm{Lip(X)}$ such that ρ is an upper gradient of u. Then to prove (2) we need only to show that

$$\mathit{Lip}\ u \leq \rho.$$

Let γ be a geodesic in X that connects two distinct points $x, y \in X$. Then since ρ is an upper gradient of u by definition we get

$$|u(x) - u(y)| \leq \int_\gamma \rho \, ds,$$

Then because X is a geodesic space and therefore a length space, we have that $d(x,y) = \ell(\gamma)$ which gives us:

$$\frac{|u(x) - u(y)|}{d(x,y)} \leq \frac{1}{\ell(\gamma)} \int_\gamma \rho \, ds.$$

Taking $x \to y$ and knowing that ρ is continuous, we can see that $Lip\ u \leq \rho$.

□

Proof that (2) ⇒ (4). Here we assume (2) holds for distinct $x, y \in X$, and restrict ourselves to functions $u, \rho \in Lip_0(X)$ such that ρ is an upper gradient for u, $u(x) = 0$, and $u(y) = 1$. We apply Proposition 2.5.2 which says that if u and ρ satisfy a p-Poincaré inequality (2.4), we must have that equation (2.15) holds so we know

$$d(x,y)^{1-p} \leq C \int_X \rho^p \, d\mu_{xy}^C.$$

Therefore we get that

$$d(x,y)^{1-p} \leq C \mathrm{cap_p(x, y; Lip_0(X), \mu_{xy}^C)}.$$

Recalling from Proposition 2.3.2 the relationship between p-capacity and p-modulus, and noting that $\mu_{xy}^C(X) < \infty$, we conclude that (4) holds. □

Proof that (4) ⇒ (1). Here we let u be a real-valued measurable function on X, and let ρ be an upper gradient for u. Then for distinct $x, y \in X$, such that $u(x) \neq u(y)$, we define $\bar{u}$ by

$$\bar{u}(z) = \left| \frac{u(z) - u(x)}{u(x) - u(y)} \right|, \qquad z \in X,$$

and then we define $\bar{\rho}$ by

$$\bar{\rho}(z) = \frac{\rho(z)}{|u(x) - u(y)|}, \qquad z \in X.$$

We would like to show that $\bar{\rho}$ is actually an upper gradient of our newly defined $\bar{u}$.

To do this we take z_1, z_2 two distinct points in X to get

$$\begin{aligned} |u(z_1) - u(z_2)| &= \left| \frac{u(z_1) - u(x)}{u(x) - u(y)} - \frac{u(z_2) - u(x)}{u(x) - u(y)} \right| \\ &= \frac{1}{|u(x) - u(y)|} |u(z_1) - u(z_2)| \\ &\leq \frac{1}{|u(x) - u(y)|} \int_\gamma \rho(z)\ ds \end{aligned} \tag{2.16}$$

for all rectifiable curves joining z_1 and z_2, and so $\bar{\rho}$ is an upper gradient of $\bar{u}$. It can be easily seen that $\bar{u}(x) = 0$ and $\bar{u}(y) = 1$. We can apply Proposition 2.3.2 and condition (4) to get

$$d(x,y)^{1-p} \leq C \mathrm{cap_p(x, y; \mu_{xy}^C)}. \tag{2.17}$$

Therefore by the definition of capacity we get

$$d(x,y)^{1-p} \leq C \int_X \bar{\rho}^p\ d\mu_{xy}^C. \tag{2.18}$$

If we re-write this in terms of u and ρ we get

$$d(x,y)^{1-p} \leq \frac{C}{|u(x) - u(y)|^p} \int_X \rho^p\ d\mu_{xy}^C. \tag{2.19}$$

and after rearranging we get exactly (2.15). Now applying Proposition (2.5.3) we get that X admits a p-Poincaré inequality for all measurable functions, which completes the proof. $\square$

$\square$

Chapter 3

Analysis on Modified Sierpiński Carpets

Here we discuss a general class of Sierpiński-type carpets as developed by Mackay, Tyson, and Wildrick in [15]. They are all doubling metric measure spaces that are homeomorphic to the standard Sierpiński carpet. Each example is derived from a subset of the plane with the Euclidean metric and Lebesgue measure. As noted previously, they are the first examples of spaces that support Poincaré inequalities for a renormalization of the Lebesgue measure that are also *compact* subsets of the Euclidean space with empty interior.

We begin in the first part of this chapter with the construction of the carpets and then proceed to introduce some definitions and notation specific to this topic. In the second section we introduce and explain the probability measure defined in [15], which is a re-normalization the Lebesgue measure that is shown to posses some very interesting properties.

In Chapter 2 we stated Proposition 2.4.3 which concluded that if a complete doubling metric measure space supports a p-Poincaré inequality then there must be some curve family that has non-trivial p-modulus. We present a new proposition in this chapter which states that on the modified Sierpiński carpets presented here, it suffices to only consider curve families joining either the left and the right, or top and bottom edges of a subcarpet.

As discussed in the introduction, each carpet depends entirely on the chosen sequence $\mathbf{a}$ and so we would like to know what impact these sequences have on the geometry of the resulting space. In their paper discussing these carpets Mackay, Tyson, and Wildrick [15] explore this topic and achieve some very interesting results relating a sequences existence in specific ℓ^q spaces and the validity of the p-Poincaré inequality on the resulting carpet $S_{\mathbf{a}}$. We present these results and their proofs in the last section of this chapter as they are very good examples of how to use modulus to prove results about Poincaré inequalities.

3.1 The Construction

We fix a sequence

$$\mathbf{a} = (a_1, a_2, a_3, \ldots),$$

where each $a_m \in \{1/3, 1/5, 1/7, \ldots\}$ (here a_m must be the reciprocal of an odd integer strictly greater than one). Each sequence then defines a corresponding Sierpiński carpet $S_{\mathbf{a}}$ which we will now describe. We begin with the unit square $T_0 = [0,1]^2$ and starting with $m = 1$ iteratively apply the following steps:

1. Divide each current square into a_m^{-2} essentially disjoint closed sub-squares of equal size, where m represents the current step of the construction.

2. Remove the middle sub-square from each square.

3. Increase the parameter m by 1.

Let $\mathcal{T}_m$ represent the collection of all the stage m squares, and then define the **level m precarpet** as

$$S_{\mathbf{a},m} := \bigcup_{T \in \mathcal{T}_m} T, \qquad m \geq 0.$$

The corresponding Sierpiński carpet $S_{\mathbf{a}}$ is then the intersection of the pre-carpets:

$$S_{\mathbf{a}} := \bigcap_{m\geq 0} S_{\mathbf{a},m}.$$

A level m **subcarpet** is the intersection $S_a \cap T$ for some $T \in \mathcal{T}_m$, i.e. it is the modified Sierpinski carpet constructed in the square T using the tail end of the sequence $(a_m, a_{m+1}, a_{m+2}, \ldots)$. The standard Sierpiński carpet can be constructed using this method with $\mathbf{a} = (1/3, 1/3, 1/3, \ldots)$, and for any $\mathbf{a}$ the corresponding set $S_{\mathbf{a}}$ is a compact, connected, locally connected subset of the plane with empty interior and containing no local cut points. This set $S_{\mathbf{a}}$ is in fact homeomorphic to the standard Sierpiński carpet and so they are referred to as *carpets*[15].

Of special interest are the sets with a fixed scaling ratio which will be denoted $S_{1/(2k+1)}$ for $k \in \mathbb{N}$. These represent the self-similar carpets $S_{\mathbf{a}}$ where $\mathbf{a}$ is the constant sequence $\left(\frac{1}{2k+1}, \frac{1}{2k+1}, \frac{1}{2k+1}, \ldots\right)$. Each carpet is then the attractor formed by, at each stage, dividing the squares into $(2k+1)^2$ sub-squares with ratio $c_i = \frac{1}{(2k+1)}$ and then removing the middle square. Taking V as the interior of the unit square we get that the open set condition, Definition (1.3.5), is satisfied, and so by Theorem 1.3.6 we get that the Hausdorff dimension of the set is given by Q_k satisfying

$$((2k+1)^2 - 1)\left(\frac{1}{2k+1}\right)^{Q_k} = 1,$$

hence

$$Q_k = \frac{\log((2k+1)^2 - 1)}{\log(2k+1)} = \frac{\log(4k^2+4k)}{\log(2k+1)} < 2. \tag{3.1}$$

It is also quite interesting to note that $S_{1/(2k+1)}$ is in fact Ahlfors regular in this dimension. To demonstrate this we must first introduce a probability measure on our carpets and so the proof will follow.

3.2 Basic Properties of the Carpets $S_{\mathbf{a}}$

We will now quickly cover some of the basic properties of the general carpets $S_{\mathbf{a}}$, and describe the natural measure that we will be using on these sets. To begin, recall that $\mathcal{T}_m$ represents the collection of all the stage m squares, and so for each m, $\mathcal{T}_m$ contains

$$\prod_{j=1}^{m} (a_j^{-2} - 1)$$

essentially disjoint closed squares, each with side length

$$s_m := \prod_{j=1}^{m} a_j.$$

3.2.1 The Natural Probability Measure on $S_{\mathbf{a}}$

We first define a measure μ_m on $[0,1]^2$ which is the Lebesgue measure restricted to our pre-carpets $S_{\mathbf{a},m}$ and renormalized to have total measure one. The sequence of measures (μ_m) then converges weakly to a probability measure μ having as its support the carpet $S_{\mathbf{a}}$ [15]. We can also see that for each square $T \in \mathcal{T}_m$ of size s_m, we have $\mu_n(T) = \mu_m(T)$ for all $n \geq m$ because later steps in the construction will only redistribute the mass into sub-squares contained in T. We therefore have the following relationship:

$$\mu(T) = \mu_m(T) = \prod_{j=1}^{m} (a_j^{-2} - 1)^{-1} =: v_m. \tag{3.2}$$

Now we examine the basic properties of μ, and the next proposition will really begin to demonstrate the relationship between the behaviour of the sequence $\mathbf{a}$ and the properties it imposes on the corresponding space $(S_{\mathbf{a}}, d, \mu)$. The notation here is taken from [15]; writing $a \lesssim b$ means that there exists $C > 0$ such that $a \leq Cb$, where C depends only on the relevant data.

Proposition 3.2.1. *(Proposition 2, [15]) The space $(S_{\boldsymbol{a}}, d, \mu)$ has the following properties:*

(i) For any $\boldsymbol{a}$, μ is a doubling measure.

(ii) *For any* $\boldsymbol{a}$*, we have* $\mu(B(x,r)) \gtrsim r^2$ *for all* x *and* $r \leq 1$*.*

(iii) *If* $\boldsymbol{a} \in c_0$*, then for any* $Q < 2$ *we have* $\mu(B(x,r)) \lesssim r^Q$ *for all* x *and* $r > 0$*, hence* $\dim S_{\boldsymbol{a}} = 2$*.*

(iv) *If* $\boldsymbol{a} \in \ell^2$*, then* μ *is comparable to Lebesgue measure with a constant depending only on* $||\boldsymbol{a}||_2$*. Moreover, in this case,* μ *is an Ahlfors 2-regular measure on* $S_{\boldsymbol{a}}$*.*

(v) *If* $\boldsymbol{a} = (a_m)$ *is eventually constant (and equal to* $\frac{1}{2k+1}$*), then* μ *is comparable to Hausdorff measure* $\mathcal{H}^{Q_k}$ *and is an Ahlfors* Q_k*-regular measure on* $S_{\boldsymbol{a}}$*.*

To prove this proposition we require the following definitions which will allow us to further understand the relationship between the steps in the construction of $S_{\mathbf{a}}$ and the side length of our squares. So for $x \in S_{\mathbf{a}}$ and $r > 0$ we define $m(x,r)$ and $m(r)$ in the following way:

1. $m(x,r)$ is the smallest integer m such that there is a $T \in \mathcal{T}_m$ with $x \in T \subset B(x,r)$;
2. $m(r)$ is the smallest integer m such that $s_m \leq r$.

We will also require the following:

Lemma 3.2.2. *For any* x *and* r*,* $m(\sqrt{2}r) \leq m(x,r) \leq m(\frac{r}{\sqrt{2}}) + 1$*.*

Proof of Lemma. Take $T \in \mathcal{T}_{m(x,r)}$ so that $T \subset B(x,r)$. Then

$$\sqrt{2}s_{m(x,r)} = \mathrm{diam}T \leq \mathrm{diam}B(x,r) \leq 2r.$$

So we get that $s_{m(x,r)} \leq \sqrt{2}r$, but we know that by definition, $m(\sqrt{2}r)$ is the smallest integer such that $s_m \leq \sqrt{2}r$, therefore

$$m(\sqrt{2}r) \leq m(x,r).$$

Now, we can always find a $T \in \mathcal{T}_{m(\frac{r}{\sqrt{2}})+1}$ such that $x \in T$. Knowing that $m(\frac{r}{\sqrt{2}})$ is the smallest integer with

$$s_{m(\frac{r}{\sqrt{2}})} \leq \frac{r}{\sqrt{2}},$$

we get that $m(\frac{r}{\sqrt{2}}) + 1$ is the smallest integer such that

$$s_{m(\frac{r}{\sqrt{2}})+1} \leq (\frac{r}{\sqrt{2}})(\frac{1}{3}).$$

This means

$$\text{diam}T \leq \frac{r}{3},$$

and so $T \subset B(x, r)$. But $m(x, r)$ is the smallest integer m such that $T \in \mathcal{T}_m$ with $x \in T \subset B(x, r)$ so we get that $m(x, r) \leq m(\frac{r}{\sqrt{2}}) + 1$.

□

Claim 3.2.3. *For all $r > 0$, $m(r) \leq m(2r) + 1$.*

Proof of Claim. We assume not, so that $m(r) > m(2r)+1$ which by definition implies $m(r) \geq m(2r) + 2$, and so $m(r) - 1 \geq m(2r) + 1$. Therefore we get the following string of inequalities:

$$\begin{aligned} r &< s_{m(r)-1} \\ &\leq s_{m(2r)+1} \\ &\leq \frac{1}{3} s_{m(2r)} \\ &\leq \left(\frac{1}{3}\right)(2r) \end{aligned} \tag{3.3}$$

which is obviously a contradiction. The first step follows from the definition of $m(r)$, the second because if $n \geq m$ we know that $s_n \leq s_m$, and then by definition we know that $S_{m(2r)} \leq 2r$. □

Claim 3.2.4. *For all $r > 0$, $m(r) \leq -\log_2(r) + 1$*

Proof of Claim. Take n to be the largest integer such that $2^n r \leq 1$. Then since $m(1) = 0$ we get that:

$$m(r) \leq m(2r) + 1 \leq \ldots \leq m(2^{n+1}r) + n + 1 \leq m(1) + n + 1 = n + 1,$$

and we then have the following string of equivalences:

$$\begin{aligned} 2^n r \leq 1 &\Longleftrightarrow \log_2(2^n r) \leq 0 \\ &\Longleftrightarrow n + \log_2(r) \leq 0 \\ &\Longleftrightarrow n \leq -\log_2(r) \\ &\Longleftrightarrow n + 1 \leq -\log_2(r) + 1. \end{aligned}$$

Therefore we have that $m(r) \leq n + 1 \leq -\log_2(r) + 1$ as desired. □

The last tool we will require to prove Proposition 3.2.1 is as follows:

Proposition 3.2.5 (Proposition 3, [15])**.** *For each $x \in S_{\boldsymbol{a}}$ and $0 < r \leq 1$,*

$$\mu(B(x,r)) \asymp h(r) := r^2 \prod_{j=1}^{m(r)} \left(\frac{1}{1 - {a_j}^2} \right). \tag{3.4}$$

Proof. We wish to bound $\mu(B(x,r))$ from above and so we cover $B(x,r)$ by squares from $\mathcal{T}_{m(r)}$. Knowing that each square has side length $s_{m(r)}$, $B(x,r)$ can then be covered by a larger square of sidelength $2r + 2s_{m(r)}$ formed as a collection of squares in $\mathcal{T}_{m(r)}$. The number of squares needed to cover $B(x,r)$ is then

$$\frac{(2r + 2s_{m(r)})^2}{s_{m(r)}^2}.$$

Recalling that $v_{m(r)}$ is the renormalized measure of a single square $T \in \mathcal{T}_{m(r)}$, we get that:

$$\begin{aligned}
\mu(B(x,r)) &\leq \frac{(2r+2s_{m(r)})^2}{{s_{m(r)}}^2} \cdot v_{m(r)} \qquad (3.5)\\
&\leq \frac{(4r)^2}{s_{m(r)}^2} \prod_{j=1}^{m(r)} \left(\frac{1}{a_j^{-2}-1} \right)\\
&= \frac{16r^2}{s_{m(r)}^2} \prod_{j=1}^{m(r)} \left(\frac{a_j^2}{1-a_j^2} \right)\\
&\leq \frac{16r^2}{s_{m(r)}^2} \prod_{j=1}^{m(r)} {a_j}^2 \prod_{j=1}^{m(r)} \left(\frac{1}{1-a_j^2} \right)\\
&\leq \frac{16r^2}{s_{m(r)}^2} \cdot s_{m(r)}^2 \prod_{j=1}^{m(r)} \left(\frac{1}{1-a_j^2} \right)\\
&\lesssim h(r)
\end{aligned}$$

The proof required to bound $\mu(B(x,r))$ from below depends on the size of r and so must be split into two cases. The value 100 used to differentiate the cases in this proof is a loose bound. The first case deals with values of r which relative to the side length $s_{m(x,r)}$ are small, and the second case where r is large enough so that we can easily bound $\mu(B(x,r))$ from below by finding a square inside the ball with positive measure. It can also be seen that although sufficient for this proof, some of the other bounds used here are also loose.

Case 1. $r \leq 100s_{m(x,r)}$

We know $B(x,r)$ must contain at least one square of side $s_{m(x,r)}$ and so we know $\mu(B(x,r)) \geq v_{m(x,r)}$. From Claim 3.2.3 preceding this proof, we know that $m(r) < m(2r)+1$ and so $m(\frac{r}{\sqrt{2}}) \leq m(\sqrt{2}r)$. Combining this with the fact that $m(r)$ is increasing and the results of Lemma 1 we get that

$$m(r) - 1 \leq m(\frac{r}{\sqrt{2}}) - 1 \leq m(\sqrt{2}r) \leq m(x,r).$$

Therefore,

$$v_{m(x,r)} = \prod_{j=1}^{m(x,r)} \frac{1}{a_j^{-2}-1} \qquad (3.6)$$

$$
\begin{aligned}
&= s_{m(x,r)}^2 \prod_{j=1}^{m(x,r)} \left(\frac{1}{1-a_j^2}\right) \\
&\geq \left(\frac{1}{100}\right)^2 r^2 (1-a_{m(r)}^2) \prod_{j=1}^{m(r)} \left(\frac{1}{1-a_j^2}\right) \\
&\gtrsim h(r)
\end{aligned}
$$

Case 2. $r > 100 s_{m(x,r)}$

Although the proof of this case may appear more difficult, the idea is that if the ball is large enough we may bound $\mu(B(x,r))$ quite loosely from below by simply finding a small enough square which is entirely contained inside $B(x,r)$. We choose $T \in \mathcal{T}_{m(x,r)-1}$ such that $x \in T$. Then we know from the definition of $m(x,r)$ that if $x \in T$ we must have that $T \not\subseteq B(x,r)$, so we get that $\operatorname{diam} T \geq r$. This means the side length of T is at least $\frac{r}{\sqrt{2}}$. As a result we can find a smaller square V' inside $T \cap B(x,r)$ that has side length $\frac{r}{4}$. Now, because $s_{m(x,r)} \leq \frac{r}{100}$ and because we know that at most one square of generation $m(x,r)$ is going to be deleted from T, V' must contain an even smaller square V with side length $s_v \in [\frac{r}{32}, \frac{r}{16}]$ which will consist of $\frac{s_v}{s_{m(x,r)}}$ squares from $\mathcal{T}_{m(x,r)}$. So, putting it all together we get

$$
\begin{aligned}
\mu(B(x,r)) &\geq \mu(v) \\
&\geq \left(\frac{s_v}{s_{m(x,r)}}\right)^2 v_{m(x,r)} \\
&= s_v^2 \cdot \prod_{j=1}^{m(x,r)} \left(\frac{1}{1-a_j^2}\right) \\
&\geq s_v^2 \cdot \prod_{j=1}^{m(r)-1} \left(\frac{1}{1-a_j^2}\right) \\
&= \frac{r^2}{32^2} (1-a_{m(r)}^2) \prod_{j=1}^{m(r)} \left(\frac{1}{1-a_j^2}\right) \\
&\gtrsim h(r)
\end{aligned}
$$

The third inequality holds because we know $m(r) - 1 \leq m(x,r)$ and the terms being multiplied are greater than 1. $\square$

The following is the proof of Proposition 3.2.1 as found in [15] with some minor elaborations.

Proof of Proposition 3.2.1. We first note that $m(r)$ is a decreasing function and that $\left(\frac{1}{1-a_j{}^2}\right) \geq 1$ because $a_j{}^2 < 1$. As a result, the following inequalities hold to prove (i):

$$\begin{aligned}\mu(B(x,2r)) &\lesssim (2r)^2 \prod_{j=1}^{m(2r)} \left(\frac{1}{1-a_j{}^2}\right) \\ &\leq (4r^2) \prod_{j=1}^{m(r)} \left(\frac{1}{1-a_j{}^2}\right) \\ &\lesssim \mu(B(x,r)).\end{aligned}$$

Part (ii) follows from the definition of $h(r)$ because as noted,

$$\prod_{j=1}^{m(r)} \left(\frac{1}{1-a_j{}^2}\right) \geq 1.$$

So now we prove part (iii). Assume that $\mathbf{a} \in c_0$, so that $a_m \to 0$. It suffices to show that for $Q < 2$,

$$\limsup_{r\to 0} \frac{\mu(B(x,r))}{r^Q} < \infty$$

uniformly in x. This equates to showing that

$$\limsup_{r\to 0} \frac{h(r)}{r^Q} < \infty$$

Recalling that $m(r) \leq -\log_2(r) + 1$, and since $a_j \to 0$, we know that for any $\epsilon > 0$ we can find C_ϵ depending only on ϵ (i.e. C_ϵ depends on how fast a_j goes to zero), such that the following holds:

$$\begin{aligned}\frac{h(r)}{r^2} = \prod_{j=1}^{m(r)} \left(\frac{1}{1-a_j^2}\right) &\leq C_\epsilon (1+\epsilon)^{m(r)} \\ &\leq C_\epsilon (1+\epsilon)^{-\log_2(r)+1} \\ &\leq C_\epsilon (1+\epsilon)^{-\log_2(r)} \\ &\leq C_\epsilon \left(2^{\log_2(1+\epsilon)}\right)^{-\log_2(r)} \\ &= C_\epsilon r^{-\log_2(1+\epsilon)}\end{aligned}$$

Since this holds for any $\epsilon > 0$, we can find ϵ such that $2 - Q > \log_2(1 + \epsilon)$. This means

$$Q < 2 - \log_2(1 + \epsilon),$$

and so for $r \leq 1$ we have

$$r^Q \geq r^2 r^{-\log_2(1+\epsilon)}.$$

Putting it all together we get that:

$$h(r) \leq C_\epsilon r^2 r^{-\log_2(1+\epsilon)} \leq C_\epsilon r^Q.$$

The proof of (iv) follows from Proposition 3.2.5. Here we need only to show that $h(r) \asymp r^2$. For any $a_j > 0$ a reciprocal of an odd integer we have that $1/(1 - a_j^2) > 1$ and so by definition we know

$$\begin{aligned} h(r) &= r^2 \prod_{j=1}^{m(r)} \frac{1}{1 - a_j{}^2} \\ &\leq r^2 \prod_{j=1}^{\infty} \frac{1}{1 - a_j{}^2}. \end{aligned}$$

We know $a \in \ell^2$ and so the product on the right must converge, therefore we immediately get that

$$h(r) \leq Cr^2,$$

where C is a constant depending only on the ℓ^2 norm of $\mathbf{a}$. Inequality in the other direction is trivial. As before we know

$$\prod_{j=q}^{m(r)} \frac{1}{1 - a_j^2} > 1$$

and so we immediately get that

$$h(r) \gtrsim r^2.$$

This shows that μ is an Alhfors 2-regular measure on $S_{\mathbf{a}}$, and μ is comparable to the Lebesgue measure with constant depending only on $||\mathbf{a}||_2$.

To prove (v) it suffices to show that $h(r) \asymp r^{Q_k}$ We know that $\mathbf{a} = (a_m)$ is eventually constant and equal to $\frac{1}{2k+1}$, so we define $N \in \mathbb{N}$ to be the point at which the sequence becomes constant. In other words, for $n \geq N$ we have that $a_n = \frac{1}{2k+1}$. To facilitate the notation we will as well define the following two values.

$$M := \max\{a_i\}_{i=1}^{N} \qquad m := \min\{a_i\}_{i=1}^{N}$$

We would also like some way to control the size of $m(r)$ and so we will first prove the following bounds

$$N + N\log_{2k+1} m - \log_{2k+1} r \leq m(r) < N + 1 + N\log_{2k+1} M - \log_{2k+1} r.$$

We know by definition that given r the inequality $s_{m(r)} \leq r$ holds and so we get that

$$\begin{aligned} \prod_{j=1}^{m(r)} a_j &\leq r \\ \left(\frac{1}{2k+1}\right)^{m(r)-N} m^N &\leq r \\ -(m(r) - N) + N\log_{2k+1} m &\leq \log_{2k+1} r \\ m(r) &\geq N + N\log_{2k+1} m - \log_{2k+1} r. \end{aligned} \tag{3.7}$$

Similarly, we know that $s_{m(r)-1} > r$ and so

$$\begin{aligned} \prod_{j=1}^{m(r)-1} a_j &> r \\ \left(\frac{1}{2k+1}\right)^{(m(r)-1)-N} M^N &> r \\ (N + 1 - m(r)) + N\log_{2k+1} M &> \log_{2k+1} r \\ m(r) &< N + 1 + N\log_{2k+1} M - \log_{2k+1} r. \end{aligned} \tag{3.8}$$

We first want to bound from above, and to do so we consider two cases depending on the size of N.

Case 1. $m(r) \leq N$

This case is trivial; considering that $Q_k < 2$ we have that

$$\begin{aligned} h(r) &= r^2 \prod_{j=1}^{m(r)} \frac{1}{1-a_j^2} \\ &\leq r^{Q_k} \prod_{j=1}^{N} \frac{1}{1-M^2} \\ &\lesssim r^{Q_k}. \end{aligned}$$

Case 2. $m(r) > N$

Here the proof relies on inequality (3.8). Applying this, we get

$$\begin{aligned} h(r) &= r^2 \prod_{j=1}^{m(r)} \frac{1}{1-a_j^2} \\ &\leq r^2 \prod_{j=1}^{m(r)-N} \frac{1}{1-\frac{1}{(2k+1)^2}} \prod_{j=1}^{N} \frac{1}{1-a_j^2} \\ &\lesssim r^2 \prod_{j=1}^{-\log_{2k+1} r} \frac{(2k+1)^2}{4k^2+4k} \\ &= r^2 \left(\frac{(2k+1)^2}{(2k+1)^2-1} \right)^{-\log_{2k+1} r} \\ &= r^2 \left(\frac{e^{\log(2k+1)^2-1}}{e^{\log((2k+1)^2)}} \right)^{\frac{\log r}{\log 2k+1}} \\ &= r^2 \cdot r^{\frac{\log((2k+1)^2-1)}{\log(2k+1)}} \cdot r^{\frac{-2\log(2k+1)}{\log(2k+1)}} \\ &= r^2 \cdot r^{Q_k} \cdot r^{-2} \\ &= r^{Q_k} \end{aligned}$$

We must now bound $h(r)$ from below which will again require two cases, both of which are very similar to the arguments already discussed here.

Case 1. $m(r) \leq N$

Similar to Case 1 when bounding from above; this case is now trivial as we know that $r^{Q_k} > r^2 > 0$ and so $r^{2-Q_k} < 1$. Recalling also that the product in the definition of

$h(r)$ is greater than 1 we can do the following

$$\begin{aligned} h(r) &= r^2 \prod_{j=1}^{m(r)} \frac{1}{1-a_j^2} \\ &\gtrsim r^2 \cdot r^{2-Q_k} \\ &= r^{Q_k} \end{aligned}$$

Case 2. $m(r) > N$

This case is the same as Case 2; however, this time we will use the inequality $m(r) \geq N + N \log_{2k+1} m - \log_{2k+1} r$ to get

$$h(r) \gtrsim r^2 \left(\frac{(2k+1)^2}{(2k+1)^2 - 1} \right)^{-\log_{2k+1} r}$$

from which we will in the same way conclude that $h(r) \gtrsim r^{Q_k}$. □

In Chapter 2, Proposition 2.4.3, we showed that if a complete, doubling, metric measure space supports a p-Poincaré inequality, then $\mathrm{mod_p}(\Gamma)$ is non-trivial for some family of curves Γ. Restricting ourselves to modified Sierpiński carpets $S_{\mathbf{a}}$ and only considering curve families joining either the left and right or top and bottom edges of a carpet we get the following result.

Proposition 3.2.6. *If $(S_{\mathbf{a}}, d, \mu)$ supports a p-Poincaré inequality, then for some $m \geq 1$ there exists a subcarpet T such that $\mathrm{mod}_p \Gamma > 0$, where Γ is the family of curves joining one of the following:*

- *the left and right hand edges of T,*
- *the top and bottom edges of T,*
- *the left and right hand edges of T and an adjacent subcarpet of the same level,*
- *the top and bottom edges of a T and an adjacent subcarpet of the same level.*

Proof of Proposition 3.2.6. We would like to show that Proposition 2.4.3 implies Proposition 3.2.6. We will do so using a proof by contradiction. We assume first that $S_{\mathbf{a}}$ supports a p-Poincaré inequality, but that for any subcaarpet of any level, the four types of curve families considered above are all trivial. We know that if $(S_{\mathbf{a}}, d, \mu)$ supports a p-Poincaré inequality then by Proposition 2.4.3 there exists a curve family Γ with positive p-moduli. We will therefore achieve a contradiction by showing that any curve family must actually have zero modulus.

By Proposition 2.2.2 part (iii), it suffices to prove this statement for families Γ that satisfy $\inf_{\gamma\in\Gamma} \operatorname{diam}\gamma \geq \epsilon$ for a fixed $\epsilon > 0$. So we choose m large enough to ensure that

$$\sqrt{2} \cdot \prod_{i}^{m-1} a_i < \epsilon,$$

which means that the diameter of a square at step $m-1$ is smaller than epsilon. Let

$$\Gamma^0 = \cup_i \Gamma_m^i,$$

the union of all the four types of level m curve families described above. There are a countable number of level m subsquares all of which have zero modulus, and so by Proposition 2.2.2 part (iii) we get that

$$\operatorname{mod}_p \Gamma^0 \leq \sum_i \operatorname{mod}_p \Gamma_m^i = 0.$$

We now interpret Γ as the following union,

$$\Gamma = \cup_w \Gamma_w,$$

where

$$\Gamma_w = \{\gamma \in \Gamma : \gamma \cap \mathcal{T}_w \neq \emptyset\},$$

and $\mathcal{T}_w$ ranges over all the level m subsquares of $S_{\mathbf{a}}$. We apply Proposition 2.2.2 part (iii) again which says it suffices to prove that $\mathrm{mod_p \Gamma_w} = 0$ for all such $\mathcal{T}_w$. Now let K_w be the union of $\mathcal{T}_w$ and all the other level m squares that border $\mathcal{T}_w$. Then, because of our choice of m, we know that for ever $\gamma \in \Gamma_w$ we must have that γ is not contained entirely in any of the K_w. Therefore, each γ must have a sub-curve γ_0 inside either some square, or a pair of adjacent squares in $\mathcal{T}_w$, such that γ_0 joins either the left and right edges, or the top and bottom edges to each other. So by definition, $\gamma_0 \in \Gamma_m^i$ for some i and we get that $\gamma_0 \in \Gamma^0$. What this means is that ever curve in Γ has a subcurve in Γ^0, so applying Proposition 2.2.2 part (vi) we get that

$$\text{mod}_p\Gamma \leq \text{mod}_p\Gamma^0 = 0$$

which completes the proof. □

This proof has been adapted from a proof used by Mackay and Tyson in [14] where they prove a more powerful statement but for the standard Sierpiński carpet. They make use of Lemma 2.2.3 which allows them to relate the moduli of the level m subsquare to that of the entire carpet $S_{\mathbf{1/3}}$. This lemma however relies on the self-similarity of the standard Sierpiśki carpet, which as we have already discussed does not hold for the modified carpets. The proposition as we have stated it here is sufficient for our purposes and is used in the following section to prove some very interesting results.

3.3 How ℓ^q Spaces Impact the Validity of p-Poincaré Inequalities

We are now in a position to begin analysing when and why a carpet $S_{\mathbf{a}}$ will support a p-Poincaré inequality, and also for which values of p. We know the construction of the carpets depend entirely on the sequence selected; therefore a strong relationship between the behaviour of this sequence and the geometric properties of the resulting carpet is expected. The results included here are all general results presented in [15].

We will include a brief discussion on the standard Sierpiński and why it fails to support a Poincaré inequality for any $p \geq 1$. This statement is true not only for the standard Sierpiński carpet, but for any carpet $S_{1/(2k+1)}$ constructed from a constant sequence.

Proposition 3.3.1 (Proposition 1, [15]). *For each k, the carpet $S_{1/(2k+1)}$, equipped with Euclidean metric and Hausdorff measure in its dimension Q_k, does not support any Poincaré inequality.*

A proof of Proposition 3.3.1 can be found in [1], [3], or [14]. Specifically a proof using modulus computations can be found in [14]. We now include a lemma from [14] that applies only to the standard Sierpiński carpet, for which we will include the proof.

Lemma 3.3.2 (Lemma 4.3.4, [14]). *Let Γ denote the family of curves joining the left and right hand edges of the Sierpiński carpet. Then* $\mathrm{mod}_p \Gamma_0 = 0$ *for every $p \geq 1$.*

The immediate consequence of this lemma is that the Sierpiński carpet does not support a p-Poincaré inequality for any $p \geq 1$. For this proof we will refer to the standard Sierpiński carpet as $S_{1/3}$.

Proof. Define a function $\rho : S_{1/3} \to (0, \infty)$ in the following way. Let $\rho_0 = \alpha$ on the six first level squares adjacent to the left and right hand edges of $S_{1/3}$, and let $\rho_0 = \beta$ on the two remaining level one squares. We let α and β be constants and we will determine their values shortly. We get that

$$\int_{S_{1/3}} \rho_0 \, d\mu = \frac{2\alpha + \beta}{3},$$

and so ρ_0 is admissible for Γ when

$$\frac{2\alpha + \beta}{3} \geq 1. \tag{3.9}$$

We can also bound the p-modulus of Γ above by

$$\int_{S_{1/3}} \rho_0^p \, d\mu = \frac{6}{8}\alpha^p + \frac{2}{8}\beta^p = \frac{3\alpha^p + \beta^p}{4}.$$

We can now define a family of admissible functions ρ_m on $S_{1/3}$ by an iterative procedure, and we would like to have that $\int \rho_m^p \; d\mu \to 0$. Inequlality (3.9) guarantees that every ρ_m is an admissible function for Γ, and we also get that the p-modulus of Γ is bounded above by

$$\int_{S_{1/3}} \rho_m^p \; d\mu = \left(\frac{3\alpha^p + \beta^p}{4}\right)^m.$$

It remains to show that there are positive constants α and β satisfying both

$$\frac{2\alpha + \beta}{3} \geq 1 \tag{3.10}$$

and

$$\frac{3\alpha^p + \beta^p}{4} < 1.$$

If $p = 1$ then we can take any $\alpha < 1$ and $\beta > 1$ however for $p > 1$ it becomes slightly more difficult. Using Lagrange multipliers Mackay and Tyson in [14] determine that the optimal values are

$$\alpha = \left(\frac{8\lambda}{3p}\right)^{1/(p-1)} \qquad \textit{and} \qquad \beta = \left(\frac{4\lambda}{p}\right)^{1/(p-1)},$$

where

$$\lambda = \frac{p3^p}{4(2^{p/(p-1)} + 3^{1/(p-1)})^{p-1}}.$$

This gives

$$\frac{3\alpha^p + \beta^p}{4} = \frac{3^{p+1}}{4(2^{p/(p-1)} + 3^{1/(p-1)})^{p-1}},$$

which has value strictly less than one for any $p > 1$. □

We will now consider some results from [14], the first of which is as follows.

Proposition 3.3.3. *If $\boldsymbol{a} \notin \ell^1$, then $S_{\boldsymbol{a}}$ does not support a 1-Poincaré inequality.*

Proof. We know from Proposition 3.2.6 that to disprove the validity of the 1-Poincaré inequality it suffices to show that for $m \geq 1$, $\mathrm{mod}_1 \Gamma_{\mathrm{m}} = 0$ where Γ_m is any one of the four types of curve families detailed in Proposition 3.2.6. We first consider that even though $S_{\mathbf{a}}$ may not be self-similar, we have that every subsquare defines a carpet $S_{\hat{\mathbf{a}}}$ such that $\hat{\mathbf{a}}$ is a tail of the sequence $\mathbf{a}$. Therfore, we know that

$$\mathbf{a} \notin \ell^1 \Leftrightarrow \hat{\mathbf{a}} \notin \ell^1.$$

Furthermore, the following proof considers the case where Γ_m is the family of curves joining the left and right hand edges of a level m subsquares, however we note that by symmetry we can easily adapt this proof to work for the curve family joining the top and bottom edges. The same proof can also be adapted just as easily to work for two adjacent subsquares. We would need only to extend the definition of the functions ρ_m to both squares which would not change at all the admissible functions and because the resulting modulus has value zero would not change that either.

It therefore suffices to prove the result for a carpet $S_{\mathbf{a}}$ such that $\mathbf{a} \notin \ell^1$, and for the faimly of curves joining only the left and the right hand edges. We now define the functions $\rho_m : S_{\mathbf{a}} \to [0, \infty]$ for $m \geq 1$ in the following way. Let $\rho_m = \frac{1}{s_m}$ on the middle strip of the carpet with width s_m and then let $\rho_m \equiv 0$ elsewhere. So we know that

$$\int_\gamma \rho_m \, d\mu = 1$$

for any $\gamma \in \Gamma$ and so ρ_m is admissible for Γ. Now we see what will happen if we integrate one such function ρ_m over the entire carpet.

$$\begin{aligned}
\int \rho_m \, d\mu &= \prod_{i=1}^{m} a_i^{-1} \prod_{i=1}^{m} \left(\frac{a_i(1-a_i)}{1-a_i^2} \right) \\
&= \prod_{i=1}^{m} \left(a_i^{-1} \cdot \frac{a_i(1-a_i)}{(1+a_i)(1-a_i)} \right) \\
&= \prod_{1}^{m} \frac{1}{1+a_i}
\end{aligned}$$

To understand why the first equality holds we will examine its various components. First, we note that $\prod_{i=1}^{m} a_i{}^{-1} = s_m^{-1}$ so this is the value of ρ_m on the middle strip. Then the pre-renormalized measure of the middle strip is $a_i(1 - a_i)$ were a_i is the width, and $1-a_i$ is the height with the appropriate squares removed. Finally we renormalise this measure of the strip by dividing by $1-a_i^2$ as we did in Proposition 3.2.5. The rest follows.

Now to finish the proof, we would like to have that as $m \to \infty$ the product on the right goes to zero. We know that $a \notin \ell^1$ and so $\sum_{i=1}^{\infty} a_i$ diverges. This is equivalent to the divergence of the series $\sum_{i=1}^{\infty} \log(1+a_i)$ This means for any $M \in \mathbb{N}$ we can find an $m \in \mathbb{N}$ such that

$$\sum_{i=1}^{m} \log(1+a_i) > M$$

and so the following string of inequalities will show that consequently $\prod_1^m \frac{1}{1+a_i} \to 0$ as $m \to \infty$.

$$\begin{aligned}\sum_{i=1}^{m} \log(1+a_i) > M &\Rightarrow -\sum_{i=1}^{m} \log(1+a_i) < -M \\ &\Rightarrow \prod_{i=1}^{m} \frac{1}{1+a_i} < e^{-M}.\end{aligned}$$

Therefore, we have shown that for any $m \in \mathbb{N}$ there is a $\delta = e^{-M} > 0$ such that $\prod_{i=1}^{m} \frac{1}{1+a_i} < \ \delta$ and hence $\mathrm{mod}_1\Gamma = 0$ as desired. □

The proof of the next result uses a very similar technique as the proof of Lemma 3.3.2.

Proposition 3.3.4. *If $\boldsymbol{a} \notin \ell^3$, then $S_{\boldsymbol{a}}$ does not support a p-Poincaré inequality for any $p \geq 1$.*

Proof. Recalling the discussion preceding the proof of Proposition 3.3.3 we note that we must again prove for $m \geq 1$, and $p \geq 1$, $\mathrm{mod}_p\Gamma_m = 0$ where Γ_m is any one of the four types of curve families detailed in Proposition 3.2.6. We know from Proposition 3.3.3 that $S_{\mathbf{a}}$ does not support a 1-Poincaré inequality and so it suffices to only consider the case $p > 1$. Similarly, it also suffices to prove that $\mathrm{mod_p}\Gamma = 0$ where Γ is the family of rectifiable curves joining the left and the right sides of $S_{\mathbf{a}}$, and $\mathbf{a} \notin \ell^3$. To do this we construct a family of upper gradients, ρ_m by iterating the following case: let $\rho_0 \equiv 1$ on the unit square $[0,1]^2$. Then for $a \in \{\frac{1}{3}, \frac{1}{5}, \dots\}$ we will remove the centre square of side length a, and then the remaining closed set we name S. Similarly again to the proof of Proposition 3.2.5, we will renormalise the measure on S by multiplying by $(1-a^2)^{-1}$.

Now we take parameters $\alpha \geq 0, \beta \geq 0$, and define ρ on S in the following way. Let $\rho \equiv \beta$ on the middle strip of width **a**, and let $\rho = \alpha$ everywhere else on S. We must ensure that $\int_\gamma \rho \; ds = 1$ and so we assume that

$$\alpha(1-a) + \beta a = 1, \tag{3.11}$$

which means

$$\beta = \frac{1}{a}(1 - (1-a)\alpha).$$

So we now need to see what happens if we integrate our new function ρ over our new set S. By construction we know that

$$\int_S {\rho_0}^p \; d\mu = 1$$

and now we have

$$\int_S \rho^p \; d\mu = \frac{(1-a)\alpha^p + a(1-a)\beta^p}{1-a^2} = \frac{\alpha^p + a^{1-p}(1-(1-a)\alpha)^p}{1+a} \tag{3.12}$$

We take the derivative of the right hand side with respect to α and for fixed a and p. We then find the value of α for which the function is minimized:

$$\alpha = \frac{(1-a)^{1/(p-1)}}{a + (1-a)^{p/(p-1)}}. \tag{3.13}$$

For this value of α we get the following expansion

$$\int_S \rho^p \; d\mu = 1 - \frac{pa^3}{2(p-1)} + \frac{(p-2)pa^4}{6(p-1)^2} + O[a]^5, \tag{3.14}$$

and it is important to note here that the a^3 coefficient is strictly negative.

We are now ready to construct our function ρ_m on $S_\mathbf{a}$ which we do by iterating the above steps. The first iteration for $a = a_1$ will produce ρ_1 with $\alpha = \alpha_1$.

For the second iteration we know that on each square of side length s_1, ρ_1 is constant. We let $a = a_2$ and then we find $\alpha = \alpha_2$ using 3.13. We then construct ρ_2 by redistributing the value of ρ_1 across each square using the new proportions α, β. If we iterate this procedure m times we will get a function ρ_m that is admissible for Γ, and by construction we know

$$\int_{S_{\mathbf{a}}} \rho_m^p \, d\mu = \prod_{i=1}^{m} \left[\alpha_i^p + a_i^{1-p}(1-(1-a_i)\alpha_i)^p\right] \frac{1}{1+a_i} \tag{3.15}$$

Therefore, using 3.12 and 3.14, and knowing that $a \notin \ell^3$ we can see that value on the right here goes to zero as m goes to infinity, forcing $\mathrm{mod_p}\Gamma = 0$ □

We note here that the ℓ^3 requirement in Proposition 3.3.4 is a loose bound and that Proposition 8 of [15] states the following result which is a sharp requirement.

Proposition 3.3.5. *If $a \notin \ell^2$, then $S_{\boldsymbol{a}}$ does not support a p-Poincaré inequality for any $p \geq 1$.*

The proof employs a similar strategy to those we have seen already in this paper and makes use of some very specific geometric properties of the Sierpiński carpets. It does so in a very lengthy and technical manner and so the complete proof is omitted. The essential idea is to begin by building an admissible function ρ for Γ, the curve family joining the left and the right hand edges of S_a. The function ρ has essential supremum zero, which means that $\mathrm{mod_p}\Gamma = 0$ for all p, and so by Proposition 3.2.6 does not support a p-Poincaré inequality for any $p \geq 1$.

We note here some additional results appearing in [15] that are very applicable, but have proofs that are beyond the scope of this thesis. The first states that the converse to Proposition 3.3.3 holds:

Theorem 3.3.6. *The carpet $(S_{\boldsymbol{a}}, d, \mu)$ supports a 1-Poincaré inequality if and only if $\boldsymbol{a} \in \ell^1$.*

Theorem 3.3.7. *The following are equivalent:*

- $(S_{\boldsymbol{a}}, d, \mu)$ *supports a p-Poincaré inequality for each* $p > 1$*,*
- $(S_{\boldsymbol{a}}, d, \mu)$ *supports a p-Poincaré inequality for some* $p > 1$
- $\boldsymbol{a} \in \ell^2$.

The proofs of these theorems require the application of Theorem 2.4.1 and a concatenation argument on curve families of positive modulus. They are included in detail in Mackay, Tyson, and Wildrick's paper on the subject [15]. We note that this means that for $\mathbf{a} \in \ell^2 \backslash \ell^1$, the space $(S_{\mathbf{a}}, d, \mu)$ will support a p-Poincaré inequality for $p > 1$ but will not support a 1-Poincaré inequality.

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