UNIVERSAL $\mathcal{R}$−MATRICES FOR GENERALIZED JORDANIAN r−MATRICES

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Abstract

Universal $R-$matrices for generalized Jordanian $r-$matrices.

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Quantization of classical integrable models by the Quantum Inverse Scattering Method requires transition from classical $r-$matrices to the quantum ones. The twists are the special elements in the algebra of observables, which help to build new classical and quantum $r-$matrices. In this thesis we develop an approach to explicit derivation of quasiclassical twists for higher dimensional analogs of Jordanian $r-$matrices. The twists are obtained as limits of more general quantum twists which allow a simple description. The considered class of $r-$matrices includes the skew-symmetric Cremmer-Gervais $r-$matrices as well as the extended Jordanian ones. The quantum analogs for both twists are obtained.

To my mother Galina

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Introduction

The motivation for study of classical $r-$matrices comes from the theory of classical integrable systems [10, 24]. The integrable systems admit a complete involutive set of integrals of motion $\mathcal{I}_n$:

$$\{\mathcal{I}_n, \mathcal{I}_m\} = 0.$$

The classical Inverse Scattering Method (ISM) allows to build an infinite family of pairwise commuting integrals of motion for partial differential equations in one spacial and one time variable representable in the form of zero curvature condition

$$[\nabla_x, \nabla_t] = \frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [U, V],$$

where

$$\nabla_x = \frac{\partial}{\partial x} - U(x, t, \lambda), \qquad \nabla_t = \frac{\partial}{\partial t} - V(x, t, \lambda).$$

U and V are $m \times m-$matrix valued functions; λ is a spectral parameter. The zero curvature condition is a necessary and sufficient condition for solvability of the associated linear system of differential equations

$$\begin{cases} \nabla_x F = 0 \\ \nabla_t F = 0, \end{cases} \tag{1}$$

where F stands for the matrix of fundamental solutions. The central object of classical ISM is the monodromy matrix. It shows how F changes over the whole segment $[-L, L]$ for a fixed time moment $t = t_0$, where U and V are assumed to be periodic with respect to the first argument

$$U(x+L,t_0)=U(x,t_0), \qquad V(x+L,t_0)=V(x,t_0).$$

The monodromy matrix is given by the following expression

$$T(\lambda,t_0)=F(L,t_0,\lambda)F(-L,t_0,\lambda)^{-1}=\mathcal{P}\exp(\int_{-L}^{L}U(x,t_0,\lambda)dx).$$

The ordered exponent $\mathcal{P}\exp$ comes from solving the following equation

$$\frac{\mathrm{d}F}{\mathrm{d}x}=U(x,t_0,\lambda)F,$$

and the ordering is taken from $-L$ to L. The fundamental Poisson brackets of the model can typically be expressed in terms of monodromy matrix using a classical $r-$matrix

$$\{T(\lambda)\underset{,}{\otimes}T(\mu)\}=[r(\lambda-\mu),T(\lambda)\otimes T(\mu)],$$

where

$$(\{T(\lambda)\underset{,}{\otimes}T(\mu)\})_{ij,kl}:=\{T(\lambda)_{ik},T(\mu)_{jl}\}.$$

The classical $r-$matrix $r(\lambda)$ acts in the tensor product of two copies of $\mathbb{C}^m$ and satisfies certain additional conditions implied by the Jacobi identity. A set of integrals of motion in involution is generated by $trT^k(\lambda)$:

$$\{trT^k(\lambda),trT^l(\mu)\}=0.$$

As a corollary of Jacobi identity, the $r-$matrix satisfies the famous classical Yang-Baxter equation

$$[r_{12}(\lambda-\mu),r_{13}(\lambda)]+[r_{12}(\lambda-\mu),r_{23}(\mu)]+[r_{13}(\lambda),r_{23}(\mu)]=0, \tag{2}$$

where the following notations are used: if $r(\lambda)=\sum_i r_i'\otimes r_i''(\lambda)$, then

$$\begin{aligned} r^{12}(\lambda-\mu) &= \textstyle\sum_i r_i'(\lambda-\mu)\otimes r_i''(\lambda-\mu)\otimes 1,\\ r^{23}(\mu) &= \textstyle\sum_i 1\otimes r_i'(\mu)\otimes r_i''(\mu),\\ r^{13}(\lambda) &= \textstyle\sum_i r_i'(\lambda)\otimes 1\otimes r_i''(\lambda). \end{aligned}$$

A typical example of an integrable model is the Heisenberg magnetic:

$$\frac{\partial \vec{S}}{\partial t} = \frac{\partial^2 \vec{S}}{\partial x^2} \times \vec{S}; \qquad \vec{S}(x,t) \in \mathbb{R}^3, \qquad \vec{S}\cdot\vec{S} = 1,$$

$$V = -\frac{1}{2\lambda} S \frac{\partial S}{\partial x} + \frac{1}{2i\lambda^2} S, \qquad U = \frac{i}{2\lambda} S, \qquad S =< \vec{S}, \vec{\sigma} >;$$

$\frac{1}{2i}\sigma_a$, $a = 1,2,3$, are Pauli matrices. The Poisson brackets and the Hamiltonian of the model are

$$\{S_a(x), S_b(y)\} = \epsilon_{abc} S_c(y) \delta(x-y),$$

$$H = \frac{1}{2} \int_{-L}^{L} \left\langle \frac{\partial \vec{S}}{\partial x}, \frac{\partial \vec{S}}{\partial x} \right\rangle dx,$$

where ϵ_{abc} are the structure constants of su_2. The $r-$matrix is given by

$$r(\lambda) = \Pi / 2\lambda,$$

where Π is the permutation operator in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

The quantization of an integrable system requires also the quantization of a classical $r-$matrix, this quantization we denote by $R(\lambda)$, which defines the quantum commutation relations between the entries of the quantum monodromy matrix $\mathbf{T}(\lambda)$ (Faddeev-Reshetikhin-Takhtajan relations):

$$R(\lambda-\mu)(\mathbf{T}(\lambda) \otimes \mathbf{T}(\mu)) = (\mathbf{T}(\lambda) \otimes \mathbf{T}(\mu)) R(\lambda - \mu).$$

The quantum $R-$matrix satisfies the quantum Yang-Baxter equation which provides associativity of the commutation relations:

$$(I \otimes R(\lambda-\mu))(R(\lambda) \otimes I)(I \otimes R(\mu)) = (R(\mu) \otimes I)(I \otimes R(\lambda))(R(\lambda-\mu) \otimes I).$$

In this thesis we shall consider a simpler case of the systems for which an $r-$matrix is an element of $\mathfrak{g} \otimes \mathfrak{g}$, where $\mathfrak{g}$ is a simple Lie algebra. Then the classical $r-$matrix $r = \sum_i r_i' \otimes r_i''$ is subject to condition

$$\mathrm{yb}(r) = [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] \in (\bigwedge^3 \mathfrak{g})^{\mathfrak{g}}, \qquad (3)$$

and $(\bigwedge^3 \mathfrak{g})^{\mathfrak{g}}$ denotes the ad−invariant part of $\bigwedge^3 \mathfrak{g}$. Usually one resticts oneself to considering a simple Lie algebra $\mathfrak{g}$, which has the advantage that $\dim(\bigwedge^3 \mathfrak{g})^{\mathfrak{g}} = 1$, and it is sufficient to consider solutions to (3) of two types:

- *unitary* or skew-symmetric: $r_{12} = -r_{21}$, $\mathrm{yb}(r) = 0$
- *non-unitary*: $r_{12} + r_{21} = t$, where t is a canonical element in $\mathfrak{g}$ corresponding to the Killing form, and $\mathrm{yb}(r) = 0$

Other solutions to (3) are obtained from the solutions of the Classical Yang-Baxter Equation (CYBE)

$$\mathrm{yb}(r) = [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0,$$

in view of the following correspondence: if r is a solution of the second type, then $-t/2 + r$ satisfies (3) and is skew-symmetric. Non-skew-symmetric solutions to (3) are not of interest to this thesis, because only the skew-symmetric part of $r-$matrix influences the Poisson bracket. We notice that the spectral parameter λ which was essential for the construction of an infinite family of commuting quantum integrals of motion can be easily taken into account. Namely, If r is a skew-symmetric solution to the CYBE without spectral parameter, then $t/\lambda + r$ satisfies (2). While the classification of non-unitary solutions to CYBE is a well-established fact due to Belavin and Drinfeld [2, 3], whose fundamental classification theorem we review in Chapter 1, the combinatorial approach to the classification of the solutions from the first family has not yet been reached, and the study is largely based on the direct search. We can, however, consider some solutions from the family of skew-symmetric $r-$matrices as coming from the degeneration of the $r-$matrices from the non-skew-symmetric family. For example, the Jordanian $r-$matrix $H \wedge E$ appears in this way.

The main motivation for the investigation undertaken in this thesis was to enlighten the quasiclassical limits of $q-$deformations, especially in view of the fact that the quantum problems can give insight into the classical ones and are sometimes easier to study because of their unbroken symmetry. In what follows, we will provide a step-by-step introduction to the notions we need from the Quantum group theory. Most of the material from the introductory chapters can be found in textbooks such as [4]. In Chapter 1, we review Poisson Lie groups and their relation to the classical $r-$matrices. We finish by exposing the Belavin-Drinfeld classification of non-skew-symmetric $r-$matrices and give some examples that are relevant to this thesis. We did some long calculations that are usually omitted from text books in order to show the richness of the information encoded into the Yang-Baxter equation. In Chapter 2, we deal with quantum analogs of the objects from Chapter 1. We show that Hopf algebra is a natural framework for working with quantization. The central notion of Chapter 2 is that of a twist as an element that allows the quantum $R-$matrix to be built. The problem of quantization of an arbitrary Lie bialgebra was solved in a series of works [7, 8]. Although explicit formulas of twists allowing the calculation of the deformed Hopf algebra structure were not given, it is possible to calculate the deformation with the desired accuracy; however, we cannot think of the deformed algebra in a global way. In particular, the theory of elliptic quantum groups, which is currently under investigation, makes use of interplay between different realizations of such algebras. Therefore, we hope that the investigation of explicit twists will help to elucidate the interrelations between the different realizations of more complicated objects, such as elliptic quantum groups. Chapter 3 and Chapter 4 introduce the main objects of further consideration. Starting from the second part of Chapter 3 we describe the results obtained in [23, 27]. The central result of this thesis is the elementary parabolic twist and its quantum version, as well as the expression of the quantum analogs of extended jordanian twists in infinite families of simple Lie algebras and the calculation of asymptotics where we make use of quantum dilogarithm [9], thus simplifying many calculations.

Chapter 1

What is an $r-$matrix?

In this chapter we introduce an $r-$matrix from a more natural point of view as an element defining the co-Poisson bracket on a Lie algebra $\mathfrak{g}$. We give this statement precise meaning and show how the Classical Yang-Baxter equation arises in this framework. What is an $r-$matrix? The notion of $r-$matrix originates from the problem of the quantization of Poisson brackets on a Lie group G. To set up the mathematical environment for the problems considered in this thesis, we recall the basic definitions and facts about Poisson manifolds and Poisson-Lie groups.

Definition 1.1. A smooth manifold M is called Poisson if there is a bilinear skew-symmetric mapping on the algebra $\mathrm{Fun}(M)$ of the functions on M

$$\{,\} : \mathrm{Fun}(M) \otimes \mathrm{Fun}(M) \to \mathrm{Fun}(M),$$

satisfying the following Jacobi identity

$$\{\{\phi, \psi\}, \chi\} + \{\{\psi, \chi\}, \phi\} + \{\{\chi, \phi\}, \psi\} = 0,$$

and being itself a differentiation with respect to each argument:

$$\{\phi, \psi\chi\} = \{\phi, \psi\}\chi + \psi\{\phi, \chi\}.$$

The map $\{,\}$ is called Poisson bracket, while $A = \mathrm{Fun}(M)$ is called Poisson algebra. It is not necessary to include M in the definition of Poisson algebra; therefore, omitting it from the definition, we speak of Poisson algebra as a pair $(A, \{,\})$ and of Poisson manifold as a pair $(M, \{,\}_M)$. This elimination of M from the definition of Poisson algebra is natural from the point of view of noncommutative geometry where noncommutative analog of M is not supplied. The map $g \longrightarrow \{g, f\}$ is a derivation of $\mathrm{Fun}(M)$ and it means that there is a vector field X_f on M such that $X_f(g) = \{g, f\}$. In particular, $\{g, f\}$ depends only on df. There is a well-defined map $B_M : T^*M \longrightarrow TM$, such that $B_M(df) = X_f$. Thus there is a skew-symmetric 2-tensor $w_M \in TM^{\otimes 2}$ called the *Poisson bivector*, such that

$$\{f, g\}_M =< (df \otimes dg), w_M > . \tag{1.1}$$

In terms of bivector w_M, we obtain a nice geometric interpretation of the skew-symmetric solutions to CYBE.

Proposition 1. *Let w_M be a bivector defining a Poisson structure by (1.1), then w_M solves CYBE.*

Proof. For the proof, see [4]. ☐

If M and N are Poisson manifolds, their product $M \times N$ is a Poisson manifold in a natural way: for $f_1, f_2 \in \mathrm{Fun}(M \times N)$, and $x \in M, y \in N$, the *product Poisson structure* is given by

$$\{f_1, f_2\}_{M\times N}(x, y) = \{f_1(\ , y), f_2(\ , y)\}_M(x) + \{f_1(x,\ ), f_2(x,\ )\}_N(y). \tag{1.2}$$

From this point, we restrict ourselves to $M = G$, where G is a Lie algebra and we assume that the base field is $\mathbb{C}$, so that we do not go into discussions of the problems related to complexification.

Definition 1.2. A Poisson manifold $(G, \{,\})$ is called a Poisson-Lie group if multiplication map

$$\begin{array}{rccc} \mu : & G \times G & \longrightarrow & G \\ & (g, g') & \longmapsto & gg'. \end{array}$$

is a morphism of Poisson manifolds.

If we introduce translation operators L_g, R_g:

$$(f \circ L_g)(g') = f(gg'), \quad f \circ R_{g'}(g) = f(gg'),$$

then (1.2) can be written as

$$\{f_1, f_2\}(gg') = \{f_1 \circ L_g, f_2 \circ L_g\}(g') + \{f_1 \circ R_{g'}, f_2 \circ R_{g'}\}(g),$$

or in terms of bivector field w_G

$$\begin{aligned} &< w_{gg'}, (df_1)_{gg'} \otimes (df_1)_{gg'} >= \\ &< w_{g'}, d(f_1 \circ L_g)_{g'} \otimes d(f_2 \circ L_g)_{g'} > + < w_{g'}, d(f_1 \circ R_{g'})_{g'} \otimes d(f_2 \circ R_{g'})_g >, \end{aligned}$$

which in turn is equivalent to

$$w_{gg'} = ((L_g)'_{g'} \otimes (L_g)'_{g'})(w_{g'}) + ((R_{g'})'_g \otimes (R_{g'})'_g)(w_g). \tag{1.3}$$

Thus we have proved the following:

Proposition 2. *A Poisson structure on a Lie group G is a Poisson-Lie group structure if and only if, for all g, $g' \in G$, the value of its Poisson bivector at gg' is the sum of the left translate by g of its value at g' and the right translate by g' of its value at g.*

Taking g and g' as equal to the identity element e of G, we deduce

Corollary 1.1. *The rank of the Poisson structure, which equals the rank of w_g, is zero at the identity element of the group. In particular, the Poisson structure of a Poisson-Lie group is never symplectic (i.e. w_g is not of maximal rank everywhere).*

The structure of a Poisson-Lie group on G defines an additional structure map on its Lie algebra $\mathfrak{g}$. There is a canonical Lie algebra structure on $\mathfrak{g}^*$: if $\xi_1, \xi_2 \in \mathfrak{g}^*$, choose $f_1, f_2 \in \mathrm{Fun}(G)$ with $(df_i)_e = \xi_i$ and set

$$[\xi_1, \xi_2]_{\mathfrak{g}^*} = (d\{f_1, f_2\})_e.$$

The defined bracket is obviously skew-symmetric and satisfies the Jacobi identity because the Poisson bracket does. The independence of choices f_1, f_2 follows from an alternative description of the introduced Lie algebra structure on $\mathfrak{g}^*$, which we give below. Take the right translate w^R of the Poisson bivector w of G to the identity, and define $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ to be the tangent linear map of w^R at e. Then we have

$$[\xi_1, \xi_2]_{\mathfrak{g}^*} = \delta^*(\xi_1 \otimes \xi_2).$$

By the definition of the right translate of the Poisson bivector field w, we have

$$\{f_1, f_2\}(g) =< w^R(g), ((R_g)'_e \otimes (R_g)'_e)^*((df_1)_g \otimes (df_2)_g) > .$$

Now, if we differentiate this equation at $\mathfrak{g} = e$ in the direction $X \in \mathfrak{g}$ and use the fact $w^R(e) = 0$ following from Corollary 1.1, we get

$$< X, d\{f_1, f_2\} >=< \delta(X), \xi_1 \otimes \xi_2 > .$$

Let us, then, rewrite (1.2) in terms of w^R. First we have

$$w(g) = ((R_g)'_e \otimes (R_g)'_e) w^R(g).$$

Thus, (1.3) can be written as

$$((R_{gg'})'_e \otimes (R_{gg'})'_e) w^R(gg') =$$
$$((L_g)'_{g'} \otimes (L_g)'_{g'}) \circ ((R_{g'})'_e \otimes (R_{g'})'_e) w^R(g') + ((R_{g'})'_g \otimes (R_{g'})'_g) \circ ((R_g)'_e \otimes (R_g)'_e) w^R(g)$$

We have the following properties of $(L_g)'$ and $(R_g)'$ stemming from similar properties of L_g and R_g operators:

$$(R_{gg'})'_e = (R_{g'})'_g \circ (R_{g'})'_e, \quad (R_{g'})'_g \circ (L_g)'_e = (L_g)'_{g'} \circ (R_{g'})'_e.$$
$$(R_{g^{-1}})'_g \circ (R_g)'_e = 1, \qquad (R_{g^{-1}} \circ L_g)'_e = \mathrm{Ad}_g.$$

Last formula means that G acts on vector fields $X \in T_eG$ by

$$(\mathrm{Ad}_g X)|_h(f) = \frac{d}{dt} f(g e^{tX} g^{-1} \cdot h)|_{t=0},$$

for any $f \in Fun(G)$. Therefore, (1.3) becomes

$$w^R(gg') = (\mathrm{Ad}_g \otimes \mathrm{Ad}_g)(w^R(g')) + w^R(g).$$

This means that w^R is a *1-cocycle* of G, with values in $\mathfrak{g} \otimes \mathfrak{g}$ (on which G acts by the adjoint representation in each factor). Its derivative δ at e is a *1-cocycle* of $\mathfrak{g}$ with values in $\mathfrak{g} \otimes \mathfrak{g}$,

$$\delta[X,Y] = (\mathrm{ad}_X \otimes 1 + 1 \otimes \mathrm{ad}_X)\delta(Y) - (\mathrm{ad}_Y \otimes 1 + 1 \otimes \mathrm{ad}_Y)\delta(X).$$

We can now give the definition *of bialgebra.*

Definition 1.3. Let $\mathfrak{g}$ be a Lie algebra. A Lie bialgebra structure on $\mathfrak{g}$ is a skew-symmetric linear map $\delta_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$, called the cocommutator, such that

- $\delta_{\mathfrak{g}}^*$ is skew-symmetric
- $\delta_{\mathfrak{g}}^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ is a Lie bracket on $\mathfrak{g}^*$
- $\delta_{\mathfrak{g}}$ is a 1-cocycle of $\mathfrak{g}$ with values in $\mathfrak{g} \otimes \mathfrak{g}$

Among the 1-cocycles of $\mathfrak{g}$ with their value in $\mathfrak{g} \otimes \mathfrak{g}$, the coboundaries are those for which we have

$$\delta(X) = (\mathrm{ad}_X \otimes 1 + 1 \otimes \mathrm{ad}_X)(r) = X.r, \tag{1.4}$$

for some $r \in \mathfrak{g} \otimes \mathfrak{g}$ and all $X \in \mathfrak{g}$.

Proposition 3. *Let $\mathfrak{g}$ be a Lie bialgebra and let $r \in \mathfrak{g} \otimes \mathfrak{g}$. The map $\delta_{\mathfrak{g}}$ defined by (1.10) is the cocommutator of a Lie bialgebra structure on $\mathfrak{g}$ if and only if the following conditions are satisfied:*

- $r_{12} + r_{21}$ *is a* $\mathfrak{g}-$*invariant element of* $\mathfrak{g} \otimes \mathfrak{g}$
- $\mathrm{yb}(r) \equiv [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]$ *is a* $\mathfrak{g}-$*invariant element of* $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$. *(The action of* $\mathfrak{g}$ *is the adjoint representation in each factor.)*

Remark 1.1. It follows from Proposition 3 that any coboundary bialgebra structure can be obtained from a skew-symmetric $r-$matrix, a fact that we mentioned in the Introduction.

Proof. The first property of $r-$matrix follows from

$$\delta^{op}(X) = X.r_{21} = -\delta(X) = -X.r_{12},$$

where δ is skew-symmetric. To prove the second property, we first introduce some notations

$$\mathrm{Jac}_\delta(X) = \sum_{c.p.} (\delta \otimes \mathrm{id})\delta(X),$$

where c.p. means the summation over cyclic permutations of tensor factors. If we define a Lie bracket on $\mathfrak{g}^*$ by

$$[\xi, \eta]_{\mathfrak{g}^*} = \delta^*(\xi \otimes \eta),$$

then, since

$$[[\xi, \eta]_{\mathfrak{g}^*}, \zeta]_{\mathfrak{g}^*} = \delta^*(\delta^* \otimes \mathrm{id})(\xi \otimes \eta \otimes \zeta),$$

it becomes clear that $[,]_{\mathfrak{g}^*}$ satisfies the Jacobi identity if and only if Jac_δ is the zero map. To complete the proof, we need

Lemma 1.1. *Let $\mathfrak{g}$ be a Lie algebra and let $r \in \mathfrak{g} \otimes \mathfrak{g}$ have a $\mathfrak{g}-$invariant symmetric part. Define $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ as a 1-coboundary. Then*

$$\mathrm{Jac}_\delta(X) + X.\mathrm{yb}(r) = 0 \tag{1.5}$$

for all $X \in \mathfrak{g}$.

Proof. We can write (1.5) as

$$\begin{aligned}
&[[X, a_i], a_j] \otimes b_j \otimes b_i + [[X, a_i], b_j] \otimes b_i \otimes a_j + [X, [a_i, a_j]] \otimes b_i \otimes b_j \\
&+[X, b_i] \otimes [a_i, a_j] \otimes b_j + [X, a_i] \otimes [b_i, a_j] \otimes b_j \\
&+[X, b_i] \otimes a_j \otimes [a_i, b_j] + [X, a_i] \otimes a_j \otimes [b_i, b_j] \\
&+[a_i, b_j] \otimes [X, b_i] \otimes a_j + [a_i, a_j] \otimes [X, b_i] \otimes b_j \\
&+[a_i, a_j] \otimes b_j \otimes [X, b_i] + [a_i, a_j] \otimes b_i \otimes [X, b_j]
\end{aligned}$$

$$+a_j \otimes [[X,a_i],b_j] \otimes b_i + a_j \otimes [a_i,b_j] \otimes [X,b_i] + a_i \otimes [X,[b_i,a_j]] \otimes b_j$$
$$+a_i \otimes [b_i,a_j] \otimes [X,b_j] + a_i \otimes [X,a_j] \otimes [b_i,b_j] + a_i \otimes a_j \otimes [X,[b_i,b_j]]$$
$$+b_j \otimes b_i \otimes [[X,a_i],a_j] + b_j \otimes [X,b_i] \otimes [a_i,a_j]$$
$$+b_i \otimes [[X,a_i],a_j] \otimes b_j + b_i \otimes a_j \otimes [[X,a_i],b_j],$$

where we used notation $r = a_i \otimes b_i$ and assumed the summation over the repeated indices. To simplify, we first note that

$$[[X,a_i],a_j] \otimes b_j \otimes b_i + [[X,a_i],b_j] \otimes b_i \otimes a_j + [X,[a_i,a_j]] \otimes b_i \otimes b_j$$
$$+b_j \otimes b_i \otimes [[X,a_i],a_j] = -a_j \otimes b_i \otimes [[X,a_i],b_j],$$

where we used the invariance of $r_{12} + r_{21}$ and the Jacobi identity after interchanging indices $i \leftrightarrow j$ in the first summand. Next, because of the invariance of the symmetric part of r, we have

$$[[X,b_i] \otimes a_i \otimes b_j + b_i \otimes [X,a_i] \otimes b_j$$
$$+[X,a_i] \otimes b_i \otimes b_j + a_i \otimes [X,b_i] \otimes b_j, 1 \otimes a_j \otimes 1] = 0, \tag{1.6}$$

which leads to

$$[X,b_i] \otimes [a_i,a_j] \otimes b_j + b_i \otimes [[X,a_i],a_j] \otimes b_j$$
$$+[X,a_i] \otimes [b_i,a_j] \otimes b_j + a_i \otimes [[X,b_i],a_j] \otimes b_j = 0.$$

The Jacobi identity allows additional simplifications:

$$a_j \otimes [[X,a_i],b_j] \otimes b_i + a_i \otimes [X,[b_i,a_j]] \otimes b_j = a_i \otimes [[X,b_i],a_j] \otimes b_j.$$

Now (1.5) simplifies after applying the indicated relations and canceling similar terms:

$$-a_j \otimes b_i \otimes [[X,a_i],b_j] + [X,b_i] \otimes a_j \otimes [a_i,b_j] + [X,a_i] \otimes a_j \otimes [b_i,b_j]$$
$$+[a_i,b_j] \otimes [X,b_i] \otimes a_j + [a_i,a_j] \otimes [X,b_i] \otimes b_j$$
$$+a_i \otimes [X,a_j] \otimes [b_i,b_j] + a_i \otimes a_j \otimes [X,[b_i,b_j]]$$
$$+b_j \otimes [X,b_i] \otimes [a_i,a_j] + b_i \otimes a_j \otimes [[X,a_i],b_j].$$

Like to (1.6), we have

$$[X,b_i] \otimes a_j \otimes [a_i,b_j] + b_i \otimes a_j \otimes [[X,a_i],b_j]$$
$$+[X,a_i] \otimes a_j \otimes [b_i,b_j] + a_i \otimes a_j \otimes [[X,b_i],b_j] = 0$$

which brings us to

$$-a_j \otimes b_i \otimes [[X, a_i], b_j] + [a_i, b_j] \otimes [X, b_i] \otimes a_j + [a_i, a_j] \otimes [X, b_i] \otimes b_j$$
$$+a_i \otimes [X, a_j] \otimes [b_i, b_j] + a_i \otimes a_j \otimes [b_i, [X, b_j]] + b_j \otimes [X, b_i] \otimes [a_i, a_j];$$

exploiting the idea of (1.6), we have

$$\begin{aligned} &a_j \otimes b_i \otimes [[X, a_i], b_j] + a_j \otimes [X, b_i] \otimes [a_i, b_j] \\ &+a_j \otimes a_i \otimes [[X, b_i], b_j] + a_j \otimes [X, a_i] \otimes [b_i, b_j] = 0, \end{aligned} \tag{1.7}$$

and after applying (1.7), (1.5) is

$$a_j \otimes [X, b_i] \otimes [a_i, b_j] + [a_i, b_j] \otimes [X, b_i] \otimes a_j$$
$$+[a_i, a_j] \otimes [X, b_i] \otimes b_j + b_j \otimes [X, b_i] \otimes [a_i, a_j] = 0,$$

by the invariance of $r_{12} + r_{21}$. □

□

The Introduction mentioned two cases: the Classical Yang-Baxter Equation (CYBE), when $\mathrm{yb}(r) = 0$; and the Modified Yang-Baxter Equation (MYBE), when $\mathrm{yb}(r) = -\omega$ and ω is the canonical $\mathfrak{g}$ element of $\bigwedge^3 \mathfrak{g}$ corresponding to the 3-linear form

$$(X, Y, Z) \mapsto ([X, Y], Z)$$

where $(,)$ is the Killing form. Note that $\mathrm{yb}(r) \in \bigwedge^3 g$ because of the invariance of $r_{12} + r_{21}$. Indeed, by applying any transposition of tensor factors, for example (23), to

$$\mathrm{yb}(r) = [a_i, a_j] \otimes b_i \otimes b_j + a_i \otimes [b_i, a_j] \otimes b_j + a_i \otimes a_j \otimes [b_i, b_j],$$

we get

$$(23) \circ \mathrm{yb}(r) = [a_i, a_j] \otimes b_j \otimes b_i + a_i \otimes b_j \otimes [b_i, a_j] + a_i \otimes [b_i, b_j] \otimes a_j,$$

and if we change $i \leftrightarrow j$ in the first term and use the invariance of $r_{12}+r_{21}$ in the second and third terms, we see that $(23) \circ \mathrm{yb}(r) = -\mathrm{yb}(r)$. This works for any transposition; therefore, $\mathrm{yb}(r) \in \bigwedge^3 g$.

Example 1.1. There are two non-trivial Lie bialgebra structures on $\mathfrak{g} = \mathfrak{sl}(2)$. Choosing a basis in a Borel subalgebra $\mathfrak{b}_+ \subset \mathfrak{g}$ $\{H, E\}$, such that $[H, E] = E$, the structures are given by

$$\delta(H) = H \wedge E, \qquad \delta(E) = 0$$

and

$$\delta(H) = 0, \qquad \delta(E) = H \wedge E.$$

The first structure is skew-symmetric with $r = H \wedge E$, while the second one is not skew-symmetric.

In general, for any simple Lie algebra $\mathfrak{g}$ non-skew-symmetric solutions to CYBE

$$\mathrm{yb}(r) = 0, \qquad r_{12} + r_{21} = t, \tag{1.8}$$

where t is the Casimir element of $\mathfrak{g} \otimes \mathfrak{g}$ corresponding to the Killing form, as classified by Belavin and Drinfeld. To parametrize the set of equivalent solutions to (1.8) (the equivalence here is up to an automorphism $r' = (\sigma \otimes \sigma) \circ r$, for $\sigma \in \mathrm{Aut}(\mathfrak{g})$) the quadruple $(\Pi_1, \Pi_0, \tau, r^0)$ where $\Pi_{0(1)}$ are arbitrary subsets of the set of all the simple roots of $\mathfrak{g}$ was introduced.

Definition 1.4. A quadruple $(\Pi_1, \Pi_0, \tau, r^0)$, where Π_1, $\Pi_0 \subset \Pi$, τ is a bijection $\Pi_1 \to \Pi_0$ and $r^0 \in \mathfrak{h} \otimes \mathfrak{h}$, is said to be admissible if it satisfies the following conditions:

- $(\tau(\alpha), \tau(\beta)) = (\alpha, \beta)$ for all $\alpha, \beta \in \Pi_1$ by $(,)$ we denote the inner product on $\mathfrak{h}^*$ induced by the Killing form on $\mathfrak{g}$)
- for every $\alpha \in \Pi_1$ there exists $m \in \mathbb{N}$, such that $\alpha, \tau(\alpha), \ldots, \tau^{m-1}(\alpha) \in \Pi_1$ but $\tau^m(\alpha) \notin \Pi_1$
- $r^0_{12} + r^0_{21} = t^0$, where the upper script ^{0} denotes the component in $\mathfrak{h} \otimes \mathfrak{h}$
- $(\tau(\alpha) \otimes 1)(r^0) + (1 \otimes \alpha)(r^0) = 0$ for all $\alpha \in \Pi_1$

To formulate the Theorem of Belavin and Drinfeld, we introduce the partial order on Π. Namely, we say $\alpha < \beta$ if there is a positive integer m, such that $\beta = \tau^m(\alpha)$.

Theorem 1 (Belavin and Drinfeld). *If* $(\Pi_1, \Pi_0, \tau, r^0)$ *is an admissible quadruple, then, with the above notation,*

$$r = r^0 + \sum_{\alpha\in\Pi} E_{-\alpha}\otimes E_\alpha + \sum_{\alpha,\beta\in\Pi,\alpha<\beta} E_{-\alpha}\wedge E_\beta$$

is a solution to (1.8) (here, $E_{-\alpha}\wedge E_\beta = E_{-\alpha}\otimes E_\beta - E_\beta\otimes E_{-\alpha}$*). Conversely every solution of* (1.8) *is equivalent to a solution of this type.*

Proof. See [2, 3] □

Let us see how the solutions of interest to us appear in this classification.

Example 1.2. If $\mathfrak{g} = \mathfrak{sl}(2)$, then the set of simple roots encoded into the Dynkin diagram consists of a single root $\Pi = \{\alpha\}$. There are two possible choices for the sets Π_0 and Π_1 :

$$\Pi_0 = \emptyset,\ \ \Pi_1 = \emptyset, \qquad \Pi_0 = \Pi,\ \ \Pi_1 = \Pi. \tag{1.9}$$

Note that $\Pi_0\cup\Pi_1$ need not be equal Π. The second choice in (1.9) does not satisfy the second condition from the definition of admissible triple. Thus, if we fix the invariant product on $\mathfrak{g}$ to be

$$(X, Y) = \mathrm{trace}(XY),$$

then

$$r = \frac{1}{4}H\otimes H + X^-\otimes X^+ \tag{1.10}$$

in the notations

$$X^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\quad X^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The $r-$matrix (1.10) corresponds to the second bialgebra structure on $\mathfrak{sl}(2)$ with r being non-skew-symmetric.

Our next example is the so-called Cremmer-Gervais $r-$matrix, which has its origin in the integrable models of 2-dimensional gravity.

Example 1.3. Take $\mathfrak{g} = \mathfrak{sl}(3)$. The positive roots here are α, β, and $\alpha+\beta$, which correspond to the following generators of the root subspaces

$$\begin{array}{llll} E_\alpha = E_{12}, & E_{-\alpha} = E_{21}, & E_\beta = E_{23}, & E_{-\beta} = E_{32}, \\ E_{\alpha+\beta} = E_{13}, & E_{-\alpha-\beta} = E_{31}, & H_\alpha = E_{11} - E_{22}, & H_\beta = E_{22} - E_{33}. \end{array}$$

Then,

$$t_0 = \frac{2}{3} H_\alpha \otimes H_\alpha + \frac{1}{3} H_\alpha \otimes H_\beta + \frac{1}{3} H_\beta \otimes H_\alpha + \frac{2}{3} H_\beta \otimes H_\beta.$$

(a) Take $\Pi_0 = \Pi_1 = \emptyset$. Then, the most general solution is

$$r^0 = \frac{1}{2} t_0 + \lambda H_\alpha \wedge H_\beta,$$

for any $\lambda \in \mathbb{C}$. The resulting $r-$matrix is

$$r = \frac{1}{2} t_0 + \lambda H_\alpha \wedge H_\beta + E_{-\alpha} \otimes E_\alpha + E_{-\beta} \otimes E_\beta + E_{-\alpha-\beta} \otimes E_{\alpha+\beta}.$$

(b) In the case of the Cremmer-Gervais $r-$matrix, take $\Pi_0 = \{\alpha\}$, $\Pi_1 = \{\beta\}$. Then the conditions

$$r^0_{12} + r^0_{21} = t_0,$$
$$(\alpha \otimes 1)(r^0) + (1 \otimes \beta)(r^0) = 0$$

have the unique solution

$$r^0 = \frac{1}{3} H_\alpha \otimes H_\alpha + \frac{1}{3} H_\beta \otimes H_\alpha + \frac{1}{3} H_\beta \otimes H_\beta,$$

in view of the commutation relation:

$$[H_\gamma, E_\delta] = \delta(H_\gamma) E_\delta, \text{ for any } \gamma, \delta \in \Pi.$$

Thus, the resulting $r-$matrix is

$$r = \tfrac{1}{3} H_\alpha \otimes H_\alpha + \tfrac{1}{3} H_\beta \otimes H_\alpha + \tfrac{1}{3} H_\beta \otimes H_\beta + E_{-\alpha} \otimes E_\alpha + E_{-\beta} \otimes E_\beta$$
$$+ E_{-\alpha-\beta} \otimes E_{\alpha+\beta} + E_{-\beta} \wedge E_\alpha.$$

(c) $\Pi_0 = \{\beta\}$, $\Pi_1 = \{\alpha\}$. In this case, the $r-$matrix is obtained by interchanging α and β in the previouse case.

In contrast with the constructive classification of non-skew-symmetric solutions to the Classical Yang-Baxter Equation by means of Belavin-Drinfeld triples (Π_0, Π_1, τ), the description of the skew-symmetric solutions appeals to finding all quasi-Frobenius (Frobenius) Lie subalgebras. By the definition, a quasi-Frobenius Lie algebra $\mathfrak{f}$ is a Lie algebra possessing a nondegenerate skew-symmetric 2-cocycle B :

$$B([x,y],z) + B([y,z],x) + B([z,x],y) = 0.$$

A quasi-Frobenius Lie algebra $(\mathfrak{f}, B)$ is called Frobenius if B is exact, namely that there exists $l \in \mathfrak{f}^*$, such that $B(x,y) = l([x,y])$. An $r-$matrix corresponding to B is obtained by inverting r in some $\mathfrak{f}-$basis. Conversely, $\mathfrak{f}$ is fixed by the requirement of being the largest subalgebra of $\mathfrak{g}$ with the property that r is non-degenerate on $\mathfrak{f}$. In [28] Stolin gave the complete list of such subalgebras for $\mathfrak{sl}(n)$, $n \leq 3$. The special families of Frobenius subalgebras were described later for other series of simple Lie algebras as well [1, 11, 21]. Another source of quasi-Frobenius Lie algebras comes from the study of so-called filliform algebras [30, 31], the nilpotent Lie algebras of the descending central sequence, which are quasi-Frobenius in some cases.

Example 1.4. The discussed r matrix $H \wedge E$, defined on the Borel subalgebra $\mathfrak{b}_+ = \{H, E | [H,E] = E\}$, corresponds to a pair $(\mathfrak{b}_+, E^* \circ [,])$. This example generalizes for any Borel subalgebra $\mathfrak{b}_+$ in $\mathfrak{sl}(n)$, making $\mathfrak{b}_+$ a Frobenius Lie algebra $(\mathfrak{b}_+, \Lambda \circ [,])$, with $\Lambda \in \mathfrak{sl}(n)^*$. If we take $\Lambda = -E^*_{1n}$ then the corresponding solution to CYBE is of the form

$$r = (E_{11} - E_{nn}) \wedge E_{1n} + 2 \sum_{i=2}^{n-1} E_{1i} \wedge E_{in}.$$

Example 1.5 ([6]). Elashvili describes a whole class of Frobenius Lie subalgebras in $\mathfrak{g} = \mathfrak{gl}(n)$. If $\mathfrak{a}_{n,k}$ is a subalgebra of $\mathfrak{gl}(n)$ consisting of matrices with vanishing last k rows, then $\mathfrak{a}_{n,k}$ is a Frobenius Lie subalgebra if and only if k divides n. The function l such that $l \circ [,]$ is non degenerate on $\mathfrak{a}_{n,k}$ can be chosen so that

$$l(a) = \sum_{i=1}^{n-k} a_{i,i+k}.$$

The $r-$matrix is given by

$$r = \sum_{i=1}^{k}\sum_{j=1}^{k}\sum_{(a,b,c,d)\in S} E_{i+ka,j+kb} \wedge E_{j+kc,i+kd}, \tag{1.11}$$

where $m = n/k$ and

$$\begin{aligned} S = \ & \{(a,b,c,d)|a,b,c,d \in \mathbb{Z},\ \ b+d-a-c=1, \\ & 0 \le b \le a < m-1,\ \ b \le c < m-1,\ \ 0 \le d \le m\}. \end{aligned}$$

To obtain the $r-$matrix for $\mathfrak{sl}(n)$ we need to apply the map $(f \otimes f)$ to (1.11):

$$f(a) = a - \frac{1}{n}\mathrm{tr}(a)E.$$

*Example 1.6 **(Gerstenhaber-Giaquinto)**.* A family of Frobenius Lie algebras appears in connection with maximal parabolic subalgebras in $\mathfrak{sl}(n)$. To every i, where $1 \le i \le n-1$, we can associate a maximal parabolic subalgebra $\mathfrak{p}_i$, namely, that generated by Cartan subalgebra and all simple root vectors but $E_{i+1,i}$. In particular, if $n = 3$ the maximal parabolic subalgebra $\mathfrak{p}_1$ is spanned by the matrices of the form

$$\begin{pmatrix} * & * & * \\ & * & * \\ & * & * \end{pmatrix},$$

where the entries other than those marked by $*$ must be equal zero. The corresponding non-degenerate Frobenius form is given by $-(E_{12}^* + E_{23}^*) \circ [,]$. Its inverse, which defines an $r-$matrix, is given by

$$\left(\frac{2}{3}E_{11} - \frac{1}{3}E_{22} - \frac{1}{3}E_{33}\right) \wedge E_{12} + \left(\frac{1}{3}E_{11} + \frac{1}{3}E_{22} - \frac{2}{3}E_{33}\right) \wedge E_{23} + E_{13} \wedge E_{32}.$$

In general, the Frobenius form turns out to be replaced by $-(E_{12}^* + E_{23}^* + \cdots + E_{n-1,n}^*) \circ [,]$ on the maximal parabolic subalgebra in $\mathfrak{sl}(n)$. The inversion of this form defines the $r-$matrix as

$$b_{CG} = \sum_{p=1}^{n-1} d_p \wedge E_{p,p+1} + \sum_{i<j}\sum_{m=1}^{j-i-1} E_{i,j-m+1} \wedge E_{j,i+m}, \tag{1.12}$$

where

$$d_p = \frac{n-p}{n}(E_{11} + E_{22} + \cdots + E_{pp}) - \frac{p}{n}(E_{p+1,p+1} + E_{p+2,p+2} + \cdots + E_{nn}).$$

The class of $r-$matrices (1.12) is called the *Boundary solutions of CYBE* [11]. Such solutions can be obtained from non-skew-symmetric solutions to CYBE with a procedure we call *contraction*, which we describe below. The skew-symmetric part of the solutions to CYBE satisfies the so-called Modifed Yang-Baxter Equation (MYBE) (3). In accordance with the Belavin-Drinfeld classification of non-skew-symmetric solutions to CYBE, we are supposed to choose a quadruple $(\Pi_1, \Pi_0, \tau, r^0)$. We take

$$\Pi_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}, \qquad \Pi_0 = \{\alpha_2, \alpha_3, \cdots, \alpha_n\},$$

and

$$\tau(\alpha_i) = \alpha_{i+1} \text{ for } 1 \leq i \leq n-1.$$

The skew-symmetric part of the *generalized Cremmer-Gervais solution to CYBE* is given by

$$r_{CG} = \sum_{i<j} E_{ij} \wedge E_{ji} + \frac{1}{n}\sum_{i<j}(n + 2(i-j))E_{ii} \wedge E_{jj} + 2\sum_{i<j}\sum_{m=1}^{j-i-1} E_{i,j-m} \wedge E_{j,i+m}.$$

Following [11], let us act by an automorphism

$$(\exp(-tx) \otimes \exp(-tx))(r_{CG}) = r_{CG} - tb_{CG},$$

where

$$x = \frac{1}{2}[(n-1)E_{12} + (n-2)E_{23} + \cdots + E_{n-1,n}],$$

and t is arbitrary. If one tends t to infinity, then in fact

$$b_{CG} = (\mathrm{ad}_x \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}_x)(r_{CG})$$

satisfies CYBE and skew-symmetric.

Chapter 2

Quantization of bialgebra structures

This chapter provides a brief introduction to the problem of quantization of bialgebra structures. The deformation itself is performed in the category of co-Poisson Hopf algebras, the definition of which is the main goal of this chapter. Our basic example for is the universal enveloping algebra $U(\mathfrak{g})$.

The existence of group structure on a Poisson manifold G defines an additional algebraic structure on $\mathrm{Fun}(G)$:

- First, we have an operation $\Delta = \mu^*$ called *coproduct* that acts as

$$(\Delta f)(g_1, g_2) = f(g_1 g_2).$$

Remark 2.1. There is an obvious embeding

$$\mathrm{Fun}(G) \otimes \mathrm{Fun}(G) \hookrightarrow \mathrm{Fun}(G \times G), \tag{2.1}$$

but in general it fails to be an isomorphism. Therefore, one needs to extend the definition of the tensor product by performing *completion*, namely allowing the elements of $\mathrm{Fun}(G) \otimes \mathrm{Fun}(G)$ to be infinite sums convergent in appropriately chosen topology. In particular, if G is an algebraic group we do not need to complete the tensor product in view of the result that the direct product of algebraic groups is an algebraic group with a polynomial ring of regular functions. In this case the embedding (2.1) is in fact an isomorphism.

- Second, we have an operation $S = \iota^*$ called *antipode*:

$$S(f)(g) = f \circ \iota(g) = f(g^{-1}).$$

- Third, the existence of unity $e \in G$ defines the *augmentation* $\epsilon = \eta^*$:

$$\epsilon(f)(g) = f \circ \eta(g) = f(e).$$

To axiomatize the listed properties, we present them as commutative diagrams. The group associativity $(g_1 g_2) g_3 = g_1 (g_2 g_3)$, or

$$\mu \circ (\mu \times \mathrm{id}) = \mu \circ (\mathrm{id} \times \mu), \tag{2.2}$$

leads to the coassociativity of Δ after applying dualization * to (2.2):

$$\begin{array}{ccc} H & \xrightarrow{\Delta} & H \otimes H \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\Delta} \\ H \otimes H & \xrightarrow{\Delta\otimes\mathrm{id}} & H \otimes H \otimes H. \end{array} \tag{2.3}$$

The axiom $e \cdot g = g \cdot e = g$ or

$$\mu \circ (\eta \times \mathrm{id}) = \mu \circ (\mathrm{id} \times \eta) = \mathrm{id}$$

after dualization leads to

$$\begin{array}{ccccc} H \otimes H & \xleftarrow{\Delta} & H & \xrightarrow{\Delta} & H \otimes H \\ {\scriptstyle\epsilon\otimes\mathrm{id}}\downarrow & & \| & & \downarrow{\scriptstyle\mathrm{id}\otimes\epsilon} \\ k \otimes H & \xrightarrow{\approx} & H & \xleftarrow{\approx} & H \otimes k, \end{array} \tag{2.4}$$

where k denotes the base field; in our case, it is always $\mathbb{C}$. The last line in the commutative diagram indicates an equivalence up to an isomorphism.

Definition 2.1. A vector space H over a field k is called *coalgebra* $(H, \mu, \eta, \Delta, \epsilon)$ if there are maps:

$$\Delta : H \longrightarrow H \otimes H, \text{ and } \epsilon : H \to k,$$

making commutative the diagrams (2.3), and (2.4).

If, in additition, the following diagram commutes

$$\begin{array}{ccccc} H\otimes H & \xleftarrow{\Delta} & H & \xrightarrow{\Delta} & H\otimes H \\ {\scriptstyle S\otimes \mathrm{id}}\downarrow & & {\scriptstyle \epsilon}\downarrow & & {\scriptstyle \mathrm{id}\otimes S}\downarrow \\ H\otimes H & & k & & H\otimes H \\ & \searrow^{\mu} & {\scriptstyle\eta}\downarrow & \swarrow^{\mu} & \\ & & H, & & \end{array} \tag{2.5}$$

then we say that H is a Hopf algebra $(H, \mu, \eta, \Delta, \epsilon, S)$.

Example 2.1. If we take $k[\mathrm{Mat}(n)]$ to be an algebra of functions on a semi-group of matrices that is isomorphic to a commutative polynomial ring

$$k[z^i_j | i, j = 1, \cdots n],$$

then we define Δ on generators as

$$\Delta(z^j_i) = \sum_{k=1}^{n} z^k_i \otimes z^j_k,$$

and extend it as a homomorphism

$$\Delta(xy) = \Delta(x)\Delta(y), \tag{2.6}$$

to the whole algebra $k[\mathrm{Mat}(n)]$. In this setting, the coassociativity of Δ follows from the associativity of matrix multiplication. It is easy to see that, if one takes $\epsilon(z^i_j) = \delta^i_j$ and extends

$$\epsilon(xy) = \epsilon(x)\epsilon(y)$$

to the whole algebra, then (2.4) holds.

While it was not difficult to define the structure of a bialgebra on a given algebra H, a Hopf algebra structure requres attaching the invertable elements, the determinants, see Example 2.2 and [5, 25]. It is similar to what happens in semi-groups, where the inverse is not always defined.

Example 2.2. An algebra of functions on $\mathrm{GL}(n)$ is obtained from $k[\mathrm{Mat}(n)]$ by attaching a formal element D^{-1}, which is the formal inverse of the determinant

$$D = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) z_1^{\sigma(1)} \cdots z_n^{\sigma(n)}.$$

Now, by definition, $\mathrm{GL}(n) \cong k[\mathrm{Mat}(n)][D^{-1}]$. It is a direct check to show that

$$\Delta(D) = D \otimes D;$$

therefore, by

$$\Delta(D^{-1}) = D^{-1} \otimes D^{-1},\ \epsilon(D^{-1}) = 1,$$

we obtain a bialgebra structure on $\mathrm{GL}(n)$ requiring Δ, and ϵ to be homomorphisms from $\mathrm{GL}(n)$ to $\mathrm{GL}(n) \otimes \mathrm{GL}(n)$. The antipode S is given by an explicit formula

$$S(z_j^i) = D^{-1} \sum_{\substack{\sigma \in S_n \\ \sigma(i)=j}} \mathrm{sgn}(\sigma) z_1^{\sigma(1)} \cdots \hat{z_i^j} \cdots z_n^{\sigma(n)},$$

solving the linear system according to Cramer's rule

$$\sum_{k=1}^{n} S(z_i^k) z_k^j = \delta_i^j,$$

which is equivalent to (2.5).

The next example of a Hopf algebra is more important for our approach to the quantization of $r-$matrices.

Example 2.3. If $\mathfrak{g}$ is a Lie algebra, then one defines its universal enveloping algebra as

$$U\mathfrak{g} = \mathbb{T}\mathfrak{g}/(x \otimes y - y \otimes x - [x, y]),$$

where

$$\mathbb{T}\mathfrak{g} = k \bigoplus_{k \geq 1} \mathfrak{g}^{\otimes k}.$$

Making no difference between $\mathfrak{g}$ and its image in $U\mathfrak{g}$, we define the Hopf algebra structure by letting $\Delta(x) = x \otimes 1 + 1 \otimes x$ and extending it to the whole $U\mathfrak{g}$ as a homomorphism. The latter is possible due to the Poincaré-Birkhoff-Witt theorem, which states that $U\mathfrak{g}$ is generated by $\mathfrak{g}$ as an associative algebra. The counit and the antipode are fixed by

$$\epsilon(x) = 0, \ S(x) = -x, \ \text{for any } x \in \mathfrak{g}.$$

Definition 2.2. A Poisson-Hopf algebra is a Poisson algebra $(A, \{,\}_A)$, which is also a Hopf algebra $(A, \mu, \eta, \Delta, \epsilon, S)$, the two structures being compatible in the sense that for all $a_1, a_2 \in A$,

$$\{\Delta(a_1), \Delta(a_2)\}_{A\otimes A} = \Delta(\{a_1, a_2\}_A).$$

The Poisson bracket $\{,\}_{A\otimes A}$ is defined by

$$\{a_1 \otimes a_1', a_2 \otimes a_2'\}_{A\otimes A} = \{a_1, a_2\}_A \otimes a_1'a_2' + a_1a_2 \otimes \{a_1', a_2'\}_A.$$

A dual notion to Poisson-Hopf algebra is co-Poisson Hopf algebra.

Definition 2.3. A co-Poisson algebra is a cocommutative coalgebra (C, Δ, ϵ) equipped with a skew-symmetric linear map $\delta : C \to C \otimes C$; the Poisson co-bracket, satisfying the following conditions

- co-Jacobi identity

$$\mathrm{Jac}_\delta(c) = \sum_{\textit{cyclic perm.}} (\delta \otimes \mathrm{id})\delta(c) = 0, \ \text{for any } c \in C$$

- co-Leibniz identity

$$(\Delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta) \circ \Delta + \sigma_{23}(\delta \otimes \mathrm{id}) \circ \Delta$$

holds.

A co-Poisson-Hopf algebra is a *co-Poisson algebra* $(C, \Delta, \epsilon, \delta)$, which is also a Hopf algebra $(C, \mu, \eta, \Delta, \epsilon, S)$. The two structures are compatible in the sense that for all $a_1, a_2 \in C$,

$$\delta(a_1 a_2) = \delta(a_1)\Delta(a_2) + \Delta(a_1)\delta(a_2). \tag{2.7}$$

Proposition 4. *Let $\mathfrak{g}$ be a Lie algebra. If its universal enveloping algebra $U(\mathfrak{g})$ has a co-Poisson structure δ, making it a co-Poisson-Hopf algebra, then $\delta(\mathfrak{g}) \subset \mathfrak{g} \otimes \mathfrak{g}$ and $\delta|_{\mathfrak{g}}$ is a Lie bialgebra structure on $\mathfrak{g}$. Conversely, any Lie bialgebra structure $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ extends uniquely to a Poisson co-bracket on $U(\mathfrak{g})$, which turns $U(\mathfrak{g})$ into a co-Poisson Hopf algebra.*

Proof. Let $\delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ be a Poisson co-bracket on $U(\mathfrak{g})$. To show that $\delta(\mathfrak{g}) \in \mathfrak{g} \otimes \mathfrak{g}$, let $x \in \mathfrak{g}$ and take $\delta(x) = \sum_i a_i \otimes a_i'$, where $a_i, a_i' \in U(\mathfrak{g})$. We may assume that the a_i' are linearly independent. By the co-Leibniz rule, we can write

$$\sum_i \Delta(a_i) \otimes a_i' = 1 \otimes \delta(x) + a \otimes \delta(1) + \sigma_{23}(\delta(x) \otimes 1 + \delta(1) \otimes x).$$

Taking $a_1 = a_2 = 1$ in (2.7), we get $\delta(1) = 0$; therefore,

$$\sum_i \Delta(a_i) \otimes a_i' = \sum_i (a_i \otimes 1 + 1 \otimes a_i) \otimes a_i'.$$

It follows that the a_i are primitive elements of $U(\mathfrak{g})$. Hence, $\delta(\mathfrak{g}) \subset \mathfrak{g} \otimes U(\mathfrak{g})$, according to the theorem that the set of primitive elements in $U(\mathfrak{g})$ is equal to $\mathfrak{g}$ over a field of zero characteristic. Since δ is skew-symmetric,

$$\delta(\mathfrak{g}) \subset (\mathfrak{g} \otimes U(\mathfrak{g})) \cap (U(\mathfrak{g}) \otimes \mathfrak{g}) = \mathfrak{g} \otimes \mathfrak{g}.$$

Next, we need to prove that $\delta|_{\mathfrak{g}}$ is a $1 - cocycle$. Thus, let $x_1, x_2 \in \mathfrak{g}$, and compute

$$\begin{aligned} \delta([x_1, x_2]) &= \delta(x_1 x_2 - x_2 x_1) \\ &= [\Delta(x_1), \delta(x_2)] - [\Delta(x_2), \delta(x_2)] \\ &= x_1.\delta(x_2) - x_2.\delta(x_1), \end{aligned}$$

where the dot denotes the adjoint representation in $\mathfrak{g} \otimes \mathfrak{g}$:

$$x.\delta(y) = (\mathrm{ad}_x \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}_x)(y).$$

Finally, by definition of co-Poisson algebra, the Jacobi identity holds. To prove the converse, we extend the bialgebra structure δ on $\mathfrak{g}$ to $U\mathfrak{g}$ by (2.7). The co-Jacobi identity also extends because it is equivalent to saying that δ^* is a bracket on $\mathfrak{g}$ □

Definition 2.4. Let $(\mathfrak{g}, \delta)$ be a Lie bialgebra. We say that a quantized universal enveloping algebra A is a quantization of $(\mathfrak{g}, \delta)$, or that $(\mathfrak{g}, \delta)$ is the quasiclassical limit of A, if

(i) A/hA is isomorphic to $U(\mathfrak{g})$ as a Hopf algebra

(ii) For any $x_0 \in \mathfrak{g}$ and any $x \in A$ equal to $x_0 \bmod h$, one has

$$h^{-1}(\Delta(x) - \Delta^{op}(x)) \equiv \delta(x_0) \bmod h,$$

where Δ^{op} is the opposite comultiplication ($\Delta^{op} = \tau\Delta$).

First, we consider so-called *quasiclassical* deformations of bialgebras associated with skew-symmetric $r-$matrices. If we are given a Hopf algebra $(H, \mu, \eta, \Delta, \epsilon, S)$, then one can introduce a *twisting element* F, or *twist*, which is by definition an invertible element of $H \otimes H$ which satisfies the properties

$$F_{12}(\Delta \otimes \mathrm{id})(F) = F_{23}(\mathrm{id} \otimes \Delta)(F), \tag{2.8}$$

$$(\epsilon \otimes \mathrm{id})(F) = (\mathrm{id} \otimes \epsilon)(F) = 1, \tag{2.9}$$

where the standard notations

$$F_{12} = F \otimes 1, \qquad F_{23} = 1 \otimes F$$

are used.

Proposition 5. *Let F be a twist; then $(H, \mu, \eta, \Delta_F, \epsilon, S_F)$ is a Hopf algebra, where*

$$\Delta_F(x) = F \circ \Delta(x) \circ F^{-1}, \qquad S_F(x) = v \circ S(x) \circ v^{-1}, \;\; x \in H$$

and

$$v = \mu \circ (\mathrm{id} \otimes S)(F).$$

Proof. Let us prove that Δ_F is coassociative:

$$(\Delta_F \otimes \mathrm{id}) \circ \Delta_F(x) = (\mathrm{id} \otimes \Delta_F) \circ \Delta_F(x).$$

Indeed,

$$F_{12}(\Delta \otimes \mathrm{id})(F)(\Delta \otimes \mathrm{id})\Delta(x)(\Delta \otimes \mathrm{id})(F^{-1})F_{12}^{-1} =$$
$$F_{23}(\mathrm{id} \otimes \Delta)(F)(\mathrm{id} \otimes \Delta)\Delta(x)(\mathrm{id} \otimes \Delta)(F^{-1})F_{23}^{-1};$$

therefore, by the definition of the twist F, the coassociativity holds. Counit axiom

$$(\epsilon \otimes \mathrm{id}) \circ \Delta_F(x) = (\mathrm{id} \otimes \epsilon) \circ \Delta_F = \eta \circ \epsilon$$

also holds by the same definition of F. To prove that S_F is an antipode we need to check that

$$\mu \circ (S_F \otimes \mathrm{id}) \circ \Delta_F(x) = \mu \circ (\mathrm{id} \otimes S_F) \circ \Delta_F(x) = \eta \circ \epsilon. \tag{2.10}$$

Before proving (2.10), we demonstrate that an antipode is an anti-homomorphism of an arbitrary Hopf algebra $(H, \mu, \eta, \Delta, \epsilon, S)$:

$$S(xy) = S(y)S(x) \;\text{ for any } x, y \in H. \tag{2.11}$$

The main tool for proving facts about antipode is the *convolution product*, which is defined as

$$f \star g = \mu \circ (f \otimes g) \circ \Delta$$

on the set of maps $\mathrm{End}(H, H)$. Now the defining properties of the antipode S are encoded in

$$S \star \mathrm{id} = \mathrm{id} \star S = \eta \circ \epsilon.$$

It becomes obvious that the antipode is unique. Indeed, if S and S' are antipodes, then

$$S = S \star (\eta\epsilon) = S \star (\mathrm{id} \star S') = (S \star \mathrm{id}) \star S' = (\eta\epsilon) \star S' = S'.$$

Let us prove (2.11). We define maps ν, ρ in $\mathrm{End}(H \otimes H, H)$ by

$$\nu(x \otimes y) = S(y)S(x) \quad \text{and} \quad \rho(x \otimes y) = S(xy).$$

Note that $H \otimes H$ is a Hopf algebra with the following structure maps:

$$\begin{aligned} \mu^{\otimes 2} &= \mu \otimes \mu, & \eta^{\otimes 2} &= \eta \otimes \eta, \\ \Delta^{\otimes 2} &= (\mathrm{id} \otimes \tau \otimes \mathrm{id})(\Delta \otimes \Delta), & \epsilon^{\otimes 2} &= \epsilon \otimes \epsilon, \\ S^{\otimes 2} &= S \otimes S, \end{aligned}$$

where τ denotes the flip of the second and the third tensor factor. To prove (2.11) we need to show that $\rho = \nu$. It is enough to prove that

$$\rho \star \mu = \mu \star \nu = \eta \circ \epsilon.$$

First we have

$$\begin{aligned} (\rho \star \mu)(x \otimes y) &= \textstyle\sum_i \rho((x \otimes y)_i^{(1)})\mu((x \otimes y)_i^{(2)}) \\ &= \textstyle\sum_{i,j} \rho(x_i^{(1)} \otimes y_j^{(1)})\mu(x_i^{(2)} \otimes y_j^{(2)}) \\ &= \textstyle\sum_{i,j} S(x_i^{(1)} y_j^{(1)}) x_i^{(2)} y_j^{(2)} \\ &= \eta \circ \epsilon(xy), \end{aligned}$$

where the notations

$$\begin{aligned} \Delta^{\otimes 2}(x \otimes y) &= \textstyle\sum_i (x \otimes y)_i^{(1)} \otimes (x \otimes y)_i^{(2)}, \\ \Delta(x) &= \textstyle\sum_i x_i^{(1)} \otimes x_i^{(2)}, \\ \Delta(y) &= \textstyle\sum_j y_j^{(1)} \otimes y_j^{(2)} \end{aligned}$$

are used as well as the definition of $\Delta^{\otimes 2}$ and

$$\Delta(xy) = \sum_k (xy)_k^{(1)} \otimes (xy)_k^{(2)} = \Delta(x)\Delta(y) = \sum_{i,j} x_i^{(1)} y_j^{(1)} \otimes x_i^{(2)} y_j^{(2)}.$$

On the other hand, we have

$$\begin{aligned}(\mu\star\nu)(x\otimes y) &= \textstyle\sum_i \mu((x\otimes y)_i^{(1)})\nu((x\otimes y)_i^{(2)})\\ &= \textstyle\sum_{i,j} x_i^{(1)}y_j^{(1)}S(y_j^{(2)})S(x_i^{(2)})\\ &= (\eta\circ\epsilon(x))(\eta\circ\epsilon(y)).\end{aligned}$$

Now we can return to the proof of (2.10). We prove only the first identity in (2.10), the second one is done by same reasoning. Assume summation over repeated indices in $F = a_i \otimes b_i$ and $F^{-1} = c_j \otimes d_j$. We have

$$\mu(S_F\otimes \mathrm{id})\Delta_F(x) = vS(c_j)S(x_k^{(1)})S(a_i)v^{-1}b_ix_k^{(2)}d_j.$$

□

We are finished if we have proven that

$$v^{-1} = \mu(S\otimes \mathrm{id})(F^{-1}) = S(c_l)d_l. \tag{2.12}$$

The definition of twist F reads

$$a_i(a_j^{(1)})_k \otimes b_i(a_j^{(2)})_k \otimes b_j = a_j \otimes a_i(b_j^{(1)})_k \otimes b_i(b_j^{(2)})_k. \tag{2.13}$$

If we apply $(\mu\circ(S\otimes\mathrm{id}))\otimes\mathrm{id}$ to both parts of (2.13), then we get

$$v\otimes 1 = a_jS((b_j^{(1)})_k)S(a_i)\otimes b_i(b_j^{(2)})_k. \tag{2.14}$$

If we multiply (2.14) by $1\otimes S(c_l)d_l$ from the right and apply μ, then we are left with

$$v(S(c_l)d_l) = a_jS((b_j^{(1)})_k)S(a_i)(S(c_l)d_l)b_i(b_j^{(2)})_k.$$

Now, by the identity

$$c_la_i\otimes d_lb_i = 1\otimes 1,$$

which leads to

$$S(a_i)S(c_l)d_lb_i = 1,$$

(2.12) follows.

Example 2.4. Let us consider the universal enveloping algebra of the Borel subalgera $\mathfrak{b}_+ = \{H, E|[H, E] = E\}$ as a Hopf algebra. It has the following structure maps defined on generators:

$$\begin{aligned} \Delta(H) &= H \otimes 1 + 1 \otimes H, & \Delta(E) &= E \otimes 1 + 1 \otimes E. \\ \epsilon(H) &= 0, & \epsilon(E) &= 0, \\ S(H) &= -H, & S(E) &= -E. \end{aligned}$$

They extend uniquely to $U\mathfrak{b}_+$ if we require Δ to be a homomorphism (2.6) and S to be an anti-homomorphism (2.11). Let

$$F = 1 \otimes 1 + \sum_{k \geq 1} \frac{H(H-1)\cdots(H-k+1)}{k!} \otimes E^k.$$

To define this element correctly, we should take the following two steps:

- Complete $U\mathfrak{b}_+$ to a topological algebra $\bar{U}\mathfrak{b}_+$ by choosing the appropriate topology
- Complete $\bar{U}\mathfrak{b}_+ \otimes \bar{U}\mathfrak{b}_+$ up to $\bar{U}\mathfrak{b}_+ \hat{\otimes} \bar{U}\mathfrak{b}_+$, where $\hat{\otimes}$ stands for a completed tensor product in some topology in $\bar{U}\mathfrak{b}_+ \otimes \bar{U}\mathfrak{b}_+$

The first step is done by fixing a gradation

$$\deg H = 0, \quad \deg E = 1$$

and the filtration by a system of ideals in $U\mathfrak{b}_+$ given by

$$U\mathfrak{b}_+ \equiv \mathcal{K}^0 \supset \mathcal{K}^1 \supset \cdots \supset \mathcal{K}^n \supset \cdots,$$

where

$$\mathcal{K}^n = \{\sum_{k,l \geq 0} c_{kl} H^k E^l | l \geq n\}.$$

The element F is convergent if we complete the tensor product $\hat{\otimes}$ with respect to a system of ideals:

$$\bar{U}\mathfrak{b}_+ \otimes \bar{U}\mathfrak{b}_+ = \mathcal{I}^0 \supset \cdots \supset \mathcal{I}^k \supset \mathcal{I}^{k+1} \supset \cdots,$$

where

$$\mathcal{I}^k =< H \otimes E - E \otimes H >^k, \tag{2.15}$$

which means that

$$F_m - F_n \in \mathcal{I}^{n+1}, \text{ for } m > n.$$

The element F can now be written as

$$F = 1 \otimes 1 + \sum_{k \geq 1} \frac{1}{k!} H^k \otimes \sigma^k = \exp(H \otimes \ln(1 + E)),$$

where

$$\sigma = \ln(1 + E) = \sum_{k \geq 1} (-1)^{k-1} \frac{E^k}{k},$$

and

$$H^k \otimes \sigma^k = H^k \otimes E^k \frac{\sigma^k}{E^k} \in \mathcal{I}^k.$$

Let us check that F is a twist. It is obviously satisfies the property

$$(\epsilon \otimes \mathrm{id})(F) = (\mathrm{id} \otimes \epsilon)(F) = 1,$$

and we need to prove that

$$F_{12}(\Delta \otimes \mathrm{id})(F) = F_{23}(\mathrm{id} \otimes \Delta)(F). \tag{2.16}$$

Clearly,

$$(\Delta \otimes \mathrm{id})(F) = F_{13}F_{23}.$$

Once we have

$$F_{12}F_{13} = F_{23}(\mathrm{id} \otimes \Delta)(F)F_{23}^{-1} = (\mathrm{id} \otimes \Delta_F)(F),$$

the proof is finished, and the latter follows from a direct calculation:

$$F\Delta(E)F^{-1} = \exp(H \otimes \ln(1 + E))(E \otimes 1 + 1 \otimes E)\exp(-H \otimes \ln(1 + E)) =$$
$$\exp(\mathrm{ad}_{H \otimes \sigma})(E \otimes 1 + 1 \otimes E) = E \otimes (1 + E) + E \otimes 1,$$

where

$$\sigma = \ln(1 + E), \qquad \Delta_F(E) = \ln[(1 + E) \otimes (1 + E)] = \sigma \otimes 1 + 1 \otimes \sigma.$$

Chapter 3

Twists for classical universal enveloping algebra

The role of twists is to deform the coalgebraic structure of a Hopf algebra. Such deformations of bialgebras $(\mathfrak{g}, \delta)$ in the category of co-Poisson Hopf algebras, see Proposition 4, can be considered as quantizations of bialgebraic structures and the corresponding skew-symmetric $r-$matrices. The case of $r-$matrices described by the Belavin-Drinfeld classification cannot be quantized this way; quantization requires the passage to Quantum Universal Enveloping algebra (QUE), which we deal with in the next chapter. The quantization of an $r-$matrix is a *quantum $R-$matrix*, which intertwines Δ with its opposite Δ^{op} and is subject to additional relations, which we state below.

Definition 3.1. A Hopf algebra $(A, \mu, \eta, \Delta, \epsilon, S)$ is said to be almost cocommutative if there exists an invertible element $R \in A \otimes A$, such that

$$\Delta^{op}(a) = R\Delta(a)R^{-1} \tag{3.1}$$

for all $a \in A$.

In particular, an $R-$matrix is a twist that relates $(A, \mu, \eta, \Delta, \epsilon, S)$ with $(A, \mu, \eta, \Delta^{op}, \epsilon, S^{-1})$

if

$$\begin{aligned} R_{12}(\Delta \otimes \mathrm{id})(R) &= R_{23}(\mathrm{id} \otimes \Delta)(R), \\ (\epsilon \otimes \mathrm{id})(R) &= (\mathrm{id} \otimes \epsilon)(R) = 1. \end{aligned} \tag{3.2}$$

Definition 3.2. A cocommutative Hopf algebra (A, R) is said to be

- coboundary if R satisfies (3.2), $R_{21} = R^{-1}$ and $(\epsilon \otimes \epsilon)(R) = 1$
- quasitriangular if

$$(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23} \tag{3.3}$$

$$(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12} \tag{3.4}$$

- triangular if it is quasitriangular, and, in adition $R_{21} = R^{-1}$

If A is quasitriangular, the element R is called the universal $R-$matrix of (A, R). The $R-$matrix satisfies the quantum analogue of the Classical Yang-Baxter equation as it follows from

Proposition 6. *Let (A, R) be a quasitriangular Hopf algebra. Then,*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{3.5}$$

$$(\epsilon \otimes \mathrm{id})(R) = 1 = (\mathrm{id} \otimes \epsilon)(R), \tag{3.6}$$

$$(S \otimes \mathrm{id})(R) = R^{-1} = (\mathrm{id} \otimes S^{-1})(R), \tag{3.7}$$

$$(S \otimes S)(R) = R. \tag{3.8}$$

Proof. By (3.2) we have

$$R_{12}R_{13}R_{23} = R_{12}(\Delta \otimes \mathrm{id})(R) = (\Delta^{op} \otimes \mathrm{id})(R)R_{12},$$

where we used (3.1). R is quasitriangular; thus

$$(\Delta^{op} \otimes \mathrm{id})(R) = R_{23}R_{13},$$

which proves (3.5). Next, if we apply $(\epsilon \otimes \mathrm{id} \otimes \mathrm{id})$ and $(\mathrm{id} \otimes \mathrm{id} \otimes \epsilon)$ to both parts of (3.3) and (3.4), then we get (3.6). To prove the remaining two identities, we argue

$$\begin{aligned} R(S \otimes \mathrm{id})(R) &= (\mu \otimes \mathrm{id})(\mathrm{id} \otimes S \otimes \mathrm{id})(R_{13}R_{23}) \\ &= (\mu \otimes \mathrm{id})(\mathrm{id} \otimes S \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(R) \\ &= (\mu(\mathrm{id} \otimes S)\Delta \otimes \mathrm{id})(R) \\ &= (\epsilon \otimes \mathrm{id})(R) = 1, \end{aligned}$$

where we used the property of antipode. The second equality in (3.7) follows by the same reasoning applied to (A^{op}, R_{21}). Finally, (3.8) follows from (3.7). □

The equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{3.9}$$

is called the Quantum Yang-Baxter Equation (QYBE). Its meaning is that if R is presented in the form

$$R = 1 \otimes 1 + h \cdot r + \cdots,$$

where h is a formal deformation parameter, then after equating terms in (3.9) modulo h^2, we are just left with CYBE for r.

Example 3.1. Let A be a cocommutative Hopf algebra (i.e $\Delta = \Delta^{op}$) and F be a twist; then A is a triangular Hopf algebra with an R-matrix

$$R = F_{21}F^{-1}. \tag{3.10}$$

First, we see that

$$\Delta_F^{op}(a) = F_{21}\Delta^{op}(a)F_{21}^{-1} = F_{21}F^{-1}\Delta_F(a)FF_{21}^{-1}.$$

Let us check that

$$(\Delta_F \otimes \mathrm{id})(R) = R_{13}R_{23}. \tag{3.11}$$

The left hand side of (3.11)

$$F_{12}(\Delta \otimes \mathrm{id})(F_{21})(\Delta \otimes \mathrm{id})(F^{-1})F_{12}^{-1} \tag{3.12}$$

can be transformed as follows:

Consider the following identity

$$F_{31}(\Delta_{13}\otimes \mathrm{id})(F) = F_{12}(\Delta\otimes \mathrm{id})(F_{21}), \tag{3.13}$$

following from (2.8) after applying permutation (23)(12) to both parts (Δ_{13} denotes Δ taken in the first and third tensor factor of $A\otimes A\otimes A$) and using $\Delta = (12)\Delta$. Similarly, we obtain the identity

$$(\Delta\otimes \mathrm{id})(F^{-1})F_{21}^{-1} = (\mathrm{id}\otimes\Delta_{13})(F^{-1})F_{13}^{-1}, \tag{3.14}$$

after applying transposition (12) to

$$(\Delta\otimes \mathrm{id})(F^{-1})(F_{12}^{-1}) = (\mathrm{id}\otimes\Delta)(F^{-1})F_{23}^{-1}.$$

Next, we need an identity relating $(\Delta_{13}\otimes \mathrm{id})(F)$ and $(\mathrm{id}\otimes\Delta_{13})(F)$. The following identities are obtained from (2.8) after applying the (12) or (23) transposition to both parts of (2.8):

$$\begin{aligned} F_{21}(\Delta\otimes \mathrm{id})(F) &= F_{13}(\mathrm{id}\otimes\Delta_{13})(F), \\ F_{13}(\Delta_{13}\otimes \mathrm{id})(F) &= F_{32}(\mathrm{id}\otimes\Delta)(F). \end{aligned}$$

Thus, using (2.8) one more time, we get the desired relation

$$F_{23}F_{32}^{-1}F_{13}(\Delta_{13}\otimes \mathrm{id})(F) = F_{12}F_{21}^{-1}F_{13}(\mathrm{id}\otimes\Delta_{13})(F). \tag{3.15}$$

Finally, by combining (3.13), (3.14), and (3.15), we prove (3.11).

Example 3.2. Consider a Hopf algebra $(H,\mu,\eta,\Delta,\epsilon,S)$ and let $w\in \mathrm{Aut}(H)$, the change of basis by w amounts to the twisting of H by the *trivial twist*

$$(w\otimes w)\circ\Delta\circ w^{-1}.$$

In particular, if $w\in \mathrm{Inn}(H)$ (i.e. of the form $w(x) = u\cdot x\cdot u^{-1}$), then

$$(u\otimes u)\Delta(u^{-1})$$

satisfies (2.8) which can be checked directly. If (H, R) is a quasi-triangular Hopf algebra, then the twisted R matrix is of the form

$$(w \otimes w) \circ R.$$

If $A = U\mathfrak{g}$, then (3.10) allows to build quantizations of skew-symmetric $r-$matrices. In fact, any skew-symmetric $r-$matrix can be quantized this way.

Theorem 2 (Drinfeld). *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, and let $r \in \mathfrak{g} \otimes \mathfrak{g}$ be a skew-symmetric solution of the classical Yang-Baxter equation*

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0.$$

Then there exists a deformation $U_h(\mathfrak{g})$ of $U(\mathfrak{g})$ whose classical limit is $\mathfrak{g}$ with the Lie bialgebra structure defined by r. Moreover, $U_h(\mathfrak{g})$ is a triangular Hopf algebra and is isomorphic to $U(\mathfrak{g})[[h]] \otimes U(\mathfrak{g})[[h]]$.

The element

$$F = \exp(H \otimes \sigma),$$

discussed in *Example 2.4*, defines the following *Jordanian* $R-$matrix:

$$R = F_{21}F^{-1} = \exp(\sigma \otimes H)\exp(-H \otimes \sigma).$$

If we introduce the deformation parameter h by $E \mapsto -hE$, then

$$R_h = 1 \otimes 1 + hH \wedge E + \cdots,$$

thus quantizing the first bialgebraic structure from *Example 1.1.* Moreover, this element allows the calculation of the Hopf structure maps of $U_h(\mathfrak{b}_+)$ explicitly.

$$\Delta(H) = H \otimes (1 - hE)^{-1} + 1 \otimes H, \;\; \Delta(E) = E \otimes 1 + 1 \otimes E - hE \otimes E,$$

$$S(H) = -H(1 - hE), \;\; S(E) = -E(1 - hE)^{-1}.$$

The higher dimensional analogs of the Jordanian twist in one direction lead to the quantization of the *extended Jordanian* $r-$matrix

$$H_{1n} \wedge E_{1n} + \sum_{i \geq 2} E_{1i} \wedge E_{in}, \tag{3.16}$$

while another direction is the quantization of Gerstenhaber-Giaquainto boundary $r-$matrices (1.12). There is an explicit formula for the twist corresponding to (3.16). Fix canonical $\mathfrak{gl}(\mathfrak{n})$-basis $\{E_{ij}| \quad i,j = 1\dots n\}$

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj}$$

and take an element H from the $\mathfrak{sl}(n)-$Cartan subalgebra

$$\mathfrak{h}_{\mathfrak{sl}(n)} = \mathrm{span}\{E_{ii} - E_{jj} | 1 \leq i,j \leq n\}.$$

Proposition 7 ([20]). *An element*

$$\mathcal{F}_{\mathcal{EJ}} = \exp(-h\sum_{i=2}^{n-1} E_{1i} \otimes E_{in}e^{-\beta_i \sigma_{-h}})\exp(H \otimes \sigma_{-h}) \tag{3.17}$$

where

$$\beta_1 = 1, \;\; [H, E_{in}] = \beta_i E_{in}, \;\; \sigma_{-h} = \ln(1 - hE)$$

is a twist.

$\mathcal{F}_{EJ}$ is called *extended Jordanian twist*; it is defined in a similar completion as the Jordanian one. The proof follows from the general discussion of quantum twists we give later, though it is not difficult to give a direct proof, as we did for the Jordanian twist, where an explicit form of the deformed costructure was used. On the contrary, we do not have an explicit quantization of (1.12) and a corresponding universal twisting element. Below, we quantize the first non-trivial boundary $r-$matrix

$$\left(\frac{2}{3}E_{11} - \frac{1}{3}E_{22} - \frac{1}{3}E_{33}\right) \wedge E_{12} + \left(\frac{1}{3}E_{11} + \frac{1}{3}E_{22} - \frac{2}{3}E_{33}\right) \wedge E_{23} + E_{13} \wedge E_{32} \tag{3.18}$$

via introduction of the *elementary parabolic twist.* Let us introduce notations

$$\begin{aligned} H^{\perp}_{12} &= \frac{1}{3}E_{11} + \frac{1}{3}E_{22} - \frac{2}{3}E_{33}, \\ H^{\perp}_{23} &= \frac{2}{3}E_{11} - \frac{1}{3}E_{22} - \frac{1}{3}E_{33}. \end{aligned}$$

The upper script $\perp$ denotes that the functionals $H^{\perp *}_{12}$ and $H^{\perp *}_{23}$ are orthogonal to the roots λ_{12} and λ_{23} respectively. Consider the subalgebra $\mathfrak{p}_1$ generated by the Borel subalgebra $\mathfrak{b}_+$ in $\mathfrak{sl}(3)$ and the element E_{32}, i.e. by the parabolic subalgebra of $\mathfrak{sl}(3)$:

$$\mathfrak{p}_1 = V_{-\alpha_2} + \mathfrak{b}_+ = V_{-\alpha_2} + \mathfrak{h} + \mathfrak{n}_+.$$

We need the adjoint action of the Cartan subalgebra on the space $V_{-\alpha_2} + \mathfrak{n}_+$ in its explicit form:

$$\begin{aligned} \left[H^{\perp}_{12}, E_{13}\right] &= E_{13}, & \left[H^{\perp}_{23}, E_{13}\right] &= E_{13}, \\ \left[H^{\perp}_{12}, E_{12}\right] &= 0, & \left[H^{\perp}_{23}, E_{12}\right] &= E_{12}, \\ \left[H^{\perp}_{12}, E_{32}\right] &= -E_{32}, & \left[H^{\perp}_{23}, E_{23}\right] &= 0, \\ \left[H^{\perp}_{12}, E_{23}\right] &= E_{23}, & \left[H^{\perp}_{23}, E_{32}\right] &= 0. \end{aligned}$$

Note that

$$H^{\perp}_{23} \wedge E_{12} + E_{13} \wedge E_{32},$$

is quantized by the extended Jordanian twist

$$\Phi = \exp(-E_{32} \otimes E_{13}e^{-\sigma_{12}}) \exp(H^{\perp}_{23} \otimes \sigma_{12}). \tag{3.19}$$

$$\Phi : U(\mathfrak{p}_1) \longrightarrow U_{\Phi}(\mathfrak{p}_1).$$

This twist has the 4-dimensional carrier subalgebra $\mathfrak{L} \subset \mathfrak{p}_1$ generated by the set $\left\{H^{\perp}_{23}, E_{13}, E_{32}, E_{12}\right\}$. It differs from (3.17) by an automorphism.

The twisted algebra $U_\Phi(\mathfrak{p}_1)$ has the following coproducts

$$\begin{aligned}
\Delta_\Phi(H_{12}^\perp) &= H_{12}^\perp \otimes 1 + 1 \otimes H_{12}^\perp, \\
\Delta_\Phi(E_{13}) &= E_{13} \otimes e^{\sigma_{12}} + e^{\sigma_{12}} \otimes E_{13}, \\
\Delta_\Phi(E_{12}) &= E_{12} \otimes e^{\sigma_{12}} + 1 \otimes E_{12}, \\
\Delta_\Phi(E_{32}) &= E_{32} \otimes e^{-\sigma_{12}} + 1 \otimes E_{32}, \\
\Delta_\Phi(H_{23}^\perp) &= H_{23}^\perp \otimes 1 + 1 \otimes H_{23}^\perp + E_{32} \otimes E_{13}e^{-2\sigma_{12}}, \\
\Delta_\Phi(E_{23}) &= E_{23} \otimes 1 + 1 \otimes E_{23} + 2H_{12}^\perp \otimes E_{13}e^{-\sigma_{12}},
\end{aligned} \tag{3.20}$$

with three primitive generators: $H_{12}^\perp$, σ_{12}, and $E_{13}e^{-\sigma_{12}}$.

We shall now demonstrate that $U_\Phi(\mathfrak{p}_1)$ can be additionally twisted by the factor F, defined on

$$A_\Phi = \{(H_{12}^\perp)^k E_{23}^l (E_{13}e^{-\sigma_{12}})^m | k, l, m \geq 0\},$$

which is a Hopf subalgebra in $U_\Phi(\mathfrak{p}_1)$. As a result, the composition of 2 factors,

$$F_\wp = F \cdot \Phi$$

will form the *parabolic twist*, yielding the quantization of (3.18).

Proposition 8. *The algebra $U_\Phi(\mathfrak{p}_1)$ admits a twist*

$$F = (1 \otimes 1 + 1 \otimes E_{23} + H_{12}^\perp \otimes E_{13}e^{-\sigma_{12}})^{(H_{12}^\perp \otimes 1)}.$$

Proof. Consider the generalized Verma identity [9]

$$e^{x\ln(1+ta)}e^{(x+y)\ln(1+sb)}e^{y\ln(1+ta)} = e^{y\ln(1+sb)}e^{(x+y)\ln(1+ta)}e^{x\ln(1+sb)}.$$

Here, the generators $a, b \in \mathfrak{n}_+ \subset \mathfrak{sl}(3)$ are subject to the relations

$$[a,[a,b]] = 0$$

and

$$[b,[b,a]] = 0;$$

x, y, s, t are some central elements. Now, let $a = E_{12}, b = E_{23}$, $x = y = z = 1$. The generalized Verma identity leads to

$$e^{\xi\sigma_{12}} e^{\xi\sigma_{23}} e^{\xi\sigma_{12}} = (e^{\sigma_{12}} e^{\sigma_{23}} e^{\sigma_{12}})^{\xi}, \tag{3.21}$$

where $\xi = x + y$. Thus,

$$F = \exp(H_{12}^{\perp} \otimes \sigma_{12}) \exp(H_{12}^{\perp} \otimes \sigma_{23}) \exp(H_{12}^{\perp} \otimes \sigma_{12}) \exp(-2H_{12}^{\perp} \otimes \sigma_{12}).$$

Using (3.21), we transform F to

$$(\exp(1 \otimes \sigma_{12}) \exp(1 \otimes \sigma_{23}) \exp(1 \otimes \sigma_{12}))^{(H_{12}^{\perp} \otimes 1)} \exp(-2H_{12}^{\perp} \otimes \sigma_{12}) = \exp(H_{12}^{\perp} \otimes \ln(e^{\sigma_{12}} e^{\sigma_{23}} e^{\sigma_{12}})) \exp(-2H_{12}^{\perp} \otimes \sigma_{12}).$$

It is easy to verify that

$$\Delta_F(e^{\sigma_{12}} e^{\sigma_{23}} e^{\sigma_{12}}) = e^{\sigma_{12}} e^{\sigma_{23}} e^{\sigma_{12}} \otimes e^{\sigma_{12}} e^{\sigma_{23}} e^{\sigma_{12}}.$$

With the primitivity of $H_{12}^{\perp}$ in $U_{\Phi}(\mathfrak{p}_1)$, see (3.20), we prove that F satisfies (2.8) as we did for the Jordanian twist in *Example 2.4*. □

Applying F to the algebra $U_{\Phi}(\mathfrak{p}_1)$, we get the final costructure of the twisted parabolic algebra $U_{\wp}(\mathfrak{p}_1)$:

$$U_{\Phi}(\mathfrak{p}_1) \xrightarrow{F} U_{\wp}(\mathfrak{p}_1).$$

The coproducts for the generators of $\mathfrak{p}_1$ are the following,

$$\begin{aligned}
\Delta_{\wp}(H_{12}^{\perp}) &= 1 \otimes H_{12}^{\perp} + (H_{12}^{\perp} \otimes 1)(1 \otimes 1 + C)^{-1}, \\
\Delta_{\wp}(H_{23}^{\perp}) &= 1 \otimes H_{23}^{\perp} + H_{12}^{\perp} \otimes e^{-\sigma_{12}} + (E_{32} \otimes E_{13} e^{-\sigma_{12}} + \\
&\quad +((H_{23}^{\perp} - H_{12}^{\perp}) \otimes 1)(1 \otimes 1 + C))(1 \otimes e^{\sigma_{12}} e^{\sigma_{23}})^{-1}, \\
\Delta_{\wp}(E_{13}) &= E_{13} \otimes e^{\sigma_{23}} e^{\sigma_{12}} + e^{\sigma_{12}} \otimes E_{13} + H_{12}^{\perp} E_{13} \otimes E_{13}, \\
\Delta_{\wp}(E_{12}) &= (e^{\sigma_{12}} \otimes e^{\sigma_{12}} e^{\sigma_{23}})(1 \otimes 1 + C)^{-1} - 1 \otimes 1,
\end{aligned}$$

$$
\begin{aligned}
\Delta_{\wp}(E_{32}) &= (E_{32} \otimes e^{-\sigma_{12}} + H_{12}^{\perp} \otimes (2H_{12}^{\perp} - H_{23}^{\perp}) - \\
&\quad -(H_{12}^{\perp})^2 \otimes e^{-\sigma_{12}} + H_{12}^{\perp} \otimes 1)(1 \otimes 1 + C)^{-1} + \\
&\quad +(H_{12}^{\perp}(H_{12}^{\perp} - 1) \otimes 1)(1 \otimes 1 + C)^{-2} + 1 \otimes E_{32}, \\
\Delta_{\wp}(E_{23}) &= E_{23} \otimes e^{\sigma_{23}} + 1 \otimes E_{23} + \\
&\quad +(E_{23} + 2e^{\sigma_{23}} H_{12}^{\perp}) \otimes E_{13} e^{-\sigma_{12}} + \\
&\quad +(E_{23} + e^{\sigma_{23}} H_{12}^{\perp}) H_{12}^{\perp} \otimes (E_{13})^2 (e^{\sigma_{12}} e^{\sigma_{23}} e^{\sigma_{12}})^{-1},
\end{aligned}
$$

where

$$
C \equiv 1 \otimes E_{23} + H_{12}^{\perp} \otimes E_{13} e^{-\sigma_{12}}.
$$

The Hopf algebra $U_{\wp}(\mathfrak{p}_1)$ can be considered as a result of the integral twist deformation

$$
U(\mathfrak{p}_1) \xrightarrow{F_{\wp}} U_{\wp}(\mathfrak{p}_1),
$$

where the element $\mathcal{F}_{\wp}$ can be written in the form

$$
\begin{aligned}
\mathcal{F}_{\wp} &= F \cdot \Phi \\
&= \exp(H_{12}^{\perp} \otimes (2\sigma_{12} + \sigma_{23})) \exp(-E_{32} \otimes E_{13} e^{\sigma_{12}}) \exp(-H_{23} \otimes \sigma_{12}).
\end{aligned}
$$

For the shorter notation, we used the formula that followed from (3.21):

$$
\ln(e^{\sigma_{12}} e^{\sigma_{23}} e^{\sigma_{12}}) = 2\sigma_{12} + \sigma_{23}.
$$

From the beginning, we did not introduce the deformation parameters to the twist $F_{\wp}$, although such parameters are easily taken into account by rescaling the generators of $\mathfrak{p}_1$:

$$
E_{12} \longrightarrow \xi E_{12}, \quad E_{23} \longrightarrow \zeta E_{23}, \quad E_{13} \longrightarrow \xi\zeta E_{13}, \quad E_{32} \longrightarrow \frac{1}{\zeta} E_{32}.
$$

The parametrization of the parabolic twist is given by

$$
\begin{aligned}
&\mathcal{F}_{\wp}(\xi, \zeta) = \\
&\exp(H_{12}^{\perp} \otimes (2\sigma_{12}(\xi) + \sigma_{23}(\zeta))) \exp(-\xi E_{32} \otimes E_{13} e^{\sigma_{12}(\xi)}) \exp(-H_{23} \otimes \sigma_{12}(\xi))
\end{aligned}
\tag{3.22}
$$

with $\sigma_{ij}(\xi) = \ln(1 + \xi E_{ij})$.

By twisting the universal enveloping algebra $U(\mathfrak{p}_1)$ with the element (3.22), we obtain the 2-dimensional smooth variety of Hopf algebras $U_\wp(\mathfrak{p}_1;\xi,\zeta)$. The parameters are independent. In the limit points, the (undeformed) main factors of the parabolic twist are reproduced:

$$\mathcal{F}_\wp(\xi,\zeta) \xrightarrow{\zeta\to 0} \Phi(\xi),$$
$$\mathcal{F}_\wp(\xi,\zeta) \xrightarrow{\xi\to 0} J(\zeta).$$

The first limit is the parametrized extended Jordanian twist; the second is the Jordanian twist for the Borel subalgebra generated by $\left\{H_{12}^{\perp}, E_{23}\right\}$ with the twisting element $J(\zeta) = e^{H_{12}^{\perp}\otimes\sigma_{23}(\zeta)}$.

The universal R–matrix for $U_\wp(\mathfrak{p}_1;\xi,\zeta)$ is defined by the standard formula (3.10):

$$\begin{gathered}
R_\wp(\xi,\zeta) = \left(\mathcal{F}_\wp(\xi,\zeta)\right)_{21}\left(\mathcal{F}_\wp(\xi,\zeta)\right)^{-1} = \\
\exp((2\sigma_{12}(\xi)+\sigma_{23}(\zeta))\otimes H_{12}^{\perp})\exp(-\xi E_{13}e^{\sigma_{12}(\xi)}\otimes E_{32})\exp(-\sigma_{12}(\xi)\otimes H_{23})\times \\
\times\exp(H_{23}\otimes\sigma_{12}(\xi))\exp(\xi E_{32}\otimes E_{13}e^{\sigma_{12}(\xi)})\exp(-H_{12}^{\perp}\otimes(2\sigma_{12}(\xi)+\sigma_{23}(\zeta))).
\end{gathered} \tag{3.23}$$

If we choose the parameters to be proportional ($\zeta = \eta\xi$), the expression (3.23) can be considered to be a quantization of the classical r-matrix (3.18). In the fundamental representation $d(\mathfrak{sl}_2)$, the R-matrix has the following form:

$$\begin{gathered}
R_\wp = d^{\otimes 2}(\mathcal{R}_\wp(\xi,\zeta)) = 1\otimes 1+ \\
+(E_{12}\wedge(\tfrac{2}{3}E_{11}-\tfrac{1}{3}E_{22}-\tfrac{1}{3}E_{33})+E_{32}\wedge E_{13})\xi+ \\
+\tfrac{2}{9}(E_{12}\otimes E_{12})\xi^2+ \\
+(E_{23}\wedge(\tfrac{1}{3}E_{11}+\tfrac{1}{3}E_{22}-\tfrac{2}{3}E_{33}))\zeta+ \\
+\tfrac{2}{9}(E_{23}\otimes E_{23})\zeta^2+ \\
+\tfrac{1}{3}(E_{13}\otimes(\tfrac{1}{3}E_{11}+\tfrac{1}{3}E_{22}-\tfrac{2}{3}E_{33})+(\tfrac{1}{3}E_{11}+\tfrac{1}{3}E_{22}-\tfrac{2}{3}E_{33})\otimes E_{13}+ \\
+\tfrac{1}{3}(E_{12}\otimes E_{23}+E_{23}\otimes E_{12}))\zeta\xi- \\
-\tfrac{2}{81}(E_{13}\otimes E_{13})\zeta^2\xi^2+\tfrac{2}{27}(E_{13}\wedge E_{23})\xi\zeta^2+ \\
+\tfrac{1}{27}(E_{13}\wedge E_{12})\xi^2\zeta.
\end{gathered}$$

Any maximal parabolic subalgebra $\mathfrak{p}_i$ with a missing negative simple root α_i can be considered as a semidirect product of the maximal simple subalgebra in $\mathfrak{p}_i$ and of the ideal generated by the basic elements E_{λ_k} whose positive roots λ_k contain the simple component α_i. This means that the corresponding Hopf algebras $U_{\wp}(\mathfrak{p}_i)$ are examples of the algebras of motion $\mathfrak{g}$ quantized by the twists whose carriers coincide with $\mathfrak{g}$. In the considered case, $U(\mathfrak{p}_1)$ is the universal enveloping of the semidirect product $\mathfrak{p}_1 \cong \mathfrak{sl}(2) \vdash t(2)$, the algebra of two-dimensional motions. The elementary parabolic twist (3.22) can be applied to any Lie algebra that contains $\mathfrak{p}_1$. The algebras A_2 and $\mathfrak{G}_2$ are the only ones among the simple Lie algebras with the rank$(\mathfrak{g}) = 2$. Any simple algebra whose rank is greater than 2 contains $\mathfrak{p}_1$ and can consequently be twisted by $F_{\wp}$.

Chapter 4

From the quantum twists to the quasiclassical ones

As it was pointed out, the quantization of non-skew-symmetric solutions to CYBE requires passage to Quantum Universal Enveloping algebra (QUE). In this chapter, we show how to build the quantum analogs of the twists we considered in the previous one. It turns out that the extended Jordanian twist has a quantum analog that is trivial, but that reduces to nontrivial expression when the quantization parameter approaches zero. The elementary parabolic twist turns out to be equivalent to the so-called Cremmer-Gervais twist [22] on the quantum level. Its reduction to the quasiclassical case is of particular interest because of its relations with Yangians. In the Yangian limit of the quantum affine algebra, we obtain the parabolic twist. Such limit degenerations were being considered in a number of works, where the $q-$Yangian was of interest for the developing theory [26, 29]. The relation established between the elementary parabolic twist and the twisting of $Y(\mathfrak{sl}(2))$ allows a simple calculation tool for the twisted costructure in $Y(\mathfrak{sl}(2))$. The quantum analog of the extended Jordanian twist depends on some element $X \in U_q(\mathfrak{g})$. The full chain of extensions appears as we specialize X to be the highest root generator in an infinite series of simple Lie algebras. To realize the program we stated, we introduce the basic definitions from the theory of Quantum Enveloping Algebras $U_q\mathfrak{g}$, where $\mathfrak{g}$ stands for a simple Lie algebra.

A Hopf algebra $U_q(\mathfrak{g})$ for a Lie algebra $\mathfrak{g}$ with a Cartan matrix $(A)_{ij} = a_{ij}$ and a set of simple roots $\{\alpha_i\}$ is defined by the relations

$$[H_i, H_j] = 0, \qquad [H_i, X_j^{\pm}] = \pm(\alpha_i, \alpha_j)X_j^{\pm}, \tag{4.1}$$

$$X_i^+X_j^- - X_j^-X_i^+ = \delta_{ij}\frac{q^{H_i} - q^{-H_i}}{q - q^{-1}}, \tag{4.2}$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} (X_i^{\pm})^k X_j^{\pm}(X_i^{\pm})^{1-a_{ij}-k} = 0 \ \text{ for } i \neq j, \tag{4.3}$$

where $q_i = q^{\frac{(\alpha_i,\alpha_i)}{2}}$ and

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{(q)_m}{(q)_n(q)_{m-n}},$$

with the costructure given by

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \Delta(X_i^{\pm}) = q^{-\frac{H_i}{2}} \otimes X_i^{\pm} + X_i^{\pm} \otimes q^{\frac{H_i}{2}}, \tag{4.4}$$

$$S(H_i) = -H_i, \quad S(X_i^{\pm}) = -q_i^{\pm 1}X_i^{\pm}, \tag{4.5}$$

$$\varepsilon(H_i) = 0, \quad \varepsilon(X_i^{\pm}) = 0. \tag{4.6}$$

One usually introduces another set of generators

$$E_i = X_i^+ q^{-\frac{H_i}{2}}, \ \ F_i = X_i^- q^{\frac{H_i}{2}}, \tag{4.7}$$

that are more convenient for working with a q-Weyl group. The costructure reads

$$\Delta(E_i) = E_i \otimes 1 + q^{-H_i} \otimes E_i, \quad \Delta(F_i) = 1 \otimes F_i + F_i \otimes q^{H_i}, \tag{4.8}$$

$$S(E_i) = -q^{H_i}E_i, \quad S(F_i) = -F_i q^{-H_i}, \tag{4.9}$$

$$\varepsilon(E_i) = 0, \quad \varepsilon(F_i) = 0. \tag{4.10}$$

To proceed further, we need some facts on the q-Weyl group, for which we refer to [15, 16, 18, 19]. One introduces a linear order on the set of positive roots Δ_+ by taking the longest element of the Weyl group $w_0 = s_{i_1} s_{i_2} \cdots s_{i_N}$ along with its reduced decomposition. The reduced decomposition fixes a linear order from left to right as

$$\Delta_+ = \{\alpha_{i_1}, s_{i_1}\alpha_{i_2}, \cdots, s_{i_1}s_{i_2}\cdots s_{i_{N-1}}\alpha_{i_N}\}.$$

Definition 4.1. A linear order L on the set Δ_+ of the positive roots of $\mathfrak{g}$ is called *convex* or *normal* if, for any positive roots α and β such that $\alpha+\beta \in \Delta_+$ and $\alpha \prec_L \beta$, we have $\alpha \prec_L \alpha + \beta \prec_L \beta$.

It can be proven that any convex order on Δ_+ comes from some reduced decomposition of w_0. The q-Weyl group allows for the definition of composite roots.

Definition 4.2. Suppose that $\alpha = s_{i_1}s_{i_2}\cdots s_{i_{p-1}}\alpha_{i_p}$ $(p = 1, 2, \cdots, N)$. The *root vectors* are defined by

$$E_\alpha = T_{i_1}T_{i_2}\cdots T_{i_{p-1}}(E_{i_p}), \qquad E_{\alpha_i} = E_i,$$

$$F_\alpha = T_{i_1}T_{i_2}\cdots T_{i_{p-1}}(F_{i_p}), \qquad F_{\alpha_i} = F_i,$$

where $T_i(\cdot)$ are the generators of the q-Weyl group (for explicit action see [19], page 138).

Let

$$\hat{R}_i = e_{q^{-2}}((1-q^{-2})^2 E_i q^{H_i} \otimes q^{-H_i} F_i);$$

then, for any $\alpha \in \Delta_+$ such that $\alpha = s_{i_1}\cdots s_{i_{p-1}}\alpha_{i_p}$, define the elements

$$\begin{aligned} \hat{R}_\alpha &= T_{i_1}T_{i_2}\cdots T_{i_{p-1}}(\hat{R}_{i_p}) \\ \hat{R}_{\prec\beta} &= \textstyle\prod_{\alpha\prec\beta}\hat{R}_\alpha. \end{aligned}$$

These elements allow for the formulation of one more general fact about $U_q(\mathfrak{g})$ from [19]:

Theorem 3 ([19]). *Consider the canonical isomorphism $\mathfrak{h} \simeq \mathfrak{h}^*$ defined by the bilinear form $(\ ,\ )$ on $\mathfrak{h}$. Let $H_\beta \in \mathfrak{h}$ be the image of a root $\beta \in \mathfrak{h}^*$ with respect to this isomorphism. Then the following identity holds:*

$$\Delta(E_\beta) = (\hat{R}_{\prec\beta})^{-1}(E_\beta \otimes 1 + q^{-H_\beta} \otimes E_\beta)\hat{R}_{\prec\beta}.$$

Proof. For the proof see [19], Proposition 3.2.1. □

Proposition 9. *Let $X \in U_q(\mathfrak{g})$ be such that*

$$(X \otimes 1)(\Delta(X) - X \otimes 1) = q^2(\Delta(X) - X \otimes 1)(X \otimes 1), \tag{4.11}$$

then the element

$$F = e_{q^2}(-1 \otimes X) \cdot e_{q^{-2}}(-q^{-2}(\Delta(X) - X \otimes 1)), \tag{4.12}$$

satisfies the Drinfeld equation

$$F_{12}(\Delta \otimes \mathrm{id})(F) = F_{23}(\mathrm{id} \otimes \Delta)(F),$$

and has the well-defined limit when $q \longrightarrow 1$.

Proof. The constructed element F is a trivial twist:

$$F = (e_{q^2}(-X) \otimes e_{q^2}(-X))\Delta(e_{q^{-2}}(-q^{-2}X)),$$

if one uses (4.11) along with the q-multiplication property of the q-exponential function

$$e_{q^2}(b)e_{q^2}(a) = e_{q^2}(a + b), \tag{4.13}$$

where $ab = q^2ba$. Non-singularity of F follows from the representation

$$F = \exp(\sum_{n\geq 1}(-1)^n \frac{1}{n \cdot (n)_{q^2}} \frac{(1 \otimes X)^n - (\Delta(X) - X \otimes 1)^n}{1 - q}),$$

where the element $\frac{(1\otimes X)^n - (\Delta(X) - X\otimes 1)^n}{1-q}$ is well-defined in the limit $q \longrightarrow 1$. □

Choosing a different X in Proposition 9, we obtain generalizations of the known quasi-classical twists to the quantum area $q \neq 1$. The general formula of the *full quantum extended Jordanian twist* emerges if we choose X to be an element corresponding to the longest root vector in the infinite series of simple Lie algebras.

Proposition 10. *Let $\mathfrak{g}$ be a simple Lie algebra of infinite series A_N, B_N, C_N, D_N. Then, for the element associated with the longest root λ, the property holds*

$$(E_\lambda \otimes 1)(\Delta(E_\lambda) - E_\lambda \otimes 1) = q^2(\Delta(E_\lambda) - E_\lambda \otimes 1)(E_\lambda \otimes 1). \tag{4.14}$$

Proof. The proof is based on the following expansion from [19]

$$E_\alpha E_\lambda - q^{-(\alpha,\lambda)} E_\lambda E_\alpha = \sum_{\alpha \prec \gamma_1 \prec \cdots \gamma_j \prec \lambda} c_{l,\gamma} E_{\gamma_1}^{l_1} \cdots E_{\gamma_j}^{l_j}, \tag{4.15}$$

where $c_{l,\gamma} \in k[[h]]$. Non-zero terms in the sum are subject to condition

$$\alpha + \lambda = l_1\gamma_1 + l_2\gamma_2 + \cdots + l_j\gamma_j. \tag{4.16}$$

We aim to show that (4.14) holds in all infinite series of simple Lie algebras.

In A_N, we choose a convex order as

$$\alpha_1 \succ \alpha_1 + \alpha_2 \succ \cdots \succ \alpha_1 + \alpha_2 + \cdots + \alpha_N \succ \overbrace{\alpha_2 \succ \cdots \succ \alpha_{N-1} + \alpha_N \succ \alpha_N}^{\text{roots without } \alpha_1}.$$

and $\lambda = \alpha_1 + \cdots \alpha_N$. If $\lambda \succ \alpha \succ \alpha_N$, we can only satisfy (4.15) with zero coefficients, without contradicting (4.16); therefore,

$$E_\alpha E_\lambda = q^{-(\alpha,\lambda)} E_\lambda E_\alpha, \tag{4.17}$$

which means that

$$(\hat{R}_{\prec\lambda})^{-1}(E_\lambda \otimes 1) = (E_\lambda \otimes 1)(\hat{R}_{\prec\lambda})^{-1} \tag{4.18}$$

and (4.14) holds. In the remaining series of simple Lie algebras, we check (4.18) for an appropriately defined convex order.

In B_N, we have

$$\alpha_1 \succ \alpha_1 + \alpha_2 \succ \cdots \succ \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{N-1} + 2\beta \succ \overbrace{\alpha_2 \succ \alpha_2 + \alpha_3 \succ \cdots \succ \beta}^{\text{roots without } \alpha_1},$$

and $\lambda = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{N-1} + 2\beta$. As with A_N, we come to the relation (4.17).

In C_N, we fix the following convex order:

$$\overbrace{\alpha_1 \prec \alpha_1 + \alpha_2 \prec \cdots \prec \alpha_1 + \cdots + \alpha_{N-1}}^{\text{roots without } \beta} \prec 2(\alpha_1 + \cdots + \alpha_{N-1}) + \beta \prec$$
$$\alpha_1 + \cdots + \alpha_{N-1} + \beta \prec \cdots \prec \alpha_1 + 2(\alpha_2 + \cdots + \alpha_{N-1}) + \beta \prec \cdots \prec \alpha_2 \prec \cdots \prec \beta,$$

and $\lambda = 2(\alpha_1 + \cdots + \alpha_{N-1}) + \beta$. This order eliminates all nonzero terms on the r.h.s of (4.15).

In D_N, we have quite a similar situation:

$$\alpha_1 \succ \alpha_1 + \alpha_2 \succ \cdots \succ \alpha_1 + \cdots + \alpha_{N-1} \succ \alpha_1 + \cdots + \alpha_{N-1} + \beta \succ$$
$$\alpha_1 + \alpha_2 + \cdots + 2\alpha_{N-2} + \alpha_{N-1} + \beta \succ \cdots \succ \alpha_1 + 2\alpha_2 + \cdots 2\alpha_{N-2} + \alpha_{N-1} + \beta \succ$$
$$\overbrace{\alpha_2 \succ \alpha_2 + \alpha_3 \succ \cdots \succ \beta}^{\text{roots without } \alpha_1},$$

and $\lambda = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{N-2} + \alpha_{N-1} + \beta$. □

Notice that we have a problem with extension of the proof to all series of simple Lie algebras. For example, in G_2 we have the following sequence of roots

$$\alpha \succ 3\alpha + \beta \succ 2\alpha + \beta \succ 3\alpha + 2\beta \succ \alpha + \beta \succ \beta,$$

so there could be the contribution $E^3_{\alpha+\beta}$ or $E^2_{2\alpha+\beta}$ on the r.h.s of (4.15) if one chooses such an ordering where $3\alpha + 2\beta \succ \beta$ or $3\alpha + 2\beta \succ \alpha$, respectively.

Example 4.1. In the case of $\mathfrak{sl}(n+1)$, we use the following formula [13]:

$$\Delta(E_{(1,n+1)}) = q^{-H_{1,n+1}} \otimes E_{(1,n+1)} + E_{(1,n+1)} \otimes 1 + (1-q^2) \sum_{i=1}^{n-1} E_{(1,i+1)} q^{-H_{i+1,n+1}} \otimes E_{(i+1,n+1)};$$

therefore the full quantum extended Jordanian twist is

$$F^q_{EJ} =$$
$$e_{q^{-2}}((1-q^{-2}) \textstyle\sum_{i=1}^{n-1} E_{(1,i+1)} q^{-H_{i+1,n+1}} \otimes E_{(i+1,n+1)} e_{q^2}(-qE_{(1,n+1)})(e_{q^{-2}}(-E_{(1,n+1)}))^{-1}) \cdot$$
$$e_{q^2}(-1 \otimes E_{(1,n+1)}) \cdot e_{q^{-2}}(-q^{-2} \cdot q^{-H_{1,n+1}} \otimes E_{(1,n+1)}),$$

where we used (4.13) and q-Serre relations in $\mathfrak{sl}(n+1)$. The elements $E_{(ij)}$ are the quantum counterparts of E_{ij} in the defining relations of $\mathfrak{sl}(n+1)$.

This twist deforms

$$F_{qEJ}: \; U_q(\mathfrak{sl}(n+1)) \longrightarrow U_{qEJ}(\mathfrak{sl}(n+1))$$
$$\Delta \qquad\qquad F_{qEJ} \circ \Delta \circ F^{-1}_{qEJ},$$

and reduces to

$$F_{EJ} = \exp(\sum_{i=1}^{n-1} E_{1,i+1} \otimes E_{i+1,n+1} e^{-\frac{1}{2}\sigma_{1,n+1}}) \cdot \exp(\frac{1}{2} H_{1,n+1} \otimes \sigma_{1,n+1})$$

in the limit $q \to 1$, here

$$\sigma_{ij} = \ln(1 + E_{ij}) = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} E_{ij}^{k}.$$

Chapter 5

From quantum Affine algebras to Yangians

Quantum affine algebras are the deformations of affine Kac-Moody algebras. In this chapter we deal with affine $\hat{\mathfrak{sl}}(2)$. This algebra corresponds to the following Cartan matrix

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

If the set of all positive simple roots is $\{\alpha, \delta - \alpha\}$, where δ is the minimal imaginary root, then $U_q(\hat{\mathfrak{sl}}(2))$ is defined by the following relations

$$q^h e_{\pm\alpha} q^{-h} = q^{\pm 2} e_{\pm\alpha} q^{-h}, \qquad q^h e_{\delta-\alpha} q^{-h} = q^{-2} e_{\delta-\alpha},$$

$$e_{\delta-\alpha} e_{-\alpha} = e_{-\alpha} e_{\delta-\alpha}, \qquad [e_\alpha, e_{-\alpha}] = \frac{q^h - q^{-h}}{q - q^{-1}},$$

$$[[[e_\alpha, e_{\delta-\alpha}]_q, e_{\delta-\alpha}]_q, e_{\delta-\alpha}]_q = 0;$$

and the Hopf algebra structure maps are defined as follows

$$\begin{aligned} \Delta(e_{\delta-\alpha}) &= e_{\delta-\alpha} \otimes 1 + q^h \otimes e_{\delta-\alpha}, & \Delta(e_\alpha) &= q^{-h} \otimes e_\alpha + e_\alpha \otimes 1, \\ \Delta(e_{-\alpha}) &= e_{-\alpha} \otimes q^h + 1 \otimes e_{-\alpha}, & \Delta(q^h) &= q^h \otimes q^h; \end{aligned}$$

$$S(e_{\delta-\alpha}) = -q^{-h}e_{\delta-\alpha}, \qquad S(e_\alpha) = -q^h e_\alpha,$$

$$S(e_{-\alpha}) = -e_{-\alpha}q^{-h}, \qquad S(q^h) = q^{-h}\otimes q^{-h}.$$

The Yangian $Y(\mathfrak{sl}(2))$ is a special subalgebra, which can be seen as the limiting case when $q \to 1$ of its quantum analog $Y_q(\mathfrak{sl}(2))$, sitting inside of $U_q(\mathfrak{sl}(2))$, [29].

Definition 5.1. A $q-$Yangian $Y_q(\mathfrak{sl}(2))$ is the minimal Hopf subalgebra in $U_q\hat{\mathfrak{sl}}(2)$ containing

$$U_q(\mathfrak{sl}(2))\bigcup\{e_{\delta-\alpha} + \frac{\eta}{q^{-2}-1}q^{-h}e_{-\alpha}\}.$$

The elementary parabolic twist gives the quantization of $\mathfrak{sl}(3)$-boundary r-matrix [11],

$$2\xi H_{23}^{\perp} \wedge E_{12} - \eta(E_{13}\wedge E_{32} + H_{12}^{\perp}\wedge E_{23}),$$

and can be written as,

$$\exp(H_{12}^{\perp}\otimes(2\sigma_{12}^{-\eta} + \sigma_{23}^{2\xi}))\cdot\exp(\eta E_{32}\otimes E_{13}e^{\sigma_{12}})\cdot\exp(H_{32}\otimes\sigma_{12}^{-\eta}). \tag{5.1}$$

It is interesting to investigate the connection of this element to the twisting of $U_q(\hat{\mathfrak{sl}}(2))$ built in [14]

$$F = (1\otimes 1 - (2)_{q^2}(a\cdot 1\otimes e_{\delta-\alpha} + b\cdot q^{-h}\otimes q^{-h}e_{-\alpha}))_{q^2}^{(-\frac{1}{2}h\otimes 1)}, \tag{5.2}$$

where

$$(1-u)_q^{(v)} = 1 + \sum_{k>0}\frac{(-v)_q(-v+1)_q\cdots(-v+k-1)_q}{(k)_q!}u^k = e_q(u)(e_q(uq^{-v}))^{-1},$$

and $(k)_q = \frac{q^k-1}{q-1}$.

Definition 5.2. We say that $\Phi_1 \sim \Phi_2$ if

$$\Phi_2 = (Q\otimes Q)\Phi_1\Delta(Q^{-1}), \text{ for some } Q\in U_q(\mathfrak{g}).$$

Proposition 11.

$$F \sim e_{q^2}(-(2)_{q^2}^2 ab\cdot q^{-h}e_{\delta-\alpha}\otimes q^{-h}e_{-\alpha}).$$

Proof. The element F admits the representation

$$F = e_{q^2}(c \cdot 1 \otimes e_{\delta-\alpha} + d \cdot q^{-h} \otimes q^{-h}e_{-\alpha}) \cdot$$
$$e_{q^{-2}}(q^{-2} \cdot (c \cdot q^h \otimes e_{\delta-\alpha} + d \cdot 1 \otimes q^{-h}e_{-\alpha})),$$

where we have introduced the new coefficients c, d to simplify notations. If we apply (4.13), then the twist F can be written as:

$$e_{q^2}(c \cdot 1 \otimes e_{\delta-\alpha}) \cdot e_{q^{-2}}(q^{-2}d \cdot 1 \otimes q^{-h}e_{-\alpha}) \cdot e_{q^2}(d \cdot q^{-h} \otimes q^{-h}e_{-\alpha}) \cdot e_{q^{-2}}(q^{-2}c \cdot q^h \otimes e_{\delta-\alpha}).$$

Let us denote

$$VW = e_{q^2}(c \cdot e_{\delta-\alpha}) \cdot e_{q^{-2}}(q^{-2}d \cdot q^{-h}e_{-\alpha});$$

then, we have

$$F = (VW \otimes VW)e_{q^2}(-cd \cdot q^{-h}e_{\delta-\alpha} \otimes q^{-h}e_{-\alpha})\Delta((VW)^{-1}),$$

where we used the *Five term relation* [9]:

$$e_{q^2}(x)e_{q^2}(y) = e_{q^2}(y)e_{q^2}(-yx)e_{q^2}(x), \qquad xy = q^2yx.$$

The element

$$\Phi = e_{q^2}(-cd \cdot e_{\delta-\alpha}q^{-h} \otimes q^{-h}e_{-\alpha})$$

is a twist that can be checked directly by using the costructure

$$\Delta(q^{-h}e_{\delta-\alpha}) = q^{-h}e_{\delta-\alpha} \otimes q^{-h} + 1 \otimes q^{-h}e_{\delta-\alpha},$$
$$\Delta(q^{-h}e_{-\alpha}) = q^{-h}e_{-\alpha} \otimes 1 + q^{-h} \otimes q^{-h}e_{-\alpha},$$
$$\Delta(q^{-h}) = q^{-h} \otimes q^{-h},$$

thus giving an independent proof that (5.2) is a twist. □

To calculate the quasi-classical limit for $q \longrightarrow 1$ of (5.2), one needs to perform the change of variables [14]:

$$f_1 = e_{\delta-\alpha} + \frac{\eta}{q^{-2}-1}q^{-h}e_{-\alpha}, \qquad f_0 = q^{-h}e_{-\alpha}.$$

The Hopf algebra structure is now

$$[h, f_1] = -2f_1, \qquad [h, f_0] = -2f_0, \qquad f_1 f_0 - q^{-2} f_0 f_1 = -\eta f_0^2,$$

$$\Delta(f_0) = f_0 \otimes 1 + q^{-h} \otimes f_0, \qquad \Delta(f_1) = f_1 \otimes 1 + q^h \otimes f_1 + \eta q^h (h)_{q^{-2}} \otimes f_0.$$

Using this basis, (5.2) can be rewritten as

$$(1 \otimes 1 - (2)_{q^2} \xi (1 \otimes f_1 + \eta (h/2)_{q^{-2}} \otimes f_0))_{q^2}^{(-\frac{1}{2} h \otimes 1)}.$$

The contracted twist

$$(1 \otimes 1 + 2\xi (1 \otimes \hat{f}_1 + \eta \frac{\hat{h}}{2} \otimes \hat{f}_0))^{-\frac{1}{2} \hat{h} \otimes 1} \qquad (5.3)$$

is defined on the Hopf algebra (which we denote as being $Y(\mathfrak{n}_+)$) generated by $\hat{h}$, $\hat{f}_0$, $\hat{f}_1$.

Proposition 12. *There exists such a twist Φ in $U(\mathfrak{sl}(3))$ and an isomorphism ι so that*

$$\iota : Y(\mathfrak{n}_+) \cong A_\Phi,$$

and

$$A_\Phi = \{(H_{12}^\perp)^k (E_{13} e^{-\sigma_{12}^{-\eta}})^l (E_{23})^m | k, l, m \geq 0\}_\Phi$$

is a Hopf algebra with a costructure deformed by Φ.

Proof. First, following [23] we rewrite the elementary parabolic twist

$$F_\mathcal{P} = \exp(\mathrm{ad} H_{12}^\perp \otimes \sigma_{12}^{-\eta})(\exp(H_{12}^\perp \otimes \sigma_{23}^{2\xi}))\exp(\eta E_{32} \otimes E_{13} e^{-\sigma_{12}^{-\eta}})\exp(H_{23}^\perp \otimes \sigma_{12}^{-\eta})$$

in the form

$$(1 \otimes 1 + 2\xi \cdot 1 \otimes E_{23} - 2\xi\eta H_{12}^\perp \otimes E_{13} e^{-\sigma_{12}^{-\eta}})^{H_{12}^\perp \otimes 1} \exp(\eta E_{32} \otimes E_{13} e^{-\sigma_{12}^{-\eta}})\exp(H_{23}^\perp \otimes \sigma_{12}^{-\eta}).$$

The twisting factor

$$(1 \otimes 1 + 2\xi \cdot 1 \otimes E_{23} - 2\xi\eta H_{12}^\perp \otimes E_{13} e^{-\sigma_{12}^{-\eta}})^{H_{12}^\perp \otimes 1}$$

is defined on A_Φ with

$$\Phi = \exp(\eta E_{32} \otimes E_{13}e^{-\sigma_{12}^{-\eta}})\exp(H_{23}^{\perp} \otimes \sigma_{12}^{-\eta}).$$

The costructure of A_Φ is given by

$$\Delta(H_{12}^{\perp}) = H_{12}^{\perp} \otimes 1 + 1 \otimes H_{12}^{\perp},$$

$$\Delta_\Phi(E_{13}e^{-\sigma_{12}^{-\eta}}) = E_{13}e^{-\sigma_{12}^{-\eta}} \otimes 1 + 1 \otimes E_{13}e^{-\sigma_{12}^{-\eta}},$$

$$\Delta_\Phi(E_{23}) = E_{23} \otimes 1 + 1 \otimes E_{23} - 2\eta H_{12}^{\perp} \otimes E_{13}e^{-\sigma_{12}^{-\eta}}.$$

We define a Hopf algebra isomorphism ι on the generators as the follows:

$$\begin{aligned} \iota(\hat{h}) &= -2H_{12}^{\perp}, \\ \iota(\hat{f}_0) &= E_{13}e^{-\sigma_{12}^{-\eta}}, \\ \iota(\hat{f}_1) &= E_{23}. \end{aligned}$$

□

Now, we find an expression giving the quantum analogue of the elementary parabolic twist. The quantum analogue of Φ is given by

$$\Phi_q = (e_{q^2}(\eta E_{(12)}) \otimes e_{q^2}(\eta E_{(12)}))\Psi\Delta(e_{q^{-2}}(q^{-2}\eta E_{(12)})),$$

where Ψ is a twist that is chosen so that

$$\Delta_\Psi(E_{(12)}) = q^{-2H_{23}^{\perp}} \otimes E_{(12)} + E_{(12)} \otimes 1 + (1-q^{-2})q^{-2H_{13}^{\perp}}E_{(32)} \otimes E_{(13)}q^{-H_{23}}.$$

It is straightforward to check that

$$\Psi = q^{H_{12}^{\perp} \otimes H_{23}} e_{q^2}((q-q^{-1})^2 E_{(32)} \otimes E_{(23)}).$$

Applying Φ_q, we calculate

$$\begin{aligned} \Delta_{\Phi_q}(E_{(23)}q^{-H_{23}}) \;=\;& q^{2H_{12}^{\perp}} \otimes E_{(23)}q^{-H_{23}} + E_{(23)}q^{-H_{23}} \otimes 1 + \\ & \eta\frac{q^{-2H_{12}^{\perp}} - q^{2H_{12}^{\perp}}}{1-q^2} \otimes \frac{1}{1-\eta E_{(12)}} E_{(13)}q^{-H_{23}}, \end{aligned}$$

and

$$\Delta_{\Phi_q}(\frac{1}{1-\eta E_{(12)}}E_{(13)}q^{-H_{23}}) = q^{-2H_{12}^{\perp}}\otimes\frac{1}{1-\eta E_{(12)}}E_{(13)}q^{-H_{23}}+\frac{1}{1-\eta E_{(12)}}E_{(13)}q^{-H_{23}}\otimes 1.$$

It is easy to check that

$$\overline{\iota_q(h)} = -2H_{12}^{\perp},$$
$$\overline{\iota_q(f_0)} = \frac{1}{1-\eta E_{(12)}}E_{(13)}q^{-H_{23}}, \qquad \overline{\iota_q(f_1)} = E_{(23)}q^{-H_{23}}, \tag{5.4}$$

where $\bar{\iota}_q$ is an isomorphism between the q-analogues of $Y(\mathfrak{n}_+)$ and A_Φ, which can be seen as a quantum version of ι, $\overline{()}$ is the conjugation

$$\bar{q} = q^{-1}.$$

Finally, we formulate the result:

Proposition 13. *The twist*

$$F_{\mathcal{P}}^q = \overline{(\iota_q\otimes\iota_q)\Big((1\otimes 1-(2)_{q^2}\xi(1\otimes f_1+\eta(h/2)_{q^{-2}}\otimes f_0))_{q^2}^{(-\frac{1}{2}h\otimes 1)}\Big)}\Phi_q,$$

gives (5.1) in the limit $q\to 1$.

Proof. The first factor in $F_{\mathcal{P}}^q$ goes to (5.3) at $q\to 1$ due to (5.4), and the second one goes to Φ according to *Example 4.1.* □

Conclusion

The problem of reconstruction of quantum twists from quasiclassical ones is far from its completion. The further investigation could go to higher dimensions and relate skew-symmetric Cremmer-Gervais $r-$matrices to degeneration of appropriately defined affine twists. In this thesis we made the first step in this direction when we related the elementary parabolic twist with the twist built by Khoroshkin, Stolin and Tolstoy. Although it was proven by Etingof and Kazhdan [7], that any bialgebra can be quantized by some twist, the proof that any quasiclassical twist comes from degeneration of the appropriately defined quantum one is yet to be given. In particular we need to classify all singular twists leading to such degenerations. The singularity cancellation conditions guarantee that the limiting twists are nonsingular. We had an example of a singularity cancellation condition when we were investigating extended Jordanian twists. In general, the singularity cancellation conditions could be much more complicated than

$$(X \otimes 1)(\Delta(X) - X \otimes 1) = q^2(\Delta(X) - X \otimes 1)(X \otimes 1).$$

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