ORDERLY BROADCASTING IN MULTIDIMENSIONAL TORI

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Abstract

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In this thesis, we describe an ordering of the vertices of a multidimensional torus and study the upper bound on the orderly broadcast time. Along with messy broadcasting, orderly broadcasting is another model where the nodes of the network have limited knowledge about their local neighborhood. However, while messy broadcasting explores the worst-case performance of broadcast schemes, orderly broadcasting, like the classical broadcast model, is concerned with finding an ordering of the vertices of a graph that will minimize the overall broadcast time.

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Chapter 1

Introduction

Broadcasting is the process of disseminating information in an interconnection network whereby a message, orignating from one of the nodes, is transmitted to all the other nodes of the network. A *broadcast problem* is the problem of determining the minimum amount of time needed to transmit a piece of information to every node in the communication graph. In what follows, we discuss the broadcast problem of a 2-dimensional torus under the *orderly broadcasting* model.

1.1 Motivation and Definitions

Computer networks provide us with an important tool for data sharing. Substantial effort has been made recently to develop effective techniques to make such communications reliable, efficient and fast. All these properties depend highly on the performance of information dissemination between network components.

Information dissemination also plays an important role in parallel processing, where a single problem is divided into sub problems, which are solved by multiple processors simultaneously. Each of these processors needs to transfer its results to the others for further computing, and the effectiveness of the result of the whole computation process relies on the efficiency of the information exchange among these processors. Moreover, in such a multiprocessing environment each processor may have its own memory cache while sharing a common memory. *Cache coherence* is the problem that arises when copies of the same memory block are present in the caches of one or more processors. When a processor modifies a memory block, the other processors might continue to access the old copy of the block that is in their caches. The protocol that manages these caches and ensures that no data is lost or overwritten relies heavily on effectively broadcasting the information of any change in the caches of all processors.

We will model a communication network as a connected graph $G = (V, E)$ consisting of a set $V = \{v_0, v_1, \ldots v_n\}$ of vertices (nodes) and a set of edges E (communication lines) connecting these vertices. Two vertices $u \in V$ and $v \in V$ are *adjacent* or are *neighbors* when they are connected by an edge $e \in E$ such that $e = (u, v)$. We define the degree of a vertex v, $\delta(v)$, to be the number of neighbors of vertex v. A *path* P of a graph G is a sequence of vertices of the form $P = (v_1, v_2, \ldots v_k)$ $(k \geq 2)$, such that every $(v_i, v_{i+1}) \in E$, $1 \leq i \leq k$. A *cycle* C of length k is a path $(v_1, v_2, \ldots v_k)$ where $(v_1, v_k) \in E$. A graph $G = (V, E)$ is said to be *connected* if for every pair of vertices $u, v \in V$ there exists a path that connects u to v. The *distance*, $d(u, v)$, between two vertices $u \in V$ and $v \in V$ is the number of edges of the shortest path between u and v. The *diameter* $D(G)$ of a graph G is the maximum distance between all pairs of vertices of G.

We say that two vertices are in a *call* when they communicate a message through their incident edge. Communication networks can be classified into two major categories, based on the number of simulataneous calls that each vertex can make within the same time unit [17] :

- **Processor-bound** called *1-port* or *whispering* in which a vertex can call only one neighbor at a time.

- **Link-bound** called *n-ports* or *shouting* in which a vertex can call up to n of its neighbors simultaneously. Notice that the processor-bound is a special case of the link-bound, where $n = 1$.

In the discussions below, we will assume the *1-port constant* model of communication, during which one call can be made between two adjacent vertices in one time unit. We define a *round* to be a set of parallel calls that are made simultaneously by many vertices. The set of all calls made between the pairs of vertices in a graph during a sequence of time units is called a *protocol* or *communication strategy.*

There are three major types of information dissemination problems that have been widely studied. A survey on the results of some of these problems can be found in [29, 28, 17]. These types of information dissemination problems are : *gossiping, broadcasting* and *accumulation.*

1. Gossiping

Gossiping refers to the information dissemination problem in which each node of the communication network knows a unique piece of information and must transmit to the rest of the nodes in the network. More formally, given a graph $G = (V, E)$, for all $v \in V$, $I(v)$ is a piece of information residing in v. The problem is to find a communication strategy, called a *gossip protocol,* such that each vertex in V learns the whole cumulative message in minimum time possible.

2. Broadcasting

Broadcasting is the problem of determining the minimum amount of time required for one vertex of a graph to transmit a piece of information to the rest of the vertices.

More formally, given a graph $G = (V, E)$, for a certain $v \in V$, $I(v)$ is a piece of information residing in v which is unknown to all vertices in $V - \{v\}$. The problem is to find a communication strategy, called a *broadcast protocol*, such that the remaining vertices in V learn the piece of information $I(v)$ in the minimum time possible.

3. Accumulation

Accumulation is the dissemination problem in which each vertex of the communication graph knows a unique piece of information and must transmit to a single given vertex in the network. That is, given a graph $G = (V, E)$, for a certain $v \in V$, $I(v)$ is a piece of information residing in v. Let, for any $x, y \in V$, the piece of information $I(x)$ and $I(y)$ be "disjoint". The set $I(G) = \{I(u) | u \in V\}$ is called the *cumulative message* of G. The problem is to find a communication strategy, called a *accumulation protocol*, such that the vertex v learns the cumulative message of G.

The above mentioned communication problems frequently arise in various applications ranging from parallel computing to communications in computer networks. In this thesis, we are interested in the broadcasting problem, which we will discuss in more detail in the following section.

1.2 Broadcast Models

Broadcasting is a major variant of the Gossip problem and was first introduced by Slater in 1977. It is an one-to-all information dissemination process during which a vertex called the *originator* sends its message to the remaining vertices of the graph through a set of calls. Broadcasting is complete when all the vertices of the graph are informed. It is required that broadcasting is completed as fast as possible and subject to the following constraints:

i) each call involves only two vertices.

ii) each call requires one unit of time.

iii) a vertex can participate in only one call per unit of time.

iv) a vertex can only call an adjacent vertex.

Broadcast models can be divided into two major groups. The first is known as *classical broadcasting* and is the one on which initial research concentrated. Various classical broadcasting models have been introduced [8, 13, 29, 28, 17], all of which deal with the issue of finding a scheme whereby information dissemination takes the least amount of time. In classical broadcasting, it was assumed that every vertex has the knowledge of the graph topology, the originator of the message and the time at which it was sent. Based on this information, and in order to minimize the broadcast time, each vertex of the graph transmits the message in the most *clever* way.

The second major type of broadcasting is known as *messy broadcasting* [2]. Unlike the classical model, messy broadcasting deals with analyzing the worst-case performance of broadcast schemes.

Below we describe some of the broadcast models which were studied extensively in each group. We give some bounds on popular graphs structures.

1.2.1 Classical Broadcasting

Given an originator $u \in V$, we define the broadcast time of vertex u, $b(u)$ in the classical model to be the minimum number of time units required to complete broadcasting from vertex u [8, 13]. The classical broadcast time of a graph $G = (V, E)$ is:

$$b(G) = max\{b(u) | u \in V\}$$

A trivial lower bound on the broadcast time is the diameter of the graph. However, since each vertex can inform one of its adjacent vertices in one time unit, then at each time unit the number of informed vertices can at most be doubled. Thus, after k time units, the number of informed vertices is bounded by 2^k. Thus, the lower bound for broadcasting a message in any graph will be $\lceil log_2(|V|) \rceil$.

As mentioned above, in the *classical broadcasting* model every vertex is required to have knowledge of the network topology, the originator of the message, and the time at which the it was sent. It was also assumed that every vertex broadcasts the message using an optimal scheme.

The *classical broadcasting* model was studied in various types of graphs. Here we list a few results for commonly used interconnection networks :

1. Complete graph K_n : $b(K_n) = \lceil log_2 n \rceil$.

2. Hypercube H_m : $b(H_m) = m$.

3. Complete $k-$ary tree of height m, T_k^m : $b(v_0, T_k^m) = k.m$ (v_0 is the root).

4. Cube Connected Cycles of dimension k, CCC_k : $b(CCC_k) = \left\lceil \frac{5k}{2} \right\rceil$.

Determining the broadcast time of a vertex in an arbitrary graph is known to be NP-complete [39].

Special interest was given to the construction of *minimum broadcast graphs (mbg)* which are graphs on n vertices that have the minimum number of edges, denoted by $B(n)$, that required to broadcast in $\lceil log_2 n \rceil$ time. Since edges stand for the communication lines, these graphs are the cheapest possible communication networks.

It turns out that *mbg*s are extremely difficult to find. In 1979, Farley *et al* [13] determined the values of $B(n)$ for $n \leq 15$ and showed that $B(2^k) = k2^{k-1}$. Today, $B(n)$s are known for $n = 2^k$ and $n = 2^k - 2$ and for some values of $n \leq 63$ [3, 4, 6, 15, 31, 36, 38] and for $n = 127$ [40].

Also studied were different variations of the initial scheme. Below we describe some of them and give some bounds.

k-Broadcasting

k-broadcasting is a generalization of the classical broadcast model. In k-broadcasting, an informed vertex can make up to k simultaneous calls in one time unit. Since the number of informed vertices can, at most, be multiplied by $k+1$ at each round, then the k-broadcast time of a graph G of n vertices is $b_k(G) \geq \lceil log_{k+1} n \rceil$.

Most of the work in k-broadcasting concentrates on finding minimum broadcast graphs ($B_k(n)$) which allow minimum time k-broadcasting. Grigni and Peleg [18] showed that $B_k(n) \in \Theta(kL_k(n)n)$, where $L_k(n)$ is the exact number of consecutive leading k's in the $(k+1)$-ary representation of $n-1$. Lazard [33] found values for $B_2(n), B_3(n)$ and $B_4(n)$ for small n. Later, König and Lazard [30] constructed minimum k-broadcast graphs for all $k+3 \leq n \leq 2k+3$. Bounds on $B_k(n)$ were improved in [32]. Harutyunyan and Liestman [24] give the best known lower and upper bounds on $B_k(n)$ to date. They also solve the problem of constructing optimal k-broadcast trees on n vertices that have minimum possible k-broadcast time [25]. In [27], Harutyunyan and Shao describe a heuristic that works well in practice. Their heuristic gives exact time for grid, torus and hypercube graphs.

Fault-Tolerant Broadcasting

Fault-tolerant broadcasting deals with the problem of broadcasting a message in a network when one or more communication lines or nodes fail to operate. In this model, it is assumed that faults are not detected during broadcast, and in order to tolerate k faults, the broadcast scheme must consist of $k+1$ edge-disjoint calling paths from the originator to every other node. Let $T_k(n)$ be the minimum time required to broadcast in the presence of k faults in a graph G on n vertices. Then, it was shown in [33, 34] that :

- $T_{0,1}(n) = \lceil log_2 n \rceil + 1$
- $T_{0,2}(n) = \lceil log_2 n \rceil + 2$ for $n \neq 4i + 3$
- $T_{0,k}(n) \geq \lceil log_2 n \rceil + k$ for appropriate values of k and n

Constructruction of optimal fault-tolerant broadcast graphs with n vertices and k faults were studied in [5, 1].

Multiple Originator Broadcasting

Another variant of broadcasting considers the problem of finding the minimum number of message originators necessary to complete broadcasting in a specified amount of time. In 1981, Farley and Proskurowski [14] settled the multiple-originator broadcast problem in arbitrary trees in at most t time units (for any t), and presented a linear algorithm for decomposing a tree into a minimum number of subtrees such that broadcasting can be completed in at most t time units in each subtree.

Multiple Message Broadcasting

Multiple message broadcasting deals with the problem broadcasting multiple messages in a graph. It was observed that simple modifications of one-message broadcast schemes will be too inefficient for multiple message broadcasting. This model was first studied by Chinn, Hedetniemi, and Mitchell [7] in 1979 and Farley [12] in 1980 for complete graphs. The functions studied in this model were the following :

$P(m,t)$: The maximum number of vertices that can be informed of m messages in t time units.

$M(p,t)$: The maximum number of messages that can be broadcast to p people in t time units.

$T(m,p)$: The minimum number of time units necessary to broadcast m messages to p vertices.

Some of the obtained results are summarized below :

1. $P(1,t) = 2^t$

2. $P(2,t) = 2^{t-2}$ for $t > 2$

3. $P(3,t) = 2^{t-4} + 2$ for $t > 4$

4. $P(4,t) = 2^{t-6} + 2$ for $t > 6$

5. $$M(p,t) = \begin{cases} \left\lfloor \dfrac{t-q+1}{2} \right\rfloor & \text{for } p \text{ odd where } q = \lfloor log_2 p \rfloor \\ \left\lfloor \dfrac{p^{t-q} + 2^{q+1} - 2}{2(p-1)} \right\rfloor & \text{for } p \text{ even} \end{cases}$$

6. $T(m,p) = \begin{cases} 2m - 1 + \lfloor log_2 p \rfloor & \text{for } p \text{ odd} \\ 2m + \lfloor log_2 p \rfloor - \left\lfloor \dfrac{m - 1 + 2^{log_2 p}}{p/2} \right\rfloor & \text{for } p \text{ even} \end{cases}$

New results are given in multiple message broadcasting are given in [21, 22].

1.2.2 Messy Broadcasting

In their paper published in 1994, Ahlswede *et al* [2] introduced a broadcast model, called *messy broadcasting*, in which every vertex knows nothing about the network topology, the originator or the time at which the message was sent, and makes only local decisions to send the message. That is, at every round, each informed vertex broadcasts to a randomly chosen neighbor, but can receive information from any number of its neighbors simultaneously. Messy broadcasting is concerned with the worst-case broadcast performance. The authors introduced the following three models of *messy broadcasting*:

1. *Model* M_1: every informed vertex knows the state of each of its neighbors: informed or uninformed. In this model, each informed vertex transmits the message to one of its uninformed neighbors, if any, in each time unit.

2. *Model* M_2: Every informed vertex knows from which vertex (vertices) it received the broadcast message and to which neighbors it has sent the message. In this model, each informed vertex transmits the message to one of its neighbors that it has not yet informed, if any, in each time unit.

3. *Model* M_3: Every informed vertex knows to which neighbors it has sent the message. In this model, each informed vertex transmits the message to one of its neighbors that it has not yet informed, if any, in each time unit.

Given a graph $G = (V, E)$, the messy broadcast time of a vertex $u \in V$, $b_i^m(u)$ is the maximum number of time units required to complete broadcasting from vertex u under the model i (where $i \in \{1, 2, 3\}$) [2, 23, 9, 20, 19]. The messy broadcast time of a graph $G = (V, E)$ is:

$$b^m(G) = max\{b^m(u)|u \in V\}$$

Let $b_1^m(G)$, $b_2^m(G)$, $b_3^m(G)$ be the broadcast time of a graph G under the models M_1, M_2, M_3 respectively.

It was shown in [2] that $b_1^m(G) \leq b_2^m(G) \leq b_3^m(G)$ for any connected graph G. Also, the following results were discussed:

- $b_2^m(G) \leq (D(G) \times (k-1)) + 1$
- $b_3^m(G) \leq D(G) \times k$

(where k is the maximum degree over all vertices $v \in V$).

The paper also dealt with the problem of constructing optimal graphs in which $b_i^o(G)$ will yield minimum time.

In [23, 9, 20] the following results were presented:

1. Complete graph, K_n : $b_1^o(K_n) = b_2^o(K_n) = b_3^o(K_n) = n - 1$

2. Path of length n, P_n : $b_1^o(P_n) = b_2^o(P_n) = n - 1$ and $b_3^o(P_n) = 2n - 3$

3. Cycle of length n, C_n : $b_1^o(C_n) = b_2^o(C_n) = \left\lceil \frac{n}{2} \right\rceil$ and $b_3^o(C_n) = n - 1$

4. Hypercube H_m :

$$b_3^o(H_m) = \frac{m(m+1)}{2} \quad \text{and}$$
$$b_2^o(H_m) = \frac{m(m-1)}{2} + 1 \quad \text{and}$$
$$\frac{5}{2}d \leq t_1(H_m) \leq \frac{m(m-1)}{2} + 1$$

5. Directed torus of dimension k, $T_{n_1 \times n_2 \times \ldots n_k}$

$$b_3^o(T_{n_1 \times n_2 \times \ldots n_k}) = (n_1 - 1) + 2(n_2 - 2) + \ldots + k(n_k - 1),$$

where $2 \leq n_1 \leq n2 \leq \ldots \leq n_k$ and $k \geq 2$

$$b_2^o(T_{n_1 \times n_2 \times \ldots n_k}) = (n_1 - 1) + 2(n_2 - 2) + \ldots + k(n_k - 1),$$

where $3 \leq n_1 \leq n_2 \leq \ldots \leq n_k$ and $k \geq 2$

Comparing the two broadcast schemes described above, it can be observed that the major difference between messy broadcasting and the classical broadcast model is that in the messy broadcast scheme, the vertices know nothing about the topology, and at each broadcast round transmit the message to a randomly selected neighbor.

In practice however, it is not realistic to require for each node of a network to know the network topology or to make decisions based on a set of stored protocols.

In many cases, the nodes have primitive structures with small memories that cannot store such information or make intelligent decisions. On the other hand, building networks in which the nodes have no decision-making responsibility is much simpler and more robust. These were the reasons that made the study of messy broadcasting interesting, and eventually led to the idea of *orderly broadcasting.*

1.3 Orderly Broadcasting

In messy broadcasting, when a vertex u receives a message, it randomly chooses a neighbor at each round and sends the message to that chosen neighbor. Ultimately, the neighbors of vertex u will be informed in some order. This means that if we randomly number the neighbors of each vertex $u \in V$ and let u send the message first to the neighbor numbered 1, then 2, ... etc., then we would have simulated an instance of messy broadcasting. The overall broadcast time of the graph in this case will depend on the way in which the neighbors of each vertex are numbered. A different ordering of the numbers might yield a different broadcast time; and the question that can be asked here is : what is the ordering that will give the minimum possible broadcast time for a given graph?

The above question has lead to the idea of *orderly broadcasting* [10, 11, 26] , which deals with the problem of finding an ordering for each vertex u of a given graph G that will minimize the overall broadcast time of G (Figure 1).

In what follows, we will state a more formal definition of the orderly broadcast model and give known bounds on some studied graphs.

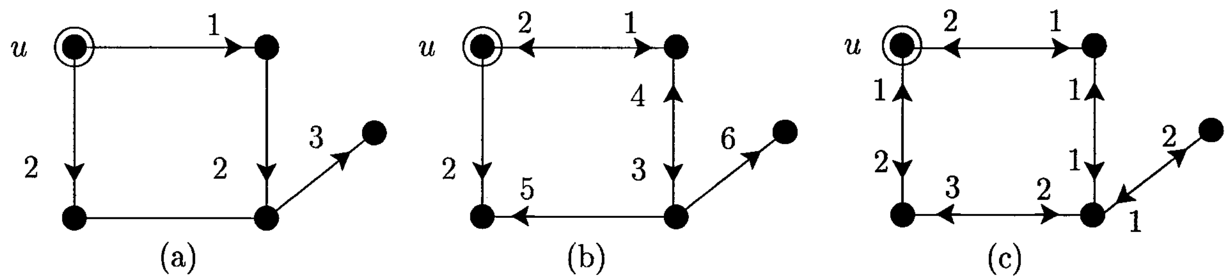

Figure 1: The three broadcast models: (a) Classical broadcasting with $b(u) = 3$ (b) Messy broadcasting with $b^m(u) = 6$ and (c) Orderly broadcasting with $b^o(u) = 4$. Note that in (c) the numbers represent the ordering of the edges and not the time at which the vertex receives the message.

A *symmetric digraph* is a graph $G = (V, E)$ where every edge $(u, v) \in E$ in the undirected graph becomes two edges (u, v), (v, u) in the digraph. We define the *out-edge* of a vertex u to be the edge $(u, v) \in E$ (Figure 2). Broadcasting in a symmetric digraph is identical to broadcasting in the associated undirected graph. Hence, in all further discussion we will model our communication network as a symmetric digraph.

Figure 2: An edge (u, v) in an undirected graph (a) becomes two edges (u, v) and (v, u) in a *symmetric digraph* (b).

Given a symmetric digraph $G = (V, E)$, an ordering Π_u of a vertex $u \in V$ of degree d is the assignment of distinct time units $1, 2, \ldots d$, called *labels*, to the outedges of u. We use $\Pi(u, v)$ to denote the label assigned to the edge from u to v. When every vertex u of a graph G has an ordering Π_u, we say that G has ordering Π.

Suppose $G = (V, E)$ is a graph with ordering Π and let $u \in V$ such that $v_1, v_2, \ldots, v_k \in V$ are the neighbors of u. Orderly broadcasting from originator u proceeds as follows: At time 0, the originator u learns the message. Vertex v_1 learns the message at time $\Pi(u, v_1)$; v_2 learns the message at time $\Pi(u, v_2); v_k$ learns the message at time $\Pi(u, v_k)$. In general, for any vertex $u \in V$, if u learns the message at time t, it sends the message along its outedges ordered i at time $t + i$, where $i = 1, 2, \ldots$. Orderly broadcasting is complete when every vertex has received the message. The time at which this happens when the originator is vertex u is denoted by $b^{\Pi}(u)$. The broadcast time of graph $G = (V, E)$ with ordering Π is :

$$b^{\Pi}(G) = max\{b^{\Pi}(u) | u \in V\}$$

The orderly broadcast time of graph G is the minimum broadcast time over all possible orderings Π of the vertices of G. That is,

$$b^{o}(G) = min_{\Pi}\{b^{\Pi}(G)\}$$

The orderly broadcast time is known only for a limited number of graphs. The problem has been studied in various types of graphs in [10, 11] and in [26] separately. In [11], the orderly broadcast model described here is referred to as *non-adaptive model of broadcasting with universal lists.* In the same paper, the authors also introduce a variant of orderly broadcasting called the *adaptive model* in which an informed vertex retransmits the message in the given order, but skips all neighbors from which it received the message. In this thesis we consider only the *non-adaptive* model descibed in [11].

Below we present some of the studied results :

1. For any tree T, $b^o(T) \leq b(T) + \left\lceil \frac{D(T)}{2} \right\rceil$

2. For a path P_n on n vertices, $b^o(P_n) = \left\lceil \frac{3n-4}{2} \right\rceil$

3. For a cycle C_n on n vertices, $b^o(C_n) = \left\lceil \frac{2n}{3} \right\rceil$

4. For a complete graph K_n on n vertices, $b^o(K_n) \leq \lceil logn \rceil + 2 \left\lceil \sqrt{logn} \right\rceil$

5. For a grid $G_{m \times n}$ on mn vertices, $b^o(G_{m \times n}) = m + n - 1$

6. For a torus $T_{m \times n}$ with m rows and n columns,

$$b^o(T_{m \times n}) \leq \begin{cases} D(T_{m \times n}) + 6 & \text{if } n \text{ is even} \\ D(T_{m \times n}) + 7 & \text{if } n \text{ is odd} \end{cases}$$

   where $D(T_{m \times n})$ is the diameter of the torus.

Contrary to what it may seem, finding an optimal ordering for the vertices of a graph is a difficult problem. Even for a simple structure like a cycle, an optimum ordering is not yet known.

In this thesis, we first describe an ordering Π for a 2-dimensional torus $T_{m \times n}$ and discuss the upper and lower bounds on the orderly broadcast time. We then generalize the result for d-dimensional tori $T_{n_1 \times n_2 \times \ldots \times n_d}$ and state the orderly broadcast time in this multidimensional graph structure.

1.4 Thesis Outline

The remainder of this thesis is structured as follows: In chapter 2 we discuss the lower bound on orderly broadcast time for 2-dimensional tori. In chapter 3 we describe an ordering for the vertices of a 2-dimensional torus, and discuss the upper bound that this ordering produces. An ordering for multidimensional tori and the upper bound for this ordering are discussed in chapter 4. Finally, in the last chapter we give a summary of the results and discuss future work in this area.

Chapter 2

Lower Bound on 2D Tori

In this chapter we show a lower bound on the orderly broadcast time $b^o(T_{m\times n})$ of a 2-dimensional torus under some optimal ordering Π.

2.1 Definitions

A 2-dimensional torus $T_{m\times n} = (V, E)$ with m rows and n columns is a connected graph on mn vertices with $2mn$ edges, such that:

$V(T_{m\times n}) = \{(i,j)|0 \leq i \leq m-1 \text{ and } 0 \leq j \leq n-1\}$

$E(T_{m\times n}) = \{((u,v),(p,q))|p = u \pm 1 \bmod m$ or

$q = v \pm 1 \bmod n$ where $(u,v) \in V, (p,q) \in V\}$

The diameter of a 2D torus is:

$$D(T_{m\times n}) = \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$$

We define $u \in V$ to be the diametric vertex of $v \in V$ if $dist(u,v) = D(T_{m\times n})$.

Given a path $P = (v_1, v_2, \ldots v_k)$, we say that a vertex $v_i \in P$ when $1 \le i \le k$

If $u \in V$ and $v \in V$ are adjacent such that $k = \Pi(u,v)$, then $u \xrightarrow{k} v$ implies that if u receives the message at time t, then it will send it to v at time $t+k$. This means that the time at which any vertex $v \in V$ receives the message from a vertex $u \in V$ is simply the sum of the labels of the edges of the path that takes the message from u to v. For example, if $u_0 \in V$ is the originator and we have the path:

$P_1 : u_0 \xrightarrow{k_1} u_1 \xrightarrow{k_2} u_2 \xrightarrow{k_3} \ldots \xrightarrow{k_n} u_n$ (for all $u_i \in V$)

then, u_n will receive the message through P_1 at time $t = \sum_{i=1}^{n} k_i$

Here we make the following observations:

Observation 2.1.1

$b(G) \le b^o(G) \le b^m(G)$ for any connected graph G.

(Here, $b^m(G)$ refers to the messy broadcast time of G under the model M_3).

Observation 2.1.2

Let $b^o_\Pi(G)$ *denote the orderly broadcast time of graph* G *under some ordering* Π*. If* $\Pi' \subset \Pi$ *then,*

$$b^o_\Pi(G) \leq b^o_{\Pi'}(G)$$

Observation 2.1.3

$b(G) \leq b(G')$ *for any connected graph* G *where* G' *is a spanning subtree of* G*.*

Observation 2.1.4

Let $u, v_1, v_2 \in V$ *such that* u *is at distance* d *from each of* v_1 *and* v_2

If u *receives the message at time* 0*, then* v_1 *can be informed at time* d *and* v_2 *can be informed not earlier than time* $d+1$

We know that at round i of any broadcast scheme, at most one vertex at distance i from the originator u can be informed. Therefore, if vertex v_1 is informed at round d, then v_2 can be informed in the next round, $d+1$

Lemma 2.1.1

Given any cycle C_i *in* $T_{m\times n}$ *of length* $2D(T_{m\times n})$*, there exists a vertex* $u \in C_i$ *such that the diametric vertex* u *is* $v \notin C_i$*.*

Proof :

Consider the torus $T_{m\times n}$ (Figure 3) where $T_{m\times n}$ is represented as a wrapped-around mesh. Let $u_{i,j}$ represent the vertex (i,j) where $0 \le i \le m-1$ and $0 \le j \le n-1$.

The diameter of $T_{m\times n}$ is $D(T_{m\times n}) = \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$, which is basically the distance between the corner vertex and the center vertex. That is,

$$D(T_{m\times n}) = dist(u_{0,0}, u_{0,\lfloor \frac{n}{2} \rfloor}) + dist(u_{0,\lfloor \frac{n}{2} \rfloor}, u_{\lfloor \frac{m}{2} \rfloor,\lfloor \frac{n}{2} \rfloor})$$

This also means that the diametric vertex of $u_{0,0}$ is the vertex $u_{\lfloor \frac{m}{2} \rfloor,\lfloor \frac{n}{2} \rfloor}$

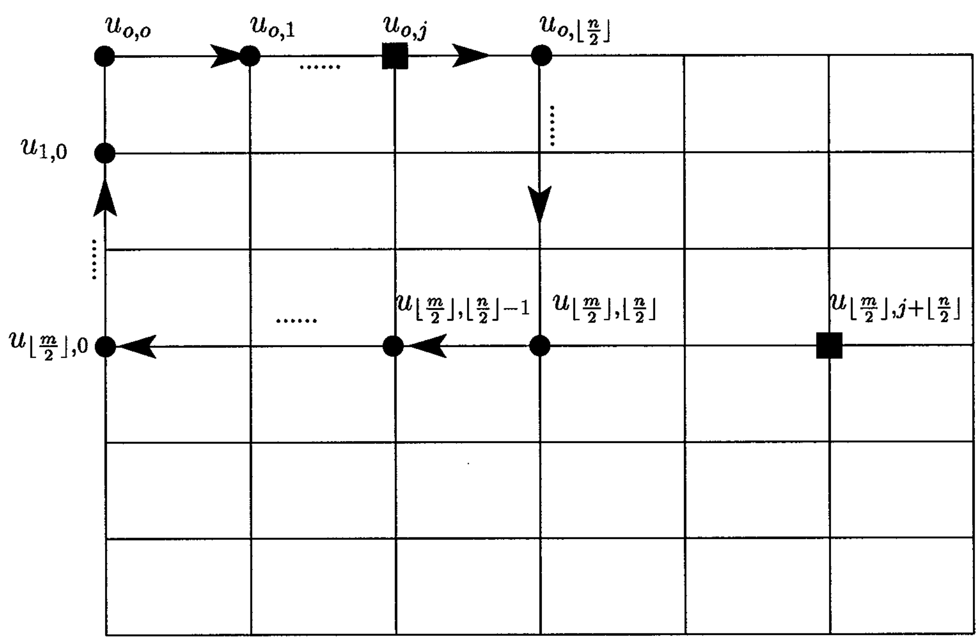

Figure 3: Cycle C_u in torus $T_{m\times n}$

Without loss of generality we will consider the cycle :

$$C_u : u_{0,0} \rightarrow u_{0,1} \rightarrow u_{0,2} \rightarrow \dots \rightarrow u_{0,\lfloor \frac{n}{2} \rfloor} \rightarrow u_{1,\lfloor \frac{n}{2} \rfloor} \rightarrow u_{2,\lfloor \frac{n}{2} \rfloor} \rightarrow \dots \rightarrow u_{\lfloor \frac{m}{2} \rfloor, \lfloor \frac{n}{2} \rfloor}$$

$$\rightarrow u_{\lfloor \frac{m}{2} \rfloor, \lfloor \frac{n}{2} \rfloor - 1} \rightarrow \dots u_{\lfloor \frac{m}{2} \rfloor, 0} \rightarrow u_{\lfloor \frac{m}{2} \rfloor - 1, 0} \rightarrow u_{\lfloor \frac{m}{2} \rfloor - 2, 0} \rightarrow \dots \rightarrow u_{0,0}$$

Consider the vertex $u_{0,j} \in C_u$ where $0 < j < \lfloor \frac{n}{2} \rfloor$. The diameter of the torus from $u_{0,j}$ will be:

$$D(T_{m \times n}) = dist(u_{0,j}, u_{0,j+\lfloor \frac{n}{2} \rfloor}) + dist(u_{0,j+\lfloor \frac{n}{2} \rfloor}, u_{\lfloor \frac{m}{2} \rfloor, j+\lfloor \frac{n}{2} \rfloor})$$

This means that the diametric vertex of $u_{0,j}$ is the vertex $u_{\lfloor \frac{m}{2} \rfloor, j+\lfloor \frac{n}{2} \rfloor}$. But since $j > 0$, then vertex $u_{\lfloor \frac{m}{2} \rfloor, j+\lfloor \frac{n}{2} \rfloor} \notin C_u$.

Therefore, we can say that for any cycle C_i in $T_{m \times n}$ of length $2D(T_{m \times n})$, there exists a vertex $u \in C_i$ such that its diametric vertex does not belong to C_i. □

2.2 Lower Bound for $T_{m \times n}$

Theorem 2.2.1

Given a torus $T_{m \times n}(V, E)$ with the ordering Π. Then,

$$b^o(T_{m \times n}) \geq \begin{cases} D(T_{m \times n}) + 1 & m \text{ if and } n \text{ are even} \\ D(T_{m \times n}) + 2 & \text{otherwise} \end{cases}$$

Proof :

Case 1: m and n are even

If m and n are both even, then for every vertex $u \in V$, there exists exactly one vertex $v \in V$ such that $dist(u, v) = D(T_{m \times n})$

Let $u, v \in V$ such that u is the diametric vertex of v. Suppose u is the originator and that there exists a path P_{uv} from u such that :

$P_{uv} : u \xrightarrow{1} u_1 \xrightarrow{1} u_2 \xrightarrow{1} \ldots \xrightarrow{1} u_k \xrightarrow{1} u_{k+1} \xrightarrow{1} \ldots \xrightarrow{1} u_{k+c} \xrightarrow{1} u_{k+c+1} \xrightarrow{1} \ldots \xrightarrow{1} u_n \xrightarrow{1} v$

(where $c \leq n - k$ is a constant and $0 \leq k \leq n$)

Then, v will receive the message through path P_{uv} at time $= D(T_{m \times n})$

Vertex v has to have a similar path (in case broadcasting starts from v) that carries the message from v to u in time $\leq D(T_{m \times n})$. Hence:

$P_{vu} : v \xrightarrow{1} v_1 \xrightarrow{1} v_2 \xrightarrow{1} \ldots \xrightarrow{1} u$

Thus, when the originator is v then vertex u will receive the message through path P_{vu} at time $= D(T_{m \times n})$. Note that the two paths P_{uv} and P_{vu} form a cycle C_u of length $2D(T_{m \times n})$.

Now consider vertex $u_k \in P_{uv}$ where the diametric vertex of u_k is $w \notin C_u$ (lemma 2.1.1). Let $P_{u_k w}$ be one of the fastest path that carries the message from u_k to its diametric vertex w. Since $w \notin C_u$, when the message originates from u_k the path $P_{u_k w}$ has to separate from C_u at one of its vertices. Without loss of generality, we will assume that this vertex is $u_{k+c} \in P_{uv}$. Then,

$P_{u_k w} : u_k \xrightarrow{1} u_{k+1} \xrightarrow{1} \ldots \xrightarrow{1} u_{k+c} \xrightarrow{\mathbf{2}} w_l \xrightarrow{1} w_{l-1} \xrightarrow{1} \ldots \xrightarrow{1} w$ (for some $l \in Z^+$)

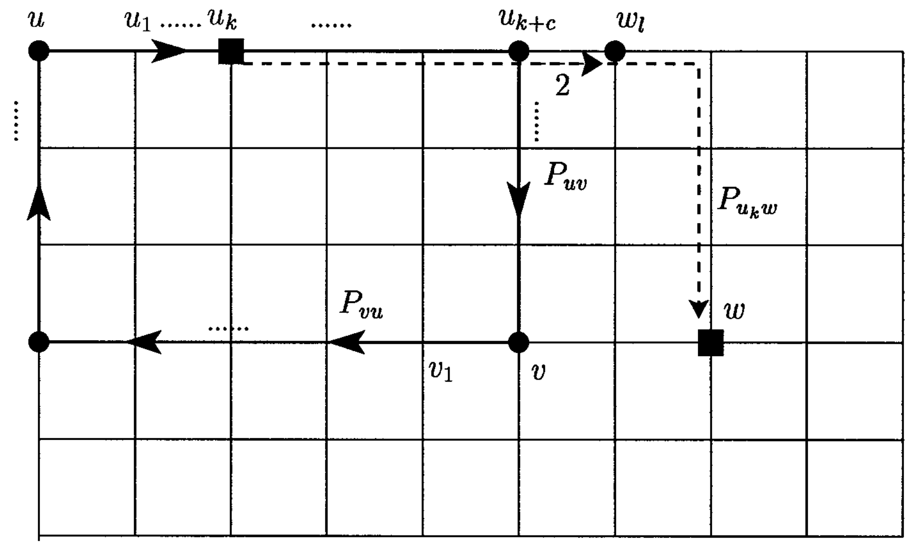

Figure 4: The paths P_{uv}, P_{vu} and $P_{u_k w}$ in $T_{m \times n}$

Suppose u_{k+c} receives the message at time t_0. Then, at time $t_0 + 1$, u_{k+c} sends the message to $u_{k+c+1} \in P_{uv}$, and at time $t_0 + 2$, u_{k+c} sends the message to $w_l \in P_{u_k w}$.

Therefore, when broadcasting starts at vertex u_k, the path that takes the message from u_k to its diametric vertex w will lose one time unit. Thus, in this case w will be informed at time $\geq D(T_{m \times n}) + 1$

Hence, $b^o(T_{m \times n}) \geq D(T_{m \times n}) + 1$ when m and n are both *even*.

Case 2: at least one of m or n is odd

If at least one of m or n is odd, then for every vertex $u \in V$, there exists at least two vertices $v_1, v_2 \in V$ such that:

$$dist(u, v_i) = D(T_{m\times n}) \qquad \text{for } i \in \{1, 2\}$$

Suppose at least one of m or n is odd and consider the path P_{uv} described above. Then, vertex $u_k \in P_u v$ will have at least two diametric vertices w and w'. Since w and w' are both at distance $D(T_{m\times n})$ from u_k, through observation 2.1.4 we know that to inform w and w', u_k needs at least $D(T_{m\times n}) + 1$ time units.

Also, from *case 1* above we know that $u_k \in P_{uv}$ will lose one time unit to inform its diametric vertex w. This implies that $b^o(T_{m\times n}) \geq D(T_{m\times n}) + 2$ when at least one of m or n is *odd*.

Therefore,

$$b^o(T_{m\times n}) \geq \begin{cases} D(T_{m\times n}) + 1 & m \text{ and } n \text{ are even} \\ D(T_{m\times n}) + 2 & \text{otherwise} \end{cases}$$

□

Chapter 3

Upper Bound on 2D Tori

3.1 Notations

Let $T_{m\times n} = (V, E)$ be a 2-dimensional torus. We denote:

- $b^{\Pi}(P_i, v)$ to be the time at which vertex $v \in V$ receives the message through path P_i under ordering Π.
- $b^{\Pi}_j(P_i, v)$ to be the time at which vertex $v \in V$ receives the message that originated from vertex $(0, j)$ through path P_i under ordering Π.
- $t^{\Pi}(v)$ or $t^{\Pi}(p, q)$ to be the time at which vertex $v = (p, q) \in V$ receives the message under ordering Π.

$$\text{Let } \ell = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor & \text{if } \left\lfloor \frac{n}{2} \right\rfloor \text{ is even} \\ \\ \left\lfloor \frac{n}{2} \right\rfloor + 1 & \text{if } \left\lfloor \frac{n}{2} \right\rfloor \text{ is odd} \end{cases}$$

3.2 An Ordering for $T_{m\times n}$

In this section we will describe the ordering Π of a torus $T_{m\times n}$. Observe that $T_{m\times n}$ is a 4-regular graph ($\delta(v) = 4$ for all $v \in V$), hence the time units assigned to the edges will be chosen from $\{1,2,3,4\}$. Let $u = (i,j) \in V, v \in V$ and $k \in Z^+$. Then,

1. $\Pi(u,v) = 1$ when:

   a) $v = (i, j+1 \bmod n)$ where $i = 2k$ and $j \notin \{0,\ell\}$

   b) $v = (i, j-1 \bmod n)$ where $i = 2k+1$ and $j \notin \{0,\ell\}$

   c) $v = (i+1 \bmod m, j)$ where $j = 0$

   d) $v = (i-1 \bmod m, j)$ where $j = \ell$

2. $\Pi(u,v) = 2$ when:

   a) $v = (i+1 \bmod m, j)$ where $j = 2k$ and $j \notin \{0,\ell\}$

   b) $v = (i-1 \bmod m, j)$ where $j = 2k+1$ and $j \notin \{0,\ell\}$

   c) $v = (i, j+1 \bmod n)$ where $i = 2k$ and $j \in \{0,\ell\}$

   d) $v = (i, j-1 \bmod n)$ where $i = 2k+1$ and $j \in \{0,\ell\}$

3. $\Pi(u,v) = 3$ when:

   a) $v = (i-1 \bmod m, j)$ where $j = 2k$ and $j \notin \{0,\ell\}$

   b) $v = (i+1 \bmod m, j)$ where $j = 2k+1$ and $j \notin \{0,\ell\}$

   c) $v = (i, j-1 \bmod n)$ where $i = 2k$ and $j \in \{0,\ell\}$

   d) $v = (i, j+1 \bmod n)$ where $i = 2k+1$ and $j \in \{0,\ell\}$

4. $\Pi(u,v) = 4$ when:

a) $v = (i, j-1 \bmod n)$ where $i = 2k$ and $j \notin \{0, \ell\}$

b) $v = (i, j+1 \bmod n)$ where $i = 2k+1$ and $j \notin \{0, \ell\}$

c) $v = (i-1 \bmod m, j)$ where $j = 0$

d) $v = (i+1 \bmod m, j)$ where $j = \ell$

Refer to Figure 5 for an example of ordering Π.

In what follows we will assume that $T_{m\times n}$ has the ordering Π described above.

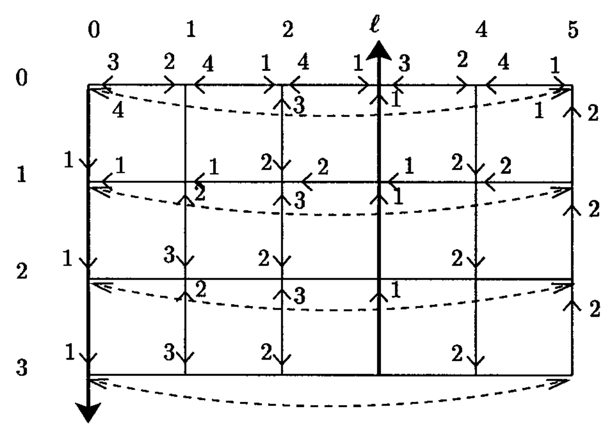

Figure 5: The ordering *Pi* for $T_{4\times 6}$. Note that some of the edges and labels are omitted to make the figure more readable.

3.3 Upper Bound on $T_{m\times n}$: An Example

In this section we will give an example and show in detail how a message propagates and eventually informs all the vertices of a given row of $T_{m\times n}$.

Consider the torus $T_{9\times 13}$. The diameter $D(T_{9\times 13}) = 10$ and $\ell = 7$. Let the originator be vertex $(0,3)$, and suppose we want to inform all the vertices on row 5. We describe four labeled paths :

- P_1 and P_2, that will inform every vertex $(5,q) \in V$ where $0 < q \leq 7$, and
- P_3 and P_4, that will inform every vertex $(5,q) \in V$ where $7 < q \leq 12$ and $q = 0$.

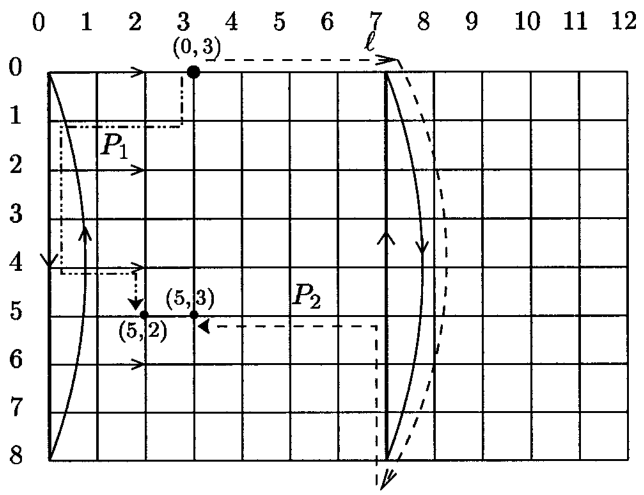

Figure 6: The two paths P_1 and P_2 originating from $(0,3)$ inform all the vertices $(5,q)$ such that $0 < q \leq 7$.

First, we concentrate on informing all the vertices $(5,q)$ where $0 \leq q \leq 7$. Consider the paths P_1 and P_2 (Figure 6):

$$P_1 : (0,3) \xrightarrow{3} (1,3) \xrightarrow{1} (1,2) \xrightarrow{1} (1,1) \xrightarrow{1} (1,0) \xrightarrow{1} (2,0) \xrightarrow{1} (3,0) \xrightarrow{1} (4,0) \xrightarrow{2} (4,1) \xrightarrow{1} (4,2) \xrightarrow{2} (5,2)$$

$$P_2 : (0,3) \xrightarrow{1} (0,4) \xrightarrow{1} (0,5) \xrightarrow{1} (0,6) \xrightarrow{1} (0,7) \xrightarrow{1} (8,7) \xrightarrow{1} (7,7) \xrightarrow{1} (6,7) \xrightarrow{1} (5,7) \xrightarrow{2} (5,6) \xrightarrow{1} (5,5) \xrightarrow{1} (5,4) \xrightarrow{1} (5,3)$$

The time that vertices $v_1 = (5,2)$ and $v_2 = (5,3)$ receive the message from each of the paths P_1 and P_2 is: $b^{\Pi}(P_1, v_1) = 14$ and $b^{\Pi}(P_2, v_2) = 13$.

Now, we look at the vertices $(5, q)$ where $7 \leq q \leq 12$. Consider the two paths (Figure 7):

$P_3 : (0,3) \xrightarrow{3} (1,3) \xrightarrow{1} (1,2) \xrightarrow{1} (1,1) \xrightarrow{1} (1,0) \xrightarrow{1} (2,0) \xrightarrow{1} (3,0) \xrightarrow{1} (4,0) \xrightarrow{1} (5,0)$
$\xrightarrow{2} (5,12) \xrightarrow{1} (5,11)$

$P_4 : (0,3) \xrightarrow{1} (0,4) \xrightarrow{1} (0,5) \xrightarrow{1} (0,6) \xrightarrow{1} (0,7) \xrightarrow{1} (8,7) \xrightarrow{1} (7,7) \xrightarrow{1} (6,7) \xrightarrow{2} (6,8)$
$\xrightarrow{1} (6,9) \xrightarrow{1} (6,10) \xrightarrow{3} (5,10)$

The time that vertices $v_3 = (5,11)$ and $v_4 = (5,10)$ receive the message from each of the paths P_1 and P_2 is: $b^{\Pi}(P_3, v_3) = 13$ and $b^{\Pi}(P_4, v_4) = 14$

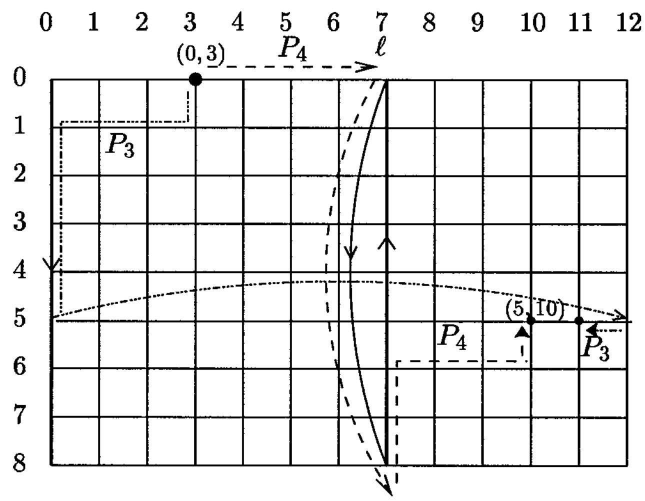

Figure 7: The two paths P_3 and P_4 originating from $(0,3)$ inform all the vertices $(5,q)$ such that $7 < q \leq 12$ and $q = 0$.

Therefore, all the vertices of row 5, except vertices $(5,1), (5,8)$ and $(5,9)$, will be informed through the above described paths P_1, P_2, P_3, P_4 in at most $D(T_{9\times 13})+4 = 14$ time units.

Vertex $(5,1)$ will be informed through the path:

$(0,3) \xrightarrow{3} (1,3) \xrightarrow{1} (1,2) \xrightarrow{1} (1,1) \xrightarrow{1} (1,0) \xrightarrow{1} (2,0) \xrightarrow{1} (3,0) \xrightarrow{1} (4,0)$
$\xrightarrow{2} (4,1) \xrightarrow{3} (5,1)$

Vertex $(5,8)$ will be informed through the path:

$(0,3) \xrightarrow{1} (0,4) \xrightarrow{1} (0,5) \xrightarrow{1} (0,6) \xrightarrow{1} (0,7) \xrightarrow{1} (8,7) \xrightarrow{1} (7,7) \xrightarrow{1} (6,7)$
$\xrightarrow{2} (6,8) \xrightarrow{3} (5,8)$

Vertex $(5,9)$ will be informed through the path:

$(0,3) \xrightarrow{1} (0,4) \xrightarrow{1} (0,5) \xrightarrow{1} (0,6) \xrightarrow{1} (0,7) \xrightarrow{1} (8,7) \xrightarrow{1} (7,7) \xrightarrow{1} (6,7)$
$\xrightarrow{2} (6,8) \xrightarrow{1} (6,9) \xrightarrow{2} (5,9)$

From the above path descriptions it is easy to see that all three vertices $(5,1), (5,8), (5,9)$ will also receive the message in less than 14 time units. Thus, we can conclude that in a $T_{9\times13}$, for a given originator $(0,3)$, all the vertices $v = (5,q) \in V$ on row 5 will be informed by time $b^{\Pi}(v) \leq D(T_{9\times13}) + 4 = 14$.

In what follows, we will state a few lemmas and observations which will help us show that the worst case for an originator (i,j) is when j is odd for any given torus $T_{m\times n}$ (this is the example that we considered above).

Observation 3.3.1

Let $v \in V$ be a vertex in $T_{m\times n}$. Then, if v receives the message twice at two different *times t_1 and t_2, then*

$$t^{\Pi}(v) \leq \left\lfloor \frac{t_1 + t_2}{2} \right\rfloor$$

Lemma 3.3.1

Given a torus $T_{m \times n}$, consider any two vertices $u = (0, j) \in V$ and $v = (0, j+1) \in V$ where $0 < j < \ell$. Let $t_u^{\Pi}(w)$ denote the time at which a vertex $w = (p, q) \in V$ receives the message under ordering Π when the originator is u. Suppose p is odd. Then, if j is odd,

$$t_v^{\Pi}(w) \leq t_u^{\Pi}(w)$$

That is, informing the vertices of an odd-numbered row in $T_{m \times n}$ will take less time if the originator is a vertex with an even-numbered column j.

Proof :

Consider the path P_1 described above and suppose we want to inform a vertex $w = (p, q) \in V$ such that $0 \leq q \leq \ell$. Then, for each of the originators $u = (0, j)$ and $v = (0, j+1)$, P_1 will proceed as follows (where $k \in \{1, 2, 3, 4\}$) :

$P_1^u : (0, j) \xrightarrow{\mathbf{3}} (1, j) \xrightarrow{1} (1, j-1) \xrightarrow{1} \ldots \xrightarrow{1} (1, 0) \xrightarrow{1} (2, 0) \xrightarrow{1} \ldots \xrightarrow{1} (p-1, q) \xrightarrow{k} (p, q)$

$P_1^v : (0, j+1) \xrightarrow{\mathbf{2}} (1, j+1) \xrightarrow{1} (1, j) \xrightarrow{1} \ldots \xrightarrow{1} (1, 0) \xrightarrow{1} (2, 0) \xrightarrow{1} \ldots \xrightarrow{1} (p-1, q)$
$\xrightarrow{k} (p, q)$

The only step that will effect the time at which vertex w recieves the message from each of the originators P_1^u and P_1^v is the first one, where $(0, j)$ informs its neighbor in P_1^u in 3 time units, while $(0, j+1)$ informs its neighbor in P_1^v in 2 time units. Hence,

$b^{\Pi}(P_1^u, w) = 3 + j + (p - 1 - 1) + 2 + (q - 1) + k = q + j + p + k + 2 \quad \text{and}$

$b^{\Pi}(P_1^v, w) = 2 + (j + 1) + (p - 1 - 1) + 2 + (q - 1) + k = q + j + p + k + 2$

This implies that $b^{\Pi}(P_1^v, w) = b^{\Pi}(P_1^u, w)$, that is vertex w will receive the message from the any of u or v at the same time.

Also for path P_2 described above,

$$P_2^u : (0, j) \xrightarrow{1} (0, j+1) \xrightarrow{1} (0, j+2) \xrightarrow{1} \ldots \xrightarrow{1} (0, \ell) \xrightarrow{1} (m-1, \ell) \xrightarrow{1} \ldots$$
$$\xrightarrow{1} (p, \ell) \xrightarrow{2} (p, \ell - 1) \xrightarrow{1} (p, \ell - 2) \xrightarrow{1} \ldots \xrightarrow{1} (p, q)$$

$$P_2^v : (0, j+1) \xrightarrow{1} (0, j+2) \xrightarrow{1} (0, j+3) \xrightarrow{1} \ldots \xrightarrow{1} (0, \ell) \xrightarrow{1} (m-1, \ell) \xrightarrow{1} \ldots$$
$$\xrightarrow{1} (p, \ell) \xrightarrow{2} (p, \ell - 1) \xrightarrow{1} (p, \ell - 2) \xrightarrow{1} \ldots \xrightarrow{1} (p, q)$$

Thus, the time that vertex (p, q) receives the message from each of P_2^u and P_2^v is:

$b^{\Pi}(P_2^u, w) = (x - j) + 1 + (m - 1 - p) + 2 + (\ell - 1 - q) = -q - j - p + m + 2\ell + 1$

$b^{\Pi}(P_2^v, w) = (x - (j + 1)) + 1 + (m - 1 - p) + 2 + (\ell - 1 - q) = -q - j - p + m + 2\ell$

This implies that $b^{\Pi}(P_2^v, w) = b^{\Pi}(P_2^u, w) - 1$

Through observation 3.3.1 we know that :

$$t_v^{\Pi}(w) \leq \left\lfloor \frac{b^{\Pi}(P_1^v, w) + b^{\Pi}(P_2^v, w)}{2} \right\rfloor$$

$$= \left\lfloor \frac{b^{\Pi}(P_1^u, w) + b^{\Pi}(P_2^u, w) - 1}{2} \right\rfloor$$

$$= \left\lfloor \frac{b^{\Pi}(P_1^u, w) + b^{\Pi}(P_2^u, w)}{2} - \frac{1}{2} \right\rfloor$$

$$= \left\lfloor t_u^{\Pi}(w) - \frac{1}{2} \right\rfloor$$

Therefore, we can see that $t_v^{\Pi}(w) \leq t_u^{\Pi}(w)$. Similarly, we can show that $t_v^{\Pi}(w) \leq t_u^{\Pi}(w)$ where $w = (p, q)$ such that $\ell \leq q \leq n - 1$.

It follows from the above discussions that for *any* vertex $w \in V$,

$$t_v^{\Pi}(w) \leq t_u^{\Pi}(w)$$

This means that, when the originator is $(0, j) \in V$ where $0 < j < \ell$ and j is odd, broadcasting to a vertex $w \in V$ that is on an odd-numbered row will take more-or-equal time than when j were even. Thus, for any vertex $(0, j) \in V$, the worst case originator will be when j is odd. □

In the next lemma we show that for any originator $(0, j) \in V$ where $0 < j < \ell$, informing all the vertices on an even-numbered row will take the same amount of time as informing all the vertices on an odd-numbered row.

3.4 Informing Even-numbered Rows

Lemma 3.4.1

Given a torus $T_{m \times n}$ and an originator $(0, j) \in V$. Then, in the worst case, all the vertices on row p will be informed by the same time as the vertices on row $p+1$. That is, for any two vertices $u = (p, q) \in V$ and $v = (p+1, q_1)$, where $0 \leq q, q_1 \leq n-1$:

$$t^{\Pi}(u) = t^{\Pi}(v)$$

Proof :

Without loss of generality, we will assume p to be odd (and thus, $p+1$ will be even). We know through lemma 3.3.1 that the worst-case for an originator is when j is odd. Thus, here we will assume that j is odd.

For each of the rows p and $p+1$, we again describe four labeled paths similar to those described in section 3.3. For row p we describe :

- P_1 and P_2, that will inform every vertex $u = (p, q) \in V$ where $0 < q \leq \ell$, and
- P_3 and P_4, that will inform every vertex $u = (p, q) \in V$ where $\ell < q \leq n-1$ and $q = 0$.

For row $p+1$ we describe :

- P_1' and P_2', that will inform every vertex $v = (p+1, q_1) \in V$ where $0 < q_1 \leq \ell$, and

- P_3' and P_4', that will inform every vertex $(p,q) \in V$ where $\ell < q \leq n-1$ and $q=0$.

First, we concentrate on informing all the vertices $u=(p,q)$ and $v=(p+1,q_1)$ such that $0 < q \leq \ell$ and $0 \leq q_1 < \ell$ (see figure 8). Suppose u receives the message through path P_1 such that (suppose $k \in \{1,2,3,4\}$):

$$P_1 : (0,j) \xrightarrow{3} (1,j) \xrightarrow{1} (1,j-1) \xrightarrow{1} \dots \xrightarrow{1} (1,0) \xrightarrow{1} (2,0) \xrightarrow{1} \dots$$
$$\xrightarrow{1} (p-1,0) \xrightarrow{2} (p-1,1) \xrightarrow{1} \dots \xrightarrow{1} (p-1,q) \xrightarrow{k} (p,q)$$

And, v receives the message through path P_1' :

$$P_1' : (0,j) \xrightarrow{3} (1,j) \xrightarrow{1} (1,j-1) \xrightarrow{1} \dots \xrightarrow{1} (1,0) \xrightarrow{1} (2,0) \xrightarrow{1} \dots$$
$$\xrightarrow{1} (p-1,0) \xrightarrow{1} (p,0) \xrightarrow{1} (p+1,0) \xrightarrow{2} (p+1,1) \xrightarrow{1} \dots \xrightarrow{1} (p+1,q_1)$$

Then, u and v will each receive the message at time:

$$b^{\Pi}(P_1,u) = 3+j+(p-1-1)+2+(q-1)+k = j+p+q+k+2$$
$$b^{\Pi}(P_1',v) = 3+j+(p+1-1)+2+(q_1-1) = j+p+q_1+4$$

Also, u will receive the message through path P_2 such that :

$$P_2 : (0,j) \xrightarrow{1} (0,j+1) \xrightarrow{1} (0,j+2) \xrightarrow{1} \dots \xrightarrow{1} (0,\ell) \xrightarrow{1} (m-1,\ell) \xrightarrow{1} \dots$$
$$\xrightarrow{1} (p,\ell) \xrightarrow{2} (p,\ell-1) \xrightarrow{1} (p,\ell-2) \xrightarrow{1} \dots \xrightarrow{1} (p,q)$$

And, v will receive the message through path P_2' (where $k' \in \{1,2,3,4\}$):

$$P_2' : (0,j) \xrightarrow{1} (0,j+1) \xrightarrow{1} (0,j+2) \xrightarrow{1} \dots \xrightarrow{1} (0,\ell) \xrightarrow{1} (m-1,\ell) \xrightarrow{1} \dots$$
$$\xrightarrow{1} (p+2,\ell) \xrightarrow{2} (p+2,\ell-1) \xrightarrow{1} (p+2,\ell-2) \xrightarrow{1} \dots \xrightarrow{1} (p+1,q_1) \xrightarrow{k'} (p+1,q_1)$$

Then, u and v will receive the message at time:

$$b^{\Pi}(P_2, u) = (\ell - j) + (m - p) + 2 + (\ell - 1 - q) = -j - p - q + 2\ell + m + 1$$

$$b^{\Pi}(P_2', v) = (\ell - j) + (m - (p+2)) + 2 + (\ell - 1 - q_1) + k' = -j - p - q_1 + 2\ell + m + k' - 1$$

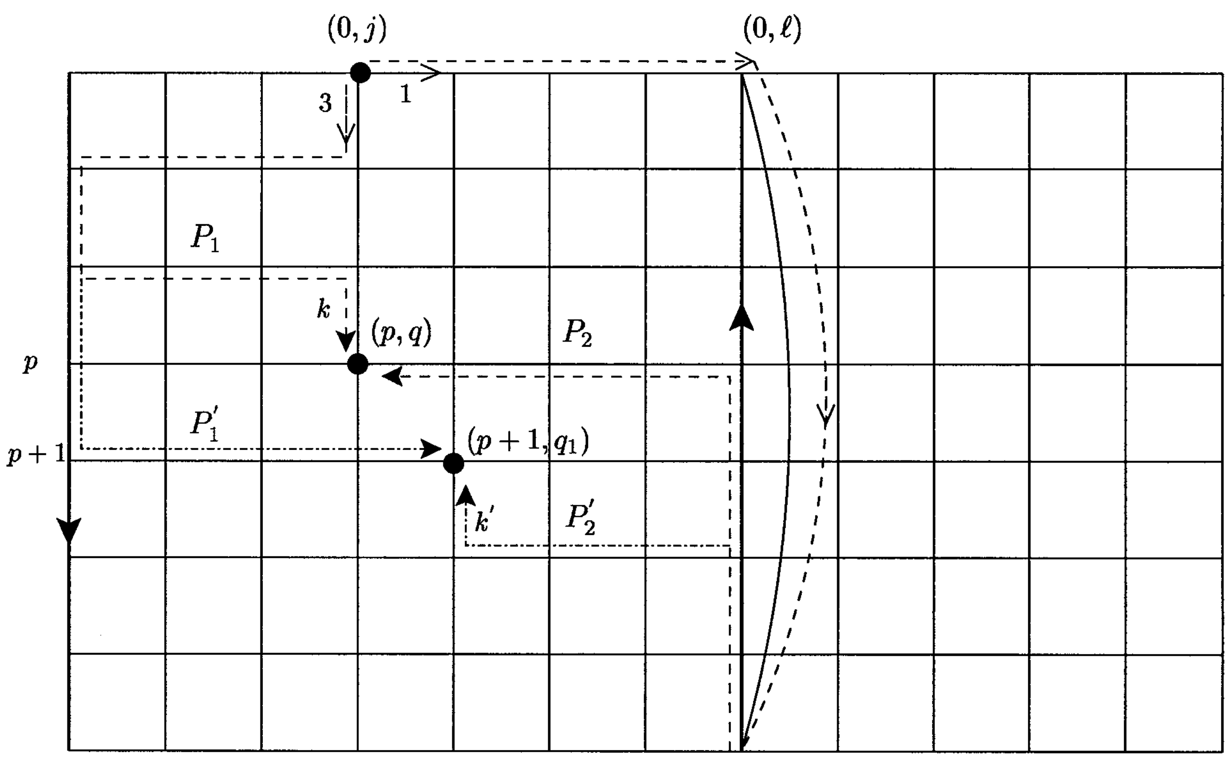

Figure 8: Originating from $(0, j)$, the paths P_1 and P_2 inform all the vertices (p, q) on row p, while the paths P_1' and P_2' inform all the vertices $(p+1, q_1)$ on row $p+1$ where $(0 < q \leq \ell)$ and $(0 \leq q_1 < \ell)$

Through observation 3.3.1 we know that any vertex $u = (p, q)$ will be informed through P_1 or P_2 at most at:

$$t^{\Pi}(u) \leq \left\lfloor \frac{b^{\Pi}(P_1, u) + b^{\Pi}(P_2, u)}{2} \right\rfloor$$

$$= \left\lfloor \frac{(j + p + q + k + 2) + (-j - p - q + 2\ell + m + 1)}{2} \right\rfloor$$

$$= \left\lfloor \frac{2\ell + m + k + 3}{2} \right\rfloor$$

Similarly, any vertex $v = (p+1, q_1) \in V$ will be informed at time :

$$t^{\Pi}(v) \leq \left\lfloor \frac{b^{\Pi}(P_1', v) + b^{\Pi}(P_2', v)}{2} \right\rfloor$$

$$= \left\lfloor \frac{(j + p + q_1 + 4) + (-j - p - q_1 + 2\ell + m + k' - 1)}{2} \right\rfloor$$

$$= \left\lfloor \frac{2\ell + m + k' + 3}{2} \right\rfloor$$

In the worst case for each of $t^{\Pi}(u)$ and $t^{\Pi}(v)$, $k = k' = 4$. This implies that $t^{\Pi}(u) = t^{\Pi}(v)$ when $0 < q \leq \ell$ and $0 \leq q_1 < \ell$, and hence, all the vertices $u = (p, q)$ and $v = (p+1, q)$ with $0 < q \leq \ell$ and p odd, will be informed at most by the same time.

Now, consider the vertices $u = (p, q)$ and $v = (p+1, q_1)$ such that $\ell < q \leq n-1$ and $q = 0$ and $\ell \leq q_1 \leq n-1$ (see figure 9). Then, u receives the message through path P_3 :

$$P_3 : (0, j) \xrightarrow{3} (1, j) \xrightarrow{1} (1, j-1) \xrightarrow{1} \ldots \xrightarrow{1} (1, 0) \xrightarrow{1} (2, 0) \xrightarrow{1} \ldots$$
$$\xrightarrow{1} (p-1, 0) \xrightarrow{1} (p, 0) \xrightarrow{2} (p, n-1) \xrightarrow{1} (p, n-2) \xrightarrow{1} \ldots \xrightarrow{1} (p, q)$$

And v receives the message through path P_3' (where $k \in \{1, 2, 3, 4\}$):

$$P_3' : (0, j) \xrightarrow{3} (1, j) \xrightarrow{1} (1, j-1) \xrightarrow{1} \ldots \xrightarrow{1} (1, 0) \xrightarrow{1} (2, 0) \xrightarrow{1} \ldots$$
$$\xrightarrow{1} (p-1, 0) \xrightarrow{1} (p, 0) \xrightarrow{2} (p, n-1) \xrightarrow{1} (p, n-2) \xrightarrow{1} \ldots \xrightarrow{1} (p, q_1) \xrightarrow{k} (p+1, q_1)$$

Then, u and v will each receive the message at time:

$$b^{\Pi}(P_3, u) = 3 + j + (p - 1) + 2 + (n - 1 - q) = j + p + n - q + 3$$

$$b^{\Pi}(P_3', v) = 3 + j + (p - 1) + 2 + (n_1 - q_1) + k = j + p + n - q_1 + k + 3$$

Also, u will receive the message through path P_4 (where $k' \in \{1, 2, 3, 4\}$):

$$P_4 : (0, j) \xrightarrow{1} (0, j + 1) \xrightarrow{1} (0, j + 2) \xrightarrow{1} \dots \xrightarrow{1} (0, \ell) \xrightarrow{1} (m - 1, \ell) \xrightarrow{1} \dots$$

$$\xrightarrow{1} (p + 1, \ell) \xrightarrow{2} (p + 1, \ell + 1) \xrightarrow{1} \dots \xrightarrow{1} (p + 1, q) \xrightarrow{k'} (p, q)$$

And, v will receive the message through path P_4' :

$$P_4' : (0, j) \xrightarrow{1} (0, j + 1) \xrightarrow{1} (0, j + 2) \xrightarrow{1} \dots \xrightarrow{1} (0, \ell) \xrightarrow{1} (m - 1, \ell) \xrightarrow{1} \dots$$

$$\xrightarrow{1} (p + 1, \ell) \xrightarrow{2} (p + 1, \ell + 1) \xrightarrow{1} \dots \xrightarrow{1} (p + 1, q_1)$$

Therfore, u and v will each receive the message at time:

$$t^{\Pi}(P_4, u) = (\ell - j) + (m - (p + 1)) + 2 + (q - (\ell + 1)) + k' = -j - p + q + m + k'$$

$$^{\Pi}(P_4', v) = (\ell - j) + (m - (p + 1)) + 2 + (q_1 - (\ell + 1)) = -j - p + q_1 + m$$

Similarly, any vertex u will be informed through P_3 or P_4 at most at:

$$t^{\Pi}(u) \leq \left\lfloor \frac{b^{\Pi}(P_3, u) + b^{\Pi}(P_4, u)}{2} \right\rfloor$$

$$= \left\lfloor \frac{(j + p + n - q + 3) + (-j - p + q + m + k')}{2} \right\rfloor$$

$$= \left\lfloor \frac{m + n + k' + 3)}{2} \right\rfloor$$

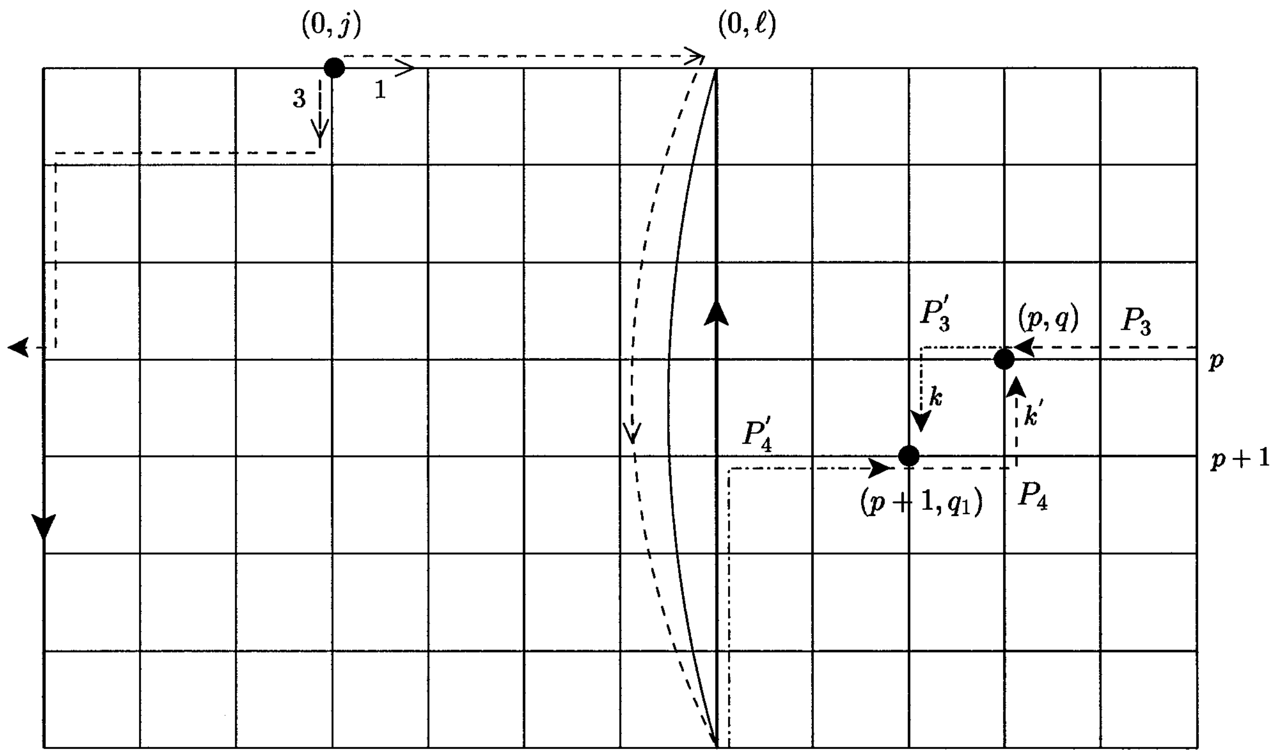

Figure 9: Originating from $(0, j)$, the paths P_3 and P_4 inform all the vertices (p, q) on row p, while the paths P_3' and P_4' inform all the vertices $(p+1, q_1)$ on row $p+1$ where $(\ell < q \leq n-1$ and $q = 0)$ and $(\ell \leq q_1 \leq n-1)$

Also, any vertex v will be informed through P_3' or P_4' at most at:

$$t^{\Pi}(v) \leq \left\lfloor \frac{b^{\Pi}(P_3', v) + b^{\Pi}(P_4', v)}{2} \right\rfloor$$

$$= \left\lfloor \frac{(j + p + n - q_1 + k + 3) + (-j - p + q_1 + m)}{2} \right\rfloor$$

$$= \left\lfloor \frac{m + n + k + 3)}{2} \right\rfloor$$

Again in the worst case for each of $t^{\Pi}(u)$ and $t^{\Pi}(v)$, $k = k' = 4$. This implies that

$t^{\Pi}(u) = t^{\Pi}(v)$ when ($\ell < q \leq n-1$ and $q = 0$) and ($\ell \leq q_1 \leq n-1$), and hence, all the vertices $u = (p, q)$ and $v = (p+1, q)$ with ($\ell < q \leq n-1$ and $q = 0$) and ($\ell \leq q_1 \leq n-1$) and p odd, will be informed at most by the same time. □

Therefore, given any originator $(0, j)$ with $0 < j < \ell$, the vertices $u \in V$ on the odd rows of $T_{m \times n}$ will be informed at most by the same time as the vertices $v \in V$ on the even rows. That is : $t^{\Pi}(u) = t^{\Pi}(v)$

Combining lemmas 3.3.1 and 3.4.1, we can say that informing the vertices of *any* row p in $T_{m \times n}$ from originator $(0, j)$ with $0 < j < \ell$ will take the most time when j is odd.

In the following two lemmas we show that the originators $(0, 0)$ and $(0, \ell)$ will yield a less-or-equal broadcast time than originator $(0, j)$ with $0 < j < \ell$ and j odd. This will mean that when j is odd, we will have the worst-case broadcast time.

3.5 Case when Originator is $(0, 0)$

Lemma 3.5.1

Given a torus $T_{m \times n}$, consider the two vertices $u = (0, 0) \in V$ and $v = (0, j) \in V$ where $0 \leq j \leq \ell$. Let $t^{\Pi}_u(w)$ denote the time at which a vertex $w = (p, q) \in V$ receives the message under ordering Π when the originator is u. Then,

$$t^{\Pi}_u(w) \leq t^{\Pi}_v(w)$$

Proof :

Through lemma 3.4.1 we know that in the worst case the parity of p does not effect the broadcast time, thus without loss of generality we will assume that p is odd. We also know through lemma 3.3.1 that the worst-case for an originator is when j is odd. Thus, here we will assume that j is odd.

We will proceed by first describing the paths that will inform all the vertices $w = (p, q)$ of a given row p when the originator is $u = (0, j)$ in one case and $v = (0, 0)$ in the other, and then compare the time by which each of these originators deliver the message to all the vertices of row p.

First, consider the vertex $w = (p, q)$ where $0 < q \leq \ell$ (assume $k, k' \in \{1, 2, 3, 4\}$). When the originator is v, w receives the message from the following two paths (see figure 10) :

$P_1^v : (0, j) \xrightarrow{3} (1, j) \xrightarrow{1} (1, j-1) \xrightarrow{1} \ldots \xrightarrow{1} (1, 0) \xrightarrow{1} (2, 0) \xrightarrow{1} \ldots$
$\quad \xrightarrow{1} (p-1, 0) \xrightarrow{2} (p-1, 1) \xrightarrow{1} \ldots \xrightarrow{1} (p-1, q) \xrightarrow{k} (p, q)$

$P_2^v : (0, j) \xrightarrow{1} (0, j+1) \xrightarrow{1} (0, j+2) \xrightarrow{1} \ldots \xrightarrow{1} (0, \ell) \xrightarrow{1} (m-1, \ell) \xrightarrow{1} \ldots$
$\quad \xrightarrow{1} (p, \ell) \xrightarrow{2} (p, \ell-1) \xrightarrow{1} (p, \ell-2) \xrightarrow{1} \ldots \xrightarrow{1} (p, q)$

When the originator is u, w receives the message from the following two paths:

$P_1^u : (0, 0) \xrightarrow{1} (1, 0) \xrightarrow{1} (2, 0) \xrightarrow{1} \ldots \xrightarrow{1} (p-1, 0) \xrightarrow{2} (p-1, 1) \xrightarrow{1} \ldots$
$\quad \xrightarrow{1} (p-1, q) \xrightarrow{k} (p, q)$

$P_2^u : (0, 0) \xrightarrow{2} (0, 1) \xrightarrow{1} (0, 2) \xrightarrow{1} \ldots \xrightarrow{1} (0, \ell) \xrightarrow{1} (m-1, \ell) \xrightarrow{1} \ldots$
$\quad \xrightarrow{1} (p+1, \ell) \xrightarrow{1} (p, \ell) \xrightarrow{2} (p, \ell-1) \xrightarrow{1} (p, \ell-2) \xrightarrow{1} \ldots \xrightarrow{1} (p, q)$

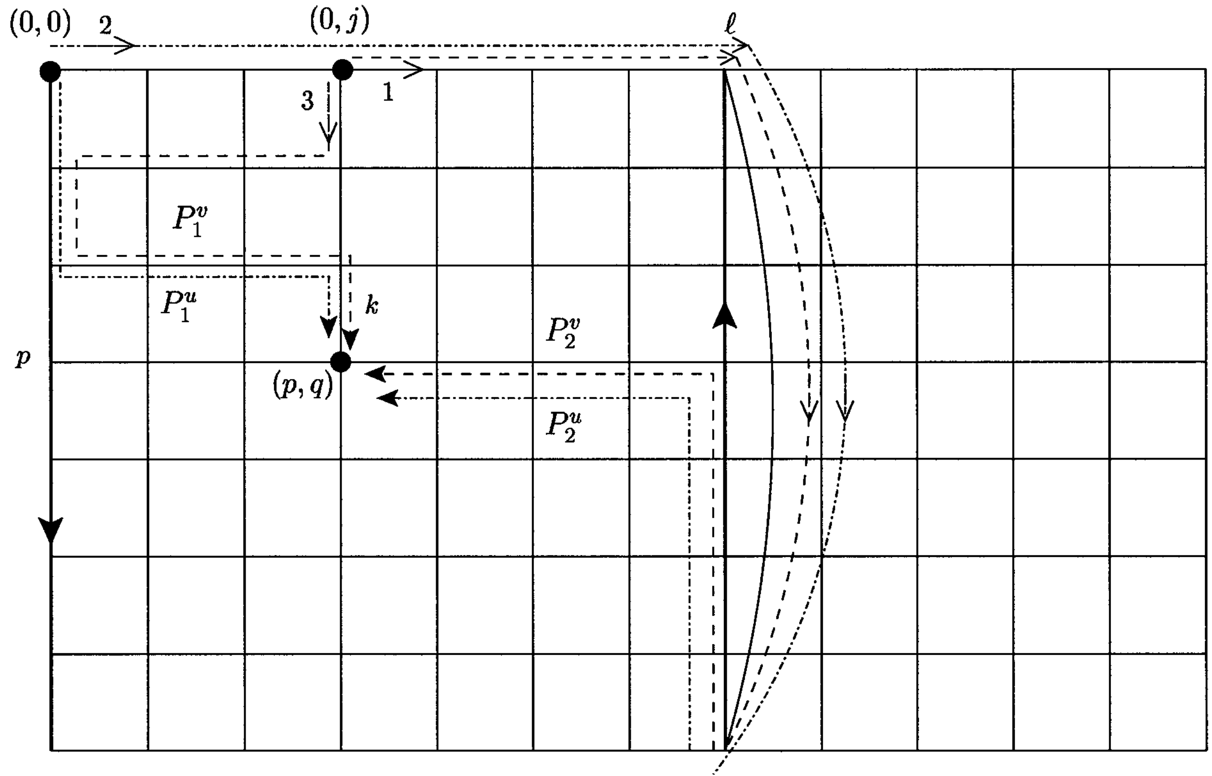

Figure 10: The paths P_1^u and P_2^u originate from $(0,0)$ and inform all the vertices (p,q) on row p, while the paths P_1^v and P_2^v originate from vertex $(0,j)$ and inform all the vertices (p,q) on row p where $(0 < j < \ell$

Vertex $w = (p,q)$ will receive the message from the two paths P_1^v and p_2^v at time:

$b^{\Pi}(P_1^v, w) = 3 + j + (p-2) + 2 + (q-1) + k = q + j + p + k + 2$

$b^{\Pi}(P_2^v, w) = (\ell - j) + (m-p) + 2 + (\ell - 1 - q) = -q - j - p + m + 2\ell + 1$

Also, w will receive the message from the two paths P_1^u and P_2^u at time:

$b\Pi(P_1^u, w) = (p-1) + 2 + (q-1) + k = q + p + k$

$b^{\Pi}(P_2^u, w) = 2 + (\ell - 1) + (m-p) + 2 + (\ell - 1 - q) = -q - p + m + 2\ell + 2$

Thus, every vertex $w = (p,q) \in V$, with $0 < q \leq \ell$, will receive the message from each of the originators u and v at most at time:

$$t_v^{\Pi}(w) \le \left\lfloor \frac{b^{\Pi}(P_1^v, w) + b^{\Pi}(P_2^v, w)}{2} \right\rfloor$$

$$= \left\lfloor \frac{(q+j+p+k+2) + (-q-j-p+m+2\ell+1)}{2} \right\rfloor$$

$$= \left\lfloor \frac{m+2\ell+k+3}{2} \right\rfloor$$

$$t_u^{\Pi}(w) \le \left\lfloor \frac{b^{\Pi}(P_1^u, w) + b^{\Pi}(P_2^u, w)}{2} \right\rfloor$$

$$= \left\lfloor \frac{(q+p+k) + (-q-p+m+2\ell+2)}{2} \right\rfloor$$

$$= \left\lfloor \frac{m+2\ell+k+2}{2} \right\rfloor$$

This implies that $t_u^{\Pi}(w) < b_v^{\Pi}(w)$ for $0 < q \le \ell$

Similarly, for $\ell < q \le n-1$ and $q = 0$. When the originator is v, w receives the message from the following two paths:

$$P_3^v : (0,j) \xrightarrow{3} (1,j) \xrightarrow{1} (1,j-1) \xrightarrow{1} \dots \xrightarrow{1} (1,0) \xrightarrow{1} (2,0) \xrightarrow{1} \dots$$
$$\xrightarrow{1} (p,0) \xrightarrow{2} (p,n-1) \xrightarrow{1} (p,n-2) \xrightarrow{1} \dots \xrightarrow{1} (p,q)$$

$$P_4^v : (0,j) \xrightarrow{1} (0,j+1) \xrightarrow{1} (0,j+2) \xrightarrow{1} \dots \xrightarrow{1} (0,\ell) \xrightarrow{1} (m-1,\ell) \xrightarrow{1} \dots$$
$$\xrightarrow{1} (p+1,\ell) \xrightarrow{2} (p+1,\ell+1) \xrightarrow{1} \dots \xrightarrow{1} (p+1,q) \xrightarrow{k'} (p,q)$$

When the originator is u, w receives the message from the following two paths:

$$P_3^u : (0,0) \xrightarrow{1} (1,0) \xrightarrow{1} (2,0) \xrightarrow{1} \ldots \xrightarrow{1} (p-1,0) \xrightarrow{1} (p,0) \xrightarrow{2} (p,n-1)$$
$$\xrightarrow{1} (p,n-2) \xrightarrow{1} \ldots \xrightarrow{1} (p,q)$$

$$P_4^u : (0,0) \xrightarrow{2} (0,1) \xrightarrow{1} (0,2) \xrightarrow{1} \ldots \xrightarrow{1} (0,\ell) \xrightarrow{1} (m-1,\ell) \xrightarrow{1} \ldots$$
$$\xrightarrow{1} (p+1,\ell) \xrightarrow{2} (p+1,\ell+1) \xrightarrow{1} \ldots \xrightarrow{1} (p+1,q) \xrightarrow{k'} (p,q)$$

Vertex $w = (p,q)$ will receive the message from the two paths P_3^v and p_4^v at time:

$$b^{\Pi}(P_3^v, w) = 3 + j + (p-1) + 2 + (n-1-q) = -q + j + p + n + 3$$

$$b^{\Pi}(P_4^v, w) = (x-j) + (m-p-1) + 1 + k' + (q-x) = q - j - p + m + k'$$

Whereas w will receive the message from the two paths P_3^u and p_4^u at time:

$$b_u^{\Pi}(P_3^u, w) = p + 2 + (n-1-q) = -q + p + n + 1$$

$$b_u^{\Pi}(P_4, w) = 2 + (x-1) + (m-(p+1)) + 2 + (q(x+1)) + k' = q - p + m + k' + 1$$

Thus, every vertex $w = (p,q) \in V$, with $\ell < q \leq n-1$ and $q = 0$, will receive the message from each of the originators u and v at most at time:

$$t_v^{\Pi}(w) \leq \left\lfloor \frac{b^{\Pi}(P_3^v, w) + b^{\Pi}(P_4^v, w)}{2} \right\rfloor$$

$$= \left\lfloor \frac{(-q+j+p+n+3) + (q-j-p+m+k')}{2} \right\rfloor$$

$$= \left\lfloor \frac{m+n+k'+3}{2} \right\rfloor$$

$$t_u^{\Pi}(w) \leq \left\lfloor \frac{b^{\Pi}(P_3^u, w) + b^{\Pi}(P_4^u, w)}{2} \right\rfloor$$

$$= \left\lfloor \frac{(-q+p+n+1)+(q-p+m+k'+1)}{2} \right\rfloor$$

$$= \left\lfloor \frac{m+n+k'+2}{2} \right\rfloor$$

Thus, $t_u^{\Pi}(w) < t_v^{\Pi}(w)$ for $\ell < q \leq n-1$ and $q = 0$.

Therefore, for any vertex $w = (p,q) \in V$ where $\ell < q \leq n-1$ and $q = 0$ and $0 < j < \ell$, $t_u^{\Pi}(w) < b_v^{\Pi}(w)$. This means that in $T_{m\times n}$, informing all the vertices $w \in V$ from originator $(0,0)$ will take less time than when the originator is $(0,j)$ with $0 < j < \ell$. □

3.6 Case when Originator is $(0, \ell)$

Lemma 3.6.1

Given a torus $T_{m\times n}$, consider the two vertices $u = (0,\ell) \in V$ and $v = (0,j) \in V$ where $0 \leq j \leq \ell$. Let $t_u^{\Pi}(w)$ denote the time at which a vertex $w = (p,q) \in V$ receives the message under ordering Π when the originator is u. Then,

$$t_u^{\Pi}(w) \leq t_v^{\Pi}(w)$$

Proof :

The proof of this lemma will proceed in a similar manner as that of lemma 3.5.1.

Thus, the parity of p does not effect the broadcasting time (lemma 3.4.1), without loss of generality we will assume that p is odd. Also, since the worst-case for an originator is when j is odd (lemma 3.3.1), we will assume j to be odd.

We first describe the paths that will inform all the vertices (p, q) of a given row p when the originator is $(0, j)$ in one case and $(0, \ell)$ in the other, and then compare the time by which each of these originators deliver the message to all the vertices of row p. Thus, consider the vertex $w = (p, q)$ where $0 \leq q \leq \ell$. When the originator is $v = (0, j)$, $w = (p, q)$ receives the message from the two paths P_1^v and P_2^v described in lemma 3.5.1. We will omit repeating the details of these two paths.

Assume $k \in \{1, 2, 3, 4\}$. When the originator is $u = (0, \ell)$, (p, q) receives the message from the following two paths (see figure 11):

$$P_1^v : (0, \ell) \xrightarrow{2} (0, \ell + 1) \xrightarrow{1} (0, \ell + 2) \xrightarrow{1} \dots \xrightarrow{1} (0, n - 1) \xrightarrow{1} (0, 0) \xrightarrow{1} (1, 0) \xrightarrow{1}$$
$$\dots \xrightarrow{1} (p - 1, 0) \xrightarrow{2} (p - 1, 1) \xrightarrow{1} \dots \xrightarrow{1} (p - 1, q) \xrightarrow{k} (p, q)$$

$$P_2^v : (0, \ell) \xrightarrow{1} (m - 1, \ell) \xrightarrow{1} (m - 2, \ell) \xrightarrow{1} \dots \xrightarrow{1} (p + 1, \ell) \xrightarrow{1} (p, \ell) \xrightarrow{2}$$
$$(p, \ell - 1) \xrightarrow{1} (p, \ell - 2) \xrightarrow{1} \dots \xrightarrow{1} (p, q)$$

Vertex $w = (p, q)$ will receive the message from the two paths P_1^v and p_2^v at time:

$$b^{\Pi}(P_1^v, w) = q + j + p + k + 2$$

$$b^{\Pi}(P_2^v, w) = -q - j - p + m + 2\ell + 1$$

Whereas w will receive the message from the two paths P_1^u and p_2^u at time :

$$b^\Pi(P_1^u, w) = 2 + (n - (\ell + 1)) + (p - 1) + 2 + (q - 1) + k = q + p + n - \ell + k + 1$$

$$b^\Pi(P_2^u, w) = (m - p) + 2 + (\ell - 1 - q) = -q - p + m + \ell + 1$$

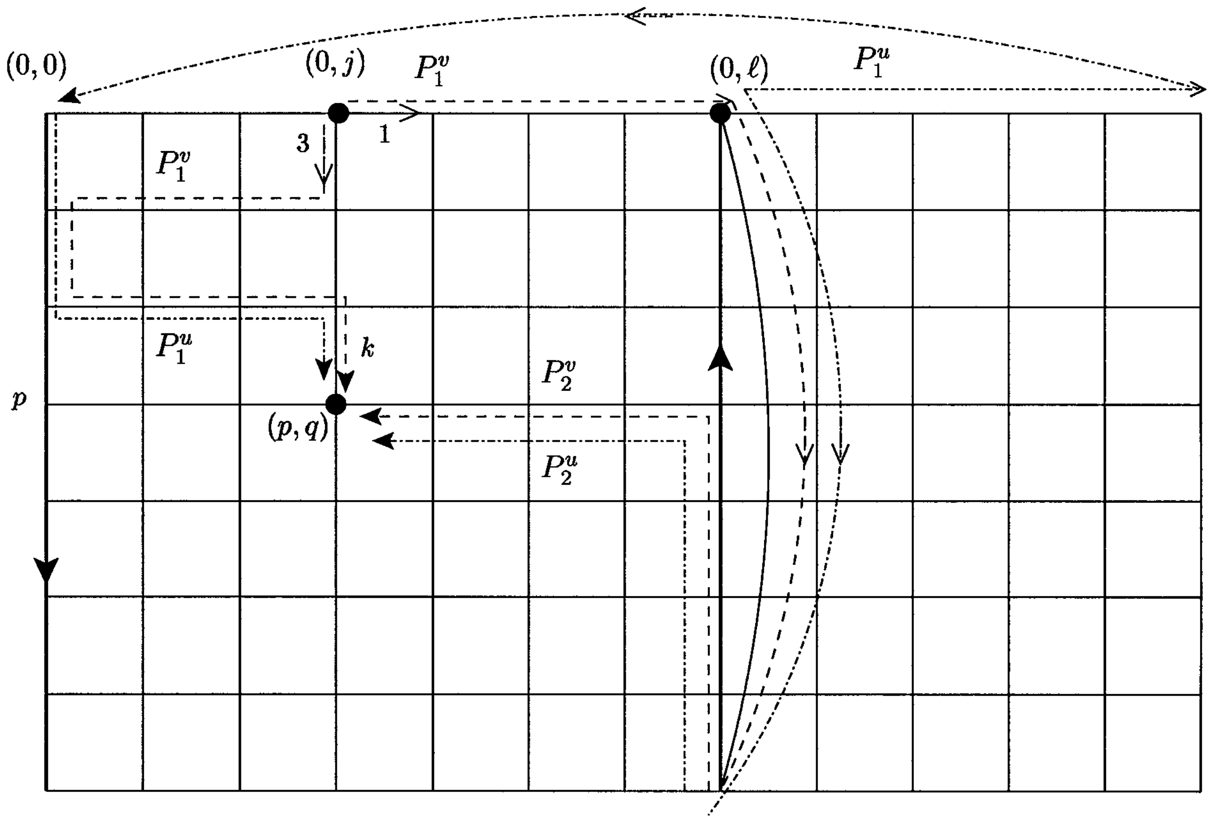

Figure 11: The paths P_1^u and P_2^u originate from $(0, \ell)$ and inform all the vertices (p, q) on row p, while the paths P_1^v and P_2^v originate from vertex $(0, j)$ and inform all the vertices (p, q) on row p where $(0 < j < \ell$

Thus, every vertex $w = (p, q) \in V$, with $0 \leq q \leq \ell$, will receive the message at most at time:

$$t_v^\Pi(w) \leq \left\lfloor \frac{m + 2\ell + k + 3}{2} \right\rfloor$$

$$t_u^\Pi(w) \leq \left\lfloor \frac{b^\Pi(P_1^u, w) + b^\Pi(P_2^u, w)}{2} \right\rfloor$$

$$= \left\lfloor \frac{(q+p+n-\ell+k+1)+(-q-p+m+\ell+1)}{2} \right\rfloor$$

$$= \left\lfloor \frac{m+n+k+2}{2} \right\rfloor$$

In the worst case, the minimum possible value of $2\ell = n-1$ (when n is odd and $\left\lfloor \frac{n}{2} \right\rfloor$ is odd). Thus,

$$t_v^{\Pi}(w) \leq \left\lfloor \frac{m+(n-1)+k+3}{2} \right\rfloor = \left\lfloor \frac{m+n)+k+2}{2} \right\rfloor$$

Which implies that $t_u^{\Pi}(w) \leq t_v^{\Pi}(w) \quad$ for $0 \leq q \leq \ell$.

Similarly, for $\ell \leq q \leq n-1$. When the originator is $v = (0,j)$, $w = (p,q)$ receives the message from the two paths P_3^v and P_4^v described in lemma 3.5.1. We will omit repeating the details of these two paths.

When the originator is $u = (0,\ell)$, $w = (p,q)$ receives the message from the following two paths:

$P_3^u : (0,\ell) \xrightarrow{2} (0,\ell+1) \xrightarrow{1} (0,\ell+2) \xrightarrow{1} \ldots \xrightarrow{1} (0,n-1) \xrightarrow{1} (0,0) \xrightarrow{1} (1,0) \xrightarrow{1}$

$\ldots \xrightarrow{1} (p-1,0) \xrightarrow{1} (p,0) \xrightarrow{2} (p,n-1) \xrightarrow{1} (p,n-2) \xrightarrow{1} \ldots \xrightarrow{1} (p,q)$

$P_4^u : (0,\ell) \xrightarrow{1} (m-1,\ell) \xrightarrow{1} (m-2,\ell) \xrightarrow{1} \ldots \xrightarrow{1} (p+1,\ell) \xrightarrow{2} (p+1,\ell+1) \xrightarrow{1}$

$\ldots \xrightarrow{1} (p+1,q) \xrightarrow{k'} (p,q)$

Vertex $w = (p,q)$ will receive the message from the two paths P_3^v and p_4^v at time:

$$b^{\Pi}(P_3^v, w) = -q + j + p + n + 3$$

$$b^{\Pi}(P_4^v, w) = q - j - p + m + k^{'}$$

Whereas w will receive the message from the two paths P_3^u and p_4^u at time :

$$b^{\Pi}(P_3^u, w) = 2 + (n - (\ell + 1)) + p + 2 + (n - 1 - q) = -q + p + 2n - \ell + 2$$

$$b^{\Pi}(P_4^u, w) = (m - (p + 1)) + 2 + (q - (\ell + 1)) + k^{'} = q - p + m - \ell + k^{'}$$

Thus, every vertex $w = (p,q) \in V$, with $\ell \leq q \leq n-1$, will receive the message at most at time:

$$t_v^{\Pi}(w) \leq \left\lfloor \frac{m + n + k^{'} + 3}{2} \right\rfloor$$

$$t_u^{\Pi}(w) \leq \left\lfloor \frac{b^{\Pi}(P_3^u, w) + b^{\Pi}(P_4^u, w)}{2} \right\rfloor$$

$$= \left\lfloor \frac{(-q + p + 2n - \ell + 2) + (q - p + m - \ell + k^{'})}{2} \right\rfloor$$

$$= \left\lfloor \frac{m + 2n - 2\ell + k^{'} + 2}{2} \right\rfloor$$

In the worst case, the minimum possible value of $2\ell = n - 1$. Thus,

$$t_u^{\Pi}(w) \leq \left\lfloor \frac{m + 2n - (n-1) + k^{'} + 2}{2} \right\rfloor = \left\lfloor \frac{m + n + k^{'} + 3}{2} \right\rfloor$$

This implies that $t_u^{\Pi}(w) \leq t_v^{\Pi}(w) \qquad$ for $\ell \leq q \leq n-1$

Therefore, for any vertex $w = (p,q) \in V$ where $0 \leq q \leq n-1$ and $0 < j < \ell$

$$t_u^{\Pi}(w) < b_v^{\Pi}(w)$$

This means that in $T_{m \times n}$, to inform all the vertices $w \in V$ from originator $(0,\ell)$ will take less time than when the originator was $(0,j)$ where $0 < j < \ell$. □

3.7 Symmetry

In this section we will show that the vertices of the rows of torus $T_{m \times n}$ have identical ordering. We will also show in lemma 3.7.2 that the paths used when broadcasting from a vertex (i,j) with $0 < j < \ell$ and j odd, are identical to the paths used when broadcasting from an originator (i,j') where $\ell < j' < n$ and j' even, for any row of the torus $T_{m \times n}$.

Observation 3.7.1

From the ordering Π *we can see that all the vertices of the odd-numbered rows* p *of* $T_{m \times n} = (V,E)$ *have identical numbering. The same can be said about the even-numbered rows* p' *of* $T_{m \times n}$. *That is, every* $(i,j) \in V$ *has an ordering identical to* $(i+2 \text{ mod } m, j) \in V$ *for all* $0 \leq j \leq n-1$.

Also, the only difference between the even-numbered rows and the odd-numbered rows is the direction of the edges labeled 1 *and* 4 *(see section 3.2 for the details).*

Thus, a vertex $(i, \ell - j) \in V$ *is symmetric to* $(i+1, j) \in V$ *where* $0 \leq j \leq \ell$. *Also, a vertex* $(i, j) \in V$ *is symmetric to :*

- $(i+1 \text{ mod } m, \ell - j) \in V$ *where* $0 \leq j \leq \ell$
- $(i+1 \text{ mod } m, \ell - j + n \text{ mod } n) \in V$ *where* $\ell \leq j \leq n-1$

This means that every vertex $(i, j) \in V$ *has a symmetric vertex* $(i', j') \in V$ *on any row* i' *of* $T_{m \times n}$. *Thus, the maximum broadcast time among all the vertices of a certain row* i *in* $T_{m \times n}$ *will also be the broadcast time of torus* $T_{m \times n}$.

Observation 3.7.2

The paths used to inform all the vertices $(p, q) \in V$ *where* $0 \leq q \leq \ell$ *from any originator* $(i, j) \in V$ *with* $0 < j < \ell$ *and* j *odd are similar to the paths used to inform vertices* $(p, q) \in V$ *where* $\ell \leq q \leq n-1$ *from any originator* $(i, j') \in V$ *where* $\ell < j' < n$ *and* j' *even. The same can be said when we take* j *to be even and* j' *to be odd.*

We illustrate the observation with figures 12 and 13. We will discuss these paths in detail later on.

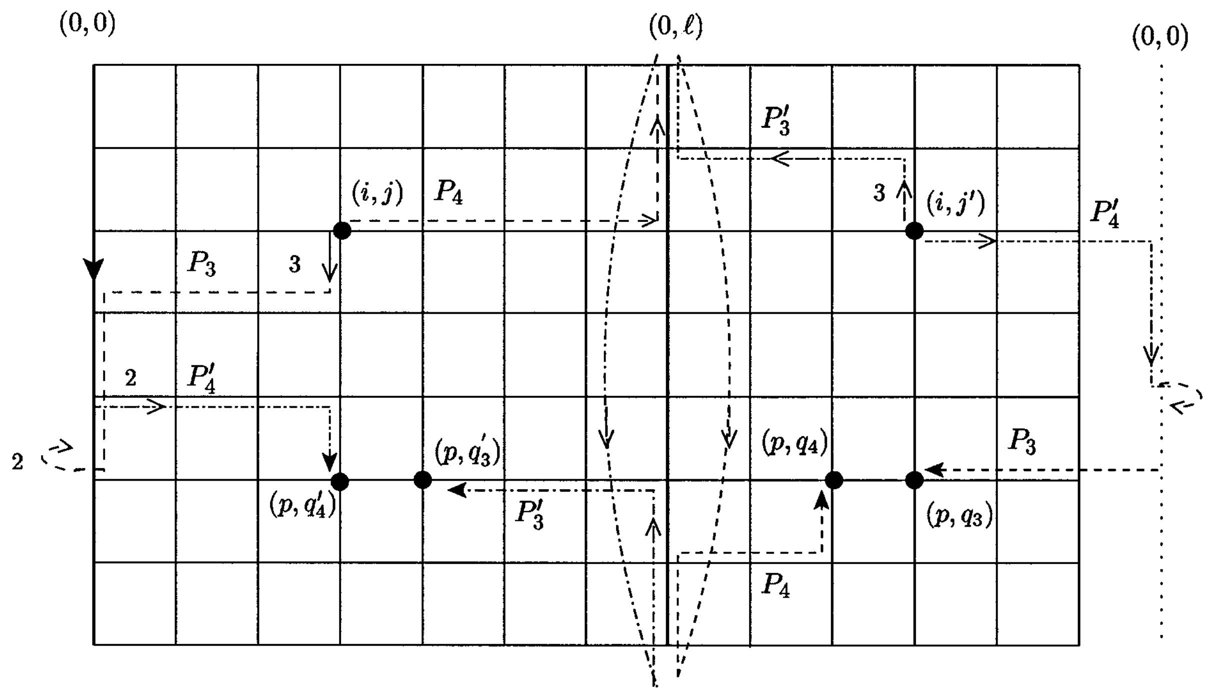

Figure 12: Path P_3 is similar to path P_3' and path P_4 is similar to path P_4'.

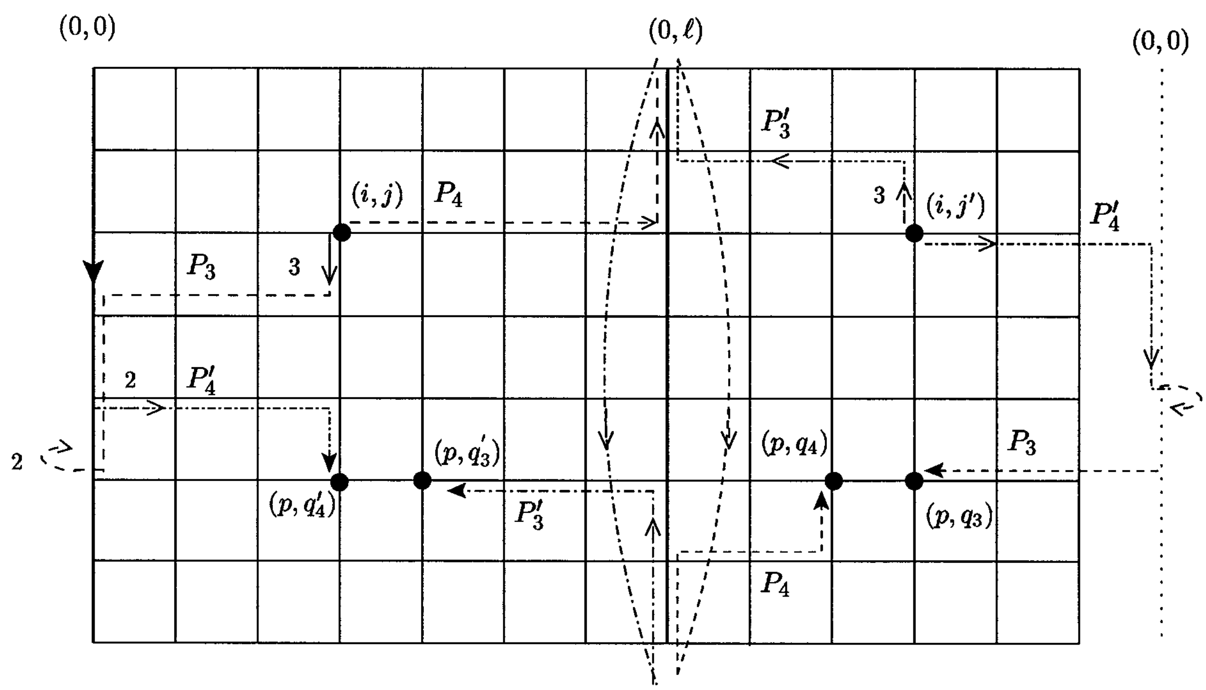

Figure 13: Path P_3 is similar to path P_3' and path P_4 is similar to path P_4'.

3.8 The Upper Bound

In this section we calculate the upper bound on $T_{m \times n}$ in two different cases : First we consider m to be even and calculate the upper bound in this case; then, we do the same calculation for m odd.

From lemma 3.3.1 we know that the worst-case originator $(0, j) \in V$ for any $0 < j < \ell$ is when j is odd. Observing the rows of $T_{m \times n}$, it is easy to see that the vertices $(i, j) \in V$ are symmetric to vertices $(i, j') \in V$ where $0 < j < \ell$ and $\ell < j' < n$, and that the paths used to broadcast from any vertex $(0, j)$ where j is odd will be identical to the paths used to broadcast from any $(0, j')$ where j' is even (and vice versa). Below, we will see these paths in more detail.

Moreover, since all the rows of $T_{m \times n}$ are symmetric (observation 3.7.1), then broadcasting from any originator $(i, j) \in V$ where $0 \leq j \leq n - 1$ will give the same result as broadcasting from a vertex $(0, j) \in V$ for the same j. Observe also that since broadcasting from originators $(i, 0) \in V$ and $(i, \ell) \in V$ yields a smaller broadcast time (lemmas 3.5.1 and 3.6.1), we will discard these originators.

Therefore, to calculate the broadcast time of torus $T_{m \times n}$, it is enough to describe the dissemination of the message from two different originators : One from vertex $u = (0, j) \in V$ where $0 < j < \ell$ and j is odd and the other from $u' = (0, j') \in V$ where $\ell < j' < n$ and j' is even.

3.8.1 Upper Bound when m is Even

In this section, we will assume m is even and discuss the upper bound on $T_{m\times n}$ for this case.

Lemma 3.8.1

Given a torus $T_{m\times n}$ where m is even, the orderly broadcast time under ordering Π *will be:*

$$b^{\Pi}(T_{m\times n}) \leq \begin{cases} D(T_{m\times n})+3 & \text{if } \frac{n}{2} \text{ is odd} \\ D(T_{m\times n})+4 & \text{otherwise} \end{cases}$$

Proof :

Through lemma 3.4.1 we know that in the parity of p does not effect the broadcast time, thus without loss of generality we will assume that p is odd.

Case A: Originator $u = (0, j) \in V$ where $0 < j < \ell$ and j odd

We will first discuss the case when the originator is vertex $u = (0, j)$. We will again use the four paths described in lemma 3.4.1.

First, we inform all the vertices (p, q) where $0 < q \leq \ell$. Let $v_1 = (p, q_1) \in V$ and $v_2 = (p, q_2) \in V$ be two vertices where $0 \leq q_1, q_2 \leq \ell$. Suppose (p, q_1) receives the message from $u = (0, j)$ through path P_1 and (p, q_2) receives the message from $u = (0, j)$ through path P_2. Let $k \in \{1, 2, 3, 4\}$. Then,

$$P_1 : (0, j) \xrightarrow{3} (1, j) \xrightarrow{1} (1, j-1) \xrightarrow{1} \dots \xrightarrow{1} (1, 0) \xrightarrow{1} (2, 0) \xrightarrow{1} \dots$$
$$\xrightarrow{1} (p-1, 0) \xrightarrow{2} (p-1, 1) \xrightarrow{1} \dots \xrightarrow{1} (p-1, q_1) \xrightarrow{k} (p, q_1)$$
$$P_2 : (0, j) \xrightarrow{1} (0, j+1) \xrightarrow{1} (0, j+2) \xrightarrow{1} \dots \xrightarrow{1} (0, \ell) \xrightarrow{1} (m-1, \ell) \xrightarrow{1} \dots$$
$$\xrightarrow{1} (p, \ell) \xrightarrow{2} (p, \ell-1) \xrightarrow{1} (p, \ell-2) \xrightarrow{1} \dots \xrightarrow{1} (p, q_2)$$

First observe that the only time when $k = 4$ is when $q_1 = \ell$. But the vertex (p, ℓ) is already being informed through path P_2. Therefore, we will assume that $q_1 < \ell$ and hence the maximum value of k is 3.

On rows p and $p-1$, the paths P_1 and P_2 will inform the following vertices:

(p, q_1) and $(p, q_2), (p, q_2+1), (p, q_2+2), \dots (p, \ell)$.

$(p-1, 0), (p-1, 1), \dots (p-1, q_1)$.

According to the ordering Π, each of the informed vertices on row $p-1$ needs at most 3 time units to inform its neighbor on row p.

Suppose vertex $(p-1, q_1)$ receives the message at time t. This means that at time t all the vertices $(p-1, 0), (p-1, 1), \dots (p-1, q_1)$ are also informed. Then, by time at most $t+3$, each of the informed vertices $(p-1, i)$ where $0 \leq i < q_1$ sends the message to its corresponding neighbor (p, i). Thus, by the time that (p, q_1) receives the message, all the vertices (p, i) where $0 \leq i < q_1$ will also be informed (Figure 14).

Now, to make sure that all the vertices (p, q) where $0 \leq q \leq \ell$ are informed, (p, q_1) and (p, q_2) have to be (at least) adjacent at a given time t.

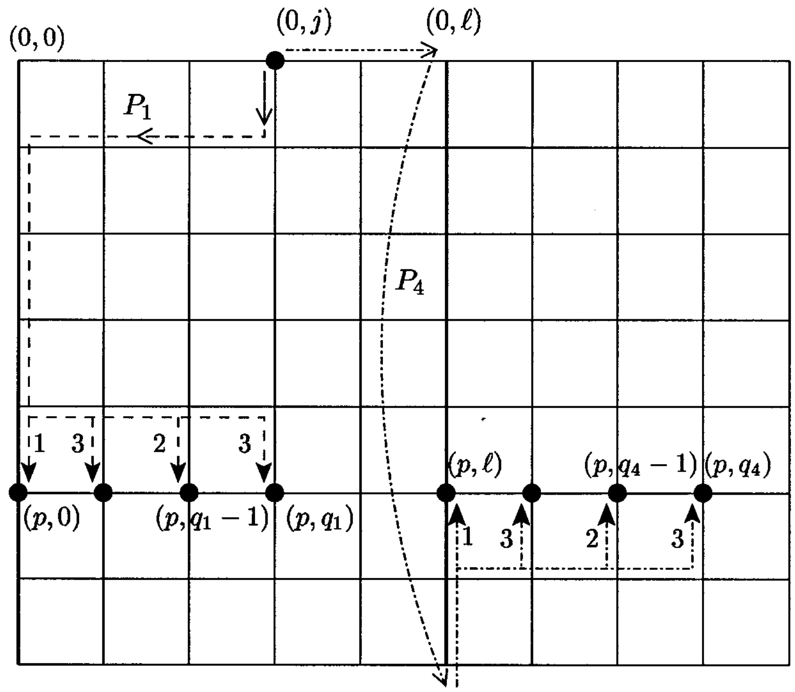

Figure 14: By the time vertex (p, q_1) receives the message through path P_1, all the vertices (p, i) where $0 \leq i < q_1$ are already informed. Similarly, by the time vertex (p, q_4) receives the message through path P_4, all the vertices (p, i) where $\ell \leq i < q_4$ are already informed.

The time that (p, q_1) and (p, q_2) receive the message from each of the paths P_1 and P_2 are:

$$b^{\Pi}(P_1, v_1) = 3 + j + (p - 1 - 1) + 2 + (q_1 - 1) + 3 = q_1 + j + p + 5$$

$$b^{\Pi}(P_2, v_2) = (\ell - j) + 1 + (m - 1 - p) + 2 + (\ell - 1 - q_2) = -q_2 - j - p + m + 2\ell + 1$$

Suppose $v_1 = (p, q_1)$ and $v_2 = (p, q_2)$ receive the message by time $D(T_{m \times n}) + c$ (where $c \in Z^+$). Then,

$$b^{\Pi}(P_1, v_1) \leq D(T_{m\times n}) + c \Rightarrow q_1 + j + p + 5 \leq D(T_{m\times n}) + c$$

$$\Rightarrow q_1 \leq D(T_{m\times n}) + c - j - p - 5$$

$$b^{\Pi}(P_2, v_2) \leq D(T_{m\times n}) + c \Rightarrow -q_2 - j - p + m + 2\ell + 1 \leq D(T_{m\times n}) + c$$

$$\Rightarrow q_2 \geq -D(T_{m\times n}) - c - j - p + m + 2\ell + 1$$

Since the path P_1 propagates from $(p, 0) \rightarrow (p, 1) \rightarrow \ldots \rightarrow (p, q_1)$ and path P_2 propagates from $(p, \ell) \rightarrow (p, \ell - 1) \rightarrow \ldots \rightarrow (p, q_2)$, then at time $D(T_{m\times n}) + c$, the last vertices that receive the message though P_1 and P_2 will be the maximum value of q_1 and the minimum value of q_2 respectively (Figure 15).

Thus, (p, q_1) and (p, q_2) are going to be (at least) adjacent when :

$$Maximum(q_1) + 1 \geq Minimum(q_2)$$

$$\Leftrightarrow D(T_{m\times n}) + c - j - p - 5 + 1 \geq -D(T_{m\times n}) - c - j - p + m + 2\ell + 1$$

$$\Leftrightarrow 2D(T_{m\times n}) + 2c - m - 2\ell - 5 \geq 0 \ldots\ldots\ldots (A)$$

Now, we look at all the vertices (p, q) such that $\ell \leq q \leq n-1$. Let $v_3 = (p, q_3) \in V$ and $v_4 = (p, q_4) \in V$ be two vertices where $\ell \leq q_3, q_4 \leq n-1$. Suppose (p, q_3) receives the message from u through path P_3 and (p, q_4) receives the message from u through path P_4. Let $k' \in \{1, 2, 3, 4\}$ (see figure 15). Then,

$$P_3 : (0, j) \xrightarrow{3} (1, j) \xrightarrow{1} (1, j-1) \xrightarrow{1} \ldots \xrightarrow{1} (1, 0) \xrightarrow{1} (2, 0) \xrightarrow{1} \ldots$$

$$\xrightarrow{1} (p-1, 0) \xrightarrow{1} (p, 0) \xrightarrow{2} (p, n-1) \xrightarrow{1} (p, n-2) \xrightarrow{1} \ldots \xrightarrow{1} (p, q_3)$$

$$P_4 : (0, j) \xrightarrow{1} (0, j+1) \xrightarrow{1} (0, j+2) \xrightarrow{1} \ldots \xrightarrow{1} (0, \ell) \xrightarrow{1} (m-1, \ell) \xrightarrow{1} \ldots$$

$$\xrightarrow{1} (p+1, \ell) \xrightarrow{2} (p+1, \ell+1) \xrightarrow{1} \ldots \xrightarrow{1} (p+1, q_4) \xrightarrow{k'} (p, q_4)$$

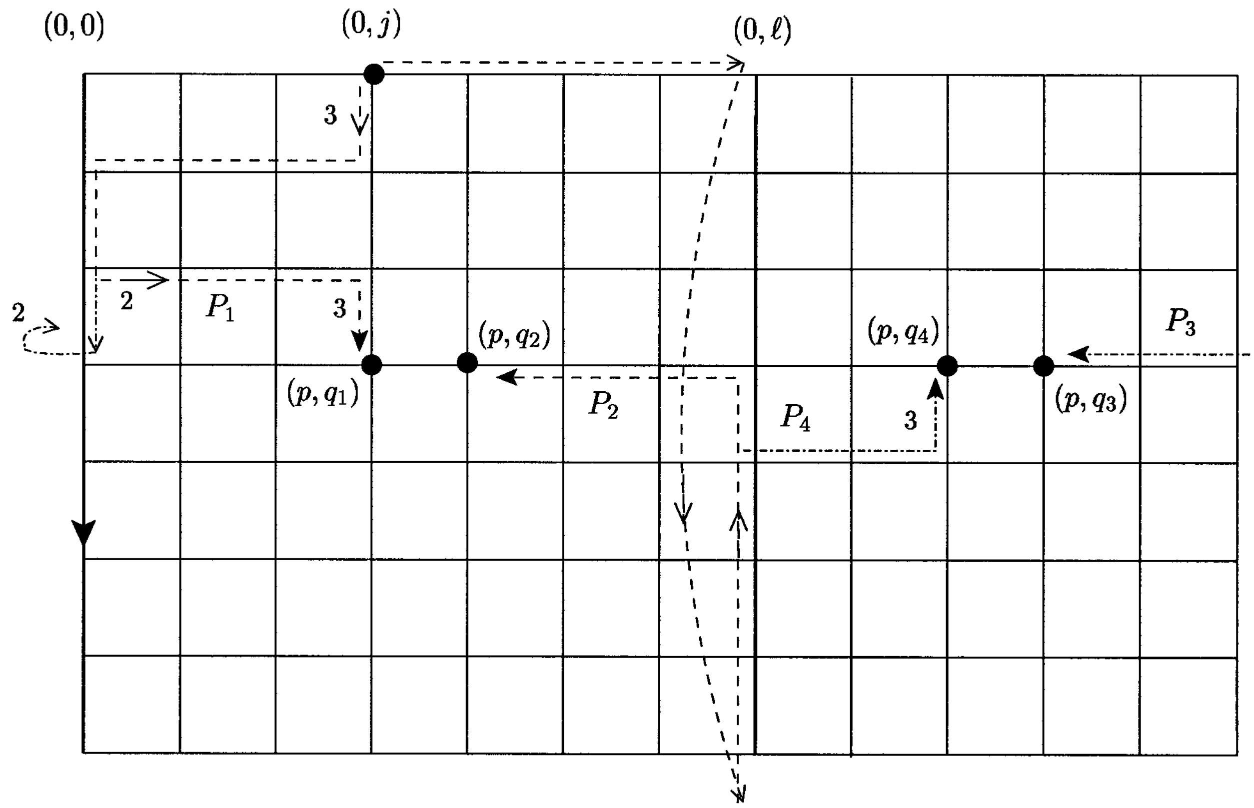

Figure 15: The four paths P_1, P_2, P_3 and P_4. Note that some of the edges are deleted in order to make the figure more readable.

Similarly, observe that the only time when $k' = 4$ is when $q_4 = 0$. But the vertex $(p, 0)$ is already being informed through path P_3. Therefore, we will assume that $q_4 < n$ and hence the maximum value of k' is 3.

On rows p and $p+1$, the paths P_3 and P_4 will inform the following vertices:

(p, q_4) and $(p, q_3), (p, q_3+1), (p, q_3+2), \ldots (p, n-1), (p, 0)$.

$(p+1, \ell), (p+1, \ell+1), \ldots (p+1, q_4)$.

According to the ordering Π, each of the informed vertices on row $p+1$ needs at most 3 time units to inform its neighbor on row p. Similarly, we can say that when $k' = 3$, then by the time that (p, q_4) receives the message, all the vertices (p, i) where

$\ell \leq i < q_4$ will also be informed (Figure 14).

To make sure that all the vertices (p, q) where $\ell \leq q \leq n-1$ are informed, (p, q_3) and (p, q_4) have to be (at least) adjacent at a given time t.

The time that (p, q_3) and (p, q_4) receive the message from each of the paths P_3 and P_4 are:

$b^{\Pi}(P_3, v_3) = 3 + j + (p-1) + 2 + (n-1-q_3) = -q_3 + j + p + n + 3$

$b^{\Pi}(P_4, v_4) = (\ell - j) + (m-p-1) + 2 + (q_4 - \ell - 1) + 3 = q_4 - j - p + m + 3$

Suppose (p, q_3) and (p, q_4) receive the message at time $D(T_{m \times n}) + c$ (where $c \in Z^+$). Then,

$$b^{\Pi}(P_3, v_3) \leq D(T_{m \times n}) + c \Rightarrow -q_3 + j + p + n + 3 \leq D(T_{m \times n}) + c$$
$$\Rightarrow q_3 \geq -D(T_{m \times n}) - c + j + p + n + 3$$

$$b^{\Pi}(P_4, v_4) \leq D(T_{m \times n}) + c \Rightarrow q_4 - j - p + m + 3 \leq D(T_{m \times n}) + c$$
$$\Rightarrow q_4 \leq D(T_{m \times n}) + c + j + p - m - 3$$

Since the path P_3 propagates from $(p, 0) \rightarrow (p, n-1) \rightarrow \ldots \rightarrow (p, q_3)$ and path P_4 propagates from $(p, \ell) \rightarrow (p, \ell + 1) \rightarrow \ldots \rightarrow (p, q_4)$, then at time $D(T_{m \times n}) + c$, the last vertices that receive the message though P_3 and P_4 will be the minimum value of q_3 and the maximum value of q_4 respectively.

Thus, (p, q_3) and (p, q_4) are going to be (at least) adjacent when :

$$Maximum(q_4) + 1 \geq Minimum(q_3)$$

$$\Leftrightarrow D(T_{m\times n}) + c + j + p - m - 3 + 1 \geq -D(T_{m\times n}) - c + j + p + n + 3$$

$$\Leftrightarrow 2D(T_{m\times n}) + 2c - m - n - 5 \geq 0 \ldots\ldots\ldots \text{(B)}$$

When the inequalities (A) and (B) hold true for some constant c, then vertex q_1 will be adjacent to q_2 and vertex q_3 will be adjacent to q_4, and hence, all the vertices on row p will be informed. In what follows we will find the value of c for each of the cases for m and n in torus $T_{m\times n}$, and derive the broadcast time when the originator is vertex u.

Case 1: m and n are both EVEN

Since m and n are both even

$$\Rightarrow D(T_{m\times n}) = \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor = \frac{m+n}{2}$$

(A) $\Rightarrow 2\left(\frac{m+n}{2}\right) + 2c - m - 2\ell - 5 \geq 0 \Rightarrow n - 2\ell + 2c - 5 \geq 0$

i) if $\left\lfloor \frac{n}{2} \right\rfloor$ is odd $\Rightarrow \ell = \left\lfloor \frac{n}{2} \right\rfloor = \frac{n}{2}$

$$\Rightarrow n - 2\ell + 2c - 5 \geq 0 \Rightarrow n - 2\left(\frac{n}{2}\right) + 2c - 5 \geq 0 \Rightarrow 2c - 5 \geq 0 \Rightarrow c \geq 3$$

ii) if $\left\lfloor \frac{n}{2} \right\rfloor$ is even $\Rightarrow \ell = \left\lfloor \frac{n}{2} \right\rfloor + 1 = \frac{n}{2} + 1$

$$\Rightarrow n - 2\ell + 2c - 5 \geq 0 \Rightarrow n - 2(\frac{n}{2} + 1) + 2c - 5 \geq 0 \Rightarrow 2c - 7 \geq 0 \Rightarrow c \geq 4$$

$$(B) \Rightarrow 2\left(\frac{m+n}{2}\right) + 2c - m - n - 5 \geq 0 \Rightarrow 2c - 5 \geq 0 \Rightarrow c \geq 3$$

Case 2: m is EVEN and n is ODD

Since m is even and n is odd

$$\Rightarrow D(T_{m\times n}) = \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor = \frac{m}{2} + \frac{n-1}{2} = \frac{m+n-1}{2}$$

$$(A) \Rightarrow 2\left(\frac{m+n-1}{2}\right) + 2c - m - 2\ell - 5 \geq 0 \Rightarrow n - 2\ell + 2c - 6 \geq 0$$

i) if $\left\lfloor \frac{n}{2} \right\rfloor$ is odd $\Rightarrow \ell = \left\lfloor \frac{n}{2} \right\rfloor = \frac{n-1}{2}$

$$\Rightarrow n - 2\ell + 2c - 6 \geq 0 \Rightarrow n - 2\left(\frac{n-1}{2}\right) + 2c - 6 \geq 0 \Rightarrow 2c - 5 \geq 0 \Rightarrow c \geq 3$$

ii) if $\left\lfloor \frac{n}{2} \right\rfloor$ is even $\Rightarrow \ell = \left\lfloor \frac{n}{2} \right\rfloor + 1 = \frac{n-1}{2} + 1 = \frac{n+1}{2}$

$$\Rightarrow n - 2\ell + 2c - 6 \geq 0 \Rightarrow n - 2(\frac{n+1}{2}) + 2c - 6 \geq 0 \Rightarrow 2c - 7 \geq 0 \Rightarrow c \geq 4$$

$$(B) \Rightarrow 2\left(\frac{m+n-1}{2}\right) + 2c - m - n - 5 \geq 0 \Rightarrow 2c - 6 \geq 0 \Rightarrow c \geq 3$$

Therefore, for any originator $(0, j)$ with $0 < j < \ell$ and j odd, every vertex $v \in V$ for a torus with even number of rows m will be informed at most by time:

$b^{\Pi}(T_{m\times n}) \leq D(T_{m\times n}) + 4$

Case B: Originator $u' = (0, j') \in V$ where $\ell < j' < n$ and j even

We will now calculate the broadcast time when the originator is $u' = (0, j') \in V$ where $\ell < j' < n$.

When j' is even, then the paths P_1', P_2', P_3' and P_4' originating from u' will be respectively symmetric to the paths P_1, P_2, P_3 and P_4 described above. Thus, consider the vertices $v_1' = (p, q_1')$, $v_2' = (p, q_2')$, $v_3' = (p, q_3')$ and $v_4' = (p, q_4')$ where $\ell \leq q_1', q_2' \leq n-1$ and $0 \leq q_3', q_4' \leq \ell$ and let $k, k' \in \{1, 2, 3, 4\}$. We define the four paths (see figure 16):

$$P_1' : (0, j') \xrightarrow{3} (m-1, j') \xrightarrow{1} (m-1, j'-1) \xrightarrow{1} \ldots \xrightarrow{1} (m-1, \ell) \xrightarrow{1} (m-2, \ell) \xrightarrow{1} \ldots$$
$$\xrightarrow{1} (p+1, \ell) \xrightarrow{2} (p+1, \ell+1) \xrightarrow{1} \ldots \xrightarrow{1} (p+1, q_1') \xrightarrow{k} (p, q_1')$$

$$P_2' : (0, j') \xrightarrow{1} (0, j'+1) \xrightarrow{1} (0, j'+2) \xrightarrow{1} \ldots \xrightarrow{1} (0, 0) \xrightarrow{1} (1, 0) \xrightarrow{1} (2, 0) \xrightarrow{1} \ldots$$
$$\xrightarrow{1} (p, 0) \xrightarrow{2} (p, n-1) \xrightarrow{1} (p, n-2) \xrightarrow{1} \ldots \xrightarrow{1} (p, q_2')$$

$$P_3' : (0, j') \xrightarrow{3} (m-1, j') \xrightarrow{1} (m-1, j'-1) \xrightarrow{1} \ldots \xrightarrow{1} (m-1, \ell) \xrightarrow{1} (m-2, \ell) \xrightarrow{1} \ldots$$
$$\xrightarrow{1} (p-1, \ell) \xrightarrow{1} (p, \ell) \xrightarrow{2} (p, \ell-1) \xrightarrow{1} (p, \ell-2) \xrightarrow{1} \ldots \xrightarrow{1} (p, q_3')$$

$$P_4' : (0, j') \xrightarrow{1} (0, j'+1) \xrightarrow{1} (0, j'+2) \xrightarrow{1} \ldots \xrightarrow{1} (0, 0) \xrightarrow{1} (1, 0) \xrightarrow{1} (2, 0) \xrightarrow{1} \ldots$$
$$\xrightarrow{1} (p-1, 0) \xrightarrow{2} (p-1, 1) \xrightarrow{1} \ldots \xrightarrow{1} (p-1, q_4') \xrightarrow{k'} (p, q_4')$$

For the same reasons mentioned in Case A above, we will assume that the worst case for k and k' is equal to 3.

The time at which each of the vertices q_1', q_2', q_3' and q_4' receives the message is :

$$b^{\Pi}(P_1', v_1') = 3 + (j' - \ell) + (m - (p+1) - 1) + 2 + (q_1' - \ell - 1) + 3 = q_1' + j' - p + m - 2\ell + 5$$

$$b^{\Pi}(P_2', v_2') = (n - j') + p + 2 + (n - q_2' - 1) = -q_2' - j' + p + 2n + 1$$

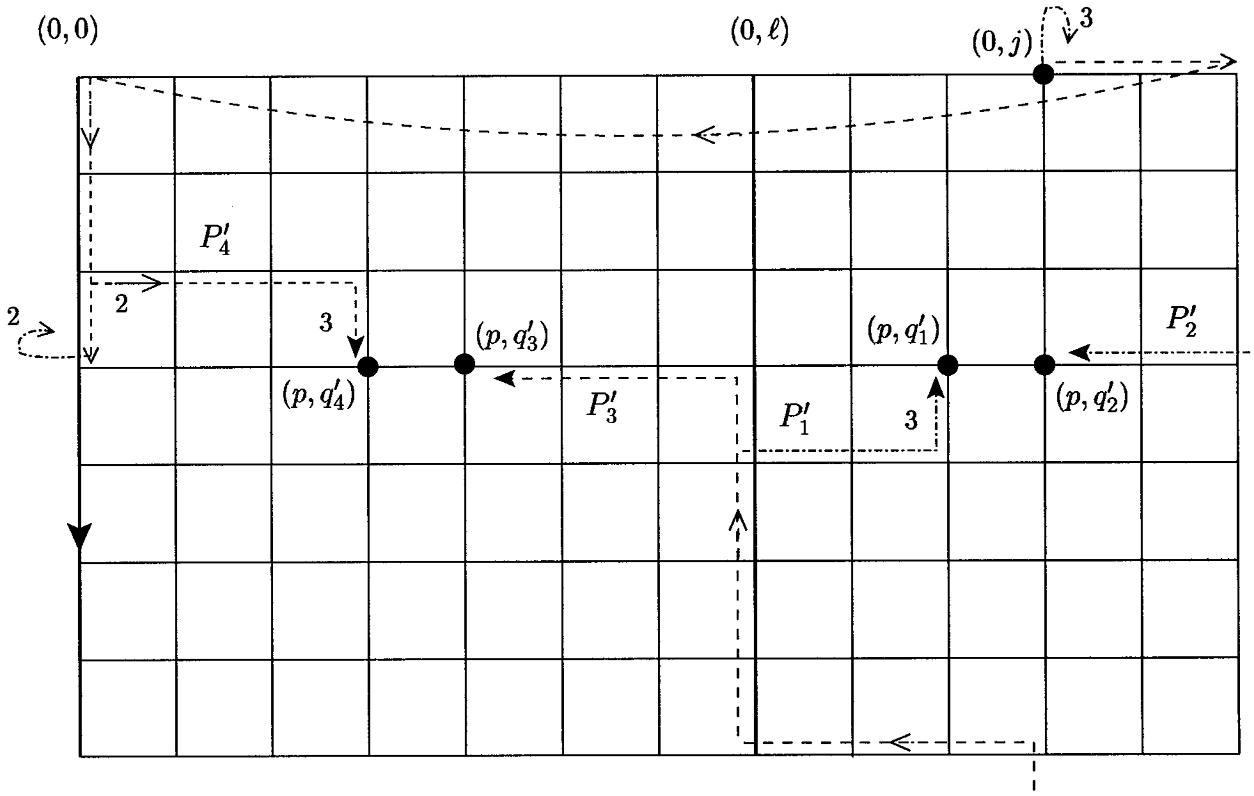

Figure 16: The four paths P_1', P_2', P_3' and P_4'. Note that some of the edges are deleted in order to make the figure more readable.

$$b^{\Pi}(P_3', v_3') = 3 + (j' - \ell) + (m - p - 1) + 2 + (\ell - 1 - q_3') = -q_3' + j' - p + m + 3$$

$$b^{\Pi}(P_4', v_4') = (n - j') + (p - 1) + 2 + (q_4' - 1) + 3 = q_4' - j' + p + n + 3$$

Similarly, we assume that each of these vertices receives the message at most by time $D(T_{m\times n}) + c$. Thus,

$$q_1' \le D(T_{m\times n}) + c - j' + p - m + 2\ell - 5$$

$$q_2' \ge -D(T_{m\times n}) - c - j' + p + 2n + 1$$

$$q_3' \ge -D(T_{m\times n}) - c + j' - p + m + 3$$

$$q_4' \le D(T_{m\times n}) + c + j' - p - n - 3$$

We know that all the vertices of $(p,q) \in V$ on row p will be informed when q_1' is adjacent to q_2' and q_3' is adjacent to q_4'. This will happen when :

$$Maximum(q_1') + 1 \geq Minimum(q_2')$$

$$\Leftrightarrow D(T_{m\times n}) + c - j' + p - m + 2\ell - 5 + 1 \geq -D(T_{m\times n}) - c - j' + p + 2n + 1$$

$$\Leftrightarrow 2D(T_{m\times n}) + 2c - m - 2n + 2\ell - 5 \geq 0 \,\ldots\ldots\ldots (A')$$

$$Maximum(q_4') + 1 \geq Minimum(q_3')$$

$$\Leftrightarrow D(T_{m\times n}) + c + j' - p - n - 3 + 1 \geq -D(T_{m\times n}) - c + j' - p + m + 3$$

$$\Leftrightarrow 2D(T_{m\times n}) + 2c - m - n - 5 \geq 0 \,\ldots\ldots\ldots (B')$$

When the inequalities (A') and (B') hold true for some constant c, then vertex q_1' will be adjacent to q_2' and vertex q_3' will be adjacent to q_4', and hence, all the vertices on row p will be informed.

In what follows we will find the value of c for each of the cases for m and n in torus $T_{m\times n}$, and derive the broadcast time when the originator is vertex u'.

Case 1: m and n are both EVEN

Since m and n are both even

$$\Rightarrow D(T_{m\times n}) = \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor = \frac{m+n}{2}$$

$$(A') \Rightarrow 2\left(\frac{m+n}{2}\right) + 2c - m - 2n + 2\ell - 5 \geq 0 \Rightarrow -n + 2\ell + 2c - 5 \geq 0$$

i) if $\left\lfloor \frac{n}{2} \right\rfloor$ is odd $\Rightarrow \ell = \left\lfloor \frac{n}{2} \right\rfloor = \frac{n}{2}$

$$\Rightarrow -n + 2\ell + 2c - 5 \geq 0 \Rightarrow -n + 2\left(\frac{n}{2}\right) + 2c - 5 \geq 0 \Rightarrow 2c - 5 \geq 0 \Rightarrow c \geq 3$$

ii) if $\left\lfloor \frac{n}{2} \right\rfloor$ is even $\Rightarrow \ell = \left\lfloor \frac{n}{2} \right\rfloor + 1 = \frac{n}{2} + 1$

$$\Rightarrow -n + 2\ell + 2c - 5 \geq 0 \Rightarrow -n + 2(\frac{n}{2} + 1) + 2c - 5 \geq 0 \Rightarrow 2c - 7 \geq 0 \Rightarrow c \geq 4$$

$$(B') \Rightarrow 2\left(\frac{m+n}{2}\right) + 2c - m - n - 5 \geq 0 \Rightarrow 2c - 5 \geq 0 \Rightarrow c \geq 3$$

Case 2: m is EVEN and n is ODD

Since m is even and n is odd

$$\Rightarrow D(T_{m \times n}) = \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor = \frac{m}{2} + \frac{n-1}{2} = \frac{m+n-1}{2}$$

$$(A') \Rightarrow 2\left(\frac{m+n-1}{2}\right) + 2c - m - 2n + 2\ell - 5 \geq 0 \Rightarrow -n + 2\ell + 2c - 6 \geq 0$$

i) if $\left\lfloor \frac{n}{2} \right\rfloor$ is odd $\Rightarrow \ell = \left\lfloor \frac{n}{2} \right\rfloor = \frac{n-1}{2}$

$$\Rightarrow -n + 2\ell + 2c - 6 \geq 0 \Rightarrow -n + 2\left(\frac{n-1}{2}\right) + 2c - 6 \geq 0 \Rightarrow 2c - 7 \geq 0 \Rightarrow c \geq 4$$

ii) if $\left\lfloor \frac{n}{2} \right\rfloor$ is even $\Rightarrow \ell = \left\lfloor \frac{n}{2} \right\rfloor + 1 = \frac{n-1}{2} + 1 = \frac{n+1}{2}$

$$\Rightarrow -n + 2\ell + 2c - 6 \geq 0 \Rightarrow -n + 2(\frac{n+1}{2}) + 2c - 6 \geq 0 \Rightarrow 2c - 5 \geq 0 \Rightarrow c \geq 3$$

$$(B') \Rightarrow 2\left(\frac{m+n-1}{2}\right) + 2c - m - n - 5 \geq 0 \Rightarrow 2c - 6 \geq 0 \Rightarrow c \geq 3$$

Thus, for any originator $(0, j')$ with $\ell < j' < n$ and j' even, every vertex $v \in V$ for a torus with even number of rows m will be informed at most by time:

$b^{\Pi}(T_{m\times n}) \leq D(T_{m\times n}) + 4$

Therefore, we can see that in the worst case, broadcasting from any vertex u or u' will give the same broadcast time. Thus, summarizing the results, we can say that when m is even, the broadcast time of a torus $T_{m\times n}$ is:

$$b^{\Pi}(T_{m\times n}) \leq \begin{cases} D(T_{m\times n}) + 3 & \text{if } \frac{n}{2} \text{ is odd} \\ D(T_{m\times n}) + 4 & \text{otherwise} \end{cases}$$

NOTE:

In the above discussion, we have assumed that the paths P_1, P_2, P_3 and P_4 will reach row p by time $D(T_{m\times n}) + c$. But what if $D(T_{m\times n}) + c$ is not enough of one of those paths to reach row p? In what follows, we will show that even in this case, all the vertices of row p will still be informed by time $D(T_{m\times n}) + 4$.

When P_1 and P_3 do not reach row p :

In this case, P_2 will inform all the vertices (p, q) where $0 \leq q \leq \ell$ and P_4 will inform all the vertices (p, q) where $\ell \leq q \leq n - 1$. To show this, we let $v_1 = v_3 = (p - 1, 0)$, $v_2 = (p, 0)$ (hence, $q_2 = 0$) and $v_4 = (p, n - 1)$ (hence $q_4 = n - 1$). Then,

$b^{\Pi}(P_1, v_1) = b^{\Pi}(P_1, v_3) = 3 + j + (p - 1 - 1) = j + p + 1$

$b^{\Pi}(P_2, v_2) = -j - p + m + 2\ell + 1$

$b^{\Pi}(P_4, v_4) = -j - p + m + n + 2$

Suppose each of the vertices v_1, v_2 and v_4 receive the message by time $D(T_{m\times n})+c$. Then,

$b^{\Pi}(P_1, v_1) \leq D(T_{m\times n}) + c \Rightarrow D(T_{m\times n}) + c - j - p - 1 \geq 0$ ....(1)

$b^{\Pi}(P_2, v_2) \leq D(T_{m\times n}) + c \Rightarrow D(T_{m\times n}) + c + j + p - m - 2\ell - 1 \geq 0$ .... (2)

$b^{\Pi}(P_3, v_3) \leq D(T_{m\times n}) + c \Rightarrow D(T_{m\times n}) + c - j - p - 1 \geq 0$ ....(3)

$b^{\Pi}(P_4, v_4) \leq D(T_{m\times n}) + c \Rightarrow D(T_{m\times n}) + c + j + p - m - n - 2 \geq 0$ .... (4)

(1) + (2) $\Rightarrow 2\left\lfloor \frac{n}{2} \right\rfloor + 2\left\lfloor \frac{m}{2} \right\rfloor + 2c - m - 2\ell - 2 \geq 0$ ....(5)

(3) + (4) $\Rightarrow 2\left\lfloor \frac{n}{2} \right\rfloor + 2\left\lfloor \frac{m}{2} \right\rfloor + 2c - m - n - 3 \geq 0$ ....(6)

In the worst case, for each of the equations (5) and (6) we will have :

- the minimum value of $2\left\lfloor \frac{n}{2} \right\rfloor = 2\left\lfloor \frac{n-1}{2} \right\rfloor = n - 1$ (when n is odd)
- the minimum value of $2\left\lfloor \frac{m}{2} \right\rfloor = 2\left\lfloor \frac{m-1}{2} \right\rfloor = m - 1$ (when m is odd)
- the maximum value of 2ℓ, which occurs when $\ell = \left\lfloor \frac{n}{2} \right\rfloor + 1$ and n is even $\Rightarrow 2\ell = n + 2$

Replacing the worst-case values in equations (4) and (5) we get :

(5) $\Rightarrow (n - 1) + (m - 1) + 2c - m - (n + 2) + 2 \geq 0 \Rightarrow c \geq 3$

(6) $\Rightarrow (n - 1) + (m - 1) + 2c - m - n - 3 \geq 0 \Rightarrow c \geq 3$

Therefore, when P_1 and P_3 do not reach row p by time $D(T_{m\times n}) + c$, P_2 and P_4 will inform all the vertices $(p, q) \in V$ where $0 \leq q \leq n-1$ if $c = 3$. Similarly, we can show that when P_2 and P_4 do not reach row p by time $D(T_{m\times n}) + c$, then P_1 and P_3 will inform all the vertices $(p, q) \in V$ where $0 \leq q \leq n-1$ if $c = 3$. The same can be shown for the paths P_1', P_2', P_3', and P_4' when the originator is vertex $u' = (0, j')$ where $0 < j' < n$ and j even.

Therefore, even when two of the pahts in each case of an originator u and u' do not reach the destination row, we can still say that the broadcast time for $T_{m\times n}$ under ordering Π will be:

$$b^{\Pi}(T_{m\times n}) \leq D(T_{m\times n}) + 4$$

□

3.8.2 Upper Bound when m is Odd

When m is odd, $m-1$ will be even and the vertices $u \in V$ on the last row $m-1$ of $T_{m\times n}$ will have the exact same numbering as the vertices $v \in V$ on row 0. That is, every vertex $(m-1, j)$ will have an ordering identical to that of vertex $(0, j)$ for any $0 \leq j \leq n-1$. This will mean that broadcasting to and from row $m-1$ will take different paths than the paths described in lemma 3.8.1.

In this section we first describe the paths used to inform all the vertices on rows 0 and $m-1$ and derive the orderly broadcast time in these two cases. We later discuss the broadcast time for the worst-case originator $u = (m-1, j) \in V$ and derive the broadcast time of $T_{m\times n}$ when m is odd.

3.8.2.1 Informing Row 0

Assuming broadcasting starts from some vertex $u = (i, j) \in V$ where i is even and j is odd, we will describe the paths that are used to inform the vertices $(0, q) \in V$ where $0 \leq q \leq n-1$

Consider a torus $T_{m\times n}$ where m is odd. Let $u = (i, j) \in V$ be the originator. Broadcasting from u will use the exact same paths described in lemma 3.8.1 to inform all the vertices $(p, q) \in V$ where $p \notin 0, m-1$. Now, suppose we want to inform all the vertices $(0, q) \in V$ on row 0. Let $v_1 = (0, q_1), v_2 = (0, q_2), v_3 = (0, q_3)$ and $v_4 = (0, q_4) \in V$ be four vertices on row 0. Consider the following paths (See figure 17):

$P_1 : (i,j) \xrightarrow{3} (i+1,j) \xrightarrow{1} (i+1,j-1) \xrightarrow{1} \ldots \xrightarrow{1} (i+1,0) \xrightarrow{1} (i+2,0) \xrightarrow{1} \ldots$
$\xrightarrow{1} (m-1,0) \xrightarrow{1} (0,0) \xrightarrow{2} (0,1) \xrightarrow{1} \ldots \xrightarrow{1} (0,q_1)$

$P_2 : (i,j) \xrightarrow{1} (i,j+1) \xrightarrow{1} (i,j+2) \xrightarrow{1} \ldots \xrightarrow{1} (i,\ell) \xrightarrow{1} (m-1,\ell) \xrightarrow{1} \ldots$
$\xrightarrow{1} (1,\ell) \xrightarrow{2} (1,\ell-1) \xrightarrow{1} (1,\ell-2) \xrightarrow{1} \ldots \xrightarrow{1} (1,q_2) \xrightarrow{k''} (0,q_2)$

$P_3 : (i,j) \xrightarrow{3} (i+1,j) \xrightarrow{1} (i+1,j-1) \xrightarrow{1} \ldots \xrightarrow{1} (i+1,0) \xrightarrow{1} (i+2,0) \xrightarrow{1} \ldots$
$\xrightarrow{1} (m-2,0) \xrightarrow{2} (m-2,n-1) \xrightarrow{1} (m-2,n-2) \xrightarrow{1} \ldots \xrightarrow{1} (m-2,q_3)$
$\xrightarrow{k} (m-1,q_3) \xrightarrow{k} (0,q_3)$

$$P_3' : (i,j) \xrightarrow{3} (i+1,j) \xrightarrow{1} (i+1,j-1) \xrightarrow{1} \ldots \xrightarrow{1} (i+1,0) \xrightarrow{1} (i+2,0) \xrightarrow{1} \ldots$$
$$\xrightarrow{1} (m-1,0) \xrightarrow{1} (0,0) \xrightarrow{1} (1,0) \xrightarrow{2} (1,n-1) \xrightarrow{1} (1,n-2) \xrightarrow{1} \ldots \xrightarrow{1} (1,q_3) \xrightarrow{k'} (0,q_3)$$
$$P_4 : (i,j) \xrightarrow{1} (i,j+1) \xrightarrow{1} (i,j+2) \xrightarrow{1} \ldots \xrightarrow{1} (i,\ell) \xrightarrow{1} (m-1,\ell) \xrightarrow{1} \ldots$$
$$\xrightarrow{1} (0,\ell) \xrightarrow{2} (0,\ell+1) \xrightarrow{1} (0,\ell+2) \xrightarrow{1} \ldots \xrightarrow{1} (0,q_4)$$

For the same reasons discussed in lemma 3.8.1, we know that $k, k', k'' \in \{1,2,3\}$. In the worst case, we will consider k'' to be equal to 3, and will discuss the values of k and k' in what follows.

The time at which each of the vertices v_1, v_2, v_3, v_4 receive the message will be :

$b^{\Pi}(P_1, v_1) = 3 + j + (m - i - 1) + 2 + (q_1 - 1) = q_1 + j + m - i + 3$

$b^{\Pi}(P_2, v_2) = (\ell - j) + (i - 1) + 2 + (\ell - 1 - q_2) + 3 = -q_2 - j + i + 2\ell + 3$

$b^{\Pi}(P_3, v_3) = 3+j+(m-2-i-1)+2+(n-1-q_3)+2k = -q_3+j-i+m+n+1+2k$

$b^{\Pi}(P_3', v_3) = 3 + j + (m - i) + 2 + (n - 1 - q_3) + k' = -q_3 + j - i + m + n + 4 + k'$

$b^{\Pi}(P_4, v_4) = (\ell - j) + i + 2 + (q_4 - \ell - 1) = q_4 - j + i + 1$

Now, consider the two paths P_3 and P_3'. Both of these paths inform vertex v_3. Observe that if $k = k' = 3$, then $b^{\Pi}(P_3', v_3) = b^{\Pi}(P_3, v_3)$. However, below we will show that when $k = 3$, then k' will be equal to 2 and vice versa.

First, suppose $k = 3$. Then,

$b^{\Pi}(P_3, v_3) = -q_3 + j - i + m + n + 4 + k$

$b^{\Pi}(P_3', v_3) = -q_3 + j - i + m + n + 4 + k'$

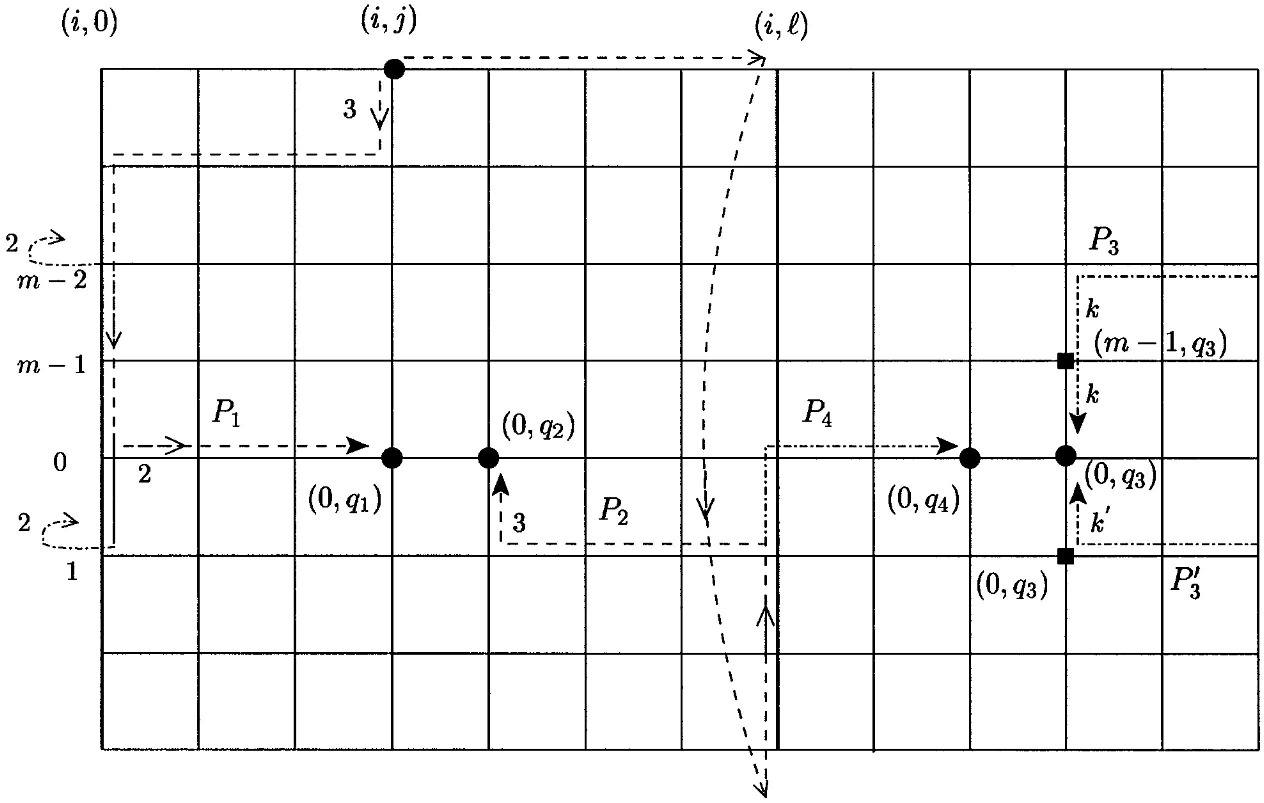

Figure 17: Informing the vertices on row 0 through the paths P_1, P_2, P_3, P_3' and P_4 when m is odd.

At time $-q_3 + j - i + m + n + 4$, the last vertex that path P_3 has informed is $(m-1, q_3)$, and the last vertex that path P_3' has informed is $(1, q_3)$ (see Figure 17). In the next step, vertex $(m-1, q_3)$ will inform $(0, q_3)$ in k time units, and vertex $(1, q_3)$ will inform $(0, q_3)$ in k' time units. Since $(m-1, q_3)$ and $(1, q_3)$ are on the same column, from the ordering Π we know that k and k' cannot have the same value.

Thus, when $k = 3$ then $k' = 2$. This implies that vertex $(0, q_3)$ will be informed through path P_3' at most by time:

$$b^{\Pi}(P_3', v_3) = -q_3 + j - i + m + n + 6$$

Similarly, when $k = 2$ then $k' = 3$. This means that vertex $(0, q_3)$ will be informed through path P_3 at most by time:

$$b^{\Pi}(P_3, v_3) = -q_3 + j - i + m + n + 5$$

Therefore, in the worst case, vertex v_3 will be informed by time:

$$b^{\Pi}(P_3', v_3) = -q_3 + j - i + m + n + 6$$

We will now proceed with the discussion in a way similar to that of lemma 3.8.1.

Since each of the informed vertices on rows 1 needs at most 3 time units to inform its neighbor on row 0, then by the time vertex $(0, q_2)$ receives the message, all the vertices $(0, i)$ where $q_2 \leq i < \ell$ will also be informed.

Now, consider vertices $(1, q_3 - 1), (1, q_3 - 2), \ldots, (1, n - 1)$ which are informed through path P_3'. All these vertices are informed by time $b^{\Pi}(P_3', v_3) - k' - 1$. Since each one of these vertices needs at most 3 time units to inform its neighbor on row 0, then their neighbors on row 0 will be informed at most by time:

$$b^{\Pi}(P_3', v_3) - k' - 1 + 3 = b^{\Pi}(P_3', v_3) - k' + 2$$

In the worst case, when $k' = 2$, all the vertices $(0, i)$ where $q_3 \leq i \leq n - 1$ will be informed by time $b^{\Pi}(P_3', v_3)$.

Thus, we know that all the vertices on row 0 are going to be informed when v_1 is at least adjacent to v_2 and v_3 is at least adjacent to v_4.

Suppose v_1, v_2, v_3 and v_4 receive the message by time $D(T_{m\times n})+c$ (where $c \in Z^+$).

Then,

$$b^{\Pi}(P_1, v_1) \le D(T_{m\times n}) + c \Rightarrow q_1 \le D(T_{m\times n}) + c - j - m + i - 3$$

$$b^{\Pi}(P_2, v_2) \le D(T_{m\times n}) + c \Rightarrow q_2 \ge -D(T_{m\times n}) - c - j + 2\ell + i + 3$$

$$b^{\Pi}(P_3', v_3) \le D(T_{m\times n}) + c \Rightarrow q_3 \ge -D(T_{m\times n}) - c + j + m + n - i + 6$$

$$b^{\Pi}(P_4, v_4) \le D(T_{m\times n}) + c \Rightarrow q_4 \le D(T_{m\times n}) + c + j - i - 1$$

Vertex v_1 will be at least adjacent to v_2 when:

$$Maximum(q_1) + 1 \ge Minimum(q_2)$$

$$\Leftrightarrow D(T_{m\times n}) + c - j - m + i - 3 + 1 \ge -D(T_{m\times n}) - c - j + 2\ell + i + 3$$

$$\Leftrightarrow 2D(T_{m\times n}) + 2c - m - 2\ell - 5 \ge 0 \ \ldots\ldots\ldots (\mathrm{A})$$

Vertex v_3 will be at least adjacent to v_4 when:

$$Maximum(q_4) + 1 \ge Minimum(q_3)$$

$$\Leftrightarrow D(T_{m\times n}) + c + j - i - 1 + 1 \ge -D(T_{m\times n}) - c + j + m + n - i + 6$$

$$\Leftrightarrow 2D(T_{m\times n}) + 2c - m - n - 6 \ge 0 \ \ldots\ldots\ldots (\mathrm{B})$$

For each of the equations (A) and (B), the worst case will occur when m and n are odd and $2\ell = 2\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) = 2\left(\frac{n+1}{2}\right) = n + 1$. Then,

$$(\mathrm{A}) \Rightarrow 2\left(\frac{n-1}{2}\right) + 2\left(\frac{m-1}{2}\right) + 2c - m - (n+1) - 5 \ge 0 \Rightarrow c \ge 4$$

$$(\mathrm{B}) \Rightarrow 2\left(\frac{n-1}{2}\right) + 2\left(\frac{m-1}{2}\right) + 2c - m - n - 6 \ge 0 \Rightarrow c \ge 4$$

Thus, when the originator is a vertex $u = (i, j) \in V$ where $0 < j < \ell$ and j odd, informing all the vertices $(0, q) \in V$ where $0 \leq q \leq n-1$ when m is odd will take at most $D(T_{m \times n}) + 4$ time units.

When the originator is a vertex $u' = (i, j') \in V$ where $\ell < j' < n$ and j' odd, then paths symmetric to P_1, P_2, P_3, P_3' and P_4 will inform all the vertices on row 0. Calculations similar to the above will lead to the conclusion that when the originator is a vertex u', all the vertices on row 0 will be informed at most by time $D(T_{m \times n}) + 4$.

Therefore, when m is odd we can conclude that for any originator $(i, j) \in V$, informing all the vertices $(0, q) \in V$ where $0 \leq q \leq n-1$ will take at most $D(T_{m \times n}) + 4$ time units.

We will later see that this is actually less than the broadcast time in $T_{m \times n}$ when m is odd.

3.8.2.2 Informing Row $m - 1$

We will now describe the paths used to inform the vertices $(m-1, q) \in V$ where $0 \leq q \leq n-1$ on row $m-1$ when m is odd and derive the broadcast time in the worst-case.

Consider a torus $T_{m \times n}$ where m is odd. Suppose we want to inform all the vertices $(m-1, q) \in V$ on row $m-1$. Broadcasting from a vertex $u = (i, j)$ where $0 < j < \ell$ and j odd will use similar paths described in section 3.8.2.1 to inform all the vertices on row $m-1$. Let $v_1 = (m-1, q_1), v_2 = (m-1, q_2), v_3 = (m-1, q_3)$ and $v_4 = (m-1, q_4) \in V$ be four vertices on row $m-1$. Consider the following paths (See figure 18):

$$P_1 : (i,j) \xrightarrow{3} (i+1,j) \xrightarrow{1} (i+1,j-1) \xrightarrow{1} \dots \xrightarrow{1} (i+1,0) \xrightarrow{1} (i+2,0) \xrightarrow{1} \dots$$
$$\xrightarrow{1} (m-1,0) \xrightarrow{2} (m-1,1) \xrightarrow{1} \dots \xrightarrow{1} (m-1,q_1)$$

$$P_2 : (i,j) \xrightarrow{1} (i,j+1) \xrightarrow{1} (i,j+2) \xrightarrow{1} \dots \xrightarrow{1} (i,\ell) \xrightarrow{1} (m-1,\ell) \xrightarrow{1} \dots$$
$$\xrightarrow{1} (1,\ell) \xrightarrow{2} (1,\ell-1) \xrightarrow{1} (1,\ell-2) \xrightarrow{1} \dots \xrightarrow{1} (1,q_2) \xrightarrow{k} (0,q_2) \xrightarrow{k} (m-1,q_2)$$

$$P_2' : (i,j) \xrightarrow{1} (i,j+1) \xrightarrow{1} (i,j+2) \xrightarrow{1} \dots \xrightarrow{1} (i,\ell) \xrightarrow{1} (m-1,\ell) \xrightarrow{1} \dots$$
$$\xrightarrow{1} (0,\ell) \xrightarrow{1} (m-1,\ell) \xrightarrow{1} (m-2,\ell) \xrightarrow{2} (m-2,\ell-1) \xrightarrow{1} (m-2,\ell-2) \xrightarrow{1} \dots$$
$$\xrightarrow{1} (m-2,q_2) \xrightarrow{k'} (m-1,q_2)$$

$$P_3 : (i,j) \xrightarrow{3} (i+1,j) \xrightarrow{1} (i+1,j-1) \xrightarrow{1} \dots \xrightarrow{1} (i+1,0) \xrightarrow{1} (i+2,0) \xrightarrow{1} \dots$$
$$\xrightarrow{1} (m-2,0) \xrightarrow{2} (m-2,n-1) \xrightarrow{1} (m-2,n-2) \xrightarrow{1} \dots \xrightarrow{1} (m-2,q_3)$$
$$\xrightarrow{k''} (m-1,q_3)$$

$$P_4 : (i,j) \xrightarrow{1} (i,j+1) \xrightarrow{1} (i,j+2) \xrightarrow{1} \dots \xrightarrow{1} (i,\ell) \xrightarrow{1} (m-1,\ell) \xrightarrow{1} \dots$$
$$\xrightarrow{1} (0,\ell) \xrightarrow{1} (m-1,\ell) \xrightarrow{2} (m-1,\ell+1) \xrightarrow{1} (m-1,\ell+2) \xrightarrow{1} \dots \xrightarrow{1} (m-1,q_4)$$

For the same reasons discussed in lemma 3.8.1, we know that $k, k', k'' \in \{1,2,3\}$. In the worst case, we will consider k'' to be equal to 3, and will discuss the values of k and k' in what follows.

The time at which each of the vertices v_1, v_2, v_3, v_4 receive the message will be :

$b^{\text{II}}(P_1, v_1) = 3 + j + (m-1-i-1) + 2 + (q_1 - 1) = q_1 + j + m - i + 2$

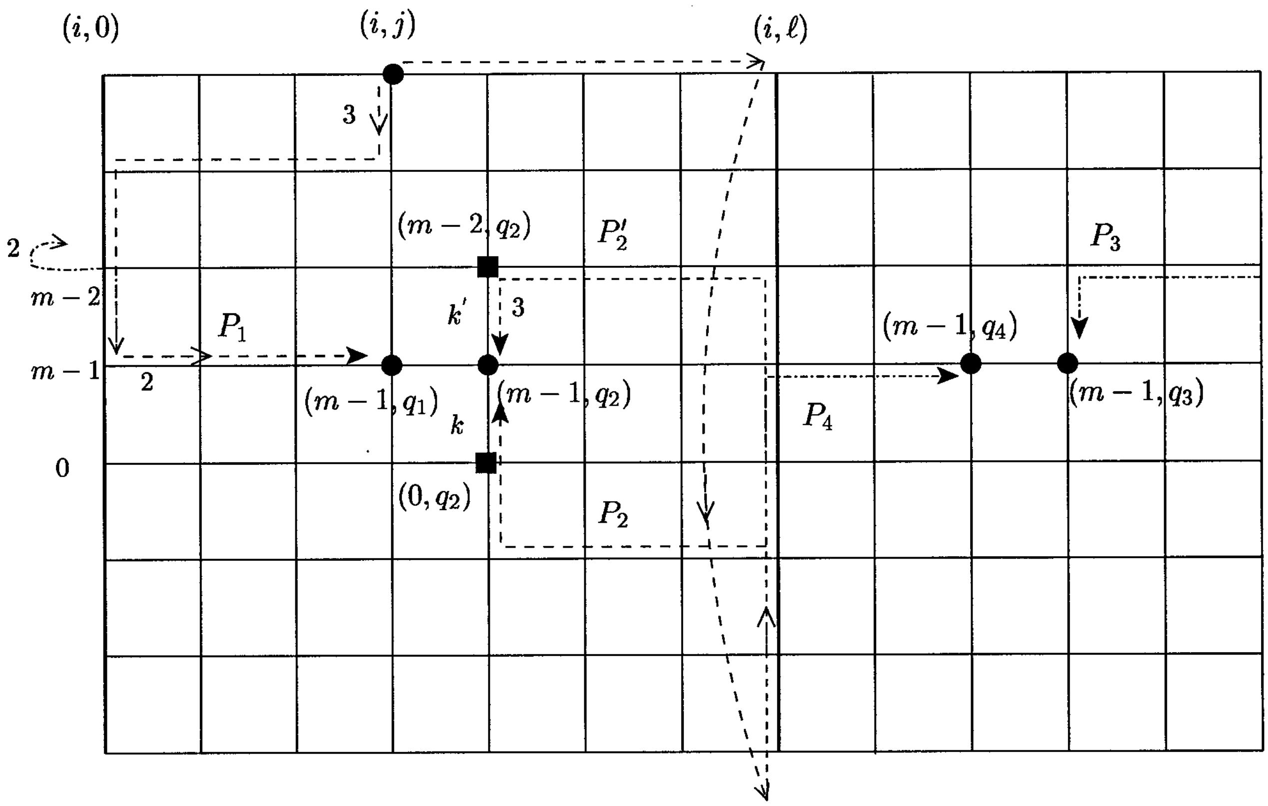

Figure 18: Informing the vertices on row $m-1$ through the paths P_1, P_2, P_2', P_3 and P_4 when m is odd.

$$b^{\Pi}(P_2, v_2) = (\ell - j) + (i-1) + 2 + (\ell - 1 - q_2) + 2k = -q_2 - j + i + 2\ell + 2k$$

$$b^{\Pi}(P_2', v_2) = (\ell - j) + (i+2) + 2 + (\ell - 1 - q_2) + k' = -q_2 - j + i + 2\ell + 3 + k'$$

$$b^{\Pi}(P_3, v_3) = 3 + j + (m - 2 - i - 1) + 2 + (n - 1 - q_3) + 3 = -q_3 + j - i + m + n + 4$$

$$b^{\Pi}(P_4, v_4) = (\ell - j) + (i+1) + 2 + (q_4 - \ell - 1) = q_4 - j + i + 2$$

Similar to the discussion in section 3.8.2.1 above, we will show that in the worst-case, vertex v_2 will receive the message by time : $b^{\Pi}(P_2', v_2) = -q_2 - j + i + 2\ell + 5$.

Thus, consider the two paths P_2 and P_2', both of which inform vertex v_2. Similarly, at time $-q_2 - j + i + 2\ell + k = -q_2 - j + i + 2\ell + 3$ the last vertex that P_2 has informed is $(0, q_2)$, and the last vertex that path P_2' has informed is $(m-2, q_2)$ (see Figure 18).

In the next step, vertex $(m-2, q_2)$ will inform $(m-1, q_2)$ in k' time units, and vertex $(0, q_2)$ will inform $(m-1, q_2)$ in k time units. Since $(m-2, q_2)$ and $(0, q_2)$ are on the same column, k and k' cannot have the same value.

Thus, when $k = 3$ then $k' = 2$. This implies that vertex $(m-1, q_2)$ will be informed through path P_2' at most by time:

$$b^{\Pi}(P_2', v_2) = -q_2 - j + i + 2\ell + 5$$

Similarly, when $k = 2$ then $k' = 3$ and hence vertex $(m-1, q_2)$ will be informed through path P_2 at most by time:

$$b^{\Pi}(P_2, v_2) = -q_2 - j + i + 2\ell + 4$$

Therefore, in the worst case, vertex v_2 will be informed by time:

$$b^{\Pi}(P_2', v_2) = -q_2 - j + i + 2\ell + 5$$

Since each of the informed vertices on rows $m-2$ needs at most 3 time units to inform its neighbor on row $m-1$, then by the time vertex $(m-1, q_2)$ receives the message, all the vertices $(m-1, i)$ where $q_2 \leq i < \ell$ will also be informed.

Now, consider vertices $(m-1, q_2+1), (m-2, q_2+2), \ldots, (m-2, \ell)$ which are informed through path P_2'. All these vertices are informed by time $b^{\Pi}(P_2', v_2) - k' - 1$. Since each one of these vertices needs at most 3 time units to inform its neighbor on row 0, then their neighbors on row 0 will be informed at most by time:

$$b^{\Pi}(P_2', v_2) - k' - 1 + 3 = b^{\Pi}(P_2', v_2) - k' + 2$$

In the worst case, when $k^{'} = 2$, all the vertices $(m-1, i)$ where $q_2 \leq i \leq \ell$ will be informed by time $b^{\Pi}(P_2', v_2)$.

Thus, we know that all the vertices on row $m-1$ are going to be informed when v_1 is at least adjacent to v_2 and v_3 is at least adjacent to v_4. Assuming each of v_1, v_2, v_3, v_4 receives the message by time $D(T_{m \times n}) + c$, calculations similar to the ones done in section 3.8.2.1 will lead to the following two equations:

Vertex v_1 will be at least adjacent to v_2 when:

$$Maximum(q_1) + 1 \geq Minimum(q_2)$$

$$\Leftrightarrow D(T_{m \times n}) + c - j - m + i - 2 + 1 \geq -D(T_{m \times n}) - c - j + 2\ell + i + 5$$

$$\Leftrightarrow 2D(T_{m \times n}) + 2c - m - 2\ell - 6 \geq 0 \ldots\ldots\ldots (A)$$

Vertex v_3 will be at least adjacent to v_4 when:

$$Maximum(q_4) + 1 \geq Minimum(q_3)$$

$$\Leftrightarrow D(T_{m \times n}) + c + j - i - 2 + 1 \geq -D(T_{m \times n}) - c + j + m + n - i + 4$$

$$\Leftrightarrow 2D(T_{m \times n}) + 2c - m - n - 5 \geq 0 \ldots\ldots\ldots (B)$$

For each of the equations (A) and (B), the worst case will occur when m and n are odd and $2\ell = 2\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) = 2\left(\frac{n+1}{2}\right) = n + 1$. Then,

$$(A) \Rightarrow 2\left(\frac{n-1}{2}\right) + 2\left(\frac{m-1}{2}\right) + 2c - m - (n+1) - 6 \geq 0 \Rightarrow c \geq 5$$

$$(B) \Rightarrow 2\left(\frac{n-1}{2}\right) + 2\left(\frac{m-1}{2}\right) + 2c - m - n - 5 \geq 0 \Rightarrow c \geq 4$$

Thus, when the originator is a vertex $u = (i, j) \in V$ where $0 < j < \ell$ and j odd, informing all the vertices $(m-1, q) \in V$ where $0 \leq q \leq n-1$ when m is odd will take at most $D(T_{m\times n}) + 4$ time units.

When the originator is a vertex $u' = (i, j') \in V$ where $\ell < j' < n$ and j' odd, then paths symmetric to P_1, P_2, P_2', P_2 and P_4 will inform all the vertices on row $m-1$. Calculations similar to the above will lead to the conclusion that when the originator is a vertex u', all the vertices on row 0 will be informed at most by time $D(T_{m\times n}) + 4$. Therefore, when m is odd we can conclude that for any originator $(i, j) \in V$, informing all the vertices $(m-1, q) \in V$ where $0 \leq q \leq n-1$ will take at most $D(T_{m\times n}) + 4$ time units.

Below we will see that this value is less than the broadcast time in $T_{m\times n}$ when m is odd.

3.8.2.3 Broadcasting from Originator$(m-1, j)$

When m is odd, broadcasting from any originator $(i, j) \in V$ when $0 < j < \ell$ and $i \neq m-1$ or when $\ell < j < n$ and $i \neq 0$, will use the exact same paths described in lemma 3.8.1 and sections 3.8.2.2 and 3.8.2.1 (for informing the vertices on rows 0 and $m-1$). In these cases, the upper bound on orderly broadcasting in $T_{m\times n}$ will be at most $D(T_{m\times n}) + 4$. In this section we will see that broadcasting from a vertex $u = (m-1, j)$ for any $0 < j < \ell$ or from a vertex $u' = (0, j')$ where $\ell < j' < n$ will yield a broadcast time of $D(T_{m\times n}) + 5$.

Lemma 3.8.2

Given a torus $T_{m\times n}$ where m is odd, the orderly broadcast time under ordering Π will be:

$$b^{\Pi}(T_{m\times n}) \leq \begin{cases} D(T_{m\times n})+5 & \text{if } (m \text{ and } n \text{ are odd }) \\ & \text{or } (m \text{ is odd and } n, \left\lfloor \frac{n}{2} \right\rfloor \text{ are even}) \\ D(T_{m\times n})+4 & \text{otherwise} \end{cases}$$

Proof :

The paths used to broadcast from vertex $(m-1, j)$ where j is odd will be identical to the paths used to broadcast from any $(0, j')$ where j' is even (and vice versa). Thus, to prove the lemma, we will describe the dissemination of the message from two different originators : One from vertex $u = (m-1, j) \in V$ where $0 < j < \ell$ and j is odd and the other from $u' = (0, j') \in V$ where $\ell < j' < n$ and j' is even.

Through lemma 3.4.1 we know that in the parity of p does not effect the broadcast time, thus without loss of generality we will assume that p is odd.

Case A: Originator $u = (m-1, j) \in V$ where $0 < j < \ell$ and j odd

Suppose broadcasting starts from vertex $u = (m-1, j)$ where $0 < j < \ell$ and j is odd. We want to inform all the vertices $(p, q) \in V$ where p is odd. The only change that the originator u will cause in the paths described in lemma 3.8.1 will be in the two paths P_1 and P_3, while P_2 and P_4 stay the same. Below we redefine the paths

P_1, P_2, P_3 and P_4 (Figure 19) that originate from $(m-1, j)$:

$$P_1 : (m-1,j) \xrightarrow{2} (m-2,j) \xrightarrow{1} (m-2,j-1) \xrightarrow{1} \ldots \xrightarrow{1} (m-2,0) \xrightarrow{1} (m-1,0)$$
$$\xrightarrow{1} (0,0) \xrightarrow{1} (1,0) \xrightarrow{1} \ldots \xrightarrow{1} (p-1,0) \xrightarrow{2} (p-1,1) \xrightarrow{1} \ldots \xrightarrow{1} (p-1,q_1) \stackrel{k=3}{\rightarrow} (p,q_1)$$
$$P_2 : (m-1,j) \xrightarrow{1} (m-1,j+1) \xrightarrow{1} (m-1,j+2) \xrightarrow{1} \ldots \xrightarrow{1} (m-1,\ell) \xrightarrow{1} (m-1,\ell) \xrightarrow{1} \ldots$$
$$\xrightarrow{1} (p,\ell) \xrightarrow{2} (p,\ell-1) \xrightarrow{1} (p,\ell-2) \xrightarrow{1} \ldots \xrightarrow{1} (p,q_2)$$
$$P_3 : (m-1,j) \xrightarrow{2} (m-2,j) \xrightarrow{1} (m-2,j-1) \xrightarrow{1} \ldots \xrightarrow{1} (m-2,0) \xrightarrow{1} (m-1,0)$$
$$\xrightarrow{1} (0,0) \xrightarrow{1} \ldots \xrightarrow{1} (p-1,0) \xrightarrow{1} (p,0) \xrightarrow{2} (p,n-1) \xrightarrow{1} (p,n-2) \xrightarrow{1} \ldots \xrightarrow{1} (p,q_3)$$
$$P_4 : (m-1,j) \xrightarrow{1} (m-1,j+1) \xrightarrow{1} (m-1,j+2) \xrightarrow{1} \ldots \xrightarrow{1} (m-1,\ell) \xrightarrow{1} (m-1,\ell) \xrightarrow{1} \ldots$$
$$\xrightarrow{1} (p+1,\ell) \xrightarrow{2} (p+,\ell+1) \xrightarrow{1} \ldots \xrightarrow{1} (p+1,q_4) \stackrel{k'=3}{\rightarrow} (p,q_4)$$

(Observe that for the same reasons mentioned in lemma 3.8.1, we assume that $k = k' = 3$ in the worst case.)

The time at which each of the vertices $v_1 = (p, q_1)$, $v_2 = (p, q_2)$, $v_3 = (p, q_3)$ and $v_4 = (p, q_4)$ receives the message will be:

$$b^{\Pi}(P_1, v_1) = 2 + j + 2 + (p-1) + 2 + (q_1 - 1) + 3 = q_1 + j + p + 7$$
$$b^{\Pi}(P_2, v_2) = (\ell - j) + (m - 1 - p) + 2 + (\ell - 1 - q_2) = -q_2 - j - p + m + 2\ell$$
$$b^{\Pi}(P_3, v_3) = 2 + j + 2 + p + 2 + (n - 1 - q_3) = -q_3 + j + p + n + 5$$
$$b^{\Pi}(P_4, v_4) = (\ell - j) + (m - 1 - p - 1) + 2 + (q_4 - \ell - 1) + 3 = q_4 - j - p + m + 2$$

After calculations similar to the discussions in lemma 3.8.1, we obtain the following two equations:

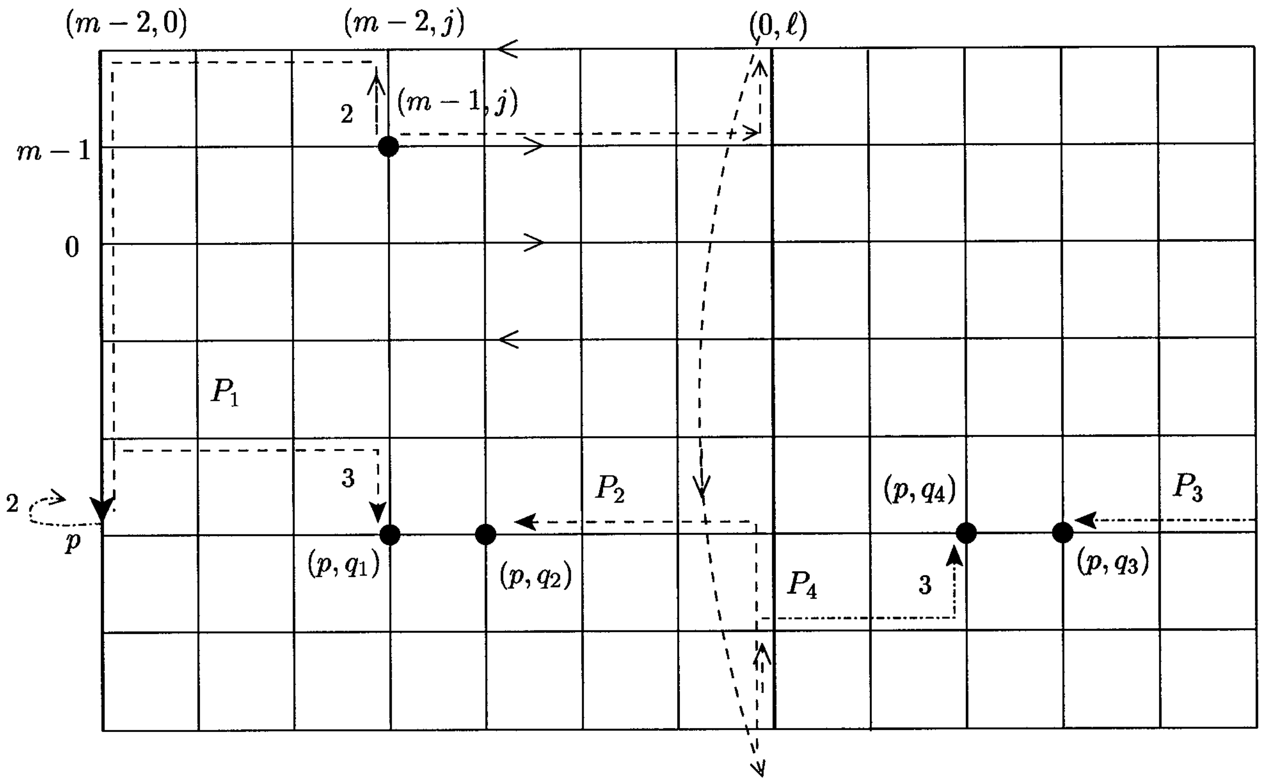

Figure 19: The four paths P_1, P_2, P_3 and P_4 originating from vertex $(m-1, j)$ when m and j are odd.

$$2D(T_{m\times n}) + 2c - m - 2\ell - 6 \geq 0 \text{ ....(A)}$$

$$2D(T_{m\times n}) + 2c - m - n - 6 \geq 0 \text{ ....(B)}$$

Case 1: m is ODD and n is EVEN

Since m is odd and n is even

$$\Rightarrow D(T_{m\times n}) = \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor = \frac{m-1}{2} + \frac{n}{2} = \frac{m+n-1}{2}$$

$$\text{(A)} \Rightarrow 2\left(\frac{m+n-1}{2}\right) + 2c - m - 2\ell - 6 \geq 0 \Rightarrow n - 2\ell + 2c - 7 \geq 0$$

i) if $\left\lfloor \frac{n}{2} \right\rfloor$ is odd $\Rightarrow \ell = \left\lfloor \frac{n}{2} \right\rfloor = \frac{n}{2}$

$$\Rightarrow n - 2\ell + 2c - 7 \geq 0 \Rightarrow n - 2\left(\frac{n}{2}\right) + 2c - 7 \geq 0 \Rightarrow 2c - 7 \geq 0 \Rightarrow c \geq 4$$

ii) if $\left\lfloor \frac{n}{2} \right\rfloor$ is even $\Rightarrow \ell = \left\lfloor \frac{n}{2} \right\rfloor + 1 = \frac{n}{2} + 1$

$$\Rightarrow n - 2\ell + 2c - 7 \geq 0 \Rightarrow n - 2(\frac{n}{2} + 1) + 2c - 7 \geq 0 \Rightarrow 2c - 9 \geq 0 \Rightarrow c \geq 5$$

$$\text{(B)} \Rightarrow 2\left(\frac{m+n-1}{2}\right) + 2c - m - n - 5 \geq 0 \Rightarrow 2c - 7 \geq 0 \Rightarrow c \geq 4$$

Case 2: m and n are both ODD

Since m and n are both odd

$$\Rightarrow D(T_{m \times n}) = \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor = \frac{m-1}{2} + \frac{n-1}{2} = \frac{m+n-2}{2}$$

$$\text{(A)} \Rightarrow 2\left(\frac{m+n-2}{2}\right) + 2c - m - 2\ell - 6 \geq 0 \Rightarrow n - 2\ell + 2c - 8 \geq 0$$

i) if $\left\lfloor \frac{n}{2} \right\rfloor$ is odd $\Rightarrow \ell = \left\lfloor \frac{n}{2} \right\rfloor = \frac{n-1}{2}$

$$\Rightarrow n - 2\ell + 2c - 8 \geq 0 \Rightarrow n - 2\left(\frac{n-1}{2}\right) + 2c - 8 \geq 0 \Rightarrow 2c - 7 \geq 0 \Rightarrow c \geq 4$$

ii) if $\left\lfloor \frac{n}{2} \right\rfloor$ is even $\Rightarrow \ell = \left\lfloor \frac{n}{2} \right\rfloor + 1 = \frac{n-1}{2} + 1 = \frac{n+1}{2}$

$$\Rightarrow n - 2\ell + 2c - 7 \geq 0 \Rightarrow n - 2(\frac{n+1}{2}) + 2c - 8 \geq 0 \Rightarrow 2c - 9 \geq 0 \Rightarrow c \geq 5$$

$$\text{(B)} \Rightarrow 2\left(\frac{m+n-2}{2}\right) + 2c - m - n - 6 \geq 0 \Rightarrow 2c - 8 \geq 0 \Rightarrow c \geq 4$$

Thus, for an originator $(m-1,j)$ with $0 < j < \ell$ and j and m odd, every vertex $v \in V$ will be informed at most by time $D(T_{m\times n}) + 5$.

Now, when j is even, the paths P_2 and P_4 will stay the same, while P_1 and P_3 will change in the following way :

$$P_1' : (m-1,j) \xrightarrow{2} (0,j) \xrightarrow{2} (1,j) \xrightarrow{1} (1,j-1) \xrightarrow{1} (1,j-2) \xrightarrow{1} \dots \xrightarrow{1} (1,0) \xrightarrow{1}$$
$$\xrightarrow{1} (2,0) \xrightarrow{1} \dots \xrightarrow{1} (p-1,0) \xrightarrow{2} (p-1,1) \xrightarrow{1} \dots \xrightarrow{1} (p-1,q_1) \xrightarrow{k} (p,q_1)$$
$$P_3' : (m-1,j) \xrightarrow{2} (0,j) \xrightarrow{2} (1,j) \xrightarrow{1} (1,j-1) \xrightarrow{1} (1,j-2) \xrightarrow{1} \dots \xrightarrow{1} (1,0) \xrightarrow{1}$$
$$\xrightarrow{1} (2,0) \xrightarrow{1} \dots \xrightarrow{1} (p-1,0) \xrightarrow{1} (p,0) \xrightarrow{2} (p,n-1) \xrightarrow{1} (p,n-2) \xrightarrow{1} \dots \xrightarrow{1} (p,q_3)$$

The time at which each of the vertices $v_1 = (p,q_1)$ and $v_3 = (p,q_3)$ receive the message through the new paths P_1' and P_3' will be:

$$b^{\Pi}(P_1',v_1) = 2+2+j+(p-2)+2+(q_1-1)+3 = q_1+j+p+6 < b^{\Pi}(P_1,v_1)$$
$$b^{\Pi}(P_3,v_3) = 2+2+j+(p-1)+2+(n-1-q_3) = -q_3+j+p+n+4 < b^{\Pi}(P_3',v_3)$$

Thus, since the time at which each of the vertices v_1 and v_3 receives the message from originator is $(m-1,j)$ when j is even is less that the time at which they receive the message when j is odd, then we can see that the worst case for an originator in this case will again be when j is odd.

Case B: Originator $u' = (0, j') \in V$ where $\ell < j' < n$ and j even

We will now calculate the broadcast time when the originator is $u' = (0, j') \in V$ where $\ell < j' < n$.

When j' is even, then the paths P_1', P_2', P_3' and P_4' originating from u' will be respectively symmetric to the paths P_1, P_2, P_3 and P_4 described above. Thus, consider the vertices $v_1' = (p, q_1')$, $v_2' = (p, q_2')$, $v_3' = (p, q_3')$ and $v_4' = (p, q_4')$ where $\ell \leq q_1', q_2' \leq n-1$ and $0 \leq q_3', q_4' \leq \ell$ and let $k, k' \in \{1, 2, 3, 4\}$. We define the four paths :

$$P_1' : (0, j') \xrightarrow{2} (1, j') \xrightarrow{1} (1, j'-1) \xrightarrow{1} \ldots \xrightarrow{1} (1, \ell) \xrightarrow{1} (0, \ell) \xrightarrow{1} (m-1, \ell) \xrightarrow{1} \ldots$$
$$\xrightarrow{1} (p+1, \ell) \xrightarrow{2} (p+1, \ell+1) \xrightarrow{1} \ldots \xrightarrow{1} (p+1, q_1') \xrightarrow{k} (p, q_1')$$
$$P_2' : (0, j') \xrightarrow{1} (0, j'+1) \xrightarrow{1} (0, j'+2) \xrightarrow{1} \ldots \xrightarrow{1} (0, 0) \xrightarrow{1} (1, 0) \xrightarrow{1} (2, 0) \xrightarrow{1} \ldots$$
$$\xrightarrow{1} (p, 0) \xrightarrow{2} (p, n-1) \xrightarrow{1} (p, n-2) \xrightarrow{1} \ldots \xrightarrow{1} (p, q_2')$$
$$P_3' : (0, j') \xrightarrow{2} (1, j') \xrightarrow{1} (1, j'-1) \xrightarrow{1} \ldots \xrightarrow{1} (1, \ell) \xrightarrow{1} (0, \ell) \xrightarrow{1} (m-1, \ell) \xrightarrow{1} \ldots$$
$$\xrightarrow{1} (p-1, \ell) \xrightarrow{1} (p, \ell) \xrightarrow{2} (p, \ell-1) \xrightarrow{1} (p, \ell-2) \xrightarrow{1} \ldots \xrightarrow{1} (p, q_3')$$
$$P_4' : (0, j') \xrightarrow{1} (0, j'+1) \xrightarrow{1} (0, j'+2) \xrightarrow{1} \ldots \xrightarrow{1} (0, 0) \xrightarrow{1} (1, 0) \xrightarrow{1} (2, 0) \xrightarrow{1} \ldots$$
$$\xrightarrow{1} (p-1, 0) \xrightarrow{2} (p-1, 1) \xrightarrow{1} \ldots \xrightarrow{1} (p-1, q_4') \xrightarrow{k'} (p, q_4')$$

(Observe that for the same reasons mentioned in lemma 3.8.1, we assume that $k = k' = 3$ in the worst case. See Figure 20 for a diagram of the four paths.)

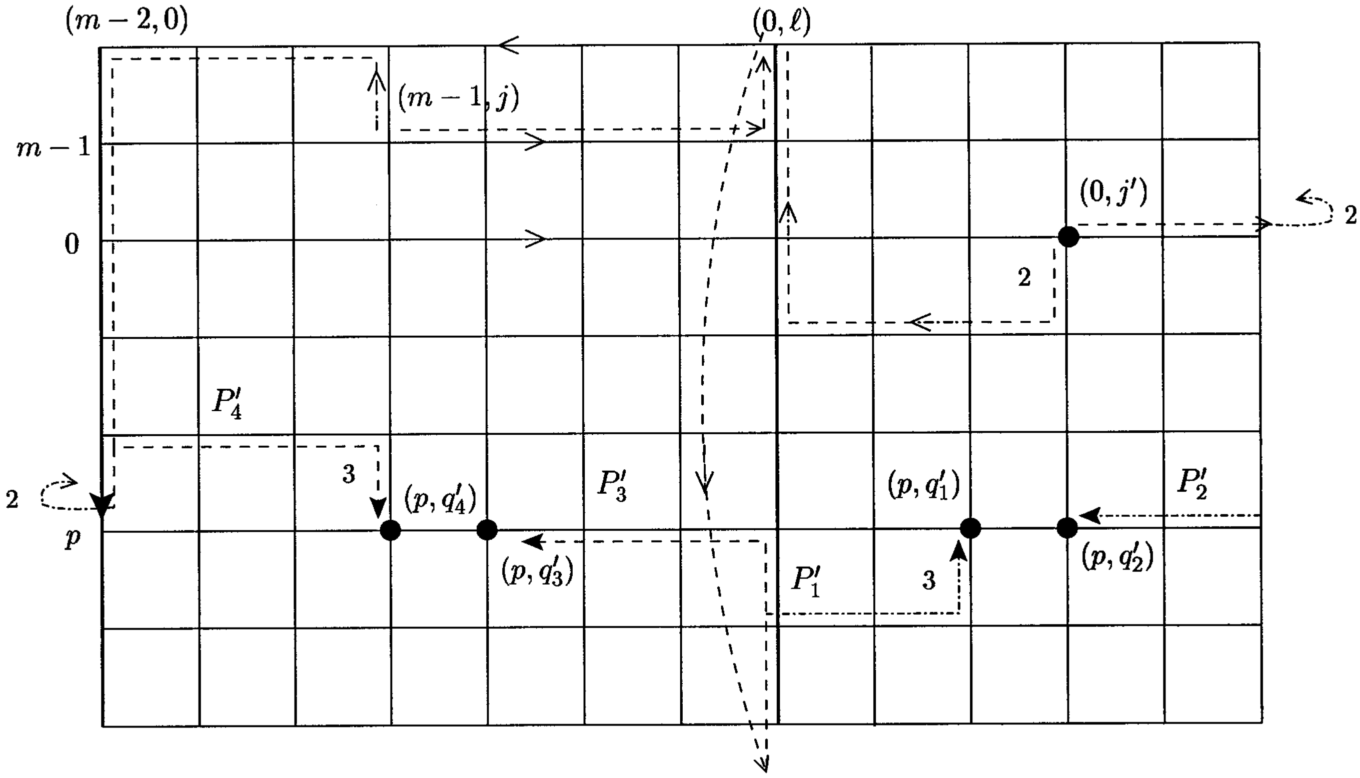

Figure 20: The four paths P_1', P_2', P_3' and P_4' originating from vertex $(0, j')$ when m is odd and j' is even.

The time at which each of the vertices $v_1' = (p, q_1')$, $v_2' = (p, q_2')$, $v_3' = (p, q_3')$ and $v_4' = (p, q_4')$ receives the message will be:

$b^{\Pi}(P_1', v_1') = 2 + (j' - \ell) + (m - (p+1) + 1) + 2 + (q_1' - \ell - 1) + 3 = q_1' + j' - p + m + 2\ell + 6$

$b^{\Pi}(P_2', v_2') = (n - j') + p + 2 + (n - q_2' - 1) = -q_2' - j' + p + 2n + 1$

$b^{\Pi}(P_3', v_3') = 2 + (j' - \ell) + (m - p + 1) + 2 + (\ell - 1 - q_3') = -q_3' + j' - p + m + 4$

$b^{\Pi}(P_4', v_4') = (n - j') + (p - 1) + 2 + (q_4' - 1) + 3 = q_4' - j' + p + n + 3$

After calculations similar to the discussions in lemma 3.8.1, we obtain the follwing two equations:

$2D(T_{m\times n}) + 2c - m - 2n + 2\ell - 6 \geq 0$ ....(A')

$2D(T_{m\times n}) + 2c - m - n - 6 \geq 0$ ....(B')

When the inequalities (A') and (B') hold true for some constant c, then vertex q_1' will be adjacent to q_2' and vertex q_3' will be adjacent to q_4', and hence, all the vertices on row p will be informed.

In what follows we will find the value of c for each of the cases for m and n in torus $T_{m\times n}$, and derive the broadcast time when the originator is vertex u'.

Case 1: m is odd n is EVEN

Since m is odd and n is even

$$\Rightarrow D(T_{m\times n}) = \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor = \frac{m-1}{2} + \frac{n}{2} = \frac{m+n-1}{2}$$

$$(A') \Rightarrow 2\left(\frac{m+n-1}{2}\right) + 2c - m - 2n + 2\ell - 6 \geq 0 \Rightarrow -n + 2\ell + 2c - 7 \geq 0$$

i) if $\left\lfloor \frac{n}{2} \right\rfloor$ is odd $\Rightarrow \ell = \left\lfloor \frac{n}{2} \right\rfloor = \frac{n}{2}$

$$\Rightarrow -n + 2\ell + 2c - 7 \geq 0 \Rightarrow -n + 2\left(\frac{n}{2}\right) + 2c - 7 \geq 0 \Rightarrow 2c - 7 \geq 0 \Rightarrow c \geq 4$$

ii) if $\left\lfloor \frac{n}{2} \right\rfloor$ is even $\Rightarrow \ell = \left\lfloor \frac{n}{2} \right\rfloor + 1 = \frac{n}{2} + 1$

$$\Rightarrow -n + 2\ell + 2c - 7 \geq 0 \Rightarrow -n + 2(\frac{n}{2} + 1) + 2c - 7 \geq 0 \Rightarrow 2c - 5 \geq 0 \Rightarrow c \geq 3$$

$$(B') \Rightarrow 2\left(\frac{m+n-1}{2}\right) + 2c - m - n - 6 \geq 0 \Rightarrow 2c - 7 \geq 0 \Rightarrow c \geq 4$$

Case 2: m and n are both ODD

Since m and n are both odd

$$\Rightarrow D(T_{m\times n}) = \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor = \frac{m-1}{2} + \frac{n-1}{2} = \frac{m+n-2}{2}$$

$$(A') \Rightarrow 2\left(\frac{m+n-2}{2}\right) + 2c - m - 2n + 2\ell - 6 \geq 0 \Rightarrow -n + 2\ell + 2c - 8 \geq 0$$

i) if $\left\lfloor \frac{n}{2} \right\rfloor$ is odd $\Rightarrow \ell = \left\lfloor \frac{n}{2} \right\rfloor = \frac{n-1}{2}$

$$\Rightarrow -n + 2\ell + 2c - 8 \geq 0 \Rightarrow -n + 2\left(\frac{n-1}{2}\right) + 2c - 8 \geq 0 \Rightarrow 2c - 9 \geq 0 \Rightarrow c \geq 5$$

ii) if $\left\lfloor \frac{n}{2} \right\rfloor$ is even $\Rightarrow \ell = \left\lfloor \frac{n}{2} \right\rfloor + 1 = \frac{n-1}{2} + 1 = \frac{n+1}{2}$

$$\Rightarrow -n + 2\ell + 2c - 8 \geq 0 \Rightarrow -n + 2(\frac{n+1}{2}) + 2c - 8 \geq 0 \Rightarrow 2c - 7 \geq 0 \Rightarrow c \geq 4$$

$$(B') \Rightarrow 2\left(\frac{m+n-2}{2}\right) + 2c - m - n - 6 \geq 0 \Rightarrow 2c - 8 \geq 0 \Rightarrow c \geq 4$$

Thus, for any originator $(0, j')$ with $\ell < j' < n$ and j' even, every vertex $v \in V$ for a torus with odd number of rows m will be informed at most by time $b^{\Pi}(T_{m\times n}) \leq D(T_{m\times n}) + 5$.

Similar to lemma 3.8.1, here we can prove that when any of the paths P_1, P_2, P_3 and P_4 or P_1', P_2', P_3' and P_4' does not reach row p, then all the vertices of row p will still be informed by time $D(T_{m\times n}) + 5$.

Therefore, we can see that in the worst case, broadcasting from any vertex u or u' will give the same broadcast time. Thus, summarizing the results, we can say that when m is odd, the broadcast time of a torus $T_{m\times n}$ is:

$$b^{\Pi}(T_{m\times n}) \leq \begin{cases} D(T_{m\times n})+5 & \text{if } (m \text{ and } n \text{ are odd }) \\ & \text{or } (m \text{ is odd and } n, \left\lfloor \frac{n}{2} \right\rfloor \text{ are even}) \\ D(T_{m\times n})+4 & \text{otherwise} \end{cases}$$

□

From the two lemmas 3.8.1 and 3.8.2 we can conclude the following theorem.

Theorem 3.8.1

A torus $T_{m\times n}$ with the ordering Π will have the following broadcast time :

$$b^{\Pi}(T_{m\times n}) \leq \begin{cases} D(T_{m\times n})+5 & \text{if } m \text{ and } n \text{ are odd} \\ D(T_{m\times n})+4 & \text{if (one of } m \text{ or } n \text{ is even) or} \\ & (m, n \text{ and } \left\lfloor \frac{n}{2} \right\rfloor \text{ are even}) \\ D(T_{m\times n})+3 & \text{otherwise} \end{cases}$$

Proof :

From lemmas 3.8.1 and 3.8.2 we know that the orderly broadcast time of $T_{m\times n}$ is equal to $D(T_{m\times n}) + 5$ when n is odd and m, $\left\lfloor \frac{n}{2} \right\rfloor$ are even, and $D(T_{m\times n}) + 4$ when otherwise one of m or n is even. But if one of the row number or the column number of $T_{m\times n}$ is even, then m can be chosen to be the even number. Thus, for any torus with either an even number of rows or an even number of columns, the broadcat time will be at most $D(T_{m\times n}) + 4$. This way, the only time when the broadcast time will be equal to $D(T_{m\times n}) + 5$ is when we don't have an even row or column. Hence, the theorem is proved. □

Chapter 4

Upper Bound on Multidimensional Tori

In this chapter we discuss the upper bound on the orderly broadcast time in multidimensional tori for a given ordering Π_d. We first describe a slight variation of ordering Π in $T_{m\times n}$ (described in section 3.2). We then use this variation to describe ordering Π_d in a d-dimensional torus and obtain an upper bound.

4.1 A Variation of Ordering Π for $T_{m\times n}$

Observation 4.1.1 *Consider the ordering* Π *described in section 3.2.*

Suppose we replace all the labels of the edges numbered 3 *with some positive integer* $k > 3$. *From theorem 3.8.1, we know that the paths* P_1, P_2, P_3, P_4 *are enough to inform* *all the vertices of $T_{m\times n}$. Observing these paths we can see that each one of them uses an edge labeled 3 at most once. Thus, replacing the label 3 with k will make the upper bound of a torus in the worst case equal to :*

$$b^{\Pi}(T_{m\times n}) \leq \begin{cases} D(T_{m\times n}) + k + 2 & \text{if } m \text{ and } n \text{ are odd} \\ D(T_{m\times n}) + k + 1 & \text{if (one of } m \text{ or } n \text{ is even) or} \\ & (m, n \text{ and } \left\lfloor \frac{n}{2} \right\rfloor \text{ are even)} \\ D(T_{m\times n}) + k & \text{otherwise} \end{cases}$$

4.2 An Ordering Π_d for $T_{n_1\times n_2\times\ldots\times n_d}$

A d-dimensional torus $T_{n_1\times n_2\times\ldots\times n_d} = (V, E)$ is a connected graph of $n_1 \cdot n_2 \ldots n_d$ vertices and $n_1 \cdot n_2 \ldots n_d$ edges, such that:

$$V(T_{n_1\times n_2\times\ldots\times n_d}) = \{(i_1, i_2, \ldots i_d) | 0 \leq i_k \leq n_k - 1 \text{ for } k \in \{1, 2, \ldots d\}\}$$

$$E(T_{n_1\times n_2\times\ldots\times n_d}) = \{((u_1, u_2, \ldots u_d), (v_1, v_2, \ldots v_d)) | u_k = v_k \pm 1 \bmod n_k$$
$$\text{where } k \in \{1, 2, \ldots d\} / (u_1, u_2, \ldots u_d) \in V, (v_1, v_2, \ldots v_d) \in V\}$$

The diameter of a d-dimensional torus is:

$$D(T_{n_1\times n_2\times\ldots\times n_d}) = \left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor + \ldots + \left\lfloor \frac{n_d}{2} \right\rfloor$$

A 3-dimensional torus is a collection of 2-dimensional tori, denoted by $T^{(i)}_{n_1 \times n_2} \subset T_{n_1 \times n_2 \times \ldots \times n_d}$, connected together (Figure 21). In general, a d-dimensional torus is a collection of $(d-1)$-dimensional tori.

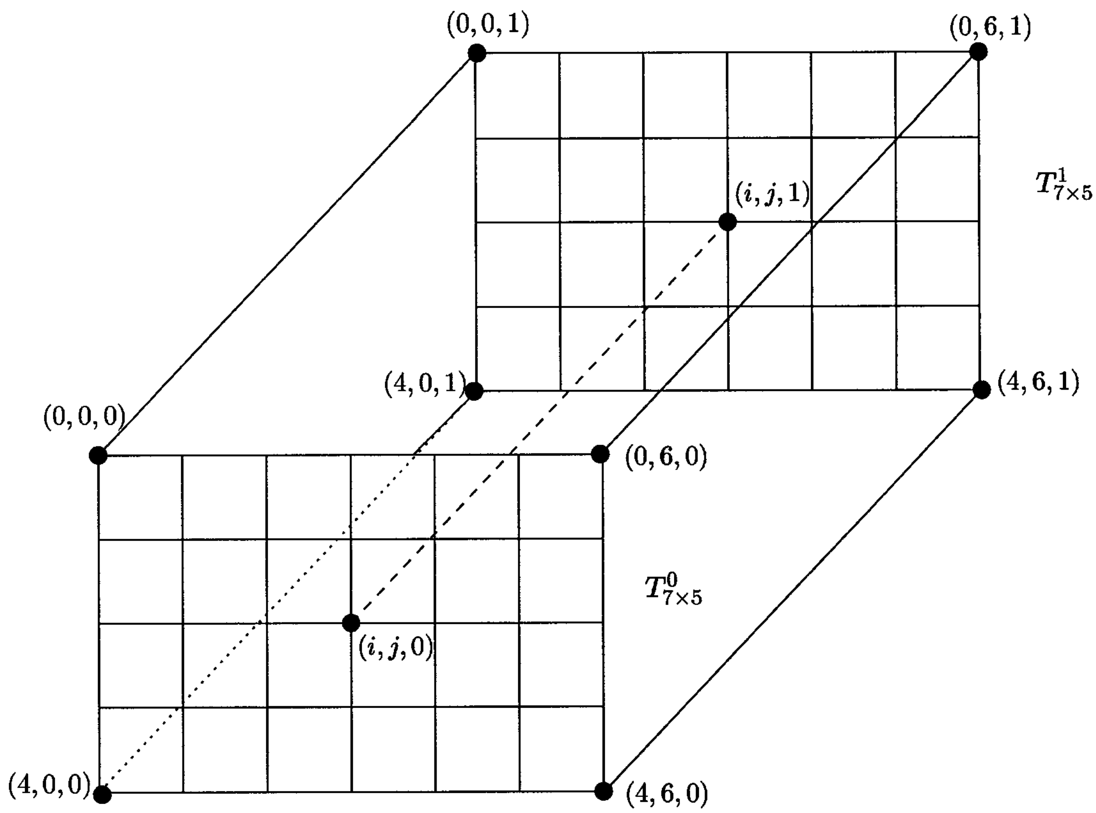

Figure 21: The three-dimensional torus $T_{7\times 5\times 2}$ is a collection of 2 two-dimensional tori $T_{7\times 5}$ where each vertex $(i, j, 0)$ is connected to $(i, j, 1)$ (with $0 \leq i \leq 4$ and $0 \leq j \leq 6$)

Let Π_d be an ordering for a d-dimensional torus $T_{n_1 \times n_2 \times \ldots \times n_d} = (V, E)$. Let $u = (i_1, i_2, \ldots, i_d) \in V$ and $v \in V$. Then,

1. $\Pi(u, v) = 1$ if:

   a) $v = (i_1, i_2 + 1 \bmod n_2, i_3 \ldots i_d)$ where $i_1 = 0 \bmod 2$ and $i_2 \notin \{0, \ell\}$

b) $v = (i_1, i_2 - 1 \bmod n_2, i_3 \dots i_d)$ where $i_1 = 0 \bmod 1$ and $i_2 \notin \{0, \ell\}$

c) $v = (i_1 + 1 \bmod n_1, i_2, i_3 \dots i_d)$ where $i_2 = 0$

d) $v = (i_1 - 1 \bmod n_1, i_2, i_3 \dots i_d)$ where $i_2 = \ell$

2. $\Pi(u, v) = 2$ if:

a) $v = (i_1 + 1 \bmod n_1, i_2, i_3 \dots i_d)$ where $i_2 = 0 \bmod 2$ and $i_2 \notin \{0, \ell\}$

b) $v = (i_1 - 1 \bmod n_1, i_2, i_3 \dots i_d)$ where $i_2 = 0 \bmod 1$ and $i_2 \notin \{0, \ell\}$

c) $v = (i_1, i_2 + 1 \bmod n_2, i_3 \dots i_d)$ where $i_1 = 0 \bmod 2$ and $i_2 \in \{0, \ell\}$

d) $v = (i_1, i_2 - 1 \bmod n_2, i_3 \dots i_d)$ where $i_1 = 0 \bmod 1$ and $i_2 \in \{0, \ell\}$

3. $\Pi(u, v) = r$ for $3 \le r \le d$ if:

a) $v = (i_1, i_2, \dots, i_r + 1 \bmod n_r, \dots i_d)$ where ($i_1 = 0 \bmod 2$ and $i_2 = 0 \bmod 1$)

or ($i_1 = 0 \bmod 1$ and $i_2 = 0 \bmod 2$)

(That is, when i_1 and i_2 have different parities)

b) $v = (i_1, i_2, \dots, i_r - 1 \bmod n_r, \dots i_d)$ where ($i_1 = 0 \bmod 2$ and $i_2 = 0 \bmod 2$)

or ($i_1 = 0 \bmod 1$ and $i_2 = 0 \bmod 1$)

(That is, when i_1 and i_2 have the same parity)

4. $\Pi(u, v) = d + 1$ if:

a) $v = (i_1 + 1 \bmod n_1, i_2, i_3 \dots i_d)$ where $i_2 = 0 \bmod 2$ and $i_2 \notin \{0, \ell\}$

b) $v = (i_1 - 1 \bmod n_1, i_2, i_3 \dots i_d)$ where $i_2 = 0 \bmod 1$ and $i_2 \notin \{0, \ell\}$

c) $v = (i_1, i_2 - 1 \bmod n_2, i_3 \dots i_d)$ where $i_1 = 0 \bmod 2$ and $i_2 \in \{0, \ell\}$

d) $v = (i_1, i_2 + 1 \bmod n_2, i_3 \dots i_d)$ where $i_1 = 0 \bmod 1$ and $i_2 \in \{0, \ell\}$

Observe that ordering Π_d for $T_{n_1 \times n_2 \times \ldots \times n_d} = (V, E)$ does not assign a label for all the edges $e \in E$. The reason for this is that during the process of message dissemination in $T_{n_1 \times n_2 \times \ldots \times n_d}$, these edges are not going to be used (the same way the edges labeled 4 were not used in $T_{m \times n}$).

Moreover, a close observation of the labeling Π_d reveals that each of the 2-dimensional tori $T^{(i)}_{n_1 \times n_2} = (V', E')$ has the ordering Π (section 3.2) where the labels 3 are replaced with $d+1$ and the labels 4 are discarded (since the edges with label 4 are not used in the broadcasting process). Thus, together with observation 4.1.1 we can conclude that when broadcasting starts from a vertex $u \in V'$, then all the vertices $v \in V'$ will be informed by time :

$$b^{\Pi}(T^{(i)}_{n_1 \times n_2}) = \begin{cases} D(T^{(i)}_{n_1 \times n_2}) + d + 3 & \text{if } n_1 \text{ and } n_2 \text{ are odd} \\ \\ D(T^{(i)}_{n_1 \times n_2}) + d + 2 & \text{if (one of } n_1 \text{ or } n_2 \text{ is even) or} \\ & \quad (n_1, n_2 \text{ and } \left\lfloor \frac{n}{2} \right\rfloor \text{ are even)} \\ \\ D(T^{(i)}_{n_1 \times n_2}) + d + 1 & \text{otherwise} \end{cases}$$

In what follows we will assume that $T_{n_1 \times n_2 \times \ldots \times n_d}$ has the ordering Π_d.

4.3 Upper Bound for $T_{n_1 \times n_2 \times \ldots \times n_d}$ with Ordering Π_d

Let $T_{n_1 \times n_2 \times \ldots \times n_d} = (V, E)$ be a d-dimensional torus with ordering Π_d. Since $T_{n_1 \times n_2 \times \ldots \times n_d}$ is a collection of 2-dimensional tori, we let $T^{(i)}_{n_1 \times n_2} = (V', E')$ represent one of these 2-dimensional tori where $1 \leq i \leq n_3 \cdot n_4 \ldots n_d$.

Before discussing the upper bound for $T_{n_1 \times n_2 \times \ldots \times n_d}$, we first make the following observation.

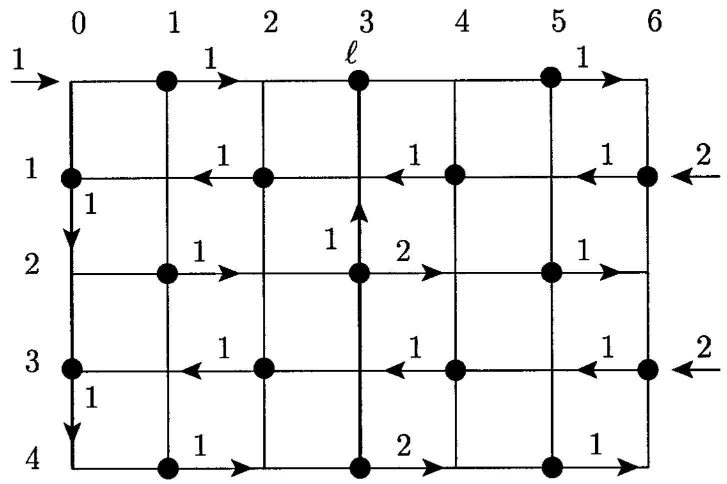

Figure 22: If all the vertices $(i, j) \in V(T_{5\times 7})$ where i and j have different parities are informed at time t, then (i, j) can inform the rest of the uninformed vertices by time $t + 2$. Note that some of the edges and lables are omitted to make the picture readable.

Observation 4.3.1

Given a torus $T_{n_1 \times n_2 \times \ldots \times n_d} = (V, E)$. Let $T^{(i)}_{n_1 \times n_2} = (V', E') \subset T_{n_1 \times n_2 \times \ldots \times n_d}$. If all the vertices $(i_1, i_2, \ldots, i_d) \in V'$, where i_1 and i_2 have different parities, are informed at time t, then every vertex $u \in V'$ will be informed at time $t + 2$ (Figure 22).

Similarly, if all the vertices $(i_1, i_2, \ldots, i_d) \in V'$, *where* i_1 *and* i_2 *have the same parity, are informed at time* t, *then every vertex* $u \in V'$ *will be informed at time* $t + 2$.

When all the vertices $(i_1, i_2, \ldots, i_d)$ with i_1 and i_2 having different parities are informed, then each row i_2 of $T^{(i)}_{n_1 \times n_2}$ will have at most $\left\lfloor \frac{n}{2} \right\rfloor + 1$ uninformed vertices. Also, every informed vertex $(i_1, i_2, \ldots, i_d) \in V'$ will have two uninformed neighbors $(i_1, i_2 - 1, \ldots, i_d)$ and $(i_1, i_2 + 1, \ldots, i_d)$ on row i_1. Thus,

- at time $t + 1$: $(i_1, i_2, \ldots, i_d) \xrightarrow{1} (i_1, i_2 + 1, \ldots, i_d)$ where i_1 is even and $i_2 \neq \ell$.

  $(i_1, i_2, \ldots, i_d) \xrightarrow{1} (i_1, i_2 - 1, \ldots, i_d)$ where i_1 is odd and $i_2 \neq 0$.

- at time $t + 2$: $(i_1, i_2 + 1, \ldots, i_d) \xrightarrow{1} (i_1, i_2 + 2, \ldots, i_d)$ where i_1 is even and $i_2 \neq \ell$.

  $(i_1, i_2 - 1, \ldots, i_d) \xrightarrow{1} (i_1, i_2 - 2, \ldots, i_d)$ where i_1 is odd and $i_2 \neq 0$.

  $(i_1, \ell, \ldots, i_d) \xrightarrow{2} (i_1, \ell + 1, \ldots, i_d)$ where i_1 is even.

This will inform all the vertices $u \in V'$

Similarly, when all the vertices $(i_1, i_2, \ldots, i_d)$ with i_1 and i_2 having the same parity are informed then:

- at time $t + 1$: $(i_1, i_2, \ldots, i_d) \xrightarrow{1} (i_1, i_2 + 1, \ldots, i_d)$ where i_1 is even and $i_2 \neq \ell$.

  $(i_1, i_2, \ldots, i_d) \xrightarrow{1} (i_1, i_2 - 1, \ldots, i_d)$ where i_1 is odd and $i_2 \neq 0$.

- at time $t + 2$: $(i_1, i_2 + 1, \ldots, i_d) \xrightarrow{1} (i_1, i_2 + 2, \ldots, i_d)$ where i_1 is even and $i_2 \neq \ell$.

  $(i_1, i_2 - 1, \ldots, i_d) \xrightarrow{1} (i_1, i_2 - 2, \ldots, i_d)$ where i_1 is odd and $i_2 \neq 0$.

$(i_1, \ell, \ldots, i_d) \xrightarrow{2} (i_1, \ell - 1, \ldots, i_d)$ where i_1 is odd.

$(i_1, i_2 + 1, \ldots, i_d) \xrightarrow{2} (i_1 + 1, i_2 + 1, \ldots, i_d)$ where i_1 is even.

This will inform all the vertices $u \in V'$

Theorem 4.3.1

Given a d-dimensional torus $T_{n_1 \times n_2 \times \ldots \times n_d} = (V, E)$ with ordering Π_d. The orderly broadcast time of $T_{n_1 \times n_2 \times \ldots \times n_d}$ will be :

$$b^{\Pi_d}(T_{n_1 \times n_2 \times \ldots \times n_d}) \leq \begin{cases} \left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor + \sum_{i=3}^{d} i \left\lfloor \frac{n_i}{2} \right\rfloor + 3d - 1 & \text{if } n_1 \text{ and } n_2 \text{ are odd} \\ \left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor + \sum_{i=3}^{d} i \left\lfloor \frac{n_i}{2} \right\rfloor + 3d - 2 & \text{if (one of } n_1 \text{ or } n_2 \text{ is even)} \\ & \text{or } (n_1, n_2 \text{ and } \left\lfloor \frac{n_1}{2} \right\rfloor \text{ are even)} \\ \left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor + \sum_{i=3}^{d} i \left\lfloor \frac{n_i}{2} \right\rfloor + 3d - 3 & \text{otherwise} \end{cases}$$

Proof :

To prove the theorem, we will explain step by step how broadcasting proceeds in $T_{n_1 \times n_2 \times \ldots \times n_d}$. We will first assume the case where n_1 is odd.

Let $u = (i_1, i_2, i_3, \ldots i_d) \in V$ be the originator where $0 \leq i_r \leq n_r$ (for $1 \leq r \leq d$).

Let $T^{(i)}_{n_1 \times n_2} = (V', E') \subset T_{n_1 \times n_2 \times \ldots \times n_d}$ for some $1 \leq i \leq n_3 \cdot n_4 \ldots n_d$. Assume $u \in V'$.

at time $t_0 = \left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor + d + 3$

- All the vertices $v \in V'$ are informed (section 4.2).

at time : $t_0 + 3$

- Vertices $(i_1, i_2, i_3, i_4 \ldots i_d)$ inform their neighbors $(i_1, i_2, i_3 + 1, i_4 \ldots i_d)$, where i_1 and i_2 have different parities (i.e. when ($i_1 = 0$ mod 2 and $i_2 = 1$ mod 2) or ($i_1 = 1$ mod 2 and $i_2 = 0$ mod 2)) for all $0 \leq i_r \leq n_r$ (where $1 \leq r \leq 2$).
- Vertices $(i_1, i_2, i_3, i_4 \ldots i_d)$ inform their neighbors $(i_1, i_2, i_3 - 1, i_4 \ldots i_d)$, where i_1 and i_2 have the same parity (i.e. when ($i_1 = 0$ mod 2 and $i_2 = 0$ mod 2) or ($i_1 = 1$ mod 2 and $i_2 = 1$ mod 2)) for all $0 \leq i_r \leq n_r$ (where $1 \leq r \leq 2$).

at time : $t_0 + 6$

- Vertices $(i_1, i_2, i_3{+}1, i_4 \ldots i_d)$ inform their neighbors $(i_1, i_2, i_3{+}2, i_4 \ldots i_d)$, where i_1 and i_2 have different parities for all $0 \leq i_r \leq n_r$ (where $1 \leq r \leq 2$).
- Vertices $(i_1, i_2, i_3{-}1, i_4 \ldots i_d)$ inform their neighbors $(i_1, i_2, i_3{-}2, i_4 \ldots i_d)$, where i_1 and i_2 have the same parity for all $0 \leq i_r \leq n_r$ (where $1 \leq r \leq 2$).

Continuing in the same manner, we obtain :

at time: $t_0 + 3\left\lfloor\frac{n_3}{2}\right\rfloor$

- Vertices $(i_1, i_2, i_3+\left\lfloor\frac{n_3}{2}\right\rfloor-1, i_4 \ldots i_d)$ inform their neighbors $(i_1, i_2, i_3+\left\lfloor\frac{n_3}{2}\right\rfloor, i_4 \ldots i_d)$, where i_1 and i_2 have different parities for all $0 \leq i_r \leq n_r$ (where $1 \leq r \leq 2$).
- Vertices $(i_1, i_2, i_3-\left\lfloor\frac{n_3}{2}\right\rfloor+1, i_4 \ldots i_d)$ inform their neighbors $(i_1, i_2, i_3-\left\lfloor\frac{n_3}{2}\right\rfloor, i_4 \ldots i_d)$, where i_1 and i_2 have the same parity for all $0 \leq i_r \leq n_r$ (where $1 \leq r \leq 2$).

At this point, we will have :

- $\left\lfloor\frac{n_3-1}{2}\right\rfloor$ tori $T^{(i)}_{n_1\times n_2} = (V', E')$ where all the vertices $(i_1, i_2, i_3, \ldots, i_d) \in V'$ are informed such that i_1 and i_2 have different parities.
- $\left\lfloor\frac{n_3-1}{2}\right\rfloor$ tori $T^{(j)}_{n_1\times n_2} = (V'', E'')$ where all the vertices $(i_1, i_2, i_3, \ldots, i_d) \in V''$ are informed such that i_1 and i_2 have the same parity.
- at most 2 informed tori $T^{(r)}_{n_1\times n_2} = (V''', E''')$ where all the vertices $v \in V'''$ are informed.

at time: $t_0 + 3\left\lfloor\frac{n_3}{2}\right\rfloor + 2$

- All the vertices $v \in V'$ are informed in each of $T^{(i)}_{n_1\times n_2}$ (observation 4.3.1).
- Similarly, all the vertices $v \in V''$ are informed in each of $T^{(j)}_{n_1\times n_2}$.

Observe that at this point, all the vertices $(i_1, i_2, i_3, i_4, \ldots, i_d)$ for all $0 \leq i_1 \leq n_1 - 1$, $0 \leq i_2 \leq n_2 - 1$ and $0 \leq i_3 \leq n_3 - 1$ are informed. This means that all the vertices of one of the 3-dimensional tori, $T^{(i)}_{n_1 \times n_2 \times n_3} \subset T_{n_1 \times n_2 \times \ldots \times n_d}$ are informed (figure 23).

at time: $t_0 + 3\left\lfloor \frac{n_3}{2} \right\rfloor + 2 + 4$

- Vertices $(i_1, i_2, i_3, i_4 \ldots i_d)$ inform their neighbors $(i_1, i_2, i_3, i_4{+}1, i_5 \ldots i_d)$, where i_1 and i_2 have different parities (i.e. when ($i_1 = 0$ mod 2 and $i_2 = 1$ mod 2) or ($i_1 = 1$ mod 2 and $i_2 = 0$ mod 2)) for all $0 \leq i_r \leq n_r$ (where $1 \leq r \leq 3$).
- Vertices $(i_1, i_2, i_3, i_4 \ldots i_d)$ inform their neighbors $(i_1, i_2, i_3, i_4{-}1, i_5 \ldots i_d)$, where i_1 and i_2 have the same parity (i.e. when ($i_1 = 0$ mod 2 and $i_2 = 0$ mod 2) or ($i_1 = 1$ mod 2 and $i_2 = 1$ mod 2)) for all $0 \leq i_r \leq n_r$ (where $1 \leq r \leq 3$).

at time: $t_0 + 3\left\lfloor \frac{n_3}{2} \right\rfloor + 2 + 8$

- Vertices $(i_1, i_2, i_3, i_4 + 1, i_5 \ldots i_d)$ inform their neighbors $(i_1, i_2, i_3, i_4 + 2, i_5 \ldots i_d)$, where i_1 and i_2 have different parities for all $0 \leq i_r \leq n_r$ (where $1 \leq r \leq 3$).
- Vertices $(i_1, i_2, i_3, i_4 - 1, i_5 \ldots i_d)$ inform their neighbors $(i_1, i_2, i_3, i_4 - 1, i_5 \ldots i_d)$, where i_1 and i_2 have the same parity for all $0 \leq i_r \leq n_r$ (where $1 \leq r \leq 3$).

Continuing in the same manner, we will get :

at time: $t_0 + 3\left\lfloor \frac{n_3}{2} \right\rfloor + 4\left\lfloor \frac{n_4}{2} \right\rfloor + 2$

- Vertices $(i_1, i_2, i_3, i_4 + \left\lfloor \frac{n_4}{2} \right\rfloor - 1, i_5 \ldots i_d)$ inform their neighbors $(i_1, i_2, i_3, i_4 + \left\lfloor \frac{n_4}{2} \right\rfloor, i_5 \ldots i_d)$, where i_1 and i_2 have different parities for all $0 \leq i_r \leq n_r$ (where $1 \leq r \leq 3$).

- Vertices $(i_1, i_2, i_3, i_4 - \left\lfloor \frac{n_4}{2} \right\rfloor + 1, i_5 \ldots i_d)$ inform their neighbors $(i_1, i_2, i_3, i_4 - \left\lfloor \frac{n_4}{2} \right\rfloor, i_5 \ldots i_d)$, where i_1 and i_2 have the same parity for all $0 \leq i_r \leq n_r$ (where $1 \leq r \leq 3$).

Similarly, at this point, we will have :

- $\left\lfloor \frac{n_4 - 1}{2} \right\rfloor$ tori $T^{(i)}_{n_1 \times n_2 \times n_3} = (V', E')$ where all the vertices $(i_1, i_2, i_3, i_4 \ldots, i_d) \in V'$ are informed such that i_1 and i_2 have different parities.

- $\left\lfloor \frac{n_4 - 1}{2} \right\rfloor$ tori $T^{(j)}_{n_1 \times n_2 \times n_4} = (V'', E'')$ where all the vertices $(i_1, i_2, i_3, i_4 \ldots, i_d) \in V''$ are informed such that i_1 and i_2 have the same parity.

- at most 2 informed tori $T^{(r)}_{n_1 \times n_2 \times n_3} = (V''', E''')$ where all the vertices $v \in V'''$ are informed.

at time: $t_0 + 3 \left\lfloor \frac{n_3}{2} \right\rfloor + 4 \left\lfloor \frac{n_4}{2} \right\rfloor + 4$

- All the vertices $v \in V'$ are informed in each of $T^{(i)}_{n_1 \times n_2 \times n_3}$ (observation 4.3.1).

- Similarly, all the vertices $v \in V''$ are informed in each of $T^{(j)}_{n_1 \times n_2 \times n_3}$.

Similarly we can say that at this point, all the vertices $(i_1, i_2, i_3, i_4, \ldots, i_d)$ for all $0 \leq i_1 \leq n_1 - 1$, $0 \leq i_2 \leq n_2 - 1$, $0 \leq i_3 \leq n_3 - 1$ and $0 \leq i_4 \leq n_4 - 1$ are informed. This means that all the vertices of one of the 4-dimensional tori, $T^{(i)}_{n_1 \times n_2 \times n_3 \times n_4} \subset T_{n_1 \times n_2 \times \ldots \times n_d}$ are informed (figure 23).

Continuing in this manner, for a d-dimensional torus $T_{n_1 \times n_2 \times \ldots \times n_d}$ with n_1 odd, the broadcast time will be :

$$b^{\Pi_d}(T_{n_1 \times n_2 \times \ldots \times n_d}) \leq \left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor + (d+3) + \sum_{i=3}^{d} i \left\lfloor \frac{n_i}{2} \right\rfloor + 2(d-2)$$

$$= \left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor + \sum_{i=3}^{d} d \left\lfloor \frac{n_i}{2} \right\rfloor + 3d - 1$$

For a torus where at least one of n_1 or n_2 is even or when n_1, n_2 and $\left\lfloor \frac{n_1}{2} \right\rfloor$ are even , the first step will take one less time unit. Thus, the broadcast time in this case will be :

$$\left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor + \sum_{i=3}^{d} i \left\lfloor \frac{n_i}{2} \right\rfloor + 3d - 2$$

In all other cases, the first step will take two less time units, making the broadcast time in this case:

$$\left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor + \sum_{i=3}^{d} i \left\lfloor \frac{n_i}{2} \right\rfloor + 3d - 3$$

Therefore,

$$b^{\Pi_d}(T_{n_1\times n_2\times\ldots\times n_d}) \leq \begin{cases} \left\lfloor\frac{n_1}{2}\right\rfloor + \left\lfloor\frac{n_2}{2}\right\rfloor + \sum_{i=3}^{d} i\left\lfloor\frac{n_i}{2}\right\rfloor + 3d-1 & \text{if } n_1 \text{ and } n_2 \text{ are odd} \\ \left\lfloor\frac{n_1}{2}\right\rfloor + \left\lfloor\frac{n_2}{2}\right\rfloor + \sum_{i=3}^{d} i\left\lfloor\frac{n_i}{2}\right\rfloor + 3d-2 & \text{if (one of } n_1 \text{ or } n_2 \text{ is even)} \\ & \text{or } (n_1, n_2 \text{ and } \left\lfloor\frac{n_1}{2}\right\rfloor \text{ are even)} \\ \left\lfloor\frac{n_1}{2}\right\rfloor + \left\lfloor\frac{n_2}{2}\right\rfloor + \sum_{i=3}^{d} i\left\lfloor\frac{n_i}{2}\right\rfloor + 3d-3 & \text{otherwise} \end{cases}$$

□

4.4 Analysis of Results

An obvious lower bound on broadcasting for a d-dimensional torus is its diameter, which is :

$$D(T_{n_1\times n_2\times\ldots\times n_d}) = \left\lfloor\frac{n_1}{2}\right\rfloor + \left\lfloor\frac{n_2}{2}\right\rfloor + \ldots + \left\lfloor\frac{n_d}{2}\right\rfloor$$

It is easy to see that we can improve the orderly broadcast time for $T_{n_1\times n_2\times\ldots\times n_d}$ by letting $n_1 \geq n_2 \geq \ldots \geq n_d$.

The ordering Π_d will yield the worst-case broadcast time when $n_1 = n_2 = \ldots = n_d$. Let $T_{n_1\times n_2\times\ldots\times n_d} = (V, E)$ be a d-dimensional torus such that $n_1 = n_2 = \ldots = n_d$, and let $N = |V|$ be the total number of vertices. Then,

$N = n_1^d \Rightarrow n_1 = n_2 = \ldots = n_d = \sqrt[d]{N}$

The diameter in terms of N will be : $D(T_{n_1 \times n_2 \times \ldots \times n_d}) \simeq \frac{\sum_{i=1}^{d} n_i}{2} = \frac{dn_1}{2} = \frac{d\sqrt[d]{N}}{2}$

Since the obvious lower bound is the diameter, then: $b^{\Pi}(T_{n_1 \times n_2 \times \ldots \times n_d}) \geq \frac{d\sqrt[d]{N}}{2}$

Now, in the worst case the broadcast time of $T_{n_1 \times n_2 \times \ldots \times n_d}$ will be:

$$b^{\Pi_d}(T_{n_1 \times n_2 \times \ldots \times n_d}) \leq \left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor + \sum_{i=3}^{d} i \left\lfloor \frac{n_i}{2} \right\rfloor + 3d - 1$$

$$\leq n_1 + \frac{n_1}{2} \sum_{i=3}^{d} i + 3d - 1$$

$$\leq n_1 + \frac{n_1}{2} \left(\frac{(d+3)(d-2)}{2} \right) + 3d - 1$$

$$\leq \frac{n_1}{2} \left(\frac{d^2 + d - 6}{2} + 2 \right) + 3d - 1$$

$$\leq \frac{n_1}{2} \left(d \cdot \frac{d+1}{2} - 1 \right) + 3d - 1$$

$$\leq \frac{d+1}{2} D(T_{n_1 \times n_2 \times \ldots \times n_d}) - \frac{n_1}{2} + 3d - 1$$

The value $\frac{n_1}{2} + 3d - 1 \ll D(T_{n_1 \times n_2 \times \ldots \times n_d})$.

Thus, we can say that: $b^{\Pi_d}(T_{n_1 \times n_2 \times \ldots \times n_d}) \leq \frac{d+1}{2} D(T_{n_1 \times n_2 \times \ldots \times n_d})$

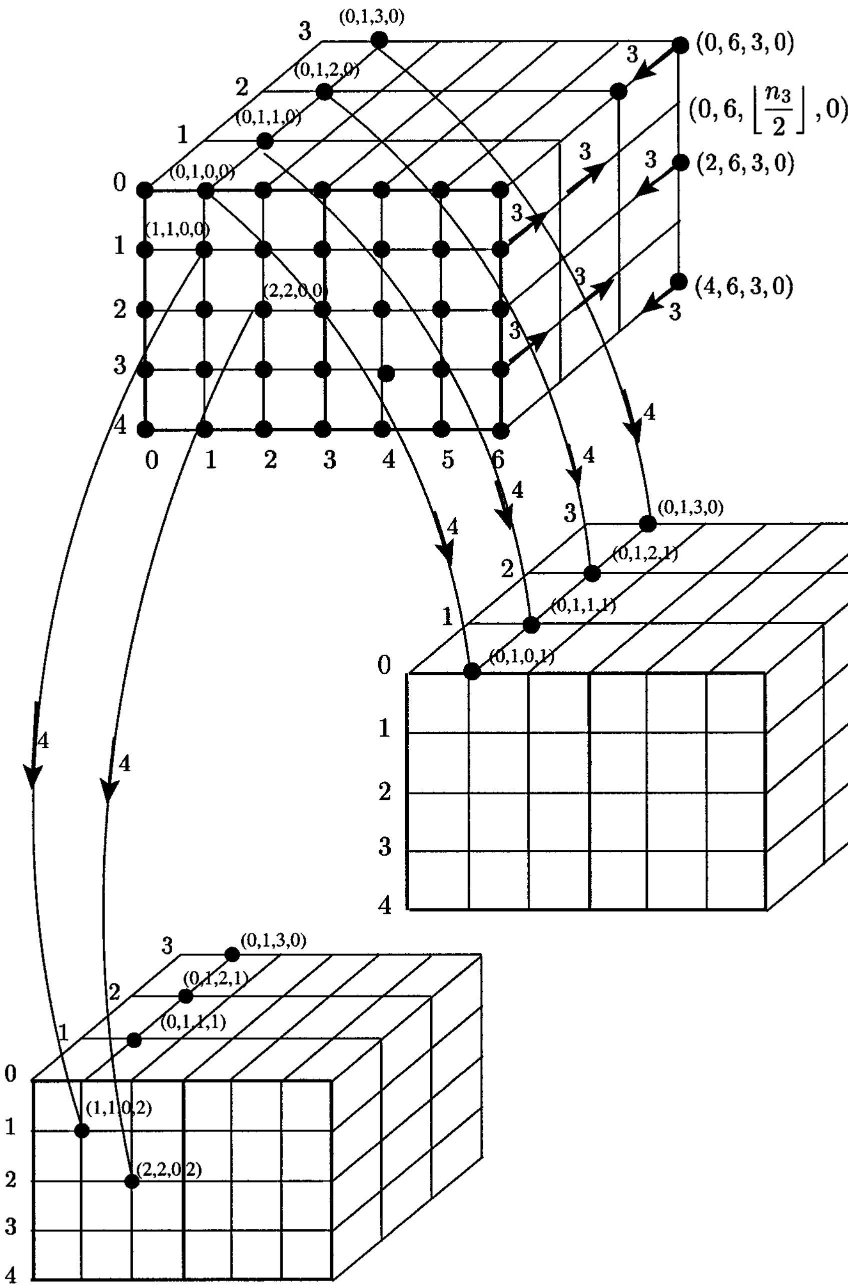

Figure 23: A 4-dimensional torus $T_{7\times5\times4\times3}$. Suppose the originator is $(0,1,0,0)$. Then at time $t_0 = 11$, all the vertices $(i_1, i_2, 0, 0)$ where $0 \leq i_1 \leq 6$ and $0 \leq i_2 \leq 4$ will be informed. At time 14, all vertices $(i_1, i_2, 1, 0)$ and $(j_1, j_2, 3, 0)$, where i_1 and i_2 have different parity while j_1 and j_2 have the same parity, will be informed. At time 16, all vertices in the first "cube" (3-dimensional torus) will be informed. The process then continues to the other 3-dimensional tori in the same manner.

Chapter 5

Conclusion and Future Work

Information dissemination problems, such as broadcasting, are problems which have numerous real-life applications, and many of the models studied have been inspired by practical needs. Orderly broadcasting is one of the models which is very simple to implement in practice. It is a model in which the nodes of a network do not have the complicated task of making "intelligent" decisions in order to optimize the broadcast time. The optimality of the broadcast time lies in the clever design of the network itself, in which the nodes transmit the message simply by following a fixed set of ordering. The challenge of this model is in finding an ordering for each node of a network that will minimize the overall broadcast time. It has turned out that coming up with an optimal ordering is not an easy task, even in networks which are as simple as cycles, for example.

In this work, we studied the orderly broadcast problem for multidimensional tori.

We first presented an ordering Π of the vertices of a 2-dimensional torus which yielded a broadcast time of : $b^{\Pi}(T_{m\times n}) \leq D(T_{m\times n}) + 5$ in the worst case. We then introduced a slight variation of the ordering Π and used it to order the vertices of a d-dimensional torus $T_{n_1\times n_2\times\ldots\times n_d}$. The broadcast time for this latter was $b^{\Pi_d}(T_{n_1\times n_2\times\ldots\times n_d}) \leq \left\lfloor\frac{n_1}{2}\right\rfloor + \left\lfloor\frac{n_2}{2}\right\rfloor + \sum_{i=3}^{d} i\left\lfloor\frac{n_i}{2}\right\rfloor + 3d - 1$

As mentioned in section 1.3, orderly broadcasting on 2-dimensional tori was discussed in [11], and the result presented was:

$$b^{o}(T_{m\times n}) \leq \begin{cases} D(T_{m\times n}) + 6 & \text{if } n \text{ is even} \\ D(T_{m\times n}) + 7 & \text{if } n \text{ is odd} \end{cases}$$

This clearly shows that our result on 2-dimensional tori has improved the upper bound by at least 2 time units.

Also in the 2-dimensional case, our result was close to the lower bound - there was a difference of only 2 time units between the upper and the lower bounds in the best case. Moreover, recall that the classical broadcast time of a 2-dimensional torus is (see [16]):

$$b(T_{m\times n}) \leq \begin{cases} D(T) & \text{if } m \text{ and } n \text{ are even} \\ D(T) + 1 & \text{otherwise} \end{cases}$$

Compared to the result mentioned above, our result on the orderly broadcast time of a 2-dimensional torus is close to the broadcast time of a 2-dimensional torus in the classical model.

As future work, the first problem that should be tackled is trying to improve the upper and lower bounds for both, 2-dimensional and multidimensional tori. Other interesting problems would include finding orderings for other graph topologies which have been studied under other broadcast models (for example, Butterfly networks, DeBruijn graphs, Cube Connected Cycles ... etc.) Another interesting problem would be trying to find a relationship between the broadcast times of a graph in the Classical, Messy and the Orderly broadcast models.

Bibliography

[1] R. Ahlswede, L. Gargano, H. Haroutunian, and L. Khachatrian. Fault-tolerant minimum broadcast networks. *Networks*, 27:293–307, 1996.

[2] R. Ahlswede, H. S. Haroutunian, and L. H. Khachatrian. Messy broadcasting in networks. *Communications and Cryptography*, pages 13–24, 1994. eds. R. E. Blahut, D. J. Costello, Jr., U. Mauter, and T. Mittelholzer(Kluwer, Boston/Dordrecht/London).

[3] J. C. Bermond, P. Fraigniaud, and J. Peters. Antepenultimate broadcasting. *Networks*, 26:125–137, 1995.

[4] J. C. Bermond, P. Hell, A. L. Liestman, and P. J. Sparse broadcast graphs. *Discrete Applied Math.*, 36:97–130, 1992.

[5] S. C. Chau and A. L. Liestman. Constructing fault-tolerant minimal broadcast networks. *Journal Comb. Inform. & System Sciences*, 10:110–122, 1985.

[6] X. Chen. An upper boun for the broadcast function $b(n)$. *Chinese J. Comput.*, 13:605–611, 1990.

[7] P. Chinn, S. Hedetniemi, and S. Mitchell. Multiple message broadcasting in complete graphs. In *Proc. Tenth Southeastern Conf. on Combinatorics, Graph Theory and Computing*, pages 251–260, Winnipeg, 1979. Utilitas Mathematica.

[8] E. J. Cockayne and S. Hedetniemi. A conjecture concerning broadcasting in m-dimensional grid graphs. Technical report, University of Oregon, 1978. Technical Report CS-TR-78-14.

[9] F. Comellas, H. A. Harutyunyan, and A. L. Liestman. Messy broadcasting in multidimensional directed tori. *J. Interconnection Networks*, 4(1):37–51, 2003.

[10] K. Diks and A. Pelc. Universal order broadcasting. *28th Annual Hawaii International Conference on System Sciences*, 2:564–573, 1995.

[11] K. Diks and A. Pelc. Broadcasting with universal lists. *Networks*, 27:183–196, 1996.

[12] A. Farley. Broadcast time in communication networks. *SIAM Journal in Applied Math.*, 39:385–390, 1980.

[13] A. Farley, S. Hedetniemi, S. Mitchell, and A. Proskurowski. Minimum broadcast graphs. *Discrete Math.*, 25:189–193, 1979.

[14] A. Farley and A. Proskurowski. Broadcasting in trees with multiple originators. *SIAM Journal of Algebraic and Discrete Methods*, 2(4):381–386, 1981.

[15] A. M. Farley. Minimal broadcast networks. *Networks*, 9:313–332, 1979.

[16] A. M. Farley and S. T. Hedetniemi. Broadcasting in grid graphs. *Proceedings 9th S-E Conference Combinatorics, Graph Theory and Computing, FL Congressus Numerantium*, XXI:275–288, 1978.

[17] P. Fraigniaud and E. Lazard. Methods and problems of communication in usual networks. *Discrete Applied Math*, 53:79–133, 1994.

[18] M. Grigni and D. Peleg. Tight bounds on minimum broadcast networks. *SIAM Journal on Discrete Math*, 4:207–222, 1991.

[19] T. Hart and H. A. Harutyunyan. Improved messy broadcasting in hypercube and complete bipartite graphs. *Congressus Numerantium*, 156:124–140, 2002.

[20] T. Hart and H. A. Harutyunyan. Messy broadcasting in common communication networks. In 21*st Biennial Symposium on Communicatins*, pages 534–538, Kingston, Canada, 2002.

[21] H. A. Harutyunyan. Minimum multiple message broadcast graphs. Submitted to Networks.

[22] H. A. Harutyunyan. Multiple message broadcasting in modified knōdel graph. In *Proc. 7th Int. Coll. on Structured Information and Communication Complexity*, pages 157–165, L'Aquila, Italy, 2000. SIROCCO 2000, Proceedings in Informatics 7.

[23] H. A. Harutyunyan and A. L. Liestman. Messy broadcasting. *Parallel Processing Letters*, 8(2):149–159, 1998.

[24] H. A. Harutyunyan and A. L. Liestman. Improved upper and lower bounds for k-broadcasting. *Networks*, 37:94–101, 2001.

[25] H. A. Harutyunyan and A. L. Liestman. k-broadcasting in trees. *Networks*, 38:163–168, 2001.

[26] H. A. Harutyunyan, A. L. Liestman, and T. C. Shermer. Orderly broadcasting. manuscript, 1999.

[27] H. A. Harutyunyan and B. Shao. A heuristic for k-broadcasting in arbitrary networks. *IEEE Conference on Applications of Graph Theory (IVO)*, pages 287–292, 2003. London, England.

[28] S. T. Hedetniemi, S. M. Hedetniemi, and A. L. Liestman. A survey of broadcasting and gossiping in communication networks. *Networks*, 18:319–349, 1988.

[29] J. Hromovic, R. Klasig, B. Monien, and R. Peine. Dissemination of information in interconnection networks (broadcasting & gossiping). *Combinatorial Network Theory*, pages 125–212, 1996. ed.: Ding-Zhu Du, D. Frank Hsu, Kluwer Academic Publishers.

[30] J. C. Kōnig and E. Lazard. Minimum k-broadcast graphs. *Discrete Applied Math*, 53:199–209, 1994.

[31] R. Labahn. A minimum broadcast graph on 63 vertices. *Discrete Applied Math.*, 53:247–250, 1994.

[32] S. Lee and J. A. Ventura. An algorithm for constructing minimal k-broadcast networks. *Networks*, 38(1):6–21, 2001.

[33] A. L. Liestman. Fault-tolerant scheduling and broadcasting problems. Technical report, University of Illinois, 1981. Technical Report UIUCDCS-R-81-1063.

[34] A. L. Liestman. Fault-tolerant broadcast graphs. *Networks*, 15:159–171, 1985.

[35] A. L. Liestman, J. Opatrny, and M. Zaragoza'. Network properties of double and triple fixed step graphs. *International Journal of Foundations of Computer Science*, 9:57–76, 1998.

[36] M. Maheo and J. F. Sacle. Some minimum broadcast graphs. *Discrete Applied Math.*, 53:275–285, 1994.

[37] Proc. Int. Symp. on Communications (ISCOM97). *Multi-broadcasting in communication networks. I: Trees*, 1997.

[38] J. F. Sacle. Lower bound for the size in four families of minimum broadcast graphs. *Discrete Math.*, 150:359–369, 1996.

[39] P. J. Slater, E. J. Cockayne, and S. T. Hedetniemi. Information dissemination in trees. Technical Report 4, SIAM Journal on Computing, 1981.

[40] X. Xu. Broadcast networks on $2^p - 1$ nodes and minimum broadcast network on 127 nodes. Master's thesis, Department of Computer Science, Concordia University, 2003.