On the Time Value of Ruin for Insurance Risk Models

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A Thesis
for
The Department
of
Mathematics and Statistics

Presented in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy at
Concordia University
Montréal, Québec, Canada

April, 2004

Your file Votre référence
ISBN: 0-612-90393-1
Our file Notre référence
ISBN: 0-612-90393-1

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ABSTRACT

On the Time Value of Ruin for Insurance Risk Models

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Concordia University, 2004

This thesis studies ruin probabilities and ruin related quantities, using a unified treatment of analysis through the celebrated Gerber-Shiu (G-S) penalty function. For different insurance risk models, a G-S function discounts a penalty due at ruin, which may depend on the surplus before ruin and the deficit at ruin. These insurance risk models include Sparre Andersen's risk model, both in a continuous and in a discrete time setting, diffusion perturbed Sparre Andersen models, as well as risk models with a constant dividend barrier. All these models are extensions of the classical risk model and of diffusion perturbed classical risk model.

These G-S penalty functions, considered as functions of initial surplus, satisfy certain integral equations or integro-differential equations, which can be solved to yield defective renewal equations. Such defective renewal equations have a natural probabilistic interpretation, which relies heavily on the roots to a generalized

Lundberg's fundamental equation that have a positive real part. These generalized Lundberg equations are from an appropriately chosen exponential martingale.

The defective renewal equations (also called recursive formulas in discrete models), that the expected penalty functions satisfy, allow the use of the existing techniques in renewal theory. They can be used to analyze many quantities associated with the time of ruin, such as explicit expressions, bounds, approximations and asymptotic formulas for ruin probabilities, the Laplace transform (or generating function in discrete models) of the time of ruin, the discounted joint and marginal distribution of the surplus immediately before ruin and the deficit at ruin, as well as their moments.

Finally, explicit results for the G-S discounted penalty function can be solved when the initial reserve is zero and when the claim sizes are rationally distributed, i.e., the Laplace transform of the claim size density is a rational function.

Acknowledgments

I wish to thank my supervisor, Prof. José Garrido, for his insightful corrections, comments, clarifications and suggestions throughout the preparation of this thesis.

I appreciate Prof. Hans Gerber and Prof. David Dickson for their numerous helpfull comments and suggestions, which resulted in the improvement of the presentations of this thesis. I also want to thank Dr. Nebebe, Dr. Sen and Dr. Zhou for their time and work in the thesis supervisor committee.

I would also like to express gratitude to Concordia University and the department of Mathematics and Statistics for their endless support.

I must point out that this project would not have gotten off the ground without the financial support of Prof. José Garrido, the Department of Mathematics and Statistics, and Concordia University. I would also like to acknowledge the important contribution of the Ph.D. Grant that I was awarded from the Casualty Actuarial Society (CAS) and the Society of Actuaries (SOA).

Finally, I wish to thank my wife, Shuyun (May), and my daughter, Lily, to whom I am infinitely grateful for their endurance, understanding, assistance and encouragement throughout the development of this academic pursuit when I was unable to live with them.

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Notation and List of Symbols

$\mathbb{R}$	$(-\infty, \infty)$
$\mathbb{R}^+$	$(0, \infty)$
$\mathbb{C}$	complex plane
$\mathbb{N}$	set of non-negative integers
$\mathbb{N}^+$	set of positive integers
Ω	the whole sample space
$\mathcal{F}$	sigma algebra on Ω
$\Re(s)$	real part of a complex number s
$\Im(s)$	imaginary part of a complex number s
$B(t)$	standard Brownian motion
$\square$	end of proof
$I(A)$	the indicator function of the set A
$F^{*n}(x)$	the n-th fold convolution of F with itself at x
$F^{*0}(x)$	0 if $x < 0$ and 1 if $x \geq 0$
$\bar{F}(x)$	$1 - F(x)$, the tail of F
$\bar{F}^{*n}(x)$	$1 - F^{*n}(x)$
$F_n(x)$	n-th order equilibirium distribution of F
T_r	an operator w.r.t. a complex number r
$h[r_1, r_2, \ldots, r_k, s]$	k-th divided difference of function h w.r.t. complex numbers $r_1, r_2, \ldots, r_n$
$\pi_k(s;\, s_1, s_2, \ldots, s_k)$	$\prod_{i=1}^{k}(s - s_k)$
$\hat{f}(s)$	Laplace transform of f, or generating function of f
r.v.	random variable
L.T.	Laplace transform
mgf	moment generating function
i.i.d.	independent and identically distributed
cdf	cumulative distribution function of a r.v.
pdf	probability density function of a r.v.

Introduction

Risk Theory, based on probability theory and statistics, stochastic processes, renewal theory, functional analysis and optimization theory, investigates fluctuations shown by incoming claims at an insurance company and has been one of the most active research areas in Actuarial Science since the beginning of the 20th century.

Ruin Theory is at the heart of Risk Theory. It discusses how an insurance portfolio may be expected to perform over an extended period of time and analyze the excess of income over outgo, or claims paid. This quantity, refereed to as the insurer's surplus, may be expected to vary with time. Ruin is said to occur if the insurer's surplus drops to a specified lower bound. One measure of risk is the probability of such an event over a finite or infinite time.

Much of the literature on ruin theory is concentrated on classical risk theory, in which the insurer starts with an initial surplus u, and collects premiums continuously at a constant rate of c, while the aggregate claims process follows a compound Poisson process. The main research goal is the evaluation of finite and infinite time ruin probabilities. Later, actuarial researchers dissected the surplus process and considered more components related to the time of ruin, like the surplus before ruin and the deficit at ruin. At each research stage, more insight has been gained. Gerber and Shiu (1998a) gives a unified treatment of three of these random variables: the surplus before ruin, the deficit at ruin and the time to ruin, by evaluating the expected discounted penalty function. Almost all classical results, e.g., the results of Gerber, Goovaerts and Kass (1987), Dufresne and Gerber

(1988a, b), Dickson (1992) and Dickson and Dos Reis (1996), are obtained as particular cases when the discount factor is zero, and almost all the previous results in classical ruin theory can be extended to the case with a positive discounting factor. Lin and Willmot (1999) proposed an approach to solve the defective renewal equation, in which the discounted penalty function is expressed in terms of a compound geometric tail. Lin and Willmot (2000) further used it to derive the moments of the surplus before ruin, the deficit at ruin, and the time of ruin.

The last few decades have witnessed an almost explosive interest in more general surplus processes, e.g, surplus models with stochastic premium income processes, classical surplus processes under economic environment (investment and inflation), surplus processes with dependent claim amounts and claim inter-occurrence time, surplus processes in which aggregate claims come from some classes of dependent or independent businesses, surplus processes with general claim number processes, or classical risk model perturbed by an independent diffusion process.

The classical risk model perturbed by a diffusion was first introduced by Gerber (1970), and subsequently further studied by numerous authors, e.g., Dufresne and Gerber (1991), Furrer and Schmidli (1994), Schmidli (1995), Gerber and Landry (1998), Wang and Wu (2000), Wang (2001), Tsai (2001, 2003), Tsai and Willmot (2002a,b), Zhang and Wang (2003), and references therein. The Gerber-Shiu penalty function in the perturbed model can be decomposed as two parts: the penalty function due to claims and that due to oscillations.

Sparre Andersen (1957) let claims occur according to a more general renewal process and derived an integral equation for the corresponding ruin probability. Since then, random walks and queueing theory have provided a more general framework, which has led to explicit results in the case where the waiting times or the claim severities have distributions related to the Erlang [e.g. see Borovkov

(1976)].

Malinovskii (1998) gives the Laplace transform of the non-ruin probability as a function of a finite time t, if claim sizes are exponentially distributed with parameter α, and waiting times have a general distribution k. Wang and Liu (2002) extends the result to claim sizes that are a mixture of two exponential distributions. In both cases, it is difficult to invert these Laplace transforms, even for special claim inter-arrival times distributions.

Dickson (1998) and Dickson and Hipp (1998, 2001) consider the case where the waiting times have a gamma(2,β) distribution. They obtain an explicit expression for the Laplace transform of the ruin probability by solving a second order integro–differential equation. More recently, Cheng and Tang (2003) complements the work of Dickson and Hipp (2001), discussing the moments of the surplus before ruin and the deficit at ruin in the Erlang(2) risk process. Li and Garrido (2004) extends the Erlang(2) risk model to Erlang(n) for any integer n, in which the expected discounted penalty function satisfies an n-th order integro-differential equation. The latter can be solved to obtain a defective renewal equation. Gerber and Shiu (2003a,b) and Gerber and Shiu (2004) further extend the Erlang risk models to generalized Erlangs, in which claim waiting times are distributed as the sum of n independent exponential random variables with possible different means.

Willmot (1999) considers the ruin probabilities for renewal risk processes where the waiting times have a K_n distribution, for which the associated Laplace–Stieltjes transform is the ratio of a polynomial of degree $m < n$ to a polynomial of degree n. This general class of distributions includes, as special cases, Erlang and phase–type distributions, as well as mixtures of these.

Stanford et al. (2000) presents a recursive method of calculating ruin probabilities for non-Poisson claim processes, by looking at the surplus process embedded at claim instants, in which claim inter-arrival times are assumed to be mixtures of exponential and Erlang(n) distributions.

Dufresne (2001) derives the Laplace transform of the integral equation given by Sparre Andersen, producing the Laplace transform of the non–ruin probability for the wide class of waiting times or severity distributions that admit a rational Laplace transform representation. Lima et al. (2002) uses Fourier/Laplace transforms to evaluate numerically quantities of interests in classical and Erlang(2) ruin theory.

Unlike the continuous model, the discrete risk models have not attracted much attention and the literature only counts a few contributions. Yet results on discrete risk models can be given a simpler understanding than their analogue in continuous time. They are also of independent interest, since formulas for discrete models are readily programmable in practice, while still reproducing the continuous analogue results as limiting cases.

The discrete time analogue of the continuous time classical risk process is the compound binomial risk process, which was first introduced in Gerber (1988) and has been further studied by Shiu (1989), Willmot (1993), Dickson (1994a), De Vylder (1996), De Vylder and Marceau (1996), Cheng et al. (2000), and Li and Garrido (2002).

Recently, some papers discuss more general discrete time risk models, e.g., Cossette and Marceau (2000) considers a discrete-time risk model with correlated classes of business, in which, the authors propose two kinds of dependence between these classes, one is a Poisson model with i.i.d. shocks (PCS model), which was first introduced in Marshall and Olkin (1967, 1988) and Kocherlakota and Kocherlakota (1992), another is a negative binomial model with common component (NBCC model). For this discrete-time risk model, Wu and Yuen (2003) proposes a new structure of dependence (IR model) and compares the results for the above PCL and NBCC models by assuming that there are two types of claims, namely, main claims and by-claims, in each class. Each main claim may produce a by-claim occuring in another class with a certain probability.

Yuen and Guo (2001) considers the ruin probability for a risk process with time-correlated claims in the compound binomial model. It is assumed that every main claim will produce a by-claim but that by-claim may be delayed one or two periods. Recursive formulas for finite ruin probabilities are obtained and explicit expressions for ultimate ruin probabilities are given in two special cases.

Cossette et al. (2003) presents a compound Markov binomial model which is an extension of the classical binomial model proposed by Gerber (1988). The compound Markov binomial model is based on the Markov Bernoulli process, introducing dependence between claim occurrences.

Wagner (2001) considers a discrete risk model governed by a two state Markov chain, in which the individual claim in each period is affected by the states of the chain; Reinhard and Snoussi (2000, 2001) study the ruin probability and the distribution of the surplus prior to ruin in a discrete semi-Markov risk model.

In this thesis, we consider the evaluation of Gerber-Shiu penalty functions, as well as their applications, for different insurance risk models, both in continuous and discrete settings.

Chapter 1 reviews the relevant results and techniques in the literature on the classical risk model, the perturbed risk model, and the Sparre Andersen model, both in continuous and discrete time settings.

Chapter 2 gives the mathematical preliminaries to the thesis, including the definition of an operator for integrable continuous real-valued functions (alternatively, for discrete functions on positive integers) and divided differences. The appendices give a review of other mathematical tools also often used in the thesis.

Chapter 3 studies the evaluation of Gerber-Shiu expected discounted penalty functions for a class of renewal risk models (Sparre Andersen models), in which the claims inter-arrival times are K_n distributed, for $n \in \mathbb{N}^+$. First the Laplace transform of the expected penalty function is given, through an integral formula derived from martingale argument. Then the expected penalty function at $u = 0$ is obtained by the initial value theorem. Third, a defective renewal equation for the penalty function for $u > 0$ is derived using the renewal structure of the Sparre Andersen risk model. With this defective renewal equation, many ruin related quantities are analyzed. Lastly, explicit results are obtained by inverting a rational Laplace transform, for the case when the claim size is rationally distributed.

Chapter 4 studies the expected discounted penalty function for the generalized Erlang (n) risk process with a constant dividend barrier. We first show that the expected discounted penalty function satisfies an n-th order integro-differential equation with certain boundary conditions. Its solution can be expressed as the expected discounted penalty function for the generalized Erlang(n) risk model without barrier plus a linear combination of n linearly independent solutions to the associated homogeneous integro-differential equation. The solution to the associated homogeneous integro-differential equation is uniquely determined by the initial conditions and satisfies a defective renewal equation. Explicit results can be obtained when the claim size distribution is of rational type.

Chapter 5 extends the classical risk process perturbed by a diffusion, and the Sparre Andersen model with generalized Erlang(n) claim waiting times studied by Gerber and Shiu (2003a, b). We analyze the expected discounted penalty function and its decomposition for a Sparre Andersen model perturbed by an independent diffusion, in which the claim inter-arrival times are generalized Erlang(n) distributed (i.e., convolution of n independent exponential distributions with possible different parameters). This leads to a generalization of the defective renewal equations for the expected discounted penalty function at the time of ruin, given by Tsai and Willmot (2002a,b) and Gerber and Shiu (2003a,b). The limiting behavior of the expected discounted penalty functions are studied, when the diffusion coefficient goes to zero (small diffusion asymptotics). Explicit results are given for the case where $n = 2$.

Finally, Chapter 6 studies the penalty function for a class of discrete Sparre Andersen risk models, in which claim inter-arrival times are discrete K_n distributed. This class of discrete risk models is the discrete analogue of the continuous Sparre Andersen risk models studied in Chapter 3. Due to the discrete nature of the models, the generating function of the expected discounted penalty function is given, through which the value of the penalty function at $u = 0$ is obtained first. Then a recursive formula for the penalty function is constructed for positive integer-valued initial surplus u. An explicit expression is obtained in terms of a compound geometric distribution. Many ruin related quantities can be obtained recursively or explicitly by specially choosing penalty functions, e.g., discounted joint and marginal distributions of the surplus before ruin and deficit at ruin, as well as their moments, the p.g.f. of the ruin time and its moments, for special claim size distributions. Explicit results are obtained by inverting the generating function when the claim size distribution belongs to two classes: the discrete K_n distribution and the discrete distribution with finite support.

Chapter 1

Review of the Literature

This chapter gives a brief review of literature on the classical risk model, the classical perturbed risk model and the Sparre Andersen risk model.

In ruin theory, both discrete time and continuous time risk models are used, requiring distinct analysis and tools.

1.1 Classical Risk Model

1.1.1 Definitions

First we review the definition of the Poisson process, the compound Poisson process (CPP) and the classical surplus process [see Feller (1971), Gerber (1979)].

Let inter-occurrence times $\{W_n : n \in \mathbb{N}^+\}$ form a sequence of independent random variables that have a common exponential distribution with parameter $\lambda > 0$. Then the counting process for the number of claims $\{N(t) : t \geq 0\}$ is called a homogenous Poisson process with constant rate or intensity λ, a special kind of renewal process. A basic property of the Poisson process is that it has independent and stationary increments, with distribution

$$P\{N(t+s) - N(s) = n\} = \frac{(\lambda\, t)^n\, e^{-\lambda\, t}}{n!}, \quad s,\, t > 0, \;\; n = 0, 1, 2 \ldots . \tag{1.1}$$

Therefore $E[N(t)] = \mathrm{Var}[N(t)] = \lambda\, t$, for $t \geq 0$, and the moment generating function (mgf) $M_{N(t)}(s) = E[e^{s\,N(t)}] = e^{\lambda\,t(e^s-1)}$.

Let the individual claims $\{X_i;\ i \in \mathbb{N}^+\}$ be i.i.d. non-negative r.v.'s, independent of $N(t)$, with common cumulative distribution function (cdf) P and corresponding probability density function (pdf) p. Assume that the k-th moment $\mu_k = E[X^k]\,(\mu_1 = \mu)$ is finite, for $k \in \mathbb{N}^+$, and denote the Laplace transform of p by $\hat{p}(s) = \int_0^\infty e^{-s\,x} p(x)dx$. Let $S(t) = \sum_{i=1}^{N(t)} X_i$, if $N(t) > 0$ (with $S(t) = 0$ if $N(t) = 0$) be the aggregate loss in $[0, t)$. The process $\{S(t);\ t \geq 0\}$ is called a compound Poisson process (CPP), denoted as $S(t) \sim C.P.[\lambda; P]$, for $t \geq 0$. The distribution of $S(t)$ is given by

$$P\{S(t) \leq x\} = \sum_{n=0}^{\infty} \frac{(\lambda\,t)^n\,e^{-\lambda\,t}}{n!} P^{*n}(x), \qquad t,\, x \geq 0, \tag{1.2}$$

where $P^{*n}(x)$ denotes the n-th convolution of P with itself at x [with $P^{*0}(x) = I(x \geq 0)$ and $P^{*1} = P$].

Now consider the surplus process

$$U(t) = u + c\,t - S(t), \qquad t \geq 0, \tag{1.3}$$

where $u \geq 0$ is the initial surplus and $c > 0$ is the constant premium rate over time. Here we can re-write $c = (1+\theta)\lambda\,\mu_1$, where $\theta > 0$ is interpreted as a relative security loading factor.

Define

$$T = \inf\{t \geq 0; U(t) < 0\} \qquad (\infty, \text{otherwise}),$$

to be the ruin time, and

$$\begin{aligned} \Psi(u,t) &= P\{T < t\,|U(0) = u\} = P\Big\{\bigcup_{s \leq t} [U(s) < 0]\,\Big|U(0) = u\Big\}, \quad u\,, t \geq 0, \\ \Psi(u) &= \{T < \infty|U(0) = u\} = P\Big\{\bigcup_{t<\infty} [U(t) < 0]\,\Big|U(0) = u\Big\}, \quad u \geq 0, \end{aligned}$$

to be the finite probability and ultimate ruin probability, respectively. Clearly, $\Psi(u) = \lim_{t\to\infty} \Psi(u,t)$.

1.1.2 Main Results on the Ruin Probability

We now give a the summary of the main results in the classical risk model.

Theorem 1.1.1. (Integro-differential equations) *The probability of ruin before time t, with initial reserve u, satisfies the following partial integro-differential equation for u, $t \geq 0$:*

$$\frac{\partial}{\partial t}\Psi(u,t) = c\frac{\partial}{\partial u}\Psi(u,t) + \lambda\bar{P}(u) - \lambda\Psi(u,t) + \lambda\int_0^u \Psi(u-x)p(x)dx, \qquad (1.4)$$

while the ultimate ruin probability $\Psi(u)$ satisfies the following integro-differential equation:

$$\Psi'(u) = \frac{\lambda}{c}\Psi(u) - \frac{\lambda}{c}\int_0^u \Psi(u-x)p(x)dx - \frac{\lambda}{c}\bar{P}(u), \qquad u \geq 0, \qquad (1.5)$$

with $\Psi(0) = \frac{1}{1+\theta}$.

Proof: See Gerber (1979) or Panjer and Willmot (1992). □

The following results show that the ultimate ruin probability $\Psi(u)$ satisfies a defective renewal equation and admits a compound geometric representation. Define $P_1(x) = \frac{\int_0^x \bar{P}(x)dx}{\mu}$, and $\bar{P}_1(x) = 1 - P_1(x)$. P_1 is called the ladder-height distribution (also integrated tail distribution, or first order equilibrium distribution) of P.

Theorem 1.1.2. (Defective renewal equation) *The ruin probability Ψ satisfies the following defective renewal equation:*

$$\Psi(u) = \frac{1}{1+\theta}\int_0^u \Psi(u-y)dP_1(y) + \frac{\bar{P}_1(u)}{1+\theta}, \quad u \geq 0. \qquad (1.6)$$

Proof: See Gerber (1979, p. 115). □

Theorem 1.1.3. (Beekman Convolution Formula) *Ψ is given by the tail of the distribution of a compound geometric sum, that is,*

$$\Psi(u) = \frac{\theta}{1+\theta}\sum_{n=1}^{\infty}\left(\frac{1}{1+\theta}\right)^n \bar{P}_1^{*n}(u), \qquad u \geq 0. \qquad (1.7)$$

Proof: See Feller (1971) or Beekman (1974). □

Another expression for Ψ can be given using a martingale approach, as in Gerber (1979). First, the concept of *adjustment coefficient*, which is a root of Lundberg's fundamental equation, is needed. In the classical risk model defined above, the adjustment coefficient, $R > 0$, is the solution, if it exists, to Lundberg's equation:

$$\int_0^\infty e^{Rx}\bar{P}(x)dx = \frac{c}{\lambda}, \tag{1.8}$$

or, equivalently,

$$\int_0^\infty e^{Rx}dP_1(x) = 1 + \theta. \tag{1.9}$$

Theorem 1.1.4. *If there exists a $R > 0$ satisfying (1.8), then*

$$\Psi(u) = \frac{e^{-Ru}}{E[e^{-RU(T)}|T < \infty]}, \quad u \geq 0. \tag{1.10}$$

Proof: See Gerber (1979). □

Although Ψ satisfies a defective renewal equation and admits representation as in (1.7) and (1.10), usually it is difficult to obtain an explicit expression for it in practical situations. The exception being perhaps for some special choices claim distribution, e.g., the exponential or mixture of exponential distributions, see Gerber (1979) and Bowers et al. (1997). In fact, explicit results can be obtained if the claim size distribution admits a rational Laplace transform. In this case, the Laplace transform of the ruin probability can be inverted by partial fraction formula, see Dufresne (2001) or Lima et al. (2002).

For more general claim amount distributions, the main results for ruin probability give bounds, asymptotic formulas, approximations, and simulations.

Theorem 1.1.5. (Lundberg's bound) *For the classical risk model defined above,*

$$\Psi(u) \leq e^{-Ru}, \quad u \geq 0, \tag{1.11}$$

where R is the adjustment coefficient defined in (1.8).

This is the first bound for the ruin probability Ψ. However, the adjustment coefficient does not exist for many distributions, such as heavy tailed distributions. Several approaches have been proposed to derive general bounds for the ruin probability. One way is to avoid reference to the adjustment coefficient, for example, De Vylder and Goovaerts (1984) gives the following bound:

$$\Psi(u) \geq \frac{\bar{P}_1(u)}{\theta + \bar{P}_1(u)}, \qquad u \geq 0. \tag{1.12}$$

Another way is to relax Lundberg's equation by replacing the exponential function by, say a new worse than used (NWU) or new better than used (NBU) distribution [see Willmot (1997b), Cai and Wu (1997)].

A third approach is to consider a truncated Lundberg's equation. That is, for a given $t > 0$, there exists a root $R(t)$ satisfying

$$\int_0^t e^{R(t)\,y} \bar{P}(y) dy = \mu(1+\theta). \tag{1.13}$$

Using $R(t)$, Dickson (1994b) gives an upper bound for a fixed t:

$$\Psi(u) \leq e^{-u\,R(t)} + \frac{\bar{P}_1(t)}{\theta + \bar{P}_1(u)}, \quad 0 \leq u \leq t. \tag{1.14}$$

In the same sprit, Cai and Garrido (1999) derives sharper two-sided bounds for ruin probability under the above truncating Lundberg equation:

Theorem 1.1.6. *For a fixed $t > 0$ and any given $0 \leq u \leq t$,*

$$\frac{\theta \alpha_1(u,t)\, e^{-u\,\mathbb{R}(t)} + \bar{P}_1(t)}{\theta + \bar{P}_1(t)} \leq \Psi(u) \leq \frac{\theta \alpha_2(u,t)\, e^{-u\,\mathbb{R}(t)} + \bar{P}_1(t)}{\theta + \bar{P}_1(t)}, \tag{1.15}$$

where

$$\alpha_1(u,t) = \inf_{0 \leq h \leq u} \alpha(h,t), \quad \alpha_2(u,t) = \sup_{0 \leq h \leq u} \alpha(h,t),$$

and

$$\alpha(h,t) = \frac{e^{h\,R(t)}[\bar{P}_1(h) - \bar{P}_1(t)]}{\int_h^t e^{y\,R(t)} dP_1(y)}.$$

Proof: See Cai and Garrido (1999). □

The above theorem implies the following simplified and practical two-sided bounds.

Corollary 1.1.1. *Under the condition of the above theorem, for any* $0 \le u \le t$,

$$\frac{\theta\, e^{-(u+t)\,R(t)} + \bar{P}_1(t)}{\theta + \bar{P}_1(t)} \le \Psi(u) \le \frac{\theta\, e^{-u\,R(t)} + \bar{P}_1(t)}{\theta + \bar{P}_1(t)}, \tag{1.16}$$

and hence, for any $t > 0$,

$$\frac{\theta\, e^{-2\,t\,R(t)} + \bar{P}_1(t)}{\theta + \bar{P}_1(t)} \le \Psi(t) \le \frac{\theta\, e^{-t\,R(t)} + \bar{P}_1(t)}{\theta + \bar{P}_1(t)}. \tag{1.17}$$

Besides bounds, another area of research interest in ruin probabilities is that of asymptotic formulas. One of the well-known result, which is referred to as *Cramér-Lundberg Asymptotic Formula*, is given below.

Theorem 1.1.7. (Cramér-Lundberg Asymptotic Formula)

$$\Psi(u) \sim \frac{\theta\, u}{M'_X(R) - \mu(1+\theta)} e^{-R\,u}, \qquad u \to \infty, \tag{1.18}$$

where R is adjustment coefficient, and M_X is the mgf of the claim amount X.

Proof: See Gerber (1979) or Panjer and Willmot (1992). □

The limitation of the above formula is that it depends on the adjustment coefficient R, which does not exist when claim size distributions are heavy tailed or even medium tailed. Embrechts and Veraverbeke (1982) have shown that if P is a subexponential distribution, then

$$\Psi(u) \sim \frac{\bar{P}_1(u)}{\theta}, \qquad u \to \infty.$$

For medium tail distributions, we note that $\int_0^\infty e^{\gamma y} dP_1(y) < 1 + \theta$, or equivalently $M_X(\gamma) < 1 + (1+\theta)\,\mu$, where $\gamma > 0$, and $M_X(t) = \infty$ for any $t > \gamma$. Hence no

R exists satisfying (1.8). In this case, Embrechts and Veraverbeke (1982) shows that

$$\Psi(u) \sim \frac{\theta\,\gamma\,\mu\,\bar{P}(u)}{[1+(1+\theta)\gamma\,\mu - M_X(\gamma)]^2}, \qquad u \to \infty.$$

Additional results on the asymptotic formulas for ruin probabilities can be found in Klüppelberg (1988, 1993).

Other results on ruin probabilities are related to approximations, numerical methods, and simulations. Since $\Psi(u)$ can be expressed as the tail of a compound geometric sum, then techniques on the evaluations of compound distributions can be used for ruin probabilities. Among all the contributions in this area, Panjer (1981) is the most famous one. It gives a recursive evaluation of the compound distribution for counting distributions in the (a, b) family. The latter includes the geometric distribution as a special case. Approximations on ruin probability can be found in Beekman and Bowers (1972) , Tijms (1994), Assmussen (1987), Kremer (1987), Garrido (1988), Dickson and Waters (1991), De Vylder (1996), and Chaubey et al. (1998). Simulations on ruin probabilities can be found in Dufresne and Gerber (1989).

1.1.3 Gerber-Shiu Penalty Function

Recently, research has focused on two other components related to the time of ruin: the surplus before ruin $U(T^-)$ and deficit at ruin $|U(T)|$. Some of the early contributions in this direction are those of Gerber, Goovaerts and Kass (1987), Dufresne and Gerber (1988a), Dickson (1992), Dickson and Dos Reis (1994) and Willmot and Lin (1998).

Gerber and Shiu (1998) gives a unified treatment of the these random variables, surplus before ruin, deficit at ruin and time of ruin, by evaluating the expected discounted penalty function. It generalizes and gives a better understanding of classical ruin theory, since many of the early results listed above are particular cases of the expected discounted penalty function, when the discount factor is zero.

For the classical risk process defined in the first section, define $F_3(x, y, t \,|\, u) = P\{U(T^-) \leq x,\ |U(T)| \leq y,\ T \leq t \,|\, U(0) = u\}$, for $x, y, t \geq 0$ and its corresponding joint probability density function, $f_3(x, y, t \,|\, u)$, of the surplus just before ruin, the deficit at ruin and the ruin time. Let $\delta \geq 0$ be the (constant) discount factor over one period and define $f_2(x, y \,|\, u) = \int_0^\infty e^{-\delta t} f_3(x, y, t \,|\, u)\, dt$ as a discounted joint p.d.f. of $U(T^-)$ and $|U(T)|$. If $f_1(x \,|\, u) = \int_0^\infty f_2(x, y \,|\, u)\, dy$, it follows that

$$f_2(x, y \,|\, u) = f_1(x \,|\, u) \frac{p(x+y)}{\bar{P}(x)}, \quad x,\ y \geq 0\,. \tag{1.19}$$

Let $w(x, y)$, for $x, y \geq 0$ be the non-negative values of a penalty function. For $\delta > 0$, define

$$\phi(u) = E\left[e^{-\delta T} w\big(U(T^-), |U(T)|\big) I(T < \infty) \;\middle|\; U(0) = u\right], \quad u \geq 0\,. \tag{1.20}$$

The quantity $w\big(U(T^-), |U(T)|\big)$ can be interpreted as the penalty at the time of ruin for the surplus $U(T^-)$ and the deficit $|U(T)|$. Then $\phi(u)$ is the expected discounted penalty if δ is viewed as the force of interest.

Many ruin related quantities can be analyzed by appropriately choosing special penalty functions, e.g., if $w(x_1, x_2) = I(x_1 = x, y_1 = y)$, then $\phi(u)$ gives the discounted joint density function of $U(T^-)$ and $|U(T)|$; if $w(x_1, x_2) = x_1^n\, x_2^m$, then $\phi(u)$ gives the discounted joint moments of $U(T^-)$ and $|U(T)|$; if $w(x_1, x_2) = 1$, then $\phi(u)$ gives the Laplace transform of T with respect to δ, and further if $\delta = 0$, $\phi(u)$ simplifies to ruin probability $\Psi(u)$.

As the ruin probability, ϕ also satisfies an integro-differential equation:

$$c\,\phi'(u) - (\delta + \lambda)\,\phi(u) + \lambda \int_0^u \phi(u-x)\, p(x) dx + \lambda\, \omega(u) = 0, \tag{1.21}$$

where $\omega(u) = \int_u^\infty w(u, x-u) p(x) dx$. A defective renewal equation satisfied by ϕ is derived by using a technique of *integrating factors* in the following theorem.

Theorem 1.1.8. [Gerber and Shiu (1998a)] *For $\delta > 0$, ϕ satisfies the following defective renewal equation,*

$$\phi(u) = \int_0^u \phi(u-y)\, g(y)dy + G(u), \qquad u \geq 0, \tag{1.22}$$

where $g(y) = \int_y^\infty e^{-\rho(x-y)}p(x)dx$, and $G(u) = \int_u^\infty e^{-\rho(x-u)}\omega(x)dx$, with $\rho > 0$ being the unique positive root to the equation:

$$l(s) := \delta + \lambda - c\, s = \lambda\, \hat{p}(s). \tag{1.23}$$

Equation (1.23) is a generalized version of *Lundberg's fundamental equation.* It has a unique positive root ρ when $\delta > 0$, and a possible negative root $-R\,(R > 0)$, if p is sufficiently regular. If $\delta = 0$, then $\rho = 0$, and R is the adjustment coefficient.

Clearly, $\phi(0) = G(0) = \hat{\omega}(\rho)$, and specially,

$$f_2(x, y|0) = \frac{\lambda}{c}e^{-\rho x}p(x+y), \qquad f_1(x\,|\,0) = \frac{\lambda}{c}e^{-\rho x}\bar{P}(x),$$
$$E[e^{-\delta T}I(T < \infty)|U(0) = 0] = \int_0^\infty g(y)dy = 1 - \frac{\delta}{c\,\rho}.$$

Similarly, if $w(x_1, x_2) = I(x_1 = x,\, x_2 = y)$, or $w(x_1,\, x_2) = I(x_1 = x)$, for $0 < x < \infty$, then $\phi(u)$ gives $f_2(x, y|u)$ and $f_1(x|u)$, respectively, i.e., they both satisfy a defective renewal equation as follows.

Corollary 1.1.2. *For $u,\, x,\, y \geq 0$,*

$$f_2(x, y\,|\,u) \;=\; \int_0^u f_2(x, y\,|u-z)\, g(z)dz + \frac{\lambda}{c}e^{-\rho(x-u)}\, p(x+y)I(x > u),$$
$$f_1(x\,|\,u) \;=\; \int_0^u f_1(x|u-z)\, g(z)dz + \frac{\lambda}{c}e^{-\rho(x-u)}\, \bar{P}(x)I(x > u).$$

The above defective renewal equations for the discounted joint and marginal densities $f_2(x, u\,|\,u)$ and $f_1(x\,|\,u)$ can be solved explicitly as in the following theorem.

Theorem 1.1.9.

$$f_1(x\,|u) = \begin{cases} f_1(x\,|\,0)\,\frac{e^{\rho\,x}\psi(u-x)-\psi(u)}{1-\psi(0)}, & 0 < x \le u, \\ f_1(x\,|\,0)\,\frac{e^{\rho\,u}-\psi(u)}{1-\psi(0)}, & x > u, \end{cases} \tag{1.24}$$

where $f_1(x|0) = \lambda\, c^{-1}\, e^{-\rho x}\bar{P}(x)$ *and* $\psi(u)$ *is defined by*

$$\psi(u) = E\left[e^{-\delta T+\rho\,U(T)}I(T<\infty)|U(0)=u\right].$$

Proof: See Gerber and Shiu (1997) or Gerber and Shiu (1998a). □

There is a probabilistic interpretation for $\psi(u)$, which is the expected present value of a payment of 1 that is made at the time of recovery, if ruin takes place. Specially, if $\delta = 0$, then $\psi(u)$ simplifies to ruin probability $\Psi(u)$, and hence above formula simplifies to the astonishing Dickson formula, which is given in Dickson (1992).

As in Lin and Willmot (1999), we can rewrite (1.22) as

$$\phi(u) = \frac{1}{1+\xi}\int_0^u \phi(u-x)\,v(x)\,dx + \frac{1}{1+\xi}H(u), \qquad u \ge 0, \tag{1.25}$$

where ξ is such that $\frac{1}{1+\xi} = \int_0^\infty g(y)dy = 1 - \frac{\delta}{c\rho}$, $v(x) = (1+\xi)g(x)$ is a proper density function and $H(u) = (1+\xi)\,G(u)$.

Specially, if $w(x_1, x_2) = 1$ in (1.20), then $\phi(u)$ simplifies to the Laplace transform of the ruin time T with respect to δ. To simplify notation, define $\phi_T(u) := E[e^{-\delta T}I(T<\infty)\,|\,U(0)=u]$. It satisfies the following defective renewal equation:

$$\phi_T(u) = \frac{1}{1+\xi}\int_0^u \phi_T(u-y)\,v(y)dy + \frac{1}{1+\xi}\int_u^\infty v(y)dy. \tag{1.26}$$

Using Laplace transforms, one obtains that $\phi_T(u)$ can also be expressed as the tail of compound distribution, i.e.,

$$\phi_T(u) = \frac{\xi}{1+\xi}\sum_{n=1}^{\infty}\left(\frac{1}{1+\xi}\right)^n \bar{V}^{*n}(u), \qquad u \ge 0, \tag{1.27}$$

where $\bar{V}(u) = \int_u^\infty v(y)dy$ is a survival function. Further if $\delta = 0$, $\phi_T(u)$ simplifies to the ruin probability $\Psi(u)$, and therefore, equations (1.26) and (1.27) simplify to (1.6) and (1.7), respectively.

Note that if we define an operator T_r w.r.t. parameter r to be such that

$$T_r\, p(x) = \int_x^\infty e^{-r(y-x)} p(y)dy, \qquad x \geq 0, \;\; r \in \mathbb{C},$$

then $v(x) = \frac{T_\rho\, p(x)}{T_\rho\, \bar{P}(0)} = (1+\xi) T_\rho\, p(x)$, and $\bar{V}(x) = T_0\, v(x) = \frac{T_\rho\, \bar{P}(x)}{T_\rho\, \bar{P}(0)} = (1+\xi) T_\rho\, \bar{P}(x)$.

The operator T_r w.r.t. a complex number r plays an important role in this thesis. Its properties will be discussed in the next chapter.

Lemma 1.1.1. *If $\rho > 0$ then the moments of $V(x)$ are given by*

$$u_n(\rho) = \int_0^\infty x^n\, dV(x) = \frac{n!}{(-\rho)^n} \left\{ 1 + \frac{\sum_{j=1}^n \frac{(-\rho)^j}{j!} \mu_j}{1 - \hat{p}(\rho)} \right\}, \quad n = 1, 2, \ldots. \tag{1.28}$$

Lin and Willmot (1999) gives a solution to the defective renewal equation (1.25), which is in terms of the compound geometric tail $\phi_T(u)$.

Theorem 1.1.10. [Lin and Willmot (1999)] *The solution $\phi(u)$ to (1.25) may be expressed as*

$$\phi(u) = -\frac{1}{\beta} \int_0^u H(u-x)\, d\phi_T(x) + \frac{1}{1+\beta} H(u), \tag{1.29}$$

or

$$\phi(u) = -\frac{1}{\beta} \int_0^u \phi_T(u-x) d\, H(x) - \frac{H(0)}{\beta} \phi_T(u) + \frac{1}{\beta} H(u). \tag{1.30}$$

If $H(u)$ is differentiable, $\phi(u)$ can be expressed as

$$\phi(u) = -\frac{1}{\beta} \int_0^u \phi_T(u-x)\, H'(x)\, dx - \frac{H(0)}{\beta} + \frac{1}{\beta} H(u), \qquad u \geq 0. \tag{1.31}$$

This theorem can be used to derive the moments of the surplus before ruin, deficit at ruin and the time of ruin. To begin with, defining the n-th order *equilibrium distribution* (survival function) of P recursively by $\bar{P}_0(x) = T_0\, p(0) = \bar{P}(x)$,

$\bar{P}_1(x) = \frac{T_0\bar{P}(x)}{T_0\bar{P}(0)} = \frac{\int_x^\infty \bar{P}(y)dy}{\mu}$, and $\bar{P}_n(x) = \frac{T_0\,\bar{p}_{n-1}(x)}{T_0\,\bar{P}_{n-1}(0)} = \frac{\int_x^\infty \bar{P}_{n-1}(x)dx}{\int_0^\infty \bar{P}_{n-1}(x)dx}$. Then

$$\bar{P}_n(x) = \frac{T_0^{n+1}p(x)}{T_0^{n+1}p(0)} = \frac{\int_x^\infty (y-x)^n p(y)dy}{\mu_n}. \tag{1.32}$$

For the distribution $v(x) = \frac{T_\rho p(x)}{T_\rho \bar{P}(0)}$, then $\bar{V}_0(x) = \bar{V}(x)$, and

$$\bar{V}_n(x) = \frac{T_0^{n+1}v(x)}{T_0^{n+1}v(0)} = \frac{T_\rho T_0^{n+1}p(x)}{T_\rho T_0^{n+1}p(0)} = \frac{T_\rho \bar{P}_n(x)}{T_\rho \bar{P}_n(0)} = \frac{\int_x^\infty e^{-\rho(y-x)}\bar{P}_n(y)dy}{\int_0^\infty e^{-\rho y}\bar{P}_n(y)dy}.$$

Define $g_n(\rho)$ to be the mean of $V_n(x)$, then $g_n(\rho) = T_0\bar{V}_n(0) = \frac{T_0^{n+2}v(0)}{T_0^{n+1}v(0)} = \frac{\mu_{n+1}(\rho)}{(n+1)\mu_n(\rho)}$.

Theorem 1.1.11. *For $k \in \mathbb{N}^+$,*

$$E\left[e^{-\delta T}|U(T)|^k | T < \infty, U(0) = u\right] = \frac{k\,\mu_{k-1}(\rho)\,\alpha_{k-1}(u,\rho) - \mu_k(\rho)\phi_T(u)}{\xi\,\Psi(u)}, \tag{1.33}$$

where $\alpha_0(u,\rho) = \beta\int_u^\infty \phi_T(x)dx$, and for $n = 1, 2, \ldots$,

$$\alpha_n(u,\rho) = \frac{\beta}{\mu_n(\rho)}\int_u^\infty (x-u)^n\phi_T(x)dx - \sum_{j=0}^{n-1}\binom{n}{j}\frac{\mu_{n-j}(\rho)}{\mu_n(\rho)}\int_u^\infty (x-u)^j\phi_T(x)dx.$$

Specially if $\delta = 0$, and $k = 1, 2, \ldots$,

$$E\left[|U(T)|^k | T < \infty, U(0) = u\right] = \frac{\mu_k}{\mu_1\,\theta}\frac{\tau_k(u)}{\Psi(u)} - \frac{\mu_{k+1}}{(k+1)\mu_1\,\theta}, \tag{1.34}$$

where $\tau_1(u) = \theta\int_u^\infty \Psi(y)dy$, and for $n = 2, 3, \ldots$,

$$\tau_n(u) = \frac{n\,\mu_1\,\theta}{\mu_n}\int_u^\infty (x-u)^{n-1}\Psi(x)dx - \sum_{j=0}^{n-2}\binom{n}{j}\frac{\mu_{n-j}}{\mu_n}\int_u^\infty (x-u)^j\,\Psi(x)dx.$$

Proof: See Lin and Willmot (2000). □

Theorem 1.1.12. *For $k, j \in \mathbb{N}$,*

$$\begin{aligned} & E\left\{[U(T^-)]^j\,|U(T)|^k\,\middle|\,T < \infty, U(0) = u\right\} \\ = \; & \frac{\mu_k}{\mu_1\,\theta\Psi(u)}\left[\int_0^u \Psi(u-x)\,x^j\bar{P}_k(x)dx + \int_u^\infty x^j\bar{P}_k(x)dx\right] - \frac{k!\,j!\,\mu_{k+j+1}}{(k+j+1)!\mu_1\,\theta}. \end{aligned}$$

The following theorem in Lin and Willmot (2000) gives the moments of the time of ruin T.

Theorem 1.1.13. *The k-th moment of the time of ruin, given that ruin has occurred, is*

$$E[T^k \,|\, T < \infty, U(0) = u] = \frac{\Psi_k(u)}{\Psi(u)}, \quad u \geq 0,$$

for $k = 1, 2, \dots$, *with* $\Psi_0(u) = \Psi(u)$, *and* $\Psi_k(u)$ *is recursively obtained by*

$$\Psi_k(u) = \frac{k}{\lambda \mu \theta} \left[\int_0^u \Psi(u-x)\, \Psi_{k-1}(x) dx + \int_u^\infty \Psi_{k-1}(x) dx - \Psi(u) \int_0^\infty \Psi_{k-1}(x) dx \right].$$

Proof: See Lin and Willmot (2000). □

1.2 Classical Risk Process Perturbed by Diffusion

The classical risk model perturbed by a diffusion was first introduced by Gerber (1970) and has been further studied by many authors during the last few years; e.g. Dufresne and Gerber (1991), Furrer and Schmidli (1994), Schmidli (1995), Gerber and Landry (1998), Wang and Wu (2000), Wang (2001), Tsai (2001, 2003), Tsai and Willmot (2002a,b), Zhang and Wang (2003), Chiu and Yin (2003), Zhou (2003), and the references therein.

1.2.1 Decomposition of the Ruin Probability

Consider the following perturbed risk process:

$$U(t) = u + ct - S(t) + \sigma B(t), \qquad u > 0, \tag{1.35}$$

where $\sigma > 0$, B is a standard Brownian motion, and all other assumptions are as in the definition of the classical risk model in the first section.

Under this perturbed model, the ultimate ruin probability $\Psi(u)$ can be decomposed as follows:

$$\Psi(u) = \Psi_d(u) + \Psi_s(u), \tag{1.36}$$

where $\Psi_d(u)$ is the probability of ruin that is caused by oscillation, and $\Psi_s(u)$ is the probability of ruin that is caused by a claim. Furthermore, denoting by $\Phi(u) = 1 - \Psi(u)$ the non-ruin probability, then $\Phi(0) = \Psi_s(0) = 0$, and $\Psi(0) = \Psi_d(0) = 1$.

Dufresne and Gerber (1991) shows that Ψ, Ψ_d and Ψ_s satisfy integro-differential equations, and thus admit a defective renewal equation representation.

Theorem 1.2.1. *Φ with $\Phi(0) = 0$ satisfies the following integro-differential equation:*

$$\frac{\sigma^2}{2}\,\Phi''(u) + c\,\Phi'(u) = \lambda\,\Phi(u) - \lambda \int_0^u \Phi(u-x)dP(x), \qquad u \geq 0, \tag{1.37}$$

and Ψ_d with $\Psi_d(0) = 1$ satisfies:

$$\frac{\sigma^2}{2}\,\Psi_d''(u) + c\,\Psi_d'(u) = \lambda\,\Psi_d(u) - \lambda \int_0^u \Psi_d(u-x)dP(x), \qquad u \geq 0. \tag{1.38}$$

Wang (2001) further proves that Ψ_d and Ψ_s are twice continuously differentiable and derives the same integro-differential equations as above using martingale techniques and Itô's formula.

Solving the above integro-differential equations gives the following result.

Theorem 1.2.2. *For $u \geq 0$,*

$$\Phi(u) = \theta\, H_1(u) + (1-\theta) \int_0^u \int_0^u \Phi(u-x)\, h_1 * h_2(x)dx, \tag{1.39}$$

and

$$\begin{aligned}
\Psi_d(u) &= 1 - H_1(u) + (1-\theta)\int_0^u \Psi_d(u-x)h_1 * h_2(x)dx, \\
\Psi_s(u) &= (1-\theta)[H_1(u) - H_1 * H_2(u)] + (1-\theta)\int_0^u \Psi_s(u-x)h_1 * h_2(x)dx,
\end{aligned}$$

where $\theta > 0$ *is the loading factor,* $h_1(x) = \nu\, e^{-\nu\, x} I(x \geq 0)$ *with* $\nu = 2\,c/\sigma^2$, $h_2(x) = \frac{1-P(x)}{\mu}$, H_1 *and* H_2 *are their corresponding distribution functions, respectively.*

Consider the aggregate loss at time t,

$$L(t) = S(t) - ct - \sigma B(t), \qquad t \geq 0, \tag{1.40}$$

and the maximal aggregate loss

$$L = \max\{L(t);\, t \geq 0\}. \tag{1.41}$$

As in the classical case

$$\Phi(u) = P(L(t) \leq u, \text{for all } t \geq 0) = P(L \leq u), \tag{1.42}$$

that is to say, Φ is the distribution of the random variable L.

Dufresne and Gerber (1991) shows that L can be decomposed as follows:

$$L = L_0^{(1)} + L_1^{(2)} + L_1^{(1)} + \cdots + L_N^{(2)} + L_N^{(1)}, \quad (L = L_0^{(1)}, \text{if} N = 0), \tag{1.43}$$

where N is the number of record highs of the process $\{L(t);\, t \geq 0\}$ that are caused by the occurrence of a claim. Let $T_1, \ldots, T_N$ denote the times when these claims occur, with $T_0 = 0$ and $T_{N+1} = \infty$. Then

$$L_k^{(1)} = \max\{L(t); t < T_{k+1}\} - L(T_k), \qquad k \in \mathbb{N}, \tag{1.44}$$

and

$$L_k^{(2)} = L(T_k) - L(T_{k-1}) - L_{k-1}^{(1)}, \qquad k \in \mathbb{N}^+. \tag{1.45}$$

Note that N is geometrically distributed as $P(N = n) = \theta(1-\theta)^n$, $n \in \mathbb{N}$, where θ is the probability that there are no record highs that are caused by a claim. By the stationarity property of the process $\{L(t);\, t \geq 0\}$, $L_0^{(1)}, L_1^{(1)}, \ldots,$ are identically distributed, and $L_1^{(2)}, L_2^{(2)}, \ldots,$ are identically distributed, with $h_i(x), H_i(x)$, for

$i = 1, 2$ as their pdf and cdf, respectively. Also $N, L_0^{(1)}, L_1^{(2)}, L_1^{(1)}, L_2^{(2)}, \ldots,$ are independent. It follows from all this that

$$\Phi(u) = \sum_{n=0}^{\infty} \theta(1-\theta)^n H_1^{*(n+1)} * H_2^{*n}(u), \qquad u \geq 0, \tag{1.46}$$

which is a generalized Beekman convolution formula for perturbed classical risk model.

Veraverbeke (1993) gives an asymptotic estimate of the ruin probability when both the adjustment coefficient exists or when it does not.

Wang and Wu (2000) studies the supremum distribution before ruin, which is defined as

$$\Gamma(u, x) = P\{\sup_{0 \leq t \leq T} U(t) \geq x,\, T < \infty \,|\, U(0) = u\}. \tag{1.47}$$

Clearly, $\Gamma(u, x) = \Psi(u)$, if $u \geq x > 0$.

$\Gamma(u, x)$ satisfies the following integro-differential equation for $x > u > 0$,

$$\frac{\sigma^2}{2} \Gamma_u''(u, x) + c\,\Gamma_u'(u, x) = \lambda\,\Gamma(u, x) - \lambda \int_0^u \Gamma(u - z, x)\, p(z)\, dz. \tag{1.48}$$

The relation between $\Gamma(u, x)$ and $\Psi(u)$ is then

$$\Gamma(u, x) = \frac{1 - \Psi(u)}{1 - \Psi(x)} \Psi(x), \qquad 0 < u \leq x.$$

1.2.2 Expected Discounted Penalty Functions

Gerber and Landry (1998) and Tsai and Willmot (2002a) analyze the above perturbed risk process using a Gerber-Shiu penalty function analysis. Define for $\delta > 0$,

$$\phi_d(u) := E[e^{-\delta T} I(T < \infty, U(T) = 0) \,|\, U(0) = u]$$

to be the penalty function due to oscillations, or equivalently, the Laplace transform of the ruin time T if ruin is caused by oscillations. For a non-negative bivariate function $w(x,y)$, define

$$\phi_s(u) := E[e^{-\delta T} w(U(T^-), |U(T)|)\, I(T < \infty, U(T) < 0) \,|\, U(0) = u]$$

to be the penalty function due to a claim. Then

$$\phi(u) = \phi_d(u) + \phi_s(u), \qquad u \geq 0,$$

is the expected discounted penalty function.

Gerber and Landry (1998) shows that $\phi_d(u)$ satisfies the following integro-differential equation:

$$\frac{\sigma^2}{2}\,\phi_d''(u) + c\,\phi_d'(u) + \lambda \int_0^u \phi_d(u-x)p(x)dx - (\lambda+\delta)\phi_d(u) = 0, \quad u \geq 0, \quad (1.49)$$

and solves it to the following defective renewal equation:

$$\phi_d(u) = \int_0^u \phi_d(u-y)g(y)dy + e^{-b\,u}, \qquad u \geq 0, \qquad (1.50)$$

where $g(y) = h * \eta(y)$, $h(y) = \frac{2\,c}{\sigma^2} e^{-by}$ with $b = \frac{2\,c}{\sigma^2} + \rho$, and $\eta(y) = \frac{\lambda}{c} T_\rho p(y) = \frac{\lambda}{c}\int_y^\infty e^{-\rho(x-y)} p(x)dx$, while ρ is the unique positive root to a generalized Lundberg equation: $\delta + \lambda - c\,s - \sigma^2\,s^2/2 = \lambda \hat{p}(s)$.

The probabilistic interpretation of $g(y)dy$ is that it gives the expected discounted value of a contingent payment of 1, made at the time of the first record low due to a jump, provided that this record low is between $u-y$ and $u-y+dy$.

Tsai and Willmot (2002a) shows that $\phi_s(u)$ satisfies the following defective renewal equation

$$\phi_s(u) = \int_0^u \phi_s(u-y)g(y)dy + h * (T_\rho \omega)(u), \qquad u \geq 0. \qquad (1.51)$$

Therefore, $\phi(u) = \phi_d(u) + \phi_s(u)$ satisfies

$$\phi(u) = \int_0^u \phi(u-y)g(y)dy + e^{-b\,u} + h * (T_\rho \omega)(u), \qquad u \geq 0. \qquad (1.52)$$

Tsai and Willmot (2002a) also discusses the limiting behavior of the penalty functions when the dispersion parameter σ goes to zero. When $\sigma \to 0$, $\phi_d(u) \to 0$, and $\phi_s(u) \to \phi_0(u)$, where $\phi_0(u)$ the expected discounted penalty function for the classical risk process without a diffusion perturbation, as defined in (1.20). Therefore $\phi(u) \to \phi_0(u)$ as $\sigma \to 0$.

Tsai (2001) studies the discounted joint and marginal distributions of the surplus before ruin and the deficit at ruin, provided that ruin is caused by a claim, by appropriately choosing the function $w(x, y)$. Tsai and Willmot (2002b) extends the results of Lin and Willmot (2000) and derives the expressions for the (discounted) moments of deficit at ruin, the joint moments of the surplus before ruin and the deficit at ruin, as well as the moments of ruin times due to claims and oscillations.

1.3 Sparre Andersen Surplus Processes

Andersen (1957) let claims occur according to a more general renewal process and derived an integral equation for the corresponding ruin probability. Since then, random walks and queuing theory have provided a more general framework in risk theory, which has led to explicit results in the case where the waiting times or the claim severities have distributions related to the Erlang [e.g. see Borovkov (1976)].

1.3.1 Model Description and Notation

Consider a continuous time Sparre Andersen surplus process

$$U(t) = u + c\,t - \sum_{i=1}^{N(t)} X_i\,, \quad t \geq 0\,, \tag{1.53}$$

where all the parameters are same as that in the classical risk model except that the process $\{N(t);\, t \geq 0\}$ is a more general counting process, defined as $N(t) =$

$\max\{n : W_1 + W_2 + \cdots + W_n \leq t\}$, where the claim waiting times W_i are assumed i.i.d. with density function $k(t)$, for $t \geq 0$, with $\hat{k}(s) = \int_0^\infty e^{-s\,x} k(x)dx$ being its Laplace transform.

Further assume that $\{W_i;\ i \in \mathbb{N}^+\}$ and $\{X_i;\ i \in \mathbb{N}^+\}$ are independent and $c\,E(W_i) > E(X_i)$ providing a positive safety loading factor.

1.3.2 Main Results

Malinovskii (1998) gives the Laplace transform of the non-ruin probability $\Phi(u,t) = 1 - \Psi(u,t)$, if the claim size is exponentially distributed with parameter α, for a general waiting times distribution k.

Theorem 1.3.1. [Malinovskii (1998)] *Let the claim size density p be exponential with parameter $\alpha > 0$, and $\hat{k}(s)$ be the Laplace transform of the waiting times density k, then*

$$\delta \int_0^\infty e^{-\delta\,t} \Phi(u,t) = 1 - \rho\, e^{-\alpha\, u(1-\rho)}, \qquad \delta > 0, \tag{1.54}$$

where ρ is the unique solution to the following equation:

$$\rho = \hat{k}[\delta + c\,\alpha(1-\rho)], \qquad \delta > 0.$$

Wang and Liu (2002) extends this result to the case when claim sizes can have a mixture of two exponential distributions, i.e.,

$$k(t) = [\zeta \alpha_1 e^{-\alpha_1\,t} + (1-\zeta)e^{-\alpha_2\,t}]\, I(t \geq 0).$$

The Laplace transform of the non-ruin probability is derived which is in terms of two positive roots to a generalized Lundberg's equation.

In both cases, it is hard to invert the Laplace transforms except for some special waiting times distributions. Also the above models are restricted to exponential claim sizes or mixtures of exponentials.

Dickson (1998) discusses the non-ruin probability $\Phi(u)$ for a Sparre Andersen risk model in which claim waiting times are Erlang(2) distributed, with density function $k(t) = \lambda^2\, t\, e^{-\lambda t} I(t \geq 0)$. It can be shown that $\Phi(u)$ satisfies the following 2-nd order integro-differential equation:

$$c^2\Phi''(u) - 2\lambda\, c\, \Phi'(u) + \lambda^2\Phi(u) = \lambda^2 \int_0^u \Phi(u-x)p(x)dx, \qquad u \geq 0. \qquad (1.55)$$

Let $\hat{\Phi}(s) = \int_0^\infty e^{-s\,u}\Phi(u)du$ be the Laplace transform of $\Phi(u)$, then taking Laplace transform to both sides of (1.55) yields,

$$\hat{\Phi}(s) = \frac{c^2\, s\, \Phi(0) + \lambda^2\, \mu - 2\lambda\, c}{c\, s^2 - 2\, c\, \lambda\, s + \lambda^2[1 - \hat{p}(s)]}, \qquad s \geq 0, \qquad (1.56)$$

where $\Phi(0) = \frac{2\lambda\, c - \lambda 62\, \mu}{c^2\, \rho}$, and ρ is the unique positive root to the equation:

$$c\, s^2 - 2\, c\, \lambda\, s + \lambda^2[1 - \hat{p}(s)] = 0.$$

Dickson and Hipp (1998) further shows that $\Phi(u)$ has a compound geometric representation:

$$\Phi(u) = \Phi(0) \sum_{n=0}^{\infty} [\Psi(0)]^n H^{*n}(u), \qquad u \geq 0, \qquad (1.57)$$

where $H(u) = \frac{\lambda^2\, \rho}{c^2\, \rho - 2\, \lambda\, c + \lambda^2\, \mu} T_\rho T_0 p(u)$.

For the same Erlang(2) risk process, Dickson and Hipp (2001) consider the Laplace transform of ruin time T w.r.t. a positive parameter δ, which is defined as

$$\phi(u) = E[e^{-\delta\, T} I(T < \infty) \,|\, U(0) = u], \qquad u \geq 0.$$

Same as the non-ruin probability $\Phi(u)$, $\phi(u)$ also satisfies an integro-differential equation for $u \geq 0$,

$$c^2\, \phi''(u) - 2(\lambda + \delta)c\, \phi'(u) + (\lambda + \delta)\phi(u) = \lambda^2 \int_0^u \phi(u-x)p(x)dx + \lambda^2\, \bar{P}(u). \quad (1.58)$$

Define $\hat{\phi}(s) = \int_0^\infty e^{-s\,u}\phi(u)du$ to be the Laplace transform of $\phi(u)$, then it is obtained by

$$\hat{\phi}(s) = \frac{\lambda^2\, \hat{\eta}(s)}{c^2 - \lambda^2\, \hat{\gamma}(s)}, \qquad s \geq 0, \qquad (1.59)$$

where $\gamma(u) = T_{\rho_1} T_{\rho_2} p(u)$, and $\eta(u) = T_0 \gamma(u) = \int_u^\infty \gamma(y) dy$, while ρ_1 and ρ_2 are the only two positive roots to a generalized Lundberg equation:

$$c^2 s^2 - 2(\delta + \lambda) c s + (\lambda + \delta)^2 = \lambda^2 \hat{p}(s).$$

Inverting the above Laplace transform yields a defective renewal equation

$$\phi(u) = \frac{\lambda^2}{c^2} \int_0^u \phi(u - x) \gamma(x) dx + \frac{\lambda^2}{c^2} \int_u^\infty \gamma(x) dx, \qquad u \geq 0, \tag{1.60}$$

with $\phi(0) = \frac{\lambda^2}{c^2} \hat{\gamma}(0) = 1 - \frac{2 \lambda \delta + \delta^2}{c^2 \rho_1 \rho_2} < 1$.

Cheng and Tang (2003) complement the work of Dickson and Hipp (2001), discussing the moments of the surplus before ruin and the deficit at ruin in the Erlang(2) risk process, and derive the asymptotic expressions for the expected penalty function for light tailed claim severity distributions and a class of heavy tail distributions.

Lin (2003) shows that the Gerber-Shiu penalty function in Erlang(2) risk model satisfies a 2-nd order integro-differential equation and further accepts a defective renewal representation.

Theorem 1.3.2 (Lin (2003)). *ϕ satisfies the following integro-differential equation:*

$$\left[(1 + \frac{\delta}{\lambda})\mathcal{I} - \frac{c}{\lambda}\mathcal{D}\right]^2 \phi(u) = \int_0^u \phi(u - x) p(x) dx + \omega(u), \quad u \geq 0, \tag{1.61}$$

where $\mathcal{I}$ and $\mathcal{D}$ are the identity and differentiation operator. It can be further solved to a defective renewal equation:

$$\phi(u) = \frac{\lambda^2}{c^2} \int_0^u \phi(u - x) \gamma(x) dx + \frac{\lambda^2}{c^2} H(u), \tag{1.62}$$

where $H(u) = T_{\rho_2} T_{\rho_1} \omega(u)$.

Dickson and Hipp (2000) considers a risk process in which the claim inter-arrival times have a phase-type (2) distribution. A phase-type distribution is such that its density satisfies a second order linear differential equation, i.e.,

$$k(t) + A_1\, k'(t) + A_2\, k''(t) = 0, \quad t > 0, \text{ and } A_2 > 0. \tag{1.63}$$

For example, linear combinations and convolutions of two exponential distributions (with possibly different means) both satisfy (1.63). In this case, the Laplace transform of $\Phi(u)$ is given by

$$\hat{\Phi}(s) = \frac{\mu - A_1\, c + A_2\, c\, k(0) + A_2\, c^2\, \Phi(0)}{A_2\, c^2\, s^2 - A_1\, c\, s + \mu\, s\, \hat{p}_1(s) + A_2\, c\, k(0)\hat{p}(s)}, \quad s \geq 0, \tag{1.64}$$

where $\Phi(0) = \frac{\theta}{A_2\, c^2\, \rho}$, and ρ is the unique positive root to the equation:

$$\mu\, \hat{p}_1(s) - A_1\, c + A_2\, c^2\, s + \hat{p}(s)\, A_2\, c\, k(0).$$

Inverting Laplace transforms yields a compound geometric formula for $\Phi(u)$,

$$\Phi(u) = \Phi(0) \sum_{n=0}^{\infty} [\Psi(0)]^n\, R^{*n}(u), \qquad u \geq 0, \tag{1.65}$$

where R is a distribution function (d.f.) with density r being

$$r(y) = \frac{\theta}{A_2\, c^2\, \rho} \left[\mu\, T_\rho p_1(y) + A_2\, c\, k(0)\, T_\rho p(y)\right]. \tag{1.66}$$

Li and Garrido (2004) studies the evaluation of the Gerber-Shiu penalty function for a Sparre Andersen risk model, in which the claim waiting times are Erlang(n) distributed with density $k_n(t) = \frac{\beta^n t^{n-1} e^{-\beta t}}{(n-1)!}$, for $t \geq 0$, $n \in \mathbb{N}^+$. Then the Gerber-Shiu penalty function $\phi(u)$ satisfies the following integro-differential equation:

Theorem 1.3.3. [Li and Garrido(2004)] *$\phi_\delta(u)$ satisfies the following equation for $u \geq 0$:*

$$\sum_{k=0}^{n} \phi^{(k)}(u) \left[\frac{-(\beta+\delta)}{c}\right]^{n-k} \binom{n}{n-k}$$
$$= \left(\frac{-\beta}{c}\right)^n \left[\int_0^u \phi(u-x)\, p(x)\, dx + \int_u^\infty w(u, x-u)\, p(x)\, dx\right]. \tag{1.67}$$

Proof: See pages 119–120 of Li (2003). □

The above equation can be solved to a defective renewal equation in terms of n roots with positive real parts to a generalized Lundberg equation. Let $l(s) = (\frac{\beta+\delta}{c} - s)^n$, for $\delta \geq 0$, $n \in \mathbb{N}^+$ and $s \in \mathbb{C}$. The equation $l(s) = \frac{\beta^n}{c^n}\,\hat{p}(s)$ is a generalized form of *Lundberg's fundamental equation*. It can be proved that for $\delta > 0$ and $n \in \mathbb{N}^+$, exactly n out of all the roots to Lundberg's equation, say $\rho_1, \rho_2, \ldots, \rho_n$, have a positive real part $\Re(\rho_j) > 0$. Moreover, if p is sufficiently regular, there is a negative root, say $-R$, then $R > 0$ is called *a generalized adjustment coefficient.*

Using the roots $\rho_1, \ldots, \rho_n$ and the operator T_{ρ_j}, the above integro-differential equation can be solved to a defective renewal equation.

Theorem 1.3.4. [Li and Garrido (2004)] *$\phi(u)$ admits a defective renewal equation representation*

$$\begin{aligned} \phi(u) &= \int_0^u \phi(u-y)\,\eta_\delta(y)\,dy + G_\delta(u)\,, \quad u \geq 0\,, && (1.68) \\ &= \frac{1}{(1+\xi_\delta)}\int_0^u \phi(u-y)\,v_\delta(y)\,dy + \frac{1}{(1+\xi_\delta)}\,H_\delta(u) && (1.69) \end{aligned}$$

where $\eta_\delta(y) = \frac{\beta^n}{c^n}\,T_{\rho_n}\,T_{\rho_{n-1}}\cdots T_{\rho_1}\,p(y)$, $G_\delta(u) = \frac{\beta^n}{c^n}\,T_{\rho_n}\,T_{\rho_{n-1}}\cdots T_{\rho_1}\,\omega(u)$, ξ_δ is such that $\frac{1}{(1+\xi_\delta)} = \int_0^\infty \eta_\delta(y)\,dy = 1 - \frac{[(\beta+\delta)^n - \beta^n]}{c^n}\frac{1}{\prod_{i=1}^n \rho_i} < 1$, $H_\delta(u) = (1+\xi_\delta)\,G_\delta(u)$ and $v_\delta(y) = (1+\xi_\delta)\,\eta_\delta(y)$ is a proper density function. Further, when $\delta \to 0^+$ then $\xi_\delta \to \xi_0$, such that $\frac{1}{(1+\xi_0)} = \int_0^\infty \eta_0(y)\,dy = 1 - \frac{\theta\,\beta^n\,\mu_1}{c^n\prod_{i=1}^{n-1}\rho_i} < 1$, if the safety loading factor θ is positive.

Using the penalty function ϕ, the discounted marginal distribution of $U(T^-)$ can be derived in the following theorem, which can be used to get the joint distribution of surplus before ruin and deficit at ruin.

Theorem 1.3.5. [Li and Garrido (2004)] *If $\rho_1, \ldots, \rho_n$ are distinct, then*

$$f_1(x|u) = \begin{cases} (-1)^{n-1}\frac{\beta^n}{c^n}\left(\frac{1+\xi_\delta}{\xi_\delta}\right)\sum_{k=1}^{n}\frac{e^{-\rho_k x}\,\bar{P}(x)\,[e^{\rho_k x}\,\Psi_k(u-x)-\Psi_k(u)]}{\tau_n'(\rho_k)}\,, & 0 \le x < u\,, \\ (-1)^{n-1}\frac{\beta^n}{c^n}\left(\frac{1+\xi_\delta}{\xi_\delta}\right)\sum_{k=1}^{n}\frac{e^{-\rho_k x}\,\bar{P}(x)\,[e^{\rho_k u}-\Psi_k(u)]}{\tau_n'(\rho_k)}\,, & x \ge u\,, \end{cases} \tag{1.70}$$

where $\Psi_k(u) = \phi_T(u) + \int_0^u \phi_T(u-t)\,\rho_k\,e^{\rho_k t}\,dt$.

Explicit results for $\phi(u)$, which is in terms of the Laplace transform of the ruin time T, can be obtained, if the claim size is rationally distributed. Since in this case, the Laplace transform $\phi_T(u)$ can be written as a rational function which can be inverted by partial fractions.

Dickson (2003) studies the density for the ruin time for Erlang(n) risk process with claim sizes being exponentially distributed.

Gerber and Shiu (2003a,b) shows that if the waiting times are generalized Erlang(n) distributed, i.e., the distribution of W is the convolution of n exponential distributions with corresponding parameters $\lambda_i > 0$, $i = 1, 2, \ldots, n$, the integro-differential equation (1.67) can be extended as follows.

Theorem 1.3.6. [Gerber and Shiu(2003a)] *Let $\mathcal{I}$ and $\mathcal{D}$ denote the identity operator and differentiation operator, respectively. Then $\phi(u)$ satisfies the following equation for $u \ge 0$*

$$\left\{\prod_{j=1}^{n}\left[\left(1+\frac{\delta}{\lambda_j}\right)\mathcal{I} - \frac{c}{\lambda_j}\mathcal{D}\right]\right\}\phi(u) = \int_0^u \phi(u-x)\,p(x)\,dx + \omega(u)\,, \tag{1.71}$$

where $\omega(u) = \int_u^\infty w(u, x-u)\,p(x)\,dx$.

Then it can be solved to a defective renewal equation

$$\phi(u) = \int_0^u \phi(u-y)g(y)dy + H(u), \qquad u \ge 0, \tag{1.72}$$

where $g(y) = \frac{\lambda_1\lambda_2\cdots\lambda_n}{c^n}T_{\rho_n}\,T_{\rho_{n-1}}\cdots T_{\rho_1}\,p(y)$, $G_\delta(u) = \frac{\lambda_1\lambda_2\cdots\lambda_n}{c^n}\,T_{\rho_n}\,T_{\rho_{n-1}}\cdots T_{\rho_1}\,\omega(u)$, and ρ_i with $\Re(\rho_i) > 0$, for $i = 1, 2, \ldots, n$ are roots to following equation:

$$\prod_{j=1}^{n}\left[\left(1+\frac{\delta}{\lambda_j}\right) - \frac{c}{\lambda_j}s\right] = \hat{p}(s), \qquad s \in \mathbb{C},\ n \in \mathbb{N}^+.$$

Dickson and Drekic (2004) studies the joint and marginal distributions of the surplus before ruin and deficit at ruin for the above Sparre Andersen risk process with generalized Erlang(n) waiting times, assuming that the claim sizes are Phase-type distributed.

Willmot (1999) studies the ruin probability of a Sparre Andersen model in which the claim waiting times are K_n distributed and the premium rate $c = 1$. It is well known that ruin probability $\Psi(u)$ may be expressed as the tail of a compound geometric distribution, i.e.,

$$\Psi(u) = \Phi(0) \sum_{n=1}^{\infty} [\Psi(0)]^n \bar{H}^{*n}(u), \qquad u \geq 0, \tag{1.73}$$

and satisfies a defective renewal equation:

$$\Psi(u) = \Psi(0) \int_0^u \Psi(u-x) dH(x) + \Psi(0)\bar{H}(u), \qquad u \geq 0. \tag{1.74}$$

The expression of H and its density function h can be recognized from the Laplace-Stieltjes transform of the non-ruin probability $\Phi(u)$.

Assume that the claim waiting times density k is from a K_n distribution, its Laplace transform is given by

$$\hat{k}(s) = \frac{\prod_{i=1}^n \lambda_i + s\,\beta(s)}{\prod_{i=1}^n (s+\lambda_i)}, \tag{1.75}$$

where $n \geq 2$, $\lambda_i > 0, i = 1, 2, \ldots, n$ and $\beta(s)$ is a polynomial of degree $n-2$ or less. This general class of d.f.'s includes (mixtures) of Erlangs and phase-type d.f.s as special cases. From De Smit (1995), or Cohen (1982) one has

$$\int_0^\infty e^{-su} d\Phi(u) = \frac{[E(X)-E(W)]\,[\prod_{i=1}^n \lambda_i] s \prod_{i=1}^{n-1}(\rho_i - s)}{[\prod_{i=1}^{n-1} \rho_i]\,[\prod_{i=1}^n (\lambda_i - s) - \prod_{i=1}^n \lambda_i \hat{p}(s) + s\,\beta(-s)\hat{p}(s)]}, \tag{1.76}$$

where $\rho_1, \rho_2, \ldots, \rho_{n-1}$ are all the roots with positive real parts to a generalized Lundberg equation: $\hat{k}(-s)\,\hat{p}(s) = 1$. The existence of ρ_i can be proved by Rouché's Theorem.

For notational convenience, let $\pi(s) = \prod_{i=1}^{n-1}(s-\rho_i)$. Here $\gamma(x)$ is such that $\hat{\gamma}(s) = \beta(-s)\,\hat{p}(s)$, $\gamma(x;r) = T_r\gamma(x)$ and $\hat{\gamma}(s;r) = T_s\gamma(0;r) = T_sT_r\gamma(0)$. While $\bar{G}(x;\,r)$ is a survival function defined by $\bar{G}(x;\,r) = \frac{T_r\bar{P}(x)}{T_r\bar{P}(0)}$, $G(x,\,r)$ and $g(x;\,r)$ are the corresponding cdf and pdf, respectively.

Theorem 1.3.7. [Willmot (1999)] *If* $\rho_1, \rho_2, \ldots, \rho_{n-1}$ *are distinct, then*

$$\Phi(0) = 1 - \frac{\prod_{i=1}^{n}\lambda_i[E(W)-E(X)]}{\prod_{i=1}^{n-1}\rho_i}, \tag{1.77}$$

and

$$\hat{h}(s) = \sum_{j=1}^{n-1}\theta_j\,\hat{g}(s;\,\rho_j) + \frac{(-1)^n}{\Psi(0)}\sum_{j=1}^{n-1}\frac{\hat{\gamma}(s;\,\rho_j)}{\pi'(\rho_j)}, \tag{1.78}$$

with

$$\theta_j = \frac{(-1)^n[\prod_{i=1}^{n}\lambda_i]E(X)[1-\hat{p}_1(\rho_j)]}{\Psi(0)\,\rho_j\,\pi'(\rho_j)}.$$

We remark that $\hat{h}(s)$ is usually difficult to invert due to the definition of $\gamma(x)$ except for some specially chosen $\beta(s)$, e.g.,

1. If $\beta(s) = \beta$, then

$$h(x) = \sum_{j=1}^{n-1}\theta_j\,g_1(x;\,\rho_j) + \sum_{j=1}^{n-1}\eta_j\,g(x;\,\rho_j), \quad x \geq 0,$$

   with $\eta_j = \frac{(-1)^n\beta\,[1-\hat{p}(\rho_j)]}{\Psi(0)\,\rho_j\,\pi'(\rho_j)}$.

2. If $\beta(s) = 0$, i.e., k is a generalized Erlang(n) distribution, then

$$h(x) = \sum_{j=1}^{n-1}\theta_j\,g_1(x;\,\rho_j), \qquad x \geq 0.$$

Stanford et al. (2000) presents a recursive method of calculating ruin probabilities for non-Poisson claim processes, by looking at the surplus process embedded at claim instants. In this paper, claim inter-arrival times are assumed to be mixtures of exponentials and Erlang(n) distributions.

Let $s_n(y)I(y \geq 0)$ be the incomplete density for the surplus after the n-th claim and $\hat{s}_n(s) = \int_0^\infty e^{-s\,y}\, s_n(y)dy$ be its Laplace transform. Also let g be the density of the increment between two consecutive claims, which is the difference between premium income and claims, and $G(y)$ be its cdf. Then g is given by

$$g(y) = \begin{cases} \int_0^\infty k(y+t)\, p(t)dt, & y \geq 0 \\ \int_0^\infty k(t)\, p(t-y)dt, & y < 0 \end{cases} \tag{1.79}$$

and its Laplace transform $\hat{g}(s) = \int_{-\infty}^\infty e^{-s\,y} g(y)dy = \hat{k}(s)\,\hat{p}(-s)$. Recursively,

$$s_n(y) = \int_0^\infty s_{n-1}(x) g(y-x)dx, \quad y \geq 0,$$

and

$$\hat{s}_n(s) = \hat{s}_{n-1}(s)\,\hat{g}(s) - \int_0^\infty e^{-s\,x}\, s_{n-1}(x) \int_x^\infty e^{s\,y}\, g(-y)dydx,$$

while its evaluation at $s = 0$ gives the probability of ruin on the n-th claim, i.e.,

$$P(n) = P(\text{ruin on the } n\text{-th claim}) = \int_0^\infty s_{n-1}(x) \int_x^\infty g(-y)\, dy\, dx.$$

For Erlang(n) inter-arrival times, the above theorem gives a recursive evaluation formula.

Theorem 1.3.8. [Stanford et al (2000)] *If the claim inter-arrival times are distributed as* $k(t) = \frac{\lambda^n t^{n-1} e^{-\lambda\,t}}{(n-1)!} I(t \geq 0)$*, and claim amounts are mixtures of* N *exponentials with density*

$$p(x) = \sum_{i=1}^N r_i\, \theta_i\, e^{-\theta_i\, x} I(x \geq 0), \tag{1.80}$$

then

$$\hat{s}_n(s) = \sum_{i=1}^N \frac{r_i\, \theta_i}{\theta_i - s} \left[\left(\frac{\lambda}{\lambda + s}\right)^n \hat{s}_{n-1}(s) - \left(\frac{\lambda}{\lambda + \theta_i}\right)^n \hat{s}_{n-1}(\theta_i) \right], \tag{1.81}$$

and

$$P(n) = \sum_{i=1}^N r_i \left(\frac{\lambda}{\lambda + \theta_i}\right)^n \hat{s}_{n-1}(\theta_i).$$

1.4 Risk Models in Discrete Time

In discrete time risk models, the surplus is examined at the end of a number of periods of equal length (usually one year). The discrete time analogue of the continuous time classical surplus process is based on the compound binomial process, which was first introduced in Gerber (1988) and has been further studied by Shiu (1989), Willmot (1993), Dickson (1994a), De Vylder (1996), De Vylder and Marceau (1996), Cheng et al. (2000), and Li and Garrido (2002).

1.4.1 Compound Binomial Process

Consider a discrete time surplus process, in which the number of insurance claims is governed by a binomial process $N(n)$, for $n \in \mathbb{N}$. In each time period, the probability of a claim is $q \in (0, 1)$, while the probability of no claim is $1 - q$. The claim occurrences in different time periods are independent events. The individual claim amounts $X_1, X_2, \ldots$ are mutually independent, identically distributed, positive integer-valued random variables. As usual, these claim severities are assumed to be independent of the counting process $\{N(n); n \in \mathbb{N}\}$. Let $p(x), x \in \mathbb{N}^+$ be their common probability function and $\mu = E(X)$. If u denotes the initial surplus, then the surplus at time n is given by

$$U(n) = u + n - \sum_{i=1}^{N(n)} X_i, \qquad u \in \mathbb{N}. \tag{1.82}$$

The discrete time risk model (1.82) is called a *compound binomial risk model.*

The compound binomial surplus process can also be used to model the case where more than one claim can occur in each time period. Then we assume that the total claims in each period are i.i.d. random variables taking non-negative integer values. Let Y_j be the sum of the claims in period j, then the surplus process $U(n)$ with $U(0) = u$ is

$$U(n) = u + n - \sum_{j=1}^{n} Y_j, \qquad u \in \mathbb{N}. \tag{1.83}$$

The process in (1.83) can be viewed as the second version of the compound binomial model, simply denoting $Y_j = I_j X_j$, where r.v. I_j's are i.i.d. with $P(I_j = 0) = P(Y_j = 0) = 1 - q$ and $P(I_j = 1) = P(Y_j > 0) = q$, while $X_j = Y_j$, given that $Y_j > 0$.

Gerber (1988) defines the ruin time to be a defective r.v. T such that

$$T = \inf\{n \geq 1 : U(n) \leq 0\}. \tag{1.84}$$

Shiu (1989) defines the ruin time T to be

$$T = \inf\{n \geq 1 : U(n) < 0\}. \tag{1.85}$$

In both cases, $\Psi(u) = P(T < \infty)$ denotes the ultimate ruin probability and $\Phi(u) = 1 - \Psi(u)$ its survival probability, while $\Psi(u, n) = P(T \leq n)$ is the finite time ruin probability, with $\Phi(u, n) = 1 - \Psi(u, n)$ being the corresponding finite time survival probability.

For the first definition of ruin time T, using two series, Gerber (1988) shows that $\Psi(0) = q\,\mu$, $P[T < \infty, U(T-1) = x, |U(T)| = y \,|\, U(0) = 0] = q\,p(x+y+1)$, for x, $y \in \mathbb{N}^+$, $y \in \mathbb{N}$, and

$$\frac{1 - \Psi(u)}{1 - \Psi(0)} = (1-q)^{-u} + \sum_{k=1}^{u-1} \frac{1}{k!} \left(\frac{q}{1-q} \right)^k E\left[(S_k - u)^{(k)} (1-q)^{S_k - u} I(S_k \leq u) \right],$$

where $a^{(k)} = k!\binom{a}{k}$ is the k-th factorial power of a, and $S_k = X_1 + X_2 + \cdots X_k$.

Using different methods, Shiu (1989) derives Gerber's results for the second definition of the ruin time T, with $\Psi(0) = \frac{1-q\mu}{1-q}$, and

$$\Phi(u) = \Phi(0) \sum_{j=0}^{\infty} \left(\frac{-q}{1-q} \right)^j E\left[\binom{u+j-S_j}{j} (1-q)^{S_j - u} I(S_j \leq u) \right], \; u \in \mathbb{N}^+, \tag{1.86}$$

which can be expressed in the form of a compound geometric sum,

$$\Phi(u) = \Phi(0) \sum_{n=0}^{\infty} [\Psi(0)]^n H^{*n}(u), \qquad u \in \mathbb{N}^+, \tag{1.87}$$

where $H(u) = \frac{\bar{P}(u)}{1-\mu}$.

Using the technique of generating functions, Willmot (1993) gives an explicit formula for the finite ruin probability based on Shiu's definition of ruin time, i.e., for $u, n \in \mathbb{N}^+$,

$$\Phi(0,n) = \frac{\sum_{m=0}^{n}(n-m+1)g_m(n+1)}{(1-q)(n+1)}, \tag{1.88}$$

$$\Phi(u,n) = G_{u+n}(n) - (1-q)\sum_{m=0}^{n-1} \Phi(0, n-1-m)\, g_{u+m+1}(m), \tag{1.89}$$

where $g_n(k) = P(S_k = n)$, and $G_n(k) = \sum_{m=0}^{n} g_m(k)$, for $n \in \mathbb{N}$. The generating function $\hat{\Psi}(z) = \sum_{u=0}^{\infty} z^u\, \Psi(u)$ of the ruin probability $\Psi(u)$ is given by

$$\hat{\Psi}(z) = \frac{1}{1-z}\left\{\frac{1-q\,\mu}{1-q\,\mu\,\hat{b}(z)}\right\}, \qquad -1 < z < 1, \tag{1.90}$$

where $\hat{b}(z) = \frac{1-\hat{p}(z)}{\mu(1-z)}$, with $\hat{p}(z) = \sum_{x=1}^{\infty} z^x p(x)$ being the p.g.f. of p.

Inverting (1.90) again gives (1.87). Explicit results for $\Psi(u)$ can be obtained when the claim sizes are constant or geometrically distributed.

Dickson (1994a) derives some above results using very elementary methods, and shows how these results can be used to approximate ruin probabilities in the continuous time classical risk model.

Using martingale techniques and a duality argument, Cheng et al. (2000) derives the discounted joint distribution functions for the surplus before ruin and the deficit at ruin, for Gerber's definition of ruin time in the compound binomial model. Define $f_3(x,y,t\,|\,u) = P[U(T-1) = x, |U(T)| = y, T = t\,|\,U(0) = u]$ and for a discount factor $0 < v < 1$, define $f_2(x,y\,|\,u) = \sum_{t=1}^{\infty} v^t f_2(x,y,t\,|\,u)$ and $f_1(x\,|\,u) = \sum_{y=0}^{\infty} f_2(x,y\,|\,u)$.

Using a duality argument, the key formula

$$f_2(x, y \mid 0) = q\, v\, \rho^x\, p(x+y+1), \qquad x,\, y \in \mathbb{N},$$

is obtained, yielding a recursive formula for $f_2(x, y \mid u)$, when $u \in \mathbb{N}^+$ and $x, y \in \mathbb{N}$:

$$f_2(x, y \mid u) = \sum_{z=0}^{u-1} f_2(x, y \mid u-z)\, g(z) + f_2(x, y \mid 0)\, \rho^{-u} I(u \leq x), \tag{1.91}$$

where $g(y) = \sum_{x=0}^{\infty} f_2(x, y \mid 0)$, and $\rho \in (0,1)$ is the unique solution to a discrete Lundberg equation $q\,\hat{p}(r) + (1-q) = \frac{r}{v}$.

Li and Garrido (2002) considers the expected penalty function for the second version of the compound binomial model (1.83), where $p(x), x \in \mathbb{N}$, is the probability mass function of Y, and $\hat{p}(s) = \sum_{x=0}^{\infty} s^x\, p(x)$ is its p.g.f..

Let $w(x, y)$ be the non-negative values of a penalty function, for $x, y \in \mathbb{N}$. For $0 < v < 1$ define

$$\phi(u) = E\left[v^T w(U(T-1), |U(T)|) I(T < \infty) \mid U(0) = u\right] \;, \quad u \in \mathbb{N}\;, \tag{1.92}$$

where ruin time $T = \min\{t \in \mathbb{N}^+ : U(t) \leq 0\}$. Then $\phi(u)$ is given recursively as follows.

Theorem 1.4.1. [Li and Garrido(2002)]

$$\phi(0) = v \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} \rho^x w(x, y) p(x+y+1)\;, \tag{1.93}$$

and

$$\phi(u) = \frac{1}{1+\beta} \sum_{x=0}^{u-1} \phi(u-x)\, l(x) + \frac{1}{1+\beta} M(u)\;, \qquad u \in \mathbb{N}^+,$$

where $0 < \rho < 1$ *is the root of the equation*

$$q(s) := \frac{\hat{p}(s)}{s} = \frac{1}{v}\;, \tag{1.94}$$

β *is such that* $\frac{1}{1+\beta} = \frac{v-\rho}{1-\rho}$, $l(x) = v\,(1+\beta) \sum_{y=0}^{\infty} \rho^y p(x+y+1)$ *is a proper probability function on* $\mathbb{N}$, *and* $M(u) = v(1+\beta) \sum_{x=u}^{\infty} \rho^{x-u} \sum_{y=0}^{\infty} w(x, y) p(x+y+1)$.

Now consider the compound geometric p.f. $k(u) = \frac{\beta}{1+\beta}\sum_{n=0}^{\infty}\left(\frac{1}{1+\beta}\right)^n l^{*n}(u)$, for $u \in \mathbb{N}$. Then $\phi(u)$ can be expressed explicitly as in the following theorem.

Theorem 1.4.2. [Li and Garrido (2002)]

$$\phi(u) = \frac{1}{\beta}\sum_{z=0}^{u-1} M(u-z)\,k(z), \qquad u \in \mathbb{N}^+. \tag{1.95}$$

An applications of Theorem 1.4.2 is to derive many quantities associated with ruin in a closed form. For example, the discounted joint distribution of the surplus before ruin and the deficit at ruin can be expressed explicitly, not only with the recursive formula in (1.91).

Theorem 1.4.3. *For $x \in \mathbb{N}^+$, and $y \in \mathbb{N}$*

$$f_2(x, y \,|\, u) = \gamma(u) f_2(x, y \,|\, 0)\ , \quad u \in \mathbb{N}^+\ , \tag{1.96}$$

where

$$\gamma(u) = \begin{cases} \frac{1+\beta}{\beta}\sum_{z=0}^{u-1} \rho^{z-u} k(z) & \text{if } 1 \le u \le x \\ \frac{1+\beta}{\beta}\sum_{z=u-x}^{u-1} \rho^{z-u} k(z) & \text{if } u > x \end{cases}. \tag{1.97}$$

The discounted joint and marginal moments of surplus before ruin $U(T-1)$ and deficit at ruin $|U(T)|$ can be obtained recursively through the penalty function $\phi(u)$. The following theorem gives a recursive formula for factorial moments $E_{(n)}(u) = E[T^{(n)} I(T < \infty) \,|\, U(0) = u]$, of the ruin time T.

Theorem 1.4.4. [Li and Garrido (2002)] *For $u \in \mathbb{N}^+$,*

$$E_{(1)}(u) = \mu_1 \sum_{z=0}^{u-1} E_{(1)}(u-z) p_1(z) + \sum_{z=u}^{\infty} \Psi(z), \tag{1.98}$$

$$\begin{aligned} E_{(n+1)}(u) &= \mu_1 \sum_{z=0}^{u-1} E_{(n+1)}(u-z) p_1(z) + (n+1) \sum_{z=0}^{u-1} E_{(n)}(u-z) p_1(z) \\ &\quad + (n+1) \sum_{z=u+1}^{\infty} E_{(n)}(z)\ , \end{aligned} \tag{1.99}$$

where $\mu_1 = E[Y]$, and $p_1(x) = \frac{\bar{P}(x)}{\mu_1}$ is the equilibrium distribution of p.

1.4.2 General Discrete Time Surplus Models

Like the continuous time surplus model, the classical discrete time binomial model can be extended in several ways.

Cossette and Marceau (2000) considers a discrete-time risk model with correlated classes of business. in which, two kinds of dependence between the classes of business are proposed, one is a Poisson model with common shock (PCS model), which was first introduced in Marshall and Olkin (1967,1988) and Kocherlakota and Kocherlakota (1992), another is a negative binomial model with common component (NBCC model). In both cases, the authors show how the dependence affects the ruin probabilities through the value of adjustment coefficient. This is illustrated by numerical examples.

For the above defined discrete time model, Wu and Yuen (2003) proposes a new structure of dependence (IR model) and compares the results with the PCL and NBCC models above. Assume that there are two types of claims, namely, main claims and by-claims. Each main claim in a class may produce a by-claim, occuring in another class, with a certain probability.

Yuen and Guo (2001) considers the ruin probability for a risk process with time-correlated claims in the compound binomial model. It is assumed that every main claim will produce a by-claim, but the occurences of the by-claims may be delayed one or two periods. Recursive formulas for the finite ruin probabilities are obtained and explicit expressions for ultimate ruin probabilities are given in two special cases.

Cossette et al. (2003) presents a compound Markov binomial model which is an extension of the classical binomial model proposed by Gerber (1988). It is based on the Markov Bernoulli process which introduces dependency between claim occurrences.

Let the surplus process $\{U_n;\, n \in \mathbb{N}\}$ be defined by

$$U_n = u + n - \sum_{i=1}^{N_n} X_i, \qquad u \in \mathbb{N}, \tag{1.100}$$

where $\{N_n;\, n \in \mathbb{N}\}$ is a binomial process defined as $N_n = I_1 + I_2 + \cdots I_n$ representing the total number of claims over n periods, $\{I_k;\, k \in \mathbb{N}\}$ is a sequence of i.i.d. Bernoulli r.v.'s with mean $q \in (0,1)$. The sequence $\{X_k;\, k \in \mathbb{N}\}$ of individual claim amounts are i.i.d..

In this discrete time compound binomial risk model, Cossette et al. (2003) assumes that $\{I_k;\, k \in \mathbb{N}\}$ is a stationary homogeneous Markov chain with state space $\{0,\, 1\}$ and with a transition matrix, making N_n a Markov binomial process. This introduces time dependency in claims occurrences.

For this compound Markov binomial risk model, recursive formulas are provided for finite and infinite time ruin probabilities. Lundberg exponential bounds are also derived and numerical examples are given in Cossette et al. (2003).

Wagner (2001) considers a discrete risk model governed by a two state Markov chain. The risk process receives a premium c at time n, if the Markov chain is in state 1, while it pays a benefit 1 at time n, if the Markov chain is in state 2. Let p_{12} be the probability of a transition from state 1 to state 2 in the interval $(n, n+1]$, while p_{21} denote the probability of transition from state 2 to state 1. Define the claim size X_n as 0, if the chain is in state 1 at time n, or as 1, if the chain is in state 2 at time n. Starting with initial surplus $u \in \mathbb{N}$ in state $i \in \{1, 2\}$, let

$$\tau_i(u) = \inf\left\{ n \in \mathbb{N} \,\middle|\, u + n\,c - (1+c)\sum_{j=1}^{n} X_j < 0 \right\}, \quad u \geq 0, \tag{1.101}$$

be the time of ruin, $\Psi_i(u) = P(\tau_i(u) < \infty)$ be the ruin probability and $\xi_i(u) = E[\tau_i(u)]$ be the expected value of ruin time. Recursive formulas for ruin probabilities and the expected ruin times are given.

Reinhard and Snoussi (2000, 2001) study the ruin probability and the joint and marginal distributions of the surplus prior to ruin and the deficit at ruin in a discrete semi-Markov risk model, by constructing a recursive system of equations. Explicit results are given for the two states model.

Chapter 2

Mathematical Preliminaries

2.1 An Operator of Integrable Real Functions

As in Dickson and Hipp (2001), we introduce the following complex operator of an integrable real-valued function f :

$$T_r f(x) = \int_x^\infty e^{-r\,(u-x)}\, f(u)\, du\ , \quad r \in \mathbb{C}\ , \quad x \geq 0\ . \tag{2.1}$$

For example, if $r = a + b\,i$, we have

$$\begin{aligned} T_r f(x) &= \int_x^\infty e^{-(a+b\,i)(u-x)}\, f(u)\, du \\ &= \int_x^\infty e^{-a(u-x)} \cos b(u-x)\, f(u)\, du - i \int_x^\infty e^{-a(u-x)} \sin b(u-x)\, f(u)\, du\ . \end{aligned}$$

The operator T_r satisfies the following properties:

1. $T_r f(0) = \int_0^\infty e^{-r\,u}\, f(u)\, du = \hat{f}(r)$, for $r \in \mathbb{C}$, is the Laplace transform of f.

2. $T_{r_1} T_{r_2}\, f(x) = T_{r_2} T_{r_1} f(x) = \frac{T_{r_1} f(x) - T_{r_2} f(x)}{r_2 - r_1}$, $\quad r_1 \neq r_2 \in \mathbb{C}\ , \quad x \geq 0$.

3. If $r = a + b\,i$ (denote by $\bar{r} = a - b\,i$, for $b \neq 0$) then

$$T_{\bar{r}}\, T_r f(x) = \frac{1}{b} \int_x^\infty e^{-a(u-x)} \sin b(u-x)\, f(u)\, du\ , \quad x \geq 0\ ,$$

   is a real number.

4. $T_r[f * g(x)] = f * [T_r g(x)] + T_r g(0) \cdot T_r f(x), \qquad x \geq 0.$

5. $T_r T_r f(x) = \int_x^\infty e^{-r(u-x)}(u-x) f(u)\,du = -\frac{d}{dr} T_r f(x) = \lim\limits_{s \to r} \frac{T_r f(x) - T_s f(x)}{s-r} = \lim_{s \to r} T_s T_r f(x)$, for $x \geq 0$.

6. $T_r^n f(x) = \underbrace{T_r \cdots T_r}_{n} f(x) = \lim\limits_{s \to r} T_s T_r^{n-1} f(x) = \int_x^\infty e^{-r(u-x)} \frac{(u-x)^{n-1}}{(n-1)!} f(u)\,du = \frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{dr^{n-1}} T_r f(x)$, while $T_s T_r^n f(0) = \frac{\hat{f}(s)}{(r-s)^n} - \sum_{j=1}^n \frac{T_r^j f(0)}{(r-s)^{n+1-j}}$, for $s \in \mathbb{C}$, is the corresponding Laplace transform.

7. If $r_1, r_2, \ldots, r_k$ are distinct complex numbers, then

$$\begin{aligned} T_{r_k} \cdots T_{r_2} T_{r_1} f(x) &= \int_x^\infty e^{-r_k(x_k - x)} \cdots \int_{x_2}^\infty e^{-r_1(x_1 - x_2)} f(x_1)\,dx_1 \cdots dx_k \\ &= (-1)^{k-1} \sum_{l=1}^k \frac{T_{r_l} f(x)}{\pi_k'(r_l)}, \qquad x \geq 0, \end{aligned} \tag{2.2}$$

where $\pi_k(r) = \prod_{l=1}^k (r - r_l)$. The corresponding Laplace transform is

$$T_s T_{r_k} \cdots T_{r_2} T_{r_1} f(0) = (-1)^k \Big[\frac{\hat{f}(s)}{\pi_k(s)} - \sum_{l=1}^k \frac{\hat{f}(r_l)}{(s - r_l)\pi_k'(r_l)} \Big], \qquad s \in \mathbb{C}.$$

8. For distinct $r_1, r_2, \ldots, r_k$, and positive integers $n_1, n_2, \ldots, n_k$,

$$\begin{aligned} T_{r_k}^{n_k} T_{r_{k-1}}^{n_{k-1}} \cdots T_{r_1}^{n_1} f(x) &= \sum_{l=1}^k \sum_{m=1}^{n_l} a_{l,m} \int_x^\infty \frac{e^{-r_l(u-x)}(u-x)^{m-1}}{(m-1)!} f(u)\,du \\ &= \sum_{l=1}^k \sum_{m=1}^{n_l} a_{l,m} T_{r_l}^m f(x), \quad x \geq 0, \end{aligned}$$

where the coefficients $\{a_{l,m}\,;\, 1 \leq l \leq k,\ 1 \leq m \leq n_l\}$ are determined by:

$$\prod_{l=1}^k \frac{1}{(s + r_l)^{n_l}} = \sum_{l=1}^k \sum_{m=1}^{n_l} \frac{a_{l,m}}{(s + r_l)^m}, \quad s \in \mathbb{C}, \tag{2.3}$$

that is

$$a_{l,m} = \frac{1}{(n_l - m)!} \frac{d^{n_l - m}}{ds^{n_l - m}} \prod_{t=1, t \neq l}^k \frac{1}{(s + r_t)^{n_t}} \Bigg|_{s = -r_l}, \quad s \in \mathbb{C}.$$

Proof: Properties 1. to 6. are trivial. 7. is proved by induction; clearly it holds for $k=1$. Assume that it also holds for an arbitrary k, then

$$\begin{aligned} T_{r_{k+1}} \cdots T_{r_1} f(x) &= (-1)^{k-1} \sum_{l=1}^{k} \frac{T_{r_{k+1}} T_{r_l} f(x)}{\pi_k'(r_l)} = (-1)^{k-1} \sum_{l=1}^{k} \frac{T_{r_{k+1}} f(x) - T_{r_l} f(x)}{(r_l - r_{k+1})\pi_k'(r_l)} \\ &= (-1)^{k-1} T_{r_{k+1}} f(x) \sum_{l=1}^{k} \frac{1}{(r_l - r_{k+1})\pi_k'(r_l)} + (-1)^k \sum_{l=1}^{k} \frac{T_{r_l} f(x)}{\pi_{k+1}'(r_l)} \\ &= (-1)^k \sum_{l=1}^{k+1} \frac{T_{r_l} f(x)}{\pi_{k+1}{}'(r_l)}, \quad x \geq 0, \end{aligned}$$

where the last step is by a property, $\sum_{l=1}^{k} \frac{\pi(s)}{(s-r_l)\pi_k'(r_l)} = 1$, of Lagrange polynomials.

To prove Property 8. note that by changing the order of integration

$$T_{r_k}^{n_k} T_{r_{k-1}}^{n_{k-1}} \cdots T_{r_1}^{n_1} f(x) = \frac{1}{r_k^{n_k} \cdots r_1^{n_1}} \int_x^\infty f(u)\, \Gamma_{(r_k,n_k)} * \cdots * \Gamma_{(r_1,n_1)}(u-x)\, du \tag{2.4}$$

holds. Here $\Gamma_{(r_k,n_k)}(x) = \frac{r_k^{n_k} x^{n_k-1} e^{-r_k x}}{\Gamma(n_k)}$ and $*$ denotes the convolution product. The Laplace transform of the convolution term is equal to $\prod_{l=1}^{k} \frac{r_l^{n_l}}{(s+r_l)^{n_l}}$, for $s \in \mathbb{C}$. On the other hand, by rational functions, we have that

$$\prod_{l=1}^{k} \frac{r_l^{n_l}}{(s+r_l)^{n_l}} = \Big(\prod_{l=1}^{k} r_l^{n_l}\Big) \sum_{l=1}^{k} \sum_{m=1}^{n_l} \frac{a_{l,m}}{(s+r_l)^m}, \quad s \in \mathbb{C},$$

where the $a_{l,m}$ coefficients are given in (2.3). Hence inverting the transform gives

$$\Gamma_{(r_k,n_k)} * \Gamma_{(r_{k-1},n_{k-1})} * \cdots * \Gamma_{(r_1,n_1)}(x) = \Big(\prod_{l=1}^{k} r_l^{n_l}\Big) \sum_{l=1}^{k} \sum_{m=1}^{n_l} a_{l,m} \frac{e^{-r_l x} x^{m-1}}{(m-1)!}.$$

Substituting into (2.4) completes the proof. □

2.2 Divided Differences

An introduction to divided differences can be found in Freeman (1960).

Definition 2.2.1. *The n-th divided difference* $f[x_0, x_1, \ldots, x_n]$ *on* $n+1$ *distinct points of a function* $f(x)$ *is defined recursively by* $f[x_0] = f(x_0)$*, and,*

$$\begin{aligned} f[x_0, x_1] &= \frac{f(x_0) - f(x_1)}{x_0 - x_1}, \\ f[x_0, x_1, x_2] &= \frac{f[x_0, x_1] - f[x_1, x_2]}{x_0 - x_2}, \\ f[x_0, x_1, \ldots, x_n] &= \frac{f[x_0, x_1, \ldots, x_{n-1}] - f[x_1, x_2, \ldots, x_n]}{x_0 - x_n}. \end{aligned}$$

For distinct numbers $x_1, x_2, \ldots, x_n$, define $\pi_n(x) = \prod_{i=1}^{n}(x - x_i)$, then by induction, the $(n-1)$-th divided difference can be expressed explicitly by

$$f[x_1, x_2, \ldots, x_n] = \sum_{i=1}^{n} \frac{f(x_i)}{\pi_n'(x_i)}. \tag{2.5}$$

If some of x_i values coincide, the divided differences with respect to repeated points in a collection can be evaluated as a derivative. For example, if a, b, c are three distinct numbers, then

$$f[a,\, a,\, a,\, b,\, b,\, c] = \frac{1}{(3-1)!}\frac{1}{(2-1)!}\frac{\partial^2}{\partial a^2}\frac{\partial}{\partial b} f[a, b, c].$$

The following identity will be used throughout this thesis, which can be proved by divided differences.

Lemma 2.2.1. *For any* $n \in \mathbb{N}^+$ *distinct complex number* $x_0, x_1, \ldots, x_n$ *and* $m \in \mathbb{Z}$ *:*

$$\sum_{i=1}^{n} \frac{(x_i - x_0)^m}{\pi_n'(x_i)} = \begin{cases} -\frac{1}{\pi_n(x_0)}, & m = -1 \\ 0, & m = 0, 1, \ldots, n-2 \\ 1, & m = n-1 \end{cases},$$

where $\pi_n(x) = \prod_{i=1}^{n}(x - x_i)$*.*

Proof: Note that $\sum_{i=1}^{n} \frac{(x_i - x_0)^m}{\pi_n'(x_i)}$ is the $(n-1)$-th divided difference of the polynomial $(x - x_0)^m$, w.r.t. the points in the collection $x_1, x_2, \ldots, x_n$. □

There is a close connection between the operator T_r and divided differences, which is stated in the following theorem.

Theorem 2.2.1. *For a collection of points* $x_1, x_2, \ldots, x_n$,

$$T_{x_n} T_{x_{n-1}} \cdots T_{x_1} f(y) = (-1)^{n-1} h_y[x_1, x_2, \ldots, x_n], \tag{2.6}$$

where $h_y(x) = T_x f(y) = \int_y^\infty e^{-x(z-y)} f(z)dz$ *is a function in* x *for a fixed parameter* y. *Specially, if* $y = 0$,

$$T_{x_n} T_{x_{n-1}} \cdots T_{x_1} f(0) = (-1)^{n-1} \hat{f}[x_1, x_2, \ldots, x_n], \tag{2.7}$$

where $\hat{f}(x) = T_x f(0) = \int_0^\infty e^{-xy} f(y)dy$ *is the Laplace transform of* f.

Proof: Using formula (2.5) and Property 7 of the operator T_r in the previous section. □

2.3 An Operator of Discrete Functions

This section gives the definition of an operator to a real valued function with domain in the positive integers (see Dickson and Hipp (2001) for the continuous version of the operator).

Define T_r to be an operator of any real valued function $f(x), x \in \mathbb{N}^+$ by

$$T_r f(y) = \sum_{x=y}^{\infty} r^{x-y} f(x) = \sum_{x=0}^{\infty} r^x f(x+y), \quad r \in \mathbb{C}, \ y \in \mathbb{N}^+. \tag{2.8}$$

Like for the continuous operator T_r in Section 2.1, its discrete restriction has many nice properties, which are helpful to simplify calculations, e.g.,

1. $T_r f(1) = \frac{\hat{f}(r)}{r}$, where $\hat{f}(r)$ is the generating function of f.

2. $T_1 f(y) = \sum_{x=y}^{\infty} f(y)$.

3. If r_1 and r_2 are distinct, then

$$T_{r_2} T_{r_1} f(y) = \frac{r_2 T_{r_2} f(y) - r_1 T_{r_1} f(y)}{r_2 - r_1}. \tag{2.9}$$

4. If r_1 is equal to r_2, then

$$\begin{aligned} T_r^2 f(y) &= T_r T_r f(y) = \lim_{r_1 \to r} T_{r_1} T_r f(y) = \lim_{r_1 \to r} \frac{r_1 T_{r_1} f(y) - r T_r f(y)}{r_1 - r} \\ &= \frac{d\,[r\,T_r f(y)]}{dr} = \sum_{x=y}^{\infty} (x - y + 1) r^{x-y} f(x). \end{aligned} \tag{2.10}$$

5. If $r_1, r_2, \ldots, r_k$ are distinct, then

$$T_{r_k} T_{r_{k-1}} \cdots T_{r_1} f(y) = \sum_{j=1}^{k} \frac{r_j^{k-1} T_{r_i} f(y)}{\pi_k'(r_j)}, \tag{2.11}$$

where $\pi_k(s) = \prod_{i=1}^{k} (s - r_i)$. While its p.g.f. transform is given by

$$s\,T_s T_{r_k} T_{r_{k-1}} \cdots T_{r_1} f(1) = \left[\prod_{i=1}^{k} \frac{s}{s - r_i} \right] \hat{f}(s) - \sum_{j=1}^{k} \left(\frac{s}{s - r_j} \right) \frac{r_j^{k-1}\,\hat{f}(r_j)}{\pi_k'(r_j)}.$$

6. If $r_i = r$, for $i = 1, 2, \ldots, k$,

$$T_r^k f(y) = \underbrace{T_r T_r \cdots T_r}_{k} f(y) = \lim_{s \to r} T_s\, T_r^{k-1} = \frac{d[r T_r^{k-1} f(y)]}{dr}. \tag{2.12}$$

Chapter 3

On a General Class of Renewal Risk Process

In this chapter, we consider the evaluation of the Gerber-Shiu penalty function for a class of renewal risk process, in which the claim inter-arrival times are K_n distributed. This general class of distributions includes, as special cases, Erlangs and phase-type distributions, as well as mixtures of these. Thus our model extends the classical risk model, the Erlang(2) model of Dickson (1998), Dickson and Hipp (1998, 2001), Cheng and Tang (2003), Lin (2003), the Erlang(n) risk model discussed by Li and Garrido (2004), generalized Erlang(n) risk model studied by Gerber and Shiu (2003a,b, 2004).

3.1 Two Classes of Continuous Distributions

In this section, we consider two classes of continuous distribution on R^+. The first is the class of K_n distributions, for $n \in \mathbb{N}^+$, while the second class is the family of $\mathcal{R}_f^+$ distributions. The first class is a subclass of the second class of distributions.

3.1.1 The K_n Class of Distributions

Definition 3.1.1. *A probability distribution is said to belong to the K_n class, $n \in \mathbb{N}^+$, if the Laplace transform $\hat{f}(s) = \int_0^\infty e^{-sx} f(x)dx$ of its density function f*

has the following form

$$\hat{f}(s) = \frac{\lambda^* + s\,\beta(s)}{\prod_{i=1}^{n}(s+\lambda_i)}, \qquad \Re(s) > \max\{-\lambda_1, -\lambda_2, \ldots, -\lambda_n\}, \tag{3.1}$$

where $\lambda_i > 0$ *for* $i = 1, 2, \ldots, n$, $\lambda^* = \prod_{i=1}^{n} \lambda_i$ *and* $\beta(s) = \sum_{i=0}^{n-2} \beta_i\, s^i$ *is a polynomial of degree* $n-2$ *or less.*

The class of K_n distributions is widely used in applied probability models [see Cohen (1982) and Tijms (1994)]. It includes, as special cases, Erlang and phase-type distributions, as well as mixtures of these. The following examples give some special distributions in the K_n family.

Example 3.1.1. The exponential distribution with density function

$$f(x) = \lambda\, e^{-\lambda x} I(x \geq 0), \qquad \lambda > 0,$$

is the only member of the K_1 family, since $\hat{f}(s) = \frac{\lambda}{(s+\lambda)}$.

Example 3.1.2. The mixture of two exponential distributions with density function

$$f(x) = [\theta\, \lambda_1\, e^{-\lambda_1 x} + (1-\theta)\lambda_2\, e^{-\lambda_2 x}] I(x \geq 0), \quad \lambda_1,\, \lambda_2 > 0, \; 0 < \theta < 1,$$

is a member of the K_2 family, since $\hat{f}(s) = \frac{\lambda_1 \lambda_2 + s[\theta\, \lambda_1 + (1-\theta)\lambda_2]}{(s+\lambda_1)(s+\lambda_2)}$.

Example 3.1.3. Phase-type (2) distributions with density functions f satisfying

$$f(x) + A_1\, f'(x) + A_2\, f''(x) = 0, \qquad A_2 > 0, \; A_1 f(0) + A_2\, f'(0) = 1,$$

are members of the K_2 family, since $\hat{f}(s) = \frac{[1 + s\, A_2 f(0)]}{[A_2\, s^2 + A_1\, s + 1]}$.

Example 3.1.4. The Erlang(n) distribution with density

$$f(x) = \frac{\lambda^n\, x^{n-1}}{(n-1)!} e^{-\lambda x} I(x \geq 0), \qquad \lambda > 0, \; n \in \mathbb{N}^+,$$

is a member of the K_n family, since $\hat{f}(s) = \frac{\lambda^n}{(s+\lambda)^n}$.

Example 3.1.5. The generalized Erlang(n) distribution being the convolution of n exponential distributions with possible different positive parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$,

$$f(x) = \sum_{i=1}^{n} \left(\prod_{j=1, j \neq i}^{n} \frac{\lambda_j}{\lambda_j - \lambda_i} \right) \lambda_i \, e^{-\lambda_i x} I(x \geq 0), \tag{3.2}$$

is a member of the K_n family, since $\hat{f}(s) = \frac{\lambda^*}{\prod_{i=1}^{n}(s+\lambda_i)}$.

Example 3.1.6. Coxian distributions with density function f having the following Laplace transform

$$\hat{f}(s) = \sum_{i=1}^{n} a_i \left(\prod_{k=1}^{i} \frac{\lambda_i}{s + \lambda_i} \right), \qquad n \in \mathbb{N}^+, \ \lambda_i > 0, \ \text{for } i = 1, 2, \ldots, n, \tag{3.3}$$

where $a_i = \left(\prod_{j=1}^{i-1} (1 - p_j) \right) p_i$, $p_i = \frac{b_i}{1 - \sum_{j=1}^{i-1} b_j}$, and $0 < b_i < 1$, with $\sum_{i=1}^{n} b_i = 1$, are members of the K_n family, since $\hat{f}(s)$ can be re-written as

$$\hat{f}(s) = \frac{\sum_{i=1}^{n} \left[a_i \, b_i \prod_{k=i+1}^{n} (s + \lambda_k) \right]}{\prod_{i=1}^{n} (s + \lambda_i)},$$

where $b_i = \prod_{k=1}^{i} \lambda_k$.

In general, the density function f of a K_n distribution can be obtained by inverting (3.1). Two cases are distinguished:

1. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct, by partial fraction one obtains:

$$\hat{f}(s) = \frac{\lambda^* + s\,\beta(s)}{\prod_{i=1}^{n} (s + \lambda_i)} = \sum_{i=1}^{n} \frac{a_i}{s + \lambda_i},$$

where $a_i = \frac{\lambda^* - \lambda_i \, \beta(-\lambda_i)}{\prod_{j=1, j \neq i}^{n} (\lambda_j - \lambda_i)}$. Inverting it gives

$$f(x) = \sum_{i=1}^{n} a_i \, e^{-\lambda_i x} \, I(x \geq 0). \tag{3.4}$$

2. If instead, some λ_i are equal, i.e., $\hat{f}(s) = \frac{\lambda^* + s\,\beta(s)}{\prod_{i=1}^{k} (s+\lambda_i)^{n_i}}$, where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct and $\sum_{i=1}^{k} n_i = n$, then by partial fractions:

$$\hat{f}(s) = \frac{\lambda^* + s\,\beta(s)}{\prod_{i=1}^{k} (s + \lambda_i)^{n_i}} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{a_{ij}}{(s + \lambda_i)^j},$$

and hence

$$f(x) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} a_{ij} \frac{x^{j-1} e^{-\lambda_i x}}{(j-1)!} I(x \geq 0), \tag{3.5}$$

where here $\lambda^* = \prod_{l=1}^{k} \lambda_l^{n_l}$ and

$$a_{ij} = \frac{1}{(n_i - j)!} \frac{d^{n_i - j}}{ds^{n_i - j}} \prod_{m=1,\, m \neq i}^{k} \frac{\lambda^* + s\,\beta(s)}{(s + \lambda_m)^{n_m}} \Bigg|_{s=-\lambda_i}, \quad s \in \mathbb{C}\,.$$

The mean and variance of a K_n distributed r.v. X can be obtained by

$$E(X) = -\hat{f}'(0) = \sum_{i=1}^{n} \frac{1}{\lambda_i} - \frac{\beta(0)}{\lambda^*}, \tag{3.6}$$

$$\mathrm{Var}(X) = \hat{f}''(0) - [-\hat{f}'(0)]^2 = \sum_{i=1}^{n} \frac{1}{\lambda_i^2} + \frac{2\beta'(0)\,\lambda^* - \beta^2(0)}{(\lambda^*)^2}, \tag{3.7}$$

3.1.2 Rational Distributions on R^+

Definition 3.1.2. *A probability distribution F on $\mathbb{R}^+$ is said to be belong to $\mathcal{R}_f^+$ (or rational distribution) if the Laplace transform of its density f is a rational function (ratio of two polynomials), i.e.,*

$$\hat{f}(s) = \frac{(\prod_{i=1}^{n} q_i) + s\,\beta(s)}{\prod_{i=1}^{n}(s + q_i)}, \quad \Re(s) > \max\{-\Re(q_i);\, i = 1, 2, \ldots, n\}, \tag{3.8}$$

where $q_1, q_2, \ldots, q_n$ are in pair of conjugate complex numbers with $\Re(q_i) > 0$ and $\beta(s) = \sum_{i=0}^{n-2} \beta_i\, s^i$ is a polynomial of degree $n-2$ or less.

The $\mathcal{R}_f^+$ is a wide class of distributions, which includes the K_n family, with all the examples above. It also includes damped sine and cosine functions like these:

Example 3.1.7. The distribution with density

$$f(x) = \frac{17}{13} e^{-x} [1 - \sin(4x)]\, I(x \geq 0), \tag{3.9}$$

has a rational Laplace transform $\hat{f}(s) = \frac{17}{13} \frac{(s^2 - 2s + 13)}{(s+1)[(s+1)^2 + 16]}$, therefore belongs to the $\mathcal{R}_f^+$ class.

Example 3.1.8. The distribution with density function

$$f(x) = \frac{(a^2+b^2)R}{(a-R)^2+b^2}\left[e^{-Rx} - e^{-a\,x}\cos(b\,x) + \frac{R-a}{b}e^{-a\,x}\sin(b\,x)\right] I(x \geq 0),$$

where $R > 0, \Re(a) > 0$, and $b \neq 0$, has a rational Laplace transform

$$\hat{f}(s) = \frac{R(a^2+b^2)}{(s+R)[(s+a)^2+b^2]},$$

and therefore belongs to $\mathcal{R}_f^+$ class.

Further discussions of rational distributions can be found in Cox (1955) and Neuts (1981, Chapter 2).

3.2 Claims Number Processes

In this section, we give the definition of a claim number process (renewal process) and a number of associated quantities.

Definition 3.2.1. *Let claim inter-arrival times W_i be i.i.d. with common distribution function K and density function k on $\mathbb{R}^+$, with $\tau_n = \sum_{i=1}^{n} W_i$ being the arrival time of the n-th claim ($\tau_0 = 0$). Then*

$$N(t) = \sup\{n \geq 0\,;\, \tau_n \leq t\}, \qquad t \geq 0, \tag{3.10}$$

is a renewal process $\{N(t)\,;\, t \geq 0\}$ called the claim number process.

The distribution of $N(t)$ for $t \geq 0$ fixed can be expressed as

$$P\{N(t) = n\} = K^{*n}(t) - K^{*(n+1)}(t), \qquad n = 0, 1, \ldots, \tag{3.11}$$

where K^{*n} denotes the n-th convolution of K with itself.

Usually, the distribution of $N(t)$ does not have an explicit formula, except for some special claim inter-arrival times distributions.

Example 3.2.1. If claim inter-arrival times are exponentially distributed with $k(x) = \lambda e^{-\lambda x}$, then $N(t)$ is called a Poisson process, and

$$P\{N(t) = n\} = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \qquad n \in \mathbb{N}. \tag{3.12}$$

Example 3.2.2. If claim inter-arrival times are Erlang(m) distributed with $k(x) = \frac{\lambda^m x^{m-1} e^{-\lambda x}}{(m-1)!}$, $m \in \mathbb{N}^+$, and then

$$P\{N(t) = n\} = \sum_{k=n\,m}^{m(n+1)-1} \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \qquad n \in \mathbb{N}. \tag{3.13}$$

In some situations, it is convenient to convert (3.11) into an equation for the probability generating function of $N(t)$. Let

$$G(t, z) = \sum_{n=0}^{\infty} z^n P\{N(t) = n\} = 1 + \sum_{n=1}^{\infty} z^{n-1}(z-1)K^{*n}(t), \quad -1 < z < 1,$$

be the generating function of $N(t)$. Taking Laplace transforms yields

$$\begin{aligned} \hat{G}(s, z) &= \int_0^{\infty} e^{-s\,t} G(t, z)\, dt = \frac{1}{s} + \frac{1}{s}\sum_{n=1}^{\infty} z^{n-1}(z-1)\widehat{K^{*n}}(s), \quad s \in \mathbb{C}, \\ &= \frac{1 - \hat{k}(s)}{s[1 - z\,\hat{k}(s)]}. \end{aligned} \tag{3.14}$$

It follows from (3.14) that whenever $\hat{k}(s)$ is a rational function, the Laplace transform of the generating function can be expended in partial functions and hence inverted in terms of elementary functions.

Moments, especially the mean value of $N(t)$, play an important role in renewal theory. The renewal function, $m(t)$, defined as $m(t) = E[N(t)]$, for $t \geq 0$, is given by

$$m(t) = \sum_{n=0}^{\infty} nP\{N(t) = n\} = \sum_{n=0}^{\infty} n[K^{*n}(t) - K^{*(n+1)}(t)] = \sum_{n=1}^{\infty} K^{*n}(t). \tag{3.15}$$

The renewal function $m(t)$ also satisfies a proper renewal equation

$$m(t) = K(t) + \int_0^t m(t-x)\, k(x)\, dx, \qquad t \geq 0. \tag{3.16}$$

Taking Laplace transforms on both sides of (3.16) yields

$$\hat{m}(s) = \frac{\hat{k}(s)}{s[1-\hat{k}(s)]}, \qquad s \in \mathbb{C}. \tag{3.17}$$

The Laplace transform in (3.17) can be inverted by partial fractions, if $\hat{k}$ is a rational function.

Example 3.2.3. If $\hat{k}(s) = \frac{\lambda}{(s+\lambda)}$, then $\hat{m}(s) = \frac{\lambda}{s^2}$. Inverting gives $m(t) = \lambda t$.

Example 3.2.4. If claim inter-arrival times are distributed as a mixture of two exponential distributions, with $\hat{k}(s) = \theta\left(\frac{\lambda_1}{s+\lambda_1}\right) + (1-\theta)\left(\frac{\lambda_2}{s+\lambda_2}\right)$, then by partial fractions

$$\hat{m}(s) = \frac{1}{s^2\,\mu} + \frac{1}{s}\frac{(\sigma^2-\mu^2)}{2\,\mu^2} - \frac{\theta(1-\theta)(\lambda_1-\lambda_2)^2}{[\theta\,\lambda_2+(1-\theta)\lambda_1]^2\{s-[\theta\,\lambda_2+(1-\theta)\lambda_1]\}}, \tag{3.18}$$

and hence

$$m(t) = \frac{t}{\mu} + \frac{(\sigma^2-\mu^2)}{2\,\mu^2} - \frac{\theta(1-\theta)(\lambda_1-\lambda_2)^2}{(1-\theta)\lambda_1+\theta\,\lambda_2}e^{-[(1-\theta)\lambda_1+\theta\,\lambda_2]t}, \quad t \geq 0, \tag{3.19}$$

where $\mu = \frac{\theta}{\lambda_1} + \frac{(1-\theta)}{\lambda_2}$ is the mean of the inter-arrival times W and $\sigma^2 = \frac{\theta}{\lambda_1^2} + \frac{(1-\theta)}{\lambda_2^2} + \theta(1-\theta)\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)^2$ is the corresponding variance.

Example 3.2.5. If claim inter-arrival times are Erlang(n) distributed with $\hat{k}(s) = \frac{\lambda^n}{(s+\lambda)^n}$, then

$$\begin{aligned}\hat{m}(s) &= \frac{\hat{k}(s)}{s[1-\hat{k}(s)]} = \frac{\lambda^n}{s[(s+\lambda)^n-\lambda^n]} = \frac{\lambda^n}{s^2\prod_{i=1}^{n-1}(s-r_i)} \\ &= \frac{1}{\mu\,s^2} + \frac{\sigma^2-\mu^2}{2\,\mu^2\,s} + \sum_{i=1}^{n-1}\frac{v_i}{(s-r_i)},\end{aligned} \tag{3.20}$$

where $r_i = \lambda[e^{\frac{2\pi k i}{n}} - 1]$, for $i = 1, 2, \ldots, n-1$ are $n-1$ non-zero roots of $(\lambda+s)^n = \lambda^n$, the mean of Erlang($n$) distribution is $\mu = \frac{n}{\lambda}$ and $\sigma^2 = \frac{n}{\lambda^2}$ is its variance, while $v_i = \frac{\lambda^n}{r_i^2\prod_{j=1,j\neq i}^{n-1}(r_j-r_i)}$. Therefore, by inverting (3.20), $m(t)$ is given by

$$m(t) = \frac{t}{\mu} + \frac{\sigma^2-\mu^2}{2\mu^2} + \sum_{i=1}^{n-1} v_i\,e^{r_i t}, \qquad t \geq 0. \tag{3.21}$$

Specially, if $n = 2$, then $\hat{m}(s) = \frac{\lambda^2}{s^2\,(s+2\,\lambda)}$. Inverting it gives

$$m(t) = \frac{t}{\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2} + \frac{1}{4}e^{-2\,\lambda\,t} = \frac{\lambda\,t}{2} - \frac{1}{4}[1 - e^{-2\lambda\,t}]. \tag{3.22}$$

If $n = 3$, $m(t)$ can be simplified to

$$m(t) = \frac{\lambda\,t}{3} - \frac{1}{3} + \frac{1}{3\sqrt{3}}e^{-\frac{3\,\lambda\,t}{2}}\left[\sin\left(\frac{\sqrt{3}\lambda\,t}{2}\right) + \sqrt{3}\cos\left(\frac{\sqrt{3}\lambda\,t}{2}\right)\right]. \tag{3.23}$$

Next, we discuss how the parameter n of Erlang(n) distribution affects the claim number process, the renewal function, and the ruin probabilities, while keeping constant the mean Erlang waiting times.

For a fixed constant μ, define $W_{(n)}$ to be an Erlang(n; $\lambda = n/\mu$) distributed r.v., with $k(t;n)$ and $K(t;\,n)$ being its density and distribution function, respectively. Also denote by $N_n(t)$ the claim number process with these Erlang($n; n/\mu$) waiting times, and let $m_n(t)$ be the corresponding renewal function. Then we have the following results.

Theorem 3.2.1. *For $n \in \mathbb{N}^+$, $W_{(n+1)}$ is smaller than $W_{(n)}$ under stop loss ordering, i.e.,*

$$W_{(n+1)} <_{sl} W_{(n)}.$$

Proof: Since $k(t;n) = \frac{n^n\,t^{n-1}\,e^{-\frac{n}{\mu}t}}{\mu^n\,(n-1)!}$, then $\frac{k(t;n)}{k(t;n+1)} = \frac{e^{t/\mu}}{t/\mu}\frac{1}{(1+1/n)^{n+1}}$. It is easy to check that there exists two numbers $t_1(n)$ and $t_2(n)$ with $0 < t_1(n) < \mu < t_2(n)$ such that $\frac{k(t;n)}{k(t;n+1)} = 1$. Further, $k(t;n+1) \geq k(t;n)$ on $\big[0, t_1(n)\big] \cup \big[t_2(n), \infty\big)$, and $k(t;n+1) \leq k(t;n)$ on $\big(t_1(n),\, t_2(n)\big)$. That is to say, there are two sign changes in densities, by Theorem 3.1.6 of Goovaerts et al. (1990), $W_{(n+1)} <_{sl} W_{(n)}$. □

Corollary 3.2.1. *For fixed $t > 0$, and $n \in \mathbb{N}^+$,*

$$N_{n+1}(t) <_{sl} N_n(t). \tag{3.24}$$

Proof: Since $P\{N_n(t) < m\} = P\{W_{(n)}^{*m} > t\}$ and $W_{(n+1)}^{*m} <_{sl} W_{(n)}^{*m}$ from Theorem 3.2.1, for any positive integer m, then (3.24) follows. □

Corollary 3.2.2. *The renewal function $m_n(t)$ for Erlang(n) waiting times is decreasing in n, for a constant mean $\mu = n/\lambda$, i.e., $m_n(t) > m_{n+1}(t)$, for a fixed $t > 0$.*

Proof: Since $N_{n+1}(t) <_{sl} N_n(t)$, by definition,

$$E[N_{n+1}(t) - d]_+ \le E[N_n(t) - d]_+, \qquad \text{for any } d \ge 0.$$

Setting $d = 0$, we have

$$m_{n+1}(t) = E[N_{n+1}(t)] \le m_n(t) = E[N_n(t)],$$

for fixed $t > 0$. □

Let $\hat{k}(s;\, n) = \int_0^\infty e^{-st}\, k(t;\, n)\, dt$ be the Laplace transform of the Erlang(n) density, and define R_n to be the adjustment coefficient, that is, $-R_n$ is the unique negative root of the Lundberg's equation $\hat{k}(-cs;\, n)\hat{p}(s) = 1$, or equivalently, of

$$(1 - c\,s/\lambda)^n = \hat{p}(s), \tag{3.25}$$

if p is sufficiently regular. The we have the following result.

Theorem 3.2.2. *The adjustment coefficient R_n is decreasing in n, for a Sparre Andersen risk model with Erlang(n) waiting times, if the mean $n/\lambda = \mu$ is kept constant.*

Proof: Setting $n/\lambda = \mu$ in Lundberg's equation (3.25), simplifies it to $(1-\frac{c\mu}{n}s)^n = \hat{p}(s)$. Since for negative s, $(1 - \frac{c\mu}{n}s)^n$ is increasing in n, then the negative root $-R_n$ is decreasing in n, and R_n is increasing in n. □

The above results show why ruin probabilities are decreasing with n when using Erlang(n) distributions to model claim waiting times, while keeping their mean constant [see the example in Section 8 of Li and Garrido (2004)]. This limitation of the two parameters Erlang(n) distribution is a drawback of this risk model.

3.3 A Class of Renewal Risk Models

Consider a continuous time Sparre Andersen surplus process as in (1.53) by

$$U(t) = u + c\,t - \sum_{i=1}^{N(t)} X_i\,, \quad t \ge 0\,.$$

In this chapter, we assume that the claim number process

$$N(t) = \max\{n : W_1 + W_2 + \cdots + W_n \le t\}$$

is a more general counting process in which the claim waiting times W_i are assumed i.i.d. distributed with distribution function K, and density function k. Denote by $\hat{k}(s) = \int_0^\infty e^{-sx} k(x)\,dx$ the Laplace transform of k.

Let $\tau_k = \sum_{j=1}^{k} W_j$ be the arrival time of the k-th claim. Consider the surplus $U_k = U(\tau_k)$ immediately after the k-th claim. Defining $\tau_0 = 0$ gives $U_0 = u$

$$U_k = U(\tau_k) = u + c\,\tau_k - \sum_{j=1}^{k} X_j = u + \sum_{j=1}^{k} [c\,W_j - X_j]\,, \qquad k \in \mathbb{N}^+.$$

We seek a function v such that the process

$$\{e^{-\delta\,\tau_k}\,v(U_k)\,;\, k \in \mathbb{N}\} \tag{3.26}$$

will form a martingale. Define $\mathcal{F}_0 = \{\emptyset, \Omega\}$, and

$$\mathcal{F}_k = \sigma\{W_1, W_2, \ldots, W_k, X_1, X_2, \ldots, X_k\}, \qquad k \in \mathbb{N}^+,$$

to be a sequence of increasing σ-algebras, representing the information of the surplus process immediately after the k-th claim. Then by definition of a discrete martingale,

$$E\left[e^{-\delta\tau_{k+1}}\,v(U_{k+1})\,\middle|\,\mathcal{F}_k\right] = e^{-\delta\tau_k}\,v(U_k), \qquad k \in \mathbb{N}. \tag{3.27}$$

Equation (3.27) is equivalent to

$$E\left[e^{-\delta(\tau_k + W_{k+1})}\,v\big(U(\tau_k + W_{k+1})\,\big|\,\mathcal{F}_k\right] = e^{-\delta\tau_k}\,v(U_k),$$

or

$$e^{-\delta\tau_k} E\left[e^{-\delta W_{k+1}}\, v\big(U(\tau_k) + (cW_{k+1} - X_{k+1})\big) \,\big|\, \mathcal{F}_k\right] = e^{-\delta\tau_k}\, v(U_k),$$

which can be simplified to

$$E\left[e^{-\delta W_{k+1}}\, v\big(U_k + (cW_{k+1} - X_{k+1})\big) \,\big|\, \mathcal{F}_k\right] = v(U_k), \qquad k \in \mathbb{N}. \tag{3.28}$$

Finally, (3.28) is equivalent to

$$\begin{aligned} v(u) &= E\left[e^{-\delta W}\, v(u + cW - X)\right], & (3.29) \\ &= \int_0^\infty e^{-\delta t}\, k(t)\, E\left[v(u + ct - X)\right] dt. & (3.30) \end{aligned}$$

Equation (3.30) is the sufficient and necessary condition for the process in (3.26) to be a martingale.

By choosing $v(u) = e^{s\,u}$ such that $\{e^{-\delta\,\tau_k + s\,U_k};\ k \in \mathbb{N}\}$ is a martingale, then (3.30) simplifies to

$$\gamma(s) := \frac{1}{\hat{k}(\delta - cs)} = \hat{p}(s), \qquad s \in \mathbb{C}, \tag{3.31}$$

which is a generalized Lundberg fundamental equation.

Now we can claim that $\{e^{-\delta\,\tau_k}\, \phi(U_k);\ k \in \mathbb{N}\}$ is a martingale. To show this, let $D = e^{-\delta T} w\big((U(T^-), |U(T)|\big)\, I(T < \infty)$, and define $M_k = E[D \,|\, \mathcal{F}_k]$, for $k \in \mathbb{N}$. It is easy to prove that $\{M_k;\ k \in \mathbb{N}\}$ is a martingale. Since $P(W_1 < \infty) = 1$, by the optional sampling theorem and renewal property of U_k, one obtains that

$$\begin{aligned} \phi(u) &= E[M_0] = E[M_1] = E[E[D \,|\, \mathcal{F}_1]] = E[e^{-\delta W_1} \phi(U_1)] \\ &= E[e^{-\delta W_1} \phi(u + cW_1 - X_1)] \\ &= \int_0^\infty e^{-\delta t}\, k(t)\, E[\phi(u + ct - X_1)]\, dt. & (3.32) \end{aligned}$$

This shows that $\{e^{-\delta\tau_k}\, \phi(U_k)\,;\ k \in \mathbb{N}\}$ is a martingale.

In the rest of this chapter, we assume that k belongs to the K_n family of distributions, i.e.,

$$\hat{k}(s) = \frac{\lambda^* + s\,\beta(s)}{\prod_{i=1}^n (s + \lambda_i)}, \tag{3.33}$$

where $\lambda^* = \prod_{i=1}^{n} \lambda_i$, $n \in \mathbb{N}^+, \lambda_i > 0$ for $i = 1, 2, \ldots, n$ and $\beta(s) = \sum_{i=0}^{n-2} \beta_i \, s^i$ is a polynomial of degree $n-2$ or less.

In this case, the generalized Lundberg equation (3.31) simplifies to

$$\gamma(s) := \frac{\prod_{i=1}^{n}(\lambda_i + \delta - cs)}{\lambda^* + (\delta - cs)\beta(\delta - cs)} = \hat{p}(s), \qquad s \in \mathbb{C}. \tag{3.34}$$

The following theorem shows that (3.34) has exactly n roots on the right half complex plane. These play an important role in this chapter.

Theorem 3.3.1. For $\delta > 0$ and $n \in \mathbb{N}^+$, Lundberg's equation in (3.34) has only n roots, say $\rho_1(\delta), \rho_2(\delta), \ldots, \rho_n(\delta)$, that a positive real part $\Re(\rho_j) > 0$.

Proof: On the half circle in the complex plane given by $z = r$ (for $r > 0$ fixed) and $\Re(z) \geq 0$, we have that $|\gamma(s)| > 1$, if r is sufficiently large. While for s on the imaginary axis ($\Re(s) = 0$) we have that $|\gamma(s)| \geq \frac{1}{|\hat{k}(\delta - c\,s)|} > 1$. That is, on the contour boundary of the half circle and the imaginary axis, we have $|\gamma(s)| > |\hat{p}(s)|$. Then we conclude that, on the right half plane, the number of the roots to Lundberg's equation equals the number of roots of $\gamma(s) = 0$. Since the latter has exactly n positive roots, we can say that equation (3.34) has exactly n roots with positive real parts, say, $\rho_1(\delta), \rho_2(\delta), \ldots, \rho_n(\delta)$. □

Remarks:

1. Define $l(s) := \hat{p}(s) - \gamma(s)$. Since $l(0) < 0$ and $\lim_{s \to -\infty} l(s) = +\infty$, then for $p(x)$ sufficiently regular, there is one negative root to $l(s) = 0$, say $-R(\delta)$. We call $R(\delta) > 0$ a *generalized adjustment coefficient*.

2. If $\delta \to 0^+$ then $-R(\delta) \to -R(0)$ and $\rho_j(\delta) \to \rho_j(0)$, for $1 \leq j \leq n$, with $\rho_n(0) = 0$, where $-R(0)$ and $\rho_j(0)$ are roots to equation:

$$\gamma_0(s) := \frac{\prod_{i=1}^{n}(\lambda_i - cs)}{\lambda^* - c\,s\,\beta(-cs)} = \hat{p}(s)\,, \qquad s \in \mathbb{C}\,.$$

3. For simplicity, write $-R$ and ρ_j for $-R(\delta)$ and $\rho_j(\delta)$, $1 \leq j \leq n$, when $\delta > 0$.

3.4 Laplace Transforms

In this section, we consider the Laplace transform of any function v such that the process $\{e^{-\delta\tau_k} v(U(\tau_k));\ k \in \mathbb{N}\}$ is a martingale when the claim waiting times are K_n distributed. Then it will lead to the Laplace transform $\hat{\phi}(s) = \int_0^\infty e^{-s\,u}\phi(u)\,du$ of the expected discounted penalty function $\phi(u)$.

By (3.30)

$$v(u) = \int_0^\infty e^{-\delta t} k(t)\, E[v(u+ct-X)]\,dt.$$

Setting $y = u + ct$ yields

$$c\,v(u) = \int_u^\infty e^{-\frac{\delta(y-u)}{c}}\, k(\frac{y-u}{c})\, E[v(y-X)]\,dy. \tag{3.35}$$

Taking Laplace transforms and inverting the order of integration gives

$$\begin{aligned} c\,\hat{v}(s) &= \int_0^\infty e^{-su} \int_u^\infty e^{-\frac{\delta(y-u)}{c}}\, k(\frac{y-u}{c})\, E[v(y-X)]\,dy\,du \\ &= \int_0^\infty e^{-\frac{\delta y}{c}} E[v(y-X)] \int_0^y e^{-(\frac{c\,s-\delta}{c})u}\, k(\frac{y-u}{c})\,du\,dy. \end{aligned} \tag{3.36}$$

First if $\lambda_1, \lambda_2, \ldots, \lambda_n$ in (3.33) are distinct, then by partial fractions,

$$\hat{k}(s) = \frac{\lambda^* + s\,\beta(s)}{\prod_{i=1}^n (s+\lambda_i)} = \sum_{i=1}^n \frac{a_i}{(s+\lambda_i)}, \tag{3.37}$$

with $a_i = \frac{\lambda^* - \lambda_i\,\beta(-\lambda_i)}{\prod_{j=1, j\neq i}^n (\lambda_j - \lambda_i)}$, this gives $k(t) = \sum_{i=1}^n a_i\, e^{-\lambda_i t} I(t \geq 0)$. Then (3.36) becomes

$$\begin{aligned} c\,\hat{v}(s) &= \int_0^\infty e^{-\frac{\delta y}{c}} E[v(y-X)] \sum_{i=1}^n a_i \int_0^y e^{-(\frac{c\,s-\delta}{c})u}\, e^{-\lambda_i(\frac{y-u}{c})}\,du\,dy \\ &= \sum_{i=1}^n a_i \int_0^\infty e^{-(\frac{\delta+\lambda_i}{c})y}\, E[v(y-X)] \int_0^y e^{-(\frac{c\,s-\delta-\lambda_i}{c})u}\,du\,dy \\ &= \sum_{i=1}^n \frac{c\,a_i}{c\,s-\delta-\lambda_i} \int_0^\infty e^{-(\frac{\delta+\lambda_i}{c})y} E[v(y-X)][1-e^{-(\frac{c\,s-\delta-\lambda_i}{c})y}]\,dy \\ &= \sum_{i=1}^n \frac{c\,a_i}{(c\,s-\delta-\lambda_i)} \Big\{\int_0^\infty e^{-(\frac{\delta+\lambda_i}{c})y} E[v(y-X)]\,dy \\ &\qquad - \int_0^\infty e^{-s\,y} E[v(y-X)]\,dy\Big\}. \end{aligned}$$

Equivalently, from (3.37)

$$\hat{v}(s) = \sum_{i=1}^{n} \frac{a_i\, e_i}{(c\,s - \delta - \lambda_i)} + \hat{k}(\delta - c\,s) \int_0^\infty e^{-s\,y} E[v(y - X)]\, dy, \tag{3.38}$$

where $e_i = \int_0^\infty e^{-(\frac{\delta+\lambda_i}{c})y} E[v(y-X)]\, dy$, for $i = 1, 2, \ldots, n$. Since

$$E[v(y - X)] = \int_0^y v(y - x)\, p(x)\, dx + \int_y^\infty v(y - x)\, p(x)\, dx,$$

equation (3.38) reduces to

$$\hat{v}(s) = \frac{\sum_{i=1}^{n} \frac{a_i\, e_i}{(c s - \delta - \lambda_i)} + \hat{k}(\delta - c\,s) \int_0^\infty e^{-s\,y} \int_y^\infty v(y - x) p(x) dx\, dy}{[1 - \hat{k}(\delta - c\,s)\, \hat{p}(s)]}. \tag{3.39}$$

If instead, some λ_i in (3.33) are not distinct, $\hat{k}(s) = \frac{\lambda^* + s\,\beta(s)}{\prod_{i=1}^{k}(s+\lambda_i)^{n_i}}$, where only $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct, and $\sum_{i=1}^{k} n_i = n$, then by partial fractions:

$$\hat{k}(s) = \frac{\lambda^* + s\,\beta(s)}{\prod_{i=1}^{k}(s + \lambda_i)^{n_i}} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{a_{i\,j}}{(s + \lambda_i)^j},$$

and hence

$$k(t) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} a_{i\,j} \frac{t^{j-1}\, e^{-\lambda_i\, t}}{(j-1)!},$$

where

$$a_{i\,j} = \frac{1}{(n_i - j)!} \frac{d^{n_i - j}}{ds^{n_i - j}} \prod_{m=1,\, m \neq i}^{k} \frac{\lambda^* + s\,\beta(s)}{(s + \lambda_m)^{n_m}} \Bigg|_{s=-\lambda_i}, \qquad s \in \mathbb{C}\,.$$

By a similar argument

$$\begin{aligned} \hat{v}(s) \;=\; & \frac{-\sum_{i=1}^{k} \sum_{j=1}^{n_i} \sum_{m=0}^{j-1} \frac{a_{i\,j}\; e_{i\,m}}{c^m\, m!(\delta+\lambda_i - c\,s)^{j-m}}}{[1 - \hat{k}(\delta - c\,s)\, \hat{p}(s)]} \\ & + \frac{\hat{k}(\delta - c\,s) \int_0^\infty e^{-s\,y} \int_y^\infty v(y - x) p(x)\, dx\, dy}{[1 - \hat{k}(\delta - c\,s)\, \hat{p}(s)]}, \end{aligned} \tag{3.40}$$

where $e_{i\,m} = \int_0^\infty y^m\, e^{-(\frac{\delta+\lambda_i}{c})y}\, E[v(y - X)]\, dy$, for $i = 1, 2, \ldots, k < n$

The following theorem shows that in both cases above, an explicit expression for $\hat{v}(s)$ can be obtained if v is chosen to be ϕ.

Theorem 3.4.1. If the density function k is a K_n distribution with $\hat{k}(s)$ being of the form in (3.33), then the Laplace transform of ϕ is given by

$$\hat{\phi}(s) = \frac{\hat{\omega}(s) - \left[\frac{q(s)}{\lambda^* + (\delta - c\,s)\beta(\delta - c\,s)}\right]}{[\gamma(s) - \hat{p}(s)]}, \tag{3.41}$$

where $\hat{\omega}(s)$ is the Laplace transform of $\omega(y) = \int_y^\infty w(y, x-y)\, p(x)\, dx$, as in (3.31), $\gamma(s)$ is given by (3.34), and $q(s)$ is a polynomial of degree $n-1$ or less, which is determined by the conditions:

$$q(\rho_j) = \hat{\omega}(\rho_j)\,[\lambda^* + (\delta - c\,\rho_j)\beta(\delta - c\,\rho_j)], \quad j = 1, 2, \ldots, n. \tag{3.42}$$

Further, if $\rho_1, \rho_2, \ldots, \rho_n$ are distinct, then

$$q(s) = \sum_{j=1}^{n} \left\{ \hat{\omega}(\rho_j)\,[\lambda^* + (\delta - c\rho_j)\beta(\delta - c\rho_j)] \left[\prod_{k=1, k\neq j}^{n} \frac{(s - \rho_k)}{(\rho_j - \rho_k)} \right] \right\}. \tag{3.43}$$

Proof: For simplicity, define $\omega(y) = \int_y^\infty w(y, x-y)\, p(x)\, dx$, then by the definition of ϕ, $E[\phi(y - X)] = \int_0^y \phi(y-x)\, p(x)\, dx + \omega(y)$. It follows that

1. If $\hat{k}(s)$ is given by (3.33) where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct, (3.39) simplifies to

$$\hat{\phi}(s) = \frac{-\sum_{i=1}^{n}\left[\frac{a_i\, e_i}{\delta + \lambda_i - c\,s}\right] + \hat{k}(\delta - c\,s)\,\hat{\omega}(s)}{[1 - \hat{k}(\delta - c\,s)\,\hat{p}(s)]}. \tag{3.44}$$

Multiplying both sides by $\gamma(s) = \frac{1}{\hat{k}(\delta - cs)}$ yields (3.41), with

$$q(s) = \left[\prod_{i=1}^{n} (\delta + \lambda_i - c\,s)\right] \left[\sum_{i=1}^{n} \frac{a_i\, e_i}{\delta + \lambda_i - c\,s}\right],$$

being a polynomial of degree $n-1$ or less. Since $\hat{\phi}(s)$ is finite for all s with $\Re(s) > 0$, and we note that ρ_j with $\Re(\rho_j) > 0, j = 1, 2, \ldots, n$ are zeros of the denominator of (3.41), then they must also be the zeros of the numerator, that is to say, condition (3.42) holds. Further, if $\rho_1, \rho_2, \ldots, \rho_n$ are distinct, then by the *Lagrange interpolation formula*, one obtains formula (3.43).

2. If $\hat{k}(s) = \frac{\lambda^* + s\,\beta(s)}{\prod_{i=1}^{k}(s+\lambda_i)^{n_i}}$, with $\sum_{i=1}^{k} n_i = n$, then the Laplace transform of ϕ can be obtained by (3.40):

$$\hat{\phi}(s) = \frac{-\sum_{i=1}^{k}\sum_{j=1}^{n_i}\sum_{m=0}^{j-1}[\frac{a_{ij}\,e_{im}}{c^m\,m!(\delta+\lambda_i-c\,s)^{j-m}}] + \hat{k}(\delta - c\,s)\,\hat{\omega}(s)}{[1-\hat{k}(\delta-c\,s)\,\hat{p}(s)]}. \tag{3.45}$$

Again, multiplying both sides by $\gamma(s) = \frac{1}{\hat{k}(\delta-cs)}$ gives (3.41), but this time with

$$q(s) = \left[\prod_{i=1}^{k}(\delta+\lambda_i - c\,s)^{n_i}\right]\left[\sum_{i=1}^{k}\sum_{j=1}^{n_i}\sum_{m=0}^{j-1}\frac{a_{ij}\,e_{im}}{c^m\,m!(\delta+\lambda_i-c\,s)^{j-m}}\right],$$

a polynomial of degree $n-1$ or less, which can also be determined by the conditions in (3.42), and determined explicitly by (3.43) if $\rho_1, \rho_2, \ldots, \rho_n$ are distinct.

This completes the proof. □

Remarks: If $\hat{k}(s) = \frac{\lambda^*}{\prod_{i=1}^{n}(s+\lambda_i)}$, that is, p is a generalized Erlang(n) distributed, then equation (3.41) simplifies to formula (7.4) in Gerber and Shiu (2003b). Moreover, for $n = 1$, this formula can be found in the discussion by D.C.M. Dickson of Gerber and Shiu (1998a).

In the evaluation of the expected discounted penalty function, usually an integro–differential equation satisfied by the expected discounted penalty function is first derived and then solves into a defective renewal equation. See Gerber and Shiu (1998a), Dickson (1998), Dickson and Hipp (1998, 2001), Cheng and Tang (2003), Li and Garrido (2004) and Gerber and Shiu (2003a,b). Here it should be pointed out that when $\beta(s) \not\equiv 0$, integro-differential equations do not exist for the expected discounted penalty function $\phi(u)$. Since otherwise there would exist a polynomial $h_m(s)$ of degree m such that $h_m(\mathcal{D})\phi(u) = \int_0^u \phi(u-x)\,p(x)\,dx + \omega(u)$, for $u \geq 0$, where $\mathcal{D}$ is a differentiation operator. Then taking Laplace transforms would give $\hat{\phi}(s) = \frac{[\hat{\omega}(s)-g_{m-1}(s)]}{[h_m(s)-\hat{p}(s)]}$, with $g_{m-1}(s)$ being a polynomial of degree $m-1$ or less. Theorem 3.4.1 shows that $h_m(s) = \gamma(s)$. When $\beta(s) \not\equiv 0$, $\gamma(s)$ is not a polynomial, contradicting the assumption that $h_m(s)$ is a polynomial of degree m.

3.5 Analysis when $u = 0$

We now turn to solving on ruin related problems when $u = 0$. For simplicity, we assume that $\rho_1, \rho_2, \ldots, \rho_n$ in Theorem 3.3.1 are distinct. First by applying the *initial value theorem,*

$$\begin{aligned}
\phi(0) &= \lim_{s\to\infty} s\,\hat{\phi}(s) = \lim_{s\to\infty} s \frac{\hat{\omega}(s) - \frac{q(s)}{[\lambda^*+(\delta-c\,s)\beta(\delta-c\,s)]}}{[\gamma(s)-\hat{p}(s)]} \\
&= \lim_{s\to\infty} \frac{\hat{\omega}(s) - \frac{\sum_{j=1}^{n}\left\{\hat{\omega}(\rho_j)\left[\lambda^*+(\delta-c\rho_j)\beta(\delta-c\rho_j)\right]\prod_{k=1,k\neq j}^{n}\frac{(s-\rho_k)}{(\rho_j-\rho_k)}\right\}}{\lambda^*+(\delta-c\,s)\beta(\delta-c\,s)}}{\left[\frac{\prod_{i=1}^{n}(\delta+\lambda_i-c\,s)}{s[\lambda^*+(\delta-c\,s)\beta(\delta-c\,s)]} - \frac{\hat{p}(s)}{s}\right]} \\
&= \frac{-\sum_{j=1}^{n}\left\{\hat{\omega}(\rho_j)\left[\lambda^*+(\delta-c\rho_j)\beta(\delta-c\rho_j)\right]\prod_{k=1,k\neq j}^{n}\frac{1}{\rho_j-\rho_k}\right\}}{(-c)^n} \\
&= \sum_{j=1}^{n}\hat{\omega}(\rho_j)\left[\frac{\lambda^*+(\delta-c\rho_j)\beta(\delta-c\rho_j)}{c^n\prod_{k=1,k\neq j}^{n}(\rho_k-\rho_j)}\right]. \qquad (3.46)
\end{aligned}$$

When $\beta(s) \equiv 0$, this is formula (8.1) in Gerber and Shiu (2003b).

Since $\omega(x) = \int_x^\infty w(x, y-x)p(y)dy = \int_0^\infty w(x,y)p(x+y)dy$, its Laplace transform is $\hat{\omega}(s) = \int_0^\infty\int_0^\infty e^{-sx}w(x,y)p(x+y)dxdy$, then $\phi(0)$ can be rewritten as

$$\phi(0) = \sum_{j=1}^{n}\frac{\lambda^*+(\delta-c\rho_j)\beta(\delta-c\rho_j)}{c^n\prod_{k=1,k\neq j}^{n}(\rho_k-\rho_j)}\int_0^\infty\int_0^\infty e^{-\rho_j x}w(x,y)\,p(x+y)dxdy.$$

On the other hand,

$$\begin{aligned}
\phi(0) &= E\left[e^{-\delta T}w(U(T^-),|U(T)|)\,I(T<\infty)\,\middle|\,U(0)=0\right] \\
&= \int_0^\infty\int_0^\infty w(x,y)\,f_2(x,y|0)\,dy\,dx\ ,
\end{aligned}$$

where f_2 is given in (1.19). Comparing these two formulas for $\phi(0)$ yields

$$f_2(x,y\,|\,0) = \sum_{j=1}^{n}\left[\frac{\lambda^*+(\delta-c\rho_j)\beta(\delta-c\rho_j)}{c^n\prod_{k=1,k\neq j}^{n}(\rho_k-\rho_j)}\right]e^{-\rho_j x}\,p(x+y), \qquad (3.47)$$

then

$$f_1(x\,|\,0) = \int_0^\infty f_2(x,y\,|\,0)\,dy = \sum_{j=1}^n \left[\frac{\lambda^* + (\delta - c\rho_j)\beta(\delta - c\rho_j)}{c^n \prod_{k=1,k\neq j}^n (\rho_k - \rho_j)}\right] e^{-\rho_j x}\bar{P}(x), \tag{3.48}$$

and

$$\begin{aligned} g(y) := g(y\,|\,0) &= \int_0^\infty f_2(x,y|0)\,dx \\ &= \sum_{j=1}^n \left[\frac{\lambda^* + (\delta - c\rho_j)\beta(\delta - c\rho_j)}{c^n \prod_{k=1,k\neq j}^n (\rho_k - \rho_j)}\right] \int_0^\infty e^{-\rho_j x} p(x+y)dx, \\ &= \sum_{j=1}^n \left[\frac{\lambda^* + (\delta - c\rho_j)\beta(\delta - c\rho_j)}{c^n \prod_{k=1,k\neq j}^n (\rho_k - \rho_j)}\right] T_{\rho_j}\, p(y), \end{aligned} \tag{3.49}$$

where T_r is an operator defined in (2.1) as

$$T_r\, p(x) = \int_x^\infty e^{-r(y-x)}\, p(y)\, dy = \int_0^\infty e^{-rx}\, p(x+y)\, dx.$$

The function $g(y)$ is a defective density which plays a very important role in this thesis. Its Laplace transform $\hat{g}(s) := \int_0^\infty e^{-s\,y}\, g(y)\, dy$ is given by

$$\begin{aligned} \hat{g}(s) &= \int_0^\infty e^{-s\,y}\, g(y\,|\,0)\, dy = T_s\, g(0) \\ &= \sum_{j=1}^n \left[\frac{\lambda^* + (\delta - c\rho_j)\beta(\delta - c\rho_j)}{c^n \prod_{k=1,k\neq j}^n (\rho_k - \rho_j)}\right] T_s\, T_{\rho_j}\, p(0) \\ &= \sum_{j=1}^n \left[\frac{\lambda^* + (\delta - c\rho_j)\beta(\delta - c\rho_j)}{c^n \prod_{k=1,k\neq j}^n (\rho_k - \rho_j)}\right] \left[\frac{\hat{p}(\rho_j) - \hat{p}(s)}{s - \rho_j}\right] \\ &= \sum_{j=1}^n \left[\frac{\prod_{i=1}^n (\delta + \lambda_i - c\rho_j)}{c^n\,(s-\rho_j) \prod_{k=1,k\neq j}^n (\rho_k - \rho_j)}\right] \\ &\qquad - \hat{p}(s) \sum_{j=1}^n \left[\frac{\lambda^* + (\delta - c\rho_j)\beta(\delta - c\rho_j)}{c^n\,(s-\rho_j) \prod_{k=1,k\neq j}^n (\rho_k - \rho_j)}\right] \\ &= \sum_{j=1}^n \left[\frac{\prod_{i=1}^n [(\delta + \lambda_i - c\,s) + c(s-\rho_j)]}{c^n\,(s-\rho_j) \prod_{k=1,k\neq j}^n (\rho_k - \rho_j)}\right] \\ &\qquad - \hat{p}(s) \sum_{j=1}^n \left[\frac{\lambda^* + \sum_{m=0}^{n-2} \beta_m [(\delta - c\,s + c(s-\rho_j)]^{m+1}}{c^n\,(s-\rho_j) \prod_{k=1,k\neq j}^n (\rho_k - \rho_j)}\right], \end{aligned}$$

where $\beta_0, \beta_1, \ldots, \beta_{n-2}$ are the coefficents of the polynomial $\beta(s)$ in (3.1).

Since

$$\prod_{i=1}^{n}[(\delta+\lambda_i-c\,s)+c(s-\rho_j)]=\sum_{l=0}^{n}\sigma_l\,c^{n-l}(s-\rho_j)^{n-l},$$

with

$$\sigma_0=1,\quad \sigma_1=\sum_{i=1}^{n}(\delta+\lambda_i-c\,s),$$

$$\sigma_2=\sum_{1\le i<j\le n}(\delta+\lambda_i-c\,s)(\delta+\lambda_j-c\,s),\;\ldots,\sigma_n=\prod_{i=1}^{n}(\delta+\lambda_i-c\,s).$$

then the first term simplifies to

$$\sum_{j=1}^{n}\left[\frac{\prod_{i=1}^{n}[(\delta+\lambda_i-c\,s)+c(s-\rho_j)]}{c^n\,(s-\rho_j)\prod_{k=1,k\neq j}^{n}(\rho_k-\rho_j)}\right]=\sum_{l=0}^{n}\sum_{j=1}^{n}\frac{\sigma_l\,c^{n-l}\,(s-\rho_j)^{n-l}}{c^n\,(s-\rho_j)\prod_{k=1,k\neq j}^{n}(\rho_k-\rho_j)}$$
$$=1-\frac{\prod_{i=1}^{n}(\delta+\lambda_i-c\,s)}{c^n\prod_{i=1}^{n}(\rho_i-s)},\tag{3.50}$$

where equation (3.50) follows from the following identities in interpolation theory (for $n\ge 2$):

$$\sum_{j=1}^{n}\frac{(s-\rho_j)^m}{\prod_{k=1,k\neq j}^{n}(\rho_k-\rho_j)}=\begin{cases}1, & m=n-1,\\ 0, & m=0,1,2,\ldots,n-2,\\ -\frac{1}{\prod_{i=1}^{n}(\rho_i-s)}, & m=-1.\end{cases}\tag{3.51}$$

The above identities can be proved by divided differences. Similarly,

$$\hat{p}(s)\sum_{j=1}^{n}\left[\frac{\lambda^*+\sum_{m=0}^{n-2}\beta_m[(\delta-c\,s+c(s-\rho_j))]^{m+1}}{c^n\,(s-\rho_j)\prod_{k=1,k\neq j}^{n}(\rho_k-\rho_j)}\right]$$
$$=-\hat{p}(s)\left[\frac{\lambda^*+\sum_{m=0}^{n-2}\beta_m(\delta-c\,s)^{m+1}}{c^n\prod_{i=1}^{n}(\rho_i-s)}\right]=-\hat{p}(s)\left[\frac{\lambda^*+(\delta-c\,s)\,\beta(\delta-c\,s)}{c^n\prod_{i=1}^{n}(\rho_i-s)}\right].$$

Finally, the Laplace transform of g is

$$\hat{g}(s)=1-\left[\frac{\prod_{i=1}^{n}(\delta+\lambda_i-c\,s)-\hat{p}(s)[\lambda^*+(\delta-c\,s)\,\beta(\delta-c\,s)]}{c^n\prod_{i=1}^{n}(\rho_i-s)}\right].\tag{3.52}$$

Setting $w(x,y) = 1$ implies

$$\begin{aligned}
\phi(0) &= E\left[e^{-\delta T} I(T<\infty) \mid U(0)=0\right] = \int_0^\infty g(y\,|\,0)dy = \lim_{s\to 0} \hat{g}(s) \\
&= 1 - \left[\frac{\prod_{i=1}^n(\lambda_i+\delta) - [\lambda^* + \delta\,\beta(\delta)]}{c^n \rho_1\rho_2\cdots\rho_n}\right] \\
&= 1 - \frac{[1-\hat{k}(\delta)]\prod_{i=1}^n(\lambda_i+\delta)}{c^n\rho_1\rho_2\cdots\rho_n} < 1. \qquad (3.53)
\end{aligned}$$

Finally,

$$\begin{aligned}
\Psi(0) &= \lim_{\delta\to 0} E\left[e^{-\delta T} I(T<\infty) \mid U(0)=0\right] \\
&= \lim_{\delta\to 0}\left[1 - \frac{\prod_{i=1}^n(\lambda_i+\delta)-\lambda^*}{c^n\rho_1\rho_2\cdots\rho_n} + \frac{\sum_{m=0}^{n-2}\beta_m\,\delta^{m+1}}{c^n\rho_1\rho_2\cdots\rho_n}\right] \\
&= 1 - \lim_{\delta\to 0}\left[\frac{\frac{\prod_{i=1}^n(\lambda_i+\delta)-\lambda^*}{\delta}}{\rho^*(0)\,\rho_n'(0)}\right] + \frac{\beta_0}{\rho^*(0)\,\rho_n'(0)} \\
&= 1 - \frac{\lambda^*(\sum_{i=1}^n \frac{1}{\lambda_i}) - \beta_0}{\rho^*(0)\,\rho_n'(0)} \\
&= 1 - \frac{\lambda^*[cE(W)-E(X)]}{\rho^*(0)} < 1, \qquad (3.54)
\end{aligned}$$

where $\rho^*(0) = \prod_{i=1}^{n-1}\rho_i(0)$. The last step follows from the fact that $E(W) = -\hat{k}'(0) = \frac{[\lambda^*(\sum_{i=1}^n \frac{1}{\lambda_i})-\beta_0]}{\lambda^*}$, and that $\rho_n'(0) = \frac{E(W)}{[c\,E(W)-E(X)]}$ by differentiating with respect to δ on both sides of $\frac{1}{\hat{k}[\delta - c\rho_n(\delta)]} = \hat{p}[\rho_n(\delta)]$, letting $\delta\to 0$ and noting that $\lim_{\delta\to 0}\rho_n(\delta)\to 0$.

Remarks:

- When $\beta(s)\equiv 0$, i.e., all coefficents $\beta_m = 0$, for $m = 0, 1, \ldots, n-2$, then (3.53) simplifies to (11.5) of Gerber and Shiu (2003b).
- (3.54) simplifies to formula (3.10) of Willmot (1999) if $c = 1$.

3.6 Defective Renewal Equation

3.6.1 General Case

By arguments similar to those in Gerber and Shiu (1998a), we condition on the first time when the surplus falls below the initial level u,

$$\begin{aligned}
\phi(u) &= \int_0^u \int_0^\infty \int_0^\infty e^{-\delta t}\,\phi(u-y)\,f_3(x,y,t\,|\,0)\,dt\,dx\,dy, \qquad u \ge 0, \\
&\quad + \int_u^\infty \int_0^\infty \int_0^\infty e^{-\delta t}\,w(x+u,y-u)\,f_3(x,y,t\,|\,0)\,dt\,dx\,dy \\
&= \int_0^u \int_0^\infty \phi(u-y)\,f_2(x,y\,|\,0)\,dx\,dy + \int_u^\infty \int_0^\infty w(x+u,y-u)\,f_2(x,y\,|\,0)\,dx\,dy \\
&= \int_0^u \phi(u-y)\,g(y)\,dy + H(u), \qquad (3.55)
\end{aligned}$$

where

$$\begin{aligned}
H(u) &= \int_u^\infty \int_0^\infty w(x+u,y-u)\,f_2(x,y\,|\,0)\,dx\,dy, \qquad u \ge 0, \\
&= \int_0^\infty \int_u^\infty w(s,t)\,f_2(s-u,t+u|0)\,ds\,dt \\
&= \sum_{j=1}^n \left[\frac{\lambda^* + (\delta - c\rho_j)\beta(\delta - c\rho_j)}{c^n \prod_{k=1,k\neq j}^n (\rho_k - \rho_j)}\right] \int_u^\infty e^{-\rho_j(s-u)} \int_0^\infty w(s,t)\,f(s+t)\,dt\,ds \\
&= \sum_{j=1}^n \left[\frac{\lambda^* + (\delta - c\rho_j)\beta(\delta - c\rho_j)}{c^n \prod_{k=1,k\neq j}^n (\rho_k - \rho_j)}\right] T_{\rho_j}\omega(u).
\end{aligned}$$

Since $\int_0^\infty g(y)\,dy < 1$, equation (3.55) is a defective renewal equation, specially if $\beta(s) \equiv 0$, it simplifies to (9.2) in Gerber and Shiu (2003b).

Setting $w(x,y) = 1$, we then obtain $\omega(u) = \bar{P}(u) = T_0\,p(u)$, and,

$$H(u) = \sum_{j=1}^n \left[\frac{\lambda^* + (\delta - c\rho_j)\beta(\delta - c\rho_j)}{c^n \prod_{k=1,k\neq j}^n (\rho_k - \rho_j)}\right] T_{\rho_j} T_0\,p(u) = T_0\,g(u) = \int_u^\infty g(y)\,dy,$$

therefore, the Laplace transform of T, $\phi_T(u) := E[e^{-\delta T} I(T < \infty)\,|\,U(0) = u]$, satisfies the following defective renewal equation:

$$\phi_T(u) = \int_0^u \phi_T(u-y)\,g(y)\,dy + \int_u^\infty g(y)\,dy, \qquad u \ge 0, \qquad (3.56)$$

further if $\delta = 0$, (3.56) gives

$$\Psi(u) = \int_0^u \Psi(u-y)\, g_0(y)\, dy + \int_u^\infty g_0(y)\, dy, \qquad u \geq 0, \tag{3.57}$$

where $g_0(y)$ can be obtained by taking limits. Since $\lim_{\delta\to 0} \rho_i(\delta) \to \rho_i(0)$, and $\lim_{\delta\to 0} \rho_n(\delta) = \rho_n(0) = 0$ then

$$\begin{aligned}
g_0(y) &= \lim_{\delta\to 0} g(y) = \lim_{\delta\to 0} \sum_{j=1}^{n} \left[\frac{\lambda^* + (\delta - c\rho_j)\beta(\delta - c\rho_j)}{c^n \prod_{k=1,k\neq j}^{n} (\rho_k - \rho_j)} \right] T_{\rho_j}\, p(y) \\
&= \sum_{j=1}^{n-1} \left[\frac{\lambda^* - c\,\rho_j(0)\beta(-c\rho_j(0))}{c^n\,[-\rho_j(0)] \prod_{k=1,k\neq j}^{n-1} [\rho_k(0) - \rho_j(0)]} \right] T_{\rho_j(0)}\, p(y) + \frac{\lambda^*\, T_0\, p(y)}{c^n\, \rho^*(0)} \\
&= -\frac{\lambda^*}{c^n} \sum_{j=1}^{n-1} \frac{T_{\rho_j(0)}\, p(y)}{\rho_j(0) \prod_{k=1,k\neq j}^{n-1} [\rho_k(0) - \rho_j(0)]} + \frac{\lambda^*\, T_0\, p(y)}{c^n\, \rho^*(0)} \\
&\quad + \frac{1}{c^{n-1}} \sum_{j=1}^{n-1} \frac{\beta(-c\,\rho_j)\, T_{\rho_j(0)}\, p(y)}{\prod_{k=1,k\neq j}^{n-1} [\rho_k(0) - \rho_j(0)]} \\
&= \frac{\lambda^*}{c^n} \sum_{j=1}^{n-1} \frac{\left[\frac{T_0\, p(y) - T_{\rho_j(0)}\, p(y)}{\rho_j(0)}\right]}{\prod_{k=1,k\neq j}^{n-1} [\rho_k(0) - \rho_j(0)]} + \frac{1}{c^{n-1}} \sum_{j=1}^{n-1} \frac{\beta(-c\,\rho_j)\, T_{\rho_j(0)}\, p(y)}{\prod_{k=1,k\neq j}^{n-1} [\rho_k(0) - \rho_j(0)]} \\
&= \frac{\lambda^*}{c^n} \sum_{j=1}^{n-1} \frac{T_{\rho_j(0)}\, \bar{P}(y)}{\prod_{k=1,k\neq j}^{n-1} [\rho_k(0) - \rho_j(0)]} + \frac{1}{c^{n-1}} \sum_{j=1}^{n-1} \frac{\beta(-c\,\rho_j)\, T_{\rho_j(0)}\, p(y)}{\prod_{k=1,k\neq j}^{n-1} [\rho_k(0) - \rho_j(0)]},
\end{aligned}$$

where the second last step follows from (3.51). Note that $T_{\rho_j(0)} T_0\, p(y) = T_{\rho_j(0)}\, \bar{P}(y)$, while $\int_0^\infty g_0(y)\, dy = \Psi(0) < 1$ is given by (3.54).

We remark that the Laplace transform of $g_0(y)$ for $c = 1$, is given by (3.11) in Willmot (1999), but the transform inversion is rather complicated, except for some special choices of β.

3.6.2 Some Subclasses

Now turn to special subclasses of distributions for two different choices of β. Alternatively, other subclasses may be considered.

1. $\beta(s) \equiv 0$ (generalized Erlang(n) distribution).

In this case, the waiting time distribution is the sum of n exponential distributions with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ and is called a generalized Erlang(n). In particular, if $\lambda_i = \lambda > 0$, for all $i = 1, 2, \ldots, n$, then it is the Erlang(n) distribution. In general,

$$\begin{aligned} H(u) &= \frac{\lambda^*}{c^n} \sum_{j=1}^{n} \frac{T_{\rho_j}\, \omega(u)}{\prod_{k=1,k\neq j}^{n}(\rho_k - \rho_j)}, \\ g(y) &= \frac{\lambda^*}{c^n} \sum_{j=1}^{n} \frac{T_{\rho_j}\, p(y)}{\prod_{k=1,k\neq j}^{n}(\rho_k - \rho_j)}, \\ g_0(y) &= \frac{\lambda^*}{c^n} \sum_{j=1}^{n-1} \frac{T_{\rho_j(0)}\, \bar{P}(y)}{\prod_{k=1,k\neq j}^{n-1}[\rho_k(0) - \rho_j(0)]}. \end{aligned}$$

The above equations can be found in Gerber and Shiu (2003b), and Li and Garrido (2004) for the $\lambda_i = \lambda$ case.

2. $\beta(s) = \beta$ (mixed exponentials). Then

$$\begin{aligned} H(u) &= \sum_{j=1}^{n} \left[\frac{(\lambda^* + \beta\,\delta) - c\,\beta\,\rho_j}{c^n \prod_{k=1,k\neq j}^{n}(\rho_k - \rho_j)} \right] T_{\rho_j}\, \omega(u), \\ g(y) &= \sum_{j=1}^{n} \left[\frac{(\lambda^* + \beta\,\delta) - c\,\beta\,\rho_j}{c^n \prod_{k=1,k\neq j}^{n}(\rho_k - \rho_j)} \right] T_{\rho_j}\, p(y), \\ g_0(y) &= \frac{\lambda^*}{c^n} \sum_{j=1}^{n-1} \frac{T_{\rho_j(0)} \bar{P}(y)}{\prod_{k=1,k\neq j}^{n-1}[\rho_k(0) - \rho_j(0)]} \\ &\quad + \frac{\beta}{c^{n-1}} \sum_{j=1}^{n-1} \frac{T_{\rho_j(0)}\, p(y)}{\prod_{k=1,k\neq j}^{n-1}[\rho_k(0) - \rho_j(0)]}. \end{aligned} \tag{3.58}$$

Further, if $n = 2$ and the density function k is the mixture of two exponential distributions, $k(x) = \theta\lambda_1 e^{-\lambda_1 x} + (1-\theta)\lambda_2 e^{-\lambda_2 x}$, then $\beta = \theta\lambda_1 + (1-\theta)\lambda_2$ and

$$g(y) = \left(\frac{\lambda_1 + \lambda_2 + \beta\,\delta - c\,\beta\rho_2}{c^2}\right) T_{\rho_2} T_{\rho_1}\, p(y) + \frac{\beta}{c}\, T_{\rho_1}\, p(y), \tag{3.59}$$

$$g_0(y) = \frac{\lambda_1\lambda_2}{c^2}\, T_{\rho_1(0)} \bar{P}(y) + \frac{\beta}{c}\, T_{\rho_1(0)}\, p(y). \tag{3.60}$$

3.7 Discounted Distributions of Surplus Before Ruin and Deficit at Ruin

In this section, we consider the discounted joint and marginal distributions of $U(T^-)$ and $|U(T)|$ using the defective renewal equation (3.55).

Theorem 3.7.1. For $x \geq 0$, $y \geq 0$, and $u \geq 0$,

$$f_2(x,y|u) = \int_0^u f_2(x,y|u-z)\, g(z)\, dz + I(u < x) f_2(x-u, u+y\,|\,0), \qquad (3.61)$$

where $f_2(x-u, y+u\,|\,0)$ can be derived by (3.47).

Proof: Setting $w(x_1, x_2) = I(x_1 = x, y_1 = y)$, then $\phi(u)$ in (3.55) simplifies to $f_2(x,y|u)$. □

Next, a closed form for the discounted marginal density $f_1(x|u)$ is obtained by inverting the discounted Laplace transform of $U(T^-)$. For notational convenience, let ξ_δ be such that $\frac{1}{1+\xi_\delta} = \int_0^\infty g(y)\, dy = \phi_T(0)$, and ξ_0 be such that $\frac{1}{1+\xi_0} = \int_0^\infty g_0(y)\, dy = \Psi(0)$.

Theorem 3.7.2.

$$f_1(x|u) = \begin{cases} \frac{1+\xi_\delta}{\xi_\delta} \sum_{j=1}^n b_j\, e^{-\rho_j x} \bar{P}(x) [e^{\rho_j x} \Psi_j(u-x) - \Psi_j(u)], & 0 \leq x < u \\ \frac{1+\xi_\delta}{\xi_\delta} \sum_{j=1}^n b_j\, e^{-\rho_j x} \bar{P}(x) [e^{\rho_j u} - \Psi_j(u)], & x \geq u \end{cases}, \qquad (3.62)$$

where $b_j = \frac{\lambda^* + (\delta - c\rho_j)\beta(\delta - c\rho_j)}{c^n \prod_{k=1, k\neq j}^n (\rho_k - \rho_j)}$, and $\Psi_j(u) := \phi_T(u) + \int_0^u \phi_T(u-t) \rho_j\, e^{\rho_j t}\, dt$.

Proof: If $w(x,y) = e^{-s\,x}$, then $\phi(u) = E[e^{-\delta T} e^{-s\,U(T^-)} I(T < \infty)\,|\,U(0) = u]$ is the discounted Laplace transform of $U(T^-)$ at s. Then $w(x) = \int_x^\infty w(x, y-x)\, p(y)\, dy = e^{-sx}\, \bar{P}(x)$ and hence by (3.55) we have:

$$\phi(u) = \frac{1}{(1+\xi_\delta)} \int_0^u \phi(u-y)[(1+\xi_\delta)\, g(y)]\, dy + \frac{1}{(1+\xi_\delta)}[(1+\xi_\delta) H(u)]\,,$$

where $H(u) = \sum_{j=1}^n b_j\, e^{\rho_k u} \int_u^\infty e^{-(\rho_k + s)\,x}\, \bar{P}(x)\, dx$, and $b_1, b_2, \ldots, b_n$ are defined as above. Using Theorem 1.1 of Lin and Willmot (1999), we have

$$\phi(u) = \frac{1}{\xi_\delta} \int_0^u [1 - \phi_T(u-x)]\, dH(x) + \frac{H(0)}{\xi_\delta}[1 - \phi_T(u)]\,, \qquad u \geq 0.$$

Substituting the expression of $H(u)$, we obtain

$$\begin{aligned}\phi(u) &= \frac{(1+\xi_\delta)}{\xi_\delta}\left\{\sum_{j=1}^n b_j \int_0^u [1-\phi_T(u-x)]\,\rho_j\, e^{\rho_j\, x} \int_x^\infty e^{-(\rho_j+s)\,t}\bar{P}(t)\,dt\,dx\right.\\ &\quad -\sum_{j=1}^n b_j \int_0^u [1-\phi_T(u-x)]\,e^{-sx}\,\bar{P}(x)dx\\ &\quad \left.+\sum_{j=1}^n b_j \int_0^\infty e^{-sx}\,e^{-\rho_j\, x}\,\bar{P}(x)\,dx\,[1-\phi_T(u)]\right\}.\end{aligned}$$

Changing the order of integration yields

$$\begin{aligned}\phi(u) &= \frac{(1+\xi_\delta)}{\xi_\delta}\left\{\sum_{j=1}^n b_j \int_0^u e^{-sx}\,e^{-\rho_j\, x}\,\bar{P}(x)\int_0^x [1-\phi_T(u-t)]\,\rho_j\, e^{\rho_j\, t}\,dt\,dx\right.\\ &\quad +\sum_{j=1}^n b_j \int_u^\infty e^{-sx}\,e^{-\rho_j\, x}\,\bar{P}(x)\int_0^u [1-\phi_T(u-t)]\,\rho_j\, e^{\rho_j\, t}\,dt\,dx\\ &\quad -\sum_{j=1}^n b_j \int_0^u [1-\phi_T(u-x)]\,e^{-sx}\,\bar{P}(x)\,dx\\ &\quad \left.+\sum_{j=1}^n b_j \int_0^\infty e^{-sx}\,e^{-\rho_j\, x}\,\bar{P}(x)\,dx\,[1-\phi_T(u)]\right\}.\end{aligned}$$

As

$$\phi(u) = E[e^{-\delta T}\,e^{-sU(T^-)}I(T<\infty)\mid u(0)=u] = \int_0^\infty e^{-sx}\,f_1(x|u)\,dx\,,$$

inverting and defining $\Psi_k(u) = \phi_T(u) + \int_0^u \phi_T(u-t)\,\rho_k\,e^{\rho_k t}\,dt$ yields the result. □

To compute $f_2(x,y\,|\,u)$ and $g(y\,|\,u)$, one uses the relation (1.19) to obtain

$$f_2(x,y\,|\,u) = \begin{cases}\frac{1+\xi_\delta}{\xi_\delta}\sum_{j=1}^n b_j\,e^{-\rho_j x}\,p(x+y)\,[e^{\rho_j x}\Psi_j(u-x)-\Psi_j(u)], & 0\le x<u\\ \frac{1+\xi_\delta}{\xi_\delta}\sum_{j=1}^n b_j\,e^{-\rho_j x}\,p(x+y)\,[e^{\rho_j u}-\Psi_j(u)], & x\ge u\end{cases}, \tag{3.63}$$

and note that $g(y\,|\,u) = \int_0^\infty f_2(x,y\,|\,u)dx$.

When $\delta\to 0$, $f_2(x,y\,|u)$, $f_1(x\,|\,u)$ and $g(y\,|\,u)$ simplify to the joint and marginal densities of $U(T^-)$ and $|U(T)|$.

3.8 Rational Claim Size Distributions

The previous section shows that the discounted joint and marginal distributions of $U(T^-)$ and $|U(T)|$ can be derived explicitly whenever there is an explicit expression for the function $\phi_T(u)$. One such case is when $\hat{\phi}_T(s) = \int_0^\infty e^{-s\,u}\,\phi_T(u)\,du$ is a rational function. By locating its poles, one can then determine $\phi_T(u)$ by partial fractions. It follows from (3.56) that $\hat{\phi}_T(s)$ is a rational function if and only if $\hat{g}(s) = \int_0^\infty e^{-sy}g(y)\,dy$ is a rational function; by (3.52), $\hat{g}(s)$ is a rational function if and only if $\hat{p}(s)$ is rational. Further, by Theorem 1.1 of Lin and Willmot (1999), the solution $\phi(u)$ to the defective renewal equation (3.55) can be expressed explicitly in terms of $\phi_T(u)$. Therefore $\phi_T(u)$ is extremely important in evaluating the expected discounted penalty function $\phi(u)$.

In this section, we assume that the claim size density function p belongs to $\mathcal{R}_f^+$, i.e., for $m \in \mathbb{N}^+$,

$$\hat{p}(s) = \frac{Q_{m-1}(s)}{Q_m(s)}, \quad \text{with } Q_m(0) = Q_{m-1}(0), \text{ and } \Re(s) \in (h_X, \infty), \tag{3.64}$$

where the abscissa of holomorphy h_X of the claim size r.v. X is defined as

$$h_X := \inf\{s \in R : E[e^{-sX}] < \infty\}.$$

Q_m is a polynomial of degree m with leading coefficient 1, and Q_{m-1} is a polynomial of degree $m-1$ or less. Further, since $\hat{p}(s)$ is finite for all s with $\Re(s) > 0$, equation $Q_m(s) = 0$ has no roots with negative real parts.

The $\mathcal{R}_f^+$ is a wide class of distributions, including the K_n and distributions with damped sine and cosine functions as part of their densities. The definition and examples can be found in Section 3.1.

We now turn to deriving $\phi_T(u)$ by inverting its Laplace transform. Taking Laplace transforms on both sides of the defective renewal equation (3.56) and using (3.52) yields

$$\hat{\phi}_T(s) = \frac{\phi_T(0) - \hat{g}(s)}{s[1-\hat{g}(s)]}$$
$$= \frac{\prod_{i=1}^{n}(\delta+\lambda_i - c\,s) - \hat{p}(s)[\lambda^* + (\delta - c\,s)\,\beta(\delta - c\,s)] - c^n[1-\phi_T(0)]\prod_{i=1}^{n}(\rho_i - s)}{s\,\{\prod_{i=1}^{n}(\delta+\lambda_i - c\,s) - \hat{p}(s)[\lambda^* + (\delta - c\,s)\,\beta(\delta - c\,s)]\}}. \tag{3.65}$$

When p is a rational density as in (3.64), $\hat{\phi}_T(s)$ can be transformed to a rational expression and we have the following results.

Denote by $q_{m-1}(s) = [\prod_{i=1}^{m}(s+R_1) - \phi_T(0)Q_m(s)]/s$, a polynomial of degree $m-1$, and by $r_i = \frac{q_{m-1}(-R_i)}{\prod_{j=1,j\neq i}^{m}(R_j - R_i)} = \frac{Q_m(-R_i)}{Q_m(0)}\prod_{j=1,j\neq i}^{m}\frac{R_j}{(R_j-R_i)}$, for $i = 1, 2, \ldots, m$, where the $-R_i$ values, with $\Re(R_i) > 0$, are all the roots with negative real parts of the equation $Q_{m,n}(s) = 0$, where

$$Q_{m,n}(s) := Q_m(s)\left[\prod_{i=1}^{n}(\delta+\lambda_i - c\,s)\right] - Q_{m-1}(s)\,[\lambda^* + (\delta - c\,s)\,\beta(\delta - c\,s)]\,.$$

Theorem 3.8.1. If the Laplace transform $\hat{p}(s)$ of the claim density is defined as in (3.64), then

$$\hat{\phi}_T(s) = \frac{q_{m-1}(s)}{(s+R_1)(s+R_2)\cdots(s+R_m)}\,. \tag{3.66}$$

Further, if $R_1, R_2, \ldots R_m$ are distinct, then

$$\hat{\phi}_T(s) = \sum_{i=1}^{m}\frac{r_i}{(s+R_i)}\,, \tag{3.67}$$

and,

$$\phi_T(u) = \sum_{i=1}^{m} r_i\, e^{-R_i u}. \tag{3.68}$$

Proof: Substituting $\hat{p}(s) = \frac{Q_{m-1}(s)}{Q_m(s)}$ into (3.65) and multiplying $Q_m(s)$ to both denominator and numerator, then

$$\hat{\phi}_T(s) = \frac{Q_{m,n}(s) - c^n\,[1-\phi_T(0)]\,Q_m(s)\prod_{i=1}^{n}(\rho_i - s)}{s\,Q_{m,n}(s)},$$

where $Q_{m,n}(s)$, defined as above, is a polynomial of degree $n+m$ with leading coefficient $(-c)^n$. It is easy to check that in this case, the generalized Lundberg equation

$$\frac{\prod_{i=1}^{n}(\delta+\lambda_i-c\,s)}{[\lambda^*+(\delta-c\,s)\beta(\delta-c\,s)]}=\frac{Q_{m-1}(s)}{Q_m(s)}$$

is equivalent to $Q_{m,n}(s)=0$, for $\Re(s)>h_X$. It has n roots with positive real parts, say $\rho_1,\rho_2,\ldots,\rho_n$ and one root with a negative real part, say $-R$, where $h_X<-R<0$. While the equation $Q_{m,n}(s)=0$ has $n+m$ roots, which are ρ_i, with $\Re(\rho_i)>0$, for $i=1,2,\ldots,n$ and $-R_i$, with $\Re(R_i)>0$, for $i=1,2,\ldots,m$, where $R=\min\{\Re(R_i)$, for $i=1,2,\ldots,m\}$. Now we can express $Q_{m,n}(s)$ as

$$Q_{m,n}(s)=c^n\prod_{i=1}^{m}(s+R_i)\prod_{j=1}^{n}(\rho_j-s)\ ,$$

substituting in the expression of $\hat{\phi}_T(s)$ and canceling out common factors yields

$$\hat{\phi}_T(s)=\frac{\prod_{i=1}^{m}(s+R_i)-[1-\phi_T(0)]\,Q_m(s)}{s\prod_{i=1}^{m}(s+R_i)}\ .$$

Since $s=0$ is a removable singularity, the above numerator must be zero if $s=0$, i.e., $1-\phi_T(0)=\frac{R_1R_2\cdots R_m}{Q_m(0)}$ and therefore $q_{m-1}(s):=\frac{\prod_{i=1}^{m}(s+R_1)-[1-\phi_T(0)]Q_m(s)}{s}$ is a polynomial of degree $m-1$ or less.

If $R_1,R_2,\ldots,R_m$ are distinct, then by partial fractions

$$\hat{\phi}_T(s)=\frac{q_{m-1}(s)}{\prod_{i=1}^{m}(s+R_i)}=\sum_{i=1}^{m}\frac{r_i}{(s+R_i)}\ ,$$

where $r_i=\frac{q_{m-1}(-R_i)}{\prod_{j=1,j\neq i}^{m}(R_j-R_i)}=\frac{Q_m(-R_i)}{Q_m(0)}\prod_{j=1,j\neq i}^{m}\frac{R_j}{(R_j-R_i)}$. Inverting the above transform gives $\phi_T(u)=\sum_{i=1}^{m}r_i\,e^{-R_iu}$. □

Remarks:

1. The fact that $P_{m-1}(s)$ must be 0 at $s=0$, shows that $1-\phi_T(0)=\frac{R_1R_2\cdots R_m}{Q_m(0)}$. Then $\phi_T(0)$ can be expressed in terms of R_i, i.e. $\phi_T(0)=1-\frac{R_1R_2\cdots R_m}{Q_m(0)}$. Of course, formula (3.53) still holds true for $\phi_T(0)$.

2. If $\hat{p}(s)$ is defined as in (3.64), $\hat{g}(s)$ is simplified in the same fashion to

$$\hat{g}(s) = \frac{Q_m(s) - \prod_{i=1}^{n}(s+R_i)}{Q_m(s)}. \tag{3.69}$$

Then we can see that g is the same type of function as the claim density p, by partial fractions.

3. $\phi_T(u)$ can also be obtained by the Theorem of Residues:

$$\phi_T(u) \quad = \quad \sum_{i=1}^{m}\{\text{residues of } e^{su}\hat{\phi}_T(s) \text{ at singularity } -R_i\}\,.$$

Example 3.8.1. Assuming that the claim waiting times distribution k is given in (3.33), and claim amounts are exponentially distributed, that is $p(x) = ae^{-ax}$, $x \geq 0$ with $\hat{p}(s) = \frac{a}{s+a}$, then $\rho_1, \rho_2, \ldots, \rho_n$ and $-R < 0$ are $n+1$ roots of the equation:

$$Q_{1,n}(s) := (s+a)\left[\prod_{i=1}^{n}(\delta+\lambda_i - c\,s)\right] - a\Big[\lambda^* + (\delta - c\,s)\,\beta(\delta - c\,s)\Big] = 0, \quad s \in \mathbb{C}.$$

Hence

$$\phi_T(u) = \int_0^u \phi_T(u-y)\,g(y)\,dy + \int_u^\infty g(y)\,dy\,, \qquad u \geq 0\,,$$

with $\phi_T(0) = 1 - \frac{R}{a}$. Formula (3.69) gives $\hat{g}(s) = (\frac{a-R}{s+a})$ and thus

$$g(y) = (a-R)\,e^{-a\,y}I(y \geq 0).$$

Theorem 3.8.1 gives $\hat{\phi}_T(s) = \frac{\phi_T(0)}{(s+R)}$ and therefore,

$$\phi_T(u) = E[e^{-\delta T}I(T<\infty)\,|\,U(0)=u] = \phi_T(0)\,e^{-Ru} = \frac{(a-R)}{a}e^{-Ru}, \qquad u \geq 0,$$

and

$$\Psi(u) = P(T<\infty|U(0)=u) = \lim_{\delta\to 0}\phi_T(u) = \frac{[a-R(0)]}{a}e^{-R(0)\,u}, \qquad u \geq 0,$$

where $R(0)$ is the negative root to the generalized Lundberg's equation with $\delta = 0$. To compute the discounted joint and marginal distributions of $U(T^-)$ and $|U(T)|$, one needs to find $\Psi_j(u)$, for $j = 1, 2, \ldots, n$, and $u \geq 0$, i.e.

$$\Psi_j(u) = \phi_T(u) + \int_0^u \phi_T(u-t)\,\rho_j\,e^{\rho_j t}\,dt = \frac{(a-R)}{a}\frac{1}{(R+\rho_j)}[Re^{-Ru} + \rho_j\,e^{\rho_j u}]\,.$$

Therefore by Theorem 3.7.2, we have for $u > 0$,

$$f_1(x|u) = \begin{cases} (a-R)e^{-(Ru+a\,x)}\sum_{j=1}^n \frac{b_j[e^{Rx}-e^{-\rho_j x}]}{(R+\rho_j)}\,, & 0 \leq x \leq u\,, \\ e^{-a\,x}\sum_{j=1}^n \left[\frac{b_j(a+\rho_j)}{(R+\rho_j)}e^{-\rho_j(x-u)} - \frac{b_j(a-R)}{(R+\rho_j)}e^{-(Ru+\rho_j\,x)}\right]\,, & x > u\,, \end{cases}$$

where $b_j = \frac{\lambda^* + (\delta - c\rho_j)\beta(\delta - c\rho_j)}{c^n \prod_{k=1,k\neq j}^n (\rho_k - \rho_j)}$.

Finally, $f_2(x, y\,|\,u) = f_1(x\,|\,u)\frac{p(x+y)}{P(x)}$ yields $f_2(x, y\,|\,u) = a\,e^{-a y} f_1(x\,|u)$, and then $g(y\,|\,u) = \int_0^\infty f_2(x, y\,|u)dx = a\,e^{-a y}\phi_T(u) = (a-R)e^{-(Ru+a\,y)}$.

Example 3.8.2. Assume that claim inter-arrival times have a mixture of exponentials distribution, with density $k(x) = [\theta\lambda_1 e^{-\lambda_1 x} + (1-\theta)\,\lambda_2 e^{-\lambda_2 x}]I(x \geq 0)$. Thus $\hat{k}(s) = \frac{\lambda_1\lambda_2 + [\theta\lambda_1 + (1-\theta)\lambda_2]s}{(s+\lambda_1)(s+\lambda_2)}$. Further, assume that the claim density is a particular member of the $\mathcal{R}_f^+$ family with $p(x) = \frac{a(a^2+b^2)}{(a^2+b^2-a\,b)}\,e^{-a\,x}\,[1-\sin(b\,x)]\,I(x \geq 0)$, and $\hat{p}(s) = \frac{Q_2(s)}{Q_3(s)} = \left[\frac{a(a^2+b^2)}{a^2+b^2-a\,b}\right]\left[\frac{s^2+(2\,a-b)s+b^2+a^2-a\,b}{(s+a)[(s+a)^2+b^2]}\right]$, where $a > 0$, and $b > 0$. Since the equation

$$Q_5(s) = (a^2+b^2-a\,b)(s+a)[(s+a)^2+b^2](\delta+\lambda_1-c\,s)(\delta+\lambda_2-c\,s)$$

$$-a(a^2+b^2)[s^2+(2\,a-b)s+b^2+a^2-a\,b][\lambda_1\lambda_2+(\theta\lambda_1+(1-\theta)\lambda_2)(\delta-c\,s)] = 0$$

has two positive roots, say ρ_1 and ρ_2, and three roots with negative real parts (at least one real), say, $-R_i$, where $\Re(R_i) > 0$, for $i = 1, 2, 3$, then

$$\phi_T(u) = \int_0^u \phi_T(u-y)\,g(y)\,dy + \int_u^\infty g(y)\,dy,$$

where $\phi_T(0) = 1 - \frac{R_1R_2R_3}{Q_3(0)} = 1 - \frac{R_1R_2R_3}{a(a^2+b^2)}$. Formula (3.69) gives

$$\hat{g}(s) = \frac{(3a - \sum_{i=1}^3 R_i)s^2 + (3a^2+b^2-R_1R_2-R_1R_3-R_2R_3)s + a(a^2+b^2) - R_1R_2R_3}{(s+a)[(s+a)^2+b^2]},$$

inverting yields

$$g(y) = e^{-ay}\,[\eta - \eta_1 \cos(b\,y) - \eta_2 \sin(b\,y)],$$

where

$$\eta = \frac{a^3 - a^2(R_1+R_2+R_3) + a(R_1R_2+R_1R_3+R_2R_3) - R_1R_2R_3}{b^2},$$

$$\eta_1 = \frac{a^3 - 3ab^2 + (b^2-a^2)(R_1+R_2+R_3) + a(R_1R_2+R_1R_3+R_2R_3) - R_1R_2R_3}{b^2}$$

and

$$\eta_2 = \frac{3a^2b - b^3 - 2ab(R_1+R_2+R_3) + b(R_1R_2+R_1R_3+R_2R_3)}{b^2}.$$

We note that g is of the same type as the claim size density function. Theorem 3.8.1 gives

$$\phi_T(u) = \phi_T(0)[z_1e^{-R_1u} + z_2\,e^{-R_2u} + z_3\,e^{-R_3u}], \qquad u \geq 0,$$

with $z_3 = 1 - z_1 - z_2$, where $z_1 = \left[\frac{(a-R_1)[(a-R_1)^2+b^2]}{a(a^2+b^2)-R_1R_2R_3}\right]\left[\frac{R_2R_3}{(R_2-R_1)(R_3-R_1)}\right]$ and $z_2 = \left[\frac{(a-R_2)[(a-R_2)^2+b^2]}{a(a^2+b^2)-R_1R_2R_3}\right]\left[\frac{R_1R_3}{(R_1-R_2)(R_3-R_2)}\right]$.

Finally

$$\begin{aligned}\Psi_j(u) &= \phi_T(u) + \int_0^u \phi_T(u-y)\,\rho_j\,e^{\rho_j\,y}dy, \qquad j=1,2,\\ &= \phi_T(0)\left[\sum_{i=1}^{3}\pi_{j,i}\,e^{-R_i\,u} + (1-\pi_{j,1}-\pi_{j,2}-\pi_{j,3})\,e^{\rho_j\,u}\right], \quad u \geq 0,\end{aligned}$$

with $\pi_{j,i} = \frac{z_i\,R_i}{R_i+\rho_j}$, for $i = 1,2,3$ and $j = 1,2$, gives

$$f_1(x\,|u) = \begin{cases} \frac{\phi_T(0)}{1-\phi_T(0)}\bar{P}(x)\sum_{j=1}^{2} b_j \sum_{i=1}^{3}\pi_{j,i}\,e^{-R_i\,u}[e^{R_i\,x} - e^{-\rho_j\,x}], & 0 \leq x < u,\\ \frac{\phi_T(0)}{1-\phi_T(0)}\bar{P}(x)\sum_{j=1}^{2} b_j e^{-\rho_j\,x}\sum_{i=1}^{3}\pi_{j,i}\left[e^{\rho_j\,u} - e^{-R_i\,u}\right], & x \geq u,\end{cases}$$

where $b_1 = \frac{\lambda_1\lambda_2+(\delta-c\,\rho_1)[\theta\lambda_1+(1-\theta)\lambda_2]}{c^2(\rho_2-\rho_1)}$, and $b_2 = \frac{\lambda_1\lambda_2+(\delta-c\,\rho_2)[\theta\lambda_1+(1-\theta)\lambda_2]}{c^2(\rho_1-\rho_2)}$.

3.9 Concluding Remarks

We have shown how the evaluation of Gerber-Shiu's expected discounted penalty function for the classical risk model can be extended to a class of renewal risk processes with claim waiting times that are K_n distributed. This leads to a defective renewal equation for the penalty function for general claim size distributions. Moreover, when the claim sizes have a rational distribution, explicit results can be obtained by partial fractions.

The defective renewal equations obtained here can be used to solve other ruin related problems; explicit expressions or bounds and asymptotic formulas for ruin probabilities, joint and marginal distributions of the three random variables, time to ruin, surplus before ruin and deficit at ruin, as well as their moments.

Further research could study the G-S function in the Sparre Andersen risk model in which claims waiting times distributions belong to the class of $\mathcal{R}_f^+$, which is wider than the K_n family.

Chapter 4

A Renewal Risk Model with Dividend Barrier

4.1 Introduction

The Sparre Anderson renewal risk model corresponds to a GI/G/1 queue in queueing theory, while the classical risk model corresponds to a M/G/1 queue. Although Andersen proposed it half of a century ago, this model remains important in risk theory research. Some recent contributions to renewal risk models are Cheng and Tang (2003), Dickson and Drekic (2004), Gerber and Shiu (2003a,b, 2004), Li and Garrido (2004), Sun and Yang (2004), and Willmot and Dickson (2003).

The barrier strategy was initially proposed by De Finetti (1957) for a binomial model. More general barrier strategies for a compound Poisson risk process have been studied in a number of papers and books. These references include Bühlmann (1970), Segerdahl (1970), Gerber (1973), Gerber (1979), Gerber (1981), Paulsen and Gjessing (1997), Albrecher and Kainhofer (2002) and Højgaard (2002). The main focus is on optimal dividend payouts and the time of ruin, under various barrier strategies and other economic conditions. This chapter, instead, considers ruin related quantities by using the Gerber-Shiu function, as most work with expected discounted penalty functions in the classical risk model and Sparre Andersen risk models without a barrier.

This chapter studies a class of Sparre Andersen risk model with generalized Erlang(n) waiting times in the presence of a constant dividend barrier, extending the paper of Lin et al. (2003). The analysis is focused on the evaluation of the function $\phi(u)$, the expected discounted penalty function at ruin, with u being the initial reserve. The evaluation of the Gerber-Shiu penalty function for generalize Erlang(n) risk model without a barrier has been studied by Gerber and Shiu (2003a,b, 2004).

The definition of the Sparre Andersen risk model is given in Section 1.3 of chapter 1. In this chapter we assume that the claim waiting time are generalized Erlang(n) distributed with its density function k being given in (3.2). The following is a summary of the results for the generalized Erlang(n) risk model without a barrier. We remark that part of these results can be obtained as special cases when $\beta(s) \equiv 0$ in Chapter 3.

Theorem 4.1.1. [Gerber and Shiu(2003)] *Let $\mathcal{I}$ and $\mathcal{D}$ denote the identity operator and differential operator, respectively. Then $\phi(u)$ satisfies the following equation for $u \geq 0$*

$$\left\{\prod_{j=1}^{n}\left[\left(1+\frac{\delta}{\lambda_j}\right)\mathcal{I}-\frac{c}{\lambda_j}\mathcal{D}\right]\right\}\phi(u)=\int_0^u \phi(u-x)\,p(x)\,dx+\omega(u)\ , \tag{4.1}$$

where $\omega(u)=\int_u^\infty w(u,x-u)\,p(x)\,dx$.

Specially, setting $\delta=0$ and $w(x,y)=1$, $\phi(u)$ reduces to the ultimate ruin probability $\Psi(u)$.

Corollary 4.1.1. *$\Psi(u)$ satisfies the following equation for $u \geq 0$,*

$$\left[\prod_{j=1}^{n}\left(\mathcal{I}-\frac{c}{\lambda_j}\mathcal{D}\right)\right]\Psi(u)=\int_0^u \Psi(u-x)\,p(x)\,dx+\bar{P}(u)\ . \tag{4.2}$$

Gerber and Shiu (2003b) and Gerber and Shiu (2004) show that the integro-differential equation in (4.1) can be solved into a defective renewal equation using two different methods: a renewal argument from a key formula derived from Laplace transforms, or the use of divided differences and the operator T_r defined by (2.1) in Chapter 2.

Theorem 4.1.2. [Gerber and Shiu (2003)]

$$\phi(u) = \int_0^u \phi(u-y)g(y)dy + H(u), \qquad u \geq 0, \tag{4.3}$$

where $g(y) = \frac{\lambda_1\lambda_2\cdots\lambda_n}{c^n} T_{\rho_n} T_{\rho_{n-1}} \cdots T_{\rho_1} p(y)$, $G_\delta(u) = \frac{\lambda_1\lambda_2\cdots\lambda_n}{c^n} T_{\rho_n} T_{\rho_{n-1}} \cdots T_{\rho_1} \omega(u)$, *and* ρ_i *with* $\Re(\rho_i) > 0$, *for* $i = 1, 2, \ldots, n$ *are roots to the following equation:*

$$\prod_{j=1}^{n} \left[\left(1 + \frac{\delta}{\lambda_j}\right) - \frac{c}{\lambda_j} s \right] = \hat{p}(s), \qquad s \in \mathbb{C}, \ \ n \in \mathbb{N}^+.$$

Additional results can be found in Gerber and Shiu (2004).

4.2 Generalized Erlang(n) Risk Model with a Constant Dividend Barrier

This section considers the expected discounted penalty function for a Sparre Andersen risk model with generalized Erlang(n) distributed waiting times under a constant dividend barrier at level $b \geq u$. If the surplus reaches this level b, then dividends are paid continuously at the full premium rate of c, until a new claim occurs.

Let U_b be the surplus process under this barrier strategy and assume that the initial surplus of $U_b(0) = u$. Then

$$dU_b(t) = \begin{cases} c\,dt - dS(t), & U_b(t) < b\,, \\ -dS(t), & U_b(t) = b\,. \end{cases} \tag{4.4}$$

Define $T_b = \inf\{t : U_b(t) < 0\}$ to be the ruin time and, for a non–negative penalty function, $w(x,y), 0 \le x, y < \infty$, define for $\delta \ge 0$

$$\phi_b(u) = E\left[e^{-\delta T_b} w\big(U(T_b^-), |U(T_b)|\big)\, I(T_b < \infty) \,\middle|\, U_b(0) = u\right] , \qquad 0 \le u \le b ,$$

to be the Gerber–Shiu discounted penalty function. Specially, if $w(x, y) = 1$, define

$$\phi_{T_b}(u) = E\left[e^{-\delta T_b}\, I(T_b < \infty) \,\middle|\, U_b(0) = u\right] , \qquad 0 \le u \le b , \tag{4.5}$$

to be the Laplace transform of the ruin time T_b with respect to δ.

Our first result shows that the integro–differential equation (2.5), with boundary condition (2.8), of Lin et al. (2003), for the classical risk model with a constant dividend barrier, can be extended to the generalized Erlang risk process.

Let $k_n(t;\, \lambda_1, \lambda_2, \ldots, \lambda_n)$ be the generalized Erlang(n) density function with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then

$$k_n(t;\, \lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{i=1}^{n} \left[\prod_{j=1, j\neq i}^{n} \frac{\lambda_j}{\lambda_j - \lambda_i} \right] \lambda_i\, e^{-\lambda_i t}. \tag{4.6}$$

It is easy to check that the following relations hold

$$k_n'(t;\, \lambda_1, \lambda_2, \ldots, \lambda_n) = \lambda_1[k_{n-1}(t;\, \lambda_2, \lambda_3, \ldots, \lambda_n) - k_n(t;\, \lambda_1, \lambda_2, \ldots, \lambda_n)], \tag{4.7}$$

$$k_{n-1}'(t;\, \lambda_2, \lambda_3, \ldots, \lambda_n) = \lambda_2[k_{n-2}(t;\, \lambda_3, \lambda_4, \ldots, \lambda_n) - k_{n-1}(t;\, \lambda_2, \lambda_3, \ldots, \lambda_n)], \tag{4.8}$$

and

$$k_2'(t;\, \lambda_{n-1}, \lambda_n) = \lambda_{n-1}[k_1(t;\, \lambda_n) - k_2(t;\, \lambda_{n-1}, \lambda_n)]. \tag{4.9}$$

The following theorem shows that the penalty function $\phi_b(u)$ satisfies an n-th order integro-differential equation with certain boundary conditions.

Theorem 4.2.1. *Let $\mathcal{I}$ and $\mathcal{D}$ denote the identity operator and differential operator, respectively. Then under the generalized Erlang(n) waiting times assumption,*

$\phi_b(u)$ satisfies the following equation for $0 \le u \le b < \infty$:

$$\left\{ \prod_{j=1}^{n} \left[\left(1 + \frac{\delta}{\lambda_j}\right) \mathcal{I} - \frac{c}{\lambda_j} \mathcal{D} \right] \right\} \phi_b(u) = \int_0^u \phi_b(u-x)\, p(x)\, dx + \omega(u) \;, \qquad (4.10)$$

with boundary conditions,

$$\phi_b^{(k)}(b) = 0, \qquad k = 1, 2, \ldots, n, \qquad (4.11)$$

where $\omega(u) = \int_u^\infty w(u, x-u)\, p(x)\, dx$.

Proof: By conditioning on the time and amount of the first claim, one finds that for $0 \le u \le b$,

$$\begin{aligned} \phi_b(u) &= \int_0^{\frac{b-u}{c}} e^{-\delta t} \gamma_b(t)\, k_n(t;\, \lambda_1, \lambda_2, \ldots, \lambda_n)\, dt \\ &\quad + \int_{\frac{b-u}{c}}^{\infty} e^{-\delta t} \gamma_b(b)\, k_n(t;\, \lambda_1, \lambda_2, \ldots, \lambda_n)\, dt, \end{aligned} \qquad (4.12)$$

where

$$\gamma_b(t) = \int_0^t \phi_b(t-y)\, p(y)\, dy + \omega(t) \;, \qquad 0 \le t \le b \,. \qquad (4.13)$$

Changing the variable in the first integral in (4.12) implies that

$$\begin{aligned} \phi_b(u) &= \frac{1}{c} \int_u^b e^{-(\frac{\delta}{c})(t-u)} k_n\Big(\frac{t-u}{c};\, \lambda_1, \lambda_2, \ldots, \lambda_n\Big) \gamma_b(t)\, dt \\ &\quad + \gamma_b(b) \int_{\frac{b-u}{c}}^{\infty} e^{-\delta t}\, k_n(t;\, \lambda_1, \lambda_2, \ldots, \lambda_n)\, dt \;. \end{aligned} \qquad (4.14)$$

Let $\lambda^* = \prod_{i=1}^n \lambda_i$ and since $e^{-\delta t} k_n(t;\, \lambda_1, \lambda_2, \ldots, \lambda_n) = \frac{\lambda^*}{\prod_{i=1}^n (\lambda_i + \delta)} \tilde{k}_n(t;\, \lambda_1 + \delta, \lambda_2 + \delta, \ldots, \lambda_n + \delta)$, where

$$\tilde{k}_n(t;\, \lambda_1 + \delta, \lambda_2 + \delta, \ldots, \lambda_n + \delta) = \sum_{i=1}^n \left[\prod_{j=1, j \ne i}^n \frac{(\lambda_j + \delta)}{(\lambda_j - \lambda_i)} \right] (\lambda_i + \delta)\, e^{-(\lambda_i + \delta)\, t}$$

is a new generalized Erlang(n) density with parameter $\lambda_1 + \delta, \lambda_2 + \delta, \ldots, \lambda_n + \delta$.

Then (4.14) can be rewritten as

$$c\,\phi_b(u) = \frac{\lambda^*}{\prod_{i=1}^n(\lambda_i+\delta)}\left\{\int_u^b \tilde{k}_n\Big(\frac{t-u}{c};\,\lambda_1+\delta,\lambda_2+\delta,\ldots,\lambda_n+\delta\Big)\gamma_b(t)\,dt\right.$$
$$\left. + c\,\gamma_b(b)\left[1-\tilde{K}_n\left(\frac{b-u}{c};\,\lambda_1+\delta,\lambda_2+\delta,\ldots,\lambda_n+\delta\right)\right]\right\}, \tag{4.15}$$

where $\tilde{K}_n$ is the distribution function of $\tilde{k}_n$. It is easy to check that the following formula holds

$$c\left[\mathcal{D}-(\frac{\lambda_1+\delta}{c})\mathcal{I}\right]\phi_b(u)$$
$$= \left[\prod_{i=1}^n \frac{\lambda_i}{\lambda_i+\delta}\right]\left\{\left(-\frac{\lambda_1+\delta}{c}\right)\int_u^b \tilde{k}_{n-1}(\frac{t-u}{c};\,\lambda_2+\delta,\lambda_3+\delta,\ldots,\lambda_n+\delta)\,\gamma_b(t)\,dt\right.$$
$$\left. -\gamma_b(t)(\lambda_1+\delta)\left[1-\tilde{K}_{n-1}(\frac{b-u}{c};\,\lambda_2+\delta,\lambda_3+\delta,\ldots,\lambda_n+\delta)\right]\right\}. \tag{4.16}$$

Recursively, for $1 \le k \le n-1$,

$$c\left\{\prod_{i=1}^k\left[\mathcal{D}-(\frac{\lambda_i+\delta}{c})\mathcal{I}\right]\right\}\phi_b(u)$$
$$= \left[\prod_{i=1}^n \frac{\lambda_i}{\lambda_i+\delta}\right]\left\{\prod_{i=1}^k\left(-\frac{\lambda_i+\delta}{c}\right)c\,\gamma_b(t)\left[1-\tilde{K}_{n-k}(\frac{b-u}{c};\,\lambda_{k+1}+\delta,\ldots,\lambda_n+\delta)\right]\right.$$
$$\left. +\prod_{i=1}^k\left(-\frac{\lambda_i+\delta}{c}\right)\int_u^b \tilde{k}_{n-k}(\frac{t-u}{c};\,\lambda_{k+1}+\delta,\ldots,\lambda_n+\delta)\,\gamma_b(t)\,dt\right\}. \tag{4.17}$$

Specially for $k = n-1$,

$$c\left\{\prod_{i=1}^{n-1}\left[\mathcal{D}-(\frac{\lambda_i+\delta}{c})\mathcal{I}\right]\right\}\phi_b(u)$$
$$= \left[\prod_{i=1}^n \frac{\lambda_i}{\lambda_i+\delta}\right]\left\{(-1)^{n-1}\frac{1}{c^{n-2}}\gamma_b(t)\left[1-\tilde{K}_1(\frac{b-u}{c};\,\lambda_n+\delta)\right]\right.$$
$$\left. +\prod_{i=1}^{n-1}\left(-\frac{\lambda_i+\delta}{c}\right)\int_u^b \tilde{k}_1(\frac{t-u}{c};\,\lambda_n+\delta)\,\gamma_b(t)dt\right\}. \tag{4.18}$$

Applying the operator $[\mathcal{D} - (\frac{\lambda_n+\delta}{c})\mathcal{I}]$ to both sides of (4.18) finally gives

$$\left\{\prod_{i=1}^{n}\left[\mathcal{D} - (\frac{\lambda_i+\delta}{c})\mathcal{I}\right]\right\}\phi_b(u) = (-1)^n\frac{\lambda^*}{c^n}\gamma_b(u), \tag{4.19}$$

which is equivalent to (4.10).

To verify the boundary conditions (4.11), setting $u = b$ in (4.15) gives $\phi_b(b) = \left(\frac{\lambda^*}{\prod_{i=1}^{n}(\lambda_i+\delta)}\right)\gamma_b(b)$, while setting $u = b$ in (4.17), successively for $k = 1, 2, \ldots, n-1$, and in (4.19) for $k = n$, gives the boundary conditions $\phi_b^{(k)}(b) = 0$, for $k = 1, 2, \ldots, n$. □

We note that if $b = \infty$, (4.10) reproduces the integro-differential equation (D10) of Gerber and Shiu (2003a), thus the above procedure gives an alternative proof.

If $\lambda_i = \lambda$, for $i = 1, 2, \ldots, n$, i.e., the claim waiting times are Erlang(n) distributed with density $k_n(t,\lambda) = \frac{\lambda^n t^{n-1}e^{-\lambda t}}{(n-1)!}$, then we have the following Corollary.

Corollary 4.2.1. If claim waiting times are Erlang(n) distributed with parameter λ, then $\phi_b(u)$ satisfies the following equation for $0 \le u \le b < \infty$:

$$\sum_{k=0}^{n}\phi_b^{(k)}(u)\left[\frac{-(\lambda+\delta)}{c}\right]^{n-k}\binom{n}{n-k}$$
$$= \left(\frac{-\lambda}{c}\right)^n\left[\int_0^u \phi_b(u-x)\,p(x)\,dx + \int_u^\infty w(u, x-u)\,p(x)\,dx\right], \tag{4.20}$$

with boundary conditions, $\phi_b^{(k)}(b) = 0$, for $k = 1, 2, \ldots, n$.

Proof: If all $\lambda_i = \lambda$ in (4.19), then

$$\left[\mathcal{D} - \frac{\lambda+\delta}{c}\mathcal{I}\right]^n = \sum_{k=0}^{n}\left[\frac{-(\lambda+\delta)}{c}\right]^{n-k}\binom{n}{n-k}\mathcal{D}^k,$$

thus (4.19) can be rewritten as (4.20). □

The solution to the integro-differential equation (4.10), with boundary conditions (4.11), heavily depends on the solution to the associated homogenous equation in v :

$$B(\mathcal{D})\,v(u)=\int_0^u v(u-x)\,p(x)\,dx,\qquad u\geq 0, \tag{4.21}$$

where $B(\mathcal{D})=\prod_{j=1}^{n}\left[\left(1+\frac{\delta}{\lambda_j}\right)\mathcal{I}-\frac{c}{\lambda_j}\mathcal{D}\right]=\sum_{k=0}^{n}B_k\,\mathcal{D}^k$ is an n-th order linear differential operator.

It follows from the general theory of differential equations that every solution to the nonhomogeneous equation can be expressed as a special solution plus a linear combination of n linearly independent solutions to the associated homogeneous equation. Then the general solution of equation (4.10) is of the form

$$\phi_b(u)=\phi(u)+\sum_{i=1}^{n}\eta_i\,v_i(u),\qquad u\geq 0, \tag{4.22}$$

where $\phi(u)$ is the G-S function for the generalized Erlang(n) risk model without a barrier, which has been studied by Gerber and Shiu (2003a,b) and Gerber and Shiu (2004). It also satisfies the defective renewal equation (3.55) as a special case; $v_i(u)$, for $i=1,2,\ldots,n$ are n linearly independent solutions to equation (4.21). To meet the boundary conditions, we choose $\eta_1,\eta_2,\ldots,\eta_n$ in a way such that the following linear equation system holds:

$$\phi_b^{(k)}(b)=0,\qquad k=1,2,\ldots,n.$$

4.3 Analysis of the Function $v(u)$

The solution to above homogenous equation is uniquely determined by the initial conditions $v^{(k)}(0)$, for $k=0,1,\ldots,n-1$, and can be solved by Laplace transforms. Taking Laplace transform on both sides of (4.21) yields

$$\hat{v}(s)=\frac{d(s)}{B(s)-\hat{p}(s)},\qquad s\in\mathbb{C}, \tag{4.23}$$

where $d(s):=\sum_{j=0}^{n-1}s^j\sum_{k=j+1}^{n}B_k\,v^{(k-1-j)}(0)=\sum_{j=0}^{n-1}d_j\,s^j$ is a polynomial of degree $n-1$. By Theorem 3.3.1, of all the roots to the equation $B(s)=\hat{p}(s)$, there are exactly n roots with positive real parts, say, $\rho_1, \rho_2, \ldots, \rho_n$. If the ρ_i are distinct, by interpolation, $d(s) = \sum_{j=1}^{n} d(\rho_j) \left[\prod_{k=1,k\neq j}^{n} \frac{(s-\rho_k)}{(\rho_j-\rho_k)}\right]$, and by divided differences

$$\begin{aligned} B(s) - \hat{p}(s) &= \prod_{j=1}^{n}(s-\rho_j)\{B[s,\rho_1,\rho_2,\ldots,\rho_n] - \hat{p}[s,\rho_1,\rho_2,\ldots,\rho_n]\} \\ &= (-1)^n \prod_{j=1}^{n}(s-\rho_j)\left[\frac{c^n}{\lambda^*} - T_s\, T_{\rho_n} T_{\rho_{n-1}} \cdots T_{\rho_1}\, p(0)\right], \end{aligned} \tag{4.24}$$

where the last step follows from relation (2.7), between divided differences and the operator T_r. Then (4.23) can be rewritten as

$$\hat{v}(s) = \frac{\frac{\lambda^*}{c^n}\sum_{j=1}^{n} \frac{1}{(s-\rho_j)} \frac{-d(\rho_j)}{\prod_{k=1,k\neq j}^{n}(\rho_k-\rho_j)}}{1 - \frac{\lambda^*}{c^n} T_s\, T_{\rho_n} T_{\rho_{n-1}} \cdots T_{\rho_1}\, p(0)}, \tag{4.25}$$

inverting yields

$$v(u) = \int_0^u v(u-y)\, g(y)\, dy + \sum_{j=1}^{n} \xi_j\, e^{\rho_j u}, \qquad u \geq 0, \tag{4.26}$$

where $g(y) = \frac{\lambda^*}{c^n} T_{\rho_n} T_{\rho_{n-1}} \cdots T_{\rho_1}\, p(y)$, and

$$\xi_j = -\frac{\lambda^*}{c^n} \frac{d(\rho_j)}{\prod_{k=1,k\neq j}^{n}(\rho_k-\rho_j)} = -\frac{\lambda^*}{c^n} \frac{\sum_{m=0}^{n-1} v^{(m)}(0) \sum_{k=m+1}^{n} B_k\, \rho_j^{k-m-1}}{\prod_{k=1,k\neq j}^{n}(\rho_k-\rho_j)}.$$

Equation (4.26) is a defective renewal equation, since $g(y)$ is a defective density function. If p is rationally distributed, v has a rational Laplace transform, therefore, it can be obtained explicitly by partial fractions as follows.

Let us assume that claim size density function p belongs to $\mathcal{R}_f^+$, i.e., for $m \in \mathbb{N}^+$:

$$\hat{p}(s) = \frac{Q_{m-1}(s)}{Q_m(s)}, \quad \text{with } Q_m(0) = Q_{m-1}(0), \text{ and } \Re(s) \in (h_X, \infty), \tag{4.27}$$

where the abscissa of holomorphy h_X of the claim size r.v. X is defined as

$$h_X := \inf\{s \in R : E[e^{-sX}] < \infty\},$$

Q_m is a polynomial of degree m with leading coefficient 1, and Q_{m-1} is a polynomial of degree $m-1$ or less. Further, since $\hat{p}(s)$ is finite for all s, with $\Re(s) > 0$, equation $Q_m(s) = 0$ has no roots with negative real parts.

Substituting (4.27) into (4.23) yields

$$\hat{v}(s) = \frac{d(s)\,Q_m(s)}{B(s)\,Q_m(s) - Q_{m-1}(s)}, \qquad s \in \mathbb{C}, \tag{4.28}$$

where $B(s)\,Q_m(s) - Q_{m-1}(s)$ is a polynomial of degree of $n+m$ with leading coefficient $(-1)^n \frac{c^n}{\lambda^*}$. Hence it can be factored as

$$B(s)\,Q_m(s) - Q_{m-1}(s) = (-1)^n \frac{c^n}{\lambda^*} \left[\prod_{i=1}^{n}(s-\rho_i)\right] \left[\prod_{i=1}^{m}(s+R_i)\right], \tag{4.29}$$

where $\rho_1, \rho_2, \ldots, \rho_n$ are all the roots with positive real parts to Lundbeg's equation $B(s) = \hat{p}(s)$. Here $R_1, R_2, \ldots, R_m$ denote the roots with negative real parts to the equation $B(s)\,Q_m(s) - Q_{m-1}(s) = 0$. Then (4.28) can be rewritten as

$$\hat{v}(s) = \frac{\lambda^*}{c^n} \frac{d(s)\,Q_m(s)}{\prod_{i=1}^{n}(\rho_i - s)\prod_{i=1}^{m}(s+R_i)}. \tag{4.30}$$

If $\rho_1, \rho_2, \ldots, \rho_n$ and $R_1, R_2, \ldots, R_m$ are all distinct, by partial fractions, we have

$$\hat{v}(s) = \sum_{i=1}^{n} \frac{\alpha_i}{(s-\rho_i)} + \sum_{j=1}^{m} \frac{\varsigma_j}{(s+R_j)}, \qquad s \in \mathbb{C}, \tag{4.31}$$

where $\alpha_i = \frac{\lambda^*}{c^n} \frac{d(\rho_i)Q_m(\rho_i)}{\prod_{j=1}^{m}(R_j+\rho_i)\prod_{k=1,k\neq i}^{n}(\rho_k-\rho_i)}$, $\varsigma_j = \frac{\lambda^*}{c^n} \frac{d(-R_j)Q_m(-R_j)}{\prod_{i=1}^{n}(R_j+\rho_i)\prod_{k=1,k\neq j}^{m}(R_k-R_j)}$. Then inverting yields

$$v(u) = \sum_{i=1}^{n} \alpha_i\, e^{\rho_i u} + \sum_{j=1}^{m} \varsigma_j\, e^{-R_j u}, \qquad u \geq 0. \tag{4.32}$$

Formula (4.32) is the general solution to the homogeneous integro-differential equation (4.21), which is uniquely determined by the initial conditions $\{v^{(k)}(0)\}_{k=0}^{n-1}$.

One can find n linearly independent solutions $v_i(u), i = 1, 2, \ldots, n$ by specifying the initial conditions $v_i^{(k)}(0) = I\{k = i-1\}$, for $k = 0, 1, 2, \ldots, n-1$. Then

$$\hat{v}_i(s) = \frac{d_i(s)\,Q_m(s)}{B(s)\,Q_m(s) - Q_{m-1}(s)}, \qquad s \in \mathbb{C}\,, \tag{4.33}$$

where $d_i(s) = \sum_{k=i}^{n} B_k s^{k-i} = \sum_{m=0}^{n-i} B_{m+i}\, s^m$, $i = 1, 2, \ldots, n$. For these special $d_i(s)$, one can use (4.32) to obtain $v_i(u)$.

To prove that the $v_i(u)$ are linearly independent, we assume that there are constants $c_1, c_2, \ldots, c_n$ such that $\sum_{i=1}^{n} c_i\, v_i(u) \equiv 0$, for any $u \geq 0$. Then $\sum_{i=1}^{n} c_i\, v_i^{(k)}(u) \equiv 0$, for $k = 0, 1, \ldots, n-1$, and any $u \geq 0$. Setting $u = 0$ and noting that $v_i^{(k)}(0) = I\{k = i-1\}$, for $k = 0, 1, 2, \ldots, n-1$, we can prove that $c_i = 0$, for all $i = 1, 2, \ldots, n$.

4.4 Examples

The following examples show how to obtain explicit results when claim sizes are rationally distributed.

Example 4.4.1. In this example, we assume that the claim sizes are exponentially distributed with parameter β, i.e., its density $p(x) = \beta\, e^{-\beta x}$ and Laplace transform $\hat{p}(s) = \frac{\beta}{(s+\beta)}$. The claim waiting times are generalized Erlang(2) distributed with parameters λ_1 and λ_2. Then the generalized Lundberg equation simplifies to

$$(\lambda_1 + \delta - c\, s)(\lambda_2 + \delta - c\, s)(s + \beta) = \lambda_1\, \lambda_2\, \beta, \tag{4.34}$$

which gives three roots, say, $\rho_1 > 0$, $\rho_2 > 0$, and $-R < 0$.

Let $v_1(u)$ with $v_1(0) = 1, v_1'(0) = 0$ and $v_2(u)$ with $v_2(0) = 0$, $v_2'(0) = 1$ be two linearly independent solutions to the homogeneous integro-differential equation (4.21). By (4.32) we have

$$\begin{aligned} v_1(u) \quad = \quad & \frac{[c\,\rho_1 - (\lambda_1 + \lambda_2 + 2\,\delta)](\rho_1 + \beta)}{c(R + \rho_1)(\rho_2 - \rho_1)} e^{\rho_1 u} + \frac{[c\,\rho_2 - (\lambda_1 + \lambda_2 + 2\,\delta)](\rho_2 + \beta)}{c(R + \rho_2)(\rho_1 - \rho_2)} e^{\rho_2 u} \\ & - \frac{(c\,R + \lambda_1 + \lambda_2 + 2\,\delta)(\beta - R)}{c(\rho_1 + R)(\rho_2 + R)} e^{-R u}, \qquad u \geq 0, \end{aligned} \tag{4.35}$$

and

$$v_2(u) = \frac{\rho_1+\beta}{(R+\rho_1)(\rho_2-\rho_1)}e^{\rho_1 u} + \frac{\rho_2+\beta}{(R+\rho_2)(\rho_1-\rho_2)}e^{\rho_2 u} + \frac{\beta-R}{(\rho_1+R)(\rho_2+R)}e^{-Ru}, \quad u \geq 0. \tag{4.36}$$

The results for the corresponding generalized Erlang(2) risk model without a barrier, are given by Theorem 3.8.1 and Example 3.8.1:

$$\phi_T(u) = \frac{\beta-R}{\beta}e^{-Ru}, \qquad u \geq 0,$$

and for $0 \leq x < u$,

$$f_1(x\,|u) = \frac{\lambda_1\,\lambda_2(\beta-R)}{c^2(\rho_2-\rho_1)(R+\rho_1)(R+\rho_2)}e^{-Ru}\Big[(\rho_2-\rho_1)e^{-(\beta-R)x} + (R+\rho_1)e^{-(\rho_2+\beta)x} - (R+\rho_2)e^{-(\rho_1+\beta)x}\Big], \tag{4.37}$$

and for $x \geq u$,

$$f_1(x\,|\,u) = \frac{\lambda_1\,\lambda_2}{c^2(\rho_2-\rho_1)}\left[\frac{\beta+\rho_1}{R+\rho_1}e^{-(\rho_1+\beta)x}e^{\rho_1 u} - \frac{\beta-R}{R+\rho_1}e^{-(\rho_1+\beta)x}e^{-Ru} - \frac{\beta+\rho_2}{R+\rho_2}e^{-(\rho_2+\beta)x}e^{\rho_2 u} + \frac{\beta-R}{R+\rho_2}e^{-(\rho_2+\beta)x}e^{-Ru}\right]. \tag{4.38}$$

The relation

$$f_2(x,\,y|\,u) = f_1(x\,|\,u)\frac{p(x+y)}{\bar{P}(x)} = \beta\,e^{-\beta y}f_1(x,\,|\,u)$$

gives

$$g(y\,|\,u) = \int_0^\infty f_2(x,\,y|\,u)dx = \beta\,e^{-\beta\,y}\phi_T(u) = (\beta-R)\,e^{-(R\,u+\beta\,y)}. \tag{4.39}$$

Then

$$\phi_{T_b}(u) = \phi_T(u) + c_1\,v_1(u) + c_2\,v_2(u), \tag{4.40}$$

where c_1, c_2 are to determined by solving equations

$$c_1\,v_1^{(i)}(b) + c_2\,v_2^{(i)}(b) = -\phi_T^{(i)}(b), \qquad i = 1, 2.$$

Let $f_{b,1}(x \mid u)$ be the discounted marginal distribution of $U_b(T_b)$, and $g_b(y \mid u)$ be the density of deficit at ruin $|U_b(T_b)|$. Then

$$f_{b,1}(x \mid u) = f_1(x \mid u) + z_1\, v_1(u) + z_2\, v_2(u), \qquad 0 \le u, \quad x \le b, \tag{4.41}$$

where z_1, z_2 are to determined by solving the equations

$$z_1\, v_1^{(i)}(b) + z_2\, v_2^{(i)}(b) = -\frac{\partial^i f_1(x \mid u)}{\partial u^i}\bigg|_{u=b}, \qquad i = 1, 2,$$

and

$$g_b(y \mid u) = g(y \mid u) + \zeta_1\, v_1(u) + \zeta_2\, v_2(u), \qquad 0 \le u \le b, \tag{4.42}$$

where ζ_1, ζ_2 are to determined by solving equations

$$\zeta_1\, v_1^{(i)}(b) + \zeta_2\, v_2^{(i)}(b) = -\frac{\partial^i g(x \mid u)}{\partial u^i}\bigg|_{u=b}, \qquad i = 1, 2.$$

Setting $c = 1.1$, $\lambda_1 = 1.0$, $\lambda_2 = 1.0$, $\beta = 0.5$ and $\delta = 0.03$ implies that the generalized Lundberg's equation reduces to

$$(1.03 - 1.1\, s)^2 (s + 0.5) = 0.5,$$

which has three roots, say $\rho_1 = 0.1199$, $\rho_2 = 1.4024$ and $-R = -0.1496$. Then

$$\begin{aligned} v_1(u) &= -1.6937\, e^{-0.1496\, u} + 3.1432\, e^{0.1199\, u} - 0.4495\, e^{1.4024\, u}, \\ v_2(u) &= 0.8375\, e^{-0.1496\, u} - 1.7933\, e^{0.1199\, u} + 0.9558\, e^{1.4024\, u}, \qquad u \ge 0, \end{aligned}$$

while

$$\phi_T(u) = 0.7008\, e^{-0.1496\, u}, \qquad u \ge 0. \tag{4.43}$$

For $0 \le x < u$,

$$f_1(x \mid u) = e^{-0.1496 u}[0.6923\, e^{-0.3504\, x} + 0.1455\, e^{-1.9024\, x} - 1.5520\, e^{-0.6199\, x}],$$

and when $x \ge u$,

$$\begin{aligned} f_1(x \mid u) &= e^{-0.6199 x}[1.4822\, e^{0.1199\, u} - 0.8378\, e^{-0.1496\, u}] \\ &\quad + e^{-1.9024 x}[0.1455\, e^{-0.1496 u} - 0.7899\, e^{1.4024\, u}]. \end{aligned}$$

By setting the constant dividend barrier to $b = 10$ and solving the boundary conditions, we obtain $c_1 = 0.02936,\ c_2 = 0.01381$, and for $0 \le u \le 10$,

$$\phi_{T_b}(u) = 0.6626\, e^{-0.1496\, u} + 0.06753\, e^{0.1199\, u} - 0.269^{-8}\, e^{1.4024\, u}. \tag{4.44}$$

If $0 \le x < u$,

$$\begin{aligned} f_{b,1}(x \mid u) &= e^{-0.1496 u}[0.6546\, e^{-0.3504\, x} + 0.1376\, e^{-1.9024\, x} - 1.4675\, e^{-0.6199\, x}] \\ &\quad + e^{0.1199 u}[0.0667\, e^{-0.3504\, x} + 0.0139\, e^{-1.9024\, x} - 0.1496\, e^{-0.6199\, x}]. \end{aligned}$$

If $u \le x \le 10$,

$$\begin{aligned} f_{b,1}(x \mid u) &= e^{-0.1496 u}[-0.03768\, e^{-0.3504\, x} + 0.1376\, e^{-1.9024\, x} - 0.7533\, e^{-0.6199\, x}] \\ &\quad + e^{0.1199 u}[0.0667\, e^{-0.3504\, x} + 0.01402\, e^{-1.9024\, x} + 1.3326\, e^{-0.6199\, x}] \\ &\quad - 0.7899\, e^{1.4024\, u}\, e^{-1.9024\, x}. \end{aligned}$$

It is easy to check that $U_b(T_b)$ has a probability mass at $b = 10$, given by

$$P(U_b(T_b^-) = 10) = 1.0837\, e^{-0.1496\, u} - 1.1521\, e^{-0.7695\, u} + 0.1211\, e^{0.1199\, u}.$$

Finally, for $0 \le u \le 10$,

$$g_b(y \mid u) = 0.5\, e^{-0.5\, y}[0.6626\, e^{-0.1496\, u} + 0.06753\, e^{0.1199\, u} - 0.269^{-8}\, e^{1.4024 u}].$$

Example 4.4.2. In this example, we assume that the claim waiting times are generalized Erlang distributed with parameters $n = 3, \lambda_1 = \lambda_2 = 0.5$ and $\lambda_3 = 2$. With these values, the claim waiting times density function is given by

$$k_3(t) = \frac{2}{9} e^{-2\, t} - \frac{2}{9} e^{-0.5\, t} + \frac{1}{3} t\ e^{-0.5\, t}, \qquad t \ge 0,$$

with mean $E(W) = 4.5$ and $\hat{k}_3(s) = \frac{0.5}{(s+0.5)^2(s+2)}$. We assume that claim sizes are distributed as a mixture of two exponential distributions with $p(x) = 0.5 \times 0.2 e^{-0.2\, x} + 0.5 \times 0.25\, e^{-0.25\, x}$, $\hat{p}(s) = \frac{Q_1(s)}{Q_2(s)} = \frac{0.05 + 0.225\, s}{(s+0.2)(s+0.25)}$ and $\mu = 4.5$. Further assuming that $c = 1.1$ gives a positive loading factor of $\theta = 0.1$.

In this example, we focus on the joint and marginal distributions of the surplus before ruin and deficit at ruin with a constant dividend barrier, therefore we assume that $\delta = 0$.

First the following generalized Lundberg equation

$$B(s)Q_2(s) - Q_1(s) = (1-2.2\,s)^2(1-0.55\,s)(s+0.2)(s+0.25) - 0.225\,s - 0.05 = 0$$

has five roots on the whole complex plane, i.e.:

$$\rho_1 = 0,\ \rho_2 = 0.7393,\ \ \rho_3 = 1.7949,\ \ -R_1 = -0.0278 \text{ and } \ -R_2 = -0.2291.$$

Now let $v_i(u)$ with $v_i^{(k)}(0) = I(k = i-1)$, for $k = 0,1,2$ and $i = 1,2,3$, be three linearly independent solutions to the homogeneous equation (4.21). Then inverting (4.33) gives

$$\begin{aligned}
v_1(u) &= 1.0999 + 0.3107\,e^{-R_1\,u} - 1.3063\,e^{-R_2\,u} - 0.1152\,e^{\rho_2\,u} + 0.0109\,e^{\rho_3\,u},\\
v_2(u) &= -1.6133 - 0.4421\,e^{-R_1\,u} + 1.5224\,e^{-R_2\,u} + 0.5876\,e^{\rho_2\,u} - 0.05453\,e^{\rho_3\,u},\\
v_3(u) &= 0.5916 + 0.1604\,e^{-R_1\,u} - 0.5149\,e^{-R_2\,u} - 0.2956\,e^{\rho_2\,u} + 0.0585\,e^{\rho_3\,u}.
\end{aligned}$$

By Theorem 3.8.1, $\Psi(0) = 1 - \frac{R_1 R_2}{Q_2(0)} = 0.8726$, and

$$\Psi(u) = 0.08709\,e^{-R_1\,u} + 0.0017\,e^{-R_2\,u}, \qquad u \geq 0.$$

By Theorem 3.7.2, for $0 \leq x < u$,

$$\begin{aligned}
f_1(x\,|\,u) &= \bar{P}(x)\Big\{e^{-0.0278\,u}\big[-0.2825 + 0.2682\,e^{0.0278\,x} - 0.0174\,e^{-0.7393\,x}\\
&\quad +0.00301\,e^{-1.7949\,x}\big] + e^{-0.2291\,u}\big[-0.00055 + 0.00037\,e^{-0.2291\,x}\\
&\quad -0.00022\,e^{-0.7393\,x} + 0.000044\,e^{-1.7949\,x}\big]\\
&\quad -0.9275\,e^{0.7393\,u}\,e^{-0.7393x} + 0.3904\,e^{1.7949\,u}\,e^{-1.7949x}\Big\},
\end{aligned}$$

for $x \geq u$,

$$\begin{aligned}
f_1(x\,|\,u) &= \bar{P}(x)\Big\{0.3244 + 0.4224\,e^{-1.7949(x-u)} - 9924\,e^{-0.7393(x-u)}\\
&\quad -e^{-0.0278\,u}\big[0.2825 + 0.0174\,e^{-0.7393\,x} - 0.0029\,e^{-1.7949\,x}\big]\\
&\quad -e^{-0.2291\,u}\big[0.00055 + 0.000218\,e^{-0.7393\,x} - 0.0000435\,e^{-1.7949\,x}\big]\Big\}.
\end{aligned}$$

Since $f_2(x, y|u) = f_1(x \,|\, u)\frac{p(x+y)}{\bar{P}(x)}$, then

$$
\begin{aligned}
g(y \,|\, u) &= \int_0^\infty f_2(x, y|u)dx \\
&= e^{-0.25y}\Big[0.1195\, e^{-0.25\,u} - 0.2825\, e^{-0.2778\,u} - 0.000056\, e^{-1.2184\,u} \\
&\qquad -0.00439\, e^{-1.0171\,u} + 5.34 \times 10^{-5}\, e^{-2.274u} \\
&\qquad +0.0003689\, e^{-2.0727\,u} - 0.0005515\, e^{-0.4291\,u}\Big] \\
&\quad +e^{-0.2y}\Big[0.1056\, e^{-0.2\,u} - 0.2825\, e^{-0.2778u} - 4.72 \times 10^{-5} e^{-1.1684u} \\
&\qquad -0.003707\, e^{-0.9671\,u} - 0.0005515\, e^{-0.4291\,u} \\
&\qquad 0.438 \times 10^{-5}\, e^{-2.224u} + 0.0003026\, e^{-2.0227\,u}\Big].
\end{aligned}
$$

Setting the constant dividend barrier at $b = 10$, then gives for $0 \leq x < u$:

$$
\begin{aligned}
f_{b,1}(x \,|\, u) &= \bar{P}(x)\Big\{-0.1237 - 0.3177\, e^{-0.0278\,u} \\
&\qquad +0.1501\, e^{-0.2291\,u} - 0.316 \times 10^{-5} e^{0.7393u} \\
&\quad +e^{0.0278x}\Big[0.1171 + 0.3014\, e^{-0.0278\,u} - 0.1426\, e^{-0.2291\,u} \\
&\qquad +0.2995 \times 10^{-5}\, e^{0.7392u} - 0.3 \times 10^{-11}\, e^{1.7949u}\Big] \\
&\quad +e^{0.2291x}\Big[0.0002516 + 0.000067\, e^{-0.2291\,u} + 0.0000714 e^{-0.0278\,u} \\
&\qquad +0.2807 \times 10^{-8} e^{0.7393u}\Big] \\
&\quad +e^{-0.7393x}\Big[-2.6396 - 0.7429\, e^{-0.0278\,u} + 2.4679\, e^{-0.2291\,u} \\
&\qquad +0.00104\, e^{0.7393\,u} + 0.5 \times 10^{-11}\, e^{1.7949u}\Big] \\
&\quad +e^{-1.7949x}\Big[0.1816 + 0.0486\, e^{-0.0278\,u} - 0.00748\, e^{-0.2291\,u} \\
&\qquad +0.8643 \times 10^{-5}\, e^{0.7393u}\Big]\Big\},
\end{aligned}
$$

and for $u \le x < 10$:

$$\begin{aligned} f_{b,1}(x \mid u) &= \bar{P}(x)\Big\{0.2007 - 0.3177\,e^{-0.0278\,u} + 0.1501\,e^{-0.2291\,u} \\ &\qquad -0.316 \times 10^{-5} e^{0.7393u} + 0.7 \times 10^{-11} e^{1.7949u} \\ &\quad + e^{0.0278x}\Big[0.1171 + 0.03324\,e^{-0.0278\,u} - 0.1426\,e^{-0.2291\,u} \\ &\qquad +0.2995 \times 10^{-5}\,e^{0.7392u} - 0.3 \times 10^{-11}\,e^{1.7949u}\Big] \\ &\quad + e^{0.2291x}\Big[0.00025 + 0.00031\,e^{-0.2291\,u} + 0.000071 e^{-0.0278\,u} \\ &\qquad +0.2807 \times 10^{-8} e^{0.7393u}\Big] \\ &\quad + e^{-0.7393x}\Big[-2.6396 - 0.7403\,e^{-0.0278\,u} + 2.4679\,e^{-0.2291\,u} \\ &\qquad -0.0869\,e^{0.7393\,u} + 0.5 \times 10^{-9}\,e^{1.7949\,u}\Big] \\ &\quad + e^{-1.7949x}\Big[0.1816 + 0.0486\,e^{-0.0278\,u} - 0.00748\,e^{-0.2291\,u} \\ &\qquad +0.8643 \times 10^{-5}\,e^{0.7393u} + 0.032\,e^{1.7949\,u}\Big]\Big\}. \end{aligned}$$

Note that $U_b(T_b^-)$ has a probability mass at $x = b = 10$, given by

$$P\{U_b(T_b^-) = 10\} = \Psi_b(u) - \int_0^u f_{b,1}(x \mid u)dx + \int_u^{10} f_{b,1}(x \mid u)dx.$$

Finally,

$$\begin{aligned} g_b(y \mid u) &= e^{0.25y}\Big[-0.0791 + 0.0963\,e^{-0.2291\,u} - 0.0224\,e^{-0.0278\,u} - 0.00055\,e^{-0.4791\,u} \\ &\qquad -0.2825\,e^{-0.2778\,u} - 0.000056\,e^{-1.2184\,u} - 0.0044\,e^{-1.0171\,u} \\ &\qquad +0.00037\,e^{-2.0727\,u} + 0.1195\,e^{-0.25\,u}\Big] \\ &\quad + e^{-0.2y}\Big[-0.07827 + 0.0953\,e^{-0.2291\,u} - 0.0222\,e^{-0.0278\,u} + 0.1506\,e^{-0.2\,u} \\ &\qquad -0.00371\,e^{-0.9671} - 0.2825\,e^{-0.2278\,u} - 0.00055\,e^{-0.4291\,u} \\ &\qquad +0.0003\,e^{-2.0227\,u} - 0.000047\,e^{-1.1684\,u}\Big]. \end{aligned}$$

Chapter 5

A Class of Renewal Risk Processes Perturbed by Diffusion

5.1 Introduction

The classical risk model perturbed by a diffusion was first introduced by Gerber (1970) and has been further studied by many authors during the last few years; e.g. Dufresne and Gerber (1991), Furrer and Schmidli (1994), Schmidli (1995), Gerber and Landry (1998), Wang and Wu (2000), Wang (2001), Tsai (2001, 2003), Tsai and Willmot (2002a,b), Zhang and Wang (2003), Chiu and Yin (2003) and the references therein.

In this chapter, we consider the expected discounted penalty function for a Sparre Andersen risk process perturbed by a diffusion. As in Gerber and Shiu (2003a,b) we assume that claim inter-arrival times have a generalized Erlang(n) distribution. Our motivation is to keep the additional variability generated by the perturbing diffusion, as in Dufresne and Gerber (1991), but for a wider class of aggregate claim processes than compound Poisson.

5.2 Model Description and Notation

Consider a time–continuous Sparre Andersen surplus process perturbed by a diffusion

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i + \sigma B(t)\,, \qquad t \geq 0\,,$$

where $u \geq 0$ is the initial reserve. The X_i are i.i.d. random variables with common probability distribution function (d.f.) P and density p, representing the i-th claim amount. Let $\mu_k = E[X^k]$ be the k-th moment of X and $\hat{p}(s) = \int_0^\infty e^{-sx}\, p(x)\, dx$ the Laplace transform of density p.

The ordinary renewal process $\{N(t)\,;\, t \geq 0\}$ denotes the number of claims up to time t, with $N(t) = \max\{k \geq 1 : W_1 + \cdots + W_k \leq t\}$, where the i.i.d. claim waiting times W_i have a common generalized Erlang(n) distribution, i.e. the W_i's are distributed as the sum of n independent and exponentially distributed r.v.'s

$$S_n := V_1 + V_2 + \cdots + V_n\,, \qquad n \in \mathbb{N}^+, \tag{5.1}$$

where the V_i may have different exponential parameters $\lambda_i > 0$.

Finally, $\{B(t)\,;\, t \geq 0\}$ is a standard Wiener process that is independent of the compound ordinary renewal process $S(t) := \sum_{i=1}^{N(t)} X_i$ and the dispersion parameter $\sigma > 0$.

Further assume that $\{W_i\,;\, i \in \mathbb{N}^+\}$ and $\{X_i;\, i \in \mathbb{N}^+\}$ are independent and $c\,E(W_i) > E(X_i)$, that is $c \sum_{i=1}^{n} \frac{1}{\lambda_i} > \mu_1$, providing a positive safety loading factor.

Now define

$$T = \inf\{t \geq 0 \,:\, U(t) \leq 0\} \quad (\infty,\ \text{otherwise})\,,$$

to be the ruin time and

$$\Psi(u) = P(T < \infty \,|\, U(0) = u)\,, \qquad u \geq 0\,,$$

to be the ultimate ruin probability. Further, define

$$\Psi_d(u) = P(T < \infty,\, U(T) = 0 \,|\, U(0) = u), \qquad u \geq 0,$$

to be the probability of ruin caused by the oscillations in $U(t)$ due to the Wiener process $B(t)$, while

$$\Psi_s(u) = P(T < \infty,\, U(T) < 0 \,|\, U(0) = u), \qquad u \geq 0,$$

is the probability of ruin caused by a claim. We have that $\Psi(u) = \Psi_d(u) + \Psi_s(u)$, with $\Psi_d(0) = 1$, and $\Psi_s(0) = 0$.

Next, for $\delta \geq 0$ define

$$\phi_d(u) = E[e^{-\delta T}\, I(T < \infty,\, U(T) = 0) \,|\, U(0) = u]\,, \quad \text{with } \phi_d(0) = 1,$$

to be the Laplace transform of the ruin time T due to the oscillations. Now let $w(x, y)$, for $x, y \geq 0$, be the non-negative values of a penalty function and define

$$\phi_s(u) = E\left[e^{-\delta T}\, w\big(U(T^-),\, |U(T)|\big)\, I(T < \infty,\, U(T) < 0) \;\big|\; U(0) = u\right], \quad u \geq 0,$$

to be the expected discounted penalty function if the ruin is caused by a claim. Then

$$\phi(u) = \phi_d(u) + \phi_s(u), \qquad u \geq 0,$$

is the expected discounted penalty function.

Our first result gives integro–differential equations for ϕ_s and ϕ_d.

Theorem 5.2.1. (Integro–differential equations)

Let $\mathcal{I}$ and $\mathcal{D}$ denote the *identity operator* and *differential operator,* respectively. Then $\phi_s(u)$ satisfies the following equation for $u \geq 0$:

$$\left\{\prod_{j=1}^{n}\left[\left(1+\frac{\delta}{\lambda_j}\right)\mathcal{I} - \frac{c}{\lambda_j}\mathcal{D} - \frac{\sigma^2}{2\,\lambda_j}\mathcal{D}^2\right]\right\}\phi_s(u) = \int_0^u \phi_s(u-x)\,p(x)\,dx + \omega(u), \quad (5.2)$$

where $\omega(u) = \int_u^\infty w(u, x-u)\, p(x)\, dx$ and $\phi_s(0) = 0$, while $\phi_d(u)$ satisfies the following similar (homogeneous) equation for $u \geq 0$:

$$\left\{\prod_{j=1}^{n}\left[\left(1+\frac{\delta}{\lambda_j}\right)\mathcal{I}-\frac{c}{\lambda_j}\mathcal{D}-\frac{\sigma^2}{2\,\lambda_j}\mathcal{D}^2\right]\right\}\phi_d(u) = \int_0^u \phi_d(u-x)\, p(x)\, dx\ , \qquad (5.3)$$

with $\phi_d(0) = 1$.

Proof: First fix the number $j = 0, \ldots, n-1$, of exponential r.v.'s of the sum $S_j := V_1 + V_2 + \cdots + V_j$ in (5.1), with $S_0 = 0$, and define

$$\phi_{s,j}(u) = E[e^{-\delta(T-t)}\, w\big(U(T^-), |U(T)|\big)\, I(T<\infty,\, U(T)<0)\, \big|\, S_j = t,\, U(t) = u]\ ,$$

with $\phi_{s,0}(u) = \phi_s(u)$ and $\phi_{s,j}(0) = 0$, for $j = 0, 1, \ldots, n-1$. Then consider the infinitesimal interval from S_j to $S_j + dt$. Conditioning, one obtains for $j = 0, 1, \ldots, n-2$ that

$$\begin{aligned}\phi_{s,j}(u) &= e^{-\delta dt}\Big\{P(V_{j+1} > dt)\, E\left[\phi_{s,j}\big(u + cdt + \sigma B(dt)\big)\right] \\ &\quad + P(V_{j+1} \leq dt)\, E\left[\phi_{s,j+1}\big(u + cdt + \sigma B(dt)\big)\right]\Big\}, \quad u \geq 0. \quad (5.4)\end{aligned}$$

Also $e^{-\delta dt} = 1 - \delta dt + o(dt)$, while

$$P(V_{j+1} > dt) = 1 - \lambda_{j+1} dt + o(dt)\ , \qquad P(V_{j+1} \leq dt) = \lambda_{j+1} dt + o(dt)$$

and

$$E\left[\phi_{s,j}\big(u + cdt + \sigma B(dt)\big)\right] = \phi_{s,j}(u) + \big[c\phi'_{s,j}(u) + \frac{\sigma^2}{2}\phi''_{s,j}(u)\big]dt + o(dt)\ .$$

Substituting these formulas into (5.4), subtracting $\phi_{s,j}(u)$ from both sides, interpreting dt and $o(dt)$ terms and canceling common factors, we obtain as $dt \to 0$, that for $j = 0, 1, \ldots, n-2$:

$$\begin{aligned}\lambda_{j+1}\phi_{s,j+1}(u) &= (\lambda_{j+1} + \delta)\,\phi_{s,j}(u) - c\,\phi'_{s,j}(u) - \frac{\sigma^2}{2}\phi''_{s,j}(u) \\ &= \big[(\lambda_{j+1} + \delta)\,\mathcal{I} - c\,\mathcal{D} - \frac{\sigma^2}{2}\mathcal{D}^2\big]\,\phi_{s,j}(u)\ , \qquad (5.5)\end{aligned}$$

where again $\mathcal{I}$ and $\mathcal{D}$ denote the identity and differential operators. Similarly for $j = n-1$, we have

$$\Big[(\lambda_n + \delta)\mathcal{I} - c\,\mathcal{D} - \frac{\sigma^2}{2}\mathcal{D}^2\Big]\phi_{s,n-1}(u) = \lambda_n \left[\int_0^u \phi_{s,0}(u-x)\,p(x)\,dx + \omega(u)\right] . \quad (5.6)$$

It follows from successive substitutions, that for $1 \le m \le n-1$,

$$\phi_{s,m}(u) = \left\{\prod_{j=1}^{m}\left[\left(1+\frac{\delta}{\lambda_j}\right)\mathcal{I} - \frac{c}{\lambda_j}\mathcal{D} - \frac{\sigma^2}{2\lambda_j}\mathcal{D}^2\right]\right\}\phi_{s,0}(u) . \quad (5.7)$$

Now let $m = n-1$ in (5.7), use (5.6) and note that $\phi_{s,0}(u) = \phi_s(u)$, to obtain

$$\left\{\prod_{j=1}^{n}\left[\left(1+\frac{\delta}{\lambda_j}\right)\mathcal{I} - \frac{c}{\lambda_j}\mathcal{D} - \frac{\sigma^2}{2\,\lambda_j}\mathcal{D}^2\right]\right\}\phi_s(u) = \int_0^u \phi_s(u-x)\,p(x)\,dx + \omega(u) .$$

Finally note that $\phi_s(0) = 0$, since $P(T<\infty, U(T)<0\,|\,U(0)=0) = 0$.

To verify the homogeneous equation for $\phi_d(u)$, define for $j = 0, 1, \ldots, n-1$:

$$\phi_{d,j}(u) = E[e^{-\delta(T-t)}\,I(T<\infty,\, U(T)=0)\;\big|\; S_j = t, U(t) = u] , \quad u \ge 0,$$

with $\phi_{d,0}(u) = \phi_d(u)$ and $\phi_{d,j}(0) = 1$, for $j = 0, 1, \ldots, n-1$. Using arguments similar to those used for (5.5) and (5.6), we have for $j = 0, 1, \ldots, n-2$:

$$\lambda_{j+1}\phi_{d,j+1}(u) = \Big[(\lambda_{j+1} + \delta)\mathcal{I} - c\,\mathcal{D} - \frac{\sigma^2}{2}\mathcal{D}^2\Big]\phi_{d,j}(u) \quad (5.8)$$

and

$$\Big[(\lambda_n + \delta)\,I - c\,\mathcal{D} - \frac{\sigma^2}{2}\mathcal{D}^2\Big]\phi_{d,n-1}(u) = \lambda_n \int_0^u \phi_{d,0}(u-x)\,p(x)\,dx . \quad (5.9)$$

Then

$$\left\{\prod_{j=1}^{n}\left[\left(1+\frac{\delta}{\lambda_j}\right)\mathcal{I} - \frac{c}{\lambda_j}\mathcal{D} - \frac{\sigma^2}{2\,\lambda_j}\mathcal{D}^2\right]\right\}\phi_d(u) = \int_0^u \phi_d(u-x)\,p(x)\,dx , \quad u \ge 0.$$

Also note that $\phi_d(0) = 1$, since $P(T<\infty,\; U(T)=0\,|\,U(0)=0) = 1$. □

Note that equation (5.2) yields (D10) of Gerber and Shiu (2003a) when $\sigma = 0$, as well as equation (2) of Li and Garrido (2003) for the special case when $\lambda_j = \beta$, a constant for all $j = 1, 2, \ldots, n$ and $\sigma = 0$.

The solution to the integro–differential equations (5.2) and (5.3) are closely related to the roots of a generalized Lundberg equation. This is discussed in the next section.

5.3 A Generalized Lundberg Equation

Let $\tau_k = \sum_{j=1}^k W_j$ be the arrival time of the k-th claim. Consider the surplus $U_k = U(\tau_k)$ immediately after k-th claim. Defining $\tau_0 = 0$ gives $U_0 = u$, and for $k = 1, 2, \ldots,$

$$\begin{aligned} U_k = U(\tau_k) &= u + c\,\tau_k - \sum_{j=1}^{k} X_j + \sigma B(\tau_k) \\ &= u + \sum_{j=1}^{k} \left[c\,W_j - X_j + \sigma B(W_j)\right] . \end{aligned}$$

We seek a number s such that the process $\{e^{-\delta\,\tau_k + s\,U_k}\,;\; k = 0, 1, 2, \ldots\}$ will form a martingale. Here this martingale condition is equivalent to

$$E\left[e^{-\delta W_1 + c\,s W_1 + s\,\sigma B(W_1) - s X_1}\right] = E\left[e^{-(\delta - c\,s) W_1 + s\,\sigma B(W_1)}\right] E\left[e^{-s X_1}\right] = 1 \,. \qquad (5.10)$$

Since

$$E\left[e^{-(\delta-cs)W_1+s\sigma B(W_1)}\right] = E\{E\left[e^{-(\delta-cs)W_1+s\sigma B(W_1)}\mid W_1\right]\} = E\left[e^{-(\delta-cs)W_1+\frac{s^2\sigma^2}{2}W_1}\right],$$

and W_1 is generalized Erlang(n), then (5.10) simplifies to

$$E\left[e^{-\left(\delta - c\,s - \frac{\sigma^2}{2}s^2\right)W_1}\right]\hat{p}(s) = \prod_{j=1}^{n} E\left[e^{-\left(\delta - c\,s - \frac{\sigma^2}{2}s^2\right)V_j}\right]\hat{p}(s) = 1\,, \qquad s \in \mathbb{C}\,. \qquad (5.11)$$

Let $\gamma(s) := \prod_{j=1}^{n}\left[\left(1+\frac{\delta}{\lambda_j}\right)-\frac{c}{\lambda_j}\,s-\frac{\sigma^2}{2\lambda_j}\,s^2\right]$, then (5.11) is equivalent to

$$\gamma(s)=\hat{p}(s)\,, \qquad \delta\geq 0,\ n\in\mathbb{N}^+ \text{ and } s\in\mathbb{C}\,, \tag{5.12}$$

which is a *Generalized Lundberg Fundamental Equation.*

Remark: Equation (5.12) simplifies to (4.2) of Gerber and Shiu (2003b) when $\sigma=0$ and to the generalized Lundberg equation in Li and Garrido (2003) for the Erlang(n) risk process, i.e. $\sigma=0$ and $\lambda_j=\beta$, a constant for all $j=1,2,\ldots,n$.

We can prove the following about the roots of this generalized Lundberg equation.

Theorem 5.3.1. For $\delta>0$ and $n\in\mathbb{N}^+$, Lundberg's equation in (5.12) has exactly n roots, say $\rho_1(\delta,\sigma),\rho_2(\delta,\sigma),\ldots,\rho_n(\delta,\sigma)$ with a positive real part $\Re(\rho_j)>0$.

Proof: Since the factors $\left(1+\frac{\delta}{\lambda_j}\right)-\frac{c}{\lambda_j}\,s-\frac{\sigma^2}{2\lambda_j}\,s^2=0$ have exactly two solutions $s_1=-\frac{c}{\sigma^2}-\sqrt{\frac{c^2}{\sigma^4}+\frac{(\lambda_j+\delta)^2}{\sigma^2}}<0$ and $s_2=-\frac{c}{\sigma^2}+\sqrt{\frac{c^2}{\sigma^4}+\frac{(\lambda_j+\delta)^2}{\sigma^2}}>0$, we see that the product $\gamma(s)=\prod_{j=1}^{n}\left[\left(1+\frac{\delta}{\lambda_j}\right)-\frac{c}{\lambda_j}\,s-\frac{\sigma^2}{2\lambda_j}\,s^2\right]$ has exactly n positive zeros.

On the half circle in the complex plane given by $z=r$ (for $r>0$ fixed) and $\Re(z)\geq 0$, we have that $|\gamma(s)|>1$, if r is sufficiently large. While for s on the imaginary axis ($\Re(s)=0$) we have that $|\gamma(s)|\geq\prod_{i=1}^{n}\frac{\lambda_j+\delta}{\lambda_j}>1$. That is, on the contour boundary of the half circle and the imaginary axis, $|\gamma(s)|>|\hat{p}(s)|$. Then we conclude that, on the right half plane, the number of the roots to Lundberg's equation equals to the number of roots of $\gamma(s)=0$. Since the later has exactly n positive roots, we can say that $\rho_1(\delta,\sigma),\rho_2(\delta,\sigma),\ldots,\rho_n(\delta,\sigma)$ are the only roots to Lundberg's equation that have a positive real part, although others may exist. □

Remarks:

1. Define $l(s):=\hat{p}(s)-\gamma(s)$. Since $l(0)<0$ and $\lim_{s\to-\infty}l(s)=+\infty$, then for $p(x)$ sufficiently regular, there is one negative root to $l(s)=0$, say $-R(\delta,\sigma)$. We call $R(\delta,\sigma)>0$ a *generalized adjustment coefficient.*

2. If $\delta \to 0^+$ then $-R(\delta,\sigma) \to -R(0,\sigma)$ and $\rho_j(\delta,\sigma) \to \rho_j(0,\sigma)$, for $1 \le j \le n$, where $-R(0,\sigma)$ and $\rho_j(0,\sigma)$ are roots to the following equation:

$$\gamma_{0,\sigma}(s) := \prod_{j=1}^{n}\Big[1 - \frac{c}{\lambda_j}\,s - \frac{\sigma^2}{2\lambda_j}\,s^2\Big] = \hat{p}(s)\,, \qquad s \in \mathbb{C}\,.$$

3. If $\sigma^2 \to 0$ then $-R(\delta,\sigma) \to -R(\delta,0)$ and $\rho_j(\delta,\sigma) \to \rho_j(\delta,0)$, for $1 \le j \le n$, where $-R(\delta,0)$ and $\rho_j(\delta,0)$ are roots of equation:

$$\gamma_{\delta,0}(s) := \prod_{j=1}^{n}\Big[\big(1 + \frac{\delta}{\lambda_j}\big) - \frac{c}{\lambda_j}\,s\Big] = \hat{p}(s)\,, \qquad s \in \mathbb{C}\,. \tag{5.13}$$

4. For simplicity, write $-R$ and ρ_j for $-R(\delta,\sigma)$ and $\rho_j(\delta,\sigma)$, $1 \le j \le n$, when $\delta > 0$ and $\sigma > 0$.

5.4 Main Results

We are now ready to solve the integro–differential equations (5.2) and (5.3). First, as in Gerber and Shiu (2003b), we use the concept of *divided differences*. For distinct numbers $r_1, r_2, \ldots, r_k$, the k-th divided difference $h[r_1, r_2, \ldots, r_k, s]$ of a function h is defined recursively as follows:

$$\begin{aligned} h(s) &= h(r_1) + (s - r_1)\,h[r_1, s]\,, \\ h[r_1, s] &= h[r_1, r_2] + (s - r_2)\,h[r_1, r_2, s]\,, \\ h[r_1, r_2, \ldots, r_{k-1}, s] &= h[r_1, r_2, \ldots, r_k] + (s - r_k)\,h[r_1, r_2, \ldots, r_k, s]\,. \end{aligned}$$

Note that if $h(s)$ is a polynomial of degree n, then $h[r_1, r_2, \ldots, r_k, s]$ is a polynomial of degree $n-k$, while $h[r_1, r_2, \ldots, r_n, s]$ is the coefficient of s^n in $h(s)$. The following result also holds:

$$h[r_1, r_2, \ldots, r_k] = \sum_{j=1}^{k} \frac{h(r_j)}{\pi_k'(r_j; r_1, r_2, \ldots, r_k)}\,, \tag{5.14}$$

where $\pi_k(s; r_1, r_2, \ldots, r_k) = \prod_{i=1}^{k}(s - r_i)$ is a polynomial. Finally, note that if two functions have the same values at points $r_1, r_2, \ldots, r_k$, then they must also share the same m-th divided differences for $m \leq k$.

Next, as in Li and Garrido (2004), we define the operator T_r of a real–valued function f, with respect to a complex number r as:

$$T_r f(x) = \int_x^{\infty} e^{-r(y-x)} f(y)\, dy\ , \qquad x \geq 0\ .$$

Using the operator T_r and divided differences, one obtains the following relations to the roots of Lundberg's equation .

Theorem 5.4.1. For $u \geq 0$, there exists a polynomial γ in terms of the differential operator $\mathcal{D}$ such that

$$(-1)^n \gamma[\rho_1, \rho_2, \ldots, \rho_n, \mathcal{D}]\, \phi_s(u) = \int_0^u \phi_s(u-y)\, \eta(y)\, dy + G(u)\ , \qquad (5.15)$$

$$(-1)^n \gamma[\rho_1, \rho_2, \ldots, \rho_n, \mathcal{D}]\, \phi_d(u) = \int_0^u \phi_d(u-y)\, \eta(y)\, dy\ , \qquad u \geq 0\ , (5.16)$$

where $\rho_1, \rho_2, \ldots, \rho_n$ are the n roots of the generalized Lundberg equation (5.12) with positive real parts, $\eta(y) = T_{\rho_n} T_{\rho_{n-1}} \cdots T_{\rho_1} p(y)$ and $G(u) = T_{\rho_n} \cdots T_{\rho_1} \omega(u)$, where ω is from (5.2).

Remark: Equation (5.15) and (5.16) are integro–differential equations of order n for ϕ_s and ϕ_d, respectively, since by (5.14)

$$\gamma[\rho_1, \rho_2, \ldots, \rho_n, s] = \sum_{j=1}^{n} \frac{\gamma(\rho_j)}{\pi_n'(\rho_j; \rho_1, \rho_2, \ldots, \rho_n)\,(\rho_j - s)} + \frac{\gamma(s)}{\pi_n(s; \rho_1, \rho_2, \ldots, \rho_n)}$$

is a polynomial of degree n.

Proof: See Section 10 of Gerber and Shiu (2003b). □

To solve equations (5.15) and (5.16) we take Laplace transforms, first on both sides of equation (5.15), to get:

$$\left\{(-1)^n \gamma[\rho_1, \rho_2, \ldots, \rho_n, s] - \hat{\eta}(s)\right\} \hat{\phi}_s(s) = \hat{G}(s) + q_{n-1}(s)\ , \qquad s \in \mathbb{C}\ , \quad (5.17)$$

where $\hat{\eta}(s) = T_s\eta(0) = T_s T_{\rho_n} \cdots T_{\rho_1} p(0)$ and $\hat{G}(s) = T_s G(0) = T_s T_{\rho_n} \cdots T_{\rho_1}\omega(0)$ are Laplace transforms of η and G, respectively, $q_{n-1}(s)$ is a polynomial of degree of $n-1$ or less, with coefficients in terms of δ, c, λ_i, ρ_i, for $i = 1, 2, \ldots, n$, and the derivatives of ϕ_s at 0, $\phi_s^{(k)}(0)$, for $k = 0, 1, 2, \ldots, n-1$.

Since $\gamma[\rho_1, \rho_2, \ldots, \rho_n, s]$ is a polynomial of degree n and the coefficient of s^n is equal to that of s^{2n} in $\gamma(s)$, which is $(-1)^n \frac{\sigma^{2n}}{2^n \lambda^*}$, where $\lambda^* = \prod_{i=1}^n \lambda_i$, then $\gamma[\rho_1, \rho_2, \ldots, \rho_n, s]$ can be factored as

$$\gamma[\rho_1, \rho_2, \ldots, \rho_n, s] = (-1)^n \frac{\sigma^{2n}(s+a_1)(s+a_2)\cdots(s+a_n)}{2^n \lambda^*}, \qquad s \in \mathbb{C}, \quad (5.18)$$

where the $a_1, a_2, \ldots, a_n$ come in pairs of conjugate complex numbers. Equation (5.17) can thus be rewritten as

$$\begin{aligned} &\hat{\phi}_s(s)\left[1 - \frac{2^n \lambda^* \hat{\eta}(s)}{\sigma^{2n}(s+a_1)(s+a_2)\cdots(s+a_n)}\right] \\ &= \frac{2^n \lambda^*}{\sigma^{2n}}\left[\frac{\hat{G}(s)}{(s+a_1)\cdots(s+a_n)} + \frac{q_{n-1}(s)}{(s+a_1)\cdots(s+a_n)}\right] \\ &= \frac{2^n \lambda^* \hat{G}(s)}{\sigma^{2n}(s+a_1)(s+a_2)\cdots(s+a_n)} + \sum_{i=1}^n \frac{b_i}{(s+a_i)}, \qquad s \in \mathbb{C}, (5.19) \end{aligned}$$

where the coefficients b_i are given by $b_i = \frac{2^n \lambda^* q_{n-1}(-a_i)}{\sigma^{2n}\left[\prod_{j=1, j\neq i}^n (a_j - a_i)\right]}$, for $i = 1, 2, \ldots, n$. Similarly, taking Laplace transforms on both sides of equation (5.16) yields

$$\{(-1)^n \gamma[\rho_1, \rho_2, \ldots, \rho_n, s] - \hat{\eta}(s)\}\hat{\phi}_d(s) = Q_{n-1}(s), \qquad s \in \mathbb{C}, \qquad (5.20)$$

where $Q_{n-1}(s)$ is a polynomial of degree $\leq n-1$, with coefficients in terms of δ, c, λ_i, ρ_i, for $i = 1, 2, \ldots, n$ and $\phi_d^{(k)}(0)$, $k = 0, 1, 2, \ldots, n-1$. Then

$$\begin{aligned} &\hat{\phi}_d(s)\left[1 - \frac{2^n \lambda^* \hat{\eta}(s)}{\sigma^{2n}(s+a_1)(s+a_2)\cdots(s+a_n)}\right] \\ &= \frac{2^n \lambda^* Q_{n-1}(s)}{\sigma^{2n}(s+a_1)(s+a_2)\cdots(s+a_n)} = \sum_{i=1}^n \frac{c_i}{(s+a_i)}, \qquad s \in \mathbb{C}, (5.21) \end{aligned}$$

where the coefficients $c_i = \frac{2^n \lambda^* Q_{n-1}(-a_i)}{\sigma^{2n}\left[\prod_{j=1, j\neq i}^n (a_j - a_i)\right]}$, for $i = 1, 2, \ldots, n$.

These can be solved for ϕ_s and ϕ_d by inverting Laplace transforms. The following Theorem shows that ϕ_s, ϕ_d, and ϕ all accept renewal equation representations.

Theorem 5.4.2. ϕ_s, ϕ_d and ϕ satisfy the following renewal equations

$$\phi_s(u) = \int_0^u \phi_s(u-y)\, g(y)\, dy + H(u) + \sum_{i=1}^n b_i e^{-a_i u}\,, \qquad u \geq 0\,, \tag{5.22}$$

$$\phi_d(u) = \int_0^u \phi_d(u-y)\, g(y)\, dy + \sum_{i=1}^n c_i e^{-a_i u}\,, \qquad u \geq 0\,, \tag{5.23}$$

$$\phi(u) = \int_0^u \phi(u-y)\, g(y)\, dy + H(u) + \sum_{i=1}^n (c_i + b_i) e^{-a_i u}\,, \quad u \geq 0\,, \tag{5.24}$$

where a_i, b_i, c_i, η and G are as above, $g(y) := h * \eta(y) := h_1 * \cdots * h_n * \eta(y)$, $H(u) := h * G(u) = h_1 * \cdots * h_n * G(u)$, with $h_i(y) = \frac{\lambda_i}{\sigma^2/2} e^{-a_i y}$, for $i = 1, 2, \ldots, n$ and $*$ denotes the convolution product.

Proof: Inverting the Laplace transforms (5.19) and (5.21) gives the renewal equations (5.22) and (5.23). Since $\phi(u) = \phi_s(u) + \phi_d(u)$, add (5.22) to (5.23) to obtain (5.24). □

Remarks:

1. When $n = 1$, equation (5.22) yields (17) of Gerber and Landry (1998), while equation (5.23) gives (2.10) of Tsai and Willmot (2002).

2. Since $\phi_s(u)$ and $\phi_d(u)$ go to 0 as u goes to ∞, we can conclude that $\Re(a_i) > 0$, for $i = 1, 2, \ldots, n$.

Moreover, the following lemmas give expressions for g and H that are useful in applications, as well as to study the limiting behavior of ϕ_s and ϕ_d when $\sigma^2 \to 0$.

Lemma 5.4.1. For $\sigma^2 > 0$, the function g can be expressed, for $y \geq 0$, as

$$g(y) = \left(\frac{2^n \lambda^*}{\sigma^{2n}}\right) \sum_{i=1}^n \frac{e^{-a_i y} T_{-a_i} T_{\rho_n} \cdots T_{\rho_1} p(0) - T_{-a_i} T_{\rho_n} \cdots T_{\rho_1} p(y)}{(-1)^{n-1} \pi_n'(a_i; a_1, a_2, \ldots, a_n)}\,, \tag{5.25}$$

while its Laplace transform $\hat{g}$ is given by

$$\hat{g}(s) = 1 - (-1)^n \frac{[\gamma(s) - \hat{p}(s)]}{\left[\prod_{i=1}^n (\rho_i - s)\right] \gamma[\rho_1, \rho_2, \ldots, \rho_n, s]} \tag{5.26}$$

$$= 1 - \left(\frac{2^n \lambda^*}{\sigma^{2n}}\right) \frac{[\gamma(s) - \hat{p}(s)]}{\left[\prod_{i=1}^n (\rho_i - s)\right] \left[\prod_{i=1}^n (s + a_i)\right]}, \qquad s \in \mathbb{C}. \tag{5.27}$$

Therefore, for $y \geq 0$

$$g(y) = h * \eta(y) \to g_0(y) := \frac{\lambda^*}{c^n} T_{\rho_n(\delta,0)} T_{\rho_{n-1}(\delta,0)} \cdots T_{\rho_1(\delta,0)} p(y),$$

as $\sigma^2 \to 0$.

Proof: Since

$$\hat{h}(s) = \prod_{i=1}^n \hat{h}_i(s) = \prod_{i=1}^n \frac{\frac{\lambda_i}{\sigma^2/2}}{(s + a_i)}$$

$$= \left(\frac{2^n \lambda^*}{\sigma^{2n}}\right) \sum_{i=1}^n \frac{1}{(-1)^{n-1} \pi_n'(a_i; a_1, a_2, \ldots, a_n)(s + a_i)}, \qquad s \in \mathbb{C},$$

then

$$h(y) = \left(\frac{2^n \lambda^*}{\sigma^{2n}}\right) \sum_{i=1}^n \frac{e^{-a_i y}}{(-1)^{n-1} \pi_n'(a_i; a_1, a_2, \ldots, a_n)}, \qquad y \geq 0. \tag{5.28}$$

Also

$$\int_0^y e^{-a_i(y-x)} \eta(x)\, dx = \int_0^\infty e^{-a_i(y-x)} \eta(x)\, dx - \int_y^\infty e^{-a_i(y-x)} \eta(x)\, dx$$

$$= e^{-a_i y} \int_0^\infty e^{a_i x} \eta(x)\, dx - \int_y^\infty e^{a_i(x-y)} \eta(x)\, dx$$

$$= e^{-a_i y} T_{-a_i} \eta(0) - T_{-a_i} \eta(y), \qquad y \geq 0, \tag{5.29}$$

implies

$$g(y) = h * \eta(y) = \left(\frac{2^n \lambda^*}{\sigma^{2n}}\right) \sum_{i=1}^n \frac{e^{-a_i y} T_{-a_i} \eta(0) - T_{-a_i} \eta(y)}{(-1)^{n-1} \pi_n'(a_i; a_1, a_2, \ldots, a_n)}, \qquad y \geq 0,$$

$$= \left(\frac{2^n \lambda^*}{\sigma^{2n}}\right) \sum_{i=1}^n \frac{e^{-a_i y} T_{-a_i} T_{\rho_n} \cdots T_{\rho_1} p(0) - T_{-a_i} T_{\rho_n} \cdots T_{\rho_1} p(y)}{(-1)^{n-1} \pi_n'(a_i; a_1, a_2, \ldots, a_n)}.$$

Now, the Laplace transform of g is given by

$$\begin{aligned}
\hat{g}(s) &= \hat{h}(s)\hat{\eta}(s) = \left(\prod_{i=1}^{n} \frac{\frac{\lambda_i}{\sigma^2/2}}{(s+a_i)}\right) T_s T_{\rho_n} \cdots T_{\rho_1} p(0) \;, \qquad s \in \mathbb{C} \;, \\
&= (-1)^n \left(\prod_{i=1}^{n} \frac{\frac{\lambda_i}{\sigma^2/2}}{(s+a_i)}\right) \hat{p}[\rho_1, \rho_2, \ldots, \rho_n, s] \;, \qquad \text{by (2.7)} \;, \\
&= (-1)^n \left(\prod_{i=1}^{n} \frac{\frac{\lambda_i}{\sigma^2/2}}{(s+a_i)}\right) \left\{\gamma[\rho_1, \rho_2, \ldots, \rho_n, s] + (\hat{p} - \gamma)[\rho_1, \rho_2, \ldots, \rho_n, s]\right\} \\
&= \frac{\gamma[\rho_1, \rho_2, \ldots, \rho_n, s] + \frac{[\hat{p}(s) - \gamma(s)]}{\prod_{i=1}^{n}(s-\rho_i)}}{\gamma[\rho_1, \rho_2, \ldots, \rho_n, s]} = 1 - \frac{(-1)^n[\gamma(s) - \hat{p}(s)]}{\prod_{i=1}^{n}(\rho_i - s)\,\gamma[\rho_1, \rho_2, \ldots, \rho_n, s]} \;,
\end{aligned}$$

this proves (5.26). Similarly, to verify (5.27) use (5.18). Finally, as $\sigma^2 \to 0$:

$$\begin{aligned}
\gamma(s) &\to \gamma_{\delta,0}(s) = \prod_{j=1}^{n}\left[\left(1 + \frac{\delta}{\lambda_j}\right) - \frac{c}{\lambda_j} s\right] \;, \qquad s \in \mathbb{C} \;, \\
\rho_i(\delta, \sigma) &\to \rho_i(\delta, 0) \;, \qquad \text{for } i = 1, 2, \ldots, n \;, \qquad \text{and} \\
\gamma[\rho_1, \rho_2, \ldots, \rho_n, s] &\to \gamma_{\delta,0}[\rho_1(\delta,0), \rho_2(\delta,0), \ldots, \rho_n(\delta,0), s] = \frac{(-c)^n}{\lambda^*} \;.
\end{aligned}$$

The first two limits follow from (5.13). The last line represents the n-th order divided difference of a polynomial, $\gamma_{\delta,0}$, of order n; hence a constant equal to the leading coefficient of the polynomial.

Accordingly,

$$\begin{aligned}
\hat{g}(s) &\to 1 - \frac{\lambda^*}{c^n} \frac{[\gamma_{\delta,0}(s) - \hat{p}(s)]}{\left\{\prod_{i=1}^{n}[\rho_i(\delta,0) - s]\right\}} \;, \qquad s \in \mathbb{C} \;, \\
&= (-1)^n \frac{\lambda^*}{c^n} \left[\prod_{i=1}^{n} \frac{-c}{\lambda_i} - \frac{[\gamma_{\delta,0}(s) - \hat{p}(s)]}{\prod_{i=1}^{n}[s - \rho_i(\delta,0)]}\right] \\
&= (-1)^n \frac{\lambda^*}{c^n} \Big\{\gamma_{\delta,0}[\rho_1(\delta,0), \ldots, \rho_n(\delta,0), s] \\
&\qquad - (\gamma_{\delta,0} - \hat{p})[\rho_1(\delta,0), \ldots, \rho_n(\delta,0), s]\Big\} \\
&= (-1)^n \frac{\lambda^*}{c^n} \hat{p}[\rho_1(\delta,0), \ldots, \rho_n(\delta,0), s] \\
&= (-1)^{2n} \frac{\lambda^*}{c^n} T_s T_{\rho_n(\delta,0)} \cdots T_{\rho_1(\delta,0)} p(0) \;,
\end{aligned}$$

then, we have

$$g(y) \to g_0(y) := \frac{\lambda^*}{c^n} T_{\rho_n(\delta,0)} \cdots T_{\rho_1(\delta,0)}\, p(y) \,, \qquad y \geq 0 \,.$$

This completes the proof. □

Similarly, the following lemma holds for the function H.

Lemma 5.4.2. For $\sigma^2 > 0$, the function H can be expressed, for $y \geq 0$, as

$$H(y) = \left(\frac{2^n \lambda^*}{\sigma^{2n}}\right) \sum_{i=1}^{n} \frac{e^{-a_i y} T_{-a_i} T_{\rho_n} \cdots T_{\rho_1}\omega(0) - T_{-a_i} T_{\rho_n} \cdots T_{\rho_1}\omega(y)}{(-1)^{n-1}\pi_n'(a_i; a_1, a_2, \ldots, a_n)} \,, \tag{5.30}$$

while its Laplace transform $\hat{H}$ is given by

$$\hat{H}(s) = \hat{h}(s)\, \hat{G}(s) = \frac{T_s T_{\rho_n} \cdots T_{\rho_1}\, \omega(0)}{(-1)^n \gamma[\rho_1, \rho_2, \ldots, \rho_n, s]} \,, \qquad s \in \mathbb{C} \,. \tag{5.31}$$

Therefore, when $\sigma^2 \to 0$

$$H(u) = h * G(u) \to H_0(u) := \frac{\lambda^*}{c^n} T_{\rho_n(\delta,0)} \cdots T_{\rho_1(\delta,0)}\, \omega(u) \,, \qquad u \geq 0 \,. \tag{5.32}$$

Remarks:

1. Since

$$\begin{aligned}
\int_0^\infty g(y)\, dy &= \hat{g}(0) = \lim_{s \to 0} g(s) \\
&= \lim_{s \to 0} \left[1 - \left(\frac{2^n \lambda^*}{\sigma^{2n}}\right) \frac{[\gamma(s) - \hat{p}(s)]}{[\prod_{i=1}^n (\rho_i - s)]\, [\prod_{i=1}^n (s + a_i)]}\right] \\
&= 1 - \left(\frac{2^n \lambda^*}{\sigma^{2n}}\right) \frac{[\prod_{i=1}^n (1 + \frac{\delta}{\lambda_i}) - 1]}{(\prod_{i=1}^n \rho_i)(\prod_{i=1}^n a_i)} < 1 \,,
\end{aligned}$$

then equations (5.22), (5.23) and (5.24) are *defective* renewal equations.

2. As $\sigma^2 \to 0$, then $\gamma[\rho_1, \rho_2, \ldots, \rho_n, s] \to \frac{(-c)^n}{\lambda^*}$ and hence, both $q_{n-1}(s)$ and $Q_{n-1}(s) \to 0$. This means that b_i and $c_i \to 0$, for $i = 1, 2, \ldots, n$, and in turn, $\phi_s(u) \to \phi_0(u)$, where ϕ_0 satisfies following defective renewal equation:

$$\phi_0(u) = \int_0^u \phi_0(u - y)\, g_0(y)\, dy + H_0(u) \,, \qquad u \geq 0 \,,$$

while $\phi_d(u) = \int_0^u \phi_d(u-y)\, g(y)\, dy + \sum_{i=1}^n c_i\, e^{-a_i u} \to 0$. We can thus finally conclude that $\phi(u) = \phi_d(u) + \phi_s(u) \to \phi_0(u)$, as $\sigma^2 \to 0$, where $\phi_0(u)$ is the expected discounted penalty function for a Sparre Andersen risk process with claim waiting times distributed as the sum of n independent exponentials with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ [see Gerber and Shiu (2003b)].

5.5 Determination of Initial Conditions

$\phi_s(u)$ and $\phi_d(u)$ are uniquely determined by the $2n$-th order integro–differential equations (5.2) and (5.3), if initial conditions $\phi_s^{(k)}(0)$ and $\phi_d^{(k)}(0)$ are given for $k = 0, 1, 2, \ldots, 2n-1$. In fact, as we will see, it is sufficient to know these initial conditions for $k = 0, 1, 2, \ldots, n-1$, to solve equations (5.15) and (5.16).

Taking Laplace transforms on both sides of the integro–differential equation (5.2) yields

$$\hat{\phi}_s(s) = \frac{[\hat{\omega}(s) + q(s)]}{[\gamma(s) - \hat{p}(s)]}\,, \qquad s \in \mathbb{C}\,,$$

where $\gamma(s) = \prod_{j=1}^n \left[\left(1 + \frac{\delta}{\lambda_j}\right) - \frac{c}{\lambda_j} s - \frac{\sigma^2}{2\lambda_j} s^2\right] = \sum_{k=0}^{2n} e_k\, s^k$ and

$$q(s) = \sum_{j=0}^{2n-1} s^j \sum_{k=j+1}^{2n} e_k\, \phi_s^{(k-1-j)}(0)\,, \qquad s \in \mathbb{C}\,,$$

is a polynomial of degree $2n-1$. Since $\hat{\phi}_s(s)$ is finite for all complex number s such that $\Re(s) > 0$, we have that

$$\hat{\omega}(\rho_i) = -q(\rho_i)\,, \qquad i = 1, 2, \ldots, n\,. \tag{5.33}$$

Changing the order of summation, this becomes

$$\sum_{m=0}^{2n-1} \phi_s^{(m)}(0) \sum_{k=m+1}^{2n} e_k\, \rho_i^{k-m-1} = -\hat{\omega}(\rho_i)\,, \qquad i = 1, 2, \ldots, n\,. \tag{5.34}$$

To determine the $\phi_s^{(k)}(0)$ values for $k = 0, 2, \ldots, 2n-1$, another n conditions are needed. Setting $u = 0$ in (5.7) yields

$$\Big\{\prod_{j=1}^{m}[(1+\frac{\delta}{\lambda_j})\mathcal{I}-\frac{c}{\lambda_j}\mathcal{D}-\frac{\sigma^2}{2\lambda_j}\mathcal{D}^2]\Big\}\phi_s(u)\Big|_{u=0} = 0\,, \qquad m = 1, 2, \ldots, n-1\,. \quad (5.35)$$

Then (5.34) and (5.35), together with $\phi_s(0) = 0$ yield a system of $2n \times 2n$ linear equations that can be solved for the unknowns $\phi_s^{(k)}(0)$, $k = 0, 1, \ldots, 2n-1$.

Similarly, the $\phi_d^{(k)}(0)$ values, $k = 0, 1, 2, \ldots, 2n-1$ and $\phi_d(0) = 1$, satisfy the following linear system:

$$\sum_{m=0}^{2n-1} \phi_d^{(m)}(0) \sum_{k=m+1}^{2n} e_k\, \rho_i^{k-m-1} \; = \; 0\,, \quad i = 1, 2, \ldots, n\,, \quad (5.36)$$

$$\Big\{\prod_{j=1}^{m}[(1+\frac{\delta}{\lambda_j})\mathcal{I}-\frac{c}{\lambda_j}\mathcal{D}-\frac{\sigma^2}{2\lambda_j}\mathcal{D}^2]\Big\}\phi_d(u)\Big|_{u=0} \; = \; 1\,, \quad m = 1, \ldots, n-1 \quad (5.37)$$

5.6 Examples

This section illustrates how to obtain the above results explicitly for the special case when $n = 2$. Then

$$\gamma(s) = \frac{1}{\lambda_1\lambda_2}\left[(\lambda_1+\delta) - cs - \frac{\sigma^2}{2}s^2\right]\left[(\lambda_2+\delta) - cs - \frac{\sigma^2}{2}s^2\right] = \sum_{k=0}^{4} e_k\, s^k\,,$$

where $e_0 = \frac{(\lambda_1+\delta)(\lambda_2+\delta)}{\lambda_1\lambda_2}$, $e_1 = -\frac{c(\lambda_1+\lambda_2+2\delta)}{\lambda_1\lambda_2}$, $e_2 = \frac{[c^2 - \frac{\sigma^2(\lambda_1+\lambda_2+2\delta)}{2}]}{\lambda_1\lambda_2}$, $e_3 = \frac{c\sigma^2}{\lambda_1\lambda_2}$ and $e_4 = \frac{\sigma^4}{4\lambda_1\lambda_2}$. Then $\gamma[\rho_1, \rho_2, s]$ can be simplified to

$$\begin{aligned}\gamma[\rho_1, \rho_2, s] &= e_4\, s^2 + \big[e_3 + e_4(\rho_2+\rho_1)\big]s + e_2 + e_3(\rho_1+\rho_2) + e_4(\rho_1^2 + \rho_1\rho_2 + \rho_2^2)\\ &= e_4(s+a_1)(s+a_2)\,,\end{aligned}$$

where ρ_1 and ρ_2 are the only positive roots (i.e $\Re(\rho_i) > 0$) of the equation $\gamma(s) - \hat{p}(s) = 0$, while a_1 and a_2 are roots obtained as in (5.18).

By (5.34) and (5.35), here $\phi_s^{(k)}(0)$, $k = 0, 1, 2, 3$, satisfy the following system of linear equations:

$$\begin{aligned} 0 &= \phi_s(0)\,, \\ -\hat{\omega}(\rho_1) &= \phi_s'(0)(e^2 + e_3\,\rho_1 + e_4\,\rho_1^2) + \phi_s''(0)(e_3 + e_4\,\rho_1) + \phi_s^{(3)}(0)\,e^4\,, \\ -\hat{\omega}(\rho_2) &= \phi_s'(0)(e^2 + e_3\,\rho_2 + e_4\,\rho_2^2) + \phi_s''(0)(e_3 + e_4\,\rho_2) + \phi_s^{(3)}(0)\,e^4\,, \\ 0 &= c\phi_s'(0) + \frac{\sigma^2}{2}\phi_s''(0)\,, \end{aligned}$$

which solves for $\phi_s(0) = 0$ and $\phi_s'(0) = \frac{T_{\rho_2}\,T_{\rho_1}\omega(0)}{[e_3+e_4\,(\rho_1+\rho_2-\frac{2c}{\sigma^2})]} = \frac{4\lambda_1\lambda_2 T_{\rho_2}\,T_{\rho_1}\omega(0)}{[2c\,\sigma^2+\sigma^4\,(\rho_1+\rho_2)]}$. Hence $q_1(s) = e_4\phi_s'(0) = \frac{T_{\rho_2}\,T_{\rho_1}\omega(0)}{[\frac{2c}{\sigma^2}+\rho_1+\rho_2]}$, while $b_1 = \frac{4\lambda_1\lambda_2 T_{\rho_2}\,T_{\rho_1}\omega(0)}{[2c\,\sigma^2+\sigma^4\,(\rho_1+\rho_2)](a_2-a_1)}$ and $b_2 = b_1\frac{(a_2-a_1)}{(a_1-a_2)}$. Then the defective renewal equation (5.22) simplifies to

$$\phi_s(u) = \int_0^u \phi_s(u-y)\,g(y)\,dy + H(u) + \frac{4\lambda_1\lambda_2 T_{\rho_2}\,T_{\rho_1}\,\omega(0)\,(e^{-a_1\,u} - e^{-a_2\,u})}{\left[2c\,\sigma^2 + \sigma^4\,(\rho_1+\rho_2)\right]\,(a_2 - a_1)}\,,$$

Similarly, the defective renewal equation for ϕ_d simplifies by first finding the initial conditions $\phi_d^{(k)}(0)$, $k = 0, 1, 2, 3$, obtained from (5.36) and (5.37) as solutions of

$$\begin{aligned} 1 &= \phi_d(0)\,, \\ -(e_1 + e_2\,\rho_1 + e_3\,\rho_1^2 + e_4\,\rho_1^3) &= \phi_d'(0)\,(e^2 + e_3\,\rho_1 + e_4\,\rho_1^2) \\ &\quad +\phi_d''(0)\,(e_3 + e_4\,\rho_1) + \phi_d^{(3)}(0)\,e^4\,, \\ -(e_1 + e_2\,\rho_2 + e_3\,\rho_2^2 + e_4\,\rho_2^3) &= \phi_d'(0)\,(e^2 + e_3\,\rho_2 + e_4\,\rho_2^2) \\ &\quad +\phi_d''(0)\,(e_3 + e_4\,\rho_2) + \phi_d^{(3)}(0)\,e^4\,, \\ \delta &= c\phi_d'(0) + \frac{\sigma^2}{2}\phi_d''(0)\,. \end{aligned}$$

Solving yields $\phi_d(0) = 1$ and $\phi_d'(0) = \frac{[2\sigma^2\,(\lambda_1+\lambda_2+\delta)-4c^2-4\,c\sigma^2\,(\rho_2+\rho_1)-\sigma^4\,(\rho_1^2+\rho_1\,\rho_2+\rho_2^2)]}{[2c\sigma^2+\sigma^4\,(\rho_1+\rho_2)]}$. Hence $Q_1(s) = \frac{\sigma^4}{4\lambda_1\lambda_2}s + \frac{[4c^2+2c\,\sigma^2(\rho_1+\rho_2)+2\,\sigma^2\,(\lambda_1+\lambda_2+\delta)+\sigma^4\rho_1\rho_2]}{4\,\lambda_1\lambda_2[\frac{2c}{\sigma^2}+\rho_1+\rho_2]}$, which in turn implies that

$$c_1 = \frac{1}{(a_2 - a_1)}\left[-a_1 + \frac{\left[4c^2 + 2c\,\sigma^2\,(\rho_1+\rho_2) + 2\,\sigma^2\,(\lambda_1+\lambda_2+\delta) + \sigma^4\rho_1\rho_2\right]}{\left[2c\,\sigma^2 + \sigma^4\,(\rho_1+\rho_2)\right]}\right]$$

and

$$c_2 = \frac{\left[(a_2 - a_1)\, c_1 + a_1 - a_2\right]}{(a_1 - a_2)}\,.$$

Then the defective renewal equation (5.23) reduces to

$$\phi_d(u) = \int_0^u \phi_d(u-y)\, g(y)\, dy + \frac{(a_2 e^{-a_2 u} - a_1 e^{-a_1 u})}{(a_2 - a_1)}$$
$$+ \frac{\left[4c^2 + 2c\,\sigma^2\,(\rho_1+\rho_2) + 2\sigma^2\,(\lambda_1+\lambda_2+\delta) + \sigma^4\,\rho_1\rho_2\right]\,(e^{-a_1 u} - e^{-a_2 u})}{\left[2c\,\sigma^2 + \sigma^4\,(\rho_1+\rho_2)\right]\,(a_2 - a_1)}\,.$$

The following example shows how to evaluate g, H and obtain explicit results for ϕ_s and ϕ_d when claim sizes are exponentially distributed.

Example 5.6.1. (Exponentially distributed claim size)

In this example, we assume exponential claim sizes, $p(x) = \beta e^{-\beta x}$, $x > 0$. Hence $\hat{p}(s) = \frac{\beta}{s+\beta}$, where $\beta\, c > \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}$ to provide a positive safety loading factor. For simplicity, set $w(x,y) = 1$, which yields $\omega(u) = \bar{P}(u) = e^{-\beta u}$. Furthermore, let ρ_1 and ρ_2 be the positive roots to Lundberg's generalized equation:

$$\gamma(s) = \frac{1}{\lambda_1\lambda_2}\left[(\lambda_1+\delta) - cs - \frac{\sigma^2}{2}s^2\right]\left[(\lambda_2+\delta) - cs - \frac{\sigma^2}{2}s^2\right] = \frac{\beta}{s+\beta}\,,$$

for $s > -\beta$, and where a_1 and a_2 are such that $e_4(s+a_1)(s+a_2) = \gamma[\rho_1, \rho_2, s]$. Then

$$\begin{aligned}
\eta(y) &= T_{\rho_2} T_{\rho_1} p(y) = \frac{T_{\rho_1} p(y) - T_{\rho_2} p(y)}{\rho_2 - \rho_1} = \frac{\beta e^{-\beta y}}{(\rho_1+\beta)(\rho_2+\beta)}\,, \qquad y \geq 0\,,\\
G(u) &= T_{\rho_2} T_{\rho_1} T_0\, p(u) = T_0\, T_{\rho_2} T_{\rho_1} p(u) = T_0\, \eta(u)\\
&= \int_u^\infty \eta(x)\, dx = \frac{e^{-\beta u}}{(\rho_1+\beta)(\rho_2+\beta)}\,, \qquad u \geq 0\,,\\
g(y) &= h_1 * h_2 * \eta(y) = \frac{4\lambda_1\lambda_2}{\sigma^4(\rho_1+\beta)(\rho_2+\beta)(\beta-a_1)(\beta-a_2)}\\
&\qquad \left[\beta e^{-\beta y} + \frac{\beta(\beta-a_2)e^{-a_1 y} - \beta(\beta-a_2)e^{-a_2 y}}{(a_2-a_1)}\right], \qquad y \geq 0
\end{aligned}$$

and $H(u) = h_1 * h_2 * G(u) = h_1 * h_2 * \left[\frac{\eta(u)}{\beta}\right] = \frac{g(u)}{\beta}$, for $u \geq 0$.

Then the Laplace transforms of the ruin time due to oscillations, ϕ_d, and of the ruin time due to a claim, ϕ_s, satisfy the above defective renewal equations which give g and H explicitly.

Furthermore, here $p(x)$ is an exponential distribution, hence $\hat{\phi}_d(s)$ and $\hat{\phi}_s(s)$ can be transformed to rational expressions. Therefore, ϕ_d and ϕ_s can also be obtained explicitly by partial fraction. By (5.17):

$$
\begin{aligned}
\hat{\phi}_s(s) &= \frac{\hat{G}(s)+q_1(s)}{\gamma[\rho_1,\rho_2,s]-\hat{\eta}(s)} = \frac{\frac{1}{(\rho_1+\beta)(\rho_2+\beta)(s+\beta)}+\frac{1}{(\rho_1+\beta)(\rho_2+\beta)(\frac{2c}{\sigma^2}+\rho_1+\rho_2)}}{e_4(s+a_1)(s+a_2)-\frac{\beta}{(\rho_1+\beta)(\rho_2+\beta)(s+\beta)}} \\
&= \left[\frac{4\lambda_1\lambda_2}{2c\sigma^2+\sigma^4(\rho_1+\rho_2)}\right]\left[\frac{\frac{1}{(\rho_1+\beta)(\rho_2+\beta)}[s+(\frac{2c}{\sigma^2}+\rho_1+\rho_2+\beta)]}{(s+a_1)(s+a_2)(s+\beta)-\frac{4\beta\lambda_1\lambda_2}{\sigma^4(\rho_1+\beta)(\rho_2+\beta)}}\right] \\
&= \left[\frac{\frac{4\lambda_1\lambda_2}{(\rho_1+\beta)(\rho_2+\beta)}}{2c\sigma^2+\sigma^4(\rho_1+\rho_2)}\right]\left[\frac{s+(\frac{2c}{\sigma^2}+\rho_1+\rho_2+\beta)}{(s+R_1)(s+R_2)(s+R_3)}\right] \\
&= \left[\frac{4\lambda_1\lambda_2}{[2c\sigma^2+\sigma^4(\rho_1+\rho_2)](\rho_1+\beta)(\rho_2+\beta)}\right]\sum_{i=1}^{3}\frac{\theta_i}{(s+R_i)}\,, \qquad s\in\mathbb{C}\,,
\end{aligned}
$$

where the $-R_i$ are roots (i.e. $\Re(R_i)>0$, for $i=1,2,3$) to the equation:

$$
(s+a_1)(s+a_2)(s+\beta)-\frac{4\beta\lambda_1\lambda_2}{\sigma^4(\rho_1+\beta)(\rho_2+\beta)}=0
$$

and $\theta_1=\frac{\frac{2c}{\sigma^2}+\beta+\rho_1+\rho_2-R_1}{(R_2-R_1)(R_3-R_1)}$, $\theta_2=\frac{\frac{2c}{\sigma^2}+\beta+\rho_1+\rho_2-R_2}{(R_1-R_2)(R_3-R_2)}$, while $\theta_3=\frac{\frac{2c}{\sigma^2}+\beta+\rho_1+\rho_2-R_3}{(R_1-R_3)(R_2-R_3)}$. Thus

$$
\begin{aligned}
\phi_s(u) &= E[e^{-\delta T}I(T<\infty,U(T)<0)|U(0)=u] \\
&= \left[\frac{4\lambda_1\lambda_2}{[2c\sigma^2+\sigma^4(\rho_1+\rho_2)](\rho_1+\beta)(\rho_2+\beta)}\right]\sum_{i=1}^{3}\theta_i e^{-R_i u}\,, \qquad u\geq 0\,.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\hat{\phi}_d(s) &= \frac{Q_1(s)}{[\gamma[\rho_1,\rho_2,s]-\hat{\eta}(s)]}\,, \qquad s\in\mathbb{C}\,, \\
&= \frac{(s+\beta)\left[s+\frac{4c^2+2c\sigma^2(\rho_1+\rho_2)+2\sigma^2(\lambda_1+\lambda_2+\delta)+\sigma^4\rho_1\rho_2}{2c\sigma^2+\sigma^4(\rho_1+\rho_2)}\right]}{\left[(s+a_1)(s+a_2)(s+\beta)-\frac{4\beta\lambda_1\lambda_2}{\sigma^4(\rho_1+\beta)(\rho_2+\beta)}\right]} \\
&= \frac{(s+\beta)\left[s+\frac{4c^2+2c\sigma^2(\rho_1+\rho_2)+2\sigma^2(\lambda_1+\lambda_2+\delta)+\sigma^4\rho_1\rho_2}{2c\sigma^2+\sigma^4(\rho_1+\rho_2)}\right]}{(s+R_1)(s+R_2)(s+R_3)} = \sum_{i=1}^{3}\frac{\zeta_i}{(s+R_i)}\,,
\end{aligned}
$$

where $\zeta_i = \frac{(-R_i+\beta)\left[-R_i+\frac{4c^2+2c\sigma^2(\rho_1+\rho_2)+2\sigma^2(\lambda_1+\lambda_2+\delta)+\sigma^4\rho_1\rho_2}{2c\sigma^2+\sigma^4(\rho_1+\rho_2)}\right]}{\prod_{j=1,j\neq i}^{3}(R_j-R_i)}$, for $i = 1, 2, 3$. Accordingly

$$\phi_d(u) = E[e^{-\delta T} I(T < \infty, U(T) = 0) | U(0) = u] = \sum_{i=1}^{3} \zeta_i \, e^{-R_i u} \,, \qquad u \geq 0 \,.$$

Finally, setting $c = 1.1$, $\lambda_1 = 1.5$, $\lambda_2 = 3$, $\beta = 1$, $\delta = 0.1$ and $\sigma = 1$, then equation $\gamma(s) - \hat{p}(s) = 0$ has five roots in the whole complex plane:

$$\rho_1 = 0.2554 \,, \quad \rho_2 = 1.8713 \,, \quad R_1 = 0.2901$$

$$\text{and} \quad R_2 = 3.6183 + 0.4219\,i \,, \quad R_3 = 3.6183 - 0.4219\,i \,.$$

Then for $u \geq 0$,

$$\phi_s(u) = .5169\, e^{-.2901u} - .5169\, e^{-3.6183u} \cos(.4219u) - 1.3339\, e^{-3.6183u} \sin(.4219u) \,,$$

$$\phi_d(u) = .2616\, e^{-.2901u} + .7384\, e^{-3.6183u} \cos(.4219u) + 1.5593\, e^{-3.6183u} \sin(.4219u) \,,$$

$$\phi(u) = .7785\, e^{-.2901u} + .2215\, e^{-3.6183u} \cos(.4219u) + .2254\, e^{-3.6183u} \sin(.4219u) \,.$$

When $\delta = 0$, then the ρ_i and R_j values change to:

$$\rho_1 = 0 \,, \quad \rho_2 = 1.8395 \,, \quad R_1 = 0.07516$$

$$\text{and} \quad R_2 = 3.5822 + 0.4321\,i \,, \quad R_3 = 3.5822 - 0.4321\,i$$

and hence the different ruin probability components become:

$$\Psi_s(u) = .6248\, e^{-.07516u} - .6248\, e^{-3.5822u} \cos(.4321u) - 1.4207\, e^{-3.5822u} \sin(.4321u) \,,$$

$$\Psi_d(u) = .3224\, e^{-.07516u} + .6776\, e^{-3.5822u} \cos(.4321u) + 1.4811\, e^{-3.5822u} \sin(.4321u) \,,$$

$$\Psi(u) = .9472\, e^{-.07516u} + .0528\, e^{-3.5822u} \cos(.4321u) + .0604\, e^{-3.5822u} \sin(.4321u) \,.$$

Figure 5.1 shows these ruin probabilities for different values of u, as well as their decomposition into the ruin probabilities due to claims and those due to oscillations. From the graph, we see that ruin probability due to oscillations is a strictly decreasing function (from 1 to 0) of the initial surplus u.. Moreover, when u is small, it decreases sharply, while it decreases slowly when u is large. By contrast, the ruin probability due to claims increases quickly at first but then decreases slowly after that.

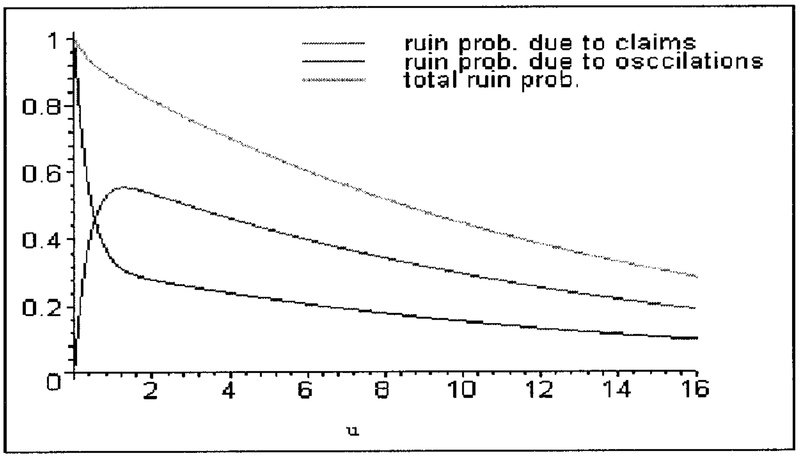

Figure 5.1: Decomposition of the ruin probability

Finally, Figures 5.2–5.4 show, as expected, that the ruin probability due to claims is decreasing in the dispersion parameter σ when u is small and increasing in σ when u is big, while that due to oscillations increases with σ. This results in an ultimate ruin probability that increases with σ.

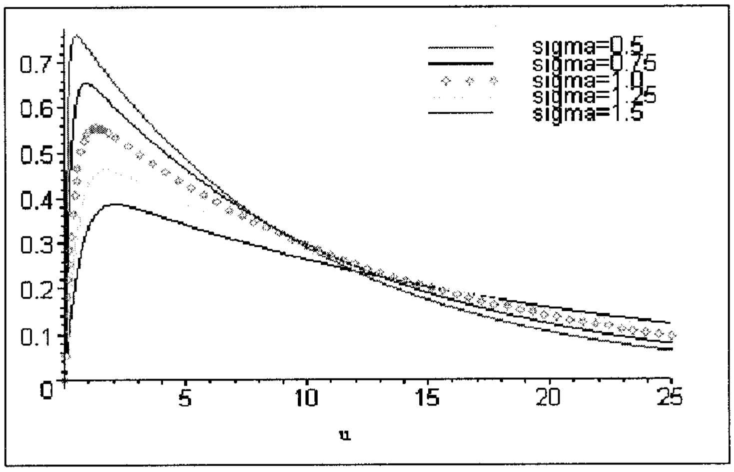

Figure 5.2: Probability of ruin caused by claims for different parameters σ

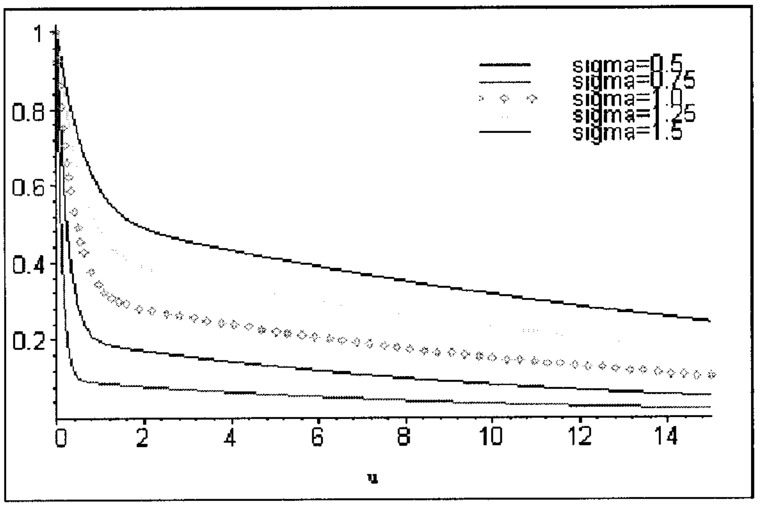

Figure 5.3: Ruin probability due to oscillations for different σ

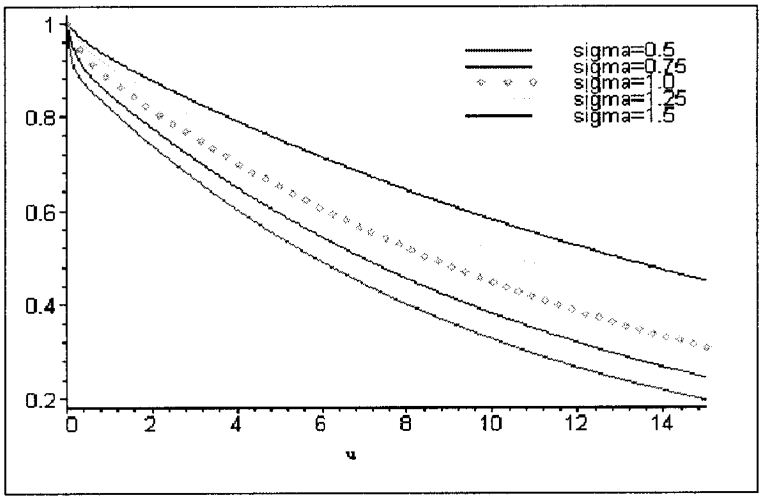

Figure 5.4: Ruin probability $\Psi(u)$ for different σ

Example 5.6.2. To illustrate the effect of the waiting time distribution, we compare here the decomposed ruin probabilities for different Erlang(n) claim waiting times.

Let $n = 1, 2, 3$ but hold the mean waiting time fixed at $n/\lambda = 1$. The first waiting time density is thus exponential, $\lambda e^{-\lambda t} I(t \geq 0)$, for $\lambda = 1$, producing Poisson(λt) claim counts. The second and third waiting time densities are given by $\lambda^2 t e^{-\lambda t} I(t \geq 0)$ and $\frac{\lambda^3}{2} t^2 e^{-\lambda t} I(t \geq 0)$, for $\lambda = 2$ and $\lambda = 3$, respectively.

All other parameters are as in Example 5.6.1, with exponential claim size density, $p(x) = \beta e^{-\beta x} I(x \geq 0)$, for $\beta = 1$. Also $c = 1.1$ and $\sigma = 1$.

For $n = 1$, the ruin probabilities can be found by equations (6.4) and (6.17) in Dufresne and Gerber (1991):

$$\begin{aligned}
\Psi_s(u) &= 0.6509\, e^{-0.06377u} - 0.6509\, e^{-3.1362u}\,, \qquad u \geq 0\,, \\
\Psi_d(u) &= 0.3047\, e^{-0.06377u} + 0.6953\, e^{-3.1362u}\,, \\
\Psi(u) &= 0.9557\, e^{-0.06377u} + 0.0443\, e^{-3.1362u}\,.
\end{aligned}$$

For $n = 2$, ruin probabilities are obtained as in Example 5.6.1, but for $\lambda_1 = \lambda_2 = 2$

and by setting $\delta = 0$. These are:

$$\begin{aligned}
\Psi_s(u) &= 0.6214\, e^{-0.0768u} - 0.6214\, e^{-3.4953u} \cos(0.5202u) \\
&\quad -1.1021\, e^{-3.4953u} \sin(0.5202u) \,, \qquad u \geq 0 \,, \\
\Psi_d(u) &= 0.3236\, e^{-0.0768u} + 0.6764\, e^{-3.4953u} \cos(0.5202u) \\
&\quad +1.1342\, e^{-3.4953u} \sin(0.5202u) \,, \\
\Psi(u) &= 0.9449\, e^{-0.0768u} + 0.05501\, e^{-3.4953u} \cos(0.5202u) \\
&\quad +0.03204\, e^{-3.4953u} \sin(0.5202u) \,.
\end{aligned}$$

Finally, the ruin probabilities for $n = 3$ can be found in a similar fashion:

$$\begin{aligned}
\Psi_s(u) &= 0.6103\, e^{-0.0825u} - 0.4389\, e^{-4.4441u} - 0.1711\, e^{-3.5649u} \cos(0.7984u) \\
&\quad -1.3242\, e^{-3.5649u} \sin(0.7984u) \,, \qquad u \geq 0 \,. \\
\Psi_d(u) &= 0.3299\, e^{-0.0825u} + 0.4508\, e^{-4.4441u} + 0.2193\, e^{-3.5649u} \cos(0.7984u) \\
&\quad +1.3631\, e^{-3.5649u} \sin(0.7984u) \,, \\
\Psi(u) &= 0.9402\, e^{-0.0825u} + 0.0191\, e^{-4.4441u} + 0.0482\, e^{-3.5649u} \cos(0.7984u) \\
&\quad +0.0397\, e^{-3.5649u} \sin(0.7984u) \,.
\end{aligned}$$

These ultimate ruin probabilities are plotted in the following graphs. Note the significant impact on $\Psi(u)$ when relaxing the Poisson assumption ($n = 1$). Ruin probabilities for Erlang waiting times ($n = 2, 3$) are substantially smaller.

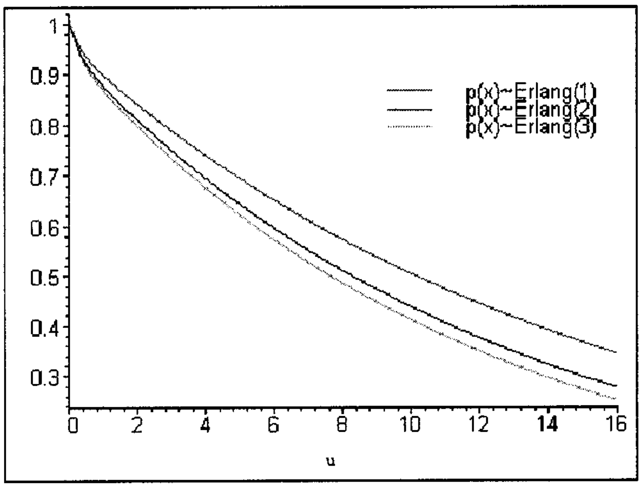

Figure 5.5: Total ruin probabilities for Erlang(n) waiting times, $n = 1, 2, 3$.

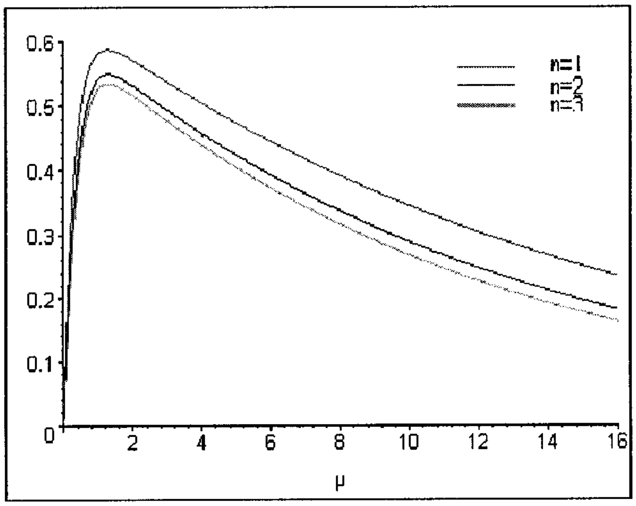

Figure 5.6: Ruin probabilities due to claims, Erlang(n) waiting times, $n = 1, 2, 3$.

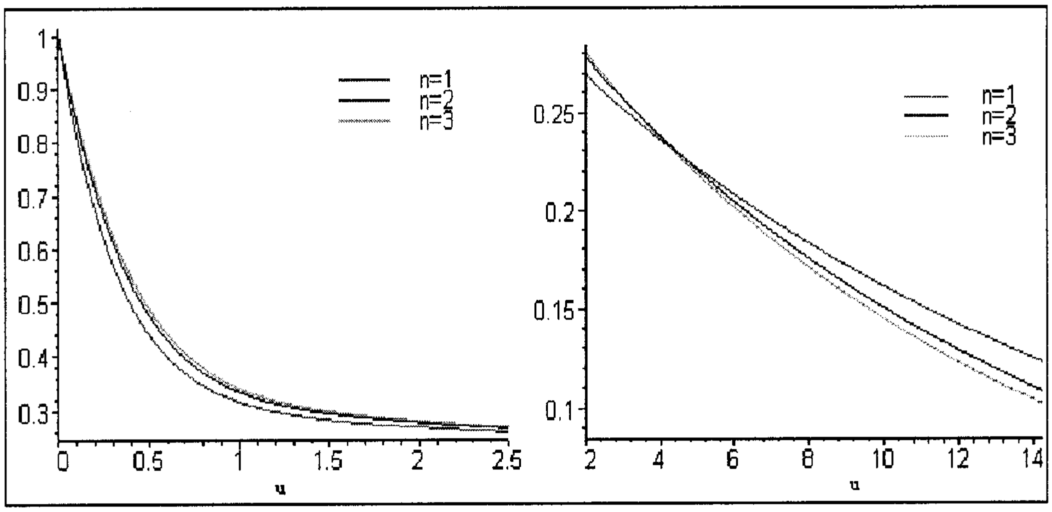

Figure 5.7: Ruin probabilities due to oscillations, Erlang(n) waiting times, $n = 1, 2, 3$.

5.7 Concluding Remarks

We have shown how the evaluation of Gerber-Shiu's expected discounted penalty function for the classical risk model perturbed by a diffusion can be extended to a perturbed Sparre Andersen risk process with generalized Erlang(n) distributed claim waiting times.

The techniques we use provide a way to solve a high order integro–differential equations that often arises in ruin theory. The defective renewal equations obtained here can be used to solve other ruin related problems; explicit expressions or bounds and asymptotic formulas for ruin probabilities, joint and marginal distributions of the three random variables, time to ruin, surplus before ruin and deficit at ruin, as well as their moments.

Chapter 6

A Class of Discrete Time Sparre Andersen Risk Processes

6.1 Introduction

Problems associated with the calculation of ultimate ruin probabilities, for the continuous time risk model, have received considerable attention in recent years. These include studies of the distribution of the ruin time (finite-time ruin probabilities), the surplus before ruin and the deficit at ruin, as well as moments of these variables.

We explore analogue problems, but in the discrete time risk model. A recursive formula for the expected discounted penalty due at ruin is given, using the tool of generating functions, instead of the Laplace transform used for the continuous time model in Chapter 3. This discounted penalty depends on the deficit at ruin and the surplus just before ruin. Hence, our recursive formula yields the joint distribution of the three random variables time to ruin, the surplus just before ruin and the deficit at ruin.

Given the discrete nature of our model, probability generating functions (p.g.f.) are used throughout to analyze the time of ruin and its associated random variables. The joint distribution for the compound binomial model is derived in

Cheng et al. (2000) using martingale techniques and a duality argument. Li and Garrido (2002) gives a recursive formula for the expected discounted penalty function for the compound binomial risk model. This chapter extends the classical compound binomial risk model to a class of discrete Sparre Andersen risk model.

These results can give a better understanding of their analogues in the continuous time model, but they are also of independent interest. They fill a gap in the scant literature on discrete time risk theory models. Our formulas are readily programmable in practice, while they can still reproduce the continuous versions as limiting cases.

6.2 Model Description and Notation

Consider the discrete time Sparre Andersen risk process

$$U(n) = u + n - \sum_{i=1}^{N(n)} X_i \,, \quad n = 1, 2, \dots \,,$$

where $u \in \mathbb{N}$ is the initial reserve. The X_i are i.i.d. random variables with common probability function (p.f.) $p(x) = P(X = x)$, for $x = 1, 2, \dots$, denoting the i-th claim amount. Denote by $\mu_k = E[X^k]$ the k-th moment of X and by $\hat{p}(s) = \sum_{i=1}^{\infty} s^i p(i)$, $s \in \mathbb{C}$ its p.g.f.. The counting process $\{N(n);\, n \in \mathbb{N}\}$ denotes the number of claims up to time n and is defined as $N(n) = \max\{k : W_1 + W_2 + \cdots + W_k \leq n\}$, where the claim waiting times W_i are assumed i.i.d. with common probability function $k(x) = P(W = x)$, for $x = 1, 2, \dots$. Denote by $\hat{k}(s) = \sum_{i=1}^{\infty} s^i k(i)$, $s \in \mathbb{C}$ its p.g.f.

We assume that $\{W_i;\, i \in \mathbb{N}^+\}$ and $\{X_i;\, i \in \mathbb{N}^+\}$ are independent, and $E(W) = (1 + \theta)E(X) = (1 + \theta)\mu$, in order to have a positive loading factor.

Now define (the possibly defective) random variable $T = \min\{n \in \mathbb{N}^+ : U(n) < 0\}$ to be the ruin time,

$$\Psi(u) = P(T < \infty \,|\, U(0) = u) \,, \quad u \in \mathbb{N} \,,$$

to be the ultimate ruin probability and

$$\psi(u,n) = P(T = n \,|\, U(0) = u)\,, \quad n = 1, 2, 3, \dots\,,$$

to be the ruin probability at time t.

Consider $f_3(x, y, t \,|\, u) = P\{U(T-1) = x,\ |U(T)| = y,\ T = t \,|\, U(0) = u\}$, $x \in \mathbb{N}$, $y \in \mathbb{N}^+$, the joint probability function of the surplus just before ruin, deficit at ruin and ruin time. Let $v \in (0,1)$ be the (constant) discount factor over one period and define $f_2(x, y \,|\, u) = \sum_{t=1}^{\infty} v^t f_3(x, y, t \,|\, u)$ as a discounted joint p.d.f. of $U(T-1)$ and $|U(T)|$. Similarly, denote by $f(x \,|\, u) = \sum_{y=0}^{\infty} f_2(x, y \,|\, u)$. The usual conditional probability formulas give the following relation:

$$f_2(x, y \,|\, u) = f(x \,|\, u) \frac{p(x+y+1)}{\bar{P}(x+1)}\,, \quad x \in \mathbb{N},\ y \in \mathbb{N}^+.$$

Let $w(x, y)$, $x, y = 0, 1, 2, \dots$ be the non-negative values of a penalty function. For $0 < v < 1$, define

$$\phi(u) = E\left[v^T w\big(U(T-1), |U(T)|\big) I(T < \infty) \;\middle|\; U(0) = u\right]\,, \quad u \in \mathbb{N}\,. \tag{6.1}$$

The quantity $w\big(U(T-1), |U(T)|\big)$ can be interpreted as the penalty at the time of ruin for the surplus $U(T-1)$ and deficit $|U(T)|$. Then $\phi(u)$ is the expected discounted penalty if v is viewed as a discount rate.

The main objective for the rest of the chapter is to evaluate the expected discounted penalty function ϕ.

6.3 On Martingales and a Generalized Lundberg Equation

Let $\tau_k = \sum_{j=1}^{k} W_j$ be the arrival time of the k-th claim and $U_k = U(\tau_k)$ be the surplus immediately after k-th claim. Defining $\tau_0 = 0$ gives $U_0 = u$, and for $k = 1, 2, \dots,$

$$U_k = U(\tau_k) = u + \tau_k - \sum_{j=1}^{k} X_j = u + \sum_{j=1}^{k} [W_j - X_j].$$

We seek a number $s \in \mathbb{C}$ such that the process:

$$\left\{ v^{\tau_k} s^{-U_k};\; k \in \mathbb{N} \right\} \tag{6.2}$$

will form a martingale. Here the martingale condition is equivalent to

$$E[v^{W_1} s^{X_1 - W_1}] = E\left[(v/s)^{W_1} s^{X_1}\right] = E\left[(v/s)^{W_1}\right] E\left[s^{X_1}\right] = 1,$$

which is

$$\hat{k}(v/s)\,\hat{p}(s) = 1. \tag{6.3}$$

Equation (6.3) is a generalized version of Lundberg equation.

In the rest of this chapter, we assume that the claim inter-arrival times have a discrete K_m distribution, i.e., the p.g.f. of $k(x)$, $x \in \mathbb{N}^+$ can be expressed as

$$\hat{k}(s) = \frac{s[\prod_{i=1}^{m}(1-q_i) + \sum_{j=1}^{m-1} \beta_j (s-1)^j]}{\prod_{i=1}^{m}(1 - s\,q_i)}, \tag{6.4}$$

where $0 < q_i < 1$, for $i = 1, 2, \ldots, m$, and the coefficients $\beta_1, \beta_2, \ldots, \beta_{m-1}$ are such that $\hat{k}'(s) > 0$, $s \in (0,1)$, to guarantee that $k(x)$, $x \in \mathbb{N}^+$ is a p.f.. The mean and second factorial moment of the claim inter-arrival times r.v.'s are thus given by

$$E(W) = \hat{k}'(1) = 1 + \sum_{i=1}^{m} \frac{q_i}{(1-q_i)} + \frac{\beta_1}{\prod_{i=1}^{m}(1-q_i)}. \tag{6.5}$$

$$\begin{aligned} E[W^{(2)}] &= \hat{k}''(1) = \frac{2\beta_2 + \beta_1 \sum_{i=1}^{m} \frac{q_i}{(1-q_i)}}{\prod_{i=1}^{m}(1-q_i)} \\ &\quad + E(W) \sum_{i=1}^{m} \frac{q_i}{(1-q_i)} + \sum_{i=1}^{m} \left(\frac{q_i}{1-q_i} \right)^2, \end{aligned} \tag{6.6}$$

where $x^{(2)} = x(x-1)$ is the second factorial power of x.

This class of distributions includes, as special cases, the shifted geometric, shifted or truncated negative binomial, as well as linear combinations (including mixture) of these. Below are some examples.

1. If $\hat{k}(s) = \frac{s(1-q)}{(1-s\,q)}$, $0 < q < 1$, then $k(x) = (1-q_1)q_1^{x-1}I(x \geq 1)$ is a shifted or truncated geometric distribution.

2. If $\hat{k}(s) = \frac{s\prod_{i=1}^m(1-q_i)}{\prod_{i=1}^m(1-s\,q_i)}$, $0 < q_i < 1$ then k is shifted distribution which is the convolution of m geometric distributions $k_i(x) = (1-q_i)q_i^x I(x \geq 0)$. Furthermore, if $q_i = q$, for all $i = 1, 2, \dots, m$, then k is a shifted negative binomial distribution with $k(x) = \binom{m+x-2}{x-1}(1-q)^m q^{x-1} I(x \geq 1)$.

3. If $\hat{k}(s) = \frac{\prod_{i=1}^m[s\,(1-q_i)]}{\prod_{i=1}^m(1-s\,q_i)}$, then $k(x) = k_1 * k_2 * \cdots * k_m(x)$ with $k_i(x) = (1-q_i)q_i^{x-1}I(x \geq 1)$, that is to say, k is the convolution of m shifted geometric distributions, furthermore, if $q_i = q$, for all $i = 1, 2, \dots, m$, then k is a negative binomial distribution starting from m, i.e.,

$$k(x) = (1-q)^m \binom{x-1}{m-1} q^{x-m}, \qquad x = m, m+1, \dots .$$

4. If $k(x) = \frac{(1-q)^m}{(1-(1-q)^m}\binom{m+x-1}{x} q^x I(x \geq 1)$ with $0 < q < 1$, then k is a truncated negative binomial distribution. Accordingly $\hat{k}(s) = \frac{(1-q)^m}{1-(1-q)^m}\frac{1-(1-s\,q)^m}{(1-s\,q)^m}$, which can be rewritten as

$$\hat{k}(s) = \frac{s[(1-q)^m + \sum_{j=1}^{m-1}\beta_j\,(s-1)^j]}{(1-s\,q)^m},$$

where $\beta_j = \frac{(1-q)^m}{1-(1-q)^m}\sum_{k=j}^{m-1}(-q)^{k+1}\binom{m}{k+1}\binom{k}{j}$.

Specially, if $q_1, q_2, \dots, q_m$ are distinct, by partial fractions, k can be expressed as a linear combination of m geometric distributions with parameters q_i :

$$k(x) = \sum_{i=1}^m \theta_i\,(1-q_i)\,q_i^{x-1}, \qquad x = 1, 2, \dots, \tag{6.7}$$

where θ_i are such that $\sum_{i=1}^m \theta_i = 1$ and given explicitly by

$$\theta_i = \frac{\sum_{k=1}^{m-1}\beta_k(1/q_i - 1)^k + \prod_{j=1}^m(1-q_j)}{(1-q_i)\left[\prod_{j=1, j\neq i}^m(1-q_j/q_i)\right]}. \tag{6.8}$$

Under the assumption that $\hat{k}(s)$ is given by (6.4), the generalized Lundberg equation $\hat{k}(v/s)\,\hat{p}(s) = 1$ simplifies to

$$\begin{aligned} \gamma(s): &= \frac{1}{\hat{k}(v/s)} \qquad\qquad (6.9)\\ &= \frac{\prod_{i=1}^{m}(s - v\,q_i)}{v\left[s^{m-1}\prod_{i=1}^{m}(1-q_i) + \sum_{j=1}^{m-1}\beta_j s^{m-1-j}\,(v-s)^j\right]} = \hat{p}(s), \quad s \in \mathbb{C}. \end{aligned}$$

The roots of the equation above play a key role in this chapter, and are discussed in the following theorem.

Theorem 6.3.1. For $0 < v < 1$, and $m \in \mathbb{N}^+$, equation (6.9) has exactly m roots, say $\rho_i(v)$, $i = 1, 2, \ldots, m$ with $0 < |\rho_i| < 1$.

Proof: Consider a unit contour $\{\Gamma : |s| = 1\}$ in $\mathbb{C}$. Since $|\hat{k}(v/s)| < \hat{k}(|v/s|) = \hat{k}(|v|) < 1$ on contour Γ, then $\left|\frac{1}{\hat{k}(v/s)}\right| > 1 = \hat{p}(1) = \hat{p}(|s|) \geq |\hat{p}(s)|$ on Γ. By Rouché's Theorem, equations $\frac{1}{\hat{k}(v/s)} = 0$ and $\frac{1}{\hat{k}(v/s)} = \hat{p}(s)$ have the same number of roots within the unit circle. By (6.9), the former has m roots within the unit circle, then Lundberg's equation has m roots within the unit contour Γ, say, $\rho_1(v), \rho_2(v), \ldots, \rho_m(v)$. It is easy to check that $s = 0$ is not a root to (6.9), we have that $0 < |\rho_i| < 1$, for $i = 1, 2, \ldots$. □

Remarks:

1. Define $l(s) := \hat{p}(s) - \frac{1}{\hat{k}(v/s)}$. Since $l(1) < 0$ and $\lim_{s\to\infty} l(s) = +\infty$, then if $p(x)$ is sufficiently regular, $l(s) = 0$ has one root greater than 1. Hence denote by $R(v)$, which can be called a generalized *adjustment coefficient.*

2. $R(v) \to R(1)$, as $v \to 1^-$, and $\rho_j(v) \to \rho_j(1)$, for $1 \leq j \leq m$, where $R(1)$ and $\rho_j(1)$ are roots to $\frac{1}{\hat{k}(1/s)} = \hat{p}(s)$.

3. For simplicity, $R(v)$ and $\rho_j(v)$ are denoted by R and ρ_j, for $1 \leq j \leq m$ and $0 < v < 1$.

6.4 Probability Generating Function

Conditioning the time and amount of the first claim, one obtains that for $u \in \mathbb{N}$:

$$\phi(u) = E[v^{W_1}\phi(U_1)] = E[v^{W_1}\,\phi(u+W_1-X_1)] = \sum_{t=1}^{\infty} v^t\, k(t) E[\phi(u+t-X_1)].$$

Now define $\hat{\phi}(s) = \sum_{u=0}^{\infty} s^u \phi(u)$ to be the p.g.f. transform of ϕ, then by (6.10),

$$\begin{aligned}
\hat{\phi}(s) &= \sum_{u=0}^{\infty} s^u \phi(u) = \sum_{u=0}^{\infty} s^u \sum_{t=1}^{\infty} v^t\, k(t) E[\phi(u+t-X_1)] \\
&= \sum_{u=0}^{\infty} s^u \sum_{y=u+1}^{\infty} v^{y-u}\, k(y-u)\, E[\phi(y-X_1)] \\
&= \sum_{y=1}^{\infty} v^y\, E[\phi(y-X_1)] \sum_{u=0}^{y-1} (s/v)^u\, k(y-u) \\
&= \sum_{y=1}^{\infty} s^y\, E[\phi(y-X_1)] \sum_{t=1}^{y} (v/s)^t\, k(t). \qquad (6.10)
\end{aligned}$$

If $q_1, q_2 \ldots, q_m$ are distinct, then $k(t)$ has the form in (6.7). Substituting it into (6.10) yields

$$\begin{aligned}
\hat{\phi}(s) &= \sum_{i=1}^{m} \frac{\theta_i\,(1-q_i)}{q_i} \sum_{y=1}^{\infty} s^y\, E[\phi(y-X_1)] \sum_{t=1}^{y} (v/s)^t\, q_i^t \\
&= \sum_{i=1}^{m} \frac{\theta_i\,(1-q_i)(v/s)}{[1-(v/s)\,q_i]} \left\{ \sum_{y=1}^{\infty} s^y\, E[\phi(y-X_1)] - \sum_{y=1}^{\infty} (v\,q_i)^y\, E[\phi(y-X_1)] \right\} \\
&= \hat{k}(v/s) \sum_{y=1}^{\infty} s^y\, E[\phi(y-X_1)] - \sum_{i=1}^{m} \frac{\theta_i(1-q_i)\,v\,b_i}{(s-v\,q_i)}, \qquad (6.11)
\end{aligned}$$

where $b_i = \sum_{y=1}^{\infty} (v\,q_i)^y\, E[\phi(y-X_1)]$. By definition of $\phi(u)$,

$$E[\phi(y-X_1)] = \sum_{x=1}^{y} \phi(y-x)\,p(x) + \sum_{x=y+1}^{\infty} w(y-1, x-y)\,p(x). \qquad (6.12)$$

For simplicity, let $\omega(y) = \sum_{x=y+1}^{\infty} w(y-1, x-y)\,p(x)$. Substituting (6.12) into (6.11) yields

$$\hat{\phi}(s) = \frac{\hat{k}(v/s)\,\hat{\omega}(s) - \sum_{i=1}^{m} \frac{\theta_i(1-q_i)\,v\,b_i}{(s-v\,q_i)}}{[1-\hat{k}(v/s)\,\hat{p}(s)]}, \qquad (6.13)$$

where $\hat{\omega}(s) = \sum_{y=1}^{\infty} s^y \omega(y)$. Multiplying both denominator and numerator by $\gamma(s) = \frac{1}{\hat{k}(v/s)}$, (6.13) can be rewritten as

$$\hat{\phi}(s) = \frac{\hat{\omega}(s) - \frac{Q_{m-1}(s)}{v\left[s^{m-1}\prod_{i=1}^{m}(1-q_i)+\sum_{j=1}^{m-1}\beta_j s^{m-1-j}(v-s)^j\right]}}{[\gamma(s) - \hat{p}(s)]}, \tag{6.14}$$

where $Q_{m-1}(s) = [\prod_{i=1}^{m}(s - v\,q_i)]\left[\sum_{i=1}^{m} \frac{\theta_i(1-q_i)\,v\,b_i}{(s-v\,q_i)}\right]$ is a polynomial of degree $m-1$ or less. Since $\hat{\phi}(s)$ is finite for all s such that $0 < |\Re(s)| < 1$, the numerator on the right hand of (6.14) must be zero whenever the denominator is zero. Then $Q_{m-1}(s)$ can be determined by the linear system for $j = 1, 2, \ldots, m$,

$$Q_{m-1}(\rho_j) = \hat{\omega}(\rho_j)\left\{v\left[{\rho_j}^{m-1}\prod_{i=1}^{m}(1-q_i) + \sum_{t=1}^{m-1}\beta_t\,{\rho_j}^{m-1-t}\,(v-\rho_j)^t\right]\right\}.$$

Further, if $\rho_1, \rho_2, \ldots, \rho_m$ are distinct, by the *Lagrange interpolation formula,* one obtains

$$Q_{m-1}(s) = \sum_{j=1}^{m} c_j\,\hat{\omega}(\rho_j)\left[\prod_{k=1,k\neq j}^{m}\frac{(s-\rho_k)}{(\rho_j-\rho_k)}\right], \tag{6.15}$$

where $c_j = v\left[{\rho_j}^{m-1}\prod_{i=1}^{m}(1-q_i) + \sum_{t=1}^{m-1}\beta_t\,{\rho_j}^{m-1-t}\,(v-\rho_j)^t\right]$, for $j = 1, 2, \ldots, m$.

We remark that if some q_i are equal, formula (6.14) still holds, and (6.15) still holds for the case where $\rho_1, \rho_2, \ldots, \rho_m$ are distinct, by the continuity property.

6.5 Analysis when $u = 0$

We now turn to finding ruin related quantities when $u = 0$. For simplicity, we assume that the $\rho_1, \rho_2, \ldots, \rho_m$ are distinct. First

$$\begin{aligned}
\phi(0) &= \lim_{s\to 0}\hat{\phi}(s) = \lim_{s\to 0}\frac{\hat{\omega}(s) - \frac{Q_{m-1}(s)}{v\left[s^{m-1}\prod_{i=1}^{m}(1-q_i)+\sum_{j=1}^{m-1}\beta_j s^{m-1-j}(v-s)^j\right]}}{[\gamma(s)-\hat{p}(s)]} \\
&= \frac{\sum_{j=1}^{m} c_j\,\hat{\omega}(\rho_j)\left[\prod_{k=1,k\neq j}^{m}\frac{\rho_k}{\rho_j-\rho_k}\right]}{v^m\prod_{i=1}^{m} q_i} \\
&= \left[\prod_{i=1}^{m}\frac{\rho_i}{v\,q_i}\right]\sum_{j=1}^{m}\frac{c_j\,\hat{\omega}(\rho_j)}{\rho_j\prod_{k=1,k\neq j}^{m}(\rho_j-\rho_k)}.
\end{aligned} \tag{6.16}$$

Since $\omega(y) = \sum_{x=y+1}^{\infty} w(y-1, x-y)\, p(x) = \sum_{t=1}^{\infty} w(y-1,t)\, p(y+t)$, and then

$$\hat{\omega}(s) = \sum_{y=1}^{\infty} s^y \omega(y) = \sum_{y=1}^{\infty}\sum_{t=1}^{\infty} s^y\, w(y-1,t)p(y+t) = \sum_{x=0}^{\infty}\sum_{y=1}^{\infty} s^{x+1} w(x,y)\, p(x+y+1),$$

therefore, (6.16) can be rewritten as

$$\phi(0) = \left[\prod_{i=1}^{m} \frac{\rho_i}{v\, q_i}\right] \sum_{j=1}^{m} \frac{c_j \sum_{x=0}^{\infty} \sum_{y=1}^{\infty} \rho_j{}^{x+1} w(x,y)\, p(x+y+1)}{\rho_j \prod_{k=1,k\neq j}^{m} (\rho_j - \rho_k)}. \tag{6.17}$$

On the other hand,

$$\begin{aligned} \phi(0) &= E[v^T w(U(T-1), |U(T)|) I(T<\infty) | U(0)=0] \\ &= \sum_{x=0}^{\infty}\sum_{y=1}^{\infty} w(x,y) f_2(x,y\,|\,0). \end{aligned} \tag{6.18}$$

Comparing these two formulas yields

$$f_2(x,y\,|\,0) = \left[\prod_{i=1}^{m} \frac{\rho_i}{v\, q_i}\right] \sum_{j=1}^{m} \frac{c_j\, \rho_j{}^{x}\, p(x+y+1)}{\prod_{k=1,k\neq j}^{m} (\rho_j - \rho_k)}, \qquad x \in \mathbb{N},\ y \in \mathbb{N}^+, \tag{6.19}$$

so

$$f_1(x\,|0) = \sum_{y=1}^{\infty} f_2(x,y|0) = \left[\prod_{i=1}^{m} \frac{\rho_i}{v\, q_i}\right] \sum_{j=1}^{m} \frac{c_j\, \rho_j{}^{x}\, \bar{P}(x+1)}{\prod_{k=1,k\neq j}^{m} (\rho_j - \rho_k)}, \qquad x \in \mathbb{N}, \tag{6.20}$$

where $\bar{P}(x+1) = P(X > x+1) = \sum_{y=x+2} p(y)$, finally,

$$\begin{aligned} g(y) := g(y\,|\,0) &= \sum_{x=0}^{\infty} f_2(x,y\,|\,0) = \left[\prod_{i=1}^{m} \frac{\rho_i}{v\, q_i}\right] \sum_{j=1}^{m} \frac{c_j \sum_{x=0}^{\infty} \rho_j{}^{x}\, p(x+y+1)}{\prod_{k=1,k\neq j}^{m} (\rho_j - \rho_k)}, \\ &= \left[\prod_{i=1}^{m} \frac{\rho_i}{v\, q_i}\right] \sum_{j=1}^{m} \frac{c_j\, T_{\rho_j} p(y+1)}{\prod_{k=1,k\neq j}^{m} (\rho_j - \rho_k)}, \quad y \in \mathbb{N}^+, \end{aligned} \tag{6.21}$$

where T_r is an operator defined by

$$T_r p(y) = \sum_{x=y}^{\infty} r^{x-y} p(x) = \sum_{x=0}^{\infty} r^x\, p(x+y), \quad r \in \mathbb{C},\ \ y \in \mathbb{N}^+.$$

Further discussion on T_r can be found in Section 2.3 of Chapter 2. The function g is a defective density function. It plays a very important role in this chapter. Define $\hat{g}(s) = \sum_{y=1}^{\infty} s^y\, g(y)$ to be the generating function of g, then we have the following Lemma.

Lemma 6.5.1. The generating function of g is given by

$$\hat{g}(s) = 1 - \frac{\prod_{i=1}^{m}(s - v\,q_i) - v\,\hat{p}(s)[s^{m-1}\prod_{i=1}^{m}(1-q_i) + \sum_{j=1}^{m-1}\beta_j s^{m-1-j}\,(v-s)^j]}{(\prod_{i=1}^{m}\frac{v\,q_i}{\rho_i})\prod_{i=1}^{m}(s-\rho_i)}. \tag{6.22}$$

Proof: Since $\hat{g}(s) = s\,T_s\,g(1)$, then

$$\begin{aligned}
\hat{g}(s) &= \left[\prod_{i=1}^{m}\frac{\rho_i}{v\,q_i}\right]\sum_{j=1}^{m}\frac{c_j\, s\, T_s T_{\rho_j} p(2)}{\prod_{k=1,k\neq j}^{m}(\rho_j-\rho_k)} \\
&= \left[\prod_{i=1}^{m}\frac{\rho_i}{v\,q_i}\right]\sum_{j=1}^{m}\frac{c_j\, s\left[\frac{s\,T_s p(2) - \rho_j\, T_{\rho_j} p(2)}{s-\rho_j}\right]}{\prod_{k=1,k\neq j}^{m}(\rho_j-\rho_k)} \\
&= \left[\prod_{i=1}^{m}\frac{\rho_i}{v\,q_i}\right]\sum_{j=1}^{m}\frac{c_j\left[\frac{\hat{p}(s)-(s/\rho_j)\hat{p}(\rho_j)}{s-\rho_j}\right]}{\prod_{k=1,k\neq j}^{m}(\rho_j-\rho_k)} \\
&= \left[\prod_{i=1}^{m}\frac{\rho_i}{v\,q_i}\right]\left\{\sum_{j=1}^{m}\frac{c_j\,\hat{p}(s)}{(s-\rho_j)\prod_{k=1,k\neq j}^{m}(\rho_j-\rho_k)}\right. \\
&\qquad \left. -\sum_{j=1}^{m}\frac{c_j\, s\,\hat{p}(\rho_j)}{\rho_j(s-\rho_j)\prod_{k=1,k\neq j}^{m}(\rho_j-\rho_k)}\right\} \\
&= \left[\prod_{i=1}^{m}\frac{\rho_i}{v\,q_i}\right]\left\{\sum_{j=1}^{m}\frac{v\hat{p}(s)\left[{\rho_j}^{m-1}\prod_{i=1}^{m}(1-q_i)+\sum_{t=1}^{m-1}\beta_t\,{\rho_j}^{m-1-t}\,(v-\rho_j)^t\right]}{(s-\rho_j)\prod_{k=1,k\neq j}^{m}(\rho_j-\rho_k)}\right. \\
&\qquad \left. -\sum_{j=1}^{m}\frac{s\prod_{i=1}^{m}(\rho_j - v\,q_i)}{\rho_j(s-\rho_j)\prod_{k=1,k\neq j}^{m}(\rho_j-\rho_k)}\right\}.
\end{aligned} \tag{6.23}$$

By a well known formula in interpolation theory,

$$\sum_{j=1}^{m}\frac{(\rho_j-s)^n}{\prod_{k=1,k\neq j}^{m}(\rho_j-\rho_k)} = \begin{cases} 1, & n = m-1, \\ 0, & n = 0,1,2,\ldots,m-2,\, , \\ -\frac{1}{\prod_{i=1}^{m}(s-\rho_i)}, & n=-1. \end{cases} \tag{6.24}$$

we have

$$\sum_{j=1}^{m} \frac{v{\rho_j}^{m-1} \prod_{i=1}^{m}(1-q_i)}{(s-\rho_j)\prod_{k=1,k\neq j}^{m}(\rho_j-\rho_k)} = -v\sum_{j=1}^{m} \frac{[(\rho_j-s)+s]^{m-1}\prod_{i=1}^{m}(1-q_i)}{(\rho_j-s)\prod_{k=1,k\neq j}^{m}(\rho_j-\rho_k)}$$

$$= -v\prod_{i=1}^{m}(1-q_i)\sum_{j=1}^{m} \frac{\sum_{l=0}^{m-1} s^{m-1-l}\binom{m-1}{l}(\rho_j-s)^l}{(\rho_j-s)\prod_{k=1,k\neq j}^{m}(\rho_j-\rho_k)}$$

$$= v\, s^{m-1}\left[\frac{\prod_{i=1}^{m}(1-q_i)}{\prod_{i=1}^{m}(s-\rho_i)}\right]. \tag{6.25}$$

In the same fashion

$$\sum_{j=1}^{m} \frac{v\left[\sum_{t=1}^{m-1}\beta_t\, {\rho_j}^{m-1-t}\,(v-\rho_j)^t\right]}{(s-\rho_j)\prod_{k=1,k\neq j}^{m}(\rho_j-\rho_k)}$$

$$= v\sum_{t=1}^{m-1}\beta_t\sum_{j=1}^{m} \frac{[(\rho_j-s)+s]^{m-1-t}\left[(v-s)+(s-\rho_j)\right]^t}{(s-\rho_j)\prod_{k=1,k\neq j}^{m}(\rho_j-\rho_k)}$$

$$= \frac{v\sum_{t=1}^{m-1}\beta_t\,(v-s)^t\, s^{m-t-1}}{\prod_{i=1}^{m}(s-\rho_i)} \tag{6.26}$$

and

$$\sum_{j=1}^{m} \frac{-\prod_{i=1}^{m}(\rho_j-v\,q_i)}{\rho_j(s-\rho_j)\prod_{k=1,k\neq j}^{m}(\rho_j-\rho_k)} = \sum_{j=1}^{m} \frac{\sum_{t=0}^{m}\sigma_{m-t}\,\rho_j^t}{\rho_j(\rho_j-s)\prod_{k=1,k\neq j}^{m}(\rho_j-\rho_k)}$$

$$= \sum_{j=1}^{m} \frac{\sigma_m}{\rho_j(\rho_j-s)\prod_{k=1,k\neq j}^{m}(\rho_j-\rho_k)} + \sum_{j=1}^{m} \frac{\sum_{t=1}^{m}\sigma_{m-t}\,\rho_j^{t-1}}{(\rho_j-s)\prod_{k=1,k\neq j}^{m}(\rho_j-\rho_k)}, \tag{6.27}$$

where $\sigma_0 = 1$, $\sigma_2 = \sum_{i=1}^{m}(-v\,q_i)$, $\sigma_3 = \sum_{1\leq i<j\leq m} v^2\, q_i\, q_j, \ldots, \sigma_m = \prod_{i=1}^{m}(-v\,q_i)$. Again, by (6.24), formula (6.27) simplifies to

$$\sum_{j=1}^{m} \frac{-\prod_{i=1}^{m}(\rho_j-v\,q_i)}{\rho_j(s-\rho_j)\prod_{k=1,k\neq j}^{m}(\rho_j-\rho_k)} = \frac{1}{s}\left[\frac{\prod_{i=1}^{m}(v\,q_i)}{\prod_{i=1}^{m}\rho_i} - \frac{\prod_{i=1}^{m}(s-v\,q_i)}{\prod_{i=1}^{m}(s-\rho_i)}\right]. \tag{6.28}$$

Substituting (6.25), (6.26) and (6.28) into (6.23) finally proves that (6.22) holds.

□

Using Lemma 6.5.1, one obtains

$$
\begin{aligned}
\phi_T(0) &= E[v^T I(T<\infty)|U(0)=0] = \sum_{y=1}^{\infty} g(y) = \lim_{s\to 1}\hat{g}(s) \\
&= 1-\left[\prod_{i=1}^{m}\frac{\rho_i}{v\,q_i}\right]\left[\frac{\prod_{i=1}^{m}(1-v\,q_i)-v\left[\prod_{i=1}^{m}(1-q_i)+\sum_{t=1}^{m-1}\beta_t\,(v-1)^t\right]}{\prod_{i=1}(1-\rho_i)}\right] \\
&= 1-\left[\prod_{i=1}^{m}\frac{\rho_i}{v\,q_i}\right]\left[\frac{\prod_{i=1}^{m}(1-v\,q_i)[1-\hat{k}(v)]}{\prod_{i=1}^{m}(1-\rho_i)}\right] < 1, \qquad (6.29)
\end{aligned}
$$

where the last step follows from the definition of $\hat{k}(s)$.

Finally,

$$
\begin{aligned}
\Psi(0) &= \lim_{v\to 1^-} E[v^T I(T<\infty)|U(0)=0] \\
&= 1-\lim_{v\to 1^-}\left[\prod_{i=1}^{m}\frac{\rho_i}{v\,q_i}\right]\left[\frac{\prod_{i=1}^{m}(1-v\,q_i)[1-\hat{k}(v)]}{\prod_{i=1}(1-\rho_i)}\right] \\
&= 1-\left(\prod_{i=1}^{m}\frac{1-q_i}{q_i}\right)\left[\prod_{i=1}^{m-1}\frac{\rho_i(1)}{1-\rho_i(1)}\right]\lim_{v\to 1^-}\frac{\left[\frac{1-\hat{k}(v)}{1-v}\right]}{\left[\frac{1-\rho_m(v)}{1-v}\right]} \\
&= 1-\left(\prod_{i=1}^{m}\frac{1-q_i}{q_i}\right)\left[\prod_{i=1}^{m-1}\frac{\rho_i(1)}{1-\rho_i(1)}\right]\left[\frac{\hat{k}'(1)}{\rho_m'(1)}\right] \\
&= 1-\left(\prod_{i=1}^{m}\frac{1-q_i}{q_i}\right)\left[\prod_{i=1}^{m-1}\frac{\rho_i(1)}{1-\rho_i(1)}\right][E(W)-E(X)], \qquad (6.30)
\end{aligned}
$$

where the last step follows from $\hat{k}'(1)=E(W)$ and $\rho_m'(1)=\frac{E(W)}{[E(W)-E(X)]}$, which is obtained by taking derivatives w.r.t. v on both sides of Lundberg's equation $\hat{k}(v/\rho_m(v))\,\hat{p}(\rho_m(v))=1$, letting $v\to 1^-$, and noting that $\lim_{v\to 1^-}\rho_m(v)=1$.

6.6 Recursive Formula for $\phi(u)$

6.6.1 General Case

In this section, a recursive formula for $\phi(u)$ is given by renewal argument, which can be used to analyze other ruin related problems. The starting point of the recursion $\phi(0)$ is given in (6.16) by

$$\phi(0) = \left[\prod_{i=1}^{m} \frac{\rho_i}{v\, q_i}\right] \sum_{j=1}^{m} \frac{c_j \sum_{x=0}^{\infty} \sum_{y=1}^{\infty} {\rho_j}^x w(x,y)\, p(x+y+1)}{\prod_{k=1,k\neq j}^{m} (\rho_j - \rho_k)}.$$

For $u \geq 1$, by similar arguments as in Gerber and Shiu (1998) for the continuous case, we condition on the first time when the surplus process drops below the initial surplus u :

$$\begin{aligned}
\phi(u) &= \sum_{y=1}^{u} \sum_{x=0}^{\infty} \sum_{t=1}^{\infty} v^t\, \phi(u-y)\, f_3(x,y,t|0) \\
&\quad + \sum_{y=u+1}^{\infty} \sum_{x=0}^{\infty} \sum_{t=1}^{\infty} v^t\, w(x+u, y-u) f_3(x,y,t|0) \\
&= \sum_{y=1}^{u} \sum_{x=0}^{\infty} \phi(u-y) f_2(x,y|0) + \sum_{y=u+1}^{\infty} \sum_{x=0}^{\infty} w(x+u, y-u) f_2(x,y|0) \\
&= \sum_{y=1}^{u} \phi(u-y) g(y) + H(u), \qquad u \in \mathbb{N}^+, \qquad (6.31)
\end{aligned}$$

where

$$\begin{aligned}
H(u) &= \sum_{y=u+1}^{\infty} \sum_{x=0}^{\infty} w(x+u, y-u) f_2(x,y|0) = \sum_{y=1}^{\infty} \sum_{x=u}^{\infty} w(x,y) f_2(x-u, y+u|0) \\
&= \left[\prod_{i=1}^{m} \frac{\rho_i}{v\, q_i}\right] \sum_{j=1}^{m} \frac{c_j}{\prod_{k=1,k\neq j}^{m} (\rho_j - \rho_k)} \sum_{x=u}^{\infty} \rho_j^{x-u} \sum_{y=1}^{\infty} w(x,y) p(x+y+1) \\
&= \left[\prod_{i=1}^{m} \frac{\rho_i}{v\, q_i}\right] \sum_{j=1}^{m} \frac{c_j}{\prod_{k=1,k\neq j}^{m} (\rho_j - \rho_k)} \sum_{x=u}^{\infty} \rho_j^{x-u} \omega(x+1) \\
&= \left[\prod_{i=1}^{m} \frac{\rho_i}{v\, q_i}\right] \sum_{j=1}^{m} \frac{c_j}{\prod_{k=1,k\neq j}^{m} (\rho_j - \rho_k)} T_{\rho_j} \omega(u+1), \qquad u \in \mathbb{N}^+. \qquad (6.32)
\end{aligned}$$

(6.31) is a recursive formula for $\phi(u)$ with the starting point $\phi(0)$. Specially, if $w(x,y) = 1$, then $\phi(u)$ simplifies to the p.g.f. transform of ruin time T w.r.t. discount factor v, which is now defined as

$$\phi_T(u) := E[v^T\, I(T < \infty)\,|\, U(0) = u], \qquad u \in \mathbb{N}.$$

In this case ω simplifies to $\omega(u) = \sum_{x=u+1}^{\infty} p(x) = \bar{P}(u) = T_1 p(u)$, and $H(u)$ simplifies to

$$\begin{aligned} H(u) &= \left[\prod_{i=1}^{m} \frac{\rho_i}{v\, q_i}\right] \sum_{j=1}^{m} \frac{c_j}{\prod_{k=1,k\neq j}^{m}(\rho_j - \rho_k)} T_{\rho_j} T_1 p(u+1) \\ &= T_1 g(u) = \sum_{y=u+1}^{\infty} g(u), \end{aligned} \tag{6.33}$$

then $\phi_T(u)$ has the following recursive formula,

$$\phi_T(u) = \sum_{y=1}^{u} \phi_T(u-y)\, g(y) + \sum_{y=u+1}^{\infty} g(y), \qquad u \in \mathbb{N}^+. \tag{6.34}$$

The ruin probability $\Psi(u)$ can thus be obtained by taking limit for $\phi_T(u)$ when $v \to 1^-$, i.e.,

$$\begin{aligned} \Psi(u) &= \lim_{v\to 1^-} E[v^T I(T<\infty)|U(0)=u] \\ &= \sum_{y=1}^{u} \Psi(u-y)\, g_1(y) + \sum_{y=u+1}^{\infty} g_1(y), \qquad u \in \mathbb{N}^+, \end{aligned} \tag{6.35}$$

where

$$\begin{aligned} g_1(y) &= \lim_{v\to 1^-} g(y) = \lim_{v\to 1^-} \left[\prod_{i=1}^{m} \frac{\rho_i}{v\, q_i}\right] \sum_{j=1}^{m} \frac{c_j\, T_{\rho_j} p(y+1)}{\prod_{k=1,k\neq j}^{m}(\rho_j - \rho_k)} \\ &= \left[\prod_{i=1}^{m} \frac{\rho_i(1)}{q_i}\right] \left[\sum_{j=1}^{m-1} \frac{c_j\, T_{\rho_j(1)} p(y+1)}{\prod_{k=1,k\neq j}^{m}[\rho_j(1) - \rho_k(1)]} + \frac{\prod_{i=1}^{m}(1-q_i) T_1 p(y+1)}{\prod_{i=1}^{m-1}[1-\rho_i(1)]}\right], \end{aligned}$$

the last step follows from the fact of $\lim_{v\to 1^-} \rho_m(v) = 1$.

6.6.2 Special Waiting Time Distributions

Now we consider some special cases of waiting time distributions by choosing $\hat{k}(s)$.

1. $\hat{k}(s) = \frac{s\,(1-q)}{(1-s\,q)}$ (Geometric), then the generalized Lundberg equation $\hat{k}(v/s)\,\hat{p}(s) = 1$ becomes

$$\frac{s - v\,q}{v(1-q)} = \hat{p}(s), \qquad v \in (0,1), \tag{6.36}$$

which has exactly one root in $(0,1)$, say ρ, under positive loading factor assumption, $\lim_{v\to 1^-}\rho = 1$. Then the recursive formula

$$\phi(u) = \sum_{y=1}^{u} \phi(u-y)g(y) + H(u), \qquad u \geq 1$$

has a starting point given in (6.17) as

$$\phi(0) = H(0) = \frac{\rho(1-q)}{q} \sum_{x=0}^{\infty}\sum_{y=1}^{\infty} \rho^x w(x,y)p(x+y+1),$$

specially, $\phi_T(0) = 1 - \frac{\rho(1-v)}{v\,q(1-\rho)}$ and $\Psi(0) = 1 - \frac{1-q}{q}[E(W) - E(X)] = 1 - \frac{\theta}{q(1+\theta)}$. While

$$g(y) = \frac{\rho(1-q)}{q} T_\rho p(y+1) = \frac{1-q}{q} \sum_{x=1}^{\infty} \rho^x p(x+y), \qquad (6.37)$$

$$H(u) = \frac{\rho(1-q)}{q} T_\rho \omega(y+1) = \frac{1-q}{q} \sum_{x=1+u}^{\infty} \rho^{x-u} \omega(u). \qquad (6.38)$$

2. If $\hat{k}(s) = \frac{s[(1-q_1)(1-q_2)+\beta(s-1)}{(1-s\,q_1)(1-s\,q_2)}$, with $0 < q_i < 1$, for $i = 1, 2$, then

$$\begin{aligned} g(y) &= \frac{\rho_1\,\rho_2[(1-q_1)(1-q_2)-\beta]}{v\,q_1\,q_2(\rho_1-\rho_2)} T_{\rho_2} T_{\rho_1} p(y+1) \\ &\quad + \left[\frac{\beta\,\rho_1\,\rho_2}{q_1\,q_2}\right] \frac{T_{\rho_1}p(y+1) - T_{\rho_2}p(y+1)}{\rho_1-\rho_2}, \qquad (6.39) \\ H(u) &= \frac{\rho_1\,\rho_2[(1-q_1)(1-q_2)-\beta]}{v\,q_1\,q_2(\rho_1-\rho_2)} T_{\rho_2} T_{\rho_1} \omega(u+1) \\ &\quad + \left[\frac{\beta\,\rho_1\,\rho_2}{q_1\,q_2}\right] \frac{T_{\rho_1}\omega(u+1) - T_{\rho_2}\omega(u+1)}{\rho_1-\rho_2}. \qquad (6.40) \end{aligned}$$

The starting point of the recursion is

$$\begin{aligned} \phi(0) = H(0) &= \frac{\rho_1\,\rho_2[(1-q_1)(1-q_2)-\beta] + \beta\,v\rho_2}{v\,q_1\,q_2(\rho_1-\rho_2)} \hat{\omega}(\rho_1) \\ &\quad + \frac{\rho_1\,\rho_2[(1-q_1)(1-q_2)-\beta] + \beta\,v\rho_1}{v\,q_1\,q_2(\rho_2-\rho_1)} \hat{\omega}(\rho_2). \qquad (6.41) \end{aligned}$$

Specially, if $w(x,y)=1$, then by (6.29) and (6.30),

$$\begin{aligned}\phi_T(0) &= 1-\frac{\rho_1\rho_2(1-v)[1-v(q_1\,q_2+\beta)]}{v^2\,q_1\,q_2\,(1-\rho_1)(1-\rho_2)}, \\ \Psi(0) &= 1-\left(\frac{(1-q_1)(1-q_2)}{q_1\,q_2}\right)\left[\frac{\rho_1(1)}{1-\rho_1(1)}\right][E(W)-E(X)].\end{aligned} \tag{6.42}$$

When $m=2$, two special cases are particularly important, these are:

- If $\beta = -[\alpha\,q_2(1-q_1)+(1-\alpha)q_1\,(1-q_2)]$, $0<\alpha<1$, then k is a mixture of two shifted geometric distributions with density $k(x) = [\alpha\,(1-q_1)q_1^{x-1}+(1-\alpha)(1-q_2)q_2^{x-1}]I(x\geq 1)$ and $0<q_i<1$, for $i=1,2$.
- If $q_1=q_2=q$ and $\beta=-\frac{(1-q)^2 q}{(2-q)}$, with $0<q<1$, then k is a truncated negative binomial distribution with density $k(x)=\frac{(1-q)^2}{[1-(1-q)^2]}(x+1)\,q^x I(x\geq 1)$.

3. If $\hat{k}(s)=\frac{s\prod_{i=1}^{n}(1-q_i)}{\prod_{i=1}^{m}(1-s\,q_i)}$, i.e., $\beta_i=0$, for $i=1,2,\ldots,m-1$. Then in this case, k is the distribution of the convolution of m geometric distributions, but shifted to the right by 1. Thus

$$\begin{aligned}g(y) &= v\left[\prod_{i=1}^{m}\frac{\rho_i(1-q_i)}{v\,q_i}\right]\sum_{j=1}^{m}\frac{\rho_j^{m-1}T_{\rho_j}p(y+1)}{\prod_{k=1,k\neq j}^{m}(\rho_j-\rho_k)} \\ &= v\left[\prod_{i=1}^{m}\frac{\rho_i(1-q_i)}{v\,q_i}\right]T_{\rho_m}T_{\rho_{m-1}}\cdots T_{\rho_1}p(y+1).\end{aligned} \tag{6.43}$$

Similarly,

$$H(u)=v\left[\prod_{i=1}^{m}\frac{\rho_i(1-q_i)}{v\,q_i}\right]T_{\rho_m}T_{\rho_{m-1}}\cdots T_{\rho_1}\omega(u+1). \tag{6.44}$$

The starting point of the recursion $\phi(0)$ is given by

$$\begin{aligned}\phi(0)=H(0) &= v\left[\prod_{i=1}^{m}\frac{\rho_i(1-q_i)}{v\,q_i}\right]\sum_{j=1}^{m}\frac{\rho_j^{m-1}T_{\rho_j}\omega(1)}{\prod_{k=1,k\neq j}^{m}(\rho_j-\rho_k)} \\ &= v\left[\prod_{i=1}^{m}\frac{\rho_i(1-q_i)}{v\,q_i}\right]\sum_{j=1}^{m}\frac{\rho_j^{m-2}\hat{\omega}(\rho_j)}{\prod_{k=1,k\neq j}^{m}(\rho_j-\rho_k)}.\end{aligned} \tag{6.45}$$

Specially, by (6.29) and (6.30),

$$\phi_T(0) = 1 - \left[\prod_{i=1}^{m} \frac{\rho_i}{1-\rho_i}\right] \frac{\prod_{i=1}^{m}(1 - v\,q_i) - v\prod_{i=1}^{m}(1-q_i)}{\prod_{i=1}^{m}(v\,q_i)}, \quad (6.46)$$

$$\Psi(0) = 1 - \left(\prod_{i=1}^{m} \frac{1-q_i}{q_i}\right)\left[\prod_{i=1}^{m-1} \frac{\rho_i(1)}{1-\rho_i(1)}\right][E(W) - E(X)]. \quad (6.47)$$

We remark that this is the discrete version of the continuous generalized Erlang(m) process.

6.7 Explicit Expression for $\phi(u)$

In this section, we show that the discounted penalty function $\phi(u)$ can be expressed explicitly in terms of a compound geometric d.f.'s. First rewrite (6.31) as

$$\phi(u) = \frac{1}{1+\xi_v}\sum_{y=1}^{u}\phi(u-y)\,l(y) + \frac{1}{1+\xi_v}M(u), \qquad u \geq 1, \qquad (6.48)$$

where ξ_v is such that $\frac{1}{1+\xi_v} = \phi_T(0)$, $l(y) = (1+\xi_v)g(y)$ is a proper d.f., $M(u) = (1+\xi_v)\,H(u)$ and $\phi(0) = \frac{1}{1+\xi_v}K(0) = H(0)$, specially, if $w(x,y) = 1$,

$$\phi_T(u) = \frac{1}{1+\xi_v}\sum_{y=1}^{u}\phi_T(u-y)\,l(y) + \frac{1}{1+\xi_v}\bar{L}(u), \qquad u \geq 1, \qquad (6.49)$$

where $\bar{L}(u) = \sum_{y=u+1}^{\infty} l(y)$ is the tail of l.

Define a compound geometric d.f. by $z(u) = \frac{\xi_v}{1+\xi_v}\sum_{n=0}^{\infty}\left(\frac{1}{1+\xi_v}\right)^n l^{*n}(u)$, for $u \in \mathbb{N}$, with $z(0) = \frac{\xi_v}{1+\xi_v}$, where $*$ denotes the convolution. Then it is easy to show, using generating functions, that $\phi_T(u)$ can be expressed as the tail of the compound geometric d.f. z as follows:

$$\phi_T(u) = \bar{Z}(u) = \sum_{y=u+1}^{\infty} z(y) = \frac{\xi_v}{1+\xi_v}\sum_{n=1}^{\infty}\left(\frac{1}{1+\xi_v}\right)^n \bar{L}^{*n}(u), \quad u \geq 0. \qquad (6.50)$$

Remarks:

1. Since the support of $l(y)$ is $\mathbb{N}^+$, then $l^{*n}(u) = 0$, if $n > u$, therefore $z(u)$ can be expressed as a finite sum by $z(u) = \frac{\xi_v}{1+\xi_v} \sum_{n=0}^{u} \left(\frac{1}{1+\xi_v}\right)^n l^{*n}(u)$.

2. $L^{*n}(u) = 0$, if $n > u$, then $\phi_T(u)$ can be expressed as the sum of finite terms by

$$\phi_T(u) = 1 - \frac{\xi_v}{1+\xi_v} \sum_{n=0}^{u} \left(\frac{1}{1+\xi_v}\right)^n L^{*n}(u), \qquad u \in \mathbb{N}. \tag{6.51}$$

The following theorem shows that, for general $w(x, y)$, the expected discounted penalty function $\phi(u)$ can be expressed explicitly in terms of the compound geometric d.f. $z(u)$.

Theorem 6.7.1.

$$\phi(u) = \frac{1}{\xi_v} \sum_{y=0}^{u} M(u-y)\, z(y), \qquad u \geq 0. \tag{6.52}$$

Proof: Let $\hat{\phi}(s) = \sum_{u=0}^{\infty} s^u \phi(u)$, $\hat{l}(s) = \sum_{y=1}^{\infty} s^y\, l(y)$, $\hat{z}(s) = \sum_{y=0}^{\infty} s^y z(y)$, and $\hat{M}(s) = \sum_{u=0}^{\infty} s^u\, M(u)$ be the generating functions of ϕ, l, z, and M, respectively. Then multiplying s^u to both sides of (6.48) and summing over u from 0 to ∞ yields

$$\hat{\phi}(s) = \frac{\hat{M}(s)}{[1+\xi_v - \hat{l}(s)]}, \qquad |\Re(s)| < 1. \tag{6.53}$$

Now $\hat{z}(s) = \frac{\xi_v}{[1+\xi_v - \hat{l}(s)]}$ implies that $\hat{\phi}(s) = \frac{\xi_v}{[1+\xi_v - \hat{l}(s])} \frac{\hat{M}(s)}{\xi_v} = \hat{z}(s)\, \hat{M}(s)$. Inverting gives (6.52). □

6.8 Distribution of the Surplus Before Ruin and the Deficit at Ruin

In this section, we study the discounted joint and marginal distributions of surplus before ruin $U(T-1)$ and deficit at ruin $|U(T)|$.

Theorem 6.8.1. For $x \geq 0$, $y \geq 1$, and $u \geq 1$:

$$
\begin{aligned}
f_2(x, y \,|\, u) &= \sum_{z=1}^{u} f_2(x,\, y \,|\, u-z) g(z) + I(u \leq x) f_2(x-u, y+u \,|\, 0), &(6.54)\\
f_1(x \,|\, u) &= \sum_{z=1}^{u} f_1(x \,|\, u-z) g(z) + I(u \leq x) \sum_{l=u+1}^{\infty} f_2(x-u,\, l \,|\, 0), &(6.55)\\
g(y \,|\, u) &= \sum_{z=1}^{u} g(y \,|\, u-z) g(z) + g(y+u \,|\, 0), &(6.56)
\end{aligned}
$$

where the starting points $f_2(x,\, y \,|\, 0)$, $f_1(x \,|\, 0)$ and $g(y \,|\, 0)$ are given by (6.19), (6.20) and (6.21), respectively.

Proof: Setting $w(x_1, x_2) = I(x_1 = x, y_1 = y)$, $w(x_1, x_2) = I(x_1 = x)$ and $w(x_1, x_2) = I(y_1 = y)$, respectively, in (6.31) gives the above recursive formulas. □

Define $Z = U(T-1) + |U(T)| + 1$ to be the claim causing ruin and let $h(z \,|\, u), z \geq 2$ be its probability distribution, then

Theorem 6.8.2. For $u \geq 1$ and $z \geq 2$,

$$
h(z \,|\, u) = \sum_{y=1}^{u} h(z \,|\, u-y)\, g(y) + I(z \geq u+2)\, p(z)\, A(u), \tag{6.57}
$$

where

$$
A(u) = \left[\prod_{i=1}^{m} \frac{\rho_i}{v\, q_i}\right] \sum_{j=1}^{m} \frac{c_j\, (1-\rho_j^{z-u-1})}{(1-\rho_j) \prod_{k=1, k\neq j}^{m} (\rho_j - \rho_k)}. \tag{6.58}
$$

While the starting point is given by

$$
h(z \,|\, 0) = p(z)\, A(0) = p(z) \left[\prod_{i=1}^{m} \frac{\rho_i}{v\, q_i}\right] \sum_{j=1}^{m} \frac{c_j\, (1-\rho_j^{z-1})}{(1-\rho_j) \prod_{k=1, k\neq j}^{m} (\rho_j - \rho_k)}. \tag{6.59}
$$

Proof: If setting $w(x, y) = I(x + y + 1 = z)$, then $\phi(u)$ simplifies to $h(z \,|\, u)$, and recursive formula (6.31) simplifies to (6.57). □

As an application of Theorem 6.7.1, we show that $f_2(x,\, y|\, u)$, $f_1(x \,|\, u)$, $g(y \,|\, u)$ and $h(z \,|\, u)$ all find explicit expressions in terms of the compound geometric p.f. z.

Theorem 6.8.3. For $x \in \mathbb{N}$, and $y \in \mathbb{N}^+$,

$$f_2(x,\, y\,|\,u) = \begin{cases} \left(\frac{1+\xi_v}{\xi_v}\right)\left(\frac{\prod_{i=1}^m \rho_i}{\prod_{i=1}^m v\, q_i}\right) p(x+y+1) \sum_{j=1}^m \frac{c_j\, \rho_j^{x-u} \sum_{n=0}^u \rho_j^n\, z(n)}{\prod_{k=1,k\neq j}^m (\rho_j-\rho_k)}, & 0 \le u \le x, \\ \left(\frac{1+\xi_v}{\xi_v}\right)\left(\frac{\prod_{i=1}^m \rho_i}{\prod_{i=1}^m v\, q_i}\right) p(x+y+1) \sum_{j=1}^m \frac{c_j\, \rho_j^{x-u} \sum_{n=u-x}^u \rho_j^n\, z(n)}{\prod_{k=1,k\neq j}^m (\rho_j-\rho_k)}, & u > x. \end{cases} \tag{6.60}$$

Proof: Rewrite (6.54) in Theorem 6.8.1 as

$$f_2(x, y\,|\,u) = \frac{1}{1+\xi_v}\left[\sum_{z=1}^u f_2(x,\, y\,|\,u-z)\, l(z) + (1+\xi_v) I(u \le x) f_2(x-u, y+u\,|\,0)\right].$$

By Theorem 6.7.1, $f_2(x,\, y\,|\,u)$ can be rewritten explicitly by

$$\begin{aligned} f_2(x, y|u) &= \frac{1+\xi_v}{\xi_v} \sum_{n=0}^u I(u-n \le x) f_2(x-u+n, y+u-n\,|\,0)\, z(n) \\ &= \left(\frac{1+\xi_v}{\xi_v}\right)\left(\frac{\prod_{i=1}^m \rho_i}{\prod_{i=1}^m v\, q_i}\right) \sum_{n=0}^u I(u-n \le x) \sum_{j=1}^m \frac{p(x+y+1)\, c_j\, \rho_j^{x-u+n}\, z(n)}{\prod_{k=1,k\neq j}^m (\rho_j - \rho_k)}. \end{aligned}$$

Changing the order of summation, the above expression simplifies to (6.60). □.

Corollary 6.8.1. For $x \in \mathbb{N}$,

$$f_1(x\,|\,u) = \begin{cases} \left(\frac{1+\xi_v}{\xi_v}\right)\left(\frac{\prod_{i=1}^m \rho_i}{\prod_{i=1}^m v\, q_i}\right) \bar{P}(x) \sum_{j=1}^m \frac{c_j\, \rho_j^{x-u} \sum_{n=0}^u \rho_j^n\, z(n)}{\prod_{k=1,k\neq j}^m (\rho_j-\rho_k)}, & 0 \le u \le x, \\ \left(\frac{1+\xi_v}{\xi_v}\right)\left(\frac{\prod_{i=1}^m \rho_i}{\prod_{i=1}^m v\, q_i}\right) \bar{P}(x) \sum_{j=1}^m \frac{c_j\, \rho_j^{x-u} \sum_{n=u-x}^u \rho_j^n\, z(n)}{\prod_{k=1,k\neq j}^m (\rho_j-\rho_k)}, & u > x. \end{cases} \tag{6.61}$$

Proof: Integrating (6.60) over y from 0 to ∞ gives formula (6.61). □

Remark: If $m = 1$, then (6.60) simplifies to

$$f_2(x, y\,|\,u) = \begin{cases} f_2(x, y\,|\,0) \frac{1+\xi_v}{\xi_v} \sum_{n=0}^u \rho^{n-u}\, z(n), & 0 \le u \le x \\ f_2(x, y\,|\,0) \frac{1+\xi_v}{\xi_v} \sum_{n=u-x}^u \rho^{n-u}\, z(n), & u > x \end{cases}, \tag{6.62}$$

which can be found in Li and Garrido (2002). Specially, if setting $v = 1$, then the joint distribution of $U(T-1)$ and $|U(T)|$ is given by

$$f_2(x, y\,|\,u) = \begin{cases} f_2(x, y\,|\,0) \frac{1-\Psi(u)}{1-\Psi(0)}, & 0 \le u \le x \\ f_2(x, y\,|\,0) \frac{\Psi(u-x)-\Psi(u)}{1-\Psi(0)}, & u > x \end{cases}, \tag{6.63}$$

which is Dickson's classical formula in the discrete model.

For the distribution of the claim causing ruin $Z = U(T-1) + |U(T)| + 1$, we have the following result.

Theorem 6.8.4. For $u \in \mathbb{N}$, and $u+2 \le z$,

$$h(z\,|\,u) = \left(\frac{1+\xi_v}{\xi_v}\right)\left(\frac{\prod_{i=1}^m \rho_i}{\prod_{i=1}^m v\,q_i}\right)p(z)\sum_{j=1}^m \frac{c_j \sum_{n=0}^u (1-\rho_j^{z-u+n-1})\,z(n)}{(1-\rho_j)\prod_{k=1,k\ne j}^m(\rho_j-\rho_k)}, \quad (6.64)$$

for $2 \le z < u+2$,

$$h(z\,|\,u) = \left(\frac{1+\xi_v}{\xi_v}\right)\left(\frac{\prod_{i=1}^m \rho_i}{\prod_{i=1}^m v\,q_i}\right)p(z)\sum_{j=1}^m \frac{c_j \sum_{n=u+2-z}^u (1-\rho_j^{z-u+n-1})\,z(n)}{(1-\rho_j)\prod_{k=1,k\ne j}^m(\rho_j-\rho_k)}. \quad (6.65)$$

Proof: Using Theorem 6.7.1 with
$M(u) = (1+\xi_v)\left[\prod_{i=1}^m \frac{\rho_i}{v\,q_i}\right]\,p(z)\,I(z \ge u+2)\sum_{j=1}^m \frac{c_j\,(1-\rho_j^{z-u-1})}{(1-\rho_j)\prod_{k=1,k\ne j}^m(\rho_j-\rho_k)}$. □

Specially, if $m = 1$, we have the following Corollary.

Corollary 6.8.2. If the claim waiting times are geometrically distributed with $k(x) = (1-q)\,q^{x-1}I(x \ge 1)$, then

$$h(z\,|\,u) = \begin{cases} \frac{v(1-\rho)(1-q)}{1-v}p(z)\sum_{n=0}^u \frac{1-\rho^{z-u+n-1}}{1-\rho}z(n), & u+2 \le z \\ \frac{v(1-\rho)(1-q)}{1-v}p(z)\sum_{n=u+2-z}^u \frac{1-\rho^{z-u+n-1}}{1-\rho}z(n), & 2 \le z < u+2 \end{cases}. \quad (6.66)$$

Further, if $v = 1$, $\rho = 1$, and $\rho'(1) = \frac{1+\theta}{\theta}$, then

$$h(z\,|\,u) = \begin{cases} \frac{(1-q)(1+\theta)}{\theta}p(z)\sum_{n=0}^u (z-u+n-1)\,z(n), & u+2 \le z, \\ \frac{(1-q)(1+\theta)}{\theta}p(z)\sum_{n=u+2-z}^u (z-u+n-1)\,z(n), & 2 \le z < u+2, \end{cases} \quad (6.67)$$

where $0 < \theta$ is the security loading factor.

6.9 Explicit Results for Two Classes of Claim Size Distributions

Theorem 6.7.1 shows that the expected discounted penalty function $\phi(u)$ can be expressed explicitly in terms of the compound geometric p.f. $z(u)$, with $z(0) = \phi_T(0)$, and $z(u) = \phi_T(u-1) - \phi_T(u)$, for $u \ge 1$, that is to say, if $\phi_T(u)$ can be obtained explicitly, then so can $\phi(u)$. One such case where $\phi_T(u)$ finds an explicit expression is when it admits a rational generating function. It follows from (6.34) that $\hat{\phi}_T(s)$ is a rational function if and only if $\hat{g}(s)$ is a rational function, while $\hat{g}(s)$ is rational function if and only if $\hat{p}(s)$ is a rational function. Another case for which $\phi_T(u)$ has an explicit expression is when $\hat{p}(s)$ is a polynomial (or $p(x)$ has a finite support), since, in this case, $\hat{\phi}_T(s)$ also has a rational generating function.

6.9.1 K_n Claim Size Distribution

From (6.34), the generating function of $\phi_T(s)$ is given by

$$\hat{\phi}_T(s) = \frac{\phi_T(0) - \hat{g}(s)}{(1-s)[1-\hat{g}(s)]} \tag{6.68}$$

$$= \frac{\prod_{i=1}^m (s - v\,q_i) - \hat{p}(s) B_{m-1}(s) - \left(\prod_{i=1}^m \frac{v\,q_i}{\rho_i}\right)[1-\phi_T(0)] \prod_{i=1}^m (s-\rho_i)}{(1-s)\{\prod_{i=1}^m (s - v\,q_i) - \hat{p}(s) B_{m-1}(s)\}},$$

where $B_{m-1}(s) = v[s^{m-1}\prod_{i=1}^m (1-q_i) + \sum_{j=1}^{m-1} \beta_j s^{m-1-j}\,(v-s)^j]$ is a polynomial of degree $m-1$, with leading coefficient $B_{m-1} = v[\prod_{i=1}^m (1-q_i) + \sum_{j=1}^{m-1}(-1)^j \beta_j$.

In this section, we assume that $p(x)$ is K_n distributed for x, $n \in \mathbb{N}^+$, i.e., its generating function is given by

$$\hat{p}(s) = \frac{E_n(s)}{\prod_{i=1}^n (1 - s\,\alpha_i)}, \qquad \Re(s) < \min\left\{\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \cdots \frac{1}{\alpha_n}\right\}, \tag{6.69}$$

where $E_n(s)$ is a polynomial of degree n with $E_n(0) = 0$, $E_n(1) = \prod_{i=1}^n (1-\alpha_i)$, and $0 < \alpha_i < 1$, for $i = 1, 2, \ldots, n$. In this case, $\hat{\phi}_T(s)$ can be transformed to a rational function, which is given in the following theorem.

Define $E_{m,n}(s) = [\prod_{i=1}^m (s - v\,q_i)][\prod_{i=1}^n (1 - s\,\alpha_i)] - E_n(s) B_{m-1}(s)$ to be a polynomial of degree $n+m$ with leading coefficient $(-1)^n(\prod_{i=1}^n \alpha_i)$. Then it is easily verified that the roots to the generalized Lundberg equation (6.3), $\rho_1, \rho_2, \ldots, \rho_m$ with $0 < |\rho_i| < 1$, are m roots to the equation $E_{m,n}(s) = 0$. Let $R_1, R_2, \ldots, R_n$ with $|R_i| \geq 1$ be the remaining n roots of $E_{m,n}(s) = 0$. We remark that there is a relation among the roots $\rho_1, \rho_2, \ldots, \rho_m$ and R_1, $R_2, \ldots, R_n$, i.e.,

$$\left[\prod_{i=1}^m \rho_i\right]\left[\prod_{i=1}^n R_i\right] = \frac{\prod_{i=1}^n v\,q_i}{\prod_{i=1}^n \alpha_i}. \tag{6.70}$$

Theorem 6.9.1. For above defined $\hat{p}(s)$, the generating function of $\phi_T(u)$ is given by

$$\hat{\phi}_T(s) = \frac{\varpi_{n-1}(s)}{(R_1 - s)(R_2 - s)\cdots(R_n - s)}, \tag{6.71}$$

where $\varpi_{n-1}(s) = \{\prod_{i=1}^n (R_i - s) - (\prod_{i=1}^m \frac{v\,q_i}{\rho_i})[1 - \phi_T(0)]\prod_{i=1}^n (1/\alpha_i - s)\}/(1-s)$ is a polynomial of degree $n-1$.

Further, if R_i are distinct, then by partial fractions,

$$\hat{\phi}_T(s) = \sum_{i=1}^n \frac{r_i}{(R_i - s)}. \tag{6.72}$$

Accordingly,

$$\phi_T(u) = \sum_{i=1}^n \left(\frac{r_i}{R_i}\right) R_i^{-u}, \qquad u \in \mathbb{N}, \tag{6.73}$$

specially,

$$\phi_T(0) = 1 - \left(\prod_{i=1}^m \frac{\rho_i}{v\,q_i}\right) \frac{\prod_{i=1}^n (R_i - 1)}{\prod_{i=1}^n (1/\alpha_i - 1)}, \tag{6.74}$$

where $r_i = \left(\prod_{k=1}^n \frac{1 - R_i\,\alpha_k}{1-\alpha_k}\right)\left(\prod_{j=1, j\neq i}^n \frac{R_j - 1}{R_j - R_i}\right)$, for $i = 1, 2, \ldots, n$.

Proof: Substituting (6.69) into (6.68) and multiplying $\prod_{i=1}^n (1 - s\,\alpha_i)$ to the both denominator and numerator yields,

$$\hat{\phi}_T(s) = \frac{E_{m,n}(s) - \left(\prod_{i=1}^m \frac{v\,q_i}{\rho_i}\right)[1 - \phi_T(0)]\prod_{i=1}^m (s - \rho_i)\prod_{i=1}^n (1 - s\,\alpha_i)}{(1-s)E_{m,n}(s)}. \tag{6.75}$$

Now that $E_{m,n}(s) = [\prod_{i=1}^n \alpha_i][\prod_{i=1}^m (s - \rho_i)][\prod_{i=1}^n (R_i - s)]$, substituting it into (6.75) and canceling out common factors gives

$$\hat{\phi}_T(s) = \frac{\prod_{i=1}^n (R_i - s) - \left(\prod_{i=1}^m \frac{v\,q_i}{\rho_i}\right)[1 - \phi_T(0)]\prod_{i=1}^n (1/\alpha_i - s)}{(1-s)\prod_{i=1}^n (R_i - s)}. \tag{6.76}$$

Since $s = 1$ is a removable singularity of $\hat{\phi}_T(s)$, the above numerator must be zero if setting $s = 1$, then there is an alternative expression for $\phi_T(0)$ given by (6.74). Also $\varpi_{n-1}(s) = \{\prod_{i=1}^n (R_i - s) - (\prod_{i=1}^m \frac{v\,q_i}{\rho_i})[1 - \phi_T(0)]\prod_{i=1}^n (1/\alpha_i - s)\}/(1-s)$ is a polynomial of degree $n-1$.

Finally, if $R_i, R_2, \ldots, R_n$ are distinct, by partial fractions, we can prove that (6.72) holds. Inverting it gives (6.73). □

Remark: If $\hat{p}(s)$ is given by (6.69), the $\hat{g}(s)$ can be simplified to

$$\hat{g}(s) = 1 - \left(\prod_{i=1}^{m} \frac{\rho_i}{v\, q_i}\right)\left(\prod_{i=1}^{n} \frac{R_i - s}{1/\alpha_i - s}\right) = 1 - \frac{\prod_{i=1}^{n}(1 - s/R_i)}{\prod_{i=1}^{n}(1 - \alpha_i\, s)}. \qquad (6.77)$$

Example 6.9.1. In this example, we assume that the claim waiting times are K_m distributed with the generating function of the claim density is given in (6.4), the claim amount is geometrically distributed with $p(s) = (1-\alpha)\,\alpha^{x-1} I(x \geq 1)$ and $\hat{p}(s) = \frac{s(1-\alpha)}{(1-s\,\alpha)}$. Then in this case, the equation

$$E_{m,1}(s) = \left[\prod_{i=1}^{m}(s - v\, q_i)\right](1 - s\,\alpha) - s\,(1-\alpha)\, B_{m-1}(s) = 0$$

has m roots, say $\rho_1, \rho_2, \ldots, \rho_m$ with $|\rho_i| < 1$, and one root $R > 1$. It is easily checked that the relation $\prod_{i=1}^{m} \frac{v\, q_i}{\rho_i} = \alpha\, R$ holds.

By (6.74), $\phi_T(0) = \frac{1-\alpha R}{R(1-\alpha)}$, and

$$\phi_T(u) = \frac{1 - \alpha\, R}{R(1-\alpha)} R^{-u} = \phi_T(0) R^{-u}, \qquad u \geq 0.$$

(6.77) gives $\hat{g}(s) = \phi_T(0)\frac{s(1-\alpha)}{(1-s\,\alpha)}$, and inverting gives $g(y) = \phi_T(0) p(y)$. Then we can use the recursive formula $\phi(u) = \sum_{y=1}^{u} \phi(u-y)\, g(y) + H(u)$ to compute the expected discounted penalty function, or use the explicit formula $\phi(u) = \frac{1}{1-\phi_T(0)} \sum_{y=0}^{u} M(u-y)\, z(y)$, with $z(0) = 1 - \phi_T(0)$, and $z(u) = \phi_T(u-1) - \phi_T(u) = \phi_T(0)\frac{R-1}{R} R^{-u} = \frac{R-1}{R}\phi_T(u)$. As an application, the discounted joint density of $U(T-1)$ and $|U(T)|$ is given for $0 \leq u \leq x$,

$$f_2(x, y|u) = \frac{1 - \alpha\, R}{\alpha\, R^2} p(x+y+1) \sum_{j=1}^{m} \frac{c_j\, \rho_j^x [R\, \rho_j^{-u} - \rho_j\, R^{-u}]}{(R - \rho_j) \prod_{k=1, k\neq j}^{m} (\rho_j - \rho_k)}, \qquad (6.78)$$

for $u > x$,

$$f_2(x, y|u) = \frac{1 - \alpha\, R}{\alpha\, R^2} p(x+y+1) \sum_{j=1}^{m} \frac{c_j\, R^{-u} [R^{x+1} - \rho_j^{x+1}]}{(R - \rho_j) \prod_{k=1, k\neq j}^{m} (\rho_j - \rho_k)}. \qquad (6.79)$$

Example 6.9.2. In this example, we assume that the claim waiting times are shifted negative binomial distributed with $k(x) = x\,(1-q)^2\,q^{x-1}I(x \geq 1)$, and $\hat{k}(s) = \frac{s(1-q)^2}{(1-s\,q)^2}$. Claim amounts have a mixture of two geometric distributions with $p(x) = \vartheta\,(1-\alpha_1)\alpha_1^{x-1} + (1-\vartheta)(1-\alpha_2)\alpha_2^{x-1}$, for $x \geq 1$ and $0 < \vartheta,\ \alpha_1,\ \alpha_2 < 1$. Then $\hat{p}(s) = \frac{s[(1-\alpha_1)(1-\alpha_2)+\beta(1-s)]}{(1-s\,\alpha_1)(1-s\,\alpha_2)}$, with $\beta = \vartheta\,\alpha_2(1-\alpha_1) + (1-\vartheta)\,\alpha_1\,(1-\alpha_2)$.

Since the equation

$$E_{2,2}(s) = (s - v\,q)^2(1 - s\,\alpha_1)(1 - s\,\alpha_2) - v\,(1-q)^2\,s^2[(1-\alpha_1)(1-\alpha_2) + \beta(1-s)] = 0,$$

has two roots, say ρ_1, ρ_2 with $|\rho_i| < 1$, and two roots, say R_1, R_2 with $|R_i| > 1$. It is easy to check that the relation $\rho_1\,\rho_2\,R_1\,R_2 = \frac{v^2\,q^2}{\alpha_1\,\alpha_2}$ holds.

By (6.74) and the above relation

$$\phi_T(0) = 1 - \frac{1}{R_1\,R_2}\frac{(R_1-1)(R_2-1)}{(1-\alpha_1)(1-\alpha_2)}, \tag{6.80}$$

and for $u \geq 0$,

$$\begin{aligned} \phi_T(u) &= \frac{(1-R_1\,\alpha_1)(1-R_2\,\alpha_2)(R_2-1)R_1^{-(u+1)}}{(1-\alpha_1)(1-\alpha_2)(R_2-R_1)} \\ &\quad - \frac{(1-R_2\,\alpha_1)(1-R_2\,\alpha_2)(R_1-1)R_2^{-(u+1)}}{(1-\alpha_1)(1-\alpha_2)(R_2-R_1)} \end{aligned} \tag{6.81}$$

(6.77) gives

$$\hat{g}(s) = \frac{s\left[(\alpha_1\,\alpha_2 - \frac{1}{R_1 R_2})s + \frac{R_1+R_2}{R_1\,R_2} - (\alpha_1 + \alpha_2)\right]}{(1 - s\,\alpha_1)(1 - s\,\alpha_2)},$$

inverting yields

$$g(y) = \varsigma_1\,\alpha_1^{y-1} + \varsigma_2\,\alpha_2^{y-1}, \qquad y \geq 1, \tag{6.82}$$

where

$$\begin{aligned} \varsigma_1 &= \frac{(R_1 R_2\,\alpha_1\,\alpha_2 - 1) + \alpha_1[R_1 + R_2 - R_1 R_2(\alpha_1 + \alpha_2)]}{R_1 R_2(\alpha_1 - \alpha_2)}, \\ \varsigma_2 &= \frac{(R_1 R_2\,\alpha_1\,\alpha_2 - 1) + \alpha_2[R_1 + R_2 - R_1 R_2(\alpha_1 + \alpha_2)]}{R_1 R_2(\alpha_2 - \alpha_1)}. \end{aligned}$$

If setting $v = 1$, and $w(x, y)$ to be $x\,y$, x and y, respectively, then $\phi(u)$ simplifies to the joint and marginal moments of $U(T-1)$ and $|U(T)|$, which can be obtained by the recursive formula $\phi(u) = \sum_{y=1}^{u} \phi(u-y)\, g(y) + H(u)$.

Now let $q = \frac{1}{3}$, $\vartheta = 0.6$, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{3}$, $v = 1$, then $E(W) = 2 > E(X) = 1.8$ means a positive loading. The equation $E_{2,2}(s) = 0$ gives four roots, $\rho_1 = 1$, $\rho_2 = 0.2183$, $R_1 = 1.1344$ and $R_2 = 2.6917$. This gives

$$\begin{aligned} \Psi(u) &= 0.7731 \times 1.1344^{-u} + 0.00342 \times 2.6917^{-u}, \qquad u \geq 0, \\ g(y) &= 0.2941\, \alpha_1^{y-1} + 0.1256\, \alpha_2^{y-1}, \qquad y \geq 1. \end{aligned}$$

Table 6.1 gives the joint and marginal moments of the surplus before ruin and deficit at ruin, together with the mean of the claim amount causing ruin. It shows that: (i) the joint moment given ruin occurs is increasing in u; (ii) the first two moments of $U(T-1)$ and $|U(T)|$ are increasing in u, while the effect of u on the first two moments of $U(T-1)$ is greater than that of $|U(T)|$; (iii) the mean of the claim causing ruin is increasing in u, and greater than the mean of claim amount r.v.'s; (iv) finally, the effect of u on all these quantities is greater for small u, and smaller for big initial surplus values u.

Table 6.2 gives the covariance and correlation coefficient of the surplus before ruin and the deficit at ruin, given that ruin occurs. It can be seen that the covariance is increasing in u, and the two random variables are positively correlated, while the smaller correlation coefficient mean that they are weakly correlated.

6.9.2 Claim Amounts Distributions with Finite Support

In this section, we assume that the claim amount distribution has a finite support, i.e., for $N \geq 2$:

$$p(n) = P(X = n) = p_n, \qquad n = 1, 2, \ldots N. \tag{6.83}$$

Table 6.1: Moments for the surplus before ruin and deficit at ruin for different u

u	A	B	C	D	E	F
0	1.9107	0.9904	1.8784	2.8557	5.2716	3.8688
1	2.95803	1.53196	1.89591	4.53027	5.37623	4.4279
2	3.53798	1.82529	1.90329	6.02392	5.42065	4.7286
3	3.86556	1.98875	1.90645	7.17367	5.43939	4.8952
4	4.05238	2.08156	1.90785	8.00108	5.44744	4.9894
5	4.15964	2.13462	1.90838	8.57300	5.45077	5.0430
6	4.22144	2.16502	1.90862	8.95754	5.45238	5.0736
7	4.25669	2.18245	1.90879	9.21084	5.45301	5.0912
8	4.27691	2.19274	1.90889	9.37486	5.45347	5.1016
9	4.28879	2.19854	1.90913	9.48004	5.45368	5.1077
10	4.29552	2.20187	1.90917	9.54631	5.45399	5.1110
11	4.29957	2.20366	1.90942	9.58805	5.45424	5.1131
12	4.30171	2.20491	1.90932	9.61360	5.45411	5.1142
13	4.30269	2.20535	1.90935	9.62939	5.45403	5.1147
14	4.30409	2.20612	1.90966	9.63979	5.45443	5.1158
15	4.30447	2.20586	1.90899	9.64538	5.45502	5.1149

A: joint moments of $U(T-1)$ and $U(T)$, given that ruin occurs
B: mean of $U(T-1)$, given that ruin occurs
C: mean of $|U(T)|$, given that ruin occurs
D: second moment of $U(T-1)$ about the origin, given that ruin occurs
E: second moment of $|U(T)|$ about the origin, given that ruin occurs
F: mean of the claim amount causing ruin, given that ruin occurs

Table 6.2: Covariance and coefficient of correlation between the surplus before ruin and the deficit at ruin for different u

u	0	1	2	3	4	5	6	7
G	0.05036	0.05356	0.06391	0.07411	0.08109	0.08599	0.08921	0.09085
H	0.02785	0.02716	0.02905	0.03075	0.03149	0.03189	0.03209	0.03202

u	8	9	10	11	12	13	14	15
G	0.09121	0.09151	0.09173	0.09185	0.09189	0.09192	0.09216	0.09349
H	0.03173	0.03156	0.03148	0.03139	0.03132	0.03131	0.03104	0.03178

G stands for the covariance, H stands for the coefficient of correlation.

Then

$$\hat{p}(s) = D_N(s) := p_1\, s + p_2\, s^2 + \cdots + p_N\, s^N, \qquad -1 < \Re(s) < 1, \tag{6.84}$$

is a polynomial of degree N. For example, the binomial distribution, the discrete uniform, and the hyper-geometric distribution all have a finite support.

Define $V(s) := D_N(s)B_{m-1}(s) - \prod_{i=1}^{m}(s - v\, q_i)$ to be a polynomial of degree $N+m-1$, with leading coefficient $V_{N+m-1} = p_N\, B_{m-1}$, where $B_{m-1} = v[\prod_{i=1}^{m}(1-q_i) + \sum_{j=1}^{m-1}(-1)^j \beta_j]$ is the leading coefficient of polynomial $B_{m-1}(s)$. Of all the $N+m-1$ roots to the equation $V(s) = 0$, $\rho_1, \rho_2, \ldots, \rho_m$ are m roots with $0 < |\rho_i| < 1$. Let $R_1, R_2, \ldots R_{N-1}$ with $|R_i| > 1$ be the remaining $N-1$ roots. Therefore, $V(s)$ can be factored as $V(s) = V_{N+m-1}\, [\prod_{i=1}^{m}(s - \rho_i)]\, [\prod_{i=1}^{N-1}(s - R_i)]$. Setting $s = 0$, it is easily shown that

$$(-1)^N V_{N+m-1}\Big(\prod_{i=1}^{m} \rho_i\Big)\Big(\prod_{i=1}^{N-1} R_i\Big) = \prod_{i=1}^{m}(v\, q_i). \tag{6.85}$$

Then (6.68) can be rewritten as

$$\hat{\phi}_T(s) = \frac{V_{N+m-1} \prod_{i=1}^{N-1}(s - R_i) + \prod_{i=1}^{m}(\frac{v\, q_i}{\rho_i})[1 - \phi_T(0)]}{V_{N+m-1}(1 - s)\prod_{i=1}^{N-1}(s - R_i)}, \tag{6.86}$$

Since $s = 1$ is a removable singularity of $\hat{\phi}_T(s)$, then we have

$$1 - \phi_T(0) = -V_{N+m-1}\left[\prod_{i=1}^{m} \frac{\rho_i}{v\, q_i}\right]\left[\prod_{i=1}^{N-1}(1 - R_i)\right] = \prod_{i=1}^{N-1} \frac{R_i - 1}{R_i} \tag{6.87}$$

and (6.86) simplifies to

$$\hat{\phi}_T(s) = \frac{F_{N-2}(s)}{\prod_{i=1}^{N-1}(R_i - s)}, \qquad -1 < \Re(s) < 1, \tag{6.88}$$

where $F_{N-2}(s) := \frac{\prod_{i=1}^{N-1}(R_i - s) - \prod_{i=1}^{N-1}(R_i - 1)}{(1-s)}$ is a polynomial of degree $N - 2$. By partial fractions,

$$\hat{\phi}_T(s) = \sum_{i=1}^{N-1} \frac{r_i}{(R_i - s)} = \sum_{i=1}^{N-1}(\frac{r_i}{R_i})\frac{1}{(1 - \frac{s}{R_i})}, \tag{6.89}$$

where $r_i = \prod_{j=1,j\neq i}^{N-1}\left(\frac{R_j-1}{R_j-R_i}\right)$. Inverting yields

$$\phi_T(u) = \sum_{i=1}^{N-1}\left(\frac{r_i}{R_i}\right) R_i^{-u}, \qquad u \in \mathbb{N}^+. \tag{6.90}$$

Finally, if $\hat{p}(s)$ is given by (6.84), then $\hat{g}(s)$ simplifies to

$$\hat{g}(s) = 1 + V_{N+m-1}\left(\prod_{i=1}^{m}\frac{\rho_i}{v\,q_i}\right)\prod_{i=1}^{N-1}(s-R_i) = 1 - \prod_{i=1}^{N-1}\frac{R_i - s}{R_i}. \tag{6.91}$$

Isolating the coefficient of s^n gives $g(n)$, for $n = 1, 2, \ldots, N-1$, e.g.,

$$\begin{aligned}
g(1) &= \left[\sum_{i=1}^{N-1}\frac{1}{R_i}\right], && (6.92)\\
g(2) &= -\sum_{1\le i<j\le N-1}\frac{1}{R_i R_j}, && (6.93)\\
\vdots &\quad \vdots \\
g(N-2) &= (-1)^{N-3}\left[\prod_{i=1}^{N-1}\frac{1}{R_i}\right]\sum_{i=1}^{N-1}R_i, && (6.94)\\
g(N-1) &= (-1)^{N-2}\prod_{i=1}^{N-1}\frac{1}{R_i}. && (6.95)
\end{aligned}$$

Example 6.9.3. In this example, we assume that claims waiting times are K_m distributed with $\hat{k}(s)$ given by (6.4), and constant claim amounts at 2, i.e., $P(X=2) = 1$ with $\hat{p}(s) = s^2$. Then $V(s) = s^2\,B_{m-1}(s) - \prod_{i=1}^{m}(s - v\,q_i)$ is a polynomial of degree $m+1$ with leading coefficient $V_{m+1} = B_{m-1} = v[\prod_{i=1}^{m}(1 - q_i) + \sum_{j=1}^{m-1}(-1)^j\,\beta_j]$. It can be factored as

$$V(s) = V_{m+1}(s-R)\prod_{i=1}^{m}(s-\rho_i), \tag{6.96}$$

where $\rho_1, \rho_2, \ldots, \rho_m$ with $|\rho_i| < 1$, and $R > 1$ are $m+1$ roots to the equation $V(s) = 0$. By (6.87)

$$\phi_T(0) = 1 - V_{m+1}\Big(\prod_{i=1}^{m}\frac{\rho_i}{v\,q_i}\Big)(R-1) = V_{m+1}\prod_{i=1}^{m}\frac{\rho_i}{v\,q_i} = \frac{1}{R}$$

and

$$\phi_T(u) = (\frac{1}{R})\, R^u = \phi_T(0)\, R^{-u} = R^{-(u+1)}, \qquad u \in \mathbb{N}.$$

Specially, if $v = 1$, then

$$\Psi(u) = [R(1)]^{-(u+1)}, \qquad u \in \mathbb{N},$$

where $R(1) = \lim_{v \to 1^-} R$.

In this case, $g(1) = \phi_T(0) = \frac{1}{R}$, and $g(i) = 0$, for $i = 2, 3, \ldots$. Then the expected discounted penalty function $\phi(u)$ admits a simple recursive formula:

$$\phi(u) = \phi(u-1)g(1) + H(u) = \frac{\phi(u-1)}{R} + H(u), \qquad u \in \mathbb{N}^+, \tag{6.97}$$

and the starting value for the recursion is $H(0)$. Explicitly, $z(0) = 1 - \phi_T(0) = \frac{R-1}{R}$, and $z(u) = \phi_T(u-1) - \phi_T(u) = (\frac{R-1}{R})R^{-u}$, $u \in \mathbb{N}^+$. Then for $u \in \mathbb{N}^+$,

$$\phi(u) = \frac{R}{(R-1)} \sum_{y=1}^{u} H(u-y)z(y) + H(u) = \sum_{y=1}^{u} H(u-y)\, R^{-y} + H(u). \tag{6.98}$$

Since in this example, $U(T-1)$ only takes value of 0, and $|U(T)|$ only takes value of 1, then by Theorem 6.8.3, the joint distribution is given by

$$\begin{aligned} f_2(0,\, 1\,|\,0) &= \left(\prod_{i=1}^{m} \frac{\rho_i}{v\, q_i}\right) \sum_{j=1}^{m} \frac{c_j}{\prod_{k=1,k\neq j}^{m} (\rho_j - \rho_k)} \\ f_2(0,\, 1\,|\,u) &= \left(\prod_{i=1}^{m} \frac{\rho_i}{v\, q_i}\right) \sum_{j=1}^{m} \frac{c_j\, R^{-u}}{\prod_{k=1,k\neq j}^{m} (\rho_j - \rho_k)}, \qquad u \in \mathbb{N}^+. \end{aligned}$$

Example 6.9.4. In this example, we assume that the claim waiting times have a negative binomial$(2, q)$ distribution, with $k(x) = x\,(1-q)^2\, q^{x-1} I(x \geq 1)$ and $\hat{k}(s) = \frac{s(1-q)^2}{(1-s\,q)^2}$. The claim sizes are uniformly distributed with $P(X = 1) = P(X = 2) = P(X = 3) = \frac{1}{3}$ and with $\hat{p}(s) = \frac{(s+s^2+s^3)}{3}$. Then $V(s) = \hat{p}(s)B_1(s) - (s - v\,q)^2 = \frac{v(1-q)^2}{3}(s^2 + s^3 + s^4) - (s - v\,q)^2$ is a polynomial of degree 4 with leading coefficient $\frac{v(1-q)^2}{3}$. It can be factored as $V(s) = \frac{v(1-q)^2}{3}(s - \rho_1)(s - \rho_2)(s - R_1)(s -$

R_2). We remark that the relation $\rho_1\,\rho_2\,R_1 R_2 = -\frac{3\,v\,q^2}{(1-q)^2}$ holds by setting $s = 0$ in the above factorization.

(6.87) together with above relation gives

$$\phi_T(0) = 1 + \frac{(1-q)^2\,\rho_1\,\rho_2}{3\,v\,q^2}(R_1 - 1)(R_2 - 1) = \frac{R_1 + R_2 - 1}{R_1\,R_2},$$

and for $u \in \mathbb{N}^+$,

$$\phi_T(u) = \frac{R_2 - 1}{R_2 - R_1} R_1^{-(u+1)} + \frac{R_1 - 1}{R_1 - R_2} R_2^{-(u+1)}. \qquad (6.99)$$

Together (6.92) and (6.95) give that

$$g(1) = \frac{R_1 + R_2}{R_1\,R_2}, \qquad g(2) = -\frac{1}{R_1\,R_2} \quad \text{and} \quad g(i) = 0, \qquad \text{for } i \geq 3. \qquad (6.100)$$

Thus the recursive formula for $\phi(u)$ simplifies to

$$\begin{aligned} \phi(0) &= H(0), \\ \phi(1) &= \phi(0)\,g(1) + H(1), \\ \phi(u) &= \phi(u-1)\,g(1) + \phi(u-2)\,g(2) + H(u), \qquad u \geq 2, \end{aligned}$$

where in this example, $H(0) = \frac{\rho_1\,\rho_2(1-q)^2}{3\,v\,q^2}[w(0,1) + w(0,2) + (\rho_1 + \rho_2)w(1,\,1)]$, $H(1) = \frac{\rho_1\,\rho_2(1-q)^2}{3\,v\,q^2} w(1,1)$, and $H(u) = 0$, for $u \geq 2$.

$\phi(u)$ can also be evaluated explicitly by

$$\phi(u) = \frac{1}{1 - \phi_T(0)} \sum_{y=1}^{u} H(u-y)\,z(y) + H(u), \qquad u \geq 1, \qquad (6.101)$$

where $\frac{1}{1-\phi_T(0)} = \frac{R_1\,R_2}{(R_1-1)(R_2-1)}$, and

$$z(u) = \phi_T(u-1) - \phi_T(u) = \frac{(R_1-1)(R_2-1)}{R_2 - R_1}[R_1^{-(u+1} - R_2^{-(u+1)}].$$

Thus (6.101) simplifies to

$$\phi(u) = \frac{R_2}{R_2 - R_1} \sum_{y=1}^{u} H(u-y)\,R_1^{-y} + \frac{R_1}{R_1 - R_2} \sum_{y=1}^{u} H(u-y)\,R_2^{-y} + H(u). \quad (6.102)$$

Since $H(u)$ is not zero only at $u = 0$ and $u = 1$, the above formula is equivalent to

$$\begin{aligned}
\phi(0) &= H(0), \qquad \phi(1) = H(1) + \frac{R_1 + R_2}{R_1 R_2} H(0), \\
\phi(u) &= \frac{R_1 R_2}{R_2 - R_1} \left\{ H(0) \left[R_1^{-(u+1)} - R_2^{-(u+1)} \right] + H(1) \left[R_1^{-u} - R_2^{-u} \right] \right\}, \quad u \geq 2.
\end{aligned}$$

Setting $w(x, y) = I(x + y + 1 = z)$, for $z = 2, 3, \ldots$, and $v = 1$, implies that $\phi(u)$ simplifies to the distribution function $h(z \,|\, u)$ of $Z = U(T-1) + |U(T)| + 1$. In particular, $z = 2$, $H(0) = \frac{(1-q)^2 \rho_1 \rho_2 (\rho_1 + \rho_2)}{3\, q^2} = -\frac{1}{R_1 R_2}$ and $H(i) = 0$, for $i = 1, 2, \ldots$. Then

$$h(2 \,|\, u) = \frac{R_1^{-(u+1)} - R_2^{-(u+1)}}{R_1 - R_2}, \qquad u \geq 0.$$

Similarly, $z = 3$, $H(0) = -\frac{1 + \rho_1 + \rho_2}{R_1 R_2}$, $H(1) = -\frac{1}{R_1 R_2}$ and $H(i) = 0$, for $i \geq 2$. Then

$$h(3 \,|\, u) = \frac{1 + \rho_1 + \rho_2}{R_1 - R_2} \left[R_1^{-(u+1)} - R_2^{-(u+1)} \right] + \frac{1}{R_1 - R_2} \left[R_1^{-u} - R_2^{-u} \right], \qquad u \geq 0.$$

Finally, for $z \geq 4$, $h(z \,|\, u) = 0$, for all $u \geq 0$.

If instead, we set $v = 1$ and $w(x, y) = xy$ (alternatively x, or y), then $\phi(u)$ simplifies to $E[U(T-1)|U(T)|I(T < \infty) \,|U(0) = u]$ ($E[U(T-1)I(T < \infty) \,|U(0) = u]$, or $E[|U(T)|I(T < \infty) \,|U(0) = u]$), and

$$\begin{aligned}
&E[U(T-1)|U(T)|I(T < \infty) \,|U(0) = u] \\
&\quad = E[U(T-1)I(T < \infty) \,|U(0) = u] \\
&\quad = \frac{(\rho_1 + \rho_2)[R_1^{-(u+1)} - R_2^{-(u+1)}] + (R_1^{-u} - R_2^{-u})}{R_1 - R_2}, \\
&E[|U(T)|I(T < \infty) \,|U(0) = u] \\
&\quad = \frac{(3 + \rho_1 + \rho_2)[R_1^{-(u+1)} - R_2^{-(u+1)}] + (R_1^{-u} - R_2^{-u})}{R_1 - R_2}.
\end{aligned}$$

Now setting $q = 0.35$, implies that $E(W) = \frac{1+q}{1-q} = 2.077 > E(X) = 2$ and equation

$$V(s) = \frac{(1-q)^2}{3} (s^2 + s^3 + s^4) - (s - q)^2 = 0$$

has four roots, say, $\rho_1 = 1$, $\rho_2 = 0.2449$, $R_1 = 1.0708$ and $R_2 = -3.3158$. The following table gives the moments of $U(T-1)$ and $|U(T)|$, as well as the covariance, for $u = 0, 1, 2, \ldots, 10$.

Table 6.3: Moments and covariance of the surplus before ruin and the deficit at ruin for different u

u	Joint Moment	$E[U(T-1)\|T<\infty]$	$E[\|U(T)\|\|T<\infty]$	Cov
0	0.3836	0.3836	1.3081	-0.1182
1	0.5856	0.5856	1.2072	-0.1213
2	0.5207	0.5207	1.2396	-0.1248
3	0.5417	0.5417	1.2291	-0.1241
4	0.5349	0.5349	1.2325	-0.1244
5	0.5371	0.5371	1.2314	-0.1243
6	0.5364	0.5364	1.23176	-0.12432
7	0.5366	0.5366	1.23165	-0.12430
8	0.53656	0.53656	1.23169	-0.12432
9	0.53657	0.53657	1.23168	-0.124312
10	0.53656	0.53656	1.23168	-0.124310

Joint Moment stands for $E[U(T-1)|U(T)||T<\infty]$.

Conclusion

The aim of this thesis was to show that ruin probabilities, and other ruin related quantities for the Sparre Anderson risk model, can be given a unified treatment through the Gerber-Shiu (G-S) function. The recent actuarial research literature already proves how the G-S function plays such a unifying role for the classical compound Poisson risk model. We hope that the results derived here will clearly show the general applicability of G-S functions.

The thesis studies expected discounted a penalty (G-S) functions and their relation with the time of ruin, the surplus before ruin and the deficit at ruin, for a variety of continuous and discrete–time risk models. In continuous–time, the G-S function is obtained for the compound renewal (Sparre Anderson) risk model with K_n–distributed claim inter–arrival times. In the special case of generalized Erlang(n) waiting times, the model is extended to accept a constant upper divided barrier or is perturbed by a diffusion.

Finally, the applicability of G-S functions is further extended in discrete–time Sparre Anderson risk model. A general approach based on recursive formulas, easily implemented on computers, should bring ruin theory very close to practice with this model.

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Appendix A

Mathematical Tools

A.1 Laplace Transform

The Laplace transform is one of the most efficient methods of solving certain ordinary, partial differential equations, and integro-differential equations. The effectiveness of the Laplace transform is due to its ability to convert the above mentioned equations into algebraic equations (see Poularikas, 1996).

Definition A.1.1. *Let $f(t)$ be a real-valued function that is defined for $t \geq 0$, and denote by $\hat{f}(s)$ its* **Laplace transform** *(L.T.), which is defined as*

$$\hat{f}(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt, \qquad s \in \mathbb{C}, \tag{A.1}$$

if the above integral converges.

Basic properties of Laplace transforms:

1. $\mathcal{L}\{a\, f(t) + b\, g(t)\} = a\, \mathcal{L}\{f(t)\} + b\, \mathcal{L}\{g(t)\} = a\, \hat{f}(s) + b\, \hat{g}(s)$.
2. $\mathcal{L}\{e^{at} f(t)\} = \hat{f}(s-a), \quad a \in \mathbb{C}$.
3. $\mathcal{L}\{f * g(t)\} = \hat{f}(s)\, \hat{g}(s)$, where $*$ denotes the convolution operation.
4. $\mathcal{L}\{f^{(n)}(t)\} = s^n\, \hat{f}(s) - s^{n-1}\, f(+0) - s^{n-2} f'(+0) - \ldots - f^{(n-1)}(+0)$.

5. $\mathcal{L}\{\int_0^t f(x)dx\} = \frac{\hat{f}(s)}{s}$.

6. $\mathcal{L}\{(-1)^n t^n f(t)\} = \hat{f}^{(n)}(s), \quad n \in \mathbb{N}^+$.

7. $\mathcal{L}\{f(t)/t\} = \int_s^\infty \hat{f}(r)\,dr$.

8. $\lim_{s\to\infty} s\,\hat{f}(s) = f(0+)$ (Initial-Value Theorem).

9. $\lim_{s\to 0} s\,\hat{f}(s) = f(+\infty)$ (Final-Value Theorem).

A.2 Generating Function (z-Transform)

The (ordinary) generating function is a powerful tool in insurance modeling (see Panjer and Willmot, 1992). It will be used throughout Chapter 5.

Definition A.2.1. *The generation function of a discrete function $f(x), x \in \mathbb{N}$ (or a sequence of real or complex number), is the function $\hat{f}(z)$ defined by the power series*

$$\hat{f}(z) = \mathcal{G}\{f(x)\} = \sum_{x=0}^{\infty} z^x f(x), \qquad z \in \mathbb{C}, \tag{A.2}$$

if the series converges. Further, if $\{f(x);\, x \in \mathbb{N}\}$ is a probability distribution, then $\hat{f}(z)$ is called probability generation function (p.g.f.).

Remarks:

- $\hat{f}(1/z) = \sum_{x=0}^{\infty} z^{-x} f(x)$ is called the *z-transform* of f.
- There is a one-to-one correspondence between a sequence $\{f(x);\, x \in \mathbb{N}\}$ and its generating function $\hat{f}(z)$. If $\hat{f}(z)$ is given, $f(x)$ can be obtained from by

$$f(x) = \frac{\hat{f}^{(x)}(0)}{x!}, \quad x \in \mathbb{N},$$

  or by a path integral on the complex plane,

$$f(x) = \frac{1}{2\pi i}\oint \frac{\hat{f}(z)}{z^{x+1}} dz, \quad x \in \mathbb{N},$$

where the path encircling the origin with poles of $\hat{f}(z)$ being outside of it.

The following properties are very important in deriving and inverting generating functions.

Basic Properties:

1. $\mathcal{G}\{a\, f(x) + b\, g(x)\} = a\, \mathcal{G}\{f(x)\} + b\, \mathcal{G}\{g(x)\} = a\, \hat{f}(z) + b\, \hat{g}(z)$.
2. $\mathcal{G}\{f(x+n)\} = z^n\, \hat{f}(z)$.
3. $\mathcal{G}\{x\, f(x)\} = z\, \hat{f}'(z)$.
4. $\mathcal{G}\{f * g(x)\} = \hat{f}(z)\, \hat{g}(z)$, where $f * g(x) = \sum_{y=0}^{x} f(x-y)\, g(y)$ is the convolution of f with g.
5. If $\{f(x);\ x \in \mathbb{N}\}$ is the probability distribution of a r.v. X, then
   - $\hat{f}(1) = 1$.
   - $\mathcal{G}\{\bar{F}(x)\} = \frac{1-\hat{f}(z)}{1-z}$, where $\bar{F}(x) = \sum_{y=x+1}^{\infty} f(y)$ is the survival function of X.
   - $E[X^{(k)}] = \frac{d^k \hat{f}(z)}{d z^k}\Big|_{z=1}$, where $x^{(k)} = x(x-1)(x-2)\cdots(x-k+1)$ is the k-th factorial power of x.

In insurance modeling, one often has to deal with positive-valued random variables, e.g., claim waiting times, claim amounts and claim causing ruin. In this situation, let $f(x), x \in \mathbb{N}^+$ be the probability distribution of a positive-valued discrete r.v., then its probability generating function is defined by

$$\hat{f}(z) = \sum_{x=1}^{\infty} z^x\, f(x), \qquad z \in \mathbb{C}, \tag{A.3}$$

with $\hat{f}(0) = 0$, $\hat{f}(1) = 1$.

Now let function g with domain $\mathbb{N}$ be such that $g(x-1) = f(x)$, $x \in \mathbb{N}^+$, then the relation between their generating functions is $\hat{f}(z) = z\, \hat{g}(z)$; on the other hand, if the generating functions of two functions f and g satisfies $\hat{f}(z) = z\,\hat{g}(z)$, then $f(x) = g(x-1)$. Then the generating function of a function with domain $\mathbb{N}^+$, can be analyzed through that of a function with domain $\mathbb{N}$, shifted to the right by one unit.

Appendix B

Renewal Theory

B.1 Renewal Equation

Definition B.1.1. *A renewal equation is an integral equation of the following form*

$$Z(x) = z(x) + q \int_0^x Z(x-y)dF(y), \qquad x \geq 0, \tag{B.1}$$

or, equivalently,

$$Z(x) = z(x) + q\, F * Z(x), \tag{B.2}$$

where $0 < q < \infty$ *is a constant,* F *is a probability distribution on* $[0, \infty)$ *with* $F(0) < 1$*,* z *and* Z *are defined on* $[0, \infty)$ *and are locally bounded.*

The renewal equation in (B.1) is called proper if $q = 1$*, defective if* $0 < q < 1$*, and excessive if* $q > 1$.

Renewal equations play an important role in probability models for insurance risks. Many ruin-related quantities in classical risk model, Sparre Anderson risk models and perturbed risk models satisfy a renewal equation. Specially ruin probabilities in the classical and Sparre Anderson risk models satisfy a certain defective renewal equation, and hence all admit a compound geometric tail representation.

However, renewal equations are rarely tractable, neither analytically nor numerically. Closed form solutions to renewal equations are only available in some special cases, e.g., if both z and F have a rational Laplace transform. Then the Laplace transform of Z can be given a rational expression, which can be inverted by partial fractions. Hence, in the literature, the main probability results for general renewal equations are either inequalities (bounds) or asymptotic formulas.

The rest of this section reviews some results for renewal equations, in particular, the elementary and key renewal theorems, which standard methods for deriving asymptotic formulas in different applied probability models including insurance risk models.

Theorem B.1.1. *Assume that* $0 \leq q\,F(0) < 1$ *and that*

$$U_q(x) = \sum_{n=0}^{\infty} q^n\, F^{*n}(x) < \infty, \qquad x \geq 0,$$

then the renewal equation (B.1) has a unique solution

$$Z(x) = U_q * z(x), \qquad x \geq 0. \tag{B.3}$$

In particular, if $q = 1$, *the proper renewal equation has a unique solution*

$$Z(x) = U * z(x), \qquad x \geq 0,$$

where $U(x) = \sum_{n=0}^{\infty} F^{*n}(x) = 1 + M(x)$, *and* $M(x) = \sum_{n=1}^{\infty} F^{*n}(x)$ *is called the renewal function.*

If $0 < q < 1$, *the defective renewal equation in (B.1) has a unique solution*

$$Z(x) = \frac{1}{1-q} G_q * z(x), \qquad x \geq 0,$$

where $G_q(x) = \sum_{n=0}^{\infty}(1-q)q^n F^{*n}(x)$ *is a compound geometric distribution function.*

Proof: See Resnick (1992). □

Theorem B.1.2. *For the proper renewal equation in (B.1), i.e., $q = 1$, suppose that $\mu = \int_0^\infty x dF(x) \leq \infty$, then*

1. (The Elementary Renewal Theorem) *If $z_0 = \lim_{x\to\infty} z(x)$,*

$$\lim_{x\to\infty} \frac{Z(x)}{x} = \frac{z_0}{\mu}.$$

2. (The Key Renewal Theorem) *If z is directly Riemann integrable and F is not lattice,*

$$\lim_{x\to\infty} Z(x) = \frac{1}{\mu}\int_0^\infty z(x)dx.$$

Proof: See Feller (1971). □

The key renewal theorem gives the limit of the solution to a proper renewal equation, which can then be used to derive asymptotic formulas for defective and excessive renewal equations.

Definition B.1.2. (Adjustment Coefficient) *Assume that F is a c.d.f. on $[0, \infty)$, with $F(0) < 1$. A constant R is called the adjustment coefficient associated with q and F if it satisfies the following equation*

$$E[e^{RX}] = \int_0^\infty e^{Ry} dF(y) = \frac{1}{q}, \tag{B.4}$$

where X is a r.v. with distribution function F.

The concept of *adjustment coefficient* is often used in risk theory and plays a critical role in studying the behavior of renewal equations.

We note that if the adjustment coefficient R exists, it is unique, since $E[e^{sX}]$ is strictly monotonic in s. If $q > 1$, R always exists, and $R < 0$ if $q > 1$, while $R = 0$ if $q = 1$. If $0 < q < 1$, R may not exists, but R is positive when it does exist, for sufficiently regular F.

Theorem B.1.3. *If F is not lattice, and assume that there exists an adjustment coefficient R, and $e^{Rx}z(x)$ is directly Riemann integrable, then the solution, Z, to the renewal equation in (B.1) has the following asymptotic formula*

$$Z(x) \sim \frac{\int_0^\infty z(y)e^{Ry}dy}{q\int_0^\infty y\, e^{Ry}dF(y)} e^{-Rx}, \qquad x \to \infty. \tag{B.5}$$

Theorem B.1.3 gives a standard method for deriving exponential asymptotic formulas for defective and excessive renewal equations, especially that defective renewal equations often arise in risk theory and queuing theory.

Besides asymptotic results for the solutions to renewal equations, another research interest is to study the two sided bounds. The following theorem gives exponential bounds for defective and excessive renewal equations.

Theorem B.1.4. *Suppose that $qF(0) < 1$ and that the adjustment coefficient R exists, then the solution, Z, to the renewal equation in (B.1) admits the following two-sided bounds*

$$l(x)\, e^{-Rx} \leq Z(x) \leq u(x)\, e^{-Rx}, \qquad x \geq 0, \tag{B.6}$$

where

$$[l(x)]^{-1} = \sup_{0\leq u\leq x} \frac{q\int_u^\infty e^{Ry}dF(y)}{z(u)\, e^{Ru}} \qquad \text{and} \qquad [u(x)]^{-1} = \inf_{0\leq u\leq x} \frac{q\int_u^\infty e^{Ry}dF(y)}{z(u)\, e^{Ru}}.$$

The adjustment coefficient R may not exist for defective renewal equations, e.g., it does not exist when F is a heavy or medium tailed distribution. There are two ways to generalize Lundberg's equation (B.4) to heavy and medium tailed distributions.

One is to replace the exponential function by a general life distribution B with $B(0) = 0$, such that $\int_0^\infty [\bar{B}(x)]^{-1}dF(x) = \frac{1}{q}$. Then upper or lower bounds can be obtained by assuming B to be in some reliability class of distributions.

See Cai and Wu (1997), Lin (1996), Willmot (1994, 1996, 1997a, 1997b) and Willmot and Lin (1997).

The other method is a truncated Lundberg's equation, which can be satisfied by general distributions. There are two versions of truncated Lundberg equations, the first version is obtained by replacing the exponential function by a truncated exponential function $\min(e^{\kappa_t y}, e^{\kappa_t t})$ that satisfies the following equation:

$$\int_0^t e^{\kappa_t y} dF(y) + e^{\kappa_t t} \bar{F}(t) = \frac{1}{q},$$

for a given $t > 0$ and κ_t. The second version is obtained by replacing the exponential function by a truncated exponential function $e^{R_t y} I(0 \leq y \leq t)$ that satisfies

$$\int_0^t e^{R_t y} dF(y) = \frac{1}{q},$$

for a given t and R_t. The bounds based on the roots to these two truncated versions of Lundberg equations can be found in Cai (1998) and Dickson (1994b).