The Distribution of the Discounted Compound PH–Renewal Process

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A Thesis

in the

Department

of

Mathematics and Statistics

Presented in Partial Fulfillment of the Requirements

For the Degree of Doctor of Philosophy at

Concordia University

Montreal, Quebec, Canada

October, 2010

CONCORDIA UNIVERSITY

School of Graduate Studies

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Entitled: **The Distribution of the Discounted Compound PH-Renewal Process**

and submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy (Mathematics)

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ABSTRACT

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by Ya Fang Wang

The family of phase–type (PH) distributions has many useful properties such as closure under convolution and mixtures, as well as rational Laplace transforms. PH distributions are widely used in applications of stochastic models such as in queueing systems, biostatistics and engineering. They are also applied to insurance risk, such as in ruin theory.

In this thesis, we extend the work of Wang (2007), that discussed the moment generating function (mgf) of discounted compound sums with PH inter–arrival times under a net interest $\delta \neq 0$. Here we focus on the distribution of the discounted compound sums. This represents a generalization of the classical risk model for which $\delta = 0$.

A differential equation system is derived for the mgf of a discounted compound sum with PH inter–arrival times and any claim severity if its mgf exists. For some PH inter–arrival times, we can further simplify this differential equation system. If inter–arrival times have a PH distribution of order 2, then second–order homogeneous differential equations are developed. By inverting the corresponding Laplace transforms, the extended density functions and cumulative distribution functions are also obtained. In addition, the series and transformation methods for solving differential equations is proposed, when the mean of inter–arrival times is small.

Applications such as stop–loss premiums, and risk measures such as VaR and CTE are investigated. These are compared for different inter–arrival times. Some numerical examples are given to illustrate the results.

Finally, asymptotical results are discussed, when the mean inter–arrival time goes to zero. For a fixed time, the asymptotic normal distribution is derived for discounted compound renewal sums.

Acknowledgments

I am heartily thankful to my supervisor José Garrido for introducing me to such an interesting research topic. His encouragement, guidance and support enable me to develop the understanding of the actuarial subjects coming from a pure mathematical background. His enthusiasm and inspiration to teaching and research work will be a benefit to my future career. I would like to thank my co-supervisor Ghislain Léveillé from Université Laval. He often came to meet with us, driving a long way from Quebec City. I am grateful to him for the discussions we had and thank him for his ideas and advice. I enjoyed the time spent working with him.

I wish to thank Dr. Zhou and Dr. Gaillardetz for being my teachers and thesis committee members, for both my Master and Ph.D theses. I appreciate their thoughtful comments on my theses and for always being able to get help from them. I would like also to thank my other teachers in our department. I have learned a lot from them. I am grateful to our graduate assistant Ann–Marie Agnew for her support during my Master and Ph.D. studies.

I also thank my student colleagues; they made my student life happy and diverse. Particularly, I thank Ramin and Jun for discussions and the sharing of ideas.

Finally I wish to thank my husband WenZhou and my daughter Yuting for their understanding. I could not have finished this thesis without their support. I owe too much time to them.

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Introduction

As a generalization of the exponential and Erlang(n) distributions, phase–type (PH) distributions were first introduced by Neuts (1975). They have been used widely in applications of stochastic models such as in queueing systems, engineering, biostatistics and reliability. PH distributions also have been applied to risk theory, in particular in ruin theory. The random variable defined as the absorption time in a continuous-time Markov chain, with n transient states $i = 1, 2, \ldots, n$ and one absorbing state 0, has a PH distribution. PH distribution and density functions are expressed in terms of a vector $\underline{\alpha}$ and a nonsingular matrix $\mathbf{A}$.

Neuts (1981) gives a detailed introduction of stochastic models in queueing theory with PH distributions. Latouche and Ramaswami (1999) discuss the application of PH in stochastic models and they also introduce the algorithms for the models. Asmussen (2003) presents more details about the properties of PH distributions and their applications to queuing theory.

In the last decade PH distributions have been applied also to insurance risk. Asmussen (2000) studies the ruin probability in the compound Poisson model with PH claim severities. Earlier Asmussen and Rolski (1991) had already introduced some numerical examples with PH claim severities to compute ruin probabilities. Frostig (2004) obtains the distribution of the time to ruin and upper bounds with PH claim size distributions. Asmussen, Avram and Pistorius (2004) show the potential of PH distributions in mathematical finance. Dickson and Hipp (2000) study a risk process in which claim inter–arrival times have a PH distribution of order 2. Li and Garrido (2005) give the expected discounted penalty (Gerber-Shiu) function for renewal risk model with rational Laplace transform of the inter–arrival times. Ren (2005) discusses the probability of ruin, the Laplace transform of the time of ruin, the expected value of the time of ruin, and the discounted moments of the deficit at ruin for the classical risk model, when claim sizes have a PH distribution.

PH distributions form an interesting family. First they can be represented using matrices and vectors, which simplifies the computations with mathematical software such as Maple, Mathematica or Matlab. Second they have many nice properties such as closure under convolutions and mixtures and rational Laplace transforms. Furthermore PH distributions generalize exponential, Erlang(n) and Cox distributions, which are already well known and widely used. Moreover they are dense in the class of all distributions defined on the non-negative real numbers, hence PH distributions enable algorithmically tractable solutions for stochastic models.

Many papers propose approximation methods using PH distributions. For instance Sangüesa (2008) introduces an approximation method for nonnegative random variables using PH distributions. Dufresne (2007) also discusses the approximation of nonnegative random variables by mixed exponentials. Fackrell (2003) presents estimation methods for PH distributions. Bladt, Gonzalez and Lauritzen (2003) consider the estimation of functionals depending on one or several PH distributions using Markov chain Monte Carlo methods. Asmussen, Avram and Usabel (2002) present a fast and simple algorithm for computing finite-horizon ruin probabilities using an Erlang (phase–type) approximation and an extrapolation scheme.

Since Andersen (1957) introduced the compound renewal sums, the model has been generalized to discounted compound sums, that consider the effect of interest and inflation. Taylor (1979), Delbaen and Haezendonck (1987) and Willmot (1989) obtain the moments of the discounted compound Poisson process. Then Léveillé and Garrido (2001a) replace the Poisson process by a renewal process and derive the first two moments of the discounted compound renewal sum. In Léveillé and Garrido (2001b) they also obtain recursive formulas for all the moments of the discounted compound renewal process. Kim and Kim (2007) discuss the moments of discounted aggregated claims in a Markovian environment, while van der Weider, Suyono and van Noorwijk (2008) get the first two moments of discounted aggregate claims for different discount functions. Finally Ren (2008) derives explicit formula for the first two moments of discounted compound renewal sums.

Considering the moment generating function (mgf) of the discounted compound renewal sums, Jang (2004) studies the Laplace transform of the discounted compound Poisson process if the claim severities are exponential or mixtures of exponentials. In 2007, Jang also derive the mgf using jump diffusion processes. Léveillé, Garrido and Wang (2010) derive analytical expressions for the mgf of the discounted compound renewal sums, in particular, they obtained a closed form for the mgf of the discounted compound Poisson sums with PH claim severities and second–order homogeneous differential equations for the mgf if the inter-arrival time is Erlang(2) distributed. Wang (2007) develops homogeneous differential equations for the mgf of discounted compound sums if the inter–arrival times have Erlang(n) distributions.

The thesis extends the work in Wang (2007) for the mgf of the discounted compound sum. The mgf is a classical technique to find the expectation and variance of a random variable, as well as its probability density function by inversion. Hence the natural question is what is the distribution of the discounted compound sum? In this thesis we generalize the above results to the PH–renewal process. First, its mgf is derived. We also discuss the technique for obtaining the extended density of the discounted compound PH renewal process and its applications. Furthermore, the asymptotic normal distribution of discounted compound renewal sums is investigated, when the mean of inter–arrival times is small.

The thesis is organized as follows. Chapter 1 gives the formal definition of PH distributions and introduces some of its basic properties and examples. Chapter 2 introduces the model and gives a differential equation for the mgf of the discounted compound renewal sums under PH inter–arrival times and general claim severities, as long as the moment generating function exists. Some corollaries and examples are also given. Chapter 4 derives the extended density function for the discounted compound sums numerically, especially the series method is used to solve differential equations. Applications of the results, such as stop–loss premiums, Value–at–Risk (VaR), Condition Tail Expectation (CTE) are studied in Chapter 4. The asymptotic normality of compound renewal sums and discounted compound renewal processes is discussed in Chapter 5.

Chapter 1

Phase–Type Distributions

Since they were introduced by Neuts (1975) in queueing theory and reliability, PH distributions have been applied also to insurance risks such as ruin probabilities. Here we mainly focus on the distribution of discounted compound PH–renewal. Hence we reproduce the definition of PH distributions, some of their basic properties and examples, that will be used later. For additional details please refer to Neuts (1981), Fackrell (2003) or Wang (2007).

1.1 The Definition of PH Distributions

A phase–type distribution is defined as a probability distribution of the time to absorption in a continuous-time Markov chain with n transient states $i = 1, 2, \ldots, n$ and one absorbing state 0. Here we consider the mathematical definition of PH distributions. For details on the probabilistic interpretation and the properties of PH distributions in Markov chains, please refer to Neuts (1981) and Asmussen (2003).

Definition 1.1.1. *Continuous phase–type distribution*
Let $\boldsymbol{A}$ *be an arbitrary non-singular square matrix of order* n *such as* $\lim_{x\to\infty} e^{\boldsymbol{A}x} = \boldsymbol{0}$ *and* $\underline{\alpha}' e^{\boldsymbol{A}x}\underline{1}$ *is a decreasing function in* x*.* $\underline{\alpha}$ *be a* n*-dimensional column vector such that* $\underline{\alpha}'\,\underline{1} = 1$*, where* $\underline{1}$ *is a* n*-dimensional column vector of* $1's$*, that is:*

$$\underline{\alpha} = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix}', \quad \sum_{i=1}^{n} \alpha_i = 1\,, \quad \alpha_i \geq 0 \quad \textit{and} \quad \underline{1} = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}'.$$

If the distribution function F_X of a random variable X can be written as:

$$F_X(x) = 1 - \underline{\alpha}' e^{\boldsymbol{A}x} \underline{1}\,, \qquad x \geq 0, \tag{1.1}$$

then we say that F_X is (or X has) a phase–type (PH) distribution with parameters $(\underline{\alpha}\,, \boldsymbol{A})$.

Remark 1.1. Note that in the original definitions of PH distributions, like that of Neuts (1975) $A = (a_{ij})$ was the rate matrix of stationary Markov chain. Consequently it was assumed that $a_{ii} < 0$ and $a_{ij} > 0$ for $i \neq j$. From Neuts (1981) we can see these conditions imply those of Definition 2.1.1.

Hence taking the derivative of F_X (see Lemma A.2.4.), we obtain the density function of X:

$$f_X(x) = -\underline{\alpha}'\, e^{\mathbf{A}x}\, \mathbf{A}\, \underline{1}\,, \qquad x \geq 0. \tag{1.2}$$

The following are some examples of PH distributions (see Neuts, 1981, Fackrell, 2003 and Wang, 2007).

Example 1.1.1. If X has an exponential distribution with density function $f_X(x) = \lambda\, e^{-\lambda x}$, for $\lambda > 0$, then it is PH with

$$\underline{\alpha}' = (1), \qquad \mathbf{A} = (-\lambda)\,.$$

Example 1.1.2. If X has an hyper–exponential distribution (also called mixed exponential) with density function

$$f_X(x) = \sum_{i=1}^{n} \alpha_i\, \lambda_i\, e^{-\lambda_i x}\,, \qquad x > 0\,, \quad \lambda_i > 0\,,$$

where $\alpha_i > 0$ and $\sum_{i=1}^{n} \alpha_i = 1$, then it is also PH with $\underline{\alpha}$ and $\mathbf{A}$ given by:

$$\underline{\alpha} = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix}'\,,$$

and

$$\mathbf{A} = \begin{pmatrix} -\lambda_1 & 0 & \cdots & 0 \\ 0 & -\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\lambda_n \end{pmatrix}.$$

Example 1.1.3. If X has an Erlang(n) distribution with density

$$f_X(x) = \frac{\lambda^n \, x^{n-1} \, e^{-\lambda x}}{(n-1)!} \,, \qquad x > 0 \,, \quad n \in \mathbb{N}^+ \,, \quad \lambda > 0,$$

then it is also PH with

$$\underline{\alpha} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}' \,,$$

a n-dimensional vector and the following matrix of order n

$$\mathbf{A} = \begin{pmatrix} -\lambda & \lambda & 0 & \cdots & 0 & 0 \\ 0 & -\lambda & \lambda & \cdots & 0 & 0 \\ 0 & 0 & -\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda & \lambda \\ 0 & 0 & 0 & \cdots & 0 & -\lambda \end{pmatrix} . \tag{1.3}$$

Example 1.1.4. If we have n different values $\lambda_i > 0$ in the previous example, then it defines a generalized Erlang(n) distribution of order n with $\underline{\alpha}$ and $\mathbf{A}$ as follows:

$$\underline{\alpha} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}' \,,$$

and

$$\mathbf{A} = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda_2 & \lambda_2 & \cdots & 0 & 0 \\ 0 & 0 & -\lambda_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda_{n-1} & \lambda_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & -\lambda_n \end{pmatrix} \,,$$

and the density function can be expressed as a mixture of exponentials $f_X(x) = \sum_{i=1}^{n} a_i e^{-\lambda_i x}$ for given polynomial coefficients a_i in terms of λ_i, where $0 < \lambda_1 \leq \lambda_2 \cdots \leq \lambda_n$.

Example 1.1.5. If X has a n-phase *Coxian* distribution with the following parameters, then it is also PH distribution:

$$\underline{\alpha} = \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \end{pmatrix}', \qquad \sum_{k=1}^{n} \alpha_k = 1\,, \qquad \alpha_i \geq 0\,,$$

and

$$\mathbf{A} = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda_2 & \lambda_2 & \cdots & 0 & 0 \\ 0 & 0 & -\lambda_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda_{n-1} & \lambda_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & -\lambda_n \end{pmatrix},$$

where $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

Example 1.1.6. If X has the following parameters $(\underline{\alpha}\,, \mathbf{A})$, it is called *unicycle* PH distribution:

$$\underline{\alpha} = \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \end{pmatrix}', \qquad \sum_{k=1}^{n} \alpha_k = 1\,, \qquad \alpha_i \geq 0\,,$$

and

$$\mathbf{A} = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda_2 & \lambda_2 & \cdots & 0 & 0 \\ 0 & 0 & -\lambda_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda_{n-1} & \lambda_{n-1} \\ \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_{n-1} & -\lambda_n \end{pmatrix},$$

where $\mu_i \geq 0$, for $i = 1, 2, \cdots, n-1$ and $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

Example 1.1.7. If $\mathbf{A}$ is an upper triangular matrix, we call X *acylic* or triangle PH (TPH). Note that in general, the parameters $(\underline{\alpha}\,, \mathbf{A})$ for PH distributions are not unique. Consider exponential PH distribution with density

$$f_X(x) = 0.01e^{-0.01x}\,, \qquad x > 0\,,$$

which can be parameterized either with ($\underline{\alpha}$, $\mathbf{A}$) or ($\underline{\beta}$, $\mathbf{B}$) given by:

$$\alpha = \begin{pmatrix} 1 \end{pmatrix}', \qquad \mathbf{A} = \begin{pmatrix} -0.01 \end{pmatrix},$$

$$\beta = \begin{pmatrix} 0.5 & 0.5 \end{pmatrix}', \qquad \mathbf{B} = \begin{pmatrix} -0.01 & 0.01 \\ 0 & -0.02 \end{pmatrix}.$$

The following subsections present the expectation and mgf of PH distributions.

1.1.1 Expectation

From (1.1) the expectation of X is given by:

$$\mathbb{E}(X) = \int_0^\infty [1 - F_X(x)]\, dx = \underline{\alpha}' \int_0^\infty e^{\mathbf{A}x}\, dx\, \underline{1}\,. \tag{1.4}$$

From the definition of the matrix exponential function (see Appendix A, Definition A.1.4), we have

$$\mathbf{A} \int_0^x e^{\mathbf{A}u}\, du = e^{\mathbf{A}x} - \mathbf{I} = \int_0^x e^{\mathbf{A}u}\, du\ \mathbf{A}\,, \tag{1.5}$$

where $\mathbf{I}$ is identity matrix of order n. Given that $\mathbf{A}^{-1}$ exists, assuming that $\lim_{x\to\infty} e^{\mathbf{A}x} = 0$, we have the following:

$$\int_0^x e^{\mathbf{A}u}\, du = \mathbf{A}^{-1}\mathbf{A} \int_0^x e^{\mathbf{A}u}\, du = \mathbf{A}^{-1}(e^{\mathbf{A}x} - \mathbf{I})\,, \tag{1.6}$$

and hence

$$\int_0^\infty e^{\mathbf{A}u}\, du = \lim_{x\to\infty} \int_0^x e^{\mathbf{A}u}\, du = \lim_{x\to\infty} \mathbf{A}^{-1}(e^{\mathbf{A}x} - \mathbf{I}) = -\mathbf{A}^{-1}\,. \tag{1.7}$$

Substituting (1.7) into (1.4) gives:

$$\mathbb{E}(X) = -\underline{\alpha}'\mathbf{A}^{-1}\underline{1}\,. \tag{1.8}$$

1.1.2 Moment Generating Function

From (1.7) one can also obtain the mgf of X :

$$M_X(t) = \int_0^\infty e^{tx} f_X(x)\, dx = -\underline{\alpha}' \int_0^\infty e^{(t\mathbf{I}+\mathbf{A})x}\, dx \mathbf{A}\, \underline{1} = \underline{\alpha}'\, (t\mathbf{I} + \mathbf{A})^{-1}\mathbf{A}\, \underline{1}\,, \qquad t \in \mathbb{R}\,. \tag{1.9}$$

The same procedure also gives the Laplace transform $\widehat{f}_X$ of X :

$$\widehat{f}_X(s) = -\underline{\alpha}'\,(\mathbf{sI}-\mathbf{A})^{-1}\mathbf{A}\,\underline{1}\,, \qquad s\in\mathbb{R}\,. \tag{1.10}$$

Consider the n-th moment of the PH distributions, we have that:

$$\begin{aligned}\mathbb{E}(X^n) &= \frac{d^n}{dt^n}M_X(t)|_{t=0} = \frac{d^n}{dt^n}\underline{\alpha}'\,(t\mathbf{I}+\mathbf{A})^{-1}\mathbf{A}\,\underline{1}|_{t=0} = \frac{d^n}{dt^n}\underline{\alpha}'\,(t\mathbf{A}\mathbf{A}^{-1}+\mathbf{A})^{-1}\mathbf{A}\,\underline{1}|_{t=0}\\ &= \frac{d^n}{dt^n}\underline{\alpha}'(t\mathbf{A}^{-1}+\mathbf{I})^{-1}\mathbf{A}^{-1}\mathbf{A}\,\underline{1}|_{t=0} = \frac{d^n}{dt^n}\underline{\alpha}'\sum_{k=0}^{\infty}(-1)^k(t\mathbf{A}^{-1})^k\underline{1}|_{t=0}\end{aligned}$$

by Definition A.1.4 of the matrix exponential, hence

$$\mathbb{E}(X^n) = \frac{d^n}{dt^n}\underline{\alpha}'\sum_{k=0}^{\infty}(-1)^k\mathbf{A}^{-k}t^k\underline{1}|_{t=0} = (-1)^n\,n!\,\underline{\alpha}'\mathbf{A}^{-n}\,\underline{1}\,, \qquad n\in\mathbb{N}^+\,.$$

For additional properties of PH distributions and detailed derivations see Neuts (1981) and Asmussen (2003).

1.2 Closure Properties

Apart from having analytical expressions for its moments and mgf, the family of PH distributions is closed under convolution and mixtures.

Property 1.2.1. If the distributions F_X of X and F_Y of Y are both continuous PH distributions with parameters $(\underline{\alpha},\mathbf{A})$ of order n and $(\underline{\beta},\mathbf{B})$ of order m respectively, then their convolution $F_X * F_Y$ is also a PH distribution with parameter $(\underline{\gamma},\mathbf{C})$. Here $\underline{\gamma}$ and $\mathbf{C}$ are given by

$$\underline{\gamma} = (\underline{\alpha}',\underline{0}'_m)' \quad \text{and} \quad \mathbf{C} = \begin{pmatrix}\mathbf{A} & -\mathbf{A}\,\underline{1}_n\,\underline{\beta}'\\ \mathbf{0} & \mathbf{B}\end{pmatrix}, \tag{1.11}$$

where $\underline{1}_k = \begin{pmatrix}1 & 1 & \cdots & 1\end{pmatrix}'$ is a vector of order $k\times 1$.

Proof. The Laplace transform of a PH $(\underline{\gamma}\,,\mathbf{C})$ random variable Z can be obtained from (1.10) as

$$
\begin{aligned}
\hat{f}_Z(s) &= -\underline{\gamma}'(s\mathbf{I}_{n+m} - \mathbf{C})^{-1}\mathbf{C}\,\underline{1} \\
&= -(\underline{\alpha}', 0'_m)\begin{pmatrix} s\mathbf{I}_n - \mathbf{A} & \mathbf{A}\underline{1}_n\underline{\beta}' \\ 0 & s\mathbf{I}_m - \mathbf{B} \end{pmatrix}^{-1}\begin{pmatrix} \mathbf{A} & -\mathbf{A}\,\underline{1}_n\underline{\beta}' \\ 0 & \mathbf{B} \end{pmatrix}\begin{pmatrix} \underline{1}_n \\ \underline{1}_m \end{pmatrix} \\
&= -(\underline{\alpha}', 0'_m)\begin{pmatrix} (s\mathbf{I}_n - \mathbf{A})^{-1} & -(s\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{A}\;\underline{\mathbf{1}}_n\underline{\beta}'(s\mathbf{I}_m - \mathbf{B})^{-1} \\ \mathbf{0} & (s\mathbf{I}_m - \mathbf{B})^{-1} \end{pmatrix} \\
&\qquad \times \begin{pmatrix} \mathbf{A} & -\mathbf{A}\underline{1}_n\underline{\beta}' \\ 0 & \mathbf{B} \end{pmatrix}\begin{pmatrix} \underline{1}_n \\ \underline{1}_m \end{pmatrix} \\
&= -\underline{\alpha}'(s\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{A}\underline{1}_n + \underline{\alpha}'(s\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{A}\underline{1}_n\underline{\beta}'\underline{1}_m \\
&\qquad +\underline{\alpha}'(s\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{A}\underline{1}_n\underline{\beta}'(s\mathbf{I}_m - \mathbf{B})^{-1}\mathbf{B}\underline{1}_m\,.
\end{aligned}
$$

Since $\underline{\beta}'\,\underline{1}_m = 1,$ then

$$
\hat{f}_Z(s) = (-\underline{\alpha}'(s\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{A}\underline{1}_n)(-\underline{\beta}'(s\mathbf{I}_m - \mathbf{B})^{-1}\mathbf{B}\underline{1}_m) = \hat{f}_X(s)\hat{f}_Y(s)\,, \qquad s \in \mathbb{C}\,, \tag{1.12}
$$

where $\hat{f}_X$ and $\hat{f}_Y$ are the Laplace transforms of a PH($\underline{\alpha}\,,\mathbf{A}$) random variable X of order n and a PH($\underline{\beta}\,,\mathbf{B}$) random variable Y of order m, respectively.

□

Remark 1.2. The convolution $F_X * F_Y$ does not have a unique PH representation. For instance, it can also be written as a PH($\underline{\gamma}\,,\mathbf{C}$) distribution with

$$
\underline{\gamma} = (\underline{\beta}',\, \underline{0}'_n)', \qquad \mathbf{C} = \begin{pmatrix} \mathbf{B} & -\mathbf{B}\underline{1}_m\underline{\alpha}' \\ \mathbf{0} & \mathbf{A} \end{pmatrix}. \tag{1.13}
$$

Property 1.2.2. The mixture $\theta F_X + (1-\theta)F_Y$, where $0 \le \theta \le 1$, is also a PH distribution with parameters $(\underline{\gamma}, C)$, where $\underline{\gamma}$ and $\mathbf{C}$ are given by:

$$
\underline{\gamma} = \big(\theta\underline{\alpha}', \quad (1-\theta)\underline{\beta}'\big)', \qquad \mathbf{C}= \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}. \tag{1.14}
$$

Proof. Let $\hat{f}_Z$ be the Laplace transform of $\theta F_X + (1-\theta)F_Y$, then:

$$\begin{aligned}
\hat{f}_Z(s) &= \theta \hat{f}_X(s) + (1-\theta)\hat{f}_Y(s) \\
&= -\theta \underline{\alpha}'(s\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{A}\underline{1}_n - (1-\theta)\underline{\beta}'(s\mathbf{I}_m - \mathbf{B})^{-1}\mathbf{B}\underline{1}_m \\
&= -(\theta\underline{\alpha}',\, (1-\theta)\underline{\beta}')' \begin{pmatrix} s\mathbf{I}_n - \mathbf{A} & \mathbf{0} \\ \mathbf{0} & s\mathbf{I}_m - \mathbf{B} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{B} \end{pmatrix} \begin{pmatrix} \underline{1}_n \\ \underline{1}_m \end{pmatrix} \\
&= -\underline{\gamma}'(s\mathbf{I}_{n+m} - \mathbf{C})^{-1}\mathbf{C}\,\underline{1}\,,
\end{aligned}$$

where $\underline{\gamma}$ and $\mathbf{C}$ are given in (1.14). □

Property 1.2.3. If the random variable X is a PH $(\underline{\alpha}\,, \mathbf{A})$, then θX also is a PH $(\underline{\gamma}, \mathbf{C})$, where

$$\underline{\gamma} = \underline{\alpha} \quad \text{and} \quad \mathbf{C} = \tfrac{1}{\theta}\mathbf{A}\,, \qquad \text{for } \theta > 0\,.$$

Proof. Let $Y = \theta X$, then

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(\theta X \le y) = \mathbb{P}(X \le \frac{1}{\theta} y) = 1 - \underline{\alpha}' e^{\frac{1}{\theta}\mathbf{A}y}\underline{1}\,, \qquad y > 0\,.$$

□

From Properties 1.2.1 and 1.2.3 we obtain the following results.

Corollary 1.2.1. *If independent random variables X and Y are both PH distributions with parameters $(\underline{\alpha}, \boldsymbol{A})$ and $(\underline{\beta}, \boldsymbol{B})$ respectively, then $aX + bY$ also is PH $(\underline{\gamma}, \boldsymbol{C})$, where*

$$\underline{\gamma} = (\underline{\alpha}' \quad 0)'\,, \quad \boldsymbol{C} = \begin{pmatrix} \frac{1}{a}\boldsymbol{A} & -\frac{1}{a}\boldsymbol{A}\underline{1}_n\underline{\beta}' \\ 0 & \frac{1}{b}\boldsymbol{B} \end{pmatrix}\,, \quad a > 0\,, \quad b > 0\,.$$

Note that the parameters $(\underline{\gamma}, \mathbf{C})$ are not unique. Another representation could be:

$$\underline{\gamma} = (\underline{\beta}' \quad 0)'\,, \quad \mathbf{C} = \begin{pmatrix} \frac{1}{b}\mathbf{B} & -\frac{1}{b}\mathbf{B}\underline{1}_m\underline{\alpha}' \\ 0 & \frac{1}{a}\mathbf{A} \end{pmatrix}\,, \quad a > 0\,, \quad b > 0\,.$$

From the definition of the Laplace transform of a PH distribution, we see that it is a rational polynomial in s. If the maximal degree of the denominator is p then the degree of the numerator is $q < p$, hence the limit of the Laplace transform goes to zero as s tends to ∞. The question then is whether a rational polynomial corresponds to a PH distribution. The answer is given in O'Cinneide (1990) and is reproduced here with the following result.

Property 1.2.4. A distribution defined on $(0, \infty)$ is a PH distribution if and only if it satisfies the following conditions:

1. it has the point mass at zero, or
2. it has
   - a strictly positive density function on $(0, \infty)$, and
   - a rational Laplace transform such that there exists a pole of maximal real part $-\gamma$ that is real, negative and such that $-\gamma > Re(-\xi)$, where $-\xi$ is any other pole.

1.3 Discrete PH Distributions

For completion, we also define discrete PH distributions. Traditionally a discrete PH random variable is defined as the absorption time of an evanescent discrete-time Markov chain $\{Y_k\}$, with $k = 0, 1, 2, \ldots$, on a finite phase space $S = \{0, 1, 2, \ldots, n\}$ where phase 0 is absorbing. Here we give an algebraic definition.

Definition 1.3.1. *Discrete phase–type distributions*
Let $\boldsymbol{A}$ *be an arbitrary square matrix of order* n*, such that* $\lim_{k\to\infty} \boldsymbol{A}^k = \boldsymbol{0}$ *and* $\boldsymbol{I} - \boldsymbol{A}$ *is non–singular and* $\underline{\alpha}$ *be a* n*-dimensional column vector such that* $\underline{\alpha}'\,\underline{1} = 1$*, where* $\underline{1}$ *is a* n*-dimensional column vector of* $1's$*, that is:*

$$\underline{\alpha} = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix}', \quad \sum_{i=1}^{n} \alpha_i = 1\,, \quad \alpha_i \geq 0 \quad \textit{and} \quad \underline{1} = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}'. \tag{1.15}$$

If the probability function $\{p_k\}$ *of a random variable* X *is given by:*

$$p_k = \underline{\alpha}'\boldsymbol{A}^{k-1}(\boldsymbol{I} - \boldsymbol{A})\underline{1}\,, \qquad k \geq 1\,. \tag{1.16}$$

Then X *is called a discrete PH distribution with parameters* $(\underline{\alpha}\,, \boldsymbol{A})$*. The cumulative distribution function, defined for* $k = 1, 2, \ldots,$ *is given by:*

$$F_k = 1 - \underline{\alpha}'\boldsymbol{A}^k\underline{1}\,.$$

From the definition of the probability function, the probability generating function is given:

$$
\begin{aligned}
G(z) &= \sum_{k=1}^{\infty} p_k z^k = \sum_{k=1}^{\infty} \underline{\alpha}' \mathbf{A}^{k-1}(\mathbf{I}-\mathbf{A})\underline{1} z^k \,, \qquad z \in \mathbb{R}\,, \\
&= z\,\underline{\alpha}' \sum_{k=1}^{\infty} (\mathbf{A}z)^{k-1}(\mathbf{I}-\mathbf{A})\underline{1} = z\,\underline{\alpha}'(\mathbf{I}-z\mathbf{A})^{-1}(\mathbf{I}-\mathbf{A})\underline{1}\,, \qquad (1.17)
\end{aligned}
$$

where $\rho(z\mathbf{A}) < 1$ (the spectral radius operator $\rho()$ is defined in Appendix A.1.3.). The expression (1.17) shows that the probability generating function is a rational function.

From the Definition A.1.4 of the matrix exponential, differentiating (1.17) with respect to z and letting $z = 1$ gives the first moment

$$
\mathbb{E}(X) = \frac{d}{dz} z\,\underline{\alpha}'(\mathbf{I}-z\mathbf{A})^{-1}(\mathbf{I}-\mathbf{A})\underline{1}\Big|_{z=1} = \frac{d}{dz} \sum_{k=1}^{\infty} \underline{\alpha}'\mathbf{A}^{k-1}(\mathbf{I}-\mathbf{A})\underline{1}z^k\Big|_{z=1}\,,
$$

which in turn implies:

$$
\mathbb{E}(X) = \underline{\alpha}'(\mathbf{I}-\mathbf{A})^{-1}\underline{1}\,.
$$

Similarly the n-th factorial moment is given by:

$$
\mathbb{E}(X^n) = n!\,\underline{\alpha}'(\mathbf{I}-\mathbf{A})^{-n}\mathbf{A}^{n-1}\underline{1}\,, \qquad n = 1, 2, \ldots\,.
$$

Some well known discrete random variables have PH distributions. For example, the geometric random variable with probability function

$$
p_k = (1-p)^{k-1}p\,, \qquad 0 \le p < 1 \quad \text{and} \quad k \ge 1\,,
$$

is a discrete PH distribution with parameters $(\underline{\alpha}\,, \mathbf{A})$ given by

$$
\underline{\alpha} = 1\,, \qquad \mathbf{A} = 1 - p\,.
$$

Remark 1.3. Latouche and Ramaswami (1999) shows that Properties 1.2.1, 1.2.2 and 1.2.3 for the continuous PH distributions are also true for discrete PH distributions. The following property shows that compound PH sums also have PH distributions if the number of terms in sum has a discrete PH distribution. For more details about properties of the discrete PH distributions please refer to Neuts (1981) and Latouche and Ramaswami (1999).

Property 1.3.1. Let p_k be the probability function of a discrete PH with parameters $(\underline{\beta}, \mathbf{S})$ and F_X be a continuous PH distribution with parameters $(\underline{\alpha}, \mathbf{A})$. Then the infinite mixture $\sum_{k=0}^{\infty} S_k\, F_X^{*k}$ is also a PH with parameter $(\underline{\gamma}, \mathbf{C})$ given by:

$$
\begin{aligned}
\gamma &= \alpha' \otimes \beta\,, \\
\mathbf{C} &= \mathbf{A} \otimes \mathbf{I} - \mathbf{A}\underline{1}\alpha' \otimes \mathbf{S}\,,
\end{aligned}
$$

where F_X^{*k} denotes the $k-$fold convolution of F_X (where $F_X^{*0} = 1[x \geq 0]$), $\mathbf{I}$ is identity matrix of order n and $\otimes$ is the Kronecker product. For a proof see Neuts (1981).

These convolution and closure properties show that the mixtures of geometric and negative binomial distributions are also discrete PH distributions. In fact, we can verify that any distribution with finite support on the nonnegative integers is a discrete PH distribution with parameters $(\underline{\alpha}, \mathbf{A})$ which are given by:

$$
\underline{\alpha}' = \begin{pmatrix} p_1 & p_2 & p_3 & \cdots & p_n \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}.
$$

Thus, binomial and hypergeometric distributions are discrete PH distributions also. However, the Poisson distribution is not a PH distribution since it does not have a rational probability generating function. For details on the properties of discrete PH distributions, see Neuts (1981), Fackrell (2003) or Wang (2007).

Chapter 2

Moment Generating Function of Compound PH–Renewal Processes

In this chapter we first revisit the compound renewal risk model. For completeness, we reproduce the model definition and the main problems tackled.

Andersen (1957) considered compound renewal sums given by:

$$S(t) = \sum_{k=1}^{N(t)} X_k \,, \qquad t \geq 0 \,,$$

where $N(t)$ is a renewal process.

In the case where interest and inflation are considered, Léveillé and Garrido (2001a) impose the following model assumptions:

- Assume that there is an inflationary impact on the risk business and the inflation rate acting on claim severities at time t is known and denoted α_t. The claim severities, $\{Y_k\}_{k\geq 1}$, are then inflated. Claim occurrence times are represented by $\{T_k\}_{k\geq 1}$.
- Let $N(t) = \sup\{k \in \mathbb{N}\,; T_k \leq t\}$ for each $t > 0$, where $\sup \emptyset = 0$ and $N(0) = 0$, count the number of claims recorded over the time interval $[0, t]$.
- β_s is the known force of interest earned at time $s \in (0, t]$. Then

$$Z(t) = \sum_{k=1}^{N(t)} e^{-B(T_k)} Y_k \,, \qquad t \geq 0 \,, \tag{2.1}$$

where $B(s) = \int_0^s \beta_u du$ for $s \in (0,t]$ and $Z(0) = 0$ if $N(0) = 0$, defines the aggregate discounted value at time 0 of all claims recorded over $[0,t]$.

The definition of the risk model:

1. the claim number process $N = \{N(t)\,, t \geq 0\}$ forms a renewal process. The inter–arrival times, denoted by $\tau_k = T_k - T_{k-1}\,,\ k \geq 2$ and $\tau_1 = T_1$ have a common distribution say F_τ.

2. The claim severities $\{Y_k\}_{k \geq 1}$ are defined as random variables. Let the deflated claim severities

$$X_k = e^{-A(T_k)} Y_k \,, \qquad k \geq 1 \,,$$

where $A(t) = \int_0^t \alpha_s \, ds$ for any $t \geq 0$, satisfy the following assumptions:

- $\{X_k\}_{k \geq 1}$ are independent and identically distributed (i.i.d.) ,
- $\{X_k \,, \tau_k\}_{k \geq 1}$ are mutually independent.

From the definition of the model, the aggregate discounted sum in (2.1) is

$$Z(t) = \sum_{k=1}^{N(t)} e^{-B(T_k)} Y_k = \sum_{k=1}^{N(t)} e^{-D(T_k)} X_k \,, \qquad t > 0 \,, \tag{2.2}$$

where $D(T_t) = B(t) - A(t) = \int_0^t (\beta_s - \alpha_s) \, ds = \int_0^t \delta_s \, ds.$

If net interest rates are constant but not zero, that is $\delta_t = \beta_t - \alpha_t = \delta > 0$, then the aggregate discounted value at time 0 of the total claims recorded over the period $[0\,, t]$ is given by

$$Z(t) = \sum_{k=1}^{N(t)} e^{-\delta T_k} X_k \,, \qquad t \geq 0 \,, \tag{2.3}$$

with $Z(t) = 0$ if $N(t) = 0$.

In this chapter, we consider the discounted compound renewal sums, where the inter–arrival times are PH distributed. An ordinary differential equation system for the mgf at time t is obtained. We also discuss the asymptotic behavior of the mgf as t goes to infinity. In addition, some corollaries and examples are given to illustrate the results.

We begin the chapter with the introduction of the PH–renewal process, especially the renewal function and the renewal density. For more details please refer to Asmussen (2003).

2.1 The PH–Renewal Process

A Poisson process is a special case of a renewal process with exponential inter–arrival times. It can been found in many probability text books. Here we consider a more general family of inter–arrival times with PH distributions.

Definition 2.1.1. *The counting process $N = \{N(t)\,; t \geq 0\}$ is said to be a PH–renewal process if the inter–arrival times $\tau_k = T_k - T_{k-1}$, $k \geq 2$ and $\tau_1 = T_1$, have a common PH distribution say F_τ, and are independent.*

In this thesis we are particularly concerned with the mean of $N(t)$. The function

$$m(t) = \mathbb{E}\big[N(t)\big]\,, \qquad t \geq 0\,, \tag{2.4}$$

is called *the renewal function*. *The renewal density* is then defined as

$$m'(t) = \lim_{\triangle t \to 0} \frac{\mathbb{E}\big[N(t + \triangle\, t)\big] - \mathbb{E}\big[N(t)\big]}{\triangle\, t}\,, \qquad t \geq 0\,.$$

For instance, the renewal function of a Poisson process is easily obtained to be

$$m(t) = \lambda t\,, \qquad t \geq 0\,,$$

where $\lambda > 0$ is the parameter of the exponential inter–arrival times.

More generally consider inter–arrival times that are PH distributed with parameters $(\underline{\alpha}\,, \mathbf{A})$, then we have the following result.

Proposition 2.1.1. *Consider a renewal process with inter–arrival times which are PH distributed with parameters* $(\underline{\alpha}\,, \boldsymbol{A})$. *Then the renewal density exists and is given by*

$$m'(t) = \underline{\alpha}' e^{\boldsymbol{A}(\boldsymbol{I}-\underline{1}\,\underline{\alpha}')t}(-\boldsymbol{A})\underline{1}\,.$$

For a proof using the Markov chain method see Neuts (1978, 1981), Latouche and Ramaswami (1999) and Asmussen (2003). Here we use the same probabilistic approach as in Wang (2007, p31) to prove the result.

Proof. From Cox (1970) we have that:

$$m(t) = \mathbb{E}\big[N(t)\big] = \sum_{k=1}^{\infty} F_\tau^{*k}(t)\,, \qquad t \geq 0\,, \tag{2.5}$$

where F_τ^{*k} denotes the $k-$fold convolution of F_τ, the distribution of inter–arrival times. When the inter–arrival times have a PH $(\underline{\alpha}, \mathbf{A})$ distribution, then

$$F_\tau(t) = 1 - \underline{\alpha}' e^{\mathbf{A}t}\underline{1}\,, \qquad t \geq 0. \tag{2.6}$$

By the closure property, the convolution of F_τ is also a PH distribution. Let F_τ^{*k}, for $k \geq 2$, take the following PH $(\underline{\alpha}_k, \mathbf{C}_k)$ form:

$$\underline{\alpha}_1 = \underline{\alpha}\,, \qquad \underline{\alpha}_k = (\underline{\alpha}'\,, \underline{0}'_{(k-1)n})'\,, \qquad \mathbf{C}_1 = \mathbf{A}\,, \qquad \mathbf{C}_k = \begin{pmatrix} \mathbf{A} & -\mathbf{A}\underline{1}\,\underline{\alpha}_{k-1} \\ \mathbf{0} & \mathbf{C}_{k-1} \end{pmatrix}\,, \tag{2.7}$$

where the order of $\underline{\alpha}$ is n and the order of $\underline{0}_{(k-1)n}$ is $(k-1)\times n$, then from (2.5) and Definition A.1.4 of Appendix A we have that

$$\begin{aligned} m(t) &= 1 - \underline{\alpha}' e^{\mathbf{A}t}\underline{1} + 1 - \underline{\alpha}_2' e^{\mathbf{C}_2 t}\underline{1}_{2n} + \cdots + 1 - \underline{\alpha}_k' e^{\mathbf{C}_k t}\underline{1}_{kn} + \cdots \\ &= 1 - \underline{\alpha}' \sum_{r=0}^{\infty} \frac{(\mathbf{A}t)^r}{r!}\underline{1} + 1 - \underline{\alpha}_2' \sum_{r=0}^{\infty} \frac{(\mathbf{C}_2 t)^r}{r!}\underline{1}_{2n} + \cdots + 1 - \underline{\alpha}_k' \sum_{r=0}^{\infty} \frac{(\mathbf{C}_k t)^r}{r!}\underline{1}_{kn} + \cdots \\ &= -\Big[\big(\underline{\alpha}'\mathbf{A}\underline{1} + \underline{\alpha}_2'\mathbf{C}_2\underline{1}_{2n}\cdots\big)t + \frac{1}{2!}\big(\underline{\alpha}'\mathbf{A}^2\underline{1} + \underline{\alpha}_2'\mathbf{C}_2^2\underline{1}_{2n} + \cdots\big)t^2 + \cdots\Big]\,. \end{aligned} \tag{2.8}$$

First let us prove the following result by induction on $j = 1, 2, \ldots$:

$$\underline{\alpha}_k' \mathbf{C}_k^j\,\underline{1}_{kn} = 0\,, \qquad \text{for} \quad k \geq j+1\,. \tag{2.9}$$

When $j = 1$ and $k \geq 2$, we have

$$\underline{\alpha}_k'\,\mathbf{C}_k\,\underline{1}_{kn} = (\underline{\alpha}' \quad \underline{0}'_{(k-1)n}) \begin{pmatrix} \mathbf{A} & -\mathbf{A}\underline{1}\,\underline{\alpha}_{k-1} \\ \mathbf{0} & \mathbf{C}_{k-1} \end{pmatrix} \begin{pmatrix} \underline{1} \\ \underline{1}_{(k-1)n} \end{pmatrix} = \underline{\alpha}'\mathbf{A}\underline{1} - \underline{\alpha}'\mathbf{A}\underline{1}\underline{\alpha}_{k-1}'\,\underline{1}_{k-1} = 0\,,$$

since $\underline{\alpha}'_{k-1}\underline{1}_{k-1} = 1$. Suppose that $j \le n-1$, then

$$\underline{\alpha}'_k \mathbf{C}_k^j \, \underline{1}_{kn} = 0\,, \qquad \text{for} \quad k \ge j+1\,. \tag{2.10}$$

Now we need to prove that (2.10) is true, when $j = n$. Note the fact that:

$$\begin{aligned}
\underline{\alpha}'_k \, \mathbf{C}_k^n \, \underline{1}_{kn} &= (\underline{\alpha}' \quad \underline{0}'_{(k-1)n}) \begin{pmatrix} \mathbf{A}^n & -\sum_{i=0}^{n-1} \mathbf{A}^{n-i}\underline{1}\,\underline{\alpha}'_{k-1}\mathbf{C}_{k-1}^i \\ \mathbf{0} & \mathbf{C}_{k-1}^n \end{pmatrix} \begin{pmatrix} \underline{1} \\ \underline{1}_{(k-1)n} \end{pmatrix} \\
&= \underline{\alpha}'\mathbf{A}^n\underline{1} - \sum_{i=0}^{n-1} \underline{\alpha}'\mathbf{A}^{n-i}\underline{1}\,\underline{\alpha}'_{k-1}\mathbf{C}_{k-1}^i\underline{1}_{(k-1)n} \\
&= -\sum_{i=1}^{n-1} \underline{\alpha}'\mathbf{A}^{n-i}\underline{1}\,\underline{\alpha}'_{k-1}\mathbf{C}_{k-1}^i\underline{1}_{(k-1)n}\,.
\end{aligned} \tag{2.11}$$

Assumption (2.10) leads to $\underline{\alpha}'_{k-1}\mathbf{C}_{k-1}^i\underline{1}_{(k-1)n} = 0$, for $k \ge n \ge i+1$, then the result $\underline{\alpha}'_k \, \mathbf{C}_k^n \, \underline{1}_{kn} = 0$ holds. Hence we have that

$$\begin{aligned}
m(t) &= -\Big\{\underline{\alpha}'\mathbf{A}\underline{1}t + \frac{1}{2!}\big[\underline{\alpha}'\mathbf{A}^2\underline{1} + \underline{\alpha}'_2\mathbf{C}_2^2\underline{1}_{2n}\big]t^2 + \frac{1}{3!}\big[\underline{\alpha}'\mathbf{A}^3\underline{1} + \underline{\alpha}'_2\mathbf{C}_2^3\underline{1}_{2n} + \underline{\alpha}'_3\mathbf{C}_3^3\underline{1}_{3n}\big]t^3 + \\
&\quad + \frac{1}{n!}\big[\underline{\alpha}'\mathbf{A}^n\underline{1} + \underline{\alpha}'_2\mathbf{C}_2^n\underline{1}_{2n} + \cdots + \underline{\alpha}'_n\mathbf{C}_n^n\underline{1}_{nn}\big]t^n + \cdots\Big\}\,.
\end{aligned} \tag{2.12}$$

We now show the following result by induction on $n = 1, 2, \cdots$:

$$\underline{\alpha}'\mathbf{A}^j\underline{1} + \underline{\alpha}'_2\mathbf{C}_2^j\underline{1}_{2n} + \cdots + \underline{\alpha}'_n\mathbf{C}_n^j\underline{1}_{nn} = \alpha'\big[\mathbf{A}(\mathbf{I} - \underline{1}\,\alpha')\big]^{j-1}\mathbf{A}\underline{1}\,. \tag{2.13}$$

When $n = 1$, obviously (2.13) holds. Suppose that when $j \le n-1$ the following result is true:

$$\underline{\alpha}'\mathbf{A}^j\underline{1} + \underline{\alpha}'_2\mathbf{C}_2^j\underline{1}_{2n} + \cdots + \underline{\alpha}'_j\mathbf{C}_j^j\underline{1}_{jn} = \alpha'\big[\mathbf{A}(\mathbf{I} - \underline{1}\,\alpha')\big]^{j-1}\mathbf{A}\underline{1}\,. \tag{2.14}$$

Then where $j = n$, from (2.11) we have that:

$$\begin{aligned}
&\underline{\alpha}'\mathbf{A}^n\underline{1} + \underline{\alpha}'_2\mathbf{C}_2^n\underline{1}_{2n} + \cdots + \underline{\alpha}'_n\mathbf{C}_n^n\underline{1}_{nn} \\
=\;& \underline{\alpha}'\mathbf{A}^n\underline{1} - \sum_{i=1}^{n-1}\underline{\alpha}'\mathbf{A}^{n-i}\underline{1}\,\underline{\alpha}'_{2-1}\mathbf{C}_{2-1}^i\underline{1}_{(2-1)n} - \cdots - \sum_{i=1}^{n-1}\underline{\alpha}'\mathbf{A}^{n-i}\underline{1}\,\underline{\alpha}'_{n-1}\mathbf{C}_{n-1}^i\underline{1}_{(n-1)n} \\
=\;& \underline{\alpha}'\mathbf{A}^n\underline{1} - \sum_{i=1}^{n-1}\underline{\alpha}'\mathbf{A}^{n-i}\underline{1}\Big[\underline{\alpha}'\mathbf{A}^i\underline{1} + \underline{\alpha}'_2\mathbf{C}_2^i\underline{1}_{2n} + \cdots + \underline{\alpha}'_{n-1}\mathbf{C}_{n-1}^i\underline{1}_{(n-1)n}\Big]\,.
\end{aligned} \tag{2.15}$$

Assumptions (2.14) and (2.9) lead (2.15) to be:

$$= \underline{\alpha}'\mathbf{A}^n\underline{1} - \sum_{i=1}^{n-1}\underline{\alpha}'\mathbf{A}^{n-i}\underline{1}\,\underline{\alpha}'\big[\mathbf{A}(\mathbf{I} - \underline{1}\,\alpha')\big]^{i-1}\mathbf{A}\underline{1} = \underline{\alpha}'\big[\mathbf{A}(\mathbf{I} - \underline{1}\,\alpha')\big]^{n-1}\mathbf{A}\underline{1}\,. \tag{2.16}$$

Hence (2.12) can be simplified as:

$$
\begin{aligned}
m(t) &= -\Big\{\underline{\alpha}'\mathbf{A}\underline{1}\,t + \frac{1}{2!}\,\underline{\alpha}'\mathbf{A}(\mathbf{I}-\underline{1}\,\underline{\alpha}')\mathbf{A}\underline{1}\,t^2 + \frac{1}{3!}\,\underline{\alpha}'\big[\mathbf{A}(\mathbf{I}-\underline{1}\,\underline{\alpha}')\big]^2\mathbf{A}\underline{1}t^3 + \cdots \\
&\qquad + \frac{1}{(k+1)!}\,\underline{\alpha}'\big[\mathbf{A}(\mathbf{I}-\underline{1}\,\underline{\alpha}')\big]^k\mathbf{A}\underline{1}\,t^{k+1} + \cdots\Big\}\,. \qquad (2.17)
\end{aligned}
$$

Differentiating (2.17) with respect to t yields

$$
\begin{aligned}
m'(t) &= -\Big\{\underline{\alpha}'\mathbf{A}\underline{1} + \underline{\alpha}'\mathbf{A}(\mathbf{I}-\underline{1}\,\underline{\alpha}')\mathbf{A}\underline{1}\,t + \frac{1}{2!}\,\underline{\alpha}'\big[\mathbf{A}(\mathbf{I}-\underline{1}\,\underline{\alpha}')\big]^2\mathbf{A}\underline{1}t^2 + \cdots \\
&\qquad + \frac{1}{k!}\,\underline{\alpha}'\big[\mathbf{A}(\mathbf{I}-\underline{1}\,\underline{\alpha}')\big]^k\mathbf{A}\underline{1}\,t^{k} + \cdots\Big\}\,. \qquad (2.18)
\end{aligned}
$$

Using Definition A.1.4. from Appendix A leads to $m'(t) = \underline{\alpha}'e^{\mathbf{A}(\mathbf{I}-\underline{1}\,\underline{\alpha}')t}(-\mathbf{A})\underline{1}$ which completes the proof. □

2.2 The Moment Generating Function of $Z(t)$

Consider now the mgf of $Z(t)$, for fixed t, when N is a renewal process. Léveillé, Garrido and Wang (2010) gives the following theorem.

Theorem 2.2.1. *For any $t > 0\,,\delta > 0$ and $s \in \Omega \subseteq \mathbb{R}$, the domain of existence of the following mgf, we have*

$$
\begin{aligned}
M_{Z(t)}(s) &= 1 + \sum_{k=0}^{\infty}\int_0^t\int_0^{t-x_1}\cdots\int_0^{t-\sum_{i=1}^k x_i}\prod_{i=1}^{k+1}\Big[M_X\big(se^{-\delta\sum_{j=1}^i x_j}\big)-1\Big] \\
&\qquad \times dm(x_{k+1})\ldots dm(x_2)\,dm(x_1)\,, \qquad (2.19)
\end{aligned}
$$

where $m(x)$ is the renewal function defined in (2.4) and $\sum_{i=1}^k x_i = 0$ for $k = 0$.

In particular, when inter–arrival times are PH $(\underline{\alpha}, \mathbf{A})$ distributed, the renewal density can be written as

$$
dm(x) = \underline{\alpha}'\,e^{\mathbf{B}x}\,(-\mathbf{A})\underline{1}\,dx\,, \qquad \text{for } \mathbf{B} = \mathbf{A}(\mathbf{I}-\underline{1}\,\underline{\alpha}').
$$

Hence from Theorem 2.2.1 the mgf $M_{Z(t)}(s)$ can be rewritten as

$$
\begin{aligned}
M_{Z(t)}(s) &= 1 + \sum_{k=0}^{\infty}\int_0^t\int_0^{t-x_1}\cdots\int_0^{t-\sum_{i=1}^k x_i}\prod_{i=1}^{k+1}\bigg(\Big[M_X\big(se^{-\delta\sum_{j=1}^i x_j}\big)-1\Big] \\
&\qquad \times \underline{\alpha}'\,e^{\mathbf{B}x_i}(-\mathbf{A})\underline{1}\bigg)\,dx_{k+1}\ldots dx_2\,dx_1\,, \qquad t > 0\,, s \in \Omega\,.
\end{aligned}
$$

Let $y_i = x_1 + x_2 + \cdots + x_i$ for $i = 1, 2, \cdots, k+1$, and $y_0 = 0$ then

$$\begin{aligned} M_{Z(t)}(s) &= 1 + \sum_{k=0}^{\infty} \int_0^t \int_{y_1}^t \cdots \int_{y_k}^t \prod_{i=1}^{k+1} \Big[M_X\big(se^{-\delta y_i}\big) - 1 \Big] \underline{\alpha}'\, e^{\mathbf{B}(y_i - y_{i-1})}(-\mathbf{A})\underline{1} \\ &\quad \times dy_{k+1} \ldots dy_2\, dy_1\,. \end{aligned}$$

Changing the order of the integrals yields:

$$\begin{aligned} M_{Z(t)}(s) &= 1 + \sum_{k=0}^{\infty} \int_0^t \int_0^{y_{k+1}} \cdots \int_0^{y_2} \prod_{i=1}^{k+1} \Big[M_X\big(se^{-\delta y_i}\big) - 1 \Big] \underline{\alpha}'\, e^{\mathbf{B}(y_i - y_{i-1})}(-\mathbf{A})\underline{1} \\ &\quad \times dy_1 \ldots dy_k\, dy_{k+1}\,, \qquad t > 0\,, s \in \Omega\,. \end{aligned} \tag{2.20}$$

Differentiating both sides of (2.20) with respect to t yields

$$\begin{aligned} \frac{\partial}{\partial t} M_{Z(t)}(s) &= \big[M_X(se^{-\delta t}) - 1\big] \underline{\alpha}'\, e^{\mathbf{B}t}(-\mathbf{A})\underline{1} + \big[M_X(se^{-\delta t}) - 1\big] \\ &\quad \times \Bigg(\sum_{k=1}^{\infty} \int_0^t \int_0^{y_k} \cdots \int_0^{y_2} \underline{\alpha}'\, e^{\mathbf{B}(t - y_k)}(-\mathbf{A})\underline{1} \prod_{i=1}^{k} \Big[M_X\big(se^{-\delta y_i}\big) - 1 \Big] \\ &\quad \times \Big[\underline{\alpha}'\, e^{\mathbf{B}(y_i - y_{i-1})}(-\mathbf{A})\underline{1} \Big]\, dy_1 \cdots dy_{k-1}\, dy_k \Bigg)\,, \qquad t > 0\,, s \in \Omega\,. \end{aligned} \tag{2.21}$$

Now let

$$\begin{aligned} f(t; s) &= \sum_{k=1}^{\infty} \int_0^t \int_0^{y_k} \cdots \int_0^{y_2} \underline{\alpha}'\, e^{\mathbf{B}(t - y_k)}(-\mathbf{A})\underline{1} \prod_{i=1}^{k} \Big[M_X\big(se^{-\delta y_i}\big) - 1 \Big] \\ &\quad \times \Big[\underline{\alpha}'\, e^{\mathbf{B}(y_i - y_{i-1})}(-\mathbf{A})\underline{1} \Big]\, dy_1 \cdots dy_{k-1}\, dy_k\,, \qquad t > 0\,, s \in \Omega\,, \end{aligned} \tag{2.22}$$

therefore (2.21) can be written in the form of a differential equation

$$\frac{\partial}{\partial t} M_{Z(t)}(s) = [M_X(se^{-\delta t}) - 1] \underline{\alpha}'\, e^{\mathbf{B}t}(-\mathbf{A})\underline{1} + [M_X(se^{-\delta t}) - 1]\, f(t; s)\,. \tag{2.23}$$

This differential equation for $M_{Z(t)}(s)$ can be solved for some PH inter–arrival times by a method of differential equation systems, as we can see in the following section.

2.3 Differential Equations for $M_{Z(t)}(s)$

Consider PH inter–arrival times with parameters $(\alpha, \mathbf{A})$, then we can obtain homogeneous differential equations or differential systems for the moment generating function of $Z(t)$.

Differentiating both sides of (2.22) with respect to t yields

$$\begin{aligned}
\frac{\partial}{\partial t} f(t;s) \;=\; & \underline{\alpha}'\mathbf{B}e^{\mathbf{B}t}\Big\{\sum_{k=1}^{\infty}\int_0^t\int_0^{y_k}\cdots\int_0^{y_2}\prod_{i=1}^{k}\Big[M_X\big(se^{-\delta y_i}\big)-1\Big]e^{-\mathbf{B}y_k}(-\mathbf{A})\underline{1} \\
& \times\underline{\alpha}'\,e^{\mathbf{B}(y_k-y_{k-1})}(-\mathbf{A})\underline{1}\cdots\underline{\alpha}'\,e^{\mathbf{B}(y_2-y_1)}(-\mathbf{A})\underline{1}\,\underline{\alpha}'\,e^{\mathbf{B}y_1}(-\mathbf{A})\underline{1} \\
& \times dy_1\cdots dy_{k-1}\,dy_k\Big\} + \underline{\alpha}'\,e^{\mathbf{B}t}\Big\{\big[M_X(se^{-\delta t})-1\big]e^{-\mathbf{B}t}\,(-\mathbf{A})\,\underline{1}\,\underline{\alpha}'\,e^{\mathbf{B}t}(-\mathbf{A})\underline{1} \\
& +\big[M_X(se^{\delta t})-1\big]\sum_{k=2}^{\infty}\int_0^t\int_0^{y_{k-1}}\cdots\int_0^{y_2}\prod_{i=1}^{k-1}\Big[M_X\big(se^{-\delta y_i}\big)-1\Big] \\
& \times e^{-\mathbf{B}t}(-\mathbf{A})\underline{1}\,\underline{\alpha}'\,e^{\mathbf{B}(t-y_{k-1})}(-\mathbf{A})\underline{1}\cdots\underline{\alpha}'\,e^{\mathbf{B}(y_2-y_1)}(-\mathbf{A})\underline{1}\,\underline{\alpha}'\,e^{\mathbf{B}y_1}(-\mathbf{A})\underline{1} \\
& \times dy_1\cdots dy_{k-2}\,dy_{k-1}\Big\}, \qquad t>0\,,s\in\Omega\,.
\end{aligned} \tag{2.24}$$

Simplifying (2.24) produces:

$$\begin{aligned}
\frac{\partial}{\partial t} f(t;s) \;=\; & \Big\{\sum_{k=1}^{\infty}\int_0^t\int_0^{y_k}\cdots\int_0^{y_2}\prod_{i=1}^{k}\Big[M_X\big(se^{-\delta y_i}\big)-1\Big]\underline{\alpha}'\,\mathbf{B}e^{\mathbf{B}(t-y_k)}(-\mathbf{A})\underline{1} \\
& \times\underline{\alpha}'\,e^{\mathbf{B}(y_k-y_{k-1})}(-\mathbf{A})\underline{1}\cdots\underline{\alpha}'\,e^{\mathbf{B}(y_2-y_1)}(-\mathbf{A})\underline{1}\,\underline{\alpha}'\,e^{\mathbf{B}y_1}(-\mathbf{A})\underline{1} \\
& \times dy_1\cdots dy_{k-1}\,dy_k\Big\} + \big[M_X(se^{-\delta t})-1\big]\underline{\alpha}'\,(-\mathbf{A})\,\underline{1}\,\underline{\alpha}'\,e^{\mathbf{B}t}(-\mathbf{A})\underline{1} \\
& +\big[M_X(se^{\delta t})-1\big]\,\underline{\alpha}'\,(-\mathbf{A})\,\underline{1}\Big\{\sum_{k=2}^{\infty}\int_0^t\int_0^{y_{k-1}}\cdots\int_0^{y_2}\prod_{i=1}^{k-1}\Big[M_X\big(se^{-\delta y_i}\big)-1\Big] \\
& \times\underline{\alpha}'\,e^{\mathbf{B}(t-y_{k-1})}(-\mathbf{A})\underline{1}\cdots\underline{\alpha}'\,e^{\mathbf{B}(y_2-y_1)}(-\mathbf{A})\underline{1}\,\underline{\alpha}'\,e^{\mathbf{B}y_1}(-\mathbf{A})\underline{1} \\
& \times dy_1\cdots dy_{k-2}\,dy_{k-1}\Big\}, \qquad t>0\,,s\in\Omega\,.
\end{aligned} \tag{2.25}$$

Substituting (2.22) into (2.25) gives:

$$\begin{aligned}
\frac{\partial}{\partial t} f(t;s) \;=\; & \sum_{k=1}^{\infty}\int_0^t\int_0^{y_k}\cdots\int_0^{y_2}\prod_{i=1}^{k}\Big[M_X\big(se^{-\delta y_i}\big)-1\Big]\underline{\alpha}'\,\mathbf{B}e^{\mathbf{B}(t-y_k)}(-\mathbf{A})\underline{1} \\
& \times\underline{\alpha}'\,e^{\mathbf{B}(y_k-y_{k-1})}(-\mathbf{A})\underline{1}\cdots\underline{\alpha}'\,e^{\mathbf{B}(y_2-y_1)}(-\mathbf{A})\underline{1}\,\underline{\alpha}'\,e^{\mathbf{B}y_1}(-\mathbf{A})\underline{1} \\
& \times dy_1\cdots dy_{k-1}\,dy_k + \big[M_X(se^{-\delta t})-1\big]\underline{\alpha}'\,(-\mathbf{A})\,\underline{1}\,\underline{\alpha}'\,e^{\mathbf{B}t}(-\mathbf{A})\underline{1} \\
& +\big[M_X(se^{-\delta t})-1\big]\,\underline{\alpha}'\,(-\mathbf{A})\,\underline{1}\,f(t;s)\,.
\end{aligned} \tag{2.26}$$

For special cases of PH distributions, we still can further simplify (2.26). We begin with a generalized Erlang(2) distribution.

2.3.1 Generalized Erlang(2) Inter–arrival Times

If inter–arrival times have generalized Erlang(2) distributions, whose parameters are given by:

$$\underline{\alpha}' = \begin{pmatrix} 1 & 0 \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ 0 & -\lambda_2 \end{pmatrix},$$

then, by Maple, we have

$$m'(t) = \underline{\alpha}'\, e^{\mathbf{B}t}(-\mathbf{A})\underline{1} = -\tfrac{\lambda_1\lambda_2}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)t} + \tfrac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\,,$$

$$\underline{\alpha}'\,\mathbf{B}e^{\mathbf{B}t}(-\mathbf{A})\underline{1} = \lambda_1\lambda_2 e^{-(\lambda_1+\lambda_2)t}\,, \quad \text{for} \quad \mathbf{B} = \mathbf{A}(\mathbf{I} - \underline{1}\underline{\alpha}')\,.$$

Hence

$$\underline{\alpha}'\,\mathbf{B}e^{\mathbf{B}t}(-\mathbf{A})\underline{1} = -(\lambda_1+\lambda_2)\,\underline{\alpha}'\,e^{\mathbf{B}t}(-\mathbf{A})\underline{1} + \lambda_1\lambda_2\,. \tag{2.27}$$

Lemma 2.3.1. *If inter–arrival times have generalized Erlang(2) distributions, then*

$$\frac{\partial}{\partial t} f(t;s) = -(\lambda_1+\lambda_2) f(t;s) + \lambda_1\lambda_2\, M_{Z(t)}(s) - \lambda_1\lambda_2\,. \tag{2.28}$$

Proof. Since the sum of the first row of $\mathbf{A}$ for generalized Erlang(2) equals to 0, that is $\underline{\alpha}'\,(-\mathbf{A})\,\underline{1} = 0$, then (2.26) can be simplified as

$$\begin{aligned} \frac{\partial}{\partial t} f(t;s) \;=\; & \sum_{k=1}^{\infty} \int_0^t \int_0^{y_k} \cdots \int_0^{y_2} \prod_{i=1}^{k} \Big[M_X\big(se^{-\delta y_i}\big) - 1\Big] \underline{\alpha}'\,\mathbf{B}e^{\mathbf{B}(t-y_k)}(-\mathbf{A})\underline{1} \\ & \times \underline{\alpha}'\,e^{\mathbf{B}(y_k-y_{k-1})}(-\mathbf{A})\underline{1} \cdots \underline{\alpha}'\,e^{\mathbf{B}(y_2-y_1)}(-\mathbf{A})\underline{1}\,\underline{\alpha}'\,e^{\mathbf{B}y_1}(-\mathbf{A})\underline{1} \\ & \times dy_1 \cdots dy_{k-1}\,dy_k\,. \end{aligned} \tag{2.29}$$

Substituting (2.27) into (2.29) yields

$$\begin{aligned} \frac{\partial}{\partial t} f(t;s) = & -(\lambda_1+\lambda_2) \sum_{k=1}^{\infty} \int_0^t \int_0^{y_k} \cdots \int_0^{y_2} \prod_{i=1}^{k} \Big[M_X\big(se^{-\delta y_i}\big) - 1\Big] \underline{\alpha}'\,e^{\mathbf{B}(t-y_k)}(-\mathbf{A})\underline{1} \\ & \times \underline{\alpha}'\,e^{\mathbf{B}(y_k-y_{k-1})}(-\mathbf{A})\underline{1} \cdots \underline{\alpha}'\,e^{\mathbf{B}(y_2-y_1)}(-\mathbf{A})\underline{1}\,\underline{\alpha}'\,e^{\mathbf{B}y_1}(-\mathbf{A})\underline{1} \\ & \times dy_1 \cdots dy_{k-1}\,dy_k + \lambda_1\lambda_2 \sum_{k=1}^{\infty} \int_0^t \int_0^{y_k} \cdots \int_0^{y_2} \prod_{i=1}^{k} \Big[M_X\big(se^{-\delta y_i}\big) - 1\Big] \\ & \times \underline{\alpha}'\,e^{\mathbf{B}(y_k-y_{k-1})}(-\mathbf{A})\underline{1} \cdots \underline{\alpha}'\,e^{\mathbf{B}(y_2-y_1)}(-\mathbf{A})\underline{1}\,\underline{\alpha}'\,e^{\mathbf{B}y_1}(-\mathbf{A})\underline{1}\,dy_1 \cdots dy_{k-1}\,dy_k\,. \end{aligned} \tag{2.30}$$

Combining (2.20) and (2.29), (2.30) gives (2.27). □

Next we show that the mgf $M_{Z(t)}(s)$ for generalized Erlang(2) inter–arrival times satisfies a homogeneous differential equation, with respect to t.

Theorem 2.3.1. *If inter–arrival times are generalized Erlang(2), then the mgf of $Z(t)$ satisfies:*

$$\frac{\partial^2}{\partial t^2} M_{Z(t)}(s) = a_1(t) \frac{\partial}{\partial t} M_{Z(t)}(s) + a_0(t) M_{Z(t)}(s) \,, \qquad t \geq 0 \,,\ s \in \Omega \,, \tag{2.31}$$

with initial values $M_{Z(0)}(s) = 1$ *and* $\frac{\partial}{\partial t} M_{Z(t)}(s)|_{t=0} = 0$*, where*

$$a_1(t) = \frac{\frac{\partial}{\partial t}\big[M_X(se^{-\delta t}) - 1\big]}{\big[M_X(se^{-\delta t}) - 1\big]} - (\lambda_1 + \lambda_2) \,, \quad a_0(t) = \lambda_1 \lambda_2 \big[M_X(se^{-\delta t}) - 1\big].$$

Proof. Differentiating both sides of (2.23) with respect to t produces

$$\begin{aligned} \frac{\partial^2}{\partial t^2} M_{Z(t)}(s) \;&=\; \frac{\partial}{\partial t}\big[M_X(se^{-\delta t}) - 1\big] \underline{\alpha}' \, e^{\mathbf{B}t}(-\mathbf{A})\underline{1} + \big[M_X(se^{-\delta t}) - 1\big] \\ &\quad \times \underline{\alpha}' \, \mathbf{B} \, e^{\mathbf{B}t}(-\mathbf{A})\underline{1} + \frac{\partial}{\partial t}\big[M_X(se^{-\delta t}) - 1\big] f(t,s) \\ &\quad + \big[M_X(se^{-\delta t}) - 1\big] \frac{\partial}{\partial t} f(t,s) \,. \end{aligned} \tag{2.32}$$

Substituting (2.28) and (2.27) into (2.32) and combining (2.20) yields Theorem 2.3.1. □

Remark 2.1. If $\delta = 0$, then the homogeneous differential equation in (2.3.1) is given by:

$$\frac{\partial^2}{\partial t^2} M_{Z(t)}(s) = a_1 \frac{\partial}{\partial t} M_{Z(t)}(s) + a_0 M_{Z(t)}(s) \,, \qquad t > 0 \,,\ s \in \mathbb{R} \,, \tag{2.33}$$

with coefficients $a_1 = -\lambda_1 - \lambda_2 \,, a_0 = \lambda_1 \lambda_2 \big[M_X(s) - 1\big]$ that are constant with respect to t. Solving this homogeneous differential equation using standard techniques (Polyanin and Zaitsev, 2003) yields the mgf of the Sparre Andersen sum with generalized Erlang(2) inter–arrival times $S(t) = \sum_{i=1}^{N(t)} X_i$:

$$M_{S(t)}(s) = e^{-\frac{1}{2}(\lambda_1 + \lambda_2)t} \Big[(\lambda_1 + \lambda_2) \sqrt{d}^{-1} \, \sinh\big(\frac{1}{2}\sqrt{d}\, t\big) + \cosh\big(\frac{1}{2}\sqrt{d}\, t\big) \Big] \,,$$

where $d = 4\lambda_1 \lambda_2 M_X(s) + (\lambda_1 - \lambda_2)^2$.

Corollary 2.3.1. *If $\lambda_1 = \lambda_2 = \lambda$, the inter–arrival times are Erlang(2) distributed, then we get*

$$\frac{\partial^2}{\partial t^2} M_{Z(t)}(s) = a_1(t) \frac{\partial}{\partial t} M_{Z(t)}(s) + a_0(t) M_{Z(t)}(s) \,, \qquad t \geq 0 \,,\ s \in \Omega \,,$$

with initial values $M_{Z(0)}(s) = 1$ *and* $\frac{\partial}{\partial t} M_{Z(t)}(s)|_{t=0} = 0$*, where* $a_1(t) = \frac{\frac{\partial}{\partial t}\big[M_X(se^{-\delta t}) - 1\big]}{\big[M_X(se^{-\delta t}) - 1\big]} - 2\lambda$, $a_0(t) = \lambda^2 \big[M_X(se^{-\delta t}) - 1\big]$ *and M_X is the mgf of the deflated claim severity X.*

This result is given by Léveillé, Garrido and Wang (2010).

2.3.2 Mixed Exponential Inter–arrival Times

Now consider the mgf of $Z(t)$, when inter–arrival times are mixed exponential of order 2 with parameters given by:

$$\underline{\alpha}' = \begin{pmatrix} a & 1-a \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix},$$

then

$$m'(t) = \underline{\alpha}'\, e^{\mathbf{B}t}(-\mathbf{A})\underline{1} = \tfrac{a(1-a)(\lambda_1-\lambda_2)^2}{(1-a)\lambda_1+a\lambda_2} e^{-[(1-a)\lambda_1+a\lambda_2]t} + \tfrac{\lambda_1\lambda_2}{(1-a)\lambda_1+a\lambda_2},$$

$$\underline{\alpha}'\, \mathbf{B}e^{\mathbf{B}t}(-\mathbf{A})\underline{1} = -a(1-a)(\lambda_1-\lambda_2)^2 e^{-[(1-a)\lambda_1+a\lambda_2]t}.$$

Hence

$$\underline{\alpha}'\, \mathbf{B}e^{\mathbf{B}t}(-\mathbf{A})\underline{1} = -\big[(1-a)\lambda_1 + a\lambda_2\big]\, \underline{\alpha}'\, e^{\mathbf{B}t}(-\mathbf{A})\underline{1} + \lambda_1\lambda_2\,. \tag{2.34}$$

Lemma 2.3.2. *If the inter–arrival times have a mixed exponential distribution of order* 2*, then*

$$\frac{\partial}{\partial t} f(t;s) = \Big\{\big[M_X(se^{-\delta t})-1\big]\underline{\alpha}'\,(-\boldsymbol{A})\underline{1} - \big[(1-a)\lambda_1 + a\lambda_2\big]\Big\} f(t;s)$$
$$+\lambda_1\lambda_2\, M_{Z(t)}(s) + \big[M_X(se^{-\delta t})-1\big]\underline{\alpha}'\,(-\boldsymbol{A})\underline{1}\,\underline{\alpha}'\, e^{\boldsymbol{B}t}(-\boldsymbol{A})\underline{1} - \lambda_1\lambda_2\,. \tag{2.35}$$

Proof. Substituting (2.34) into (2.26) produces

$$\frac{\partial}{\partial t} f(t;s) = -\big[(1-a)\lambda_1 + a\lambda_2\big]\Bigg\{\sum_{k=1}^{\infty}\int_0^t\int_0^{y_k}\cdots\int_0^{y_2}\prod_{i=1}^{k}\Big[M_X\big(se^{-\delta y_i}\big)-1\Big]\underline{\alpha}'$$
$$\times e^{\mathbf{B}(t-y_k)}(-\mathbf{A})\underline{1}\underline{\alpha}'\, e^{\mathbf{B}(y_k-y_{k-1})}(-\mathbf{A})\underline{1}\cdots\underline{\alpha}'\, e^{\mathbf{B}(y_2-y_1)}(-\mathbf{A})\underline{1}\,\underline{\alpha}'\, e^{\mathbf{B}y_1}(-\mathbf{A})\underline{1}$$
$$\times dy_1\cdots dy_{k-1}\,dy_k\Bigg\} + \lambda_1\lambda_2\Bigg\{\sum_{k=1}^{\infty}\int_0^t\int_0^{y_k}\cdots\int_0^{y_2}\prod_{i=1}^{k}\Big[M_X\big(se^{-\delta y_i}\big)-1\Big]$$
$$\times\underline{\alpha}'\, e^{\mathbf{B}(y_k-y_{k-1})}(-\mathbf{A})\underline{1}\cdots\underline{\alpha}'\, e^{\mathbf{B}(y_2-y_1)}(-\mathbf{A})\underline{1}\,\underline{\alpha}'\, e^{\mathbf{B}y_1}(-\mathbf{A})\underline{1}$$
$$\times dy_1\cdots dy_{k-1}\,dy_k\Bigg\} + \big[M_X(se^{-\delta t})-1\big]\underline{\alpha}'\,(-\mathbf{A})\,\underline{1}\,\underline{\alpha}'\, e^{\mathbf{B}t}(-\mathbf{A})\underline{1}$$
$$+\big[M_X(se^{-\delta t})-1\big]\,\underline{\alpha}'\,(-\mathbf{A})\,\underline{1}\, f(t;s)\,. \tag{2.36}$$

By (2.22) and (2.20), (2.36) implies (2.35) holds. □

Next we are ready to prove that the mgf $Z(t)$, for mixed exponential inter–arrival times, also satisfies an homogeneous differential equation.

Theorem 2.3.2. *If the inter–arrival times are mixed exponential of order 2, then the mgf of $Z(t)$ satisfies:*

$$\frac{\partial^2}{\partial t^2}M_{Z(t)}(s) = a_1(t)\frac{\partial}{\partial t}M_{Z(t)}(s) + a_0(t)M_{Z(t)}(s)\,, \qquad t \geq 0\,,\ s \in \Omega\,, \tag{2.37}$$

with initial values $M_{Z(0)}(s) = 1$ and $\frac{\partial}{\partial t}M_{Z(t)}(s)|_{t=0} = \big[M_X(s) - 1\big]\underline{\alpha}'(-\boldsymbol{A})\underline{1}$, where

$$\begin{aligned}
a_1(t) &= \frac{\frac{\partial}{\partial t}\big[M_X(se^{-\delta t}) - 1\big]}{\big[M_X(se^{-\delta t}) - 1\big]} + \big[M_X(se^{-\delta t}) - 1\big]\underline{\alpha}'(-\boldsymbol{A})\underline{1} - \big[(1-a)\lambda_1 + a\lambda_2\big]\,,\\
a_0(t) &= \lambda_1\lambda_2\big[M_X(se^{-\delta t}) - 1\big]\,.
\end{aligned}$$

Proof. Substituting (2.34) and (2.35) into (2.32) and combining (2.23) gives (2.37). □

Consider PH inter–arrival times, if the order of matrix **A** is 2, then we obtain second–order homogeneous differential equations for the mgf of $Z(t)$. This result will be discussed in the next subsection.

2.3.3 PH Inter–arrival Times of Order 2

Let a PH random variable represent the inter–arrival times with parameters ($\underline{\alpha}$, $\mathbf{A}$), such that

$$\underline{\alpha}' = \begin{pmatrix} a & 1-a \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{2.38}$$

then $\mathbf{B} = \mathbf{A}(\mathbf{I} - \underline{1}\underline{\alpha}')$ can be diagonalized, as is proved in the following lemma.

Lemma 2.3.3. *If the PH inter–arrival times have parameters given by (2.38), then matrix $\boldsymbol{B}$ has two independent eigenvectors.*

Proof. Since

$$\begin{aligned}
\mathbf{B} &= \mathbf{A}(\mathbf{I} - \underline{1}\underline{\alpha}') = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1-a & -1+a \\ -a & a \end{pmatrix}\\
&= \begin{pmatrix} (1-a)a_{11} - a\,a_{12} & (-1+a)a_{11} + a\,a_{12} \\ (1-a)a_{21} - a\,a_{22} & (-1+a)a_{21} + a\,a_{22} \end{pmatrix},
\end{aligned}$$

then the characteristic polynomial is given by

$$\det(\lambda\mathbf{I} - \mathbf{B}) = \lambda^2 - \big[a_{11} - a_{21} + a\,(a_{21} + a_{22} - a_{11} - a_{12})\big]\lambda\,,$$

which implies $\lambda_1 = 0$ and $\lambda_2 = a_{11} - a_{21} + a\,(a_{21} + a_{22} - a_{11} - a_{12})$ are eigenvalues of $\mathbf{B}$. Now $a_{11} < 0\,, a_{22} < 0$ and $a_{12} > 0\,, a_{21} > 0$, then $a_{11} - a_{21} + a\,(a_{21} + a_{22} - a_{11} - a_{12}) < 0$. Hence $\mathbf{B}$ has two different eigenvalues, which implies that eigenvectors, corresponding to eigenvalues $a_{11} - a_{21} + a\,(a_{21} + a_{22} - a_{11} - a_{12})$ and 0 are independent. $\square$

Remark 2.2. Lemma 2.3.3 not only demonstrates that $\mathbf{B}$ has two distinct eigenvalues with one being 0, but it also implies that $\mathbf{B}$ can be diagonalized, since it has two independent eigenvectors, such that

$$\begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{P}^{-1}\mathbf{B}\mathbf{P} \Rightarrow \mathbf{B} = \mathbf{P} \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \mathbf{P}^{-1}\,,$$

where $\lambda = a_{11} - a_{21} + a\,(a_{21} + a_{22} - a_{11} - a_{12})$ is the nonzero eigenvalue and the columns of $\mathbf{P}$ are eigenvectors, corresponding to eigenvalues λ and 0.

By Definition A.1.4 of the matrix exponential, we have that

$$e^{Bt} = \mathbf{P} \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{P}^{-1}\,,$$

then there exists two numbers c_{11} and c_{12}, such that

$$m'(t) = \underline{\alpha}'\, e^{\mathbf{B}t}(-\mathbf{A})\underline{1} = \underline{\alpha}'\mathbf{P} \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{P}^{-1}(-\mathbf{A})\underline{1} = c_{11}e^{\lambda t} + c_{12}\,.$$

Similar arguments can be also applied to $\underline{\alpha}'\,\mathbf{B}e^{\mathbf{B}t}(-\mathbf{A})\underline{1}$, then

$$\underline{\alpha}'\,\mathbf{B}e^{\mathbf{B}t}(-\mathbf{A})\underline{1} = \underline{\alpha}'\mathbf{P} \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{P}^{-1}(-\mathbf{A})\underline{1} = c_{21}e^{\lambda t}\,,$$

where c_{21} is a constant number. Hence there exists $c^\star = \frac{c_{21}}{c_{11}}$ and $c^{\star\star} = -\frac{c_{21}c_{12}}{c_{11}}$, such that

$$\underline{\alpha}'\,\mathbf{B}e^{\mathbf{B}t}(-\mathbf{A})\underline{1} = c^\star\,\underline{\alpha}'\,e^{\mathbf{B}t}(-\mathbf{A})\underline{1} + c^{\star\star}\,. \tag{2.39}$$

Lemma 2.3.4. *If PH inter–arrival times have parameters given by (2.38), then*

$$\begin{aligned} \frac{\partial}{\partial t}f(t;s) \;=\; & \Big\{\big[M_X(se^{-\delta t}) - 1\big]\underline{\alpha}'\,(-\boldsymbol{A})\underline{1} + c^\star\Big\}f(t;s) \\ & + c^{\star\star}\,M_{Z(t)}(s) + \big[M_X(se^{-\delta t}) - 1\big]\underline{\alpha}'\,(-\boldsymbol{A})\underline{1}\underline{\alpha}'\,e^{\boldsymbol{B}t}(-\boldsymbol{A})\underline{1} - c^{\star\star}\,. \end{aligned} \tag{2.40}$$

Proof. Substituting (2.39) into (2.26) gives

$$\begin{aligned}\frac{\partial}{\partial t} f(t;s) &= c^{\star}\Bigg\{\sum_{k=1}^{\infty}\int_0^t\int_0^{y_k}\cdots\int_0^{y_2}\prod_{i=1}^{k}\Big[M_X\big(se^{-\delta y_i}\big)-1\Big]\underline{\alpha}'\,e^{\mathbf{B}(t-y_k)}(-\mathbf{A})\underline{1}\\ &\quad\times\underline{\alpha}'\,e^{\mathbf{B}(y_k-y_{k-1})}(-\mathbf{A})\underline{1}\cdots\underline{\alpha}'\,e^{\mathbf{B}(y_2-y_1)}(-\mathbf{A})\underline{1}\,\underline{\alpha}'\,e^{\mathbf{B}y_1}(-\mathbf{A})\underline{1}\\ &\quad\times dy_1\cdots dy_{k-1}\,dy_k\Bigg\}+c^{\star\star}\Bigg\{\sum_{k=1}^{\infty}\int_0^t\int_0^{y_k}\cdots\int_0^{y_2}\prod_{i=1}^{k}\Big[M_X\big(se^{-\delta y_i}\big)-1\Big]\\ &\quad\times\underline{\alpha}'\,e^{\mathbf{B}(y_k-y_{k-1})}(-\mathbf{A})\underline{1}\cdots\underline{\alpha}'\,e^{\mathbf{B}(y_2-y_1)}(-\mathbf{A})\underline{1}\,\underline{\alpha}'\,e^{\mathbf{B}y_1}(-\mathbf{A})\underline{1}\\ &\quad\times dy_1\cdots dy_{k-1}\,dy_k\Bigg\}+\big[M_X(se^{-\delta t})-1\big]\underline{\alpha}'\,(-\mathbf{A})\,\underline{1}\,\underline{\alpha}'\,e^{\mathbf{B}t}(-\mathbf{A})\underline{1}\\ &\quad+\big[M_X(se^{\delta t})-1\big]\,\underline{\alpha}'\,(-\mathbf{A})\,\underline{1}\,f(t;s)\,. \end{aligned} \tag{2.41}$$

Combining (2.20) with (2.41) yields (2.40). □

Theorem 2.3.3. *If inter–arrival times are PH with parameters given by (2.38), then the mgf of $Z(t)$ satisfies:*

$$\frac{\partial^2}{\partial t^2}M_{Z(t)}(s)=a_1(t)\frac{\partial}{\partial t}M_{Z(t)}(s)+a_0(t)M_{Z(t)}(s)\,,\qquad t\geq 0\,,\ s\in\Omega\,, \tag{2.42}$$

with initial values $M_{Z(0)}(s)=1$ and $\frac{\partial}{\partial t}M_{Z(t)}(s)|_{t=0}=\big[M_X(s)-1\big]\underline{\alpha}'(-\boldsymbol{A})\underline{1}$, where

$$\begin{aligned} a_1(t) &= \frac{\frac{\partial}{\partial t}\big[M_X(se^{-\delta t})-1\big]}{\big[M_X(se^{-\delta t})-1\big]}+\big[M_X(se^{-\delta t})-1\big]\underline{\alpha}'\,(-\boldsymbol{A})\underline{1}+c^{\star}\,,\\ a_0(t) &= c^{\star\star}\,\big[M_X(se^{-\delta t})-1\big]\,. \end{aligned}$$

Proof. Substituting (2.39) and (2.40) into (2.32) and combining (2.23) gives (2.42). □

2.3.4 PH Inter–arrival Times of Order $n \geq 3$

Let

$$\begin{aligned} f^{\star}(t;s) &= \sum_{k=1}^{\infty}\int_0^t\int_0^{y_k}\cdots\int_0^{y_2}e^{\mathbf{B}(t-y_k)}(-\mathbf{A})\underline{1}\prod_{i=1}^{k}\Big[M_X\big(se^{-\delta y_i}\big)-1\Big]\underline{\alpha}'\,e^{\mathbf{B}(y_i-y_{i-1})}(-\mathbf{A})\underline{1}\\ &\quad\times dy_1\cdots dy_{k-1}\,dy_k\,,\qquad t>0\,,s\in\Omega\,, \end{aligned} \tag{2.43}$$

then (2.22) can be written as:

$$f(t;s)=\underline{\alpha}'f^{\star}(t;s)\,.$$

In order to obtain the mgf of $Z(t)$ when the inter–arrival time has a PH distribution, we need the following result.

Lemma 2.3.5. *For any $t \geq 0, \delta \geq 0$ and $s \in \Omega$, $f^\star$ satisfies the following differential equations:*

$$\frac{\partial}{\partial t} f^\star(t;s) = \Big\{ \boldsymbol{B} - \big[M_X(se^{-\delta t}) - 1\big]\, \boldsymbol{A}\underline{1}\,\underline{\alpha}' \Big\} f^\star(t;s) + \boldsymbol{A}\underline{1}\,\underline{\alpha}'\, e^{\boldsymbol{B}t}\boldsymbol{A}\underline{1}\big[M_X(se^{-\delta t}) - 1\big]\,. \quad (2.44)$$

Further, (2.44) has unique continuous solution in $[0,\infty)$ with initial value $f^\star(0,s) = \underline{0}$.

Proof. Differentiating both sides of (2.43) with respect to t for $s \in \Omega$ yields

$$\begin{aligned}
\frac{\partial}{\partial t} f^\star(t;s) \;=\;& \mathbf{B}e^{\mathbf{B}t}\Big\{ \sum_{k=1}^{\infty} \int_0^t \int_0^{y_k} \cdots \int_0^{y_2} e^{-\mathbf{B}y_k}(-\mathbf{A})\underline{1} \prod_{i=1}^{k} \Big[M_X\big(se^{-\delta y_i}\big) - 1\Big] \\
& \times \Big[\underline{\alpha}'\, e^{\mathbf{B}(y_i - y_{i-1})}(-\mathbf{A})\underline{1}\Big] dy_1 \cdots dy_{k-1}\, dy_k \Big\} \\
& + e^{\mathbf{B}t}\Big\{ \big[M_X(se^{-\delta t}) - 1\big] e^{-\mathbf{B}t}\,(-\mathbf{A})\,\underline{\mathbf{1}}\,\underline{\alpha}'\, e^{\mathbf{B}t}(-\mathbf{A})\underline{1} \\
& + \big[M_X(se^{\delta t}) - 1\big] \sum_{k=2}^{\infty} \int_0^t \int_0^{y_{k-1}} \cdots \int_0^{y_2} e^{-\mathbf{B}t}(-\mathbf{A})\underline{1}\,\underline{\alpha}'\, e^{\mathbf{B}(t - y_{k-1})}(-\mathbf{A})\underline{1} \\
& \times \prod_{i=1}^{k-1} \Big[M_X\big(se^{-\delta y_i}\big) - 1\Big] \underline{\alpha}'\, e^{\mathbf{B}(y_i - y_{i-1})}(-\mathbf{A})\underline{1}\, dy_1 \cdots dy_{k-2}\, dy_{k-1} \Big\}. \quad (2.45)
\end{aligned}$$

Simplifying (2.45), we have that

$$\begin{aligned}
\frac{\partial}{\partial t} f^\star(t;s) \;=\;& \mathbf{B}\Big\{ \sum_{k=1}^{\infty} \int_0^t \int_0^{y_k} \cdots \int_0^{y_2} e^{\mathbf{B}(t-y_k)}(-\mathbf{A})\underline{1} \prod_{i=1}^{k} \Big[M_X\big(se^{-\delta y_i}\big) - 1\Big] \\
& \times \Big[\underline{\alpha}'\, e^{\mathbf{B}(y_i - y_{i-1})}(-\mathbf{A})\underline{1}\Big] dy_1 \cdots dy_{k-1}\, dy_k \Big\} \\
& + \big[M_X(se^{-\delta t}) - 1\big](-\mathbf{A})\,\underline{1}\,\underline{\alpha}'\, e^{\mathbf{B}t}(-\mathbf{A})\underline{1} + \big[M_X(se^{-\delta t}) - 1\big] \\
& \times \Big\{ \sum_{k=2}^{\infty} \int_0^t \int_0^{y_{k-1}} \cdots \int_0^{y_2} (-\mathbf{A})\underline{1}\,\underline{\alpha}' e^{\mathbf{B}(t - y_{k-1})} \prod_{i=1}^{k-1} \Big[M_X\big(se^{-\delta y_i}\big) - 1\Big] \\
& \times \Big[\underline{\alpha}'\, e^{\mathbf{B}(y_i - y_{i-1})}(-\mathbf{A})\underline{1}\Big]\, dy_1 \cdots dy_{k-2}\, dy_{k-1} \Big\}, \qquad t > 0, s \in \Omega. \quad (2.46)
\end{aligned}$$

Substituting (2.43) into (2.46) yields

$$\begin{aligned}
\frac{\partial}{\partial t} f^\star(t;s) \;=\;& \mathbf{B}\, f^*(t;s) + \mathbf{A}\underline{1}\,\underline{\alpha}'\, e^{\mathbf{B}t}\mathbf{A}\underline{1}\big[M_X(se^{-\delta t}) - 1\big] \\
& - \big[M_X(se^{-\delta t}) - 1\big]\mathbf{A}\underline{1}\,\underline{\alpha}' f^\star(t;s)\,. \quad (2.47)
\end{aligned}$$

From (2.47) we have equation (2.44).

Since here we consider continuous PH distributions for the inter–arrival times and the PH claim severities, then the uniqueness of the solution of (2.44) in Lemma 2.3.5 with initial value $f^\star(0;s) = \underline{0}$, follows the results of ordinary differential equations (see Bellman, 1997). $\square$

Consider equation (2.23) and Lemma 2.3.5, then the following differential equations are obtained for the mgf of $Z(t)$:

Theorem 2.3.4. *If the inter–arrival times are PH distributed with parameters* $(\underline{\alpha}, \boldsymbol{A})$*, and* X *is the claim severity, then for any* $t \geq 0\,, \delta \geq 0$ *and* $s \in \Omega$*, the mgf of* $Z(t)$ *satisfies the following equations:*

$$\begin{aligned}
\frac{\partial}{\partial t} M_{Z(t)}(s) &= \big[M_X(se^{-\delta t}) - 1\big]\underline{\alpha}'\, e^{\boldsymbol{B}t}(-\boldsymbol{A})\underline{1} + \big[M_X(se^{-\delta t}) - 1\big]\, f(t,s)\,,\\
f(t;s) &= \underline{\alpha}'\, f^\star(t;s)\,,
\end{aligned}$$

with initial value $M_{Z(0)}(s) = 1$ *and where* $f^\star(t;s)$ *is given by*

$$\frac{\partial}{\partial t} f^\star(t;s) = \Big\{\boldsymbol{B} - \big[M_X(se^{-\delta t}) - 1\big]\,\boldsymbol{A}\underline{1}\,\underline{\alpha}'\Big\} f^\star(t;s) + \boldsymbol{A}\underline{\boldsymbol{1}}\,\underline{\alpha}'\, e^{\boldsymbol{B}t}\boldsymbol{A}\underline{1}\big[M_X(se^{-\delta t}) - 1\big]\,, \quad (2.48)$$

with initial value $f^\star(0;s) = \underline{0}$*.*

Remark 2.3. For general PH distributions the above equations are difficult to simplify further. For special cases such as the Erlang(n) and PH distributions of order 2, we can further simplify the results, but if the order of matrix **A** is large with many parameters, the relation between $f(t;s)$ and its derivatives is not clear. Instead of resorting to homogeneous differential equations or ordinary differential equations we use ordinary differential systems.

The following corollaries are already well known result; see for instance Gerber (1971), Karlin and Taylor (1975) and Willmot (1989).

Corollary 2.3.2. *If inter–arrival times have an exponential distribution with parameter* λ*, then the mgf of* $Z(t)$ *is given by:*

$$M_{Z(t)}(s) = e^{\lambda \int_0^t [M_X(se^{-\delta u}) - 1]du}\,, \qquad s \in \Omega\,, \tag{2.49}$$

where M_X *is the mgf of the claim severity* X*.*

Proof. Inter–arrival times are exponential with parameter λ, hence $\mathbf{A} = -\lambda$, with vectors $\alpha = 1$ and $\underline{1} = 1$ which implies that matrix $\mathbf{B} = 0$, then $f^{\star}(t; s) = f(t; s)$. From Theorem 2.3.4 we have that

$$\frac{\partial}{\partial t} f(t; s) = \lambda \big[M_X(se^{-\delta t}) - 1\big] f(t; s) + \lambda^2 \left[M_X(se^{-\delta t}) - 1\right], \tag{2.50}$$

and

$$\frac{\partial}{\partial t} M_{Z(t)}(s) = \big[M_X(se^{-\delta t}) - 1\big] f(t; s) + \lambda \left[M_X(se^{-\delta t}) - 1\right]. \tag{2.51}$$

Hence

$$\frac{\partial}{\partial t} f(t; s) = \lambda \frac{\partial}{\partial t} M_{Z(t)}(s), \tag{2.52}$$

then

$$f(t; s) = \lambda\, M_{Z(t)}(s) + c(s), \tag{2.53}$$

where $c(s)$ is a function in s. The initial values $M_{Z(0)}(s) = 1$ and $f(0; s) = 0$ give $c(s) = -\lambda$. The solution of first-order differential equations yields the solution of (2.50) given by:

$$f(t; s) = \lambda e^{\lambda \int_0^t [M_X(se^{-\delta u}) - 1] du} - \lambda, \tag{2.54}$$

with initial value of 0. Substituting (2.54) into (2.53) yields

$$M_{Z(t)}(s) = e^{\lambda \int_0^t [M_X(se^{-\delta u}) - 1] du} - 1 - \frac{1}{\lambda} c(s). \tag{2.55}$$

$c(s) = \lambda$ implies the result:

$$M_{Z(t)}(s) = e^{\lambda \int_0^t [M_X(se^{-\delta u}) - 1] du}.$$

□

Corollary 2.3.3. *If the inter–arrival times are Erlang(n), then the n-th derivative of the mgf $M_{Z(t)}(s)$ with respect to t is given by Wang (2007, p41):*

$$\begin{aligned} \frac{\partial^n}{\partial t^n} M_{Z(t)}(s) &= a_{n-1}(t) \frac{\partial^{n-1}}{\partial t^{n-1}} M_{Z(t)}(s) + a_{n-2}(t) \frac{\partial^{n-2}}{\partial t^{n-2}} M_{Z(t)}(s) + \cdots \\ &\quad + a_1(t) \frac{\partial}{\partial t} M_{Z(t)}(s) + a_0(t) M_{Z(t)}(s), \end{aligned} \tag{2.56}$$

with initial values

$$M_{Z(0)}(s) = 1, \frac{\partial}{\partial t} M_{Z(t)}(s)|_{t=0} = 0, \frac{\partial^2}{\partial t^2} M_{Z(t)}(s)|_{t=0} = 0, \cdots, \frac{\partial^{n-1}}{\partial t^{n-1}} M_{Z(t)}(s)|_{t=0} = 0,$$

where, for a fixed s,

$$a_k(t) = \frac{\binom{n-1}{n-k}\frac{\partial^{n-k}}{\partial t^{n-k}}M(t,s) - \binom{n}{n-k}\lambda^{n-k}M(t,s) - \sum_{i=1}^{(n-2)-(k-1)} a_{k+i}(t)\binom{k+i-1}{i}\frac{\partial^i}{\partial t^i}M(t,s)}{M(t,s)},$$

with $M(t,s) = \left[M_X(se^{-\delta t}) - 1\right]$, *for* $k = 1, 2, \cdots, n-1$ *and* $a_0(t) = \lambda^n M(t,s)$.

Note that there are 3 lemmas needed in order to prove this result in Wang (2007). Here we prove them for completeness, beginning with the second one. The proof of the first one is trivial.

Lemma 2.3.6. *If the inter–arrival times have Erlang(n) distributions with parameters* $(\underline{\alpha}, \boldsymbol{A})$*, then*

$$\underline{\alpha}'\,\boldsymbol{B}^k\,(-\boldsymbol{A})\underline{1} = 0\,, \qquad \textit{for } k \le n-2\,, \tag{2.57}$$

where $\boldsymbol{B} = \boldsymbol{A}(\boldsymbol{I} - \underline{1}\,\underline{\alpha}')$.

For the proof see Wang (2007, p42)

Lemma 2.3.7. *Let the inter–arrival times have Erlang*(n) *distributions with parameters* $(\underline{\alpha}, \boldsymbol{A})$*, then:*

$$\underline{\alpha}'\frac{\partial^{n-1}}{\partial t^{n-1}}f^\star(t,s) = \underline{\alpha}'\boldsymbol{B}^{n-1}f^\star(t;s)\,. \tag{2.58}$$

Proof. From Theorem 2.3.4

$$\frac{\partial}{\partial t}f^\star(t;s) = \left\{\mathbf{B} - \left[M_X(se^{-\delta t}) - 1\right]\mathbf{A}\underline{1}\,\underline{\alpha}'\right\}f^\star(t;s) + \mathbf{A}\underline{\mathbf{1}}\,\underline{\alpha}'\,e^{\mathbf{B}t}\mathbf{A}\underline{1}\left[M_X(se^{-\delta t}) - 1\right]. \tag{2.59}$$

Multiplying (2.59) on the right by $\underline{\alpha}'$ gives:

$$\begin{aligned}\underline{\alpha}'\frac{\partial}{\partial t}f^\star(t;s) \quad &= \quad \underline{\alpha}'\mathbf{B}f^\star(t;s) + \left[M_X(se^{-\delta t}) - 1\right]\underline{\alpha}'(-\mathbf{A})\underline{1}\,\underline{\alpha}'f^\star(t;s)\\ &\quad + \left[M_X(se^{-\delta t}) - 1\right]\underline{\alpha}'(-\mathbf{A})\underline{\mathbf{1}}\,\underline{\alpha}'\,e^{\mathbf{B}t}(-\mathbf{A})\underline{1}\,.\end{aligned} \tag{2.60}$$

Since $\underline{\alpha}'(-\mathbf{A})\underline{\mathbf{1}} = 0$, hence

$$\underline{\alpha}'\frac{\partial}{\partial t}f^\star(t;s) = \underline{\alpha}'\mathbf{B}f^\star(t;s)\,. \tag{2.61}$$

Differentiating (2.61) with respect to t yields:

$$\underline{\alpha}'\frac{\partial^2}{\partial t^2}f^\star(t;s) = \underline{\alpha}'\mathbf{B}\frac{\partial}{\partial t}f^\star(t;s)\,. \tag{2.62}$$

By (2.59) and Lemma 2.3.6, (2.62) can be written as:

$$\underline{\alpha}' \frac{\partial^2}{\partial t^2} f^\star(t;s) = \underline{\alpha}' \mathbf{B}^2 f^\star(t;s)\,. \tag{2.63}$$

Repeating the procedure with higher order derivatives in (2.62) and (2.63) we have

$$\underline{\alpha}' \frac{\partial^{n-1}}{\partial t^{n-1}} f^\star(t;s) = \underline{\alpha}' \mathbf{B}^{n-1} f^\star(t;s)\,. \tag{2.64}$$

□

Lemma 2.3.8. $(\underline{\alpha}, \boldsymbol{A})$ *are the parameters for Erlang(n) distributions and* $\boldsymbol{B} = \boldsymbol{A}(\boldsymbol{I} - \underline{1}\,\underline{\alpha}')$ *then*

$$\underline{\alpha}' e^{\boldsymbol{B}x} \boldsymbol{B}^{n-1} (-\boldsymbol{A})\underline{1} = \lambda^n - \sum_{k=1}^{n-1} \lambda^k \binom{n}{k} \underline{\alpha}' e^{\boldsymbol{B}x} \boldsymbol{B}^{n-1-k} (-\boldsymbol{A})\underline{1}\,. \tag{2.65}$$

For a proof see Wang (2007, p43).

Lemma 2.3.9. *If the inter–arrival times are Erlang(n), then* $(n-1)$*-th derivative of* f *in* t *is given by*

$$\frac{\partial^{n-1}}{\partial t^{n-1}} f(t;s) = -\sum_{k=1}^{n-1} \lambda^k \binom{n}{k} \frac{\partial^{n-1-k}}{\partial t^{n-1-k}} f(t;s) + \lambda^n M_{Z(t)}(s) - \lambda^n\,. \tag{2.66}$$

The proof of Corollary 2.3.2 follows from Lemmas 2.3.7, 2.3.8 and 2.3.9, for details please refer to Wang (2007).

Now consider the classical renewal compound risk model under the condition that $\delta = 0$. A closed form is obtained for the mgf of the compound sum when the inter–arrival times are PH distributed. In order to prove the result, first we need the following result.

Lemma 2.3.10. *Let* $\underline{y}(t)$ *be a vector with order* n *and each component be a function of* t*. If*

$$\frac{d}{dt}\underline{y}(t) = \boldsymbol{D}\underline{y}(t) + \underline{b}(t)\,, \tag{2.67}$$

with initial value $\underline{0}$*, then the solution is given by:*

$$\underline{y}(t) = e^{\boldsymbol{D}t} \int_0^t e^{-\boldsymbol{D}x}\, \underline{b}(x)dx\,, \tag{2.68}$$

where $\boldsymbol{D}$ *is a square matrix of order* n*, and* $\underline{b}(t)$ *is vector of dimension* n*.*

Proof. The definition of matrix exponential gives the derivative of $e^{\mathbf{D}t}$ to be:

$$\frac{d}{dt}\, e^{\mathbf{D}t} = \mathbf{D}e^{\mathbf{D}t} = e^{\mathbf{D}t}\mathbf{D}\,,$$

hence differentiating both sides of (2.68) with respect to t gives

$$\begin{aligned} \frac{d}{dt}\,\underline{y}(t) &= \mathbf{D}\,e^{\mathbf{D}t}\int_0^t e^{-\mathbf{D}x}\,\underline{b}(x)dx + e^{\mathbf{D}t}e^{-\mathbf{D}t}\underline{b}(t) \\ &= \mathbf{D}\,\underline{y}(t) + \underline{b}(t)\,. \end{aligned} \tag{2.69}$$

□

Corollary 2.3.4. *If the inter–arrival times have PH* $(\underline{\alpha}, \boldsymbol{A})$ *and the net interest force* $\delta = 0$, *then for any* $t \geq 0$ *and* $s \in \Omega$, *the moment generating function of* $Z(t)$ *is given by:*

$$\begin{aligned} M_{Z(t)}(s) &= m(t)\left[M_X(s) - 1\right] + \left[M_X(s)-1\right]^2 \Bigg(\underline{\alpha}'\boldsymbol{B}^{*-1}e^{\boldsymbol{B}^*t}(\boldsymbol{I}\otimes\underline{\alpha}')\big[(-\boldsymbol{B}^*)\oplus\boldsymbol{B}\big]^{-1} \\ &\quad \times\Big[e^{((-\boldsymbol{B}^*)\oplus\boldsymbol{B})t} - \boldsymbol{I}\big[(\boldsymbol{A}\underline{1})\otimes(\boldsymbol{A}\underline{1})\big] + m(t)\underline{\alpha}'\boldsymbol{B}^{*-1}\boldsymbol{A}\underline{1}\Bigg) + 1\,, \end{aligned} \tag{2.70}$$

where $m(t)$ *is a renewal function,* M_X *is the mgf of the claim severity variable* X *and* $\boldsymbol{B}^* = \boldsymbol{B} - \big[M_X(s) - 1\big]\,\boldsymbol{A}\underline{1}\,\underline{\alpha}'$.

Proof. If $\delta = 0$ by Theorem 2.3.4 we have that:

$$\frac{\partial}{\partial t} f^\star(t;s) = \Big\{\mathbf{B} - \big[M_X(s)-1\big]\,\mathbf{A}\underline{1}\,\underline{\alpha}'\Big\} f^\star(t;s) + \big[M_X(s)-1\big]\,\mathbf{A}\underline{\mathbf{1}}\,\underline{\alpha}'\,e^{\mathbf{B}t}\mathbf{A}\underline{1}\,.$$

Let $\mathbf{B}^* = \mathbf{B} - \big[M_X(s)-1\big]\,\mathbf{A}\underline{1}\,\underline{\alpha}'$ and $\underline{b}(t) = \big[M_X(s)-1\big]\,\mathbf{A}\underline{\mathbf{1}}\,\underline{\alpha}'\,e^{\mathbf{B}t}\mathbf{A}\underline{1}$, then

$$\frac{\partial}{\partial t} f^\star(t;s) = \mathbf{B}^* f^\star(t;s) + \underline{b}(t)\,. \tag{2.71}$$

Then by Lemma 2.3.10 the solution of (2.71) is given by:

$$f^\star(t;s) = e^{\mathbf{B}^*t}\int_0^t e^{\mathbf{B}^*x}\,\underline{b}(x)dx\,. \tag{2.72}$$

It follows that $f(t;s) = \underline{\alpha}' f^\star(t;s) = \underline{\alpha}' e^{\mathbf{B}^*\,t}\int_0^t e^{-\mathbf{B}^*\,x}\,\underline{b}(x)dx$. Hence from Theorem 2.3.4 we have the following expression:

$$\frac{\partial}{\partial t} M_{Z(t)}(s) = \left[M_X(s)-1\right]\underline{\alpha}'\,e^{\mathbf{B}t}(-\mathbf{A})\underline{1} + \left[M_X(s)-1\right]\underline{\alpha}'e^{\mathbf{B}^*t}\int_0^t e^{-\mathbf{B}^*x}\,\underline{b}(x)dx\,, \tag{2.73}$$

which implies that:

$$\begin{aligned} M_{Z(t)}(s) &= [M_X(s)-1]\int_0^t \underline{\alpha}'\, e^{\mathbf{B}x}(-\mathbf{A})\underline{1}dx \\ &\quad +[M_X(s)-1]\int_0^t \underline{\alpha}' e^{\mathbf{B}^* u}\int_0^u e^{-\mathbf{B}^* x}\,\underline{b}(x)dx\,du+1\,, \end{aligned} \tag{2.74}$$

since the initial value of $M_{Z(t)}(s)$ at $t=0$ is 1 and $m'(x)=\underline{\alpha}'\, e^{\mathbf{B}x}(-\mathbf{A})\underline{1}$ is a renewal density function, then the renewal function is given by

$$m(t)=\int_0^t m'(x)dx=\int_0^t \underline{\alpha}'\, e^{\mathbf{B}x}(-\mathbf{A})\underline{1}dx\,.$$

Now we consider $\int_0^t \underline{\alpha}' e^{\mathbf{B}^* u}\int_0^u e^{-\mathbf{B}^* x}\,\underline{b}(x)dx\,du$; changing the order of integration yields,

$$\begin{aligned} \int_0^t \underline{\alpha}' e^{\mathbf{B}^* u}\int_0^u e^{-\mathbf{B}^* x}\,\underline{b}(x)dx\,du &= \underline{\alpha}'\int_0^t\int_x^t e^{\mathbf{B}^* u}e^{-\mathbf{B}^* x}\,\underline{b}(x)\,du\,dx \\ &= \underline{\alpha}'\int_0^t e^{-\mathbf{B}^* x}\,\mathbf{B}^{*-1}e^{\mathbf{B}^* u}\,|_x^t\,\underline{b}(x)\,dx \\ &= \underline{\alpha}'\int_0^t \mathbf{B}^{*-1}\big(e^{\mathbf{B}^* t}-e^{\mathbf{B}^* x}\big)e^{-\mathbf{B}^* x}\,\underline{b}(x)\,dx\,. \end{aligned} \tag{2.75}$$

Simplifying (2.75) gives

$$\int_0^t \underline{\alpha}' e^{\mathbf{B}^* u}\int_0^u e^{-\mathbf{B}^* x}\,\underline{b}(x)dx\,du=\underline{\alpha}'\mathbf{B}^{*-1}e^{\mathbf{B}^* t}\int_0^t e^{-\mathbf{B}^* x}\,\underline{b}(x)\,dx-\underline{\alpha}'\mathbf{B}^{*-1}\int_0^t \underline{b}(x)\,dx\,. \tag{2.76}$$

Substituting $\underline{b}(t)=\big[M_X(s)-1\big]\,\mathbf{A}\underline{1}\,\underline{\alpha}'\,e^{\mathbf{B}t}\mathbf{A}\underline{1}$ into (2.76) yields

$$\begin{aligned} \int_0^t \underline{\alpha}' e^{\mathbf{B}^* u}\int_0^u e^{-\mathbf{B}^* x}\,\underline{b}(x)dx\,du &= \big[M_X(s)-1\big]\underline{\alpha}'\mathbf{B}^{*-1}e^{\mathbf{B}^* t}\int_0^t e^{-\mathbf{B}^* x}\mathbf{A}\underline{1}\,\underline{\alpha}'\,e^{\mathbf{B}x}\mathbf{A}\underline{1}\,dx \\ &\quad -\big[M_X(s)-1\big]\underline{\alpha}'\mathbf{B}^{*-1}\int_0^t \mathbf{A}\underline{1}\,\underline{\alpha}'\,e^{\mathbf{B}x}\mathbf{A}\underline{1}\,dx\,. \end{aligned} \tag{2.77}$$

Then using the properties of Kronecker's product $\otimes$ and sum $\oplus$ (see Definitions A.3.1 and A.3.2, as well as Properties A.3.1 and A.3.2):

$$(\mathbf{U}_1\mathbf{U}_2\cdots\mathbf{U}_n)\otimes(\mathbf{V}_1\mathbf{V}_2\cdots\mathbf{V}_n)=(\mathbf{U}_1\otimes\mathbf{V}_1)(\mathbf{U}_2\otimes\mathbf{V}_2)\cdots(\mathbf{U}_n\otimes\mathbf{V}_n)\,,\qquad n\geq 1\,,$$

and

$$\exp(\mathbf{U})\otimes\exp(\mathbf{V})=\exp(\mathbf{U}\oplus\mathbf{V})\,.$$

Since $m'(x) = \underline{\alpha}' \, e^{\mathbf{B}x}(-\mathbf{A})\underline{1}$ is a number, hence:

$$
\begin{aligned}
&\int_0^t \underline{\alpha}' e^{\mathbf{B}^* u} \int_0^u \; e^{-\mathbf{B}^* x} \, \underline{b}(x) dx \, du \\
&\quad = \big[M_X(s) - 1\big]\underline{\alpha}'\mathbf{B}^{*-1} e^{\mathbf{B}^* t} \int_0^t (\mathbf{I} \otimes \underline{\alpha}')(e^{-\mathbf{B}^* x} \otimes e^{\mathbf{B}x})(\mathbf{A}\underline{1} \otimes \mathbf{A}\underline{1}) dx \\
&\qquad + m(t)\big[M_X(s) - 1\big]\underline{\alpha}'\mathbf{B}^{*-1}\mathbf{A}\underline{1} \\
&\quad = \big[M_X(s) - 1\big]\underline{\alpha}'\mathbf{B}^{*-1} e^{\mathbf{B}^* t} \int_0^t (\mathbf{I} \otimes \underline{\alpha}') e^{((-\mathbf{B}^*)\oplus\mathbf{B})x}(\mathbf{A}\underline{1} \otimes \mathbf{A}\underline{1}) dx \\
&\qquad + m(t)\big[M_X(s) - 1\big]\underline{\alpha}'\mathbf{B}^{*-1}\mathbf{A}\underline{1} \, .
\end{aligned}
\tag{2.78}
$$

Then

$$
\begin{aligned}
&\int_0^t \underline{\alpha}' e^{\mathbf{B}^* u} \int_0^u \; e^{-\mathbf{B}^* x} \, \underline{b}(x) dx \, du \\
&\quad = \big[M_X(s) - 1\big]\underline{\alpha}'\mathbf{B}^{*-1} e^{\mathbf{B}^* t}(\mathbf{I} \otimes \underline{\alpha}')\big[(-\mathbf{B}^*) \oplus \mathbf{B}\big]^{-1}\big(e^{((-\mathbf{B}^*)\oplus\mathbf{B})x}\big|_0^t\big)\,(\mathbf{A}\underline{1} \otimes \mathbf{A}\underline{1}) \\
&\qquad + m(t)\big[M_X(s) - 1\big]\underline{\alpha}'\mathbf{B}^{*-1}\mathbf{A}\underline{1} \\
&\quad = \big[M_X(s) - 1\big]\underline{\alpha}'\mathbf{B}^{*-1} e^{\mathbf{B}^* t}(\mathbf{I} \otimes \underline{\alpha}')\big[(-\mathbf{B}^*) \oplus \mathbf{B}\big]^{-1}\Big[e^{((-\mathbf{B}^*)\oplus\mathbf{B})t} - \mathbf{I}\Big](\mathbf{A}\underline{1} \otimes \mathbf{A}\underline{1}) \\
&\qquad + m(t)\big[M_X(s) - 1\big]\underline{\alpha}'\mathbf{B}^{*-1}\mathbf{A}\underline{1} \, .
\end{aligned}
\tag{2.79}
$$

Substituting (2.79) into (2.74) gives

$$
\begin{aligned}
M_{Z(t)}(s) \;&=\; m(t)\,[M_X(s) - 1] + [M_X(s) - 1]^2 \bigg(\underline{\alpha}'\mathbf{B}^{*-1} e^{\mathbf{B}^* t}(\mathbf{I} \otimes \underline{\alpha}')\big[(-\mathbf{B}^*) \oplus \mathbf{B}\big]^{-1} \\
&\qquad \times \Big[e^{((-\mathbf{B}^*)\oplus\mathbf{B})t} - \mathbf{I}\Big]((\mathbf{A}\underline{1}) \otimes (\mathbf{A}\underline{1})) + m(t)\underline{\alpha}'\mathbf{B}^{*-1}\mathbf{A}\underline{1}\bigg) + 1 \, ,
\end{aligned}
\tag{2.80}
$$

that gives the proof of Corollary 2.3.3. □

Remark 2.4. Corollary 2.3.4 gives a closed formula for the mgf of compound PH–renewal processes, when the interest rate $\delta = 0$. There are a few special cases available in the literature for this mgf, such as exponential and Erlang(2) inter–arrival times. This generalizes classical risk models to any PH inter–arrival times.

For instance, if the inter–arrival times are Coxian PH distributed with parameters $(\underline{\alpha}, \mathbf{A})$ given by:

$$
\underline{\alpha}' = \begin{pmatrix} \alpha & 1-\alpha \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ 0 & -\lambda_2 \end{pmatrix},
$$

then from Corollary 2.3.3, the mgf is given by

$$M_{Z(t)}(s) = e^{-\frac{1}{2}at}\left(\frac{\big((-1+\alpha)M_X(s)-2\alpha+1\big)\lambda_2-\lambda_1}{\sqrt{b}}\sinh\frac{1}{2}t\sqrt{b}+\cosh\frac{1}{2}t\sqrt{b}\right),$$

where

$$\begin{aligned} a &= \lambda_2(-1+\alpha)M_X(s)+\lambda_2+\lambda_1\,, \quad \text{and} \\ b &= \big(1+(-1+\alpha)M_X(s)\big)^2\lambda_2^2+2\lambda_1\big(-1+(\alpha+1)M_X(s)\big)\lambda_2+\lambda_1^2\,. \end{aligned}$$

Instead, if we consider inter–arrival times have a PH distribution which is a mixture of exponentials with parameters $(\underline{\alpha}, \mathbf{A})$ given by:

$$\underline{\alpha}' = \begin{pmatrix} \alpha & 1-\alpha \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix},$$

then the mgf of $Z(t)$ is given by:

$$\begin{aligned} M_{Z(t)}(s) &= e^{-\frac{1}{2}ta}\Bigg(\frac{\big((2-M_X(s))\alpha+M_X(s)-1\big)\lambda_2+\big(1+(M_X(s)-2)\alpha\big)\lambda_1}{\sqrt{b}}\sinh(\frac{1}{2}t\sqrt{b}) \\ &\qquad +\cosh(\frac{1}{2}t\sqrt{b})\Bigg), \end{aligned}$$

$$\begin{aligned} a &= \big((-\lambda_2+\lambda_1)\alpha+\lambda_2\big)M_X(s)-\lambda_1-\lambda_2\,, \quad \text{and} \\ b &= \lambda_1^2\big(\alpha M_X(s)-1\big)^2-2\lambda_1\lambda_2\big(1-M_X(s)+\alpha^2M_X(s)^2-\alpha M_X(s)^2\big)\lambda_1 \\ &\qquad +\lambda_2\big(1-M_X(s)+\alpha M_X(s)\big)^2\,. \end{aligned}$$

Finally, if the inter–arrival times are PH distributed as generalized Erlang(2), the mgf of $M_{Z(t)}$ are given in Remark 2.1.

2.4 Numerical Examples

In this section some examples are considered to illustrate the results. Expectations, variances and the asymptotic behavior of $M_{Z(t)}(s)$ as time t goes to infinity are also studied.

Example 2.4.1. We consider a mixed exponential renewal process with parameters $(\underline{\alpha}, \mathbf{A})$ and exponential claim sizes with parameter θ, where

$$\underline{\alpha}' = \begin{pmatrix} 0.5 & 0.5 \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}, \qquad \theta = 1\,.$$

Here the net interest force $\delta = 0.01$, while $\lambda_1 = 0.02$ and $\lambda_2 = 0.04$.

Let $f^\star(t;s) = \begin{pmatrix} h_1(t;s) \\ h_2(t;s) \end{pmatrix}$. Using Maple, we obtain the following results:

$$\begin{aligned} h_1(t;s) &= \frac{1}{150(s-1)^3}\Big[s(s^2+3s-1)\,e^{-0.03t} + 3s^2(s-4)\,e^{-0.02t} \\ &\qquad +3s(4-s)\,e^{-0.01t} - 4s^3 + 12s - 11s\Big]\,, \\ h_2(t;s) &= \frac{1}{150(s-1)^3}\Big[s(6s^2-4s+1)\,e^{-0.04t} + 2s(1-s^2-3s)\,e^{-0.03t} \\ &\qquad +2s(4-s)\,e^{-0.01t} - 4s^3 + 12s^2 - 11s\Big]\,. \end{aligned}$$

Then we have for a fixed s:

$$\begin{aligned} f(t;s) &= \underline{\alpha}' f^\star(t;s) = 0.5h_1(t;s) + 0.5h_2(t;s) \\ &= \frac{1}{150(s-1)^3}\Big[s(6s^2-4s+1)\,e^{-0.04t} + s(1-s^2-3s)\,e^{-0.03t} + 3s^2(s-4)\,e^{-0.02t} \\ &\qquad +5s(4-s)\,e^{-0.01t} - 8s^3 + 24s^2 - 22s\Big]\,. \end{aligned}$$

By the integral differential equation in Theorem 2.3.4 with an initial value of $M_{Z(0)}(s) = 1$, it follows that for a fixed t the mgf $M_{Z(t)}(s)$ at $s < 1$ is:

$$M_{Z(t)}(s) = \frac{1}{12(1-s)^3}\left[s(4s-1-6s^2)\,e^{-0.04t} + 6(4-s)(s^2e^{-0.02t} + \frac{4}{3}se^{-0.01t} - \frac{1}{2})\right]. \quad (2.81)$$

If we differentiating (2.81) with respect to s and evaluate at let $s = 0$, we obtain moments of $M_{Z(t)}(s)$:

$$\begin{aligned} \mathbb{E}\big[Z(t)\big] &= \frac{11}{4} - \frac{8}{4}e^{-0.01t} - \frac{1}{12}e^{-0.04t}\,, \qquad t > 0 \\ \mathbb{E}\big[Z(t)^2\big] &= 51 + 33e^{-0.02t} - 84e^{-0.01t}\,, \qquad t > 0\,, \end{aligned}$$

which are consistent with results in Léveillé and Garrido (2001a).

The asymptotic mgf of $M_{Z(t)}(s)$ at t goes to infinity is :

$$M_{Z(\infty)}(s) = \frac{4-s}{4(1-s)^3} = \frac{1}{4}\frac{1}{(1-s)^2} + \frac{3}{4}\frac{1}{(1-s)^3}\,, \qquad s < 1\,, \tag{2.82}$$

which implies that the distribution of $Z(\infty)$ is mixed Erlang(2) with probability $\frac{1}{4}$ and Erlang(3) with probability $\frac{3}{4}$, while the scale parameter is 1.

The following example considers generalized Erlang(2) inter–arrival times.

Example 2.4.2. Let inter–arrival times have a generalized Erlang(2) distribution with parameters $(\underline{\alpha}, \mathbf{A})$ and claim severities be exponential parameter θ, where

$$\underline{\alpha}' = \begin{pmatrix} 1 & 0 \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ 0 & -\lambda_2 \end{pmatrix}, \qquad \theta = 1\,. \tag{2.83}$$

Let $\delta = 0.01$, here $\lambda = 0.01$ and $\lambda_2 = 0.02$, then Maple gives:

$$\begin{aligned} h_1(t;s) &= \frac{1}{150s^3}\Big[6(1-s)(-1+se^{-0.01t})\ln\big(\frac{1-se^{-0.01t}}{1-s}\big) + (3s^3-6s)e^{-0.01t} \\ &\quad +(3s^2-3s^3)e^{-0.02t} + s^3e^{-0.03t} - s^3 + 6s - 3s^2\Big]\,, \\ h_2(t;s) &= \frac{1}{150s^3(e^{0.01t}-s)}\Big[6(-1+s)(e^{0.01t}-s)\ln\big(\frac{1-se^{-0.01t}}{1-s}\big) + (-3s^2-s^3+6s)e^{0.01t} \\ &\quad +(3s^2-3s^3)e^{-0.01t} + (s^3-3s^4)e^{-0.02t} + 2s^4e^{-0.03t} + 3s^3 - 6s + s^4\Big]\,, \end{aligned}$$

which implies that:

$$f(t;s) = \underline{\alpha}' f^{\star}(t;s) = h_1(t;s)\,.$$

Then the mgf of $Z(t)$ at a fixed t is given for $s < 1$ by:

$$\begin{aligned} M_{Z(t)}(s) &= \frac{1}{s^3}\Big\{\Big[(4s-4s^2)e^{-0.01t} + 6s - 6\Big]\ln\big(\frac{1-se^{-0.01t}}{1-s}\big) \\ &\quad +(s^2-s^3)e^{-0.02t} + (2s^3+2s^2-6s)e^{-0.01t} + 6s - 3s^2\Big\}\,. \end{aligned} \tag{2.84}$$

Differentiating (2.84) with respect to s and letting $s = 0$, the moments of $Z(t)$ are also obtained:

$$\begin{aligned} \mathbb{E}\big[Z(t)\big] &= \frac{1}{8} - \frac{1}{6}e^{-0.01t} + \frac{1}{24}e^{-0.04t}\,, \\ \mathbb{E}\big[Z(t)^2\big] &= \frac{3}{5} + \frac{2}{5}e^{-0.05t} - \frac{1}{3}e^{-0.04t} - \frac{2}{3}e^{-0.01t}\,, \end{aligned}$$

which are also consistent with results in Léveillé and Garrido (2001a).

Remark 2.5. For expression (2.84), again, the asymptotic result as $t \to \infty$ is:

$$M_{Z(\infty)}(s) = \frac{1}{s^3}\Big[(6-6s)\ln(1-s) + 6s - 3s^2\Big], \qquad s < 1\,. \tag{2.85}$$

Example 2.4.3. Let the inter–arrival times have Coxian PH distributions with parameters $(\underline{\alpha}, \mathbf{A})$ and the claim severities be exponential with parameter θ, where

$$\underline{\alpha}' = \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix}', \quad \mathbf{A} = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ 0 & -\lambda_2 \end{pmatrix},$$

and here

$$\alpha_1 = \alpha_2 = 0.5\,, \quad \lambda_1 = 0.01\,, \quad \lambda_2 = 0.02 \text{ and } \theta = 1\,.$$

Under the effect of a net interest force $\delta = 0.01$, by Theorem 2.3.4 and Maple we have that

$$\begin{aligned} h_1(t;s) &= \frac{s}{100(1-s)}(1-e^{-0.01t})^2\,, \qquad s<1, \quad t>0\,, \\ h_2(t;s) &= \frac{s}{100(1-s)}(1-e^{-0.02t})\,, \qquad s<1, \quad t>0\,. \end{aligned}$$

Then the function $f(t;s)$ can be obtained from

$$f(t;s) = \underline{\alpha}' f^\star(t;s) = 0.5\,h_1(t,s) + 0.5\,h_2(t;s) = \frac{0.01s}{1-s}(1-e^{-0.01t})\,, \qquad s<1, \quad t>0\,.$$

The integrating differential equation in Theorem 3.3.1 yields the mgf of $Z(t)$ for fixed $t > 0$:

$$M_{Z(t)}(s) = \frac{1-se^{-0.01t}}{1-s}\,, \qquad s<1\,. \tag{2.86}$$

Note that (2.86) is the mgf of $Z(t)$ for a Poisson process with parameter $\lambda = 0.01$ and exponential claims with parameter $\theta = 1$.

Differentiating (2.86) with expect to s gives the first and second moments:

$$\mathbb{E}[Z(t)] = 1 - e^{-0.01t} \quad \text{and} \quad \mathbb{E}[Z(t)^2] = 2 - 2e^{-0.02t}\,,$$

which can also be proved by the results in Léveillé and Garrido (2001a).

The asymptotic result of $M_{Z(t)}(s)$ as t goes to infinity is given by:

$$M_{Z(\infty)}(s) = \frac{1}{1-s}\,, \qquad s<1\,, \tag{2.87}$$

which shows that here the asymptotic distribution of $Z(t)$ is exponential with parameter 1.

Example 2.4.4. In the previous examples we considered exponential claim sizes. Now let claim sizes have a mixed exponential distribution, which is also PH with parameters $(\beta, \mathbf{B})$, while inter–arrival times are still Coxian with parameters $(\underline{\alpha}, \mathbf{A})$, as in Example 2.4.3. The parameters are given here by:

$$\beta' = \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix}, \quad \underline{\alpha}' = \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ 0 & -\lambda_2 \end{pmatrix},$$

where $\beta_1 = \beta_2 = 0.5$, $\theta_1 = 1, \theta_2 = 2$, $\lambda_1 = 0.02, \lambda_2 = 0.04$, and the net interest force is still $\delta = 0.01$. Then Maple gives:

$$\begin{aligned} h_1(t;s) &= -\frac{s}{50}\frac{(s-1)e^{-0.04t} - 2se^{-0.02t} + 4e^{-0.01t} + s - 3}{s^2 - 3s + 2}, \qquad s < 1, \quad t > 0, \\ h_1(t;s) &= -\frac{s}{50}\frac{(1-s)e^{-0.04t} + 2e^{-0.01t} + s - 3}{s^2 - 3s + 2}, \qquad s < 1, \quad t > 0. \end{aligned}$$

Then the corresponding function $f(t;s)$ is obtained from:

$$f(t;s) = \underline{\alpha}' f^\star(t;s) = 0.5\, h_1(t;s) + 0.5\, h_2(t;s) = \frac{-0.02s(-3 + s - se^{-0.02t} + e^{-0.0tt})}{s^2 - 3s + 2}.$$

Again Maple and Theorem 2.3.4, gives the derivative equation of the mgf:

$$\frac{\partial}{\partial t} M_{Z(t)}(s) = \frac{0.01se^{-0.01t}(-2 - s^2e^{-0.02t} + 3se^{-0.01t})(-3 + 2se^{-0.01t})}{(s^2 - 3s + 2)(2 - se^{-0.01t})(1 - se^{-0.01t})}, \tag{2.88}$$

where the initial value is $M_{Z(0)}(s) = 1$. Integrating (2.88) with respective to t yields the mgf of $Z(t)$ for a fixed $t > 0$:

$$M_{Z(t)}(s) = \frac{s^2e^{-0.02t} - 3se^{-0.01t} + 2}{(1-s)(2-s)}, \qquad s < 1. \tag{2.89}$$

Remark 2.6. Differentiating (2.89) with respect to s and letting $s = 0$ yields the following moments of $Z(t)$:

$$\mathbb{E}[Z(t)] = \frac{3}{2} - \frac{3}{2}e^{-0.01t} \quad \text{and} \quad \mathbb{E}[Z(t)^2] = \frac{7}{2} + e^{-0.02t} - \frac{9}{2}e^{-0.01t}, \qquad t > 0,$$

which can be also proved by the results in Léveillé and Garrido (2001a).

Here the asymptotic result for $M_{Z(t)}(s)$ as $t \to \infty$ is given by:

$$M_{Z(\infty)}(s) = \frac{2}{(1-s)(2-s)} = 2\frac{1}{1-s} - \frac{2}{2-s}, \qquad s < 1, \tag{2.90}$$

which show the distribution of $Z(\infty)$ is a combination of exponentials with parameters 1 and 2 respectively, while the coefficients are 2 and -1.

Example 2.4.5. Finally consider a more complicated case with generalized Erlang(2) inter–arrival times and Erlang(2) claim severities. The PH distribution for the generalized Erlang(2) has parameters $(\underline{\beta}, \mathbf{B})$, while the Erlang(2) has parameters $(\underline{\alpha}, \mathbf{A})$, given here by

$$\underline{\beta}' = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -0.01 & 0.01 \\ 0 & -0.02 \end{pmatrix}, \quad \underline{\alpha}' = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

By Theorem 2.3.4 we obtain the vector $f^\star(t;s) = \begin{pmatrix} h_1(t;s) \\ h_2(t;s) \end{pmatrix}$ where $h_1(t;s)\,, h_2(t;s)$ are given for a fixed $s < 1$ and $t > 0$ by:

$$\begin{aligned} h_1(t;s) &= \frac{1}{150(s-1)(1-se^{-0.01t})^2}\Big(2s(s+1)e^{-0.01t} + (3s^3+3s^2-2s)e^{-0.03t} \\ &\quad -(s^3+5s^2)e^{-0.02t} - (s^2-s^3)e^{-0.05t} + (s^2+s-3s^3)e^{-0.04t} - s\Big), \\ h_2(t;s) &= \frac{1}{7200(1-s)(1-se^{-0.01t})^4}\Big(6s^4(s-1)e^{-0.07t} - 3s^3(5s+3s^2-7)e^{-0.06t} \\ &\quad +12s^2(3s^2-2)e^{-0.05t} + 3s(3+7s-s^3+s^4-15s^2)e^{-0.04t} \\ &\quad +6s(3s-2s^3-2+s^2)e^{-0.03t} + 3s^3(5s+3s^2-7)e^{-0.02t} - 12s^2e^{-0.01t} + 3s\Big). \end{aligned}$$

Hence

$$\begin{aligned} f(t;s) &= \underline{\alpha}' f^\star(t;s) \\ &= h_1(t;s) = \frac{1}{150(1-s)(1-se^{-0.01t})^2}\Big((2s-3s^3-3s^2)e^{-0.03t} - 2s(s+1)e^{-0.01t} \\ &\quad +(s^3+5s^2)e^{-0.02t} + (s^2-s^3)e^{-0.05t} - (s^2+s-3s^3)e^{-0.04t} + s\Big). \end{aligned}$$

Then the derivative in t of the m.g.f. of $Z(t)$ is given by:

$$\begin{aligned} \frac{\partial}{\partial t}M_{Z(t)}(s) &= \frac{se^{-0.01t}(2-se^{-0.01t})}{150(s-1)(-1+se^{-0.01t})^4}\Big(se^{-0.03t}(-2+3s^2+3s) - se^{-0.04t}(-s-1+3s^2) \\ &\quad +s^2(s-1)e^{-0.05t} + 2s^2(s+1)e^{-0.02t} - s^2 - s^2e^{-0.03t}(s+5) \\ &\quad -(s-1)(-1+se^{-0.01t})^2(e^{-0.03t}-1)\Big), \qquad s<1\,, \quad t>0\,. \end{aligned} \tag{2.91}$$

with initial value $M_{Z(0)}(s) = 1$. Integrating (2.91) yields

$$\begin{aligned} M_{Z(t)}(s) &= \frac{2}{(1-se^{-0.01t})^3(1-s)}\Big(s(s^3+\frac{1}{2}s^2-\frac{1}{2}s+\frac{1}{6})e^{-0.04t} - \frac{1}{2}s^2(s^2-s+\frac{1}{3})e^{-0.05t} \\ &\quad -\frac{13}{6}se^{-0.01t} + \frac{11}{3}s^2e^{-0.02t} - 3s^3e^{-0.03t} + \frac{1}{2}\Big), \qquad s<1\,, \quad t>0\,. \end{aligned} \tag{2.92}$$

Remark 2.7. Differentiating (2.92) with respect to s yields the moments of $Z(t)$, here the first two moments are given by.

$$\mathbb{E}[Z(t)] = 1 - \frac{4}{3}e^{-0.01t} + \frac{1}{3}e^{-0.04t} \quad \text{and} \quad \mathbb{E}[Z(t)^2] = 2 + \frac{2}{3}e^{-0.02t} + \frac{4}{3}e^{-0.05t} - \frac{8}{3}e^{-0.01t} - \frac{4}{3}e^{-0.04t}\,,$$

which implies by results in Léveillé and Garrido (2001a).

Finally, the asymptotic result for the moment generating function is:

$$M_{Z(\infty)}(s) = \frac{1}{1-s}\,, \qquad s < 1\,, \tag{2.93}$$

which is an exponential distribution with mean 1.

In this chapter we have derived homogeneous differential equations (or differential systems) for the mgf of $Z(t)$, when the inter–arrival times are PH distributed. It is difficult to get the solution to the differential equations, even for simple cases, because the coefficients are functions of t. For instance, the solution in Example 2.4.4 is not such a simple function, even for Coxian inter–arrival times. But the results are much more general than what was available in the literature.

In the next chapter we discuss the calculation of the distribution of $Z(t)$ by inversion methods. A truncated series method is proposed to approximate the solution of the differential equations in the cases where we can not get the exact solution.

Chapter 3

PH–Renewal Distribution Functions

In Chapter 2 we introduced the model of discounted compound PH–renewal sums and obtained homogeneous and differential equation system for the mgf of $Z(t)$. Some examples are also given to explain results. In this chapter a deeper analysis of the model is presented. First, the distribution function of discounted compound renewal sums is obtained by inverting Laplace transforms. Very often a simple expression for the mgf of $Z(t)$ is not possible, so an approximate solution to the differential equation is needed.

Since the solution of the differential equations is the mgf of $Z(t)$, the coefficient of differential equations of $M_{Z(t)}(s)$, with respect to t, is a function in t. Then a series approximation method gives solutions that are polynomials in t, in which each coefficient is a rational polynomial in s. Hence we can use the corresponding Laplace transform and invert it to obtain the approximate distribution function of $Z(t)$. We call this approximation the truncated series solution, by contrast to the exact solution. The accuracy of this method depends on where one truncates the series. A more accurate solution requires a longer series. Hence by controlling the time t parameter properly, we can obtain very nice results for the distribution function of discounted compound renewal sums. The truncated series method is discussed in detail in Section 3.2.

3.1 Exact PH–Renewal Density Functions

The solutions of the differential equations depend on the function $f(t;s)$ or $f^{\star}(t;s)$, the solution of the differential equation system in (2.48). This solution of the differential equation is the exact mgf of $Z(t)$. Inverting its corresponding Laplace transform produces the exact distribution function of $Z(t)$.

To illustrate this exact solution we use the same numerical examples as in Section 2.4 and derive the exact density (or extended density) and cumulative distribution functions.

Example 3.1.1. (Example 2.4.1 revisited) For a mixed exponential renewal process combined to exponential claim sizes of mean 1, consider a net interest force $\delta = 0.01$ and mixed exponential parameters $\lambda_1 = 0.02$, $\lambda_2 = 0.04$, and $\underline{\alpha}' = \begin{pmatrix} 0.5 & 0.5 \end{pmatrix}$. Example 2.4.1 gives the mgf for the discounted compound renewal sum of $Z(t)$ in (2.81).

Inverting the Laplace transform corresponding to (2.81) using Maple gives the following probability mass at $x = 0$ and density function of $Z(t)$ for $x > 0$, which is called the extended density function (edf).:

$$f(x) = \begin{cases} \frac{1}{2}(e^{-0.02t} + e^{-0.04t})\,, & \text{if} \quad x = 0\,, \\ \Big[(\frac{3}{4}x^2 - \frac{5}{2}x + \frac{1}{2})e^{-0.02t} + (\frac{11}{12}x - \frac{1}{8}x^2 - \frac{7}{6})e^{-0.04t} & \\ \quad +(\frac{4}{3}x + \frac{2}{3} - x^2)e^{-0.01t} + \frac{3}{8}x^2 + \frac{1}{4}x\Big]e^{-x}\,, & \text{if} \quad x > 0\,, \end{cases} \tag{3.1}$$

and the cumulative distribution function (cdf) is given by

$$\begin{aligned} F(x) &= 1 + (\frac{1}{2} - \frac{2}{3}x + \frac{1}{8}x^2)e^{-x-0.04t} + (\frac{1}{2} + x - \frac{3}{4}x^2)e^{-x-0.02t} + (x^2 + \frac{2}{3}x)e^{-x-0.01t}\,. \\ &\quad -(1 + x + \frac{3}{8}x^2)e^{-x}\,, \qquad x \geq 0\,. \end{aligned}$$

The two following graphs show the cdf of $Z(t)$ and its conditional density function, given $x > 0$, for different times t. From the graphs we see how fast the curves tend to the asymptotic result.

The asymptotic density function of $Z(\infty)$ given in (3.2) is also produced by inverting the Laplace transform corresponding to $M_{Z(\infty)}(s)$ given in (2.82). Integrating this density function gives the asymptotic cdf of $Z(t)$ in (3.2). The expressions for the density function and cdf of $Z(\infty)$ are given by:

$$f_\infty(x) = \frac{1}{4}\, x\, e^{-x} + \frac{3}{4}\, x^2 e^{-x} \quad \text{and} \quad F_\infty(x) = 1 - e^{-x}(1 + x + \frac{3}{8}x^2)\,, \qquad x \geq 0\,. \tag{3.2}$$

Hence the asymptotic distribution of $Z(\infty)$ is a mixture of an exponential with mean of 1 and a gamma ($\alpha = 3$ and $\beta = 1$).

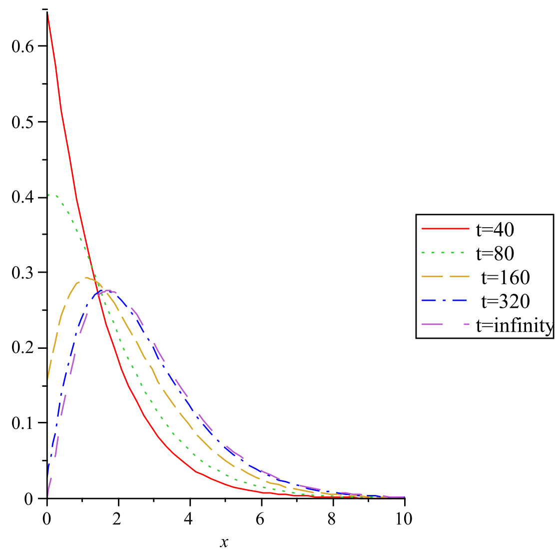


Figure 3.1: Cond. d. of $Z(t)$ in Example 3.1.1

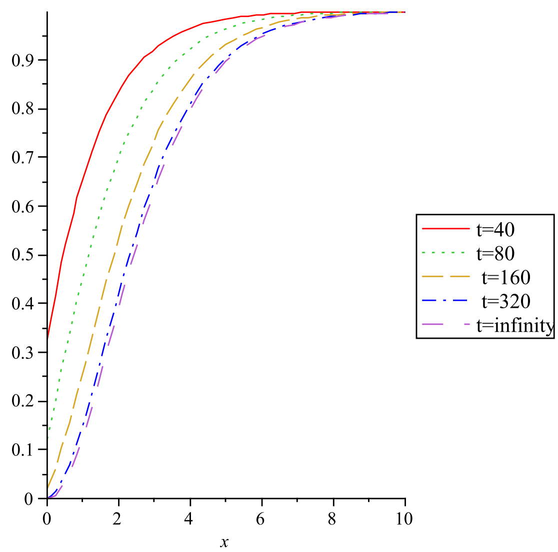


Figure 3.2: Cdf of $Z(t)$ in Example 3.1.1

Note that the extended density function has a jump at $x = 0$, hence the conditional density (cond.d.) given that $x > 0$ is plotted in Figure 3.1. Figures 3.1 and 3.2 show that the conditional density function and the cdf converge to the asymptotical results, when $t \geq 320$. The cdf takes longer to reach probability one for larger t values, since there are more claims when time t is large. The cdf graph confirms that $Z(t)$ has a heavier tail for large time t values. The graph also shows that the distribution of $Z(t)$ reaches the asymptotic value when $t > 320$ and that the point mass at $x = 0$ disappears asymptotically.

Example 3.1.2. (Example 2.4.2 revisited) The mgf of $Z(t)$ in (2.84) is given by:

$$\begin{aligned} M_{Z(t)}(s) = \frac{1}{s^3}\Bigg\{ \Big[(4s - 4s^2)e^{-0.01t} + 6s - 6\Big] \ln\left(\frac{1 - se^{-0.01t}}{1 - s}\right) \\ + (s^2 - s^3)e^{-0.02t} + (2s^3 + 2s^2 - 6s)e^{-0.01t} + 6s - 3s^2 \Bigg\}\,, \qquad s < 1\,, \end{aligned} \tag{3.3}$$

when inter–arrival times have a generalized Erlang(2) distribution and exponential claim severities with parameters in (2.83). Inverting the Laplace transform corresponding to (2.84) gives the following point mass at $x = 0$ and defective density function for $x > 0$:

$$f(x) = \begin{cases} 2e^{-0.01t} - e^{-0.02t} \quad , & \text{if} \quad x = 0\,, \\ (4e^{-0.01t} + 6x + 3x^2 + 4xe^{-0.01t})\Big[Ei(1, xe^{0.01t}) - Ei(1, x)\Big] & \\ -(3x + 6 + e^{-0.01t})e^{-xe^{0.01t} - 0.01t} + e^{-x}(3 + 3x + 4e^{-0.01t})\,, & \text{if} \quad x > 0\,, \end{cases}$$

where $Ei(1, x)$ is called exponential integral defined by:

$$Ei(1, x) = \int_x^{\infty} \frac{e^{-u}}{u}\, du\,,$$

Integrating this density function when $x > 0$ plus adding the mass at the point zero gives the following cdf of $Z(t)$:

$$F(x) = 2e^{-0.01t} - e^{-0.02t} + e^{-0.03t}\Big[(4xe^{0.02t} + 3x^2e^{0.03t} + x^3e^{0.03t} + 2x^2e^{0.02t}) \\ \Big[Ei(1, xe^{0.01t}) - Ei(1, x)\Big] - e^{-xe^{0.01t} + 0.01t}(1 + x + x^2e^{0.01t} + 3xe^{0.01t}) \\ + 2e^{-x + 0.02t}(1 + x) + e^{-x + 0.03t}(x^2 + 2x - 1) + e^{0.01t} - 2e^{0.02t} + e^{0.03t}\Big]\,, \qquad x \geq 0\,.$$

Note that the asymptotic density function is obtained by inverting the corresponding Laplace transform of the mgf given in (2.85). The asymptotic density function and cdf of $Z(\infty)$ are given by the following expressions:

$$f_\infty(x) = (3 + 3x)e^{-x} - (6x + 3x^2)Ei(1, x)\,, \qquad x \geq 0 \qquad (3.4)$$

$$F_\infty(x) = (2x + x^2 - 1)e^{-x} - (3x^2 + 3x^3)Ei(1, x)\,, \qquad x \geq 0\,. \qquad (3.5)$$

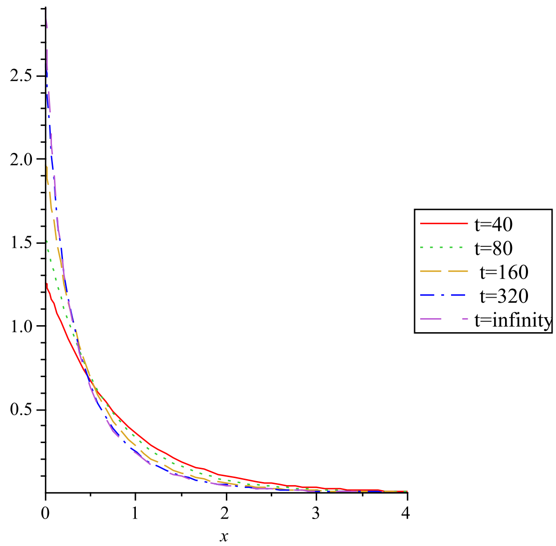


Figure 3.3: Cond. d. of $Z(t)$ in Example 3.1.2

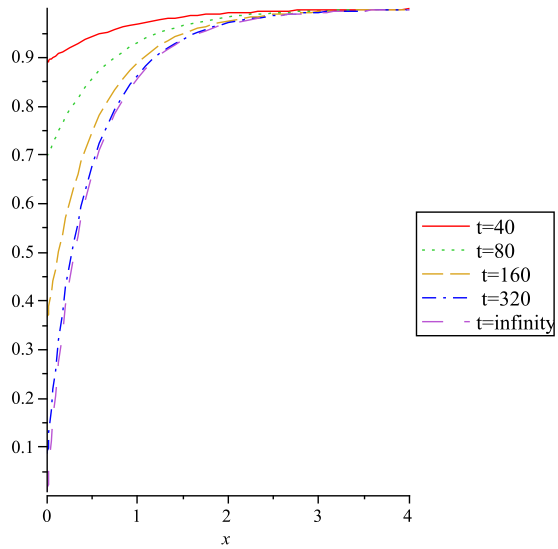


Figure 3.4: Cdf of $Z(t)$ in Example 3.1.2

Figures 3.3 and 3.4 shows that the conditional density, given that $x > 0$, and the cdf converges to their asymptotic results in (3.4) and (3.5) when $t \geq 320$. Again the distribution $Z(t)$ has a heavier tail when t is large.

Example 3.1.3. (Example 2.4.3 revisited) Coxian PH inter–arrival times with exponential claim sizes. Here the mgf of $Z(t)$ for a fixed t is given by (2.86) to be:

$$M_{Z(t)}(s) = \frac{1 - se^{-0.01t}}{1-s}\,, \qquad s < 1\,. \tag{3.6}$$

Inverting the corresponding Laplace transform gives the following probability mass and defective density function:

$$f(x) = \begin{cases} e^{-0.01t}\,, & \text{if} \quad x = 0\,, \\ e^{-x}(1 - e^{-0.01t})\,, & \text{if} \quad x > 0\,, \end{cases} \tag{3.7}$$

and cdf of $Z(t)$ is:

$$F(x) = 1 - e^{-x} + e^{-x-0.01t}\,, \qquad x \geq 0\,. \tag{3.8}$$

Note that the extended density function in (3.7) and the cdf in (3.8) are very simple, which is consistent with corresponding mgf, because this discounted compound sum $Z(t)$ is a compound Poisson process with rate 0.01 and exponential claim severities with means 1. There is also an exponential jump mass at zero.

The asymptotic density function of $Z(\infty)$ given in Figure in 3.5 is obtained by inverting the Laplace transform corresponding to mgf in (2.82). Integrating this asymptotic density function yields the asymptotic cdf of $Z(\infty)$. Hence the density function of $Z(\infty)$ is e^{-x}, which show that $Z(\infty)$ is exponential with mean of 1.

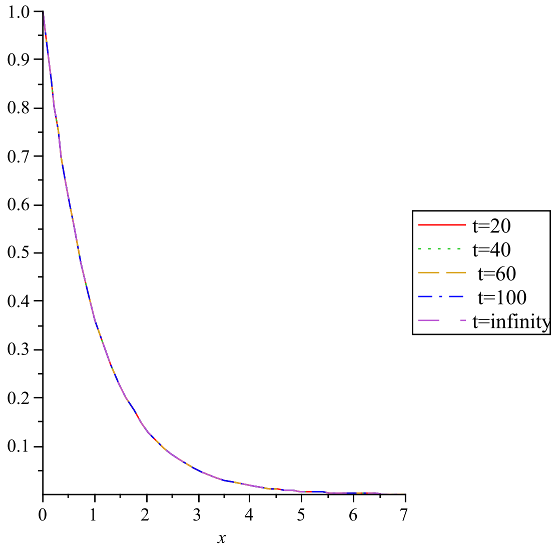


Figure 3.5: Cond. d. of $Z(t)$ in Example 3.1.3

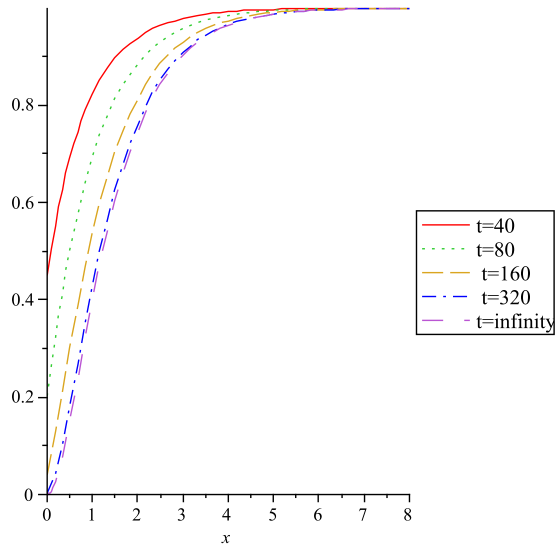


Figure 3.6: Cdf of $Z(t)$ in Example 3.1.3

It is not a surprise that the conditional densities have the same shape, since $f(x) = e^{-x}(1 - e^{-0.01t})$ and the probability mass $e^{-0.01t}$, hence the conditional density function is $f_c(x) = \frac{e^{-x}(1-e^{-0.01t})}{1-e^{-0.01t}} = e^{-x}$. This conditional density function is exponential with parameter 1. Figure 3.6 also shows that the distribution of $Z(t)$ converges to its asymptotic values when $t \geq 320$ and that the tail is heavier as t increases.

Example 3.1.4. (Example 2.4.4 revisited) Coxian inter–arrival times with mixed exponential claim sizes. In this case the mgf of the discounted compound sum is given in (2.89) that is:

$$M_{Z(t)}(s) = \frac{s^2 e^{-0.02t} - 3se^{-0.01t} + 2}{(1-s)(2-s)}, \qquad s < 1 .$$

Inverting the corresponding Laplace transform gives the following point mass and defective density function:

$$f(x) = \begin{cases} e^{-0.02t}, & \text{if} \quad x = 0, \\ 2e^{-x} - 2e^{-2x} - 3e^{-0.01t}(e^{-x} - 2e^{-2x}) + e^{-0.02t}(e^{-x} - 4e^{-2x}), & \text{if} \quad x > 0, \end{cases}$$

while the cdf is given by:

$$F(x) = 1 - 2e^{-x} + e^{-2x} + 3e^{-0.01t-x}(1 - e^{-x}) + e^{-0.02t-x}(2e^{-x} - 1), \qquad x \geq 0 .$$

The asymptotic density function of $Z(\infty)$ reproduced in Figure 3.7 is given by inverting the Laplace transform corresponding to the mgf in (2.90), while integrating the density function in (3.9) yields the corresponding cdf:

$$f_\infty(x) = 2e^{-x}(1-e^{-x}), \qquad x>0, \tag{3.9}$$

$$F_\infty(x) = 1-e^{-x}(2-e^{-x}), \qquad x \geq 0. \tag{3.10}$$

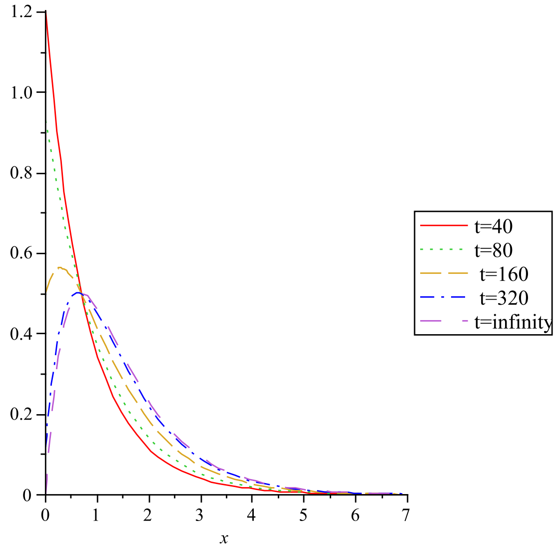


Figure 3.7: Cond. d. of $Z(t)$ in Example 3.1.4

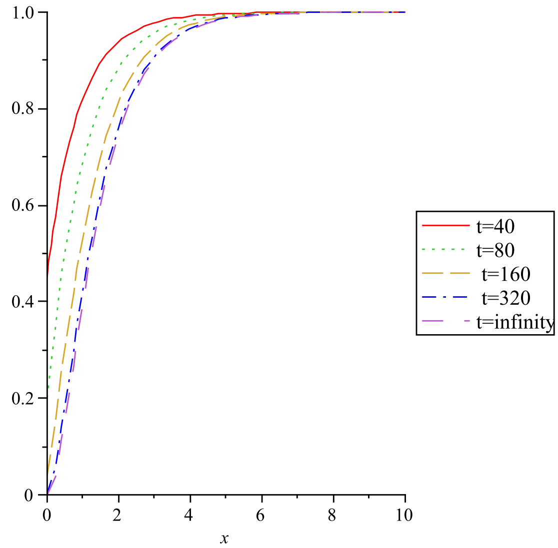


Figure 3.8: Cdf of $Z(t)$ in Example 3.1.4

Figures 3.7 and 3.8 show again that the distributions reach the asymptotic values for $t \geq 320$.

Example 3.1.5. Reconsider the last example in Section 2.4. For the discounted compound sum with generalized Erlang(2) inter–arrival times and Erlang(2) claim severities, the mgf is given in (2.92) to be:

$$M_{Z(t)}(s) = \frac{2}{(1-se^{-0.01t})^3(1-s)}\left[s\left(s^3+\frac{1}{2}s^2-\frac{1}{2}s+\frac{1}{6}\right)e^{-0.04t}-\frac{1}{2}s^2\left(s^2-s+\frac{1}{3}\right)e^{-0.05t}\right.$$
$$\left.-\frac{13}{6}se^{-0.01t}+\frac{11}{3}s^2e^{-0.02t}-3s^3e^{-0.03t}+\frac{1}{2}\right], \qquad s<1.$$

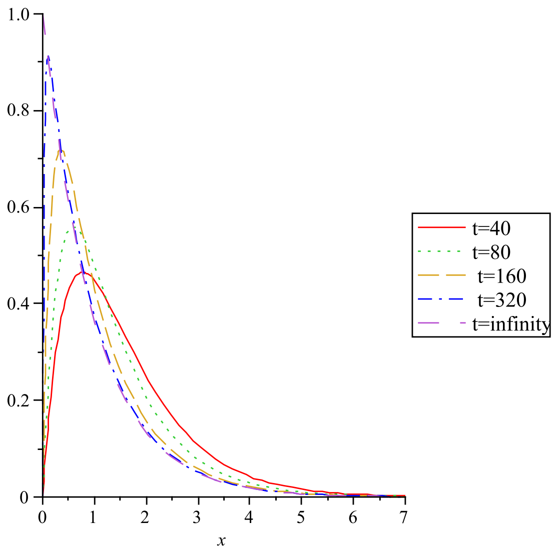
Inverting the corresponding Laplace transform yields the following point mass and defective density function:

$$f(x) = \begin{cases} 2e^{-0.01t}-e^{-0.02t}, & \text{if} \quad x=0, \\ \frac{1}{(1-e^{-0.01t})^3}\left[\frac{1}{3}(e^{-x}-e^{-xe^{0.01t}})(3-e^{0.05t}+22e^{-0.02t}-13e^{-0.01t}+7e^{-0.04t}-18e^{-0.03t})\right] & \\ +\frac{1}{(1-e^{-0.01t})^2}\left[\frac{1}{3}xe^{-xe^{0.01t}}(e^{0.01t}-4+e^{-0.03t}-4e^{-0.02t}+6e^{-0.01t}\right., & \text{if} \quad x>0. \end{cases}$$

while the cdf of $Z(t)$ for $x \geq 0$:

$$
\begin{aligned}
F(x) \;=\; & 2e^{-0.01t} - e^{-0.02t} - \frac{\frac{1}{3}e^{-0.02t}}{(1-e^{-0.01t})^3}\Big[(xe^{0.02t}+10x+8)e^{-xe^{0.01t}} + 15e^{0.01t} \\
& -(5x+2)e^{-xe^{0.01t}+0.01t} - (10x+12)e^{-xe^{0.01t}-0.01t} + (5x+8)e^{-xe^{0.01t}-0.02t} \\
& +(3e^{0.02t} - 13e^{0.01t} - e^{-0.03t} + 7e^{-0.02t} - 18e^{-0.01t})e^{-x} + 3e^{-0.03t} \\
& +30e^{-0.01t} + 22e^{-x} - 3e^{0.02t} - (x+2)e^{-xe^{0.01t}-0.03t} - 15e^{-0.02t} - 30\Big]. \quad (3.11)
\end{aligned}
$$

Figures 3.9 and 3.10 show the conditional density and cdf of $Z(t)$ for $x > 0$ and its cdf:



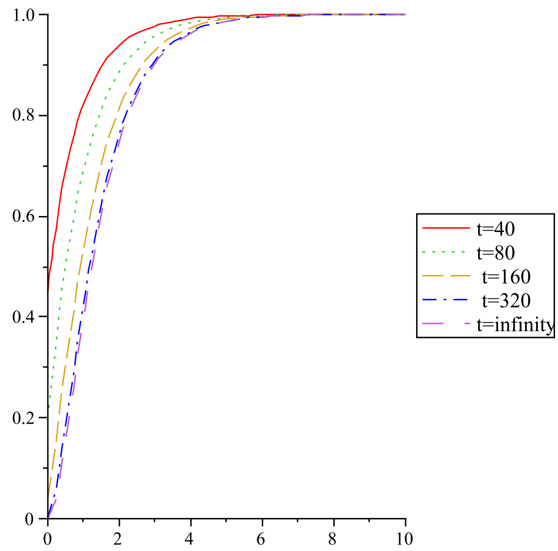


Figure 3.9: Cond. d. of $Z(t)$ in Example 3.1.5   Figure 3.10: Cdf of $Z(t)$ in Example 3.1.5

Note that the asymptotic density function of $Z(\infty)$ in Figure 3.9 is obtained from the mgf in (2.93). The expressions for the density function and the cdf of $Z(\infty)$ here are:

$$f_\infty(x) = e^{-x}, \qquad x \geq 0, \tag{3.12}$$

$$F_\infty(x) = 1 - e^{-x}, \qquad x \geq 0. \tag{3.13}$$

Again here for $t > 320$, the conditional density function and cdf converge to the asymptotic ones and the point mass vanishes.

The above examples show that the extended density function of $Z(t)$ can be obtained by inverting the Laplace transform, at least in cases where the differential equation can be solved analytically. In other cases, instead of looking for an exact solution, we propose a series approximation method discussed in detail in the next section.

3.2 Numerical Approximation for PH Renewal Density Functions

In many cases it is difficult, if not impossible, to find exact solutions to the differential equations presented above. In particular when a differential equation has non–constant coefficients or more than one argument s, t, it increases the difficulty in finding a solution. In the previous section we discussed a few exact solutions of these differential equations that greatly depended on the parameters chosen. For other parameters or in other cases we cannot generally find an exact solution. An alternative is to solve the differential equation by a series approximation method. If we consider relatively small parameter t, then the series method can work very well. In this section we let time $t \leq 100$, in such case the series can be truncated accurately after 15 terms.

First we discuss the basic ideas about the truncated series solution to differential equations. The coefficients in differential equations or differential systems are functions of the parameters s and t. They are expanded in term of t by the series. Let the solution to the differential equation be series in the term of t, then coefficients of this series solution can be written as functions of coefficients in expanded series. We explain this method by the following simple case. Consider the first–order homogeneous differential equation

$$\frac{d}{dt}f(t) = a(t;s)f(t)\,, \tag{3.14}$$

with initial value $f(0) = c_0$.
The coefficient $a(t;s)$ is written as a series form given by

$$a(t;s) = a_0(s) + a_1(s)t + a_2(s)t^2 + \cdots + a_n(s)t^n + \cdots\,, \tag{3.15}$$

Let the function $f(t)$ be the series form in term of t

$$f(t) = b_0(s) + b_1(s)t + b_2(s)t^2 + \cdots + b_n(s)t^n + \cdots\,. \tag{3.16}$$

Differentiating (3.16) yields

$$\frac{d}{dt}f(t) = b_1(s) + 2b_2(s)t + 3b_3(s)t^2 + \cdots + nb_n(s)t^{n-1} + \cdots\,. \tag{3.17}$$

Substituting (3.17), (3.16) and (3.15) into (3.14) gives

$$b_0(s) = c_0\,, \quad b_1(s) = a_0(s)b_0(s)\,, \quad b_2(s) = \frac{1}{2}\big[a_0(s)b_1(s) + a_1(s)b_0(s)\big]\,,$$
$$\ldots\,, b_n = \frac{1}{n}\sum_{k=0}^{n-1} a_k(s)b_{n-1-k}(s)\,, \ldots\,,$$

hence the coefficients of the series solutions are functions in s. If we control the time parameter t, we can truncate the series. Inverting the Laplace transform yields an approximate distribution3 function at the chosen time.

The following example gives a comparison between exact and series solutions.

Example 3.2.1. Revisit Examples 3.1.1, where the inter–arrival times have a mixed exponential distribution and claims are exponential. The exact extended density function of $Z(t)$, for $x > 0$, is given in (3.1):

$$f(x) = \Big[(\frac{3}{4}x^2 - \frac{5}{2}x + \frac{1}{2})e^{-0.02t} + (\frac{11}{12}x - \frac{1}{8}x^2 - \frac{7}{6})e^{-0.04t} + (\frac{4}{3}x + \frac{2}{3} - x^2)e^{-0.01t} + \frac{3}{8}x^2 + \frac{1}{4}x\Big]e^{-x}\,.$$

For the series solution, the mgf of $Z(t)$ is truncated at order 15, thus the extended density function produced by inverting the Laplace transform has a long and messy expression, we only include the graphs of the conditional density function, given $x > 0$ and the point mass at $x = 0$.

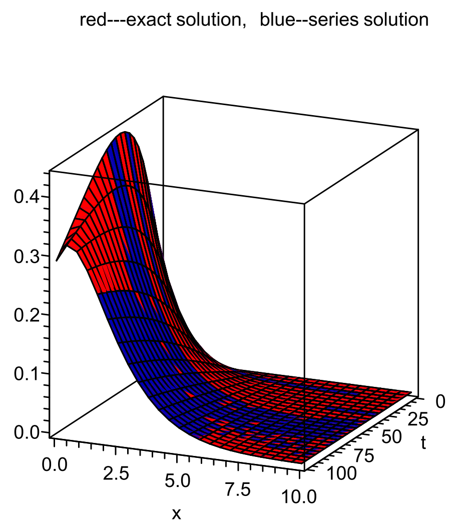


Figure 3.11: Cond. d. in Example 3.2.1

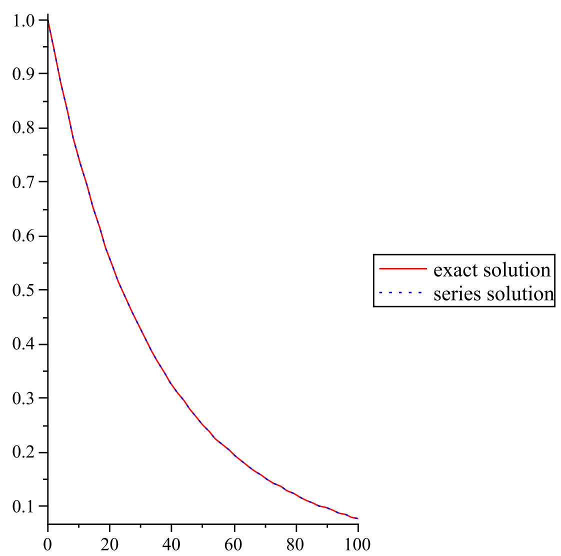


Figure 3.12: Prob. at $x = 0$ in Example 3.2.1

Note that there is a jump at $x = 0$ in the distribution of $Z(t)$. Hence Figure 3.11 only shows the cond.d., given $x > 0$.

Figures 3.11 show that the exact and series solutions are exactly the same if we control the time parameter $t \leq 100$, including the probability mass (prob.) at $x = 0$ in Figure 3.12. For large t the asymptotic distribution provides an accurate approximation. Hence we conclude that the truncated series method can serve as an accurate approximation method to solve the differential equations at fixed points of time t in some cases, where the exact solutions cannot be obtained.

A natural question is if the series method can be used for large time t values? First let us look at the following example.

Example 3.2.2. Reconsider Example 2.4.1, the asymptotic result for the mgf is:

$$M_{Z(\infty)}(s) = \frac{4-s}{4(1-s)^3}\,, \qquad s < 1\,.$$

Inverting its corresponding Laplace transform yields the asymptotic density function of $Z(\infty)$:

$$f(x) = \frac{1}{8}e^{-x}(3x^2 + 2x)\,, \qquad x \geq 0\,.$$

The following graph shows defective density functions obtained by the series method, as well as the asymptotic results when $x > 0$. The masses at $x = 0$ are 0.0004, 0.0004 and 0 for the exact, series and asymptotic results, respectively.

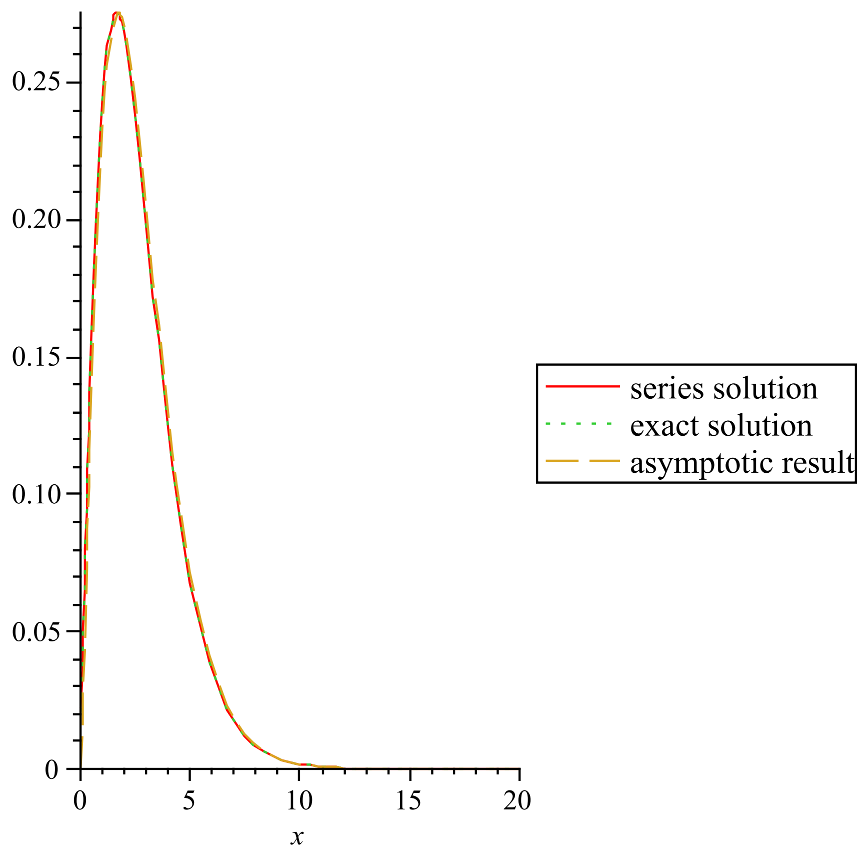


Figure 3.13: Defective density function of $Z(t)$ for $x > 0$ in Example 3.2.2

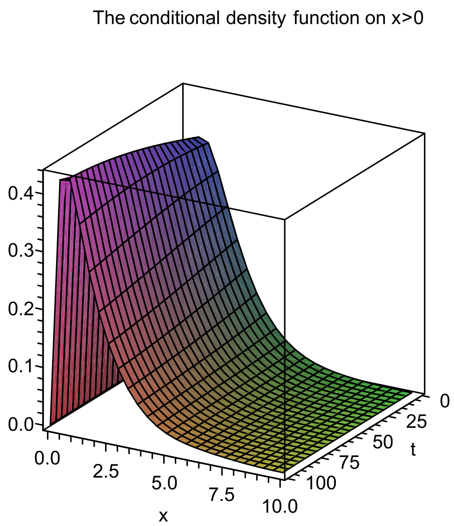
In the above series solution, the series solutions for the mgf was truncated at 50 terms and the time parameter is fixed at 350, which reaches the asymptotic result. Hence we again conclude that for large t value we do not need the high order terms in the series. The truncated series method is a suitable approach to solve differential equations (or differential equation systems) when the exact solution is difficult to obtain.

The following examples illustrate further the series solution and its applications.

Example 3.2.3. (Revisit Example 2.4.1) For exponential claim sizes, the exact solution can be obtained. However, as soon as we drop the exponential assumption, even for Erlang(2) claim sizes and the same parameters as in Example 2.4.1, the differential equations no longer yield an simple form for the exact solution. The truncated series method provides a simple solution. Hence the time parameter is set to values $t \leq 100$, while the parameters for the inter–arrival times mixture of exponentials are

$$\underline{\alpha}' = \begin{pmatrix} 0.5 & 0.5 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} -\frac{1}{100} & 0 \\ 0 & -\frac{1}{300} \end{pmatrix},$$

while the Erlang(2) claim severities have a mean of 2. The resulting extended density function is again tedious as it takes a long polynomial form. 3D graphs of the conditional density function, given $x > 0$ and the cdf are given below:



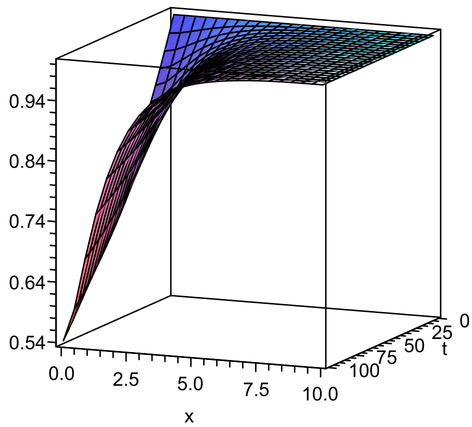


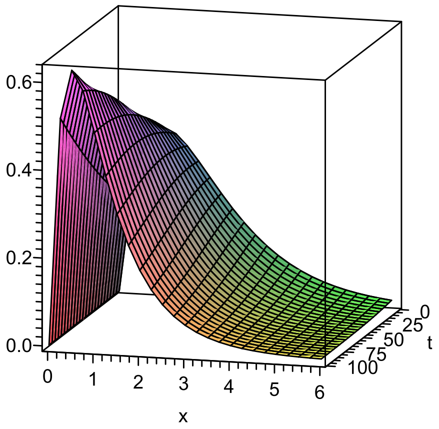
Figure 3.14: Cond. d. of $Z(t)$ in Example 3.2.3   Figure 3.15: Cdf of $Z(t)$ in Example 3.2.3

Note that the distribution of $Z(t)$ gets a heavier tail as t increases. As a result it takes a longer time to reach probability 1.

Example 3.2.4. (Revisit the example 2.4.5) For generalized Erlang(2) inter–arrival times and Erlang(2) claim severities, we have the exact solution to the differential equations for the parameters chosen in the example. But this is not always this case. For example if the generalized Erlang(2) parameters are:

$$\underline{\alpha}' = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} -\frac{2}{300} & \frac{2}{300} \\ 0 & -0.02 \end{pmatrix},$$

it becomes difficult to get the exact solution. By the series method, the extended density function can be approximated by a polynomial in time t and x. The following graphs show the conditional density function, given $x > 0$ and the cdf.



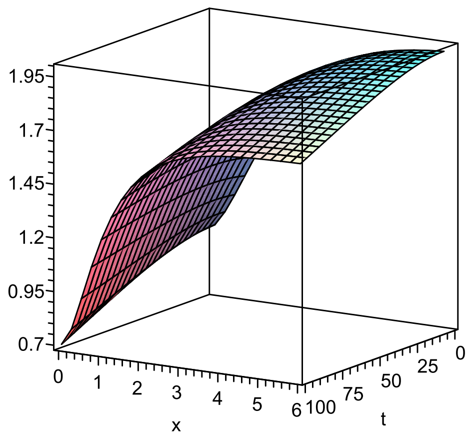


Figure 3.16: Cond. d. of $Z(t)$ in Example 3.2.4 Figure 3.17: Cdf of $Z(t)$ in Example 3.2.4

These results will be compared to those of the previous model in the following section.

Remark 3.1. In the truncated series method, there are no constrains on the parameters. The solution to the differential equations (or differential systems) exist for any parameters. For small time t, for example $t \leq 10$, our series are very short and compact. In Example 2.4.1 if $t \leq 10$, the approximate conditional density function is $0.510^{-6}\, te^{-x}\Big[60000 + (\frac{70}{3} - \frac{40}{3}x + x^2)t^2 + (600x - 1600)t\Big]$. For large t, as in Examples 3.2.3 and 3.2.4 the solutions form large polynomials with 15 terms to approximate accurately the solution. Still these series expressions for the density function are written as simple polynomials without any special functions, making their integration and differentiation possible.

3.3 Comparison of the Models

In this section we compare two models; the discounted compound Poisson sums and the discounted compound Erlang(n) sums.

This is possible here as both models for $Z(t)$ are special cases of the compound PH–renewal process, for which we obtained the mgf in Chapter 2 and the distribution in the previous section. Different compound sums are usually compared through their mean and variances. Since here we have the extended density function of $Z(t)$, the model comparisons can be based on both, their extended density functions and cdf's.

In our numerical examples, the mean of the inter–arrival times is kept constant at 200 and the claim severities are Erlang(2) with mean 2. We compare the cases where the inter–arrival times are exponential, Erlang(2), Erlang(3) and Erlang(4). Since some of the series approximations include many terms, we report only the mass at $x = 0$ and show the graphs of the conditional density functions, given $x > 0$ and the cdf's.

Note that the point mass $x = 0$ with the series method is the same as the exact value $\mathbb{P}\big(N(t) = 0\big)$ for time $t \leq 100$. These exact expressions for the masses at $x = 0$ are reported below.

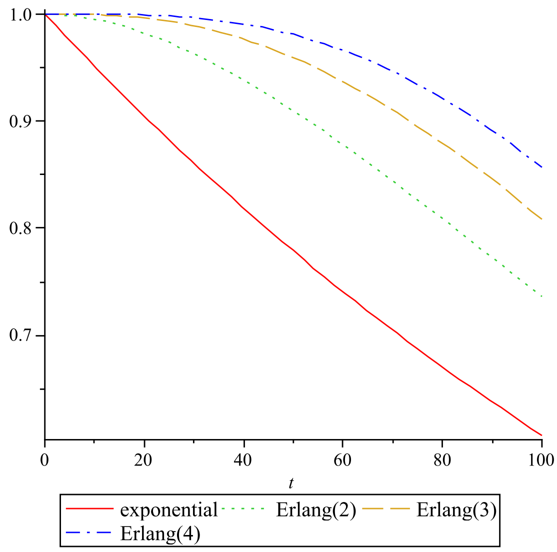


Figure 3.18: Probability mass at $x = 0$

Inter–arrival times	Probability mass at $x = 0$
Exponential	$e^{-0.005t}$
Erlang(2)	$e^{-0.01t} + 0.01te^{-0.01t}$
Erlang(3)	$e^{-0.03t} + \frac{3}{200}te^{-0.03t} + \frac{9}{80000}t^2e^{-0.03t}$
Erlang(4)	$e^{-0.02t} + \frac{1}{50}te^{-0.02t} + \frac{1}{5000}t^2e^{-0.02t} + \frac{1}{750000}t^3e^{-0.02}$

Figure 3.19: Probability mass $x = 0$

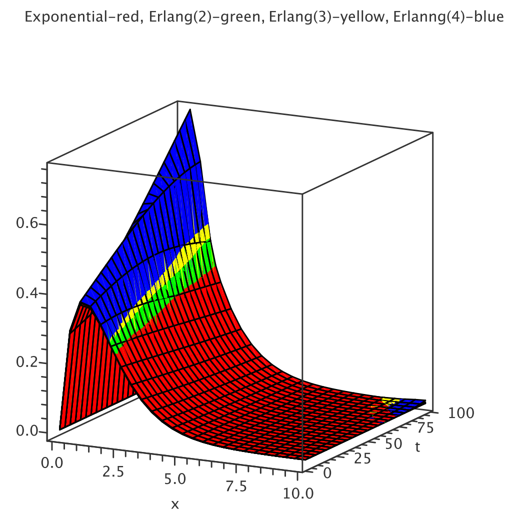


Figure 3.20: Cond. d. of $Z(t)$

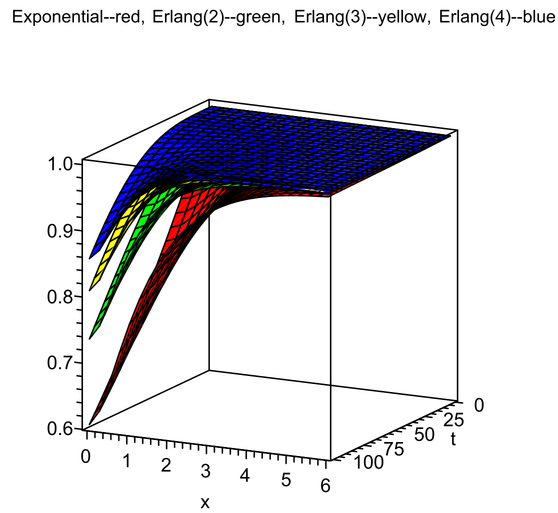


Figure 3.21: Cdf of $Z(t)$

Figures (3.20) and (3.21) show that:

- The density of $Z(t)$ for the exponential inter–arrival times (Poisson model) has the heaviest tail and the mass at $x = 0$ is the smallest in Figure (3.18).
- The difference between curves for different Erlang(n) models gets smaller as n increases and the tail of the density of $Z(t)$ decreases resulting in an increasing mass at 0.

Hence we conclude that the Poisson process is riskier than th Erlang(n)-renewal process, for $n = 2, 3$ and 4.

3.4 Relationship Between Poisson and Erlang(n) Processes

The previous section compares graphically the riskiness between Erlang($n = 1, 2, 3, 4$) claim renewal models with Erlang(2) claim severities. In this section we analyze the mathematical relationships between models, as initiated in Léveillé, Garrido and Wang (2010) which studies compound renewal processes with discounted PH claim severities. Here we generalize these results to the discounted compound renewal sum with any Erlang(n) inter–arrival times

Let $M^P_{Z(t)}(s)$ be the mgf of $Z^P(t)$ when inter–arrival times are exponential (Poisson model), $M^{E(n)}_{Z(t)}(s)$ be the mgf of $Z^{E(n)}(t)$ when inter–arrival times are Erlang(n). Then

$$M^P_{Z(t)}(s) = M^{E(n)}_{Z(t)}(s)\,\widehat{h}(t;s)\,, \tag{3.18}$$

and $\widehat{h}(t;s)$ is a function of t at fixed s. The idea is to solve (3.18) for $\widehat{h}(t;s)$ and show that whether or not function $\widehat{h}(t;s)$ is the mgf of a random variable, which means that the Poisson process can be written as sum of two independent variables.

First consider simple cases where the claim severity has an exponential distribution with parameter θ. Hence

$$M^P_{Z(t)}(s) = \left(\frac{\theta - se^{-\delta t}}{\theta - s}\right)^{\frac{\lambda}{\delta}}\,, \qquad s < \theta\,, \tag{3.19}$$

where λ is a Poisson rate. This result can be obtained by simplifying Corollary 2.3.2

Let the Poisson and Erlang(2) processes have the same mean inter–arrival time, hence the Poisson rate is 0.005 and the Erlang(2) parameter is 0.01. Under the assumption that exponential claim severities parameter $\theta = 1$, the net interest $\delta = 0.01$, (3.19) gives

$$M^P_{Z(t)}(s) = \left(\frac{1 - se^{-0.01t}}{1 - s}\right)^{0.5}\,, \qquad s < 1,$$

and the solution of second–order homogeneous differential equation (2.56) is:

$$M^{E(2)}_{Z(t)}(s) = \frac{1}{s^2}\left\{(s-1)\left(se^{-0.01t} - 2\right)\ln\left[\frac{1-s}{1-se^{-0.01t}}\right] + se^{-0.01t}(s-2) + 2s\right\}.$$

Thus the function $\widehat{h}(t;s)$ in (3.18) is written as:

$$\begin{aligned}\widehat{h}(t;s) &= \frac{M^P_{Z(t)}(s)}{M^{E(2)}_{Z(t)}} \\ &= \frac{s^2\left(1 - se^{-0.01t}\right)^{0.5}}{(1-s)^{0.5}\left[(s-1)\left(se^{-0.01t} - 2\right)\ln\left[\frac{1-s}{1-se^{-0.01t}}\right] + se^{-0.01t}(s-2) + 2s\right]}.\end{aligned} \tag{3.20}$$

Since the expression of $\widehat{h}(t;s)$ is complicated, inverting the corresponding Laplace transform is impossible. However the series approximation in terms of t gives a nice result. The function denoted by $h^{(2)}(x)$, which corresponds to the inverted Laplace transform for the time parameter $t \leq 40$; is given by the series method to be

$$h^{(2)}(x) = \begin{cases} 0.8769 \quad , & \text{if} \quad x = 0\,, \\ (0.1357 - 0.0001667x^3 + 0.002533x^2 - 0.0167x)e^{-x}\,, & \text{if} \quad x > 0\,. \end{cases}$$

We also have

$$\int_0^{\infty} (0.1357 - 0.0001667x^3 + 0.002533x^2 - 0.0167x)e^{-x} dx + 0.8769 = 1\,, \tag{3.21}$$

which implies that the function $h^{(2)}$ is an extended density function for a certain random variable and $\widehat{h}(t;s)$ is the mgf of this random variable. Thus random variable $Z^P(t)$ can also be written as the sum of two independent random variables. That is (in distribution)

$$Z^P(t) \stackrel{\mathcal{D}}{=} Z^{E(2)}(t) + Z^{(2)}(t)\,, \tag{3.22}$$

where the random variables $Z^{E(2)}(t)$ and $Z^{(2)}(t)$ are independent.

We have checked that the above result is also true for Erlang(3) inter–arrival times with parameter $\lambda = 0.015$. Comparing to the previous Erlang(2) inter–arrival times, a closed form solution for $M_{Z(t)}^{E(3)}$ is difficult to get, however the series method gives $M_{Z(t)}^{E(3)}$, hence (3.18) yields the function $\widehat{h}(t;s)$. Since the expression is tedious, here we give the inverse Laplace transform corresponding to $\widehat{h}(t;s)$ as follows:

$$h^{(3)}(x) = \begin{cases} 0.7920 \quad , & \text{if} \quad x = 0\,, \\ 0.23063 + 0.00586x^2 - 0.00041x^3 - 0.0319x\,, & \text{if} \quad x > 0\,. \end{cases}$$

From

$$0.7920 + \int_0^{\infty} 0.23063 + 0.00586x^2 - 0.00041x^3 - 0.0319x dx = 1\,,$$

thus $h^{(3)}(x)$ is an extended density function of a random variable denoted by $Z^{(3)}(t)$, and hence

$$Z^P(t) \stackrel{\mathcal{D}}{=} Z^{E(3)}(t) + Z^{(3)}(t)\,,$$

where the variables $Z^{E(3)}(t)$ and $Z^{(3)}(t)$ are independent.

We have checked with Maple that it is also true that there exist independent random variables $Z^{E(4)}(t)$ and $Z^{(4)}(t)$, for a fixed t such that

$$Z^P(t) \stackrel{\mathcal{D}}{=} Z^{E(4)}(t) + Z^{(4)}(t)\,.$$

We conjecture that for Erlang(n) for a large $n = 5, 6, \cdots$, the following relation holds

$$Z^P(t) \stackrel{\mathcal{D}}{=} Z^{E(n)}(t) + Z^{(n)}(t)\,,$$

where the random variables $Z^{E(n)}(t)$ and $Z^{(n)}(t)$ are independent, when claim severities are exponentially distributed.

The natural question now is if that this relationship holds true also for the other claim severities? Consider Erlang(2) claim severities with mean of 2, under a net interest rate $\delta = 0.01$, hence:

$$M_X(se^{-\delta t}) = \frac{1}{(1 - se^{-\delta t})^2}\,. \tag{3.23}$$

Substituting (3.23) into (2.49) produces

$$M^P_{Z(t)}(s) = \left(\frac{1 - se^{-0.01t}}{1 - s}\right)^{0.5} \exp\left\{\frac{1}{1-s} - \frac{1}{1 - se^{-0.01t}}\right\}, \qquad s < 2\,. \tag{3.24}$$

The series approximation gives $M^{Er(2)}_{Z(t)}(s)$, hence $\widehat{h}(t;s)$ can be written in the form of $\frac{M^P_{Z(t)}(s)}{M^{Er(2)}_{Z(t)}(s)}$. The approximation of $M^{Er(2)}_{Z(t)}(s)$ produces long polynomials with many terms. Hence, here, only the corresponding inverted Laplace transform of $\widehat{h}(t;s)$ is given by:

$$h^{(2)}(x) = \begin{cases} 0.8307, & \text{if } x = 0, \\ 0.217e - 4x(x - 10.0117)(x - 47.0697)(x^2 - 2.9187x + 22.1542)e^{-x}, & \text{if } x > 0\,. \end{cases}$$

Finally, we can check that the following integral

$$\int_0^\infty 0.217e - 4x(x - 10.0117)(x - 47.0697)(x^2 - 2.9187x + 22.1542)e^{-x}dx + 0.8307 = 1\,,$$

which implies that the function $h^{(2)}$ is the extended density function of some random variable, denoted $Z^{(2)}(t)$ as before, and $\widehat{h}(t;s)$ is its mgf.

For Erlang($n = 3, 4$) inter–arrival times with Erlang(2) claim severities, $Z^P(t)$ can also be written as the sum of two independent variables as follows:

$$Z^P(t) \stackrel{\mathcal{D}}{=} Z^{E(n)}(t) + Z^{(n)}(t)\,. \tag{3.25}$$

Remark 3.2. From the above numerical results, we conjecture that (3.25) is true for any claim severities.

Chapter 4

Applications

Reinsurance is a common mechanism to share riskiness among insurance companies. Stop–loss and excess–of–loss reinsurance treaties are examples of popular contracts. However general formulas to calculate exact stop–loss premiums under net interest $\delta \neq 0$ are not available for discounted compound sums $Z(t)$.

Kaas, Vanneste and Goovaerts (1992) find the maximal stop–loss premium for a given retention of compound Poisson (λ) risk models, with known λ and known means and variance of the claim severity distribution. Xu, Bricker and Kortanek (1998) give an approximation for both upper and lower bounds on the stop-loss premium, when the claim distribution is not known. Genest, Marceau and Mesfioui (2002) present two different approaches to calculate upper bounds for stop-loss premiums, when the mean and variance of claims are known, but there is no information concerning the dependence on claim severities. For the discounted compound renewal models, Léveillé and Garrido (2001a, 2001b) obtain the first two moments and recursive formulas for higher moments.

In the next Section 4.1 we discuss the calculation of stop–loss premiums in detail for discounted PH–renewal sums. A transformation method is introduced in order to solve the differential equations for the mgf of $Z(t)$, when the mean of inter–arrival times is very small. With it we can calculate the stop-loss premiums numerically.

A second application of results on discounted compound renewal sums is to compare the riskiness of different aggregate claim models. A common method for such comparisons of financial risks is the use of risk measures. In Section 4.2, some well-known risk measures such as Value-at-Risk (VaR), Conditional Tail Expectation (CTE) are discussed. In practice the number of claims received by insurance or reinsurance companies may be large. Two numerical examples are given to illustrate the results for large numbers of claims. The stop–loss premium, VaR and CTE are compared for different discounted compound renewal models and net interest rates.

4.1 Stop–Loss Premiums

In previous chapters, we have obtained ordinary differential equations or homogeneous differential equations for the mgf of $Z(t)$. In obtaining the density function of $Z(t)$, two problems are encountered. The first one obviously is to solve the differential equation to obtain the mgf, and the second is to then get density functions by inverting the Laplace transform.

Theoretically, the series method can provide a solution to the differential equations, however when the mean of the inter–arrival time is very small and time is counted in years, then a large number of terms are needed in the series to approximate the solution accurately. Hence computer programs require more time and more memory to reach a solution. For example, for Erlang(n) inter–arrival times with parameter λ for the discounted compound sum $Z(t)$, if the mean inter–arrival time is $\frac{n}{\lambda} = 0.01$ (i.e. $\lambda = 100n$) the number of terms required by the series method to solve the differential equations for $t = 2$ years is at least $1,800$. For such cases, where λ is large, we propose the use of an exponential factor, such as $e^{-\lambda t}$ or $e^{\lambda t}$, applied to the differential equations before seeking a solution. When $\lambda = 100n$ and t is 1 or 2 , then $e^{-\lambda t}$ is very small and it takes longer to approximate the solution by the series method. In some circumstances we can obtain exact solutions to these differential equations, but in such cases the solution is written in terms of special functions. These special functions are not always available in closed form. The mgf of $Z(t)$, while written in series form, can be inverted easily. Hence we focus here on the series method to solve on differential equations.

4.1.1 Exponential Inter–Arrival Times

Consider Poisson claim arrivals at rate λ. Even thought a closed form for the mgf of $Z(t)$ is available, it cannot be inverted easily, especially for large values of the parameter λ. Hence for a given function g, and a fixed s, the transformation

$$M_{Z(t)}(s) = e^{-\lambda t} g(t; s) \tag{4.1}$$

will be faster and simpler to invert numerically.

First consider a simpler case, where inter–arrival times have an exponential distribution with parameter λ. Hence (2.49) gives a first-order differential equation as follows:

$$\frac{\partial}{\partial t} M_{Z(t)}(s) = a_0(t; s, 1) M_{Z(t)}(s) \,, \tag{4.2}$$

where, for a fixed s, the coefficient $a_0(t; s, 1) = \lambda\Big[M_X(se^{-\delta t}) - 1\Big]$ and M_X is the mgf of the claim severity distribution F_X.

Differentiating both sides of (4.1) with respect to t gives:

$$\frac{\partial}{\partial t} M_{Z(t)}(s) = e^{-\lambda t}\Big[- \lambda g(t; s) + \frac{\partial}{\partial t} g(t; s)\Big] \,. \tag{4.3}$$

From (4.2) we have:

$$a_0(t; s, 1) M_{Z(t)}(s) = e^{-\lambda t}\Big[- \lambda g(t; s) + \frac{\partial}{\partial t} g(t; s)\Big] \,, \tag{4.4}$$

which implies the following homogeneous differential equation in t for $g(t; s)$ at a fixed s:

$$\frac{\partial}{\partial t} g(t; s) = \lambda M_X(se^{-\delta t}) g(t; s) \,. \tag{4.5}$$

Solving the differential equation in (4.5) for $g(t; s)$ is simpler than the original one in (4.2) for $M_{Z(t)}(s)$. The solution of (4.2) is then given by multiplying $g(t; s)$ by $e^{-\lambda t}$. This is illustrated in the following cases.

4.1.2 Erlang(2) Inter–Arrival Times

For Erlang(2) inter–arrival times, Léveillé and Garrido and Wang (2010) obtained the following homogeneous differential equation for $M_{Z(t)}(s)$ in t at a fixed s:

$$\frac{\partial^2}{\partial t} M_{Z(t)}(s) = a_1(t; s, 2) \frac{\partial}{\partial t} M_{Z(t)}(s) + a_0(t; s, 2) M_{Z(t)}(s) \,, \qquad t \geq 0 \,, \tag{4.6}$$

with initial values $M_{Z(0)}(s) = 1$ and $\frac{\partial}{\partial t} M_{Z(t)}(s)|_{t=0} = 0$, where $a_0(t;s,2) = \lambda^2 \big[M_X(se^{-\delta t}) - 1\big]$ and $a_1(t;s,2) = \frac{\frac{\partial}{\partial t}\big[M_X(se^{-\delta t})-1\big]}{\big[M_X(se^{-\delta t})-1\big]} - 2\lambda$.

To obtain the function $g(t;s)$, differentiate both sides of (4.1) with respect to t gives (4.3). Differentiating a second time gives:

$$\frac{\partial^2}{\partial t^2} M_{Z(t)}(s) = e^{-\lambda t}\Big[\lambda^2 g(t;s) - 2\lambda \frac{\partial}{\partial t} g(t;s) + \frac{\partial^2}{\partial t^2} g(t;s)\Big] . \tag{4.7}$$

Substituting expressions (4.3) and (4.7) into (4.6) yields the following differential equation for $g(t;s)$:

$$\frac{\partial^2}{\partial t^2} g(t;s) = b_1(t;s,2) \frac{\partial}{\partial t} g(t;s) + b_0(t;s,2) g(t;s) , \tag{4.8}$$

with coefficients $b_1(t;s,2) = 2\lambda + a_1(t;s,2)$ and $b_0(t;s,2) = -\lambda^2 - a_1(t;s,2)\lambda - a_0(t;s,2)$.

A pattern on the order n of the Erlang(n) assumption seems to emerge when going from $n = 1$ in (4.5) to $n = 2$ in (4.8). The next section further explores this pattern for $n \geq 3$.

4.1.3 Erlang(3) Inter–Arrival Times

When inter–arrival times are Erlang(3), from Corollary 2.3.3, we have the following homogeneous differential equation in t at a fixed s:

$$\frac{\partial^3}{\partial t^3} M_{Z(t)}(s) = a_2(t;s,3) \frac{\partial^2}{\partial t^2} M_{Z(t)}(s) + a_1(t;s,3) \frac{\partial}{\partial t} M_{Z(t)}(s) + a_0(t;s,3) M_{Z(t)}(s) , \quad t \geq 0 , \tag{4.9}$$

with initial values $M_{Z(0)}(s) = 1$, $\frac{\partial}{\partial t} M_{Z(t)}(s)|_{t=0} = 0$, $\frac{\partial^2}{\partial t^2} M_{Z(t)}(s)|_{t=0} = 0$, and where

$$\begin{aligned}
a_2(t;s,3) &= \frac{2 \frac{\partial}{\partial t}\big[M_X(se^{-\delta t}) - 1\big]}{\big[M_X(se^{-\delta t}) - 1\big]} - 3\lambda , \\
a_1(t;s,3) &= \frac{\frac{\partial^2}{\partial t^2}\big[M_X(se^{-\delta t}) - 1\big] - 3\lambda^2 \big[M_X(se^{-\delta t}) - 1\big] - a_2(t;s,3) \frac{\partial}{\partial t}\big[M_X(se^{-\delta t}) - 1\big]}{\big[M_X(se^{-\delta t}) - 1\big]} , \\
a_0(t;s,3) &= \lambda^3 \big[M_X(se^{-\delta t}) - 1\big] .
\end{aligned}$$

To get $g(t;s)$ in (4.1), differentiate (4.1) with respect to t, which yields (4.3). Differentiating a second time gives (4.7) and the third time we have at a fixed s:

$$\frac{\partial^3}{\partial t^3} M_{Z(t)}(s) = e^{-\lambda t}\Big[-\lambda^3 g(t;s) + 3\lambda^2 \frac{\partial}{\partial t} g(t;s) - 3\lambda \frac{\partial^2}{\partial^2 t} g(t;s) + \frac{\partial^3}{\partial t^3} g(t;s)\Big] . \tag{4.10}$$

Substituting expressions (4.3), (4.7) and (4.10) into (4.9) also yields a third-order homogeneous differential equation for $g(t;s)$:

$$\frac{\partial^3}{\partial t^3}g(t;s) = b_2(t;s,3)\frac{\partial^2}{\partial t^2}g(t;s) + b_1(t;s,3)\frac{\partial}{\partial t}g(t;s) + b_0(t;s,3)g(t;s)\,, \tag{4.11}$$

with initial values $g(0;0) = 1$, $\frac{\partial}{\partial t}g(t;s)|_{t=0} = \lambda$ and $\frac{\partial^2}{\partial t^2}g(t;s)|_{t=0} = \lambda^2$ and coefficients :

$$\begin{aligned} b_2(t;s,3) &= 3\lambda + a_2(t;s,3)\,, \\ b_1(t;s,3) &= -3\lambda^2 - 2a_2(t;s,3) + a_1(t;s,3)\,, \\ b_0(t;s,3) &= \lambda^3 + \lambda^2 a_2(t;s,3) - \lambda a_1(t;s,3) + a_0(t;s,3)\,. \end{aligned}$$

Clearly a pattern for $g(t;s)$, when the inter–arrival times has Erlang($n = 4, 5, \cdots$) distribution emerges and is given in the following remark.

Remark 4.1. Corollary 2.3.3 gives an n-th homogeneous differential equation for $M_{Z(t)}(s)$, when the inter–arrival times are Erlang(n). The differential equation emerges and is given by:

$$\begin{aligned} \frac{\partial^n}{\partial t^n}M_{Z(t)}(s) &= a_{n-1}(t;s,n)\frac{\partial^{n-1}}{\partial t^{n-1}}M_{Z(t)}(s) + a_{n-2}(t;s,n)\frac{\partial^{n-2}}{\partial t^{n-2}}M_{Z(t)}(s) + \cdots \\ &\quad + a_1(t;s,n)\frac{\partial}{\partial t}M_{Z(t)}(s) + a_0(t;s,n)M_{Z(t)}(s)\,, \end{aligned} \tag{4.12}$$

with initial values

$$M_{Z(0)}(s) = 1\,, \frac{\partial}{\partial t}M_{Z(t)}(s)|_{t=0} = 0\,, \frac{\partial^2}{\partial t^2}M_{Z(t)}(s)|_{t=0} = 0\,, \cdots\,, \frac{\partial^{n-1}}{\partial t^{n-1}}M_{Z(t)}(s)|_{t=0} = 0\,,$$

where

$$a_k(t;s,n) = \frac{\binom{n-1}{n-k}\frac{\partial^{n-k}}{\partial t^{n-k}}M(t;s) - \binom{n}{n-k}\lambda^{n-k}M(t;s) - \sum_{i=1}^{(n-2)-(k-1)} a_{k+i}(t;s,n)\binom{k+i-1}{i}\frac{\partial^i}{\partial t^i}M(t;s)}{M(t;s)}\,,$$

with $M(t;s) = \left[M_X(se^{-\delta t}) - 1\right]$, for $k = 1, 2, \cdots, n-1$ and $a_0(t;s,n) = \lambda^n M(t;s)$.
Following the same procedure as for exponential, Erlang(2) and Erlang(3) inter–arrival times, a n-th order homogeneous differential equation of $g(t;s)$ is obtained when the inter–arrival times are Erlang(n). This homogeneous differential equation is given by:

$$\begin{aligned} \frac{\partial^n}{\partial t^n}g(t;s) &= b_{n-1}(t;s,n)\frac{\partial^{n-1}}{\partial t^{n-1}}g(t;s) + b_{n-2}(t;s,n)\frac{\partial^{n-2}}{\partial t^{n-2}}g(t;s) + \cdots \\ &\quad + b_1(t;s,n)\frac{\partial}{\partial t}g(t;s) + b_0(t;s,n)g(t;s)\,, \end{aligned}$$

with initial value

$$g(0;s) = 1\,,\ \frac{\partial}{\partial t}g(t;s)|_{t=0} = \lambda\,,\ \frac{\partial^2}{\partial t^2}g(t;s)|_{t=0} = \lambda^2\,,\cdots,\ \frac{\partial^{n-1}}{\partial t^{n-1}}g(t;s)|_{t=0} = \lambda^{n-1}\,,$$

where the coefficients are

$$b_j(t;s,n) = (-1)^{n+j+1}\binom{n}{j}\lambda^{n-j} + \sum_{k=j}^{n-1}(-1)^{k-j}\binom{k}{j}a_k(t;s,n)\lambda^{k-j}$$

and $a_k(t;s,n)$ are the coefficients of the differential equation (4.12).

The following numerical examples illustrate how to solve differential equations by the transformation and series methods. The time parameter is kept small here, at typical levels in practice of $t = 1$ or 2 years.

Example 4.1.1. Consider inter–arrival times to be exponential, Erlang(2) and Erlang(3), while claim severities are Erlang(2). The mean of inter–arrival times are 0.02 (*i.e.* $\lambda = 50n$), while the mean of claim severities is 1000. Under a net interest rate of $\delta = 0.04$ ($\beta = 0.08$ and $\alpha = 0.04$), the mean of $Z(1)$ for each model is given in the following table.

Inter–arrival times	Exponential	Erlang(2)	Erlang(3)
Mean of $Z(1)$	$49,013.20$	$48,763.25$	$48,679.93$

Table 4.1: Mean of $Z(1)$

The extended density function of $Z(1)$ can be found by solving differential equations (4.5), (4.8) and (4.11) for each model, respectively. These, first, yields the function $g(t;s)$. Then the mgf of $Z(1)$ is obtained by (4.1). Hence inverting the corresponding Laplace transforms gives the distribution of $Z(1)$, with the probability of point masses at $x = 0$ is $e^{-\lambda t}$, $e^{-\lambda t}(1+\lambda t)$ and $e^{-\lambda t}(1+\lambda t+\frac{1}{2}\lambda^2 t^2)$, respectively, for the inter–arrival times exponential, Erlang(2) and Erlang(3). Since the parameter λ is large, the probability at $x = 0$ is close to 0.

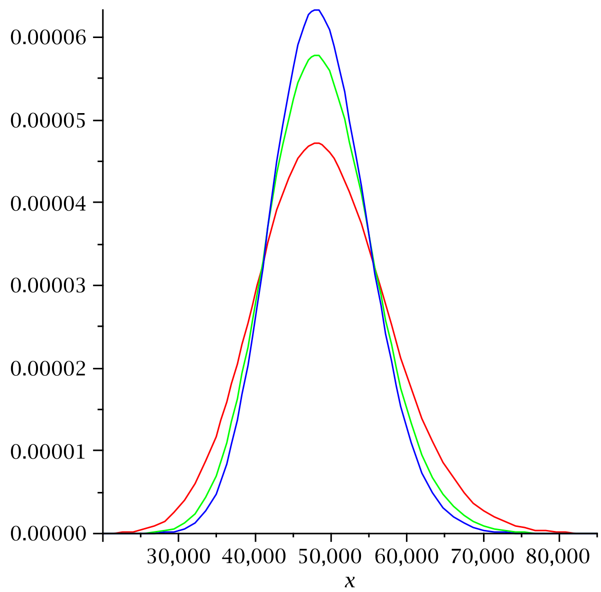


Figure 4.1: Conditional density function of $Z(1)$: exponential [in red], Erlang(2) [green], Erlang(3) [blue]

The conditional density functions in Figure 4.1 have the shape of normal distributions. We also see that right and left tails of $Z(1)$ are again heavier for exponentials inter–arrival times than for Erlang(n), when $n = 2$ and 3. $Z(1)$ is more concentrated around its mean as n increases.

In the next example, the mean inter–arrival time is twice that of Example 4.1.1 and t extends to 2, while claim severities are exponential.

Example 4.1.2. Again here, we assume that the inter–arrival times are Erlang(n) distributed ($n = 1, 2, 3$), but exponential claim severities. The mean of inter–arrival time is $0.01(\lambda = 100n)$ and mean claim severities 1000. Again the net interest $\delta = 0.04$ (where $\beta = 0.08$ and $\alpha = 0.04$). The graph of corresponding densities follows.

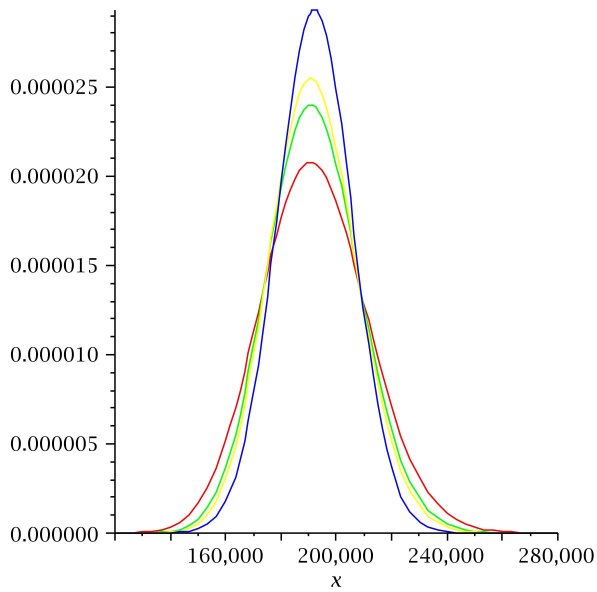


Figure 4.2: Conditional density function of $Z(2)$ in Example 4.1.2: exponential [in red], Erlang(2) [green], Erlang(3) [yellow], Erlang($n \to \infty$) [blue]

Note that in this example, the discounted compound sum is calculated at time $t = 2$. The conditional density functions in Figure 4.2 confirm the trend observed in Example 4.1.1. Again, the shape of the distributions seems normal. As n increases the aggregate claims in $Z(2)$ are more concentrated around the mean and the tail of the distribution is thicker for exponential inter–arrival times.

Remark 4.2. From Figures 4.1 and 4.2, we claim that the random sum $Z(t)$ goes to a normal distribution as the parameter n of the Erlang(n) inter–arrival times distribution tends to infinity. This idea can be verified analytically.

Let the mean and variance of inter–arrival times τ be $\mu = n\beta$ and $\sigma^2 = n\beta^2$, respectively, then

$$\sigma^2 = n(\frac{\beta}{n})^2 \longrightarrow 0\,, \quad \text{as} \quad n \longrightarrow \infty.$$

Chebyshev's inequality shows that

$$\tau \xrightarrow{\mathbb{P}} \mu\,,$$

hence, in the limit, inter–arrival times are constant (deterministic) and the number of claims is $\frac{t}{\mu}$ up to time t. By Lyapunov's central limit theorem, we conclude that $Z(t)$ goes to a normal distribution, whose graph (in blue) is given in Figure 5.1.2.

4.1.4 Application to Stop–loss Premiums

As an application of the above results, we now discuss the calculation of stop–loss premiums for discounted sums $Z(t)$. For the classical renewal risk model $S(t) = \sum_{k=1}^{N(t)} X_k$, the stop–loss premium for duration t is written as:

$$\pi_d(t) = \mathbb{E}\Big[\Big(S(t) - dt\Big)_+\Big] = \mathbb{E}\Big[\Big(\sum_{k=1}^{N(t)} X_k - dt\Big)_+\Big], \tag{4.13}$$

where the notation $(X-d)_+$, for any random variable X and non–negative value d:

$$(X-d)_+ = \begin{cases} 0\,, & \text{if} \quad x \le d\,, \\ X-d\,, & \text{if} \quad x > 0\,, \end{cases} \tag{4.14}$$

This model does not consider the effect of interest and inflation. When these parameters are included in accumulated retention limits, then dt becomes a non–trivial function of the contract duration t. Under constant rates of inflation β and interest α (and hence the net interest $\delta = \beta - \alpha$), the accumulated retention d for a period of t years is no longer linear, as in $d \cdot t$, but rather $d(t) = d\int_0^t e^{\alpha(t-s)}ds = d\bar{s}_{\overline{t}|\alpha}$ in actuarial notation. Hence the difference between the accumulated claims and this barrier, if positive, can be written as

$$\mathbb{E}\Big[\Big(e^{\beta t}Z(t) - d\bar{s}_{\overline{t}|\alpha}\Big)_+\Big] = \mathbb{E}\Big[\Big(\sum_{k=1}^{N(t)} e^{\beta t-\delta T_k}X_k - d\int_0^t e^{\alpha(t-s)}ds\Big)_+\Big]. \tag{4.15}$$

Discounting (4.15) back to the present value and taking expectation yields to

$$\pi_d(t) = e^{-\beta t}\,\mathbb{E}\Big[\Big(\sum_{k=1}^{N(t)} e^{\beta t-\delta T_k}X_k - d\int_0^t e^{\alpha(t-s)}ds\Big)_+\Big], \tag{4.16}$$

which implies that the stop-loss premium can be calculated as:

$$\pi_d(t) = \mathbb{E}\Big[\Big(\sum_{k=1}^{N(t)} e^{-\delta T_k}X_k - d\int_0^t e^{-\delta t-\alpha s}ds\Big)_+\Big] = \mathbb{E}\Big[\Big(Z(t) - d\,e^{-\delta t}\Big(\frac{1-e^{-\alpha t}}{\alpha}\Big)\Big)_+\Big] \tag{4.17}$$

Example 4.1.3. Revisit the numerical Example 3.1.1, with Erlang$(n, n = 1,2,3)$ inter–arrival times and Erlang(2) claim severities. The density functions of $Z(1)$ are continuous for $x > 0$, and (4.17) gives stop–loss premiums, reported in the following table, for a retention $d = 58,974.86$, which is calculated by the formula $d = \mathbb{E}[Z(1)] + 2\sqrt{\mathbb{V}(Z(1))}$.

Inter–arrival times	$\delta = 0$	$\delta = 0.04$	$\delta = 0.08$
Exponential	750.65	577.41	438.73
Erlang(2)	372.46	263.50	182.98
Erlang(3)	259.35	175.03	115.51

Table 4.2: Stop–loss premiums of $Z(1)$ in Example 4.1.3

Table 4.2 shows that stop–loss premiums are largest for exponential inter–arrival times, at any value of $\delta = 0$, $\delta = 0.04$ or $\delta = 0.08$. In each column, stop–loss premiums decrease for increasing Erlang(n) parameters $n = 1$ to $n = 3$. In each row, stop–loss premiums also decrease with increasing δ as expected. We also see that there is a large difference between $\delta = 0$ and $\delta > 0$. Hence considering the effect of interest is important, even for a one-year discounted compound sum. These conclusions are consistent with the prior observation that the random variable $Z(1)$ has a heavier tail, when inter–arrival times are exponential.

Example 4.1.4. Similarly for Example 4.1.2, with Erlang($n = 1, 2, 3$) inter–arrival times and exponential claim severities, the stop–loss premiums for $Z(2)$ are given in the following table for a retention $d = 22,524.82$, again calculated by the same formula $d = \mathbb{E}[Z(2)]+2\sqrt{\mathbb{V}(Z(2))}$ as in Example 4.1.3.

Inter–arrival times	$\delta = 0$	$\delta = 0.04$	$\delta = 0.08$
Exponential	$3,431.45$	$1,605.89$	662.80
Erlang(2)	2,497.62	1,001.03	338.99
Erlang(3)	2,173.75	809.35	249.27

Table 4.3: Stop–loss premiums for $Z(2)$ in Example 4.1.4

Note the substantial difference in stop–loss premiums due to the presence of the discounted rates $\delta = 0$, $\delta = 0.04$ or $\delta = 0.08$.

Remark 4.3. The solutions of our homogeneous differential equations are approximated by the series method. Hence, the length of the approximating series is determined by the choice of time t and parameter λ. For the calculation of stop–loss premiums as well as in the next section, these approximate results are rounded to two decimals.

4.2 Risk Measures

Financial institutions and insurance companies use single-index values to describe the riskiness of their businesses. Risk measures provide a useful analysis tool. We review here, the definition of a risk measure and discuss some common risk measures such as VaR and CTE. As an application of the previous results, VaR and CTE are calculated for different discounted compound sums $Z(t)$, for short time values $t = 1$ or $t = 2$. A comparison among the different values obtained for VaR and CTE is made.

Definition 4.2.1. *A risk measure is defined as a mapping from the set of random variables representing the risks at hand to the real line.*

A common notation for the risk measure associated with a random variable X is $\rho[X]$. Here we consider nonnegative random variables X representing insurance losses.

4.2.1 VaR

Value-at-risk (VaR) has become one of the most popular risk measures. In statistical terms, VaR is a quantile of the distribution of aggregate risks. VaR measures the worst loss, under normal market conditions for a specific time interval, at a given confidence level. Using the notation of Cai and Tan (2007), we define VaR as follows.

Definition 4.2.2. *The Value-at-Risk (VaR) of a random variable X at a confidence level $1-\alpha$, $0 < \alpha < 1$ is given by:*

$$VaR_X(\alpha) = \inf\left\{x : \mathbb{P}(X > x) \leq \alpha\right\} = \inf\left\{x : \mathbb{P}(X \leq x) \geq 1-\alpha\right\}. \tag{4.18}$$

By the definition $\mathbb{P}\big(X > VaR_X(\alpha)\big) \leq \alpha$ while for any $x < VaR_X(\alpha)$, $\mathbb{P}(X > x) > \alpha$. If the cdf of X is a continuous function then VaR can be defined by

$$VaR_X(\alpha) = \inf\{x :\, \mathbb{P}(X > x) = \alpha\}, \tag{4.19}$$

or

$$VaR_X(\alpha) = \inf\{x :\, \mathbb{P}(X \leq x) = 1-\alpha\} = F_X^{-1}(1-\alpha), \tag{4.20}$$

where $F_X^{-1}(1-\alpha)$ is the inverse cdf corresponding to the confidence level α, which is also called tolerance probability. In practice, α is often selected to be a small value such as

$\alpha \leq 0.05$.

As a illustration, we consider the previous numerical examples in Examples 4.1.1 and 4.1.2. The random variable $Z(t)$ is continuous on the positive real axis for the net interest rates $\delta = 0$, $\delta = 0.04$ and $\delta = 0.08$. Hence the risk measure can be written as

$$VaR_{Z(1)}(\alpha) = F^{-1}_{Z(1)}(1-\alpha)\,.$$

The following VaR results were calculated by Maple at a confidence level of $\alpha = 0.05$.

Inter–arrival times	$\delta = 0$	$\delta = 0.04$	$\delta = 0.08$
Exponential	64,793.81	63,516.11	62,274.00
Erlang(2)	61,786.10	60,561.81	59,371.30
Erlang(3)	60,639.15	59,435.41	58,264.70

Table 4.4: VaR of $Z(1)$ in Example 4.1.1

Inter–arrival times	$\delta = 0$	$\delta = 0.04$	$\delta = 0.08$
Exponential	233,729.62	224,634.10	216,025.43
Erlang(2)	228,924.71	220,004.20	211,559.10
Erlang(3)	227,161.53	218,305.54	209,920.90

Table 4.5: VaR of $Z(2)$ in Example 4.1.2

From Tables 4.4 and 4.5, we see that the largest VaR measures are for exponential inter–arrival times. In each column VaR decreases as n increases, and in each row VaR also decreases as the net interest δ increases. The percentage difference is almost 2% for each column in Table 4.4 and 4% in Table 4.5, which shows the effect of interest and inflation on $Z(1)$ and $Z(2)$ is non–negligible. The calculation of VaR also confirms that $Z(t)$, with exponential inter–arrival times is more risky than with Erlang(2 or 3) inter–arrival times.

Even though VaR is widely used by financial institutions, it is less popular in the academic world for not being sub-additive, i.e. the riskiness of a portfolio as a whole can be larger than the sum of the stand-alone risks of its components when measured by VaR. Consequently, VaR fails to justify diversification in a portfolio as a risk reducing measure. A popular alternative risk measure is Conditional Tail Expectation (CTE), discussed in the next subsection.

4.2.2 CTE

VaR measures the "worst case" losses, defined as the tail events with $1 - \alpha$ probability. However it does not account for the severities of losses above VaR, if the worst case actually occurs. Hence an alternative risk measure, Conditional Tail Expectation (CTE) has been proposed. Intuitively CTE captures the worst expected losses, given that they are greater or equal to VaR at a fixed confidence level.

Definition 4.2.3. *CTE*
The Conditional Tail Expectation (CTE) of a random variable X at its $VaR_X(\alpha)$ is formally defined as

$$CTE_X(\alpha) = \mathbb{E}[X|X > VaR_X(\alpha)]\,, \tag{4.21}$$

or

$$CTE_X(\alpha) = \mathbb{E}[X|X \geq VaR_X(\alpha)]\,. \tag{4.22}$$

If the random variable is continuous, then equations (4.21) and (4.22) are equivalent. Clearly CTE is larger than VaR. The calculation for CTE equivalent to (4.22) is

$$CTE_X(\alpha) \quad = \quad \frac{\mathbb{E}[XI_{X \geq VaR_X(\alpha)}]}{\mathbb{P}\big(X \geq VaR_X(\alpha)\big)}\,.$$

If the random variable X is continuous, then this formula can be further simplified as:

$$CTE_X(\alpha) \quad = \quad \frac{1}{\alpha}\int_{VaR_X(\alpha)}^{\infty} yf(y)dy\,,$$

where f is density function of X. Again we revisit Examples 4.1.1 and 4.1.2. The following CTE values were computed at the same confidence level $\alpha = 0.05$.

Inter–arrival times	$\delta = 0$	$\delta = 0.04$	$\delta = 0.08$
Exponential	$68,964.53$	$67,604.19$	$66,283.79$
Erlang(2)	$65,157.73$	$63,866.97$	$62,612.36$
Erlang(3)	$63,707.42$	$62,443.22$	$61,214.01$

Table 4.6: CTE of $Z(1)$ in Example 4.1.1

Inter–arrival times	$\delta = 0$	$\delta = 0.04$	$\delta = 0.08$
Exponential	$242,925.56$	$233,474.86$	$224,535.12$
Erlang(2)	$236,862.95$	$227,535.51$	$218,904.26$
Erlang(3)	$234,639.44$	$225,494.18$	$216,839.93$

Table 4.7: CTE of $Z(2)$ in Example 4.1.2

In both Tables 4.6 and 4.7, CTE decreases as n increases, in each column. It also decreases as the net interest δ increases. The comparison with Tables 4.4-4.5, we see that CTE is larger than VaR. We also see a substantial decrease in CTE, as δ goes from 0 to 0.08.

In this chapter we studied stop–loss reinsurance premiums. Since we now know the distribution function of the discounted compound sums, we can compute these premiums for different models. To calibrate the riskiness for insurance businesses, two risk measures are used here, VaR and CTE. We compared these model applications for different inter–arrival times and differential net interest rates. We conclude that interest and inflation play an important role in practical applications. Stop–loss premium, VaR and CTE values vary greatly for different inter–arrival time distributions and also when discounting is introduced.

From our numerical examples we also see that the shape of the conditional density function of $Z(t)$ looks like that of a normal distribution, as the mean inter–arrival time decreases. This is be proved in the next chapter.

Chapter 5

Asymptotics for Discounted Compound Renewal Processes

Insurance companies or financial institutions can have a large number of claims each year. For such large-scale portfolios, researchers use the heavy traffic method to approximate risk processes and get their distributions, as was initially done for an infinite-server system in the queueing literature.

The first treatment of diffusion approximations based on heavy traffic in risk theory is due to Iglehart (1969) and then Grandell (1977). The basic idea is to let the number of claims grow in a unit time interval. Garrido (1988) derived weak convergence theorems for modified compound renewal processes of large insurance portfolios. Furrer, Michna and Weron (1997) and Michna (2005) suggested an approximation based on $\alpha-$stable Lévy motion. Sarkar and Sen (2005) obtained the ruin probability and expected discounted penalty function for a diffusion perturbed model as the limit of compound Poisson risk models. Lam and Blanchet (2010) discuss the modeling of large–scale insurance portfolios, especially in the context of life insurance, from a heavy traffic perspective. Heavy traffic means that the arrival rate is large in an asymptotic sense, which resembles the typical scenario of large insurance companies with a high number of policyholders.

In the previous chapter, we developed a transformation method to solve differential equations when the mean of inter–arrival time is very small. Figures 4.1 and 4.2 show that the distribution of discounted compound renewal processes looks like a normal distribution. In this chapter, we investigate analytically the asymptotical normal properties for a portfolio with a large number of claims.

The approach that we use here is a normal approximation. We assume that the mean inter–arrival times goes to 0, which implies that the rate of arrival is large. We study the asymptotic distribution for both compound Poisson processes and compound renewal processes, separately. Furthermore, the asymptotic distributions of discounted compound Poisson and renewal processes are also obtained.

5.1 Compound Poisson Processes

For the Poisson process with rate λ, the inter–arrival times are exponentially distributed with mean $\frac{1}{\lambda}$. If the mean inter–arrival time goes to zero, it implies that λ goes to ∞. That is

$$\frac{1}{\lambda} \longrightarrow 0 \Leftrightarrow \lambda \longrightarrow \infty \,.$$

The following results can be obtained by heavy traffic methods, since the role played by λ and t is symmetric in the probability function of the number of claims (see Remark 5.1).

Let $S(t) = \sum_{k=1}^{N(t)} X_k$ and X_k be iid claim severities, where $N(t)$ forms a Poisson process. Consider the normalized process:

$$S^{\star}(t) = \frac{1}{\sigma_X \sqrt{\lambda t}} \Big[S(t) - N(t)\mu_X \Big] = \frac{1}{\sigma_X \sqrt{\lambda t}} \Big[\sum_{k=1}^{N(t)} X_k - N(t)\mu_X \Big] \,,$$

where μ_X and σ_X are the mean and standard deviation of claim severities X, and hence $\sigma_X \sqrt{\lambda t} = \sqrt{\mathbb{V}(S(t) - N(t)\mu_X)}$. Next we prove that $S^{\star}(t)$ is asymptotically normally distributed.

Proposition 5.1.1. *Let the mean of inter–arrival times be* $\frac{1}{\lambda}$*, then for a fixed* $t > 0$

$$S^{\star}(t) \xrightarrow{D} N(0,1) \,, \quad \textit{if} \quad \lambda \longrightarrow \infty \,. \tag{5.1}$$

This result is given by Bening and Korolev (2002). Here we reproduce a proof that uses the mgf method.

Proof.

$$S^{\star}(t) = \frac{1}{\sigma_X\sqrt{\lambda t}}\Big[\sum_{k=1}^{N(t)} X_k - N(t)\mu_X\Big] = \sum_{k=1}^{N(t)} \frac{Y_k}{\sigma_X\sqrt{\lambda t}}\,, \tag{5.2}$$

where $Y_k = X_k - \mu_X$. Let the mgf of $S^{\star}(t)$ be denoted by $M_{S^{\star}(t)}(s)$, then we have

$$M_{S^{\star}(t)}(s) = e^{\lambda t\left[M_Y(\frac{s}{\sigma_X\sqrt{\lambda t}})-1\right]}\,. \tag{5.3}$$

Since the mgf $M_Y(\frac{s}{\sigma_X\sqrt{\lambda t}})$ can be written in the following form:

$$M_Y(\frac{s}{\sigma_X\sqrt{\lambda t}}) \;=\; 1+\frac{s}{\sigma_X\sqrt{\lambda t}}\mathbb{E}[Y]+\frac{1}{2}(\frac{s}{\sigma_X\sqrt{\lambda t}})^2\mathbb{E}[Y^2]+o(\frac{1}{\lambda})\,, \tag{5.4}$$

where $\mathbb{E}[Y] = 0$ and $\mathbb{E}[Y^2] = \mathbb{V}[X] = \sigma_X^2$, hence

$$\lambda t\Big[M_Y(\frac{s}{\sigma_X\sqrt{\lambda t}})-1\Big] = \lambda t\Big[\frac{s^2}{2\lambda t}+o(\frac{1}{\lambda})\Big] \longrightarrow \frac{s^2}{2}\,, \quad \text{as} \quad \lambda\longrightarrow\infty. \tag{5.5}$$

Substituting (5.5) into (5.3) gives:

$$\lim_{\lambda\to\infty} M_{S^{\star}(t)}(s) = e^{\frac{s^2}{2}}\,. \tag{5.6}$$

Thus

$$S^{\star}(t) \xrightarrow{D} N(0,1)\,, \quad \text{as} \quad \lambda\longrightarrow\infty\,. \tag{5.7}$$

□

Proposition 5.1.2. *Let $N(t)$ be a Poisson process with rate λ, then $N(t)$ is approximately asymptotically normal with mean λt and variance λt, when λ goes to ∞. That is*

$$\frac{N(t)-\lambda t}{\sqrt{\lambda t}} \xrightarrow{D} N(0,1)\,, \qquad \textit{as} \quad \lambda\longrightarrow\infty\,. \tag{5.8}$$

This is a classical result, for example see Bowers et al. (1997).

Proof. Let $N^{\star}(t) = \frac{1}{\sqrt{\lambda t}}(N(t)-\lambda t)$, since the mgf of $N(t)$ is

$$M_{N(t)}(s) = e^{\lambda t(e^s-1)}\,. \tag{5.9}$$

This implies

$$M_{N^{\star}(t)}(s) = e^{-\sqrt{\lambda t}s}M_{N(t)}(\frac{s}{\sqrt{\lambda t}}) = e^{-\sqrt{\lambda t}s}e^{\lambda t(e^{\frac{s}{\sqrt{\lambda t}}}-1)} = e^{-\sqrt{\lambda t}s+\lambda t(e^{\frac{s}{\sqrt{\lambda t}}}-1)} \tag{5.10}$$

By Taylor's series, we have

$$e^{\frac{s}{\sqrt{\lambda t}}} - 1 = 1 + \frac{s}{\sqrt{\lambda t}} + \frac{1}{2}(\frac{s}{\sqrt{\lambda t}})^2 + o(\frac{1}{\lambda}) - 1\,, \tag{5.11}$$

this gives, as $\lambda \longrightarrow \infty$,

$$-\sqrt{\lambda t}s + \lambda t(e^{\frac{s}{\sqrt{\lambda t}}} - 1) = -\sqrt{\lambda t}s + \lambda t[\frac{s}{\sqrt{\lambda t}} + \frac{1}{2}(\frac{s}{\sqrt{\lambda t}})^2 + o(\frac{1}{\lambda})] \longrightarrow \frac{1}{2}s^2\,. \tag{5.12}$$

Hence

$$\lim_{\lambda\to\infty} M_{N^\star(t)}(s) = e^{\frac{s^2}{2}}\,, \tag{5.13}$$

which shows that

$$\frac{1}{\sqrt{\lambda t}}(N(t) - \lambda t) \xrightarrow{D} N(0,1)\,, \qquad \text{as} \quad \lambda \longrightarrow \infty\,. \tag{5.14}$$

□

Corollary 5.1.1. *For a fixed $t > 0$,*

$$\frac{1}{\sqrt{\lambda t(\mu_X^2 + \sigma_X^2)}}\Big(\sum_{k=1}^{N(t)} X_k - \lambda t \mu_X\Big) \xrightarrow{D} N(0,1)\,, \qquad \textit{as} \quad \lambda \longrightarrow \infty\,. \tag{5.15}$$

Proof. Based on Propositions 5.1.1 and 5.1.2, composition gives (5.15). For details see Englund (1983). □

Remark 5.1. The result in Proposition 5.1.1 is a special case of the weak convergence heavy traffic results for the compound Poisson process by Iglehart (1969), Grandell (1977), Garrido (1988), Furrer, Michna and Weron (1997) and Michna (2005):

$$\frac{1}{\sqrt{\lambda nt(\mu_X^2 + \sigma_X^2)}}\Big(\sum_{k=1}^{N(nt)} X_k - \lambda nt \mu_X\Big) \xrightarrow{D} W(t)\,, \qquad \text{as} \quad n \longrightarrow \infty\,, \tag{5.16}$$

where λ is the Poisson rate and W is a Wiener process. Since the distribution of $N(nt)$ is:

$$\mathbb{P}\big(N(nt) = k\big) = \frac{(\lambda nt)^k}{k!}e^{-\lambda nt}\,, \quad \text{for } k = 0, 1, 2, ...$$

then $N(nt)$ has the same asymptotic probability distribution, if $\lambda \longrightarrow \infty$ or $n \longrightarrow \infty$, for fixed t, respectively. Let $n = \lambda$ in Proposition 5.1.1 and $\lambda = 1$ in (5.16), then $N(nt)$ in (5.16) and $N(t)$ Proposition 5.1.1 have the same asymptotic distribution as $n \longrightarrow \infty$ and $\lambda \longrightarrow \infty$, respectively. We then consider the distributions (5.16) and (5.15) to be equivalent.

Before we study the asymptotic normality of discounted compound Poisson sums, the following result is needed.

Remark 5.2. In general, for any renewal process, if the mean inter–arrival time λ^{-1} goes to 0, it implies that inter–arrival times $\tau_n \xrightarrow{\mathbb{P}} 0$ as $\lambda \longrightarrow \infty$. Then it can be seen that

$$T_{N(t)} \xrightarrow{\mathbb{P}} t\,, \quad \text{as} \quad \lambda \longrightarrow \infty\,. \tag{5.17}$$

Because $\lim_{\lambda\to\infty} T_{N(t)} \leq t$ for any t, hence we conclude that if $\lim_{\lambda\to\infty} T_{N(t)} < t$, then with probability one, there would be at least one claim between times $\lim_{\lambda\to\infty} T_{N(t)}$ and t. This would contradict the definition of renewal process $N(t)$. Hence (5.17) holds. This demonstrates that asymptotically there exits a claim at any time t. This also implies that the process $N(t)$ asymptotically restarts at any time, hence $N(t)$ has independent increments. The results extend asymptotically to compound renewal sums.

Now consider the discounted compound renewal process:

$$Z(t) = \sum_{k=0}^{N(t)} e^{-\delta T_k} X_k\,,$$

for a fixed t. Let $t_0 = 0, t_1, t_2, \cdots, t_m = t$ be an equally–spaced partition on $[0, t]$, such that the length of interval $(t_1 - 0)$ is small enough that the effect of interest and inflation can be negligible. That is

$$e^{-\delta t_1} \approx 1\,.$$

For example if $\delta = 0.04$ and $t_1 = 0.01$, then $e^{-\delta t_1} = 0.9996$. For $T_k \leq t_1$,

$$\mathbb{P}\big(e^{-\delta T_k} \geq e^{-\delta t_1}\big) = 1 \Rightarrow e^{-\delta T_k} \xrightarrow{\mathbb{P}} 1\,.$$

Hence $Z(t_1)$ can be approximated by $S(t_1)$, that is

$$Z(t_1) \approx S(t_1)\,.$$

Since

$$Z(t) = \sum_{k=0}^{N(t)} e^{-\delta T_k} X_k = Z(t_1) + Z(t_2) - Z(t_1) + \cdots + Z(t) - Z(t_{m-1})\,, \tag{5.18}$$

and Remark 5.2 show that at each point t_k, for $k = 1, 2, \cdots, m$, we have

$$T_{N(t_k)} \xrightarrow{\mathbb{P}} t_k\,, \quad \text{as} \quad \lambda \longrightarrow \infty\,.$$

Hence we rewrite (5.18) as follows:

$$\lim_{\lambda\to\infty} Z(t) = \lim_{\lambda\to\infty} Z(t_1) + e^{-\delta t_1} \lim_{\lambda\to\infty} Z(t_2 - t_1) + \cdots + e^{-(m-1)\delta t_1} \lim_{\lambda\to\infty} Z(t - t_{m-1}), \quad (5.19)$$

this is because Remark 5.2 shows that asymptotically $N(t)$ has independent increments, hence so does $Z(t)$. Since $S(t_1)$ approximates $Z(t_1)$ and Proposition 5.1.1 shows that $S(t_1)$ is asymptotic normal with mean $\lambda t_1 \mu_X$ and variance $\sqrt{\lambda t_1(\mu_X^2 + \sigma_X^2)}$, hence $Z(t)$ is also asymptotically normal with mean

$$(1 + e^{-\delta t_1} + e^{-2\delta t_1} + \cdots + e^{-(m-1)\delta t_1})\lambda t_1 \mu_X = \frac{1 - e^{-m\delta t_1}}{1 - e^{-\delta t_1}} \lambda t_1 \mu_X = \frac{1 - e^{-\delta t}}{1 - e^{-\delta t_1}} \lambda t_1 \mu_X \,. \quad (5.20)$$

Taylor's series gives, for small t,

$$1 - e^{-\delta t_1} \approx 1 - 1 + \delta t_1 \,. \quad (5.21)$$

Substituting (5.21) into (5.20) yields

$$\frac{1 - e^{-\delta t}}{1 - e^{-\delta t_1}} \lambda t_1 \mu_X = \frac{1 - e^{-\delta t}}{\delta t_1} \lambda t_1 \mu_X = \frac{1 - e^{-\delta t}}{\delta} \lambda \mu_X \,. \quad (5.22)$$

Similarly arguments applied to the variance $Z(t)$ give an approximate asymptotic variance

$$\begin{aligned}(1 + e^{-2\delta t_1} + \cdots + e^{-2(m-1)\delta t_1})\lambda t_1(\mu_X^2 + \sigma_X^2) &= \frac{1 - e^{-2m\delta t_1}}{1 - e^{-2\delta t_1}} \lambda t_1 (\mu_X^2 + \sigma_X^2) \\ = \frac{1 - e^{-2\delta t}}{1 - e^{-2\delta t_1}} \lambda t_1 (\mu_X^2 + \sigma_X^2) \approx \frac{1 - e^{-2\delta t_1}}{2\delta t_1} \lambda t_1 (\mu_X^2 + \sigma_X^2) &= \frac{1 - e^{-2\delta t}}{2\delta} \lambda (\mu_X^2 + \sigma_X^2) \,.\end{aligned} \quad (5.23)$$

Hence we have the following theorem for the discounted compound Poisson processes.

Theorem 5.1.1. *For a fixed* $t > 0$,

$$\frac{\sqrt{2\delta}}{\sqrt{\lambda(1 - e^{-2\delta t})(\mu_X^2 + \sigma_X^2)}} \left[Z(t) - \frac{\lambda \mu_X}{\delta}(1 - e^{-\delta t}) \right] \xrightarrow{D} N(0,1) \,, \quad \text{as} \quad \lambda \longrightarrow \infty \,. \quad (5.24)$$

Remark 5.3. For both the classical compound Poisson process and the discounted compound Poisson process, the approximations of the mean and variance in Proposition 5.1.1 and Theorem 5.1.1 are unbiased.

Example 5.1.1. Consider a compound Poisson process with exponential claim severities, and let the mean inter–arrival time and mean claim size be 0.01 and 1000, respectively. The exact distribution of the classical compound Poisson process and discounted compound Poisson process are obtained by the transformation method discussed in Section 4.1.1. The following graphs show the conditional density functions for the compound Poisson sums.

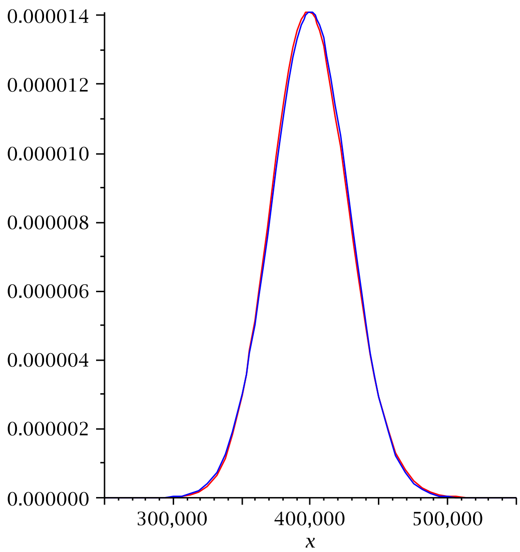


Figure 5.1: Cond. d. of $S(4)$ in Example 5.1.1: exact [red], asymptotic [blue]

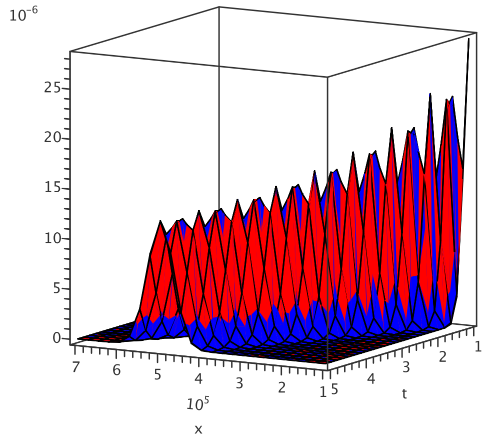


Figure 5.2: Cond. d. of $S(t)$ in Example 5.1.1: exact [red], asymptotic [blue]

Figure 5.1 shows their close match at time $t = 4$. The same accuracy is obtained in Figure 5.2, for 3-dimensional density function graphs. In similar comparisons the asymptotic and exact conditional densities remain close, when the mean inter–arrival time is equal or smaller than 0.01.

In the next section we generalize the above results to a discounted compound renewal process. Normality still exits if the mean inter–arrival time goes to zero.

5.2 Compound Renewal Processes

To prove the asymptotic normality for discounted compound renewal processes, we need the following assumptions on inter–arrival times τ_n with mean $\frac{1}{\lambda}$:

1. $\mathbb{E}[\tau_n^k] < \infty$, for $k = 1, 2, 3$ and 4,

2. $\lim_{\lambda\to\infty} \mathbb{E}[\tau_n^k]\lambda^k = \gamma_k$, for $k = 1, 2, 3$ and 4, where γ_k is some finite number .

Remark 5.4. Exponential, Erlang(n), generalized Erlang(n), mixture of exponentials, log-normal, Weilbull and inverse Gaussian random variables satisfy these assumptions. These are reasonable assumptions, in particular since any continuous positive random variable can be approximated by a mixture of exponential or of Erlang(n).

The following lemma is given by Loève (1977).

Lemma 5.2.1. *Lyapunov's central limit theorem*
Let ξ_k, for $k \in \mathbb{N}$, be independent random variables, $m_k = \mathbb{E}[\xi_k]$ and $\sigma_k^2 = \mathbb{V}[\xi_k]$. Denote $S_n = \sum_{k=1}^n \xi_k$ and $s^2 = \sum_{k=1}^n \sigma_k^2$. If Lyapunov's condition is satisfied:

$$\frac{1}{s^{2+\varepsilon}} \sum_{k=1}^{n} \mathbb{E}|\xi_k - m_k|^{2+\varepsilon} \longrightarrow 0\,, \qquad as \quad n \longrightarrow \infty\,,$$

for some $\varepsilon > 0$, then the central limit theorem:

$$\frac{S_n - \mathbb{E}[S_n]}{\sqrt{Var[S_n]}} \xrightarrow{D} N(0,1)\,, \quad as \quad n \longrightarrow \infty.$$

holds.

Proposition 5.2.1. *Let τ_n be the inter–arrival times with mean $\frac{1}{\lambda}$ and variance $\sigma^2(\lambda)$ that we will denote σ^2 for simplicity. If $N(t)$ forms a renewal process, then*

$$\frac{1}{\sigma\sqrt{\lambda t}}\big[\frac{N(t)}{\lambda} - t\big] = \frac{1}{\sigma\sqrt{\lambda^3 t}}\big[N(t) - \lambda t\big] \xrightarrow{D} N(0,1)\,, \quad as \quad \lambda \longrightarrow \infty\,. \tag{5.25}$$

Proof. Let $m_t = \lambda t + y\sigma(\lambda)\sqrt{\lambda^3 t}$. From the definition of a renewal process, we have

$$\begin{aligned} \mathbb{P}\big(N(t) < m_t\big) &= \mathbb{P}\big(T_{m_t} > t\big) \\ &= \mathbb{P}\big(\frac{T_{m_t} - m_t\frac{1}{\lambda}}{\sigma\sqrt{m_t}} > \frac{t - m_t\frac{1}{\lambda}}{\sigma\sqrt{m_t}}\big)\,. \end{aligned} \tag{5.26}$$

Since

$$t - m_t\frac{1}{\lambda} = t - (\lambda t + y\sigma\sqrt{\lambda^3 t})\frac{1}{\lambda} = -y\sigma\sqrt{\lambda t}\,,$$

then

$$\frac{t - m_t\frac{1}{\lambda}}{\sigma\sqrt{m_t}} = \frac{-y\sigma\sqrt{\lambda t}}{\sigma\sqrt{\lambda t + y\sigma\sqrt{\lambda^3 t}}} = \frac{-y}{\sqrt{1 + y\sigma(\lambda t^{-1})^{\frac{1}{2}}}}\,. \tag{5.27}$$

Now

$$\sigma^2\lambda^2 = \Big[\mathbb{E}[\tau_n^2] - (\frac{1}{\lambda})^2\Big]\lambda^2 = \mathbb{E}[\tau_n^2]\lambda^2 - 1\,, \tag{5.28}$$

hence Assumption 2 gives that (5.28) is a finite number. Then taking square root implies that

$$\sigma(\lambda)\sqrt{\lambda} = \sigma(\lambda)\sqrt{\lambda} = \frac{\sigma(\lambda)\lambda}{\sqrt{\lambda}} \longrightarrow 0\,, \quad \text{as} \quad \lambda \longrightarrow \infty\,, \tag{5.29}$$

which implies that

$$\frac{-y}{\sqrt{1+\sigma(\lambda)(\lambda t^{-1})^{\frac{1}{2}}}} \longrightarrow -y\,, \quad \text{as} \quad \lambda \longrightarrow \infty. \tag{5.30}$$

We can show that

$$\frac{1}{\sigma\sqrt{m_t}}(T_{m_t} - m_t\frac{1}{\lambda}) \xrightarrow{D} N(0,1)\,, \quad \text{as} \quad \lambda \longrightarrow \infty\,. \tag{5.31}$$

Let $s^2 = \sigma^2 m_t$ and $a^4 = \mathbb{E}[(\tau_n - \frac{1}{\lambda})^4]m_t$, then

$$\begin{aligned}
\frac{a^4}{s^4} &= \frac{\mathbb{E}[(\tau_n - \frac{1}{\lambda})^4]m_t}{\sigma(\lambda)^4 m_t^2} \\
&= \frac{\mathbb{E}[\tau_n^4] - 4\frac{1}{\lambda}\mathbb{E}[\tau_n^3] + 6\frac{1}{\lambda^2}\mathbb{E}[\tau_n^2] - 4\frac{1}{\lambda^3}\mathbb{E}[\tau_n] + \frac{1}{\lambda^4}}{\sigma(\lambda)^4 m_t} \\
&= \frac{\lambda^4\big(\mathbb{E}[\tau_n^4] - 4\frac{1}{\lambda}\mathbb{E}[\tau_n^3] + 6\frac{1}{\lambda^2}\mathbb{E}[\tau_n^2] - 4\frac{1}{\lambda^3}\mathbb{E}[\tau_n] + \frac{1}{\lambda^4}\big)}{\lambda^4\sigma(\lambda)^4 m_t} \\
&= \frac{\lambda^4\mathbb{E}[\tau_n^4] - 4\lambda^3\mathbb{E}[\tau_n^3] + 6\lambda^2\mathbb{E}[\tau_n^2] - 4\lambda\mathbb{E}[\tau_n] + 1}{\lambda^4\big(\mathbb{E}[\tau_n^2] - \frac{1}{\lambda^2}\big)^2 m_t}\,.
\end{aligned} \tag{5.32}$$

By Assumption 2, simplifying (5.32) produces

$$\lim_{\lambda\to\infty}\frac{a^4}{s^4} = \lim_{\lambda\to\infty}\frac{\gamma_4 - 4\gamma_3 + 6\gamma_2 - 4\gamma_1 + 1}{(\gamma_2 - 1)^2 m_t}\,. \tag{5.33}$$

Since γ_n, for $n = 1, 2, 3$ and 4 are finite and $m_t \longrightarrow \infty$, as $\lambda \longrightarrow \infty$, then

$$\frac{a^4}{s^4} \longrightarrow 0\,, \quad \text{as} \quad \lambda \longrightarrow \infty. \tag{5.34}$$

(5.34) satisfies Lyapounov's central limit theorem condition, where $\varepsilon = 2$, then (5.31) holds. Combining (5.27), (5.30) and (5.31), we conclude that (5.25) holds. □

Remark 5.5. Note that in the above proof uses $\epsilon = 2$ in Lyapounov's condition, however this power could be reduced to $\epsilon = 1$ by changing Assumption 2 to:

$$\lim_{\lambda\to\infty}\mathbb{E}[|\tau_n - \frac{1}{\lambda}|^3]\lambda^3 = \gamma_3^\star\,,$$

where $\gamma_3^\star$ is a finite number. Exponential, Erlang(n), generalized Erlang(n), mixture of exponentials, lognormal, Weilbull and inverse Gaussian random variables satisfy these assumptions.

The variance of the number of claims $N(t)$ is then asymptotically $\sigma^2\lambda^3 t$, since

$$\lim_{\lambda\to\infty} \frac{\mathbb{V}[N(T)]}{\sigma^2\lambda^3 t} = 1\,,$$

for Exponential, Erlang(n), generalized Erlang(n) or mixture of exponential inter–arrival times. The conditions can even apply more generally as any continuous positive random variable can be approximated by a mixture of exponentials or of Erlang(n).

Proposition 5.2.2. *For a fixed $t > 0$,*

$$\frac{N(t)}{\lambda} \xrightarrow{\mathbb{P}} t\,, \quad \textit{as} \quad \lambda \longrightarrow \infty\,. \tag{5.35}$$

Proof. Proposition 5.2.1 shows that $N(t)$ is an asymptotically normal with mean λt and variance $\sigma(\lambda)^2\lambda^3 t$, then $\frac{N(t)}{\lambda}$ is also asymptotically normal with mean t and variance $\sigma(\lambda)^2\lambda t$. Chebyshev's inequality gives

$$\mathbb{P}\big(|\frac{N(t)}{\lambda} - \frac{\mathbb{E}[N(t)]}{\lambda}| > \varepsilon\big) \leq \frac{\mathbb{V}[\frac{N(t)}{\lambda}]}{\varepsilon^2}\,, \quad \text{for any } \varepsilon > 0\,, \tag{5.36}$$

which implies that

$$\mathbb{P}\big(|\frac{N(t)}{\lambda} - t| > \varepsilon\big) \leq \frac{\sigma(\lambda)^2\lambda t}{\varepsilon^2}\,, \quad \text{for any } \varepsilon > 0\,. \tag{5.37}$$

From (5.29) we have $\lim_{\lambda\longrightarrow\infty}\sigma(\lambda)^2\lambda = 0$, then (5.37) goes to 0 as λ goes to ∞. Thus (5.35) holds. □

Remark 5.6. Since $\frac{1}{\lambda}$ is the mean inter–arrival time and $N(t)$ is the total number of claims up to time t, then $\frac{1}{\lambda}N(t)$ is the average arrival time, which asymptotically equals to t. This proposition also demonstrates that $\lim_{\lambda\to\infty}\mathbb{P}\big(N(t) = \infty\big) = 1$ and $N(t) \xrightarrow{\mathbb{P}} \mathbb{E}[N(t)]$.

Theorem 5.2.1. *Let X_k, $k \in \mathbb{N}$, be iid claim severities with mean μ_X and variance σ_X^2, Define*

$$S^\star(t) = \frac{1}{\sigma_X\sqrt{\mathbb{E}[N(t)]}}\Big[S(t) - N(t)\mu_X\Big]\,,$$

where $S(t) = \sum_{k=1}^{N(t)} X_k$, then

$$S^\star(t) \xrightarrow{D} N(0,1)\,, \quad \textit{as} \quad \lambda\longrightarrow\infty. \tag{5.38}$$

Proof. Let $Y_i = X_i - \mu_X$, then

$$S^\star(t) = \frac{1}{\sigma_X\sqrt{\mathbb{E}[N(t)]}}\sum_{k=1}^{N(t)} Y_k\,.$$

The mgf of $S^\star(t)$ is denoted by $M_{S^\star(t)}(s)$, then

$$\begin{aligned}
M_{S^\star(t)}(s) &= \mathbb{E}\Big[e^{sS^\star(t)}\Big] \\
&= \sum_{n=0}^{\infty}\mathbb{E}\Big[e^{sS^\star(t)} \mid N(t)=n\Big]\mathbb{P}\big(N(t)=n\big) \\
&= \sum_{n=0}^{\infty}\mathbb{E}\Big[e^{s\frac{1}{\sigma_X\sqrt{n}}\sum_{k=1}^{n}Y_k} \mid N(t)=n\Big]\mathbb{P}\big(N(t)=n\big) \\
&= \sum_{n=0}^{\infty}\Big[1+\frac{s}{\sigma_X\sqrt{n}}\mathbb{E}[Y_1]+\frac{s^2}{2\sigma_X^2 n}\mathbb{E}[Y_1^2]+o(\frac{1}{\lambda})\Big]^n\mathbb{P}\big(N(t)=n\big) \\
&= \sum_{n=0}^{\infty}\Big[1+\frac{s^2}{2n}+o(\frac{1}{\lambda})\Big]^n\mathbb{P}\big(N(t)=n\big)\,. \qquad (5.39)
\end{aligned}$$

Since for any $x \in \mathbb{R}$

$$\lim_{n\to\infty}(1+\frac{x}{n})^n = e^x\,,$$

and $(1+\frac{x}{n})^n$ is increasing function in n, then for $\epsilon > 0$, there exists a number $M > 0$, such that, when $n > M$

$$(1+\frac{1}{n})^n > e - \varepsilon\,. \qquad (5.40)$$

Hence (5.39) can be written as:

$$M_{S^\star(t)}(s) = \sum_{n=0}^{M}\Big[1+\frac{s^2}{2n}+o(\frac{1}{\lambda})\Big]^n\mathbb{P}\big(N(t)=n\big) + \sum_{n=M+1}^{\infty}\Big[1+\frac{s^2}{2n}+o(\frac{1}{\lambda})\Big]^n\mathbb{P}\big(N(t)=n\big)$$

Proposition 5.2.2 and Remark 5.6 give that $\lim_{\lambda\to\infty}\mathbb{P}\big(N(t)=\infty\big)=1$, which implies that $\lim_{\lambda\to\infty}\mathbb{P}\big(N(t)=k\big)=0$, for $k<\infty$, then

$$\begin{aligned}
\lim_{\lambda\to\infty} M_{S^\star(t)}(s) &= \lim_{\lambda\to\infty}\sum_{n=0}^{M}\Big[1+\frac{s^2}{2n}+o(\frac{1}{\lambda})\Big]^n\mathbb{P}\big(N(t)=n\big) \\
&\quad + \lim_{\lambda\to\infty}\sum_{n=M+1}^{\infty}\Big[1+\frac{s^2}{2n}+o(\frac{1}{\lambda})\Big]^n\mathbb{P}\big(N(t)=n\big). \qquad (5.41)
\end{aligned}$$

Combining (5.40) and (5.41) yields

$$e^{\frac{s^2}{2}} - \varepsilon \le \lim_{\lambda\to\infty} M_{S^\star(t)}(s) \le e^{\frac{s^2}{2}}$$

Hence

$$\lim_{\lambda\to\infty} M_{S^\star(t)}(s) = e^{\frac{1}{2}s^2}\,,$$

which implies that (5.38) holds. □

Theorem 5.2.2. *Let X_k, $k \in \mathbb{N}$, be iid claim severities, $N(t)$ forms a renewal process, then*

$$\frac{1}{\sqrt{\lambda t(\sigma_x^2 + \sigma^2\lambda^2\mu_X^2)}}\Big[\sum_{k=1}^{N(t)} X_k - \lambda\mu_X t\Big] \xrightarrow{D} N(0,1)\,, \quad as \quad \lambda \longrightarrow \infty\,. \tag{5.42}$$

Proof. Based on Proposition 5.2.1 and Theorem 5.2.1, composition gives (5.42). □

Now consider the relationship between the classical compound Poisson process and compound renewal process. Let $S^P(t)$ and $S^R(t)$ be compound Poisson and compound renewal processes, respectively.

Corollary 5.2.1. *For a fixed $t > 0$, $S^P(t)$ can be approximated asymptotically by the sum of $S^R(t)$ and $\mu_X\sqrt{\lambda t(1-\sigma(\lambda)^2\lambda^2)}N(0,1)$, if $1-\sigma(\lambda)^2\lambda^2 > 0$, where $S^R(t)$ and $N(0,1)$ are independent; or $S^R(t)$ can be approximated asymptotically by the sum of $S^P(t)$ and $\mu_X\sqrt{\lambda t(\sigma(\lambda)^2\lambda^2-1)}N(0,1)$, if $\sigma(\lambda)^2\lambda^2 - 1 > 0$, where $S^P(t)$ and $N(0,1)$ are independent.*

Proof. Proposition 5.1.1 shows that $S^P(t)$ is asymptotically normal with mean $\lambda\mu_X t$ and variance $\lambda t(\mu_X + \sigma_X^2)$. Theorem 5.2.2 demonstrates that $S^R(t)$ is also the same, asymptotically normal with mean $\lambda\mu_X t$ and variance $\sqrt{\lambda t(\sigma_x^2 + \sigma(\lambda)^2\lambda^2\mu_X^2)}$. They have the same asymptotic mean with different asymptotic variances. By adding the difference between two variances, we have Corollary 5.2.1. □

Remark 5.7. Theorem 5.2.2 gives the asymptotic distribution of compound renewal risk processes. If the inter–arrival time is exponentially distributed with parameter λ, then $\sigma(\lambda)^2\lambda^3 = \lambda$ and the compound renewal process reduces to the compound Poisson processes. As we mentioned before, the asymptotic mean and variance of compound Poisson processes is unbiased. Since asymptotically $\lim_{\lambda\longrightarrow\infty}\lambda t$ and $\lim_{\lambda\longrightarrow\infty}(\lambda t - c)$ are equal, with c a finite number, the same idea is applied to the variance of compound renewal risk processes. The result in Theorem 5.2.2 may increase or decrease the mean and variance of $S(t)$, compared to its real mean and variance. For example, if inter–arrival times are Erlang(2), we have that $\mathbb{E}[N(t)] = \lambda t - \frac{1}{4} + \frac{1}{4}e^{-4\lambda t}$ and $\mathbb{V}[N(t)] = \frac{1}{2}\lambda t + \frac{1}{16} - \lambda t e^{-4\lambda t} - \frac{1}{16}e^{-8\lambda t}$, then the difference between the asymptotic mean in Theorem 5.2.2 and the real mean is $-\frac{1}{4}\mu_X$ and the difference between the asymptotic variance is $\frac{1}{16}\mu_X^2 - \frac{1}{4}\sigma_X^2$. In practice, λ can not be ∞, which means that the mean and variance of the distribution in Theorem 5.2.2 is biased for some compound renewal processes. Hence we can rewrite Theorem 5.2.2 in the form of a central limit theorem for adjusted mean and variance to make them unbiased.

Theorem 5.2.3.

$$\frac{1}{\mathbb{V}[S(t)]}\Big(S(t) - \mathbb{E}[S(t)]\Big) \xrightarrow{D} N(0,1)\,, \quad as \quad \lambda \longrightarrow \infty\,.$$

5.3 Discounted Compound Renewal Sums

In this section, the normality of the discounted compound renewal risk processes is investigated. The asymptotic distribution for the discounted compound renewal sum is given, when the mean of inter–arrival times is small enough.

The asymptotic approximation of $Z(t_1)$ by $S(t_1)$ is discussed in Section 5.1, for a very small value of t_1. Theorem 5.2.2 shows that $S(t_1)$ are approximately asymptotic normal with mean $\lambda\mu_X t_1$ and variance $\lambda t_1(\sigma_x^2 + \sigma(\lambda)^2\lambda^2\mu_X^2)$. Hence the asymptotic mean of $Z(t_1)$ is $\lambda\mu_X t_1$, then $Z(t)$ has the same mean as (5.22). For the variance, Theorem 5.2.2 and (5.23) give

$$\frac{1-e^{-2\delta t}}{2\delta t_1}\lambda t_1(\sigma_X^2 + \sigma(\lambda)^2\lambda^2\mu_X^2) = \frac{1-e^{-2\delta t}}{2\delta}\lambda(\sigma_X^2 + \sigma(\lambda)^2\lambda^2\mu_X^2)\,. \tag{5.43}$$

This leads us to the following central limit theorem for the discounted compound renewal process.

Theorem 5.3.1. *For a fixed $t > 0$,*

$$\frac{\sqrt{2\delta}}{\sqrt{\lambda(1-e^{-2\delta t})(\sigma_X^2 + \sigma(\lambda)^2\lambda^2\mu_X^2)}}\Big[Z(t) - \frac{\lambda\mu_X}{\delta}(1-e^{-\delta t})\Big] \xrightarrow{D} N(0,1)\,, \quad as \quad \lambda \longrightarrow \infty\,. \tag{5.44}$$

Let $Z^P(t)$ and $Z^R(t)$ be the discounted compound Poisson and discounted compound renewal processes, respectively. By similar arguments to those of Corollary 5.1.2, we have the following result.

Corollary 5.3.1. *$Z^P(t)$ can be approximated asymptotically by the sum of $Z^R(t)$ and $\mu_X\sqrt{\lambda\frac{1-e^{-2\delta t}}{2\delta}(1-\sigma(\lambda)^2\lambda^2)}\,N(0,1)$, if $1-\sigma(\lambda)^2\lambda^2>0$, where $Z^R(t)$ and $N(0,1)$ are independent; or $Z^R(t)$ can be approximated asymptotically by the sum of random variable $Z^P(t)$ and $\mu_X\sqrt{\lambda\frac{1-e^{-2\delta t}}{2\delta}(\sigma(\lambda)^2\lambda^2-1)}\,N(0,1)$, if $\sigma(\lambda)^2\lambda^2-1>0$, where $S^P(t)$ and $N(0,1)$ are also independent.*

Proof. Theorem 5.1.1 shows that $Z^P(t)$ is asymptotically normal with mean $\lambda\mu_X\frac{1-e^{-\delta t}}{\delta}$ and variance $\lambda(\mu_X^2+\sigma_X^2)\frac{1-e^{-2\delta t}}{2\delta}$. Theorem 5.2.2 demonstrates that $Z^R(t)$ is also asymptotically normal with mean $\lambda\mu_X\frac{1-e^{-\delta t}}{\delta}$ and variance $\lambda(\sigma_x^2+\sigma(\lambda)^2\lambda^2\mu_X^2)(\frac{1-e^{-2\delta t}}{2\delta})$. They have the same asymptotic mean, but different asymptotic variances. By adding the difference between the two variances, we get Corollary 5.3.1. □

A similar argument applied to Theorem 5.3.1 gives a central limit theorem with adjusted mean and variance, for the discounted compound renewal process.

Theorem 5.3.2.

$$\frac{1}{\mathbb{V}[Z(t)]}\Big(Z(t)-\mathbb{E}[Z(t)]\Big)\xrightarrow{D}N(0,1)\,,\quad as\quad \lambda\longrightarrow\infty\,.$$

Remark 5.8. Note that Theorems 5.3.1 and 5.3.2 prove that the distribution of normalized discounted compound renewal process converges to the normal distribution, which is consistent with Figures 4.1 and 4.2.

Example 5.3.1. Consider Erlang(2) inter–arrival times with mean 100 and $\delta=0.04$, while claim severities are exponential with mean $\mu_X = 1000$. By the transformation method discussed in Section 4.1.2, we have

$$M_{Z(t)}(s)=e^{-200t}g_1(t;s)\,,$$

where the function $g_1(t;s)$ is given by the following second–order differential equation:

$$\frac{\partial^2}{\partial t^2}g_1(t;s)=b_1(t;s,2)\frac{\partial}{\partial t}g_1(t;s)+b_0(t;s,2)g_1(t;s)\,,$$

and the coefficients are given by:

$$b_1(t;s,2)=\frac{0.00008}{-0.001+se^{-0.04t}}\,,\quad b_0(t;s,2)=-\frac{40.016}{-0.001+se^{-0.04t}}\,,\tag{5.45}$$

with initial values $g_1(0, s) = 1$ and $\frac{\partial}{\partial t} g_1(t; s)|_{t=0} = 200$.

For the distribution of $S(t)$, without considering the effect of interest and inflation, we also have

$$M_{S(t)}(s) = e^{-200t} g_2(t; s)\,,$$

where the function $g_2(t; s)$ is given by the following second–order differential equation:

$$\frac{\partial^2}{\partial t^2} g_2(t; s) = b_1(t; s, 2) \frac{\partial}{\partial t} g_2(t; s) + b_0(t; s, 2) g_2(t; s)\,. \tag{5.46}$$

Here the coefficients are different from (5.45) and are given by:

$$b_1(t; s, 2) = 0\,, \quad b_0(t; s, 2) = -\frac{40}{-0.001 + s}\,,$$

with initial values $g_2(0, s) = 1$ and $\frac{\partial}{\partial t} g_2(t; s)|_{t=0} = 200$. Solving (5.46) by the series method gives the function $g_2(t; s)$. Inverting the corresponding Laplace transform of $e^{-200t} g_2(t; s)$ gives the distribution of $S(t)$. Similarly the same method can be applied to get the distribution of $Z(t)$. The following means and variances are obtained for $S(2)$.

	Mean	Variance
Exact distribution	$199,750$	$299,812,500$
Theorem 5.2.2	$200,000$	$300,000,000$
Theorem 5.2.3	$199,750$	$299,812,500$

Table 5.1: Mean and variance of $S(2)$ for different models in Example 5.3.1

In Table 5.1, the asymptotic distribution of Theorem 5.2.2 has larger mean and variance, compared to those of the exact distribution or those given by Theorem 5.2.3. The differences in mean and variance between results in Theorems 5.2.2 and 5.2.3 can be used to adjust the mean, decreasing it by $\frac{1}{4}\mu_X = 250$, and the variance by $\frac{1}{4}\sigma(\lambda)^2 - \frac{1}{16}\mu_X^2 = 187,500$.

	Mean	Variance
Exact distribution	$191,959$	$277,033,690$
Theorem 5.3.1	$192,209$	$277,230,395$
Theorem 5.3.2	$191,959$	$277,042,895$

Table 5.2: Mean and variance of $Z(2)$ for different models in Example 5.3.1

Table 5.2 shows that the asymptotic distribution in Theorem 5.3.1, for the discounted compound sum $Z(t)$, produces a mean that is larger by $\frac{1}{4}\mu_X = 250$ than the one for the distribution in Theorem 5.3.2. For the variance, the asymptotic distribution in Theorem 5.3.2 is closer to the exact distribution, than the asymptotic distribution in Theorem 5.3.1

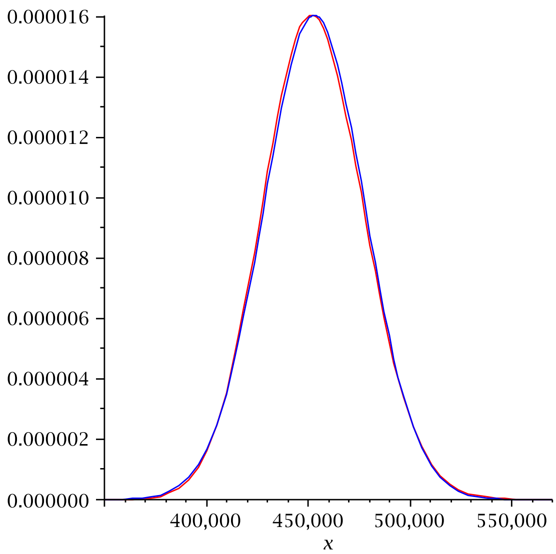


Figure 5.3: Cond. d. of $Z(5)$ in Example 5.3.1: exact [red], asymptotic [blue]

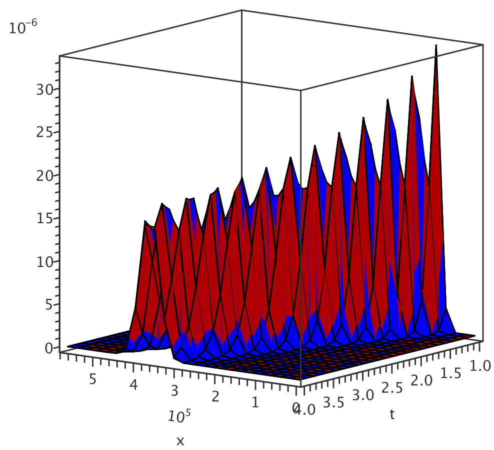


Figure 5.4: Cond. d. of $Z(t)$ in Example 5.3.1: exact [red], asymptotic [blue]

Figures 5.3 and 5.4 report the conditional density function of $Z(t)$ in red, and the normal approximation in Theorem 5.3.1 From Figure 5.3, we can conclude that the two density functions match well for time $t = 5$, when the mean inter–arrival time is 0.01 per year. The same conclusion seems hold for other t values in Figure 5.4, which shows the conditional density functions at different t from 1 to 5. Hence if the mean inter–arrival time is greater than or equal to 0.01 per year, the distribution of the discounted compound renewal sum can be approximated by the result in Theorem 5.3.1.

In conclusion, this chapter studies the normal limit distribution of the compound renewal sum and the discounted compound renewal sum, when the mean inter–arrival time is small. This limit distribution is then used as an approximation of the compound renewal sum and the corresponding discounted compound sum. This result is consistent with our observations in Figures 4.1 and 4.2. In addition, we prove that the compound Poisson sum can be written in a convolution formulas as the sum of the compound renewal sum and an independent normal distribution. This result also holds for the discounted compound sums, which gives an analytical justification to our numerical results in Section 3.4.

Conclusion

This thesis investigates the distribution of the discounted compound PH–renewal process. PH distributions form an interesting family of distributions, since they are represented by matrices and vectors, which are easily computed numerically. Using basic properties of matrices and vectors, we derive formulas for the mgf of the discounted compound PH–renewal process. Particularly we obtain homogeneous and ordinary differential equations or differential systems for the mgf of $Z(t)$. Some numerical examples illustrate these results.

In addition, the truncated series method is proposed to help solve the differential equations. The coefficients of these truncated series solutions are rational polynomial functions in s. These can be inverted easily to approximate the distribution of $Z(t)$. To reduce computing time, the transformation method for differential equations is used. From our numerical examples, we see that this method produces accurate and fast results.

In applications, we calculate stop–loss premiums for reinsurance contracts. Since risk measures are used by insurance companies or financial institutions to analyze the riskiness for their businesses, then Value-at-Risk (VaR) and Conditional Tail Expectation (CTE) are studied in the thesis. We compare the riskiness under different net interest rates for different risk models. We conclude that the discounted compound Poisson model is more risky than any other discounted compound PH–renewal models.

In this last chapter we prove the asymptotic normality of discounted compound renewal sums, when the mean inter–arrival time is very small. This normal limit distribution was observed empirically in Figures 4.1 and 4.2. Furthermore the asymptotic relationship between the discounted compound Poisson sum and the discounted compound renewal sum is studied.

For future research, a natural extension would be to consider the stochastic process properties of $Z(t)$. For instance, an asymptotic diffusion result could be established to generalize the results in Chapter 5.

Another interesting aspect would be to study the distribution of discounted compound PH–renewal sums with heavy tail claim severities. This is very important for insurance applications. Since PH distributions are dense in the family of densities on the positive real line, these could be used to fit even heavy tailed distributions.

For the financial study of insurance portfolios, it would also be interesting to extend the results in this thesis to random interest rates δ. This would help assess the combined effect of the insurance and financial risks on the solvency of insurance companies.

Finally, the dependence structure between inter–arrival times and claim sizes is a problem of interest for these discounted compound renewal models. In all the above mentioned extensions the technical difficulties have limited, so far, the results that can be found in the literature. Further research in these directions in definitely needed.

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Appendix A

Matrix Exponential

Matrices play a crucial role in PH distributions. The characteristics of the distribution depend on a matrix $\mathbf{A}$. This is illustrated by the examples of Chapter 1. Hence we give here a brief introduction to matrices. We discuss some basic properties and definitions that are used in this thesis.

A.1 Definition

In this section first we introduce some definitions, especially that of the matrix exponential function and some of its properties. We also prove theorems that are used in the thesis. For more details on matrix theory, readers can refer to the book Bernstein (2005), that gives formulas and applications to the theory of linear systems. An alternative choice is Ortega (1987).

Definition A.1.1. *Nonsingular and singular matrices*

If the determinant $|\boldsymbol{A}| \neq 0$*, we call* $\boldsymbol{A}$ *nonsingular, otherwise it is called singular.*

Note that throughout this thesis, we assume that $\mathbf{A}$ is a nonsingular matrix, hence its inverse $\mathbf{A}^{-1}$ exists.

Definition A.1.2. *Eigenvalues and Eigenvectors*

An eigenvalue of a square matrix $\boldsymbol{A}$ *of order* n *is a real or complex scalar* λ *satisfying the equation:*

$$\boldsymbol{A}\underline{x} = \lambda \underline{x}\,,$$

for some nonzero vector $\underline{x}$*, we call* λ *an eigenvalue and* $\underline{x}$ *an eigenvector of* $\boldsymbol{A}$*.*

Definition A.1.3. *Spectral radius of a matrix* $\boldsymbol{A}$ *of order* n

Let $\lambda_1, \lambda_1, \ldots, \lambda_n$ *be eigenvalues of* $\boldsymbol{A}$*, we define the spectral radius of* $\boldsymbol{A}$ *as*

$$\rho(\boldsymbol{A}) = \max\{|\lambda_i|\,,\, 1 \le i \le n\}\,.$$

Definition A.1.4. *Matrix exponential*

Let $\boldsymbol{A}$ *be square matrix of order* n*, then we call matrix exponential, denoted* $e^{\boldsymbol{A}}$ *or* $\exp(\boldsymbol{A})$*, the matrix:*

$$e^{\boldsymbol{A}} = \sum_{k=0}^{\infty} \frac{1}{k!} \boldsymbol{A}^k\,, \tag{A.1}$$

with $e^{\boldsymbol{0}} = \boldsymbol{I}_n$*, where* $\boldsymbol{0}$ *is a zero matrix of order* n*.*

Definition A.1.5. *Logarithm of* $\boldsymbol{A}$

Let $\boldsymbol{A}$ *be square matrix with order* n *, then we call* $\boldsymbol{B}$ *a logarithm of* $\boldsymbol{A}$ *if matrix* $\boldsymbol{B}$ *satisfies:*

$$e^{\boldsymbol{B}} = \boldsymbol{A}\,. \tag{A.2}$$

Then if $sprad(\boldsymbol{A} - \boldsymbol{I}) < 1$*, we can define*

$$\boldsymbol{B} = \ln \boldsymbol{A} = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} (\boldsymbol{A} - \boldsymbol{I})^i\,. \tag{A.3}$$

This leads to the following notion of $\ln(\mathbf{I} - \mathbf{A})$ and $\ln(\mathbf{I} + \mathbf{A})$

$$\ln(\mathbf{I} - \mathbf{A}) = -\sum_{i=1}^{\infty} \frac{\mathbf{A}^i}{i}\,, \tag{A.4}$$

and

$$\ln(\mathbf{I} + \mathbf{A}) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} \mathbf{A}^i\,, \tag{A.5}$$

if $sprad(\mathbf{A}) < 1$.

A.2 Lemmas

Lemma A.2.1. *If λ is an eigenvalue of $\boldsymbol{A}$, then λ^{-1} is eigenvalue of $\boldsymbol{A}^{-1}$.*

Proof. Since $\mathbf{A}$ is nonsingular, then λ^{-1} exits, and we have:

$$|\lambda\mathbf{I} - \mathbf{A}| = |\lambda\mathbf{A}\mathbf{A}^{-1} - \mathbf{A}| = |\mathbf{A}||\lambda\mathbf{A}^{-1} - \mathbf{I}| = -|\mathbf{A}||\lambda||\lambda^{-1}\mathbf{I} - \mathbf{A}^{-1}|\,.$$

Hence λ^{-1} is an eigenvalue of $\mathbf{A}^{-1}$. □

Lemma A.2.2. *If $\boldsymbol{A}$ is a matrix of order n with $sprad(\boldsymbol{A}) < 1$, then $(\boldsymbol{I} - \boldsymbol{A})^{-1}$ exists and we have*

$$(\boldsymbol{I} - \boldsymbol{A})^{-1} = \lim_{k\to\infty} \sum_{i=0}^{k} \boldsymbol{A}^k = \sum_{k=0}^{\infty} \boldsymbol{A}^k\,.$$

Proof. See Ortega (1987). □

Lemma A.2.3. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be square matrices of order n. Then*

$$e^{t\boldsymbol{A}} e^{t\boldsymbol{B}} = e^{t(\boldsymbol{A}+\boldsymbol{B})}\,, \qquad t \in R\,, \tag{A.6}$$

if $\boldsymbol{AB} = \boldsymbol{BA}$.

Proof. See Bernstein (2005). □

Lemma A.2.4. *The derivative of a matrix exponential function is given by:*

$$\frac{d}{dt} e^{\boldsymbol{A}t} = e^{\boldsymbol{A}t}\boldsymbol{A}\,, \qquad t \in \Omega\,. \tag{A.7}$$

Proof. Take a derivative term by term in the series expansion in Definition A.1.4. □

Lemma A.2.5. *Let $\boldsymbol{A}$ be a square nonsingular matrix of order n and $\rho(\boldsymbol{I} - \boldsymbol{A}) < 1$. Then we have the following result:*

$$-\ln \boldsymbol{A} = \ln \boldsymbol{A}^{-1}\,. \tag{A.8}$$

Proof.

$$\begin{aligned}
\ln \mathbf{A} &= (\mathbf{A} - \mathbf{I}) - \frac{1}{2}(\mathbf{A} - \mathbf{I})^2 + \frac{1}{3}(\mathbf{A} - \mathbf{I})^3 + \cdots + (-1)^{k+1}\frac{1}{k}(\mathbf{A} - \mathbf{I})^k + \cdots \\
-\ln \mathbf{A} &= -[(\mathbf{A} - \mathbf{I}) - \frac{1}{2}(\mathbf{A} - \mathbf{I})^2 + \frac{1}{3}(\mathbf{A} - \mathbf{I})^3 + \cdots \\
&\quad + (-1)^{k+1}\frac{1}{k}(\mathbf{A} - \mathbf{I})^k + \cdots]\,,
\end{aligned} \tag{A.9}$$

then

$$\ln \mathbf{A}^{-1} = (\mathbf{A}^{-1} - \mathbf{I}) - \frac{1}{2}(\mathbf{A}^{-1} - \mathbf{I})^2 + \frac{1}{3}(\mathbf{A}^{-1} - \mathbf{I})^3 + \cdots + (-1)^{k+1}\frac{1}{k}(\mathbf{A}^{-1} - \mathbf{I})^k + \cdots .$$

Since $\mathbf{A}^{-1} = \big[\mathbf{I} + (\mathbf{A} - \mathbf{I})\big]^{-1}$, from Lemma A.2.2 we have:

$$\begin{aligned} A^{-1} &= [\mathbf{I} + (\mathbf{A} - \mathbf{I})]^{-1} \\ &= \mathbf{I} - (\mathbf{A} - \mathbf{I}) + (\mathbf{A} - \mathbf{I})^2 - (\mathbf{A} - \mathbf{I})^3 + \cdots + (-1)^k(\mathbf{A} - \mathbf{I})^k + \cdots , \end{aligned}$$

and hence

$$\begin{aligned} \ln\ \mathbf{A}^{-1} \;=\; & \big[- (\mathbf{A} - \mathbf{I}) + (\mathbf{A} - \mathbf{I})^2 - (\mathbf{A} - \mathbf{I})^3 + \cdots + (-1)^k(\mathbf{A} - \mathbf{I})^k + \cdots \big] \\ & - \frac{1}{2}\big[- (\mathbf{A} - \mathbf{I}) + (\mathbf{A} - \mathbf{I})^2 - (\mathbf{A} - \mathbf{I})^3 + \cdots + (-1)^k(\mathbf{A} - \mathbf{I})^k + \cdots \big]^2 \\ & + \frac{1}{3}\big[- (\mathbf{A} - \mathbf{I}) + (\mathbf{A} - \mathbf{I})^2 - (\mathbf{A} - \mathbf{I})^3 + \cdots + (-1)^k(\mathbf{A} - \mathbf{I})^k + \cdots \big]^3 \\ & + \cdots + (-1)^{k+1}\frac{1}{k}\big[- (\mathbf{A} - \mathbf{I}) + (\mathbf{A} - \mathbf{I})^2 - (\mathbf{A} - \mathbf{I})^3 + \cdots \\ & + (-1)^k(\mathbf{A} - \mathbf{I})^k + \cdots \big]^k + \cdots . \end{aligned} \qquad \text{(A.10)}$$

By comparing the polynomial series in (A.9) and (A.10) for $(\mathbf{A} - \mathbf{I})$, we see that they are exactly the same, hence

$$- \ln \mathbf{A} = \ln \mathbf{A}^{-1}.$$

□

Lemma A.2.6. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be square nonsingular matrices. If $\boldsymbol{AB} = \boldsymbol{BA}$, then we have*

$$\ln \boldsymbol{A} - \ln \boldsymbol{B} = \ln \boldsymbol{AB}^{-1} .$$

Proof.

$$e^{\ln \mathbf{A} - \ln \mathbf{B}} = e^{\ln \mathbf{A}} e^{-\ln \mathbf{B}} = A e^{\ln \mathbf{B}^{-1}} = \mathbf{AB}^{-1} ,$$

from Lemmas A.2.4 and A.2.5 and using that $e^{\ln \mathbf{AB}^{-1}} = \mathbf{AB}^{-1}$, then $\ln \mathbf{A} - \ln \mathbf{B} = \ln \mathbf{AB}^{-1}$.

□

Lemma A.2.7. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be the same order of square matrices, if the inverse of $\boldsymbol{A} + s\boldsymbol{B}$ exists then*

$$\frac{d}{ds}(\boldsymbol{A} + s\boldsymbol{B})^{-1} = -(\boldsymbol{A} + s\boldsymbol{B})^{-1}\boldsymbol{B}(\boldsymbol{A} + s\boldsymbol{B})^{-1} . \qquad \text{(A.11)}$$

For the proof see Bernstein (2005).

A.3 Kronecker Product and Sum

In this thesis we use the Kronecker product and sum, hence here we introduce these definitions. Some properties are also given. For more detail please refer to Bernstein (2005).

Definition A.3.1. *Kronecker Product*
Consider matrix $\boldsymbol{A} = (a_{ij})$ *where* $a_{ij} \in \mathbb{R}$ *for* $i = 1, 2, \cdots, n$ *and* $j = 1, 2, \cdots, m$ *and matrix* $\boldsymbol{B} \in \mathbb{R}^{l\times k}$. *Then the Kronecker product* $\boldsymbol{A} \otimes \boldsymbol{B} \in R^{nl\times mk}$ *of* $\boldsymbol{A}$ *is the partitioned matrix*

$$\boldsymbol{A} \otimes \boldsymbol{B} = \begin{pmatrix} a_{11}\boldsymbol{B} & a_{12}\boldsymbol{B} & \cdots & a_{1m}\boldsymbol{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}\boldsymbol{B} & a_{n2}\boldsymbol{B} & \cdots & a_{nm}\boldsymbol{B} \end{pmatrix}.$$

For example, let matrices **A** and **B** be given as follows:

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 1 & -5 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 & 4 & 6 \\ 2 & 0 & 9 \\ 0 & 3 & -4 \end{pmatrix},$$

then the order of Kronecker product is 4×6, and by the Definition A.3.1, it is

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} 3\mathbf{B} & 2\mathbf{B} \\ 1\mathbf{B} & -5\mathbf{B} \end{pmatrix}.$$

Because

$$3\mathbf{B} = \begin{pmatrix} 3 & 12 & 18 \\ 6 & 0 & 27 \\ 0 & 9 & -12 \end{pmatrix}, \quad 2\mathbf{B} = \begin{pmatrix} 2 & 8 & 12 \\ 4 & 0 & 18 \\ 0 & 6 & -8 \end{pmatrix}, \text{ and } -5\mathbf{B} = \begin{pmatrix} 5 & 20 & 30 \\ 10 & 0 & 45 \\ 0 & 15 & -20 \end{pmatrix}$$

we have

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} 3 & 12 & 18 & 2 & 8 & 12 \\ 6 & 0 & 27 & 4 & 0 & 18 \\ 0 & 9 & -12 & 0 & 6 & -8 \\ 1 & 4 & 6 & 5 & 20 & 30 \\ 2 & 0 & 9 & 10 & 0 & 45 \\ 0 & 3 & -4 & 0 & 15 & -20 \end{pmatrix}$$

Definition A.3.2. *Kronecker Sum*

Let matrix $\boldsymbol{A} \in \Omega^{n\times n}$ *and* $\boldsymbol{B} \in \mathbb{R}^{m\times m}$. *Then the Kronecker sum* $\boldsymbol{A} \oplus \boldsymbol{B} \in \Omega^{nm\times nm}$ *of* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *is*

$$\boldsymbol{A} \oplus \boldsymbol{B} = \boldsymbol{A} \otimes \boldsymbol{I}_m + \boldsymbol{I}_n \otimes \boldsymbol{B}.$$

Consider **A** and **B** in previous example, then we have

$$\mathbf{A} \otimes \mathbf{I}_3 = \begin{pmatrix} 3 & 0 & 0 & 2 & 0 & 0 \\ 0 & 3 & 0 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 & 0 & 2 \\ 1 & 0 & 0 & -5 & 0 & 0 \\ 0 & 1 & 0 & 0 & -5 & 0 \\ 0 & 0 & 1 & 0 & 0 & -5 \end{pmatrix}, \quad \text{and} \quad \mathbf{I}_2 \otimes \mathbf{B} = \begin{pmatrix} 1 & 4 & 6 & 0 & 0 & 0 \\ 2 & 0 & 9 & 0 & 0 & 0 \\ 0 & 3 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 2 & 0 & 9 \\ 0 & 0 & 0 & 0 & 3 & -4 \end{pmatrix}.$$

Hence the Kronecker sum is

$$\mathbf{A} \oplus \mathbf{B} = \begin{pmatrix} 4 & 4 & 6 & 2 & 0 & 0 \\ 2 & 3 & 9 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 & 0 & 2 \\ 1 & 0 & 0 & -4 & 4 & 6 \\ 0 & 1 & 0 & 2 & -5 & 9 \\ 0 & 0 & 1 & 0 & 3 & -9 \end{pmatrix}.$$

Proposition A.3.1. *Let* $\boldsymbol{A} = (a_{ij})$ *for* $i = 1, 2, \cdots, n$ *and* $j = 1, 2, \cdots, m$, $\boldsymbol{B} \in \mathbb{R}^{l\times k}$, $\boldsymbol{C} = (c_{ij})$ *for* $i = 1, 2, \cdots, m$ *and* $j = 1, 2, \cdots, k$ *and* $\boldsymbol{D} \in \mathbb{R}^{k\times p}$. *Then*

$$\big(\boldsymbol{A} \otimes \boldsymbol{B}\big)\big(\boldsymbol{C} \otimes \boldsymbol{D}\big) = \boldsymbol{AC} \otimes \ \boldsymbol{BD}.$$

Proof. The ij block of $\big(\mathbf{A} \otimes \mathbf{B}\big)\big(\mathbf{C} \otimes \mathbf{D}\big)$ is given by

$$\begin{aligned} \Big(\big(\mathbf{A} \otimes \mathbf{B}\big)\big(\mathbf{C} \otimes \mathbf{D}\big)\Big)_{ij} &= \begin{pmatrix} a_{i1}\mathbf{B} & \cdots & a_{im}\mathbf{B} \end{pmatrix} \begin{pmatrix} c_{1j}\mathbf{D} \\ \vdots \\ c_{mj}\mathbf{D} \end{pmatrix} \\ &= \sum_{k=1}^{m} a_{ik}c_{kj}\mathbf{BD} = \big(\mathbf{AC}\big)_{ij}\mathbf{BD} \\ &= \big(\mathbf{AC} \otimes \ \mathbf{BD}\big)_{ij}. \end{aligned}$$

□

Proposition A.3.2. *Let* $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ *and* $\boldsymbol{B} \in \mathbb{R}^{m\times m}$*. Then*

$$e^{\boldsymbol{A}\otimes\boldsymbol{I}_m} = e^{\boldsymbol{A}} \otimes \boldsymbol{I}_m\,, \tag{A.12}$$

$$e^{\boldsymbol{I}_n\otimes\boldsymbol{B}} = \boldsymbol{I}_n \otimes e^{\boldsymbol{B}}\,, \tag{A.13}$$

$$e^{\boldsymbol{A}\oplus\boldsymbol{B}} = e^{\boldsymbol{A}} \otimes e^{\boldsymbol{B}}\,. \tag{A.14}$$

Proof. From Definition A.1.4 we have that

$$\begin{aligned} e^{\mathbf{A}\otimes\mathbf{I}_m} &= \mathbf{I}_{nm} + \mathbf{A}\otimes\mathbf{I}_m + \frac{1}{2!}\big(\mathbf{A}\otimes\mathbf{I}_m\big)^2 + \cdots \\ &= \mathbf{I}_n\otimes\mathbf{I}_m + \mathbf{A}\otimes\mathbf{I}_m + \frac{1}{2!}\big(\mathbf{A}^2\otimes\mathbf{I}_m\big) + \cdots \\ &= \Big(\mathbf{I}_n + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \cdots\Big)\otimes\mathbf{I}_m \\ &= e^{\mathbf{A}}\otimes\mathbf{I}_m\,, \end{aligned}$$

and similarly for (A.13). To prove (A.14), from Proposition A.3.1 we have that

$$\begin{aligned} \big(\mathbf{A}\otimes\mathbf{I}_m\big)\big(\mathbf{I}_n\otimes\mathbf{B}\big) &= \mathbf{A}\otimes\mathbf{B}\,, \\ \big(\mathbf{I}_n\otimes\mathbf{B}\big)\big(\mathbf{A}\otimes\mathbf{I}_m\big) &= \mathbf{A}\otimes\mathbf{B}\,, \end{aligned}$$

which shows that $\mathbf{A}\otimes\mathbf{I}_m$ and $\mathbf{I}_n\otimes\mathbf{B}$ commute. Hence, by Lemma A.2.3

$$e^{\mathbf{A}\oplus\mathbf{B}} = e^{\mathbf{A}\otimes\mathbf{I}_m + \mathbf{I}_n\otimes\mathbf{B}} = \big(e^{\mathbf{A}\otimes\mathbf{I}_m}\big)\big(e^{\mathbf{I}_n\otimes e^{\mathbf{B}}}\big) = e^{\mathbf{A}}\otimes\mathbf{B}\,.$$

□