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FORMULAE OF BEURLING-DENY AND LEJAN FOR NON-SYMMETRIC DIRICHLET FORMS

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Abstract

By the classical Beurling-Deny formula, any regular *symmetric* Dirichlet form is decomposed into the diffusion, jumping and killing parts. Further, the diffusion part is characterized by LeJan's formula. In this paper, both the Beurling-Deny formula and LeJan's formula are extended to regular *non-symmetric* Dirichlet forms. In addition, a counterexample is presented to show the gap in the Beurling-Deny formula for non-symmetric Dirichlet forms in the existing literatures.

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1 Introduction and Preliminary

The classical Beurling-Deny formula given in Beurling and Deny [2] tells us that any regular symmetric Dirichlet form is decomposed into the diffusion, jumping and killing parts. Further, LeJan's formula given in LeJan [13] characterizes the diffusion part. These structure results are particularly important because they give us an analytic description of the sample path properties of the associated Markov processes. We refer the readers to Fukushima et al. [8] for a nice representation of the formulae and to Bliedtner [3], Kim [11], Chen and Zhao [5] and Mataloni [15] for some attempts of extending the formulae to the non-symmetric case (cf. Remarks 1.7 and

2.5 below). For some applications of the formulae, we refer the interested readers to Fukushima et al. [8, Ch.5], Chen et al. [4] and Mosco [16], etc.

In this paper, we extend both the Beurling-Deny formula and LeJan's formula to regular non-symmetric Dirichlet forms. In addition, we use a counterexample to show that there is a gap in the Beurling-Deny formula for non-symmetric Dirichlet forms in some existing literatures. For the notions and notations used in this paper, we refer to Fukushima et al. [8] and Ma and Röckner [14].

Let E be a locally compact separable metric space and m a positive Radon measure on E with $\mathrm{supp}[m] = E$. Hereafter, we use $\mathrm{supp}[\cdot]$ to denote the support of a measure or a function on E and use $C_0(E)$ to denote the set of all continuous functions on E with compact supports. Throughout this paper, we assume that $(\mathcal{E}, D(\mathcal{E}))$ is a regular (non-symmetric) Dirichlet form on $L^2(E;m)$. Denote by $(\cdot,\cdot)$ the inner product of $L^2(E;m)$ and denote by $(G_\alpha)_{\alpha>0}$ and $(\hat{G}_\alpha)_{\alpha>0}$ the resolvents associated with $(\mathcal{E}, D(\mathcal{E}))$ and its dual form $(\hat{\mathcal{E}}, D(\mathcal{E}))$, respectively. Define

$$\mathcal{E}^{(\beta)}(u,v) = \beta(u - \beta G_\beta u, v). \tag{1.1}$$

It is known that (cf., e.g. Ma and Röckner [14, Theorem I.2.13 (iii)])

$$\lim_{\beta\to\infty} \mathcal{E}^{(\beta)}(u,v) = \mathcal{E}(u,v) \text{ for all } u,v \in D(\mathcal{E}). \tag{1.2}$$

Lemma 1.1. If S is a positive linear bounded operator on $L^2(E;m)$, then there is a unique positive Radon measure σ on the product space $E \times E$ satisfying that for $u,v \in L^2(E;m)$, $(Su,v) = \int_{E\times E} u(x)v(y)\sigma(dx,dy)$. If in addition S is sub-Markovian, then $\sigma(E \times A) \le m(A)$ for all $A \in \mathcal{B}(E)$.

Proof. The proof is similar to Fukushima et al. [8, Lemma 1.4.1] and the only difference is that the measure σ given here is non-symmetric in general. □

Corollary 1.2. There exists a unique positive Radon measure σ_β on $E \times E$ satisfying

$$(\beta G_\beta u, v) = \int_{E\times E} u(x)v(y)\sigma_\beta(dx,dy) \quad \text{for } u,v \in L^2(E;m). \tag{1.3}$$

Moreover,

$$\sigma_\beta(E \times A) \le m(A) \quad \text{for all } A \in \mathcal{B}(E). \tag{1.4}$$

□

Lemma 1.3. Let U be a relatively compact open subset of E. Then, for $u,v \in C_0(E) \cap D(\mathcal{E})$ with supports contained in U,

$$\mathcal{E}^{(\beta)}(u,v) = \beta \int_{U\times U} (u(y) - u(x))v(y)\sigma_\beta(dx,dy) + \beta \int_U u(x)v(x)(1 - \beta G_\beta I_U(x))m(dx). \tag{1.5}$$

Proof. Direct consequence of (1.1), (1.3) and (1.4). □

Lemma 1.4. The following assertions hold:
(i) For $u \in C_0(E)$, there exists a sequence $\{u_n\}_{n\in\mathbf{N}} \subset C_0(E) \cap D(\mathcal{E})$ such that $\mathrm{supp}[u_n] \subset \{x \in E | u(x) \neq 0\}$, $n \in \mathbf{N}$, and u_n converges to u uniformly as $n \to \infty$.

(ii) For any compact set F and relatively compact open set G with $F \subset G$, there exists $u \in C_0(E) \cap D(\mathcal{E})$ such that $0 \leq u \leq 1, u|_F = 1$ and $u|_{E \backslash G} = 0$.

Proof. By the regularity of $(\mathcal{E}, D(\mathcal{E}))$ and Kuwae [12, Lemma 2.1 (ii)], this lemma can be proved similarly to the case of symmetric Dirichlet forms. $\square$

Definition 1.5. Denote by d the diagonal of $E \times E$.
(i) A subset $A \subset E \times E \backslash d$ is said to be symmetric if its indicator function I_A is symmetric, i.e. $I_A(x, y) = I_A(y, x)$ for all $(x, y) \in E \times E \backslash d$.
(ii) Let J be a Radon measure on $E \times E \backslash d$. A measurable function f on $E \times E \backslash d$ is said to be integrable w.r.t. (with respect to) J in the sense of *symmetrical principle value* (abbreviated by *S.P.V. integrable*), if f is integrable on each relatively compact symmetric subset $A \subset E \times E \backslash d$ and for any increasing sequence of relatively compact symmetric sets $\{A_n\}_{n \geq 1}$ with $\cup_{n=1}^{\infty} A_n = E \times E \backslash d$, the limit

$$S.P.V. \int_{E \times E \backslash d} f(x, y) J(dx, dy) := \lim_{n \to \infty} \int_{A_n} f(x, y) J(dx, dy)$$

exists and is independent of the specific choice of the sequence $\{A_n\}_{n \geq 1}$.

Theorem 1.6. (i) There exist a unique positive Radon measure J on $E \times E \backslash d$ and a unique positive Radon measure K on E such that for $v \in C_0(E) \cap D(\mathcal{E})$ and $u \in I(v)$,

$$\mathcal{E}(u, v) = \int_{E \times E \backslash d} 2(u(y) - u(x)) v(y) J(dx, dy) + \int_E u(x) v(x) K(dx), \tag{1.6}$$

where $I(v) := \{u \in C_0(E) \cap D(\mathcal{E}) : u$ is constant on a neighbourhood of $\text{supp}[v]\}$.
(ii) Define $\mathcal{A}(v) := \{u \in C_0(E) \cap D(\mathcal{E}) | (u(y) - u(x)) v(y)$ is S.P.V. integrable w.r.t. $J\}$. Then for $v \in C_0(E) \cap D(\mathcal{E})$ and $u \in \mathcal{A}(v)$, we have the following unique decomposition:

$$\begin{aligned} \mathcal{E}(u, v) &= \mathcal{E}^c(u, v) + S.P.V. \int_{E \times E \backslash d} 2(u(y) - u(x)) v(y) J(dx, dy) \\ &\quad + \int_E u(x) v(x) K(dx), \end{aligned} \tag{1.7}$$

where $\mathcal{E}^c(u, v)$ satisfies the *left strong local property* in the sense that $I(v) \subset \mathcal{A}(v)$ and $\mathcal{E}^c(u, v) = 0$ whenever $v \in C_0(E) \cap D(\mathcal{E})$, $u \in I(v)$.

Proof. (i) The uniqueness of J and K satisfying (1.6) can be proved in the same way as in Fukushima et al. [8, Theorem 3.2.1] by virtue of Lemma 1.4 (i). The existence of J can be proved similarly to Fukushima et al. [8, Theorem 3.2.1]. Moreover,

$$\frac{\beta}{2} \sigma_\beta \to J \text{ vaguely on } E \times E \backslash d \text{ as } \beta \to \infty. \tag{1.8}$$

To show the existence of K, we fix a relatively compact open set U. For any compact subset F of U, by Lemma 1.4 (ii), there exist $u, v \in C_0(E) \cap D(\mathcal{E})$ satisfying $\text{supp}[u] \cup \text{supp}[v] \subset U$, such that $v|_F \equiv 1$, $v \geq 0$, $u|_{\text{supp}[v]} \equiv 1$ and $0 \leq u \leq 1$. Then, we get by (1.5) that

$$\int_F \beta(1 - \beta G_\beta I_U(x)) m(dx) \leq \beta \int_U u(x) v(x) (1 - \beta G_\beta I_U(x)) m(dx)$$

$$\begin{aligned}
&\le \beta \int_U u(x)v(x)(1-\beta G_\beta I_U(x))m(dx) \\
&\quad +\beta \int_{U\times U} (u(y)-u(x))v(y)\sigma_\beta(dx,dy) \\
&= \mathcal{E}^{(\beta)}(u,v). \qquad (1.9)
\end{aligned}$$

Now it follows from (1.9) that the family of measures $\{\beta(1-\beta G_\beta I_U(x))m(dx)\}$ are uniformly bounded on any compact subset of U. Let $\bar{\rho}$ be a metric compatible with the topology of E, $\{U_l\}_{l\ge 1}$ an increasing sequence of relatively compact open sets satisfying $\cup_{l=1}^{\infty} U_l = E$, and $\{\delta_l\}_{l\ge 1}$ ($\delta_l \downarrow 0$) a decreasing sequence of positive numbers such that $U_l \times U_l \backslash \{(x,y)|\bar{\rho}(x,y) < \delta_l\}$ is a continuous set of J for each l. Note that such $\{U_l\}$ and $\{\delta_l\}$ always exist. Then, there exist an increasing sequence $\{\beta_n\}_{n\in\mathbf{N}}$ satisfying $\beta_n \to \infty$ as $n\to\infty$, and a positive Radon measure K_l on U_l such that for each $l \ge 1$,

$$\beta_n(1-\beta_n G_{\beta_n} I_{U_l})\cdot m \to K_l \text{ vaguely on } U_l \text{ as } n\to\infty. \qquad (1.10)$$

Extend K_l to E by setting $K_l(A) := K_l(A\cap U_l)$ for any Borel subset A of E. By (1.10), for each compact subset F of E, there exists l_0 such that $\{K_l(F)\}_{l\ge l_0}$ is non-increasing. Consequently, there exists a Radon measure K on E such that

$$K_l \to K \text{ vaguely on } E \text{ as } l\to\infty. \qquad (1.11)$$

Denote $\Gamma_l := U_l \times U_l \backslash \{(x,y)|\bar{\rho}(x,y) < \delta_l\}$. Let $v \in C_0(E)\cap D(\mathcal{E})$ and $u \in I(v)$. Suppose that $u(x) = \alpha$ on a neighborhood of supp$[v]$ for some constant α. Then, we get by (1.2) and (1.5) that

$$\begin{aligned}
\mathcal{E}(u,v) &= \lim_{n\to\infty} \frac{\beta_n}{2} \int_{U_l\times U_l, \bar{\rho}(x,y)<\delta_l} 2(u(y)-u(x))v(y)\sigma_{\beta_n}(dx,dy) \\
&\quad + \int_{\Gamma_l} 2(u(y)-u(x))v(y)J(dx,dy) + \int_{U_l} u(x)v(x)K_l(dx)
\end{aligned}$$

provided $l \ge l_1$ for some large enough l_1. Letting $l\to\infty$, we obtain

$$\mathcal{E}(u,v) = \int_{E\times E\backslash d} 2(u(y)-u(x))v(y)J(dx,dy) + \int_E u(x)v(x)K(dx),$$

where the integrability of $(u(y)-u(x))v(y)$ follows from the fact that for any $y \in$ supp$[v]$,

$$(u(y)-u(x))v(y) = (\alpha - u(x))v(y) = (\alpha-u(x))^+ v(y) - (\alpha-u(x))^- v(y),$$

and either supp$[(\alpha-u(x))^+v(y)]$ or supp$[(\alpha-u(x))^-v(y)]$ must be contained in Γ_{l_1} for some large l_1, since u has a compact support. Thus, the measure K constructed in (1.11) satisfies (1.6), which in turn implies that K is independent of the specific choice of $\{U_l\}_{l\ge1}$ and $\{\delta_l\}_{l\ge 1}$ by the uniqueness of K.

(ii) For $v \in C_0(E)\cap D(\mathcal{E})$ and $u \in \mathcal{A}(v)$, define

$$\mathcal{E}^c(u,v) := \lim_{n\to\infty} \frac{\beta_n}{2} \int_{U_l\times U_l, \bar{\rho}(x,y)<\delta_l} 2(u(y)-u(x))v(y)\sigma_{\beta_n}(dx,dy). \qquad (1.12)$$

Then, we obtain decomposition (1.7) by the proof of (i) above. The uniqueness is obvious by (i) and the left strong local property of $\mathcal{E}^c(u,v)$ follows from (1.12). The proof is complete. $\square$

Remark 1.7. (i) Note that if $v \in C_0(E)\cap D(\mathcal{E})$ and $u \in I(v)$, then $\mathcal{E}^c(u,v) = 0$ since $I(v) \subset \mathcal{A}(v)$. In this case, decomposition (1.7) has been obtained in Kim [11, Lemma 2.14] . Further, Chen and Zhao [5, (A.15)] extended the result to non-symmetric Dirichlet forms in the extended sense that only the sub-Markovian property of the dual semigroup of the α-subprocess is assumed for some $\alpha > 0$, rather than that for the original process (that is $\alpha = 0$).
(ii) Mataloni [15, Theorems 2.7 and 2.8] has obtained decompositions like (1.7) but without introducing the notion of S.P.V. integral and the constraint that $u \in \mathcal{A}(v)$. This condition is essential and cannot be dropped. See Example 4.5. We thank Kazuhiro Kuwae for drawing our attention to the paper [15].
(iii) As in the symmetric case, J and K respectively represent the *jumping* and *killing* measures of the Markov process associated with $(\mathcal{E}, D(\mathcal{E}))$. For any $\mathcal{E}$-exceptional set N, $J(E \times N\backslash d) = J(N \times E\backslash d) = 0$ and $K(N) = 0$ (cf. e.g. Hu [9]).

The rest of this paper is organized as follows. In Section 2, we extend the Beurling-Deny formula to a regular non-symmetric Dirichlet form (cf. Theorem 2.3 and Remark 2.4 below). In Section 3, we give some representations of the diffusion part $\mathcal{E}^c$ (cf. (1.7)) of a regular non-symmetric Dirichlet form, which are extensions of LeJan's formula for a regular symmetric Dirichlet form. In Section 4, we give an example and a counterexample.

2 Decomposition of Regular Non-Symmetric Dirichlet Form

We denote the diffusion part, jumping and killing measures of $\hat{\mathcal{E}}$ (cf. Theorem 1.6) by $\hat{\mathcal{E}}^c$, $\hat{J}$ and $\hat{K}$, respectively. One can see that $\hat{J}(dx,dy) = J(dy,dx)$. By Theorem 1.6,

$$\begin{aligned}\hat{\mathcal{E}}(u,v) &= \hat{\mathcal{E}}^c(u,v) + S.P.V.\int_{E\times E\backslash d} 2(u(y)-u(x))v(y)\hat{J}(dx,dy) \\ &\quad + \int_E u(x)v(x)\hat{K}(dx) \quad \text{for all } v \in C_0(E)\cap D(\mathcal{E}) \text{ and } u \in \hat{\mathcal{A}}(v), \qquad (2.1)\end{aligned}$$

where $\hat{\mathcal{A}}(v) := \{u \in C_0(E)\cap D(\mathcal{E}) | (u(y)-u(x))v(y)$ is S.P.V. integrable w.r.t. $\hat{J}\}$.

Theorem 2.1. (i) There always exists a metric ρ on E compatible with the topology of E satisfying the following properties:
(ρ.1) $\int_{E\times F\backslash d}\rho^2(x,y)/(1+\rho^2(x,y))J(dx,dy) < \infty$ for any compact subset $F \subset E$.
(ρ.2) There is a *special standard core* (cf. Fukushima et al. [8, p.6]) $D_\rho \subset C_0(E)\cap D(\mathcal{E})$ such that any $u \in D_\rho$ is ρ-Lipschitz in the sense that for some constant $C > 0$,

$$|u(y)-u(x)| \le C\rho(x,y) \text{ for all } x,y \in E.$$

(ii) Let ρ be a metric satisfying (ρ.1). Then for any $a > 0$, $\mathcal{E}$ can be expressed for $u, v \in C_0(E)\cap D(\mathcal{E})$ as follows

$$\mathcal{E}(u,v) = \mathcal{E}^{a,\rho}(u,v) + \int_{E\times E\backslash d} 2\left(1 - \frac{1}{1+a\rho^2(x,y)}\right)(u(y)-u(x))v(y)J(dx,dy)$$

$$+\int_E u(x)v(x)K(dx). \tag{2.2}$$

Here $\mathcal{E}^{a,\rho}$ is a bilinear form with domain $C_0(E)\cap D(\mathcal{E})$ and satisfies

$$\mathcal{E}^{a,\rho}(u,v)=\int_{E\times E\backslash d}\frac{2(u(y)-u(x))v(y)}{1+a\rho^2(x,y)}J(dx,dy)\ \ \text{for } v\in C_0(E)\cap D(\mathcal{E}) \text{ and } u\in I(v). \tag{2.3}$$

J is a positive Radon measure on $E\times E\backslash d$ and K is a positive Radon measure on E. Furthermore, such $\mathcal{E}^{a,\rho}$, J and K are uniquely determined by $\mathcal{E}$.

Proof. (i) A metric ρ satisfying Theorem 2.1 (i) is not unique. Below we will construct such a ρ. By the regularity of $(\mathcal{E},D(\mathcal{E}))$, we can always find a countable family $D_0\subset C_0(E)\cap D(\mathcal{E})$, say $D_0=\{f_i\}_{i\geq 1}$, which constitutes a *core* of $\mathcal{E}$ (cf. Fukushima et al. [8, p.6]) and separates the points of E.

For any relatively compact open set U and $u,v\in C_0(E)\cap D(\mathcal{E})$ with supports contained in U, similar to (1.5), we get

$$\begin{aligned}\mathcal{E}^{(\beta)}(u,v) &= \frac{\beta}{2}\int_{U\times U}(u(y)-u(x))(v(y)-v(x))\sigma_\beta(dx,dy)\\ &\quad+\frac{\beta}{2}\int_{U\times U}(u(y)v(x)-u(x)v(y))\sigma_\beta(dx,dy)\\ &\quad+\frac{\beta}{2}\int_U u(x)v(x)[(1-\beta G_\beta I_U(x))+(1-\beta\hat{G}_\beta I_U(x))]m(dx).\end{aligned} \tag{2.4}$$

In particular,

$$\begin{aligned}\mathcal{E}^{(\beta)}(u,u) &= \frac{\beta}{2}\int_{U\times U}(u(y)-u(x))^2\sigma_\beta(dx,dy)\\ &\quad+\frac{\beta}{2}\int_U u^2(x)[(1-\beta G_\beta I_U(x))+(1-\beta\hat{G}_\beta I_U(x))]m(dx).\end{aligned} \tag{2.5}$$

By (1.2), (1.8) and (2.5), we find that

$$\int_{E\times E\backslash d}(u(y)-u(x))^2J(dx,dy)\leq\mathcal{E}(u,u)<\infty \text{ for all } u\in C_0(E)\cap D(\mathcal{E}). \tag{2.6}$$

For $x,y\in E$, we define

$$\rho(x,y)=\left(\sum_{i=1}^{\infty}2^{-i}\left(\frac{|f_i(x)-f_i(y)|}{1+\|f_i\|_\infty+\|f_i\|_{\tilde{\mathcal{E}}_1}}\right)^2\right)^{1/2}. \tag{2.7}$$

Since $\{f_i\}_{i\geq 1}$ separates the points of E, one can check that ρ is a metric on E. Moreover, the topology induced by ρ coincides with the trace topology on each compact subset F of E, since $\{f_i\}_{i\geq 1}$ are all continuous functions. Thus, ρ is compatible with the trace topology of E since E is locally compact. By (2.6), ρ satisfies $(\rho.1)$. By (2.7), each $f_i\in D_0$ is ρ-Lipschitz. Denote $D_\rho:=\{u\in C_0(E)\cap D(\mathcal{E})|u \text{ is } \rho\text{-Lipschitz}\}$. Since $D_0\subset D_\rho$, D_ρ is a core of $(\mathcal{E},D(\mathcal{E}))$. Moreover, if f is ρ-Lipschitz, then $f^+ \wedge 1$ is also ρ-Lipschitz. Therefore, D_ρ is a special standard core and $(\rho.2)$ is satisfied.

(ii) Let ρ be a metric satisfying $(\rho.1)$. Then $(1-1/(1+a\rho^2))$ is integrable w.r.t. J on $E\times F\backslash d$ for any $a>0$ and any compact subset $F\subset E$. Since for $u,v\in C_0(E)$, $(u(y)-u(x))v(y)$ is a bounded function such that $\{|(u(y)-u(x))v(y)|>0\}\subset E\times F\backslash d$ for some compact subset $F\subset E$, the second term of (2.2) is well defined. Therefore, decomposition (2.2) can be obtained by simply setting

$$\begin{aligned}\mathcal{E}^{a,\rho}(u,v) \quad &:= \quad \mathcal{E}(u,v)-\int_{E\times E\backslash d} 2\left(1-\frac{1}{1+a\rho^2(x,y)}\right)(u(y)-u(x))v(y)J(dx,dy)\\ &\quad -\int_E u(x)v(x)K(dx). \end{aligned}\tag{2.8}$$

By (1.6) and (2.8), (2.3) is obvious. The uniqueness of decomposition (2.2) under conditions $(\rho.1)$ and (2.3) is a direct consequence of the uniqueness of decomposition (1.6). □

Note that when a tends to ∞, $(1-1/(1+a\rho^2(x,y)))$ tends to 1, and thus (2.2) approximates (1.7). Indeed, $(u(y)-u(x))v(y)$ is S.P.V. integrable w.r.t. J if and only if $(u(y)-u(x))v(y)$ $/(1+a\rho^2(x,y))$ is S.P.V. integrable w.r.t. J. In this case,

$$\mathcal{E}^{a,\rho}(u,v)=\mathcal{E}^c(u,v)+S.P.V.\int_{E\times E\backslash d}\frac{2(u(y)-u(x))v(y)}{1+a\rho^2(x,y)}J(dx,dy),$$

and (2.2) coincides with (1.7).

In general, J is not symmetric and $J-\hat{J}$ is a signed Radon measure, which is well defined on each compact subset of $E\times E\backslash d$. Denote by $J_1:=(J-\hat{J})^+$ the positive part of the Jordan decomposition of $(J-\hat{J})$. Set $J_0:=J-J_1$. One can check that J_0 is the largest symmetric positive Radon measure dominated by J. In particular, if J is symmetric then $J=J_0$. In the next theorem, we will give some sufficient conditions with which decomposition (1.7) holds for all $u,v\in C_0(E)\cap D(\mathcal{E})$, or for all u,v in a special standard core.

Theorem 2.2. Let J be as in Theorem 2.1 and write $J=J_0+J_1$ as above.
(i) If $J_1(E\times F\backslash d)<\infty$ for any compact subset $F\subset E$, then $\mathcal{A}(v)=C_0(E)\cap D(\mathcal{E})$ for any $v\in C_0(E)\cap D(\mathcal{E})$ and hence decomposition (1.7) holds for all $u,v\in C_0(E)\cap D(\mathcal{E})$. In particular, if J is symmetric then decomposition (1.7) holds for all $u,v\in C_0(E)\cap D(\mathcal{E})$.
(ii) If we can find a metric ρ satisfying $(\rho.1)$, $(\rho.2)$ (cf. Theorem 2.1 (i)), and

$$(\rho.3)\qquad \int_{E\times F\backslash d}\frac{\rho(x,y)}{1+\rho^2(x,y)}J_1(dx,dy)<\infty \text{ for any compact subset } F\subset E,$$

then there is a special standard core $D_1\supset D_\rho$ such that decomposition (1.7) holds for all $u,v\in D_1$.

Proof. First, we will show that for any $u,v\in C_0(E)\cap D(\mathcal{E})$, $(u(y)-u(x))v(y)$ is S.P.V. integrable w.r.t. J_0 on $E\times E\backslash d$, moreover,

$$\begin{aligned}S.P.V.\int_{E\times E\backslash d} & 2(u(y)-u(x))v(y)J_0(dx,dy)\\ &=\int_{E\times E\backslash d}(u(y)-u(x))(v(y)-v(x))J_0(dx,dy),\end{aligned}\tag{2.9}$$

where the right hand side of (2.9) is a usual integral. Note that J_0 is the largest symmetric positive Radon measure dominated by J. For any relatively compact symmetric subset $A \subset E \times E\backslash d$, we get by the symmetry of J_0 that

$$\begin{aligned}
&\int_A 2(u(y)-u(x))v(y)J_0(dx,dy) \\
&\quad = \int_A (u(y)-u(x))v(y)J_0(dx,dy) + \int_A (u(x)-u(y))v(x)J_0(dx,dy) \\
&\quad = \int_A (u(y)-u(x))(v(y)-v(x))J_0(dx,dy). \qquad (2.10)
\end{aligned}$$

By Hölder inequality, (2.6) and the fact that $J_0 \leq J$, we find that

$$\begin{aligned}
&\int_{E\times E\backslash d} |(u(y)-u(x))(v(y)-v(x))|J_0(dx,dy) \\
&\quad \leq \left(\int_{E\times E\backslash d} (u(y)-u(x))^2 J(dx,dy)\right)^{\frac{1}{2}} \cdot \left(\int_{E\times E\backslash d} (v(y)-v(x))^2 J(dx,dy)\right)^{\frac{1}{2}} \\
&\quad < \infty.
\end{aligned}$$

Thus, for any increasing sequence $\{A_n\}_{n\geq 1}$ of relatively compact symmetric subsets of $E \times E\backslash d$ satisfying $\cup_{n=1}^{\infty} A_n = E \times E\backslash d$,

$$\lim_{n\to\infty} \int_{A_n} (u(y)-u(x))(v(y)-v(x))J_0(dx,dy) = \int_{E\times E\backslash d} (u(y)-u(x))(v(y)-v(x))J_0(dx,dy).$$

Therefore, (2.9) holds by (2.10).

Now we have shown that $u \in \mathcal{A}(v)$ if and only if $u \in C_0(E) \cap D(\mathcal{E})$ and $(u(y)-u(x))v(y)$ is S.P.V. integrable w.r.t. J_1. Assertion (i) follows immediately. Note that for $u, v \in D_\rho$ (D_ρ is specified in condition (ρ.2)), we get by (ρ.1) and (ρ.3) that

$$\begin{aligned}
\int_{E\times E\backslash d} |(u(y)-u(x))v(y)|J_1(dx,dy) &\leq \int_{E\times F\backslash d} \frac{C\rho(x,y)}{1+\rho^2(x,y)}|v(y)|J_1(dx,dy) \\
&\quad + \int_{E\times F\backslash d} \frac{\rho^2(x,y)}{1+\rho^2(x,y)}|(u(y)-u(x))v(y)|J_1(dx,dy) \\
&< \infty,
\end{aligned}$$

where $F = \text{supp}[v]$ and C is the ρ-Lipschitz constant of u. Therefore, at least D_ρ satisfies assertion (ii).

$\square$

Theorem 2.3. (i) Let ρ be a metric satisfying (ρ.1) or, more generally, satisfying (ρ.1)$'$ below:

$$(\rho.1)' \quad \int_{F\times F\backslash d} \frac{\rho^2(x,y)}{1+\rho^2(x,y)} J(dx,dy) < \infty \text{ for any compact subset } F \subset E.$$

Then for any $a > 0$, $\mathcal{E}$ can be expressed for $u, v \in C_0(E) \cap D(\mathcal{E})$ as follows

$$\begin{aligned}
\mathcal{E}(u,v) &= \tilde{\mathcal{E}}^c(u,v) + \int_{E\times E\backslash d} (u(y)-u(x))(v(y)-v(x))J(dx,dy) + \int_E u(x)v(x)\tilde{K}(dx) \\
&\quad + \check{\mathcal{E}}^{a,\rho}(u,v) + \int_{E\times E\backslash d} \left(1 - \frac{1}{1+a\rho^2(x,y)}\right)(u(y)v(x)-u(x)v(y))J(dx,dy). \qquad (2.11)
\end{aligned}$$

Here $\tilde{\mathcal{E}}^c(u,v)$ is a symmetric, nonnegative definite form satisfying the strong local property (i.e., $u \in I(v)$ implies $\tilde{\mathcal{E}}^c(u,v)=0$), J is a positive Radon measure on $E\times E\backslash d$, $\tilde{K}$ is a positive Radon measure on E, and $\check{\mathcal{E}}^{a,\rho}$ is an anti-symmetric form satisfying

$$\check{\mathcal{E}}^{a,\rho}(u,v)=\int_{E\times E\backslash d}\frac{1}{1+a\rho^2(x,y)}(u(y)v(x)-u(x)v(y))J(dx,dy) \text{ if } \mathrm{supp}[u]\cap\mathrm{supp}[v]=\emptyset. \quad (2.12)$$

Furthermore, such $\tilde{\mathcal{E}}^c$, J, $\tilde{K}$ and $\check{\mathcal{E}}^{a,\rho}$ are uniquely determined by $\mathcal{E}$.
(ii) Let $u,v\in C_0(E)\cap D(\mathcal{E})$ be such that

$$(u(y)v(x)-u(x)v(y)) \text{ is S.P.V. integrable w.r.t. } J. \quad (2.13)$$

Then

$$\begin{aligned}\mathcal{E}(u,v) &= \tilde{\mathcal{E}}^c(u,v)+\int_{E\times E\backslash d}(u(y)-u(x))(v(y)-v(x))J(dx,dy)+\int_E u(x)v(x)\tilde{K}(dx)\\ &\quad +\check{\mathcal{E}}^c(u,v)+S.P.V.\int_{E\times E\backslash d}(u(y)v(x)-u(x)v(y))J(dx,dy), \end{aligned}\quad (2.14)$$

where $\tilde{\mathcal{E}}^c$, J and $\tilde{K}$ are the same as in (2.11), $\check{\mathcal{E}}^c$ is an anti-symmetric form satisfying the local property, i.e. $\mathrm{supp}[u]\cap\mathrm{supp}[v]=\emptyset$ implies $\check{\mathcal{E}}^c(u,v)=0$. Moreover, such $\tilde{\mathcal{E}}^c$, J, $\tilde{K}$ and $\check{\mathcal{E}}^c$ are uniquely determined by $\mathcal{E}$ if decomposition (2.14) holds for all u, v in some special standard core (cf. Theorem 2.6 below).

Proof. (i) Let $\{U_l\}_{l\geq 1}$ be as in (1.10), taking a subsequence of $\{\beta_n\}$ if necessary, we may assume that for each l,

$$\beta_n(1-\beta_n\hat{G}_{\beta_n}I_{U_l})\cdot m\to\hat{K}_l \text{ vaguely on } U_l \text{ as } n\to\infty, \quad (2.15)$$

where $\hat{K}_l$ is the corresponding object of K_l w.r.t. $(\hat{\mathcal{E}},D(\mathcal{E}))$ and

$$\hat{K}_l\to\hat{K} \text{ vaguely on } E \text{ as } l\to\infty. \quad (2.16)$$

Let $\{\delta_l\}$ and $\{\Gamma_l\}$ be the same as in the proof of Theorem 1.6, by (1.8), $(\rho.1)'$, (1.10), (1.11), (2.15) and (2.16), the following limits exist.

$$\begin{aligned}I_1(u,v) &:= \lim_l\lim_n\frac{\beta_n}{2}\int_{\Gamma_l}(u(y)-u(x))(v(y)-v(x))\sigma_{\beta_n}(dx,dy)\\ &= \int_{E\times E\backslash d}(u(y)-u(x))(v(y)-v(x))J(dx,dy),\\ I_2(u,v) &:= \lim_l\lim_n\frac{\beta_n}{2}\int_{\Gamma_l}\frac{a\rho^2(x,y)}{1+a\rho^2(x,y)}(u(y)v(x)-u(x)v(y))\sigma_{\beta_n}(dx,dy)\\ &= \int_{E\times E\backslash d}\left(1-\frac{1}{a\rho^2(x,y)}\right)(u(y)v(x)-u(x)v(y))J(dx,dy),\\ I_3(u,v) &:= \lim_l\lim_n\beta_n\int_{U_l}u(x)v(x)\left(\frac{1-\beta G_{\beta_n}I_{U_l}(x)}{2}+\frac{1-\beta_n\hat{G}_{\beta_n}I_{U_l}(x)}{2}\right)m(dx)\\ &= \int_E u(x)v(x)\frac{1}{2}(K(dx)+\hat{K}(dx)).\end{aligned}$$

By (1.2) and (2.4), we get

$$\begin{aligned}\tilde{\mathcal{E}}(u,v) &:= \frac{1}{2}(\mathcal{E}(u,v)+\mathcal{E}(v,u))\\ &= \lim_l \lim_n \frac{\beta_n}{2}\int_{U_l\times U_l}(u(y)-u(x))(v(y)-v(x))\sigma_{\beta_n}(dx,dy)\\ &\quad +\lim_l\lim_n \beta_n\int_{U_l} u(x)v(x)\cdot\frac{1}{2}\{(1-\beta_n G_{\beta_n}I_{U_l}(x))+(1-\beta_n\hat{G}_{\beta_n}I_{U_l}(x))\}m(dx).\end{aligned}$$

Define $\tilde{\mathcal{E}}^c(u,v) := \tilde{\mathcal{E}}(u,v) - I_1(u,v) - I_3(u,v)$, then $\tilde{\mathcal{E}}^c(u,v)$ admits the following expression

$$\tilde{\mathcal{E}}^c(u,v) = \lim_l\lim_n\frac{\beta_n}{2}\int_{U_l\times U_l,\rho(x,y)<\delta_l}(u(y)-u(x))(v(y)-v(x))\sigma_{\beta_n}(dx,dy).$$

From this expression, it is clear that $\tilde{\mathcal{E}}^c(u,v)$ is a symmetric, nonnegative definite form satisfying the strong local property. Then, we obtain (2.11) by setting

$$\check{\mathcal{E}}^{a,\rho}(u,v) := \mathcal{E}(u,v) - \tilde{\mathcal{E}}^c(u,v) - I_1(u,v) - I_2(u,v) - I_3(u,v).$$

It follows that

$$\begin{aligned}\check{\mathcal{E}}^{a,\rho}(u,v) &= \lim_l\lim_n\left\{\frac{\beta_n}{2}\int_{U_l\times U_l,\rho(x,y)<\delta_l}(u(y)v(x)-u(x)v(y))\sigma_{\beta_n}(dx,dy)\right.\\ &\quad \left.+\int_{\Gamma_l}\frac{1}{1+a\rho^2(x,y)}(u(y)v(x)-u(x)v(y))J(dx,dy)\right\}. \end{aligned}\tag{2.17}$$

Since $u(y)v(x)-u(x)v(y) = (u(y)-u(x))v(y)-(v(y)-v(x))u(y)$, if $\mathrm{supp}[u]\cap\mathrm{supp}[v]=\emptyset$, then

$$\lim_l\lim_n\frac{\beta_n}{2}\int_{U_l\times U_l,\rho(x,y)<\delta_l}(u(y)v(x)-u(x)v(y))\sigma_{\beta_n}(dx,dy) = 0. \tag{2.18}$$

Therefore, (2.12) follows from (2.17) and (2.18).

Now let us show the uniqueness of the decomposition. For $u,v\in C_0(E)\cap D(\mathcal{E})$ satisfying $\mathrm{supp}[u]\cap\mathrm{supp}[v]=\emptyset$, we get by the strong local property of $\tilde{\mathcal{E}}^c(u,v)$ and (2.12) that

$$\begin{aligned}\mathcal{E}(u,v) &= \int_{E\times E\backslash d}(-u(y)v(x)-u(x)v(y))J(dx,dy)\\ &\quad +\int_{E\times E\backslash d}(u(y)v(x)-u(x)v(y))J(dx,dy)\\ &= -2\int_{E\times E\backslash d}u(x)v(y)J(dx,dy).\end{aligned}$$

Hence the measure J is unique (cf. Fukushima et al. [8, Theorem 3.2.1]).

Since $\tilde{\mathcal{E}}^c(u,v)$ is symmetric and $\check{\mathcal{E}}^{a,\rho}(u,v)$ is anti-symmetric, for $u,v\in C_0(E)\cap D(\mathcal{E})$,

$$\begin{aligned}\mathcal{E}(v,u) &= \tilde{\mathcal{E}}(u,v)+\int_{E\times E\backslash d}(v(y)-v(x))(u(y)-u(x))J(dx,dy)\\ &\quad +\int_E v(x)u(x)\tilde{K}(dx)-\check{\mathcal{E}}^{a,\rho}(u,v)\\ &\quad -\int_{E\times E\backslash d}\left(1-\frac{1}{1+a\rho^2(x,y)}\right)(u(y)v(x)-u(x)v(y))J(dx,dy).\end{aligned}\tag{2.19}$$

Set $\hat{J}(dx,dy)=J(dy,dx)$. By (2.11) and (2.19), for $u,v\in C_0(E)\cap D(\mathcal{E})$,

$$\begin{aligned}
\tilde{\mathcal{E}}(u,v) &= \tilde{\mathcal{E}}^c(u,v)+\int_{E\times E\backslash d}(u(y)-u(x))(v(y)-v(x))J(dx,dy)\\
&\quad+\int_E u(x)v(x)\tilde{K}(dx,dy)\\
&= \tilde{\mathcal{E}}^c(u,v)+\int_{E\times E\backslash d}(u(y)-u(x))(v(y)-v(x))\frac{J+\hat{J}}{2}(dx,dy)\\
&\quad+\int_E u(x)v(x)\tilde{K}(dx,dy).
\end{aligned}$$

Thus the uniqueness of $\tilde{\mathcal{E}}^c$ and $\tilde{K}$ follows from the uniqueness of the classical Beurling-Deny formula for symmetric Dirichlet forms. Finally, the uniqueness of $\check{\mathcal{E}}^{a,\rho}$ follows from (2.11) and the uniqueness of J, $\tilde{\mathcal{E}}^c$ and $\tilde{K}$.

(ii) Note that for any $u,v\in C_0(E)$, condition $(\rho.1)'$ implies that $(1-1/(1+a\rho^2(x,y)))(u(y)v(x)-u(x)v(y))$ is always integrable on $E\times E\backslash d$ w.r.t. J. Thus, in this case, $(u(y)v(x)-u(x)v(y))$ is S.P.V. integrable w.r.t. J if and only if $(1/(1+a\rho^2(x,y)))(u(y)v(x)-u(x)v(y))$ is S.P.V. integrable w.r.t. J. From the above proof of part (i), we know that $\check{\mathcal{E}}^{a,\rho}(u,v)$ can be expressed as

$$\begin{aligned}
\check{\mathcal{E}}^{a,\rho}(u,v) &= \lim_l\lim_n\left\{\frac{\beta_n}{2}\int_{U_l\times U_l,\rho(x,y)<\delta_l}(u(y)v(x)-u(x)v(y))\sigma_{\beta_n}(dx,dy)\right.\\
&\quad\left.+\int_{\Gamma_l}\frac{1}{1+a\rho^2(x,y)}(u(y)v(x)-u(x)v(y))J(dx,dy)\right\}\\
&:= \lim_l\lim_n\{A_{l,n}(u,v)+B_{l,n}(u,v)\}.
\end{aligned}$$

Suppose now that $(u(y)v(x)-u(x)v(y))$ is S.P.V. integrable w.r.t. J. Then $\check{\mathcal{E}}^c(u,v):=\lim_l\lim_n A_{l,n}(u,v)$ exists and we get decomposition (2.14). By the expression of $A_{l,n}(u,v)$, we see that $\check{\mathcal{E}}^c(u,v)$ is an anti-symmetric form satisfying the local property. Under the additional assumption that decomposition (2.14) holds for all u, v in some special standard core, the uniqueness of $\tilde{\mathcal{E}}^c$, J, $\tilde{K}$ and $\check{\mathcal{E}}^c$ follows similarly to part (i). □

Remark 2.4. If $u\in\mathcal{A}(v)$ and $v\in\mathcal{A}(u)$, then condition (2.13) is fulfilled. In this case,

$$\tilde{\mathcal{E}}^c(u,v) = \frac{1}{2}(\,\mathcal{E}^c(u,v)+\hat{\mathcal{E}}^c(u,v)\,) \tag{2.20}$$

and

$$\check{\mathcal{E}}^c(u,v)=\frac{1}{2}(\,\mathcal{E}^c(u,v)-\mathcal{E}^c(v,u)\,)=\frac{1}{2}(\,\hat{\mathcal{E}}^c(v,u)-\hat{\mathcal{E}}^c(u,v)). \tag{2.21}$$

In general, it is not necessary that $\check{\mathcal{E}}^c(u,v)=\frac{1}{2}(\mathcal{E}^c(u,v)-\hat{\mathcal{E}}^c(u,v))$.

If $(\mathcal{E},D(\mathcal{E}))$ is symmetric, then for any $u,v\in C_0(E)\cap D(\mathcal{E})$, $u\in\mathcal{A}(v)$ and $v\in\mathcal{A}(u)$ (cf. the proof of Theorem 2.2). In this case, the dual form $(\hat{\mathcal{E}},D(\mathcal{E}))$ coincides with $(\mathcal{E},D(\mathcal{E}))$, and thus $\mathcal{E}^c(u,v)=\hat{\mathcal{E}}^c(u,v)$ for any $u,v\in C_0(E)\cap D(\mathcal{E})$. By (2.20) and (2.21), we find that $\tilde{\mathcal{E}}^c(u,v)=\mathcal{E}^c(u,v),\check{\mathcal{E}}^c(u,v)=0$. By the uniqueness of (2.14), we know that the measure J in (2.14) is symmetric. Then, it is easy to check that $S.P.V. \int_{E\times E\backslash d}(u(y)v(x) - u(x)v(y))J(dx, dy) = 0$. Hence (2.14) is just the classical Beurling-Deny formula for symmetric forms.

Remark 2.5. In Bliedtner [3, (9.2)], the author gave a representation which is essentially the same as (2.14) but without introducing the notion of S.P.V. integral and the crucial condition (2.13). We point out that condition (2.13) cannot be dropped. See Example 4.5.

Theorem 2.6. Let $J = J_0 + J_1$ be as in Theorem 2.2.
(i) If $J_1(F \times F\backslash d) < \infty$ for any compact subset $F \subset E$, then condition (2.13) is always fulfilled and hence decomposition (2.14) holds for all $u, v \in C_0(E)\cap D(\mathcal{E})$. In particular, if J is symmetric, then (2.14) holds for all $u, v \in C_0(E) \cap D(\mathcal{E})$.
(ii) If we can find a metric ρ satisfying (ρ.1)-(ρ.3). Then, there is a special standard core $D_1 \supset D_\rho$ such that decomposition (2.14) holds for all $u, v \in D_1$.

Proof. (i) is clear. Note that

$$u(y)v(x) - u(x)v(y) = (u(y) - u(x))v(y) - (v(y) - v(x))u(y).$$

Then, condition (2.13) is fulfilled if $u \in \mathcal{A}(v)$ and $v \in \mathcal{A}(u)$. Therefore, assertion (ii) follows directly from Theorem 2.2 and Theorem 2.3 (ii). □

3 Extension of LeJan's Formula to Non-Symmetric Dirichlet Form

Let J be the jumping measure of $(\mathcal{E}, D(\mathcal{E}))$. Decompose $J = J_0 + J_1$ as in Section 2. Throughout this section, we make the following assumption.

Assumption 3.1. For any compact subset $F \subset E$, $J_1(E \times F\backslash d) < \infty$ and $J_1(F \times E\backslash d) < \infty$.

Note that if J is symmetric, then $J_1 = 0$ and Assumption 3.1 is automatically satisfied. Proposition 4.4 (i) below provides another example for which Assumption 3.1 is satisfied.

By Assumption 3.1 and Theorem 2.2, (1.7) and (2.1) hold for any $u, v \in C_0(E)\cap D(\mathcal{E})$. Then, it follows that

$$\begin{aligned}
\check{\mathcal{E}}(u, v) &:= \frac{1}{2}(\mathcal{E}(u, v) - \mathcal{E}(v, u)) \\
&= \frac{1}{2}(\mathcal{E}^c(u, v) - \hat{\mathcal{E}}^c(u, v)) + S.P.V. \int_{E\times E\backslash d} 2(u(y) - u(x))v(y)\frac{J - \hat{J}}{2}(dx, dy) \\
&\quad + \int_E u(x)v(x)\frac{K - \hat{K}}{2}(dx) \\
&:= \check{\mathcal{E}}^c(u, v) + S.P.V. \int_{E\times E\backslash d} 2(u(y) - u(x))v(y)\check{J}(dx, dy) + \int_E u(x)v(x)\check{K}(dx). \quad (3.1)
\end{aligned}$$

Here we point out that the definition of $\check{\mathcal{E}}^c(u, v)$ is different from that given in Section 2 (cf. Remark 2.4). Throughout this section, we define $\check{\mathcal{E}}^c(u, v)$ by (3.1).

In the following, we will give some structure results for the diffusion part $\mathcal{E}^c$ of $(\mathcal{E}, D(\mathcal{E}))$. Note that $\mathcal{E}^c(u, v) = \tilde{\mathcal{E}}^c(u, v) + \check{\mathcal{E}}^c(u, v)$, where $\tilde{\mathcal{E}}^c(u, v) = \frac{1}{2}(\mathcal{E}^c(u, v) + \hat{\mathcal{E}}^c(u, v))$. Consequently, we only need to concentrate on the structure of $\check{\mathcal{E}}^c$ below.

3.1 Linear Functional $L(u,v)$

For $u,v\in C_0(E)\cap D(\mathcal{E})$, we define a linear functional $L(u,v)$ on $C_0(E)\cap D(\mathcal{E})$ by

$$< L(u,v),f>:=\frac{1}{2}(\mathcal{E}^c(u,vf)-\hat{\mathcal{E}}^c(u,vf))\ \text{ for all } f\in C_0(E)\cap D(\mathcal{E}). \tag{3.2}$$

Then, $\check{\mathcal{E}}^c(u,v)=< L(u,v),f>$ for arbitrary $f\in C_0(E)\cap D(\mathcal{E})$ satisfying $f|_{\text{supp}[v]}=1$. It follows from (3.1) that

$$\begin{aligned}< L(u,v),f> &= \frac{1}{2}(\mathcal{E}(u,vf)-\hat{\mathcal{E}}(u,vf))-S.P.V.\int_{E\times E\backslash d}2(u(y)-u(x))v(y)f(y)\check{J}(dx,dy)\\ &\quad -\int_E u(x)v(x)f(x)\check{K}(dx).\end{aligned} \tag{3.3}$$

Theorem 3.2. Let $u,v,w,f\in C_0(E)\cap D(\mathcal{E})$. Then
(i) $< L(u,v),f>$ is bilinear w.r.t. u and v.
(ii) $< L(u,vw),f>=< L(u,w),vf>$.
(iii)$< L(u^2,v),f>=< L(u,v),2uf>$.
(iv) $< L(uv,w),f>=< L(u,w),vf>+< L(v,w),uf>$.

Proof. (i) and (ii) are obvious by (3.2). (iv) follows by applying (iii) to $(u+v)^2$, u^2 and v^2. In the following, we prove (iii). Suppose $\text{supp}[u]\cup\text{supp}[v]\cup\text{supp}[f]\subset G_1\subset\bar{G}_1\subset G_2$, where G_1,G_2 are two relatively compact open sets. Then,

$$\begin{aligned}\mathcal{E}^{(\beta)}(u,vf) &= \beta(u-\beta G_\beta u,vf)\\ &= \beta\int_{G_2\times G_2}(u(y)-u(x))v(y)f(y)\sigma_\beta(dx,dy)\\ &\quad +\beta\int_{G_2}(1-\beta G_\beta I_{G_2}(x))u(x)v(x)f(x)m(dx)\\ &\to \mathcal{E}(u,vf)\ \text{as}\ \beta\to\infty.\end{aligned} \tag{3.4}$$

Similarly,

$$\begin{aligned}\hat{\mathcal{E}}^{(\beta)}(u,vf) &= \beta\int_{G_2\times G_2}(u(y)-u(x))v(y)f(y)\hat{\sigma}_\beta(dx,dy)\\ &\quad +\beta\int_{G_2}(1-\beta\hat{G}_\beta I_{G_2}(x))u(x)v(x)f(x)m(dx)\\ &\to \hat{\mathcal{E}}(u,vf)\ \text{as}\ \beta\to\infty,\end{aligned} \tag{3.5}$$

where $\hat{\sigma}_\beta$ is the measure associated with $\beta\hat{G}_\beta$ (cf. Corollary 1.2). Note that $\hat{\sigma}_\beta(dx,dy)=\sigma_\beta(dy,dx)$.

By (3.3), (3.4) and (3.5) (cf. the proof of Theorem 1.6), we get

$$\begin{aligned}< L(u,v),f> &= \Big[\frac{1}{2}\lim_{\beta\to\infty}\beta\int_{G_2\times G_2}(u(y)-u(x))v(y)f(y)(\sigma_\beta-\hat{\sigma}_\beta)(dx,dy)\\ &\quad +\frac{1}{2}\int_E u(x)v(x)f(x)K(dx)-\frac{1}{2}\int_E u(x)v(x)f(x)\hat{K}(dx)\Big]\end{aligned}$$

$$
\begin{aligned}
&-S.P.V.\int_{E\times E\backslash d}2(u(y)-u(x))v(y)f(y)\check{J}(dx,dy)-\int_E u(x)v(x)f(x)\check{K}(dx)\\
=\ & \frac{1}{2}\lim_{\beta\to\infty}\beta\int_{G_2\times G_2}(u(y)-u(x))v(y)f(y)(\sigma_\beta-\hat{\sigma}_\beta)(dx,dy)\\
&-S.P.V.\int_{E\times E\backslash d}2(u(y)-u(x))v(y)f(y)\check{J}(dx,dy). \qquad (3.6)
\end{aligned}
$$

It follows from (3.6) that

$$
\begin{aligned}
&< L(u^2,v),f> - < L(u,v),2uf>\\
&\quad = -\frac{1}{2}\lim_{\beta\to\infty}\beta\int_{G_2\times G_2}(u(y)-u(x))^2v(y)f(y)(\sigma_\beta-\hat{\sigma}_\beta)(dx,dy)\\
&\quad\quad +S.P.V.\int_{E\times E\backslash d}(u(y)-u(x))^2v(y)f(y)(J-\hat{J})(dx,dy). \qquad (3.7)
\end{aligned}
$$

We now analyze the limit term in (3.7). Note that

$$
\begin{aligned}
&\frac{1}{2}\lim_{\beta\to\infty}\beta\int_{G_2\times G_2}(u(y)-u(x))^2v(y)f(y)(\sigma_\beta-\hat{\sigma}_\beta)(dx,dy)\\
&\quad = \frac{1}{2}\lim_{\beta\to\infty}\beta\int_{G_2\times G_2}(u(y)-u(x))^2(v(y)f(y)-v(x)f(x))\sigma_\beta(dx,dy)\\
&\quad = \frac{1}{2}\lim_{\beta\to\infty}\beta\left[\int_{G_2\times G_2}(u(y)-u(x))^2(v(y)f(y)-v(x)f(x))\phi(x)\phi(y)\sigma_\beta(dx,dy)\right.\\
&\quad\quad +\int_{G_2\times G_2}u^2(y)v(y)f(y)(1-\phi(x))\sigma_\beta(dx,dy)\\
&\quad\quad \left.-\int_{G_2\times G_2}u^2(x)v(x)f(x)(1-\phi(y))\sigma_\beta(dx,dy)\right]\\
&\quad := \frac{1}{2}\lim_{\beta\to\infty}[I_1(\beta)+I_2(\beta)-I_3(\beta)], \qquad (3.8)
\end{aligned}
$$

where $\phi\in C_0(E)\cap D(\mathcal{E})$ satisfying $\phi\geq 0,\phi|_{G_1}=1$ and $\mathrm{supp}[\phi]\subset G_2$. Note that $(v(y)f(y)-v(x)f(x))\phi(x)\phi(y)\in C_\infty(E\times E\backslash d)$, where $C_\infty(E\times E\backslash d)$ denotes the set of all continuous functions on $E\times E\backslash d$ vanishing at the infinity, and the measures $\beta(u(y)-u(x))^2\sigma_\beta(dx,dy)$ are uniformly bounded by $2\mathcal{E}(u,u)$ and converge vaguely to $2(u(y)-u(x))^2J(dx,dy)$ on $E\times E\backslash d$ as $\beta\to\infty$ (cf. (2.5)). Then, we get

$$
\lim_{\beta\to\infty}I_1(\beta)=2\int_{E\times E\backslash d}(u(y)-u(x))^2(v(y)f(y)-v(x)f(x))\phi(x)\phi(y)J(dx,dy). \qquad (3.9)
$$

For $I_2(\beta)$, noting that $\phi|_{G_1}=1$, by Corollary 1.2, (1.2), (1.10), (1.11) and Theorem 1.6, we get

$$
\begin{aligned}
I_2(\beta) &= \beta\int_{G_2\times G_2}u^2(y)v(y)f(y)\sigma_\beta(dx,dy)-\beta\int_{G_2\times G_2}u^2(y)v(y)f(y)\phi(x)\sigma_\beta(dx,dy)\\
&= -\beta(\beta G_\beta\phi,u^2vf)+\beta(\beta G_\beta I_{G_2},u^2vf)
\end{aligned}
$$

$$
\begin{aligned}
&= \beta(\phi - \beta G_\beta \phi, u^2 v f) - \beta(1 - \beta G_\beta I_{G_2}, u^2 v f) \\
&\to \mathcal{E}(\phi, u^2 v f) - \int_E u^2(x) v(x) f(x) K(dx) \\
&= \mathcal{E}^c(\phi, u^2 v f) + S.P.V. \int_{E\times E\backslash d} 2(\phi(y) - \phi(x)) u^2(y) v(y) f(y) J(dx, dy) \\
&\quad + \int_E u^2(x) v(x) f(x) \phi(x) K(dx) - \int_E u^2(x) v(x) f(x) K(dx) \\
&= S.P.V. \int_{E\times E\backslash d} 2u^2(y) v(y) f(y)(1 - \phi(x)) J(dx, dy). \qquad (3.10)
\end{aligned}
$$

Similarly, we get

$$
\lim_{\beta\to\infty} I_3(\beta) = S.P.V. \int_{E\times E\backslash d} 2u^2(x) v(x) f(x)(1 - \phi(y)) J(dx, dy). \qquad (3.11)
$$

Therefore, $< L(u^2, v), f >=< L(u, v), 2uf >$ by (3.7)-(3.11). □

Lemma 3.3. Let $u, v, w \in C_0(E) \cap D(\mathcal{E})$. Then
(i) If u is constant on a relatively compact open set G, then for any $f \in C_0(G) \cap D(\mathcal{E})$, $< L(u, v), f >= 0$.
(ii) If $v = w$ m-a.e. on a relatively compact open set G, then for any $f \in C_0(G) \cap D(\mathcal{E})$, $< L(u, v), f >=< L(u, w), f >$.

Proof. (i) Direct consequence of (3.2) and the left strong local property of $\mathcal{E}^c$ and $\hat{\mathcal{E}}^c$.
(ii) It is obvious by (3.2). □

Proposition 3.4. Let $u_1, \dots, u_m,\ v \in C_0(E) \cap D(\mathcal{E})$. If $\phi \in C^2(\mathbf{R}^m)$ with $\phi(0) = 0$, then $\phi(u) := \phi(u_1, \dots, u_m) \in C_0(E) \cap D(\mathcal{E})$, and for any $f \in C_0(E) \cap D(\mathcal{E})$,

$$
< L(\phi(u), v), f >= \sum_{i=1}^{m} < L(u_i, v), \phi_{x_i}(u) f > . \qquad (3.12)
$$

Proof. The proof follows from that of Fukushima et al. [8, Theorem 3.2.2]. Let $\phi \in C^2(\mathbf{R}^m)$ with $\phi(0) = 0$. By the inequality (cf. Fukushima et al. [8, (3.2.27)])

$$
\sqrt{\mathcal{E}_\alpha(\phi(u), \phi(u))} \le \sum_{i=1}^{m} \|\phi_{x_i}\|_{L^\infty(S)} \sqrt{\mathcal{E}_\alpha(u_i, u_i)}, \quad \alpha \ge 0, \qquad (3.13)
$$

noting that $\phi(0) = 0$, one can check that $\phi(u) \in C_0(E) \cap D(\mathcal{E}), \phi_{x_i}(u) \in D(\mathcal{E}), 1 \le i \le m$. Let $\mathcal{A} := \{\phi \in C^2(\mathbf{R}^m) | \phi(0) = 0$ and (3.12) holds$\}$. By Theorem 3.2 (iv) and (i), one finds that $\phi\psi \in \mathcal{A}$ if $\phi, \psi \in \mathcal{A}$. Since $\mathcal{A}$ contains the coordinate functions, it contains all the polynomials vanishing at the origin.

Let S be a finite cube containing the range of $u(x) = (u_1(x), \dots, u_m(x))$. Then there exists a sequence $\{\phi^k(x)\}$ of polynomials vanishing at the origin such that $\phi^k \to \phi, \phi^k_{x_i} \to \phi_{x_i}, \phi^k_{x_i x_j} \to \phi_{x_i x_j}, 1 \le i, j \le m$, uniformly on S (cf. Courant and Hilbert [6, II §4]). Due to (3.13), $\phi^k(u)$ is $\mathcal{E}_1$-convergent to $\phi(u)$ and $\phi^k_{x_i}(u)$ is $\mathcal{E}_1$-convergent to $\phi_{x_i}(u), 1 \le i \le m$, as $k \to \infty$. And

$\phi^k(u(x)), \phi^k_{x_i}(u(x)), 1 \le i \le m$ are uniformly bounded and respectively converge to $\phi(u(x))$ and $\phi_{x_i}(u(x)), 1 \le i \le m, x \in E$, as $k \to \infty$. Thus

$$\begin{aligned}
< L(\phi(u), v), f > &= \frac{1}{2}(\mathcal{E}(\phi(u), vf) - \mathcal{E}(vf, \phi(u)) \\
&\quad -S.P.V. \int_{E\times E\setminus d} 2(\phi(u)(y) - \phi(u)(x))v(y)f(y)\frac{J_1 - \hat{J}_1}{2}(dx, dy) \\
&\quad - \int_E \phi(u)(x)v(x)f(x)\check{K}(dx) \\
&= \lim_{k\to\infty}\Bigg[\frac{1}{2}(\mathcal{E}(\phi^k(u), vf) - \mathcal{E}(vf, \phi^k(u)) \\
&\quad -S.P.V. \int_{E\times E\setminus d} 2(\phi^k(u)(y) - \phi^k(u)(x))v(y)f(y)\frac{J_1 - \hat{J}_1}{2}(dx, dy) \\
&\quad - \int_E \phi^k(u)(x)v(x)f(x)\check{K}(dx)\Bigg] \\
&= \lim_{k\to\infty} < L(\phi^k(u), v), f > \\
&= \sum_{k=1}^m \lim_{k\to\infty} < L(u_i, v), \phi^k_{x_i}(u)f > \\
&= \sum_{i=1}^m \lim_{k\to\infty}\Bigg[\frac{1}{2}(\mathcal{E}(u_i, \phi^k_{x_i}(u)f) - \mathcal{E}(\phi^k_{x_i}(u)f, u_i) \\
&\quad -S.P.V. \int_{E\times E\setminus d} 2(u_i(y) - u_i(x))\phi^k_{x_i}(u)(y)f(y)\frac{J_1 - \hat{J}_1}{2}(dx, dy) \\
&\quad - \int_E u_i(x)\phi^k_{x_i}(u)(x)f(x)\check{K}(dx)\Bigg] \\
&= \sum_{i=1}^m \Bigg[\frac{1}{2}(\mathcal{E}(u_i, \phi_{x_i}(u)f) - \mathcal{E}(\phi_{x_i}(u)f, u_i) \\
&\quad -S.P.V. \int_{E\times E\setminus d} 2(u_i(y) - u_i(x))\phi_{x_i}(u)(y)f(y)\frac{J_1 - \hat{J}_1}{2}(dx, dy) \\
&\quad - \int_E u_i(x)\phi_{x_i}(u)(x)f(x)\check{K}(dx)\Bigg] \\
&= \sum_{i=1}^m < L(u_i, v), \phi_{x_i}(u)f > .
\end{aligned}$$

The proof is complete. □

Define

$$\begin{aligned}
\mathcal{F}^c_{loc} := \{u| &\text{ for any relatively compact open set } G, \text{ there exists a function} \\
&v \in C_0(E) \cap D(\mathcal{E}) \text{ such that } u = v \ m\text{-a.e. on } G\}. \qquad (3.14)
\end{aligned}$$

Remark 3.5. (i) Since $(\mathcal{E}, D(\mathcal{E}))$ is regular, it is easy to show that $1 \in \mathcal{F}^c_{loc}$ (cf. Lemma 1.4 (ii)).

(ii) If $v \in \mathcal{F}^c_{loc}$ and $f \in C_0(E) \cap D(\mathcal{E})$, then by the definition of $\mathcal{F}^c_{loc}$ there exists a function $\bar{v} \in C_0(E) \cap D(\mathcal{E})$ such that $v = \bar{v}$ m-a.e. on supp$[f]$ and thus $vf = \bar{v}f$ m-a.e. Consequently, $vf \in C_0(E) \cap D(\mathcal{E})$.

Take a sequence $\{G_n\}_{n\in\mathbf{N}}$ of relatively compact open sets such that $\bar{G}_n \subset G_{n+1}, \forall n \geq 1$, and $\cup_{n=1}^{\infty} G_n = E$. For $u, v \in \mathcal{F}^c_{loc}$, there exist two sequences $\{u_n\}_{n\in\mathbf{N}}, \{v_n\}_{n\in\mathbf{N}}$ with $u_n, v_n \in C_0(E) \cap D(\mathcal{E}), \forall n \geq 1$, such that $u_n = u, v_n = v$ m-a.e. on G_n. By Lemma 3.3, we can define the linear functional $< L(u,v), f >:=< L(u_n, v_n), f >$, $\forall f \in C_0(G_n) \cap D(\mathcal{E})$. This definition is independent of the specific choice of $\{u_n\}_{n\in\mathbf{N}}$ and $\{v_n\}_{n\in\mathbf{N}}$. We call such $\{u_n\}_{n\in\mathbf{N}}$ an *approximating sequence* of u.

By Theorem 3.2, one can easily prove the following corollary.

Corollary 3.6. Let $u, v, w \in \mathcal{F}^c_{loc}$ and $f \in C_0(E) \cap D(\mathcal{E})$. Then
(i) $< L(u,v), f >$ is bilinear w.r.t. u and v .
(ii) $< L(u, vw), f >=< L(u,w), vf >$.
(iii) $< L(uv, w), f >=< L(u,w), vf > + < L(v,w), uf >$.

Proposition 3.7. Let $u_1, \ldots, u_m, v \in \mathcal{F}^c_{loc}, \phi \in C^2(\mathbf{R}^m)$, and $u(x) := (u_1(x), \ldots, u_m(x))$. Then $\phi(u), \phi_{x_i}(u) \in \mathcal{F}^c_{loc}, 1 \leq i \leq m$, and for any $f \in C_0(E) \cap D(\mathcal{E})$,

$$< L(\phi(u), v), f >= \sum_{i=1}^{m} < L(u_i, v), \phi_{x_i}(u) f > . \tag{3.15}$$

Proof. Define $\psi(x) := \phi(x) - \phi(0)$. Then $\psi \in C^2(\mathbf{R}^m)$ with $\psi(0) = 0$. If $w_1, \ldots, w_m \in C_0(E) \cap D(\mathcal{E})$, one can show that $\psi(w_1, \ldots, w_m) \in C_0(E) \cap D(\mathcal{E})$ following the proof of Proposition 3.4. Then $\psi(u) \in \mathcal{F}^c_{loc}$. Since the constant function $\phi(0) \in \mathcal{F}^c_{loc}$ by Remark 3.5 (i), one finds that $\phi(u) \in \mathcal{F}^c_{loc}$. Similarly, one can show that $\phi_{x_i}(u) \in \mathcal{F}^c_{loc}, 1 \leq i \leq m$.

By Corollary 3.6 (i) and Lemma 3.3 (i),

$$< L(\phi(u), v), f >=< L(\psi(u), v), f > + < L(\phi(0), v)), f >=< L(\psi(u), v), f > . \tag{3.16}$$

Take the sequence $\{G_n\}_{n\in\mathbf{N}}$ of relatively compact open sets as above. Let $\{u_1^{(k)}\}_{k\in\mathbf{N}}, \ldots, \{u_m^{(k)}\}_{k\in\mathbf{N}}$ and $\{v^{(k)}\}_{k\in\mathbf{N}}$ be approximating sequences of $u_1, \ldots, u_m$ and v, respectively. One can show that $\{\psi(u_1^{(k)}, \ldots, u_m^{(k)})\}_{k\in\mathbf{N}}$ is an approximating sequence of $\psi(u_1, \ldots, u_m)$. Then, for any $f \in C_0(G_k) \cap D(\mathcal{E})$, we get by Proposition 3.4 that

$$\begin{aligned} < L(\psi(u), v), f > &= < L(\psi(u_1^{(k)}, \ldots, u_m^{(k)}), v^{(k)}), f > \\ &= \sum_{i=1}^{m} < L(u_i^{(k)}, v^{(k)}), \psi_{x_i}(u_1^{(k)}, \ldots, u_m^{(k)}) f > \\ &= \sum_{i=1}^{m} < L(u_i, v), \psi_{x_i}(u_1, \ldots, u_m) f > \\ &= \sum_{i=1}^{m} < L(u_i, v), \psi_{x_i}(u) f > . \end{aligned} \tag{3.17}$$

Since $\phi_{x_i} = \psi_{x_i}, 1 \leq i \leq m$, (3.15) follows form (3.16) and (3.17). The proof is complete. $\square$

3.2 Functional Representation of $\check{\mathcal{E}}^c(u,v)$

In this subsection, we assume that U is a domain of $\mathbf{R}^d$ and $(\mathcal{E},D(\mathcal{E}))$ is a regular (non-symmetric) Dirichlet form on $L^2(U,m)$ satisfying $C_0^\infty(U)\subset D(\mathcal{E})$. Further, we make the following assumption.

Assumption 3.1'. For any compact set $K\subset U$, $J(U\times K\backslash d)<\infty$ and $J(K\times U\backslash d)<\infty$.

Note that if Assumption 3.1' is satisfied, then Assumption 3.1 is automatically satisfied since $J=J_0+J_1$. As an example, Assumption 3.1' is satisfied with $U=\mathbf{R}^d$ in the situation of Proposition 4.4 (i) below.

For $u,v\in C_0^\infty(U)$ and $f\in C_0(U)\cap D(\mathcal{E})$, we get by Corollary 3.6 (ii) and Proposition 3.7 that

$$\check{\mathcal{E}}^c(u,vf)=<L(u,v),f>=<L(u,1),vf>=\sum_{i=1}^d<L(x_i,1),\frac{\partial u}{\partial x_i}vf>.$$

Taking an f satisfying $f|_{\mathrm{supp}[v]}=1$, we get

$$\check{\mathcal{E}}^c(u,v)=\sum_{i=1}^d<L(x_i,1),\frac{\partial u}{\partial x_i}v>. \tag{3.18}$$

In order to give an explicit representation of $<L(x_i,1),\frac{\partial u}{\partial x_i}v>$, we need the following two lemmas. In the sequel, we let $(\tilde{\mathcal{E}},D(\tilde{\mathcal{E}}))$ be the symmetric part of $(\mathcal{E},D(\mathcal{E}))$ and respectively denote by $\tilde{\mathcal{E}}^c$, $\tilde{J}$ and $\tilde{K}$, the diffusion part, jumping and killing measures of $(\tilde{\mathcal{E}},D(\tilde{\mathcal{E}}))$.

Lemma 3.8. For $u\in C_0(U)\cap D(\mathcal{E})$, there exists a unique $F(u)\in D(\mathcal{E})$ such that

$$\check{\mathcal{E}}^c(u,v)=\tilde{\mathcal{E}}_1(F(u),v)-\int_{U\times U\backslash d}2(u(y)-u(x))v(y)\check{J}(dx,dy)\ \text{ for all } v\in C_0(U)\cap D(\mathcal{E}).$$

Proof. By Assumption 3.1', Cauchy-Schwartz inequality and (2.6), we find that $2(u(y)-u(x))v(y)$ is integrable w.r.t. $\check{J}$ on $U\times U\backslash d$. Then, for any $v\in C_0(U)\cap D(\mathcal{E})$, we get by (3.1) that

$$\begin{aligned}\check{\mathcal{E}}^c(u,v)\ &+\int_{U\times U\backslash d}2(u(y)-u(x))v(y)\check{J}(dx,dy)\\ &=\ \frac{1}{2}(\mathcal{E}(u,v)-\mathcal{E}(v,u))-\int_U u(x)v(x)\check{K}(dx)\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\left|\int_U u(x)v(x)\check{K}(dx)\right|\ &\leq\ \int_U|u(x)v(x)|\frac{1}{2}(K+\hat{K})(dx)\\ &=\ \int_U|u(x)v(x)|\tilde{K}(dx)\\ &\leq\ \left(\int_U u^2(x)\tilde{K}(dx)\right)^{\frac{1}{2}}\cdot\left(\int_U v^2(x)\tilde{K}(dx)\right)^{\frac{1}{2}}\\ &\leq\ \tilde{\mathcal{E}}^{\frac{1}{2}}(u,u)\tilde{\mathcal{E}}^{\frac{1}{2}}(v,v).\end{aligned} \tag{3.20}$$

By (3.19), (3.20) and the weak sector condition of Dirichlet forms, there exists a constant $C(u) > 0$ such that $|\check{\mathcal{E}}^c(u,v) + \int_{U\times U\backslash d} 2(u(y)-u(x))v(y)\check{J}(dx,dy)| \leq C(u)\tilde{\mathcal{E}}_1^{1/2}(v,v)$. Then, the proof is complete by noting that $C_0(U)\cap D(\mathcal{E})$ is dense in $D(\mathcal{E})$. $\square$

Lemma 3.9. Let $v \in D(\mathcal{E}), u_1,\dots,u_m \in D(\mathcal{E})_b$, and $\phi \in C^1(\mathbf{R}^m)$. Denote $\phi(u) := \phi(u_1,\dots,u_m)$. Then

$$d\tilde{\mu}^c_{<\phi(u),v>} = \sum_{i=1}^m \phi_{x_i}(\tilde{u})d\tilde{\mu}^c_{<u_i,v>}, \tag{3.21}$$

where for any $w \in D(\mathcal{E})$, $\tilde{\mu}^c_{<w,v>}$ is the local part of the energy measure $\tilde{\mu}_{<w,v>}$ associated with the regular symmetric Dirichlet form $(\tilde{\mathcal{E}}, D(\mathcal{E}))$ (cf. Fukushima et al. [8, §3.2]).

Proof. Let $\{v_n\}_{n\in\mathbf{N}} \subset D(\mathcal{E})_b$ be an approximating sequence of v, i.e. v_n is $\mathcal{E}_1$-convergent to v as $n\to\infty$. Then by the remark right before Fukushima et al. [8, Theorem 3.2.3] and the classical LeJan's formula (cf. Fukushima et al. [8, Theorem 3.2.2 and p.117]), we find that for any $f \in C_0(U)\cap D(\mathcal{E})$,

$$\begin{aligned}
\int_U f(x)d\tilde{\mu}^c_{<\phi(u),v>} &= \lim_{n\to\infty}\int_U f(x)d\tilde{\mu}^c_{<\phi(u),v_n>}\\
&= \lim_{n\to\infty}\sum_{i=1}^m\int_U f(x)\phi_{x_i}(\tilde{u})d\tilde{\mu}^c_{<u_i,v_n>}\\
&= \sum_{i=1}^m\lim_{n\to\infty}\int_U f(x)\phi_{x_i}(\tilde{u})d\tilde{\mu}^c_{<u_i,v_n>}\\
&= \sum_{i=1}^m\int_U f(x)\phi_{x_i}(\tilde{u})d\tilde{\mu}^c_{<u_i,v>}
\end{aligned}$$

and therefore (3.21) follows. $\square$

By the regularity of $(\mathcal{E}, D(\mathcal{E}))$, it is easy to show that $1, x_i \in \mathcal{F}^c_{loc}, 1 \leq i \leq d$ ($\mathcal{F}^c_{loc}$ is defined in (3.14)). Take a sequence $\{G_k\}_{k\in\mathbf{N}}$ of relatively compact open sets such that $\bar{G}_n \subset G_{n+1}, \forall n \geq 1$, and $\cup_{n=1}^\infty G_n = U$. Let $\{w_k^{(x_i)}\}_{k\in\mathbf{N}}, 1\leq i\leq d, \{w_k^{(1)}\}_{k\in\mathbf{N}}$ be approximating sequences of $x_i, 1 \leq i \leq d$, and 1, respectively. There exists a $k_0 \in \mathbf{N}$ such that $\cup_{i=1}^d \mathrm{supp}[\frac{\partial u}{\partial x_i}] \cup \mathrm{supp}[v] \subset G_{k_0}$. Then for any $k \geq k_0$,

$$\begin{aligned}
< L(x_i,1), \frac{\partial u}{\partial x_i}v > &= < L(w_k^{(x_i)}, w_k^{(1)}), \frac{\partial u}{\partial x_i}v >\\
&= \check{\mathcal{E}}^c(w_k^{(x_i)}, w_k^{(1)}\frac{\partial u}{\partial x_i}v)\\
&= \check{\mathcal{E}}^c(w_k^{(x_i)}, \frac{\partial u}{\partial x_i}v).
\end{aligned} \tag{3.22}$$

By Lemma 3.8, for each $k \in \mathbf{N}$ and $1 \leq i \leq d$, there exists a function $F(w_k^{(x_i)}) \in D(\mathcal{E})$ such that

$$\begin{aligned}
\check{\mathcal{E}}^c(w_k^{(x_i)}, \frac{\partial u}{\partial x_i}v) &= \tilde{\mathcal{E}}_1(F(w_k^{(x_i)}), \frac{\partial u}{\partial x_i}v)\\
&\quad - \int_{U\times U\backslash d} 2(w_k^{(x_i)}(y) - w_k^{(x_i)}(x))\frac{\partial u}{\partial x_i}(y)v(y)\check{J}(dx,dy).
\end{aligned} \tag{3.23}$$

By the classical Beurling-Deny formula (cf. Fukushima et al. [8, Lemma 4.5.4]), we find that

$$\begin{aligned}
&\tilde{\mathcal{E}}_1(F(w_k^{(x_i)}),\ \frac{\partial u}{\partial x_i}v) \\
&\quad = \int_U \frac{\partial u}{\partial x_i} v F(w_k^{(x_i)}) dm + \tilde{\mathcal{E}}^c(\frac{\partial u}{\partial x_i}v, F(w_k^{(x_i)})) \\
&\qquad + \int_{U\times U\backslash d} \left(\widetilde{F(w_k^{(x_i)})}(y) - \widetilde{F(w_k^{(x_i)})}(x)\right)\left(\frac{\partial u}{\partial x_i}(y)v(y) - \frac{\partial u}{\partial x_i}(x)v(x)\right)\tilde{J}(dx,dy) \\
&\qquad + \int_U \widetilde{F(w_k^{(x_i)})}(x)\frac{\partial u}{\partial x_i}(x)v(x)\tilde{K}(dx). \qquad (3.24)
\end{aligned}$$

Since $\tilde{J} = 1/2(J(dx,dy) + J(dy,dx))$, $\tilde{J}(U \times K\backslash d) < \infty$ for any compact subset K of U by Assumption 3.1'. Then we get by the symmetry of $\tilde{J}$ that

$$\begin{aligned}
&\int_{U\times U\backslash d} \left(\widetilde{F(w_k^{(x_i)})}(y) - \widetilde{F(w_k^{(x_i)})}(x)\right)\left(\frac{\partial u}{\partial x_i}(y)v(y) - \frac{\partial u}{\partial x_i}(x)v(x)\right)\tilde{J}(dx,dy) \\
&\quad = \int_{U\times U\backslash d} 2\left(\widetilde{F(w_k^{(x_i)})}(y) - \widetilde{F(w_k^{(x_i)})}(x)\right)\frac{\partial u}{\partial x_i}(y)v(y)\tilde{J}(dx,dy). \qquad (3.25)
\end{aligned}$$

Extend $\tilde{J}$ to a measure on $U\times U$, which we still denote by $\tilde{J}$, by setting $\tilde{J}\{(x,y)\in U\times U | x = y\} = 0$. Set $\mu(dy) := \tilde{J}(U,dy)$. Note that $\tilde{J}(U\times K) < \infty$ for any compact subset K of U. Following the proof of Ethier and Kurtz [7, Appendixes, Theorem 8.1], there exists a kernel $\eta^{(k)}(y,dx)$ $(k\geq k_0)$ such that

$$\begin{aligned}
&\int_{U\times U} 2\left(\widetilde{F(w_k^{(x_i)})}(y) - \widetilde{F(w_k^{(x_i)})}(x)\right)\frac{\partial u}{\partial x_i}(y)v(y)\tilde{J}(dx,dy) \\
&\quad = 2\int_U \frac{\partial u}{\partial x_i}(y)v(y)\int_U \left(\widetilde{F(w_k^{(x_i)})}(y) - \widetilde{F(w_k^{(x_i)})}(x)\right)\eta^{(k)}(y,dx)\mu(dy). \qquad (3.26)
\end{aligned}$$

Extend $\check{J}$ to a measure on $U\times U$, which we still denote by $\check{J}$, by setting $\check{J}\{(x,y)\in U\times U | x = y\} = 0$. Set $\check{\mu}(dy) := \check{J}(U,dy)$. Similar to $\eta^{(k)}(y,dx)$, there exists a kernel $\check{\eta}^{(k)}(y,dx)$ $(k\geq k_0)$ such that

$$\begin{aligned}
&\int_{U\times U} 2(w_k^{(x_i)}(y) - w_k^{(x_i)}(x))\frac{\partial u}{\partial x_i}(y)v(y)\check{J}(dx,dy) \\
&\quad = 2\int_U \frac{\partial u}{\partial x_i}(y)v(y)\int_U (w_k^{(x_i)}(y) - w_k^{(x_i)}(x))\check{\eta}^{(k)}(y,dx)\check{\mu}(dy). \qquad (3.27)
\end{aligned}$$

By Lemma 3.9, we get

$$\tilde{\mathcal{E}}^c(\frac{\partial u}{\partial x_i}v, F(w_k^{(x_i)})) = \frac{1}{2}\sum_{j=1}^d \int_U \frac{\partial}{\partial x_j}\left(\frac{\partial u}{\partial x_i}v\right) d\tilde{\mu}_{<x_j, F(w_k^{(x_i)})>}\ . \qquad (3.28)$$

For each $k\in\mathbf{N}$, and $1\leq i,j\leq d$, define

$$\begin{aligned}
\mu_i^{(k)}(dx) \ :=\ & F(w_k^{(x_i)})(x)m(dx) + \widetilde{F(w_k^{(x_i)})}(x)\tilde{K}(dx) \\
& +2\int_U \left(\widetilde{F(w_k^{(x_i)})}(x) - \widetilde{F(w_k^{(x_i)})}(y)\right)\eta^{(k)}(x,dy)\mu(dx) \\
& -2\int_U (w_k^{(x_i)}(x) - w_k^{(x_i)}(y))\check{\eta}^{(k)}(x,dy)\check{\mu}(dx) \qquad (3.29)
\end{aligned}$$

and

$$\mu_{ij}^{(k)}(dx) \quad := \quad \frac{1}{2}\tilde{\mu}^c_{<x_j, F(w_k^{(x_i)})>}(dx). \tag{3.30}$$

Therefore, it follows from (3.23)-(3.30) that

$$\check{\mathcal{E}}^c(w_k^{(x_i)}, \frac{\partial u}{\partial x_i}v) = \int_U \frac{\partial u}{\partial x_i} v d\mu_i^{(k)} + \sum_{j=1}^d \int_U \frac{\partial}{\partial x_j}\left(\frac{\partial u}{\partial x_i}v\right) d\mu_{ij}^{(k)}. \tag{3.31}$$

We now give the main result of this subsection.

Theorem 3.10. There exist unique generalized functions $\{F_i\}_{1\le i\le d}$ on U such that for any $u, v \in C_0^\infty(U)$,

$$\check{\mathcal{E}}^c(u,v) = \sum_{i=1}^d < \frac{\partial u}{\partial x_i}v, F_i > . \tag{3.32}$$

Moreover, for any relatively compact open set $G \subset U$, there exist signed Radon measures $\{\mu_i^G\}_{1\le i\le d}$ and $\{\mu_{ij}^G\}_{1\le i,j\le d}$ on G such that

$$< v, F_i >= \int_G v d\mu_i^G + \sum_{j=1}^d \int_G \frac{\partial v}{\partial x_j} d\mu_{ij}^G \ \text{ for all } v \in C_0^\infty(G). \tag{3.33}$$

Proof. For $1 \le i \le d$, we define the generalized function F_i by

$$< v, F_i >:= \lim_{k\to\infty}\left[\int_U v\, d\mu_i^{(k)} + \sum_{j=1}^d \int_U \frac{\partial v}{\partial x_j}\, d\mu_{ij}^{(k)}\right], \quad v \in C_0^\infty(U), \tag{3.34}$$

where $\mu_i^{(k)}$, $\mu_{ij}^{(k)}$ are given by (3.29), (3.30), respectively. First, we show that F_i is well defined. Suppose that $\mathrm{supp}[v] \subset G_{k_0}$ for some $k_0 \in \mathbf{N}$. Let $w \in C_0^\infty(U)$ be a function satisfying $w|_{\mathrm{supp}[v]} = x_i$. Then for any $k \ge k_0$, we get by (3.31) and (3.22) that

$$\begin{aligned}\int_U v\, d\mu_i^{(k)} + \sum_{j=1}^d \int_U \frac{\partial v}{\partial x_j}\, d\mu_{ij}^{(k)} \quad &= \quad \int_U \frac{\partial w}{\partial x_i} v\, d\mu_i^{(k)} + \sum_{j=1}^d \int_U \frac{\partial}{\partial x_j}\left(\frac{\partial w}{\partial x_i}v\right) d\mu_{ij}^{(k)} \\ &= \quad < L(x_i, 1), v > . \end{aligned} \tag{3.35}$$

Thus, F_i is well defined and (3.32) follows from (3.18), (3.35) and (3.34). The uniqueness of F_i is obvious by noting that $< v, F_i >=< L(x_i,1), v >$ for any $v \in C_0^\infty(U)$. Furthermore, we fix a $k \in \mathbf{N}$ such that $\overline{G} \subset G_k$ and respectively define $\{\mu_i^G\}_{1\le i\le d}$, $\{\mu_{ij}^G\}_{1\le i,j\le d}$ to be $\{\mu_i^{(k)}\}_{1\le i\le d}$, $\{\mu_{ij}^{(k)}\}_{1\le i,j\le d}$. Then (3.33) holds. □

3.3 Measure Representation of $\check{\mathcal{E}}^c(u,v)$

Let $(\mathcal{E}, D(\mathcal{E}))$ and $L(\cdot,\cdot)$ be as in Subsection 3.1. For $v \in C_0(E) \cap D(\mathcal{E})$, we define

$$\begin{aligned}\mathcal{L}(v) \quad := \{u \in C_0(E)\cap D(\mathcal{E}) |\, & f \in C_0(E)\cap D(\mathcal{E}) \to < L(u,v), f > \\ & \text{is continuous w.r.t. the uniform norm } \|\cdot\|_\infty\}.\end{aligned}$$

Proposition 3.11. (i) If $u \in I(v)$, then $u \in \mathcal{L}(v)$, where $I(v)$ is defined in Theorem 1.6.
(ii) If $u \in D(A) \cap D(\hat{A}) \cap C_0(E)$, then $u \in \mathcal{L}(v)$, where $(A, D(A))$ and $(\hat{A}, D(\hat{A}))$ are the generator and co-generator of the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$, respectively.
(iii) $\mathcal{L}(v)$ is an algebra.

Proof. (i) In this case, $< L(u,v), f >= \frac{1}{2}(\mathcal{E}^c(u, vf) - \hat{\mathcal{E}}^c(u, vf)) = 0$ for any $f \in C_0(E) \cap D(\mathcal{E})$.
(ii) If $u \in D(A) \cap D(\hat{A}) \cap C_0(E)$,

$$\mathcal{E}(u, vf) - \hat{\mathcal{E}}(u, vf) = (-Au, vf) - (-\hat{A}u, vf) = \int_U (\hat{A}u(x) - Au(x))v(x)f(x)m(dx), \qquad (3.36)$$

and

$$\begin{aligned} & S.P.V. \int_{E\times E\backslash d} 2(u(y) - u(x))v(y)f(y)\frac{J - \hat{J}}{2}(dx, dy) \\ = \; & S.P.V. \int_{E\times E\backslash d} 2(u(y) - u(x))v(y)f(y)\frac{J_1 - \hat{J}_1}{2}(dx, dy), \end{aligned} \qquad (3.37)$$

where $\hat{J}_1(dx, dy) = J_1(dy, dx)$.

By (3.3), (3.36), (3.37) and Assumption 3.1, we find that $< L(u,v), f >$ is a continuous linear functional of f on $C_0(E) \cap D(\mathcal{E})$ w.r.t. the uniform norm $\|\cdot\|_\infty$.
(iii) Direct consequence of Theorem 3.2 (i) and (iv). □

For $v \in C_0(E) \cap D(\mathcal{E})$ and $u \in \mathcal{L}(v)$, $< L(u,v), \cdot >$ can be extended to become a continuous linear functional on $C_0(E)$ since $C_0(E) \cap D(\mathcal{E})$ is dense in $C_0(E)$. Then, by Riesz representation theorem, there exists a finite signed measure, denoted by $\breve{\mu}^c_{<u,v>}$, such that

$$< L(u,v), f >= \int_E f(x) d\breve{\mu}^c_{<u,v>} \quad \text{for any } f \in C_0(E) \cap D(\mathcal{E}).$$

Remark 3.12. (i) If $v \in C_0(E) \cap D(\mathcal{E})$ and $u \in I(v)$, then $\breve{\mu}^c_{<u,v>} = 0$.
(ii) If $v \in C_0(E) \cap D(\mathcal{E})$ and $u \in \mathcal{L}(v)$, we get by taking a function $f \in C_0(E) \cap D(\mathcal{E})$ with $f|_{\mathrm{supp}[v]} = 1$ that $\check{\mathcal{E}}^c(u, v) = \int_E d\breve{\mu}^c_{<u,v>}$.
(iii) Let $v \in C_0(E) \cap D(\mathcal{E})$ and $u \in \mathcal{L}(v)$. Then for any $w \in C_0(E) \cap D(\mathcal{E})$, one can easily get by Theorem 3.2 (ii) that $u \in \mathcal{L}(vw)$ and $d\breve{\mu}^c_{<u,vw>} = w\, d\breve{\mu}^c_{<u,v>}$.
(iv) For $w \in C_0(E) \cap D(\mathcal{E})$, if $u, v \in \mathcal{L}(v)$, then $uv \in \mathcal{L}(v)$ by Proposition 3.11 (iii) and we get by Theorem 3.2 (iv) that $d\breve{\mu}^c_{<uv,w>} = u d\breve{\mu}^c_{<v,w>} + v d\breve{\mu}^c_{<u,w>}$.
(v) Let $v \in C_0(E) \cap D(\mathcal{E})$ and $u \in \mathcal{L}(v)$. If u is constant on a relatively compact open set G, then $d\breve{\mu}^c_{<u,v>} = 0$ on G by Lemma 3.3 (i).
(vi) For $v_1, v_2 \in C_0(E) \cap D(\mathcal{E})$, if $u \in \mathcal{L}(v_1) \cap \mathcal{L}(v_2)$ and $v_1 = v_2$ m-a.e. on a relatively compact open set G, then $d\breve{\mu}^c_{<u,v_1>} = d\breve{\mu}^c_{<u,v_2>}$ on G by Lemma 3.3 (ii).

Proposition 3.13. Let $v \in C_0(E) \cap D(\mathcal{E})$. If $u_1, \ldots, u_m \in \mathcal{L}(v), \phi \in C^1(\mathbf{R}^m)$ with $\phi(0) = 0$, and $\phi(u) \in \mathcal{L}(v)$, then

$$d\breve{\mu}^c_{<\phi(u),v>} = \sum_{i=1}^m \phi_{x_i}(u) d\breve{\mu}^c_{<u_i,v>}. \qquad (3.38)$$

Proof. The proof is similar to that of Proposition 3.4. Define

$$\mathcal{B} := \{\phi \in C^1(\mathbf{R}^m) | \phi(0) = 0, \phi(u) \in \mathcal{L}(v) \text{ and (3.38) holds}\}.$$

Similar to Proposition 3.4, one can show that $\mathcal{B}$ contains all the polynomials vanishing at the origin.

Suppose $\phi \in C^1(\mathbf{R}^m)$ with $\phi(0) = 0$ and $\phi(u) \in \mathcal{L}(v)$. Let S be a finite cube containing the range of $u(x) := (u_1(x), \dots, u_m(x))$. Similar to Proposition 3.4, there exists a sequence $\{\phi^k(x)\}_{k\in\mathbf{N}}$ of polynomials vanishing at the origin such that $\phi^k \to \phi, \phi^k_{x_i} \to \phi_{x_i}, 1 \leq i, \leq m$, uniformly on S and $\phi^k(u)$ is $\mathcal{E}_1$-convergent to $\phi(u)$ as $k \to \infty$. And $\phi^k(u(x)), \phi^k_{x_i}(u(x)), 1 \leq i \leq m$, are uniformly bounded and respectively converge to $\phi(u(x))$ and $\phi_{x_i}(u(x)), x \in E$, as $k \to \infty$. Now for any $f \in C_0(E) \cap D(\mathcal{E})$, we get similarly to Proposition 3.4 that

$$\int_E f(x) d\check{\mu}^c_{<\phi(u),v>} = \sum_{i=1}^m \int_E f(x)\phi_{x_i}(u) d\check{\mu}^c_{<u_i,v>}.$$

Therefore, (3.38) holds and the proof is complete. □

Define

$$\begin{aligned}\tilde{\mathcal{F}}^c_{loc} := \{u | & \text{ for any relatively compact open set } G, \text{ there exists a function} \\ & w \in C_0(E) \cap D(A) \cap D(\hat{A}) \text{ such that } u = w \ m\text{-a.e. on } G\}.\end{aligned}$$

Take a sequence $\{G_n\}_{n\in\mathbf{N}}$ of relatively compact open sets such that $\bar{G}_n \subset G_{n+1}, \forall n \geq 1$, and $\cup_{n=1}^\infty G_n = E$. For $v \in \mathcal{F}^c_{loc}$ and $u \in \tilde{\mathcal{F}}^c_{loc}$, there exist two sequences $\{v_n\}_{n\in\mathbf{N}}, \{u_n\}_{n\in\mathbf{N}}$ with $v_n \in C_0(E) \cap D(\mathcal{E})$ and $u_n \in C_0(E) \cap D(A) \cap D(\hat{A})$ such that $v = v_n, u = u_n$ m-a.e. on G_n for any $n \geq 1$. (Note that by Proposition 3.11 (ii), $u_n \in \mathcal{L}(w)$ for any $w \in C_0(E) \cap D(\mathcal{E})$.) By Remark 3.12 (v) and (vi), we can define the measure $\check{\mu}^c_{<u,v>}$ by $\check{\mu}^c_{<u,v>} := \check{\mu}^c_{<u_n,v_n>}$ on G. Further, by Remark 3.12 (iii) and (iv), we get
(i) For any $v_1, v_2 \in \mathcal{F}^c_{loc}, u \in \tilde{\mathcal{F}}^c_{loc}$,

$$\check{\mu}^c_{<u,v_1 v_2>} = v_1 \check{\mu}^c_{<u,v_2>}. \tag{3.39}$$

(ii) For any $v \in \mathcal{F}^c_{loc}, u_1, u_2 \in \tilde{\mathcal{F}}^c_{loc}$, $d\check{\mu}^c_{<u_1 u_2,v>} = u_1 d\check{\mu}^c_{<u_2,v>} + u_2 d\check{\mu}^c_{<u_1,v>}$.

Proposition 3.14. Let $v \in \mathcal{F}^c_{loc}$. If $u_1, \dots, u_m \in \tilde{\mathcal{F}}^c_{loc}, \phi \in C^2(\mathbf{R}^m)$ with $\phi(u) := \phi(u_1, \dots, u_m) \in \tilde{\mathcal{F}}^c_{loc}$, then

$$d\check{\mu}^c_{<\phi(u),v>} = \sum_{i=1}^m \phi_{x_i}(u) d\check{\mu}^c_{<u_i,v>}. \tag{3.40}$$

Proof. Since $\tilde{\mathcal{F}}^c_{loc} \subset \mathcal{F}^c_{loc}$, by Proposition 3.7, for any $f \in C_0(E) \cap D(\mathcal{E})$,

$$\begin{aligned}\int_E f(x) d\check{\mu}^c_{<\phi(u),v>} &= < L(\phi(u), v), f > \\ &= \sum_{i=1}^m < L(u_i, v), \phi_{x_i}(u) f > \\ &= \sum_{i=1}^m \int_E f(x)\phi_{x_i}(u)(x) d\check{\mu}^c_{<u_i,v>}.\end{aligned}$$

Therefore, (3.40) holds and the proof is complete. $\square$

Theorem 3.15. Let U be a domain of $\mathbf{R}^d$ and $(\mathcal{E}, D(\mathcal{E}))$ a regular (non-symmetric) Dirichlet form on $L^2(U;m)$ with the generator $(A, D(A))$ and the co-generator $(\hat{A}, D(\hat{A}))$. Suppose that $J_1(U \times F\backslash d) < \infty, J_1(F \times U\backslash d) < \infty$ for any compact set F of U, and $C_0^\infty(U) \subset D(A) \cap D(\hat{A})$. Then for $u, v \in C_0^\infty(U)$, $\mathcal{E}(u,v)$ can be expressed as follows

$$\begin{aligned}\mathcal{E}(u,v) &= \sum_{i,j=1}^{d} \int_U \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} d\nu_{ij} + \sum_{i=1}^{d} \int_U \frac{\partial u}{\partial x_i} v d\nu_i \\ &\quad + S.P.V. \int_{U\times U\backslash d} 2(u(y)-u(x))v(y)J(dx,dy) + \int_U u(x)v(x)K(dx), \qquad (3.41)\end{aligned}$$

where $\{\nu_{ij}\}_{1\le i,j\le d}$ and $\{\nu_i\}_{1\le i\le d}$ are Radon measures on U, J and K are the jumping and killing measures of $(\mathcal{E}, D(\mathcal{E}))$, respectively.

Proof. By Theorem 2.2, for $u, v \in C_0(U) \cap D(\mathcal{E})$, we get

$$\mathcal{E}(u,v) = \mathcal{E}^c(u,v) + S.P.V. \int_{U\times U\backslash d} 2(u(y)-u(x))v(y)J(dx,dy) + \int_U u(x)v(x)K(dx), \quad (3.42)$$

where

$$\mathcal{E}^c(u,v) = \tilde{\mathcal{E}}^c(u,v) + \check{\mathcal{E}}^c(u,v), \quad \tilde{\mathcal{E}}^c(u,v) = \frac{1}{2}(\mathcal{E}^c(u,v) + \hat{\mathcal{E}}^c(u,v)) \qquad (3.43)$$

(cf. (2.20) and (3.1)). Moreover, by LeJan's formula for regular symmetric Dirichlet forms (cf. Fukushima et al. [8, Theorem 3.2.3]),

$$\tilde{\mathcal{E}}^c(u,v) = \frac{1}{2}\sum_{i,j=1}^{d} \int_U \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} d\tilde{\mu}^c_{<x_i,x_j>}, \qquad (3.44)$$

where $\tilde{\mu}^c_{<x_i,x_j>}$ is the local part of the energy measure of x_i, x_j, associated with the symmetric Dirichlet form $(\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}}))$.

Since $C_0^\infty(U) \subset D(A) \cap D(\hat{A})$, one can show that the constant function $1 \in \tilde{\mathcal{F}}^c_{loc}$. If $u \in C_0^\infty(U)$, then $u_i := x_i \cdot u \in C_0^\infty(U)$. Hence $x_i \in \tilde{\mathcal{F}}^c_{loc}, 1 \le i \le d$. By (3.39) and Proposition 3.14, we get

$$\check{\mathcal{E}}^c(u,v) = \sum_{i=1}^{d} \int_U \frac{\partial u}{\partial x_i} v d\check{\mu}^c_{<x_i,1>}. \qquad (3.45)$$

Set $\nu_i := \check{\mu}^c_{<x_i,1>}$, $\nu_{ij} := \frac{1}{2}\tilde{\mu}^c_{<x_i,x_j>}$, $1 \le i,j \le d$. Then (3.41) follows from (3.42)-(3.45). The proof is complete. $\square$

4 Example and Counterexample

Example 4.1. Lévy Process

Let $X = (X_t)_{t\ge 0}$ be a Lévy process on $\mathbf{R}^d$ with the characteristic exponent η, i.e. $E\{\exp(i\langle\lambda, X_t\rangle)\} = \exp(-t\eta(\lambda))$ for $\lambda \in \mathbf{R}^d$ and $t \ge 0$. Hereafter, $\mathbf{R}^d$ is equipped with the standard product $\langle\cdot,\cdot\rangle$

and Euclidean norm $|\cdot|$. The celebrated Lévy-Khintchine formula (cf. e.g. Bertoin [1, p.3] or Sato [17, p.37]) tells us that

$$\eta(\lambda) = i\langle b, \lambda\rangle + \frac{1}{2}Q(\lambda) + \int_{\mathbf{R}^d} \left(1 - e^{i\langle \lambda, x\rangle} + i\langle \lambda, x\rangle 1_{\{|x|\le 1\}}\right) \mu(dx),$$

where $b = (b_1, \dots, b_d) \in \mathbf{R}^d$, Q is a symmetric, nonnegative definite quadratic form on $\mathbf{R}^d$, and μ is a Lévy measure satisfying $\mu(\{0\}) = 0$ and

$$\int_{\mathbf{R}^d} \frac{|x|^2}{1+|x|^2}\mu(dx) < \infty. \tag{4.1}$$

Or equivalently, the infinitesimal generator A of $(X_t)_{t\ge 0}$ is characterized by (cf. Sato [17, Theorem 31.5])

$$\begin{aligned} Au(y) &= \sum_{i=1}^d (-b_i)\partial_i u(y) + \frac{1}{2}\sum_{i,j=1}^d Q_{ij}\partial_i\partial_j u(y) \\ &\quad + \int_{\mathbf{R}^d} \left(u(y+x) - u(y) - \sum_{i=1}^d x_i \partial_i u(y) 1_{\{|x|\le 1\}}(x) \right) \mu(dx) \end{aligned} \tag{4.2}$$

for $u \in C_0^\infty(\mathbf{R}^d)$. Hereafter, $C_0^\infty(\mathbf{R}^d)$ denotes the set of all infinitely differentiable functions on $\mathbf{R}^d$ with compact supports. If in addition μ satisfies $\int_{|x|\le 1}|x|\mu(dx) < \infty$, then (4.2) can be written as

$$Au(y) = \sum_{i=1}^d (-\bar{b}_i)\partial_i u(y) + \frac{1}{2}\sum_{i,j=1}^d Q_{ij}\partial_i\partial_j u(y) + \int_{\mathbf{R}^d} (u(y+x) - u(y))\,\mu(dx) \tag{4.3}$$

with $\bar{b}_i = b_i + \int_{|x|\le 1} x_i \mu(dx)$, $1 \le i \le d$.

Let $(A, D(A))$ be the $L^2(\mathbf{R}^d; dx)$-generator of $(X_t)_{t\ge 0}$. It is known that (cf. Jacob [10, Example 4.7.32])

$$D(A) = \mathcal{H}_\eta(\mathbf{R}^d) = \left\{ f \in L^2(\mathbf{R}^d; dx) \,\middle|\, \int_{\mathbf{R}^d} |\eta(\lambda)|^2 |\hat{f}(\lambda)|^2 d\lambda < \infty \right\}, \tag{4.4}$$

where $\hat{f}(\lambda)$ stands for the Fourier transform of f, i.e. $\hat{f}(\lambda) = (2\pi)^{-d/2}\int_{\mathbf{R}^d} \exp\{-i\langle \lambda, x\rangle\} f(x)dx$. Since $|\eta(\lambda)| \le C(1+|\lambda|^2), \forall \lambda \in \mathbf{R}^d$, for some constant $C > 0$. One can check that $S(\mathbf{R}^d)$, the space of rapidly decreasing functions, is contained in D_A, and (4.2) holds for all $u \in C_0^\infty(\mathbf{R}^d)$. By Jacob [10, Example 4.7.32], $(X_t)_{t\ge 0}$ is associated with a Dirichlet form on $L^2(\mathbf{R}^d; dx)$ if and only if there exists a constant $C > 0$ such that

$$|Im(\eta(\lambda))| \le C(1 + Re(\eta(\lambda))),\ \forall \lambda \in \mathbf{R}^d, \tag{4.5}$$

where $Im(\eta(\lambda))$ and $Re(\eta(\lambda))$ stand for the imaginary and real parts of $\eta(\lambda)$, respectively.

In what follows, we assume that the characteristic exponent η satisfies condition (4.5) and $(\mathcal{E}, D(\mathcal{E}))$ is the Dirichlet form associated with $(X_t)_{t\ge 0}$. More precisely, $(\mathcal{E}, D(\mathcal{E}))$ is the unique coercive closed form on $L^2(\mathbf{R}^d; dx)$ such that (cf. e.g. Ma and Röckner [14]):

$$\mathcal{E}(u, v) = (-Au, v) \quad \text{for all } u \in D(A), v \in D(\mathcal{E}). \tag{4.6}$$

By [J 01, Example 4.7.32],

$$
\begin{aligned}
D(\mathcal{E}) &= \{f \in L^2(R^d;dx)| \int_{\mathbf{R}^d} Re(\eta(\lambda))\,|\hat{f}(\lambda)|^2 d\lambda < \infty\},\\
\mathcal{E}(f,g) &= \int_{\mathbf{R}^d} \eta(\lambda)\hat{f}(\lambda)\overline{\hat{g}(\lambda)}d\lambda,\ \forall f,g \in D(\mathcal{E}).
\end{aligned} \tag{4.7}
$$

Moreover, by (4.7), one can check that $(\mathcal{E}, D(\mathcal{E}))$ is a regular Dirichlet form with the special standard core $C_0^\infty(\mathbf{R}^d)$.

Proposition 4.2. Set $J(dx,dy) = \frac{1}{2}\mu(dx-y)dy,\ K(dx) = 0$, and $\rho(x,y) = |x-y|$. Then J and K are the unique measures specified by Theorem 1.6, and ρ satisfies conditions $(\rho.1)$ and $(\rho.2)$ of Theorem 2.1 (i) with $D_\rho = C_0^\infty(\mathbf{R}^d)$. Therefore, all the decompositions given in Theorems 1.6, 2.1 and 2.3 (i) hold for $(\mathcal{E}, D(\mathcal{E}))$ with these specified J, K, and ρ.

Proof. Set $J(dx,dy) = \frac{1}{2}\mu(dx-y)dy$ and $K(dx) = 0$. By (4.2) and (4.6), one can check that (1.6) holds for $u,v \in C_0^\infty(\mathbf{R}^d)$ with $u \in I(v)$. Since $C_0^\infty(\mathbf{R}^d)$ is a special standard core of $\mathcal{E}$, the conclusions of Theorem 1.6 follow. Set $\rho(x,y) = |x-y|$. Then $(\rho.1)$ is satisfied by virtue of (4.1). $(\rho.2)$ is trivially satisfied with $D_\rho = C_0^\infty(\mathbf{R}^d)$. Hence the conclusions of Theorems 2.1 and 2.3 (i) follow. □

Remark 4.3. By (4.2) and (4.6), one can check that for $u,v \in C_0^\infty(\mathbf{R}^d)$,

$$
\begin{aligned}
\mathcal{E}^{a,\rho}(u,v) &= \int_{\mathbf{R}^d}\left[\sum_{i=1}^d b_i\partial_i u(y) - \frac{1}{2}\sum_{i,j=1}^d Q_{ij}\partial_i\partial_j u(y)\right]v(y)dy + \int_{\mathbf{R}^d\times\mathbf{R}^d\backslash d}\left[\frac{2(u(y)-u(x))v(y)}{1+a|x-y|^2}\right.\\
&\quad \left. +2\left(\sum_{i=1}^d (x-y)_i\partial_i u(y)1_{\{|x-y|\le 1\}}(x)\right)v(y)\right] J(dx,dy).
\end{aligned}
$$

Proposition 4.4. Set $J(dx,dy) = \frac{1}{2}\mu(dx-y)dy$ and $K(dx) = 0$.
(i) If $\mu(\{|x| \le 1\}) < \infty$, then decomposition (1.7) holds for all $u,v \in C_0(\mathbf{R}^d) \cap D(\mathcal{E})$.
(ii) If $\int_{|x|\le 1}|x|\mu(dx) < \infty$, then decomposition (1.7) holds for all $u,v \in C_0^\infty(\mathbf{R}^d)$.
(iii) In the situation of either (i) or (ii), for $u,v \in C_0^\infty(\mathbf{R}^d)$,

$$
\mathcal{E}^c(u,v) = \int_{\mathbf{R}^d}\left[\sum_{i=1}^d \bar{b}_i\partial_i u(y) - \frac{1}{2}\sum_{i,j=1}^d Q_{ij}\partial_i\partial_j u(y)\right]v(y)dy,
$$

where $\bar{b}_i = b_i + \int_{|x|\le 1} x_i\mu(dx)$, $1 \le i \le d$.

Proof. Taking account of (4.1), one can easily check that the conditions of (i) and (ii) in this proposition imply conditions (i) and (ii) of Theorem 2.2, respectively. Therefore, Assertions (i) and (ii) follow from Theorem 2.2. Assertion (iii) follows from (4.3), (4.6) and the uniqueness of decomposition (1.7). □

Example 4.5. Counterexample

Let $(X_t)_{t\ge 0}$ be a Lévy process on $\mathbf{R}^1$ with the characteristic exponent

$$
\eta(\lambda) = C|\lambda|^\alpha(1 - i\,\mathrm{sgn}(\lambda)\tan(\frac{\alpha\pi}{2})), \lambda \in (-\infty,\infty),
$$

where $C > 0, 1 < \alpha < 2$. Then $(X_t)_{t\geq 0}$ is an α-stable process with Lévy measure

$$\mu(dx) = \frac{c}{|x|^{\alpha+1}} 1_{\{x>0\}} dx,$$

where $c > 0$ is a constant.

Since

$$|Im(\eta(\lambda))| = |\tan(\frac{\alpha\pi}{2})| \cdot C|\lambda|^\alpha \leq |\tan(\frac{\alpha\pi}{2})|(1 + Re(\eta(\lambda))),$$

η satisfies (4.5). Then $(X_t)_{t\geq 0}$ is associated with a regular Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(\mathbf{R}^1, dx)$ and

$$D(\mathcal{E}) \;\; = \;\; \left\{ f \in L^2(\mathbf{R}^1; dx) \,\middle|\, \int_{\mathbf{R}^1} |\lambda|^\alpha \, |\hat{f}(\lambda)|^2 d\lambda < \infty \right\}.$$

It follows from Proposition 4.2 that the jumping measures J and $\hat{J}$ have the forms

$$J(dx, dy) = \frac{1}{2}\mu(dx - y)dy, \;\; \hat{J}(dx, dy) = \frac{1}{2}\mu(dy - x)dx.$$

For $B \subset \mathbf{R}^1 \times \mathbf{R}^1 \backslash d$,

$$J(B) \;\; = \;\; \int_{\mathbf{R}^1}\int_{\mathbf{R}^1} I_B(x, y)\frac{1}{2}\mu(dx - y)dy = \int_{\mathbf{R}^1}\int_{\mathbf{R}^1} I_B(x + y, y)\frac{1}{2}\mu(dx)dy.$$

Define $u(x) = (|x| - 1)1_{\{|x|<1\}}$ and $v(x) = (|x| - 2)1_{\{|x|<2\}}$. We will show that $u, v \in C_0(\mathbf{R}^1) \cap D(\mathcal{E})$. Obviously, $u, v \in C_0(\mathbf{R}^1)$. Since

$$\hat{u}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}^1} e^{-i\xi x} u(x) dx = -\frac{2}{\sqrt{2\pi}} \times \frac{1 - \cos(\xi)}{\xi^2}, \text{ if } \xi \neq 0$$

and

$$\left| -\frac{2}{\sqrt{2\pi}} \times \frac{1 - \cos(\xi)}{\xi^2} \right| \leq \frac{4}{\sqrt{2\pi}} \times \frac{1}{\xi^2}, \;\; \left| -\frac{2}{\sqrt{2\pi}} \times \frac{1 - \cos(\xi)}{\xi^2} \right| \leq \frac{1}{\sqrt{2\pi}},$$

we get

$$|\hat{u}(\xi)|^2 \leq \left(\frac{4}{\sqrt{2\pi}}\right)^2 \cdot \left(1 \wedge \frac{1}{\xi^4}\right).$$

Since $\hat{u}(\xi)$ is continuous, it follows that $\int_{\mathbf{R}^1} |\xi|^\alpha |\hat{u}(\xi)|^2 d\xi < \infty$, noting that $1 < \alpha < 2$. Thus, $u \in D(\mathcal{E})$. Similarly, we can show that $v \in D(\mathcal{E})$. In fact, one can further check that $u, v \in D(A)$ if $1 < \alpha < 3/2$ (cf. (4.4)), where $(A, D(A))$ is the $L^2(\mathbf{R}^1; dx)$-generator of $(X_t)_{t\geq 0}$.

Through direct computation we get

$$\int_{|x-y|>\delta} (u(y)v(x) - u(x)v(y))J(dx, dy) = f(\alpha, \delta) + c \cdot \frac{4(\alpha - 1)^2 + 3(2 - 3\alpha)\delta}{3\alpha(\alpha - 1)^2\delta^\alpha},$$

where $\lim_{\delta\downarrow 0} f(\alpha, \delta)$ exists and the limit is finite. Since $1 < \alpha < 2$,

$$\lim_{\delta\downarrow 0} \frac{4(\alpha - 1)^2 + 3(2 - 3\alpha)\delta}{3\alpha(\alpha - 1)^2\delta^\alpha} = \infty$$

and therefore

$$\lim_{\delta \downarrow 0} \int_{|x-y|>\delta} (u(y)v(x) - u(x)v(y))J(dx, dy) = \infty.$$

Remark 4.6. (i) Example 4.5 shows that condition (2.13) in Theorem 2.3 (ii) can not be dropped. (ii) Example 4.5 also implies that there exist $u, v \in C_0(\mathbf{R}^1) \cap D(\mathcal{E})$ such that the limit

$$\lim_{\delta \downarrow 0} \int_{|x-y|>\delta} 2(u(y) - u(x))v(y)J(dx, dy)$$

doesn't exist, since $u(y)v(x) - u(x)v(y) = (u(y) - u(x))v(y) - (v(y) - v(x))u(y)$. Therefore, the constraint that $u \in \mathcal{A}(v)$ in (1.7) cannot be dropped. Similarly, the constraint that $u \in \hat{\mathcal{A}}(v)$ in (2.1) cannot be dropped either.

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