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ASYMPTOTIC RUIN PROBABILITIES OF THE RENEWAL MODEL WITH CONSTANT INTEREST FORCE AND REGULAR VARIATION

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Abstract

Klüppelberg and Stadtmüller (1998, Scand. Actuar. J., no. 1, 49–58) obtained a simple asymptotic formula for the ruin probability of the classical model with constant interest force and regularly varying tailed claims. This short note extends their result to the renewal model. The proof is based on a result of Resnick and Willekens (1991, Comm. Statist. Stochastic Models 7, no. 4, 511–525).

Keywords: Asymptotics, regular variation, ruin probability, stochastic premiums.

1 The model

We investigate the ruin probability of the renewal model. In this model the claims, X_n, $n \geq 1$, form a sequence of independent, identically distributed (i.i.d.), and nonnegative random variables with common distribution function F, and the interarrival times, Y_n, $n \geq 1$, form another sequence of i.i.d. nonnegative random variables, which are independent of the random variables X_n, $n \geq 1$, and are not degenerate at 0. The locations of the successive claims, $\tau_n = \sum_{k=1}^{n} Y_k$, constitute a renewal counting process

$$N(t) = \#\{n \geq 1 : \tau_n \in [0, t]\}, \qquad t \geq 0,$$

where, by convention, the cardinal number of the empty set is 0. Therefore, the total amount of claims accumulated up to time $t \geq 0$ is represented as a compound sum

$$S(t) = \sum_{n=1}^{N(t)} X_n, \qquad t \geq 0,$$

with $S(t) = 0$ when $N(t) = 0$. Let $C(t)$, $t \geq 0$, be a nonnegative and nondecreasing stochastic process, denoting the total amount of premiums accumulated up to time $t \geq 0$, let $\delta > 0$ be the constant interest force (that is, after time t a capital x becomes $x\mathrm{e}^{\delta t}$), and let $x \geq 0$ be the initial surplus of the insurance company. Then the total surplus up to time t, denoted by $U(t)$, satisfies the equation

$$U(t) = x\mathrm{e}^{\delta t} + \int_{[0,t]} \mathrm{e}^{\delta(t-y)} C(\mathrm{d}y) - \int_{[0,t]} \mathrm{e}^{\delta(t-y)} S(\mathrm{d}y), \qquad t \geq 0. \tag{1}$$

Assume that the total discounted amount of premiums is finite, that is,

$$\widetilde{C} = \int_{[0,\infty)} \mathrm{e}^{-\delta y} C(\mathrm{d}y) < \infty \qquad \text{almost surely.} \tag{2}$$

The ruin probability is defined by

$$\psi(x) = \Pr\left(U(t) < 0 \text{ for some } t \geq 0\right).$$

If $C(t) = Ct$, $t \geq 0$, with $C > 0$ a deterministic constant and $N(t)$, $t \geq 0$, is a Poisson process with intensity $\lambda > 0$, then the model above is reduced to the classical one.

The asymptotic behavior of the ruin probability of the classical model has been extensively investigated in the literature. In particular, Klüppelberg and Stadtmüller (1998) considered the asymptotic behavior of the ruin probability for the case of regularly varying tailed claims. We say that $\overline{F} = 1 - F$ is regularly varying with index $-\alpha < 0$, denoted by $F \in \mathcal{R}_{-\alpha}$, if there is some slowly varying function $L(\cdot)$ such that

$$\overline{F}(x) = x^{-\alpha} L(x), \qquad x > 0.$$

For this case, Klüppelberg and Stadtmüller (1998, Corollary 2.4) proved that

$$\psi(x) \sim \frac{\lambda}{\alpha\delta} \overline{F}(x). \tag{3}$$

[Hereafter, all limit relationships are for $x \to \infty$ unless stated otherwise; for two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) \sim b(x)$ if $\lim a(x)/b(x) = 1$ and write $a(x) \gtrsim b(x)$ if $\liminf a(x)/b(x) \geq 1$.] In doing so, they applied a quite sophisticated L_p transform technique. However, their approach does not work now since in the current general case we can not obtain Sundt and Teugels' (1995) integral equation (2), which is the starting point of Klüppelberg and Stadtmüller (1998).

Furthermore, Asmussen (1998, Corollary 4.1(ii)), Kalashnikov and Konstantinides (2000), and Konstantinides et al. (2002) also obtained results similar to (3) for some larger classes of heavy-tailed distributions. We also refer the interested reader to Tang (2004) for some parallel discussions in a discrete time model.

2 An important preliminary

We denote a randomly weighted series by

$$W = \sum_{n=1}^{\infty} \theta_n X_n, \tag{4}$$

where $\{X_n, n \geq 1\}$ is a sequence of i.i.d. nonnegative random variables with common distribution function F, and $\{\theta_n, n \geq 1\}$ is another sequence of nonnegative random variables, independent of $\{X_n, n \geq 1\}$. The following result is the one-dimensional version of Theorem 2.1 of Resnick and Willekens (1991):

Lemma 1. *Consider the randomly weighted series (4) above with $F \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$. We have*

$$\Pr(W > x) \sim \overline{F}(x) \sum_{n=1}^{\infty} \mathrm{E}\theta_n^{\alpha}$$

if one of the following assumptions holds:

1. $0 < \alpha < 1$ *and*

$$\sum_{n=1}^{\infty} \mathrm{E}\left(\theta_n^{\alpha+\varepsilon} + \theta_n^{\alpha-\varepsilon}\right) < \infty \qquad \textit{for some } \varepsilon > 0;$$

2. $1 \leq \alpha < \infty$ *and*

$$\sum_{n=1}^{\infty} \mathrm{E}\left(\theta_n^{\alpha+\varepsilon} + \theta_n^{\alpha-\varepsilon}\right)^{\frac{1}{\alpha+\varepsilon}} < \infty \qquad \textit{for some } \varepsilon > 0.$$

The merit of this lemma is that no information about the dependence structure of the sequence $\{\theta_n, n \geq 1\}$ is requested.

3 The main result

Theorem 1. *Consider the renewal model introduced in Section 1 with $F \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$. We have*

$$\psi(x) \sim \frac{\mathrm{E}e^{-\delta\alpha Y_1}}{1 - \mathrm{E}e^{-\delta\alpha Y_1}} \overline{F}(x) \tag{5}$$

if one of the following assumptions holds:

1. *the premium process $\{C(t), t \geq 0\}$ is independent of $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$;*
2. *the total discounted amount of premiums, defined by (2), satisfies that*

$$\Pr\left(\widetilde{C} > x\right) = o\left(\overline{F}(x)\right).$$

Remark. Comparing (5) with (3), we have successfully extended the result of Klüppelberg and Stadtmüller (1998) to the renewal model. Assumption 1 above has been used by Glukhova and Kapustin (2001) and Boikov (2002), while assumption 2, which does not require the independence between the premium process and the claim process, allows for a more realistic case that the premium rate varies as a deterministic or stochastic function of the current surplus, as that considered by Petersen (1989), Michaud (1996), and Jasiulewicz (2001).

Proof of Theorem 1. We define the discounted values of the surplus process (1) as

$$\widetilde{U}(t) = \mathrm{e}^{-\delta t}U(t) = x + \int_{[0,t]} \mathrm{e}^{-\delta y} C(\mathrm{d}y) - \sum_{n=1}^{\infty} X_n \mathrm{e}^{-\delta\tau_n} 1_{(\tau_n \le t)}, \qquad t \ge 0,$$

where 1_A denotes the indicator function of an event A. It is clear that

$$\psi(x) = \Pr\left(\widetilde{U}(t) < 0 \text{ for some } t \ge 0\right)$$

and that

$$x - \sum_{n=1}^{\infty} X_n \mathrm{e}^{-\delta\tau_n} \le \widetilde{U}(t) \le x + \widetilde{C} - \sum_{n=1}^{\infty} X_n \mathrm{e}^{-\delta\tau_n} 1_{(\tau_n \le t)}, \qquad t \ge 0. \tag{6}$$

Using the first inequality of (6) and Lemma 1, we have

$$\psi(x) \le \Pr\left(\sum_{n=1}^{\infty} X_n \mathrm{e}^{-\delta\tau_n} > x\right) \sim \overline{F}(x) \sum_{n=1}^{\infty} \mathrm{E}\mathrm{e}^{-\delta\alpha\tau_n} = \frac{\mathrm{E}\mathrm{e}^{-\delta\alpha Y_1}}{1 - \mathrm{E}\mathrm{e}^{-\delta\alpha Y_1}} \overline{F}(x).$$

Thus, in order to complete the proof of Theorem 1 it suffices to prove that

$$\psi(x) \gtrsim \frac{\mathrm{E}\mathrm{e}^{-\delta\alpha Y_1}}{1 - \mathrm{E}\mathrm{e}^{-\delta\alpha Y_1}} \overline{F}(x). \tag{7}$$

For this purpose, by the second inequality of (6) we derive

$$\begin{aligned} \psi(x) &\ge \Pr\left(\sum_{n=1}^{\infty} X_n \mathrm{e}^{-\delta\tau_n} 1_{(\tau_n \le t)} > x + \widetilde{C} \text{ for some } t \ge 0\right) \\ &= \Pr\left(\sum_{n=1}^{\infty} X_n \mathrm{e}^{-\delta\tau_n} > x + \widetilde{C}\right). \end{aligned} \tag{8}$$

Under assumption 1 of Theorem 1, by conditioning on $\widetilde{C}$ and applying Fatou's lemma and

Lemma 1 in turn, we obtain

$$\begin{aligned}
\liminf_{x\to\infty}\frac{\psi(x)}{\overline{F}(x)} &\geq \liminf_{x\to\infty}\frac{1}{\overline{F}(x)}\int_{[0,\infty)}\Pr\left(\sum_{n=1}^{\infty}X_n\mathrm{e}^{-\delta\tau_n}>x+y\right)\Pr\left(\widetilde{C}\in\mathrm{d}y\right)\\
&\geq \int_{[0,\infty)}\liminf_{x\to\infty}\frac{\Pr\left(\sum_{n=1}^{\infty}X_n\mathrm{e}^{-\delta\tau_n}>x+y\right)}{\overline{F}(x+y)}\frac{\overline{F}(x+y)}{\overline{F}(x)}\Pr\left(\widetilde{C}\in\mathrm{d}y\right)\\
&= \frac{\mathrm{E}\mathrm{e}^{-\delta\alpha Y_1}}{1-\mathrm{E}\mathrm{e}^{-\delta\alpha Y_1}}.
\end{aligned}$$

Hence, relation (7) holds. Under assumption 2 of Theorem 1, from (8) we have that for an arbitrarily fixed number $l>0$,

$$\psi(x)\geq\Pr\left(\sum_{n=1}^{\infty}X_n\mathrm{e}^{-\delta\tau_n}>(1+l)x\right)-\Pr\left(\widetilde{C}>lx\right).$$

Then, applying Lemma 1 again,

$$\begin{aligned}
\liminf_{x\to\infty}\frac{\psi(x)}{\overline{F}(x)} &\geq \liminf_{x\to\infty}\frac{1}{\overline{F}(x)}\Pr\left(\sum_{n=1}^{\infty}X_n\mathrm{e}^{-\delta\tau_n}>(1+l)x\right)-\limsup_{x\to\infty}\frac{\Pr\left(\widetilde{C}>lx\right)}{\overline{F}(x)}\\
&= \frac{\mathrm{E}\mathrm{e}^{-\delta\alpha Y_1}}{1-\mathrm{E}\mathrm{e}^{-\delta\alpha Y_1}}\liminf_{x\to\infty}\frac{\overline{F}((1+l)x)}{\overline{F}(x)}-\limsup_{x\to\infty}\frac{\Pr\left(\widetilde{C}>lx\right)}{\overline{F}(lx)}\frac{\overline{F}(lx)}{\overline{F}(x)}\\
&= \frac{\mathrm{E}\mathrm{e}^{-\delta\alpha Y_1}}{1-\mathrm{E}\mathrm{e}^{-\delta\alpha Y_1}}(1+l)^{-\alpha}.
\end{aligned}$$

Hence, relation (7) also holds since the number l above can be arbitrarily close to 0. □

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