Generation of Polynomials for Application in the Design of Stable 2-Dimensional Filters

Shahrokh H. Fallah

A Thesis

in

The Department

of

Electrical and Computer Engineering

Presented in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy at
Concordia University
Montréal, Québec, Canada

April, 1988

ISBN 0-315-44853-9

ABSTRACT

Shahrokh H. Fallah, Ph.D.

Concordia University, 1988

It is well-known that one of the main difficulties in the design of 2-D stable recursive digital filters is to ensure it's stability. Instability could be caused by the occurrence of the non-essential singularities of the second kind, which are not present in the case of 1-D. The other difficulty is that a multivariable polynomial cannot be factorized in general. Hence, the derivation of a guaranteed stable 2-D recursive transfer function is a major problem. This thesis discusses the design of 2-D stable recursive digital filters derived from the property of the slope of an n-variable reactance function.

In order to avoid the non-essential singularities of the second kind, the denominator of a stable 2-D transfer function shall be a Very Strictly Hurwitz Polynomial (VSHP). In this thesis, this is generated by first considering the slope of a multivariable reactance function, which is known to be positive on the imaginary axes. The starting point is an n-port gyrator terminated in n-variable capacitive or inductive reactances. Since this is a physically realizable lossless network, its input immittance is a reactance function. Various properties of the slopes of such a reactance function on the imaginary axes are determined. These are utilized in the generation of VSHP.

The VSHP so generated is used as the denominator of a 2-D discrete transfer function. This contains the elements of the n-port gyrator matrix as variables. The numerator polynomial is properly assigned. The resulting transfer function is utilized in the design of the two types of recursive filters, namely: (i) denominator separable, and (ii) denominator nonseparable which will meet certain sym-

metry constraints. The design is carried out using nonlinear optimization techniques in order to minimize the error between the prescribed and the actual magnitudes and/or group delay responses. After obtaining filters with high precision coefficients, 2-D filters using integer coefficients are obtained. A number of examples are given.

Finally, the sensitivity performance of these filters have been carried out and it is verified that they are low.

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ACKNOWLEDGMENTS

I would like to express my deep sense of gratitude to my thesis supervisor Professor V. Ramachandran for his excellent guidance during the course of this research.

I also would like to record my deep appreciation of the excellent help received from Professor M. Ahmadi during the course of this investigation.

Special thanks are due to my mother, Nasrin, Djavid, Fahime, Farrokh, and all my friends for keeping my spirits alive which made this thesis possible.

TO

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LIST OF ABBREVIATIONS AND SYMBOLS

A	Skew symmetric matrix of order 2n.
A	Determinant of the matrix **A**
$\sqrt{A}$	Pfaffian.
$A_{i_1 i_2 \cdots i_k}$	Determinant of the submatrix of **A** obtained by deleting $i_1 i_2 \cdots i_k$ rows and columns.
$\mathbf{A}_{2n}$, $\mathbf{A}_{2n+1}$	Skew-symmetric matrices of order $2n$ or $2n+1$.
a_{ij}	The element of the i^{th} row and j^{th} column of the determinant A.
a_{i_N}	N is an integer chosen from the ascending inequality $b \leq m_{l+1} < m_{l+2} \cdots < m_{k-b+1} \leq k$, and then the subscript of a in a_{i_N} is obtained from $1 \leq i_b < i_{b+1} \cdots < i_k \leq 2n$.
a_{2t}	$a_{2t} = \gamma(g)\sqrt{g}$
a'_{2t}	$a'_{2t} = \gamma(g')\sqrt{g'}$
A'	$A' = \left\{a_{12}, a_{13}, \ldots, a_{16}, \ldots, a_{23}, \ldots, a_{26}, \ldots, a_{56}\right\}$
B	$B \equiv \lvert B_1 \rvert$
B_1	$B_1 = a_{24}\sqrt{A_{13}} + a_{23}\sqrt{A_{14}} + a_{14}\sqrt{A_{23}} + a_{13}\sqrt{A_{24}}$
B_{2t+1}	$B_{2t+1} = a_{ii_{j_{p+1}}} \sum_{h=1}^{2t-p+1} \alpha_{B_{2t+1}} a_{ji_{m_h}} \times \sqrt{A_{iji_{j_1} \ldots i_{j_{p+1}} i_{m_1} \ldots i_{m_{2t-p+1}}} A_{iji_{j_1} \ldots i_{j_{p+1}} i_{m_{2t-p+2}} \ldots i_{m_{2t-2p-1}} i_{m_h}}}$
b_{2t+1}	$b_{2t+1} = \gamma(h)\sqrt{h}$ $b'_{2t+1} = \gamma(h')\sqrt{h'}$
C	$C \equiv \lvert C_1 \rvert$
C_1	$C_1 = \left(a_{13}^2 + a_{23}^2\right) a_{45} a_{46}$

C_{2l}	$M_{2n} \doteq C_0+C_2+\ldots+C_{2l}+\ldots+C_{2n}$
c_{2t}	$c_{2t} = \gamma(p)\sqrt{p}$
C_{2t}	$C_{2t} = a_{ii_{j_p+1}} \sum_{s_1} \alpha_{C_{2t}} a_{js_1} \times \sqrt{A_{iji_{j_1} \ldots i_{j_p+1} i_{m_1} \ldots i_{m_{2t-p}} s_1} A_{iji_{j_1} \ldots i_{j_p+1} i_{m_{2t-p+1}} \ldots i_{m_{2t-2p-1}}}}$
c'_{2t}	$c'_{2t} = \gamma(p')\sqrt{g'}$
D	$D \equiv \lvert D_1 \rvert$
D_1	$D_1 = \left(a_{35}a_{36}+a_{45}a_{46}\right) a_{12}^2$
D_{2l-2}	$D_{2l-2} \doteq \sum_{\substack{1\le i_1<i_2<\cdots<i_{2l-2}\le 2n \\ i_k \neq i,j}} \mu_{i_1}\mu_{i_2} \cdots \mu_{i_{2l-2}} A_{iji_1i_2 \cdots i_{2l-2}}$
d_{2t+1}	$d_{2t+1} = \gamma(q)\sqrt{q}$
d'_{2t+1}	$d'_{2t+1} = \gamma(q')\sqrt{q'}$
D_{2t+1}	$D_{2t+1} = a_{ji_{j_p+1}} \sum_{h=1}^{2t-p+1} \alpha_{D_{2t+1}} a_{ii_{m_h}} \times \sqrt{A_{iji_{j_1} \ldots i_{j_p+1} i_{m_1} \ldots i_{m_{2t-p+1}}} A_{iji_{j_1} \ldots i_{j_p+1} i_{m_{2t-p+2}} \ldots i_{m_{2t-2p-1}} i_{m_h}}}$
DSP	Digital signal processing.
E	$E \equiv \lvert E_1 \rvert$
E_1	$E_1 = a_{25}a_{26}+a_{15}a_{16} = \dfrac{a_{35}a_{36}}{a_{34}^2}\left(a_{14}^2+a_{24}^2\right)$
E_{2t}	$E_{2t} \doteq a_{ji_{j_p+1}} \sum_{s_1} \alpha_{E_{2t}} a_{is_1} \times \sqrt{A_{iji_{j_1} \ldots i_{j_p+1} i_{m_1} \ldots i_{m_{2t-p}} s_1} A_{iji_{j_1} \ldots i_{j_p+1} i_{m_{2t-p+1}} \ldots i_{m_{2t-2p-1}}}}$
E_{2l-1}	$E_{2l-1} = \sum_{\substack{1\le i_1<i_2<\cdots<i_{2l-1}\le 2n \\ i_k \neq i,j}} \mu_{i_1}\mu_{i_2} \cdots \mu_{i_{2l-1}} A_{ii_1i_2 \cdots i_{2l-1}}$
E_{τ_1}	The error between the group delay responses of the designed and ideal filter.

F	$F \equiv \mid F_1 \mid$
F_1	$F_1 = a_{12}^2 \sqrt{A_{12}} a_{34}$
F_{2l-1}	$F_{2l-1} = \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_{2l-1} \le 2n \\ i_k \ne i, j}} \mu_{i_1}\mu_{i_2} \cdots \mu_{i_{2l-1}} A_{ji_1i_2 \cdots i_{2l-1}}$
F_{2t+3}	$F_{2t+3} = \sum_{h=1}^{2t-p+3} \sum_{\substack{h'=1 \\ h' \ne h}}^{2t-p+3} \sum_{\substack{h''=1 \\ h'' \ne h, h'}}^{2t-p+3} \alpha_{F_{2t+3}} a_{ii_{m_h}} a_{ji_{m_{h'}}} a_{i_{j_{p+1}} i_{m_{h''}}} \times \sqrt{A_{iji_{j_1} \ldots i_{j_{p+1}} i_{m_1} \ldots i_{m_{2t-p+3}}} A_{iji_{j_1} \ldots i_{j_{p+1}} i_{m_{2t-p+4}} \ldots i_{m_{2k-2p-1}} i_{m_h} i_{m_{h'}} i_{m_{h''}}}}$
F'_{2t+1}	$F'_{2t+1} = \epsilon_{ij} F_{2t+1}$
FIR	Finite impulse response.
g	$g = A_{i_{j_1} \ldots i_{j_p} i_{m_1} \ldots i_{m_{2t-p}}} A_{iji_{j_1} \ldots i_{j_p} i_{j_{p+1}} i_{m_{2t-p+1}} \ldots i_{m_{2k-2p-1}}}$
G	$G \equiv \mid G_1 \mid$
G_1	$G_1 = a_{12} \sqrt{A_{34}}$
G_{2t}	$G_{2t} = \sum_{h=1}^{2t-p} \sum_{\substack{h'=1 \\ h' \ne h}}^{2t-p} \sum_{s_1} \alpha_{G_{2t}} a_{ii_{m_h}} a_{ji_{m_{h'}}} a_{i_{j_{p+1}} s_1} \times \sqrt{A_{iji_{j_1} \ldots i_{j_{p+1}} i_{m_1} \ldots i_{m_{2t-p}} s_1} A_{iji_{j_1} \ldots i_{j_{p+1}} i_{m_{2t-p+1}} \ldots i_{m_{2k-2p-1}} i_{m_h} i_{m_{h'}}}}$
g'	$g' = A_{iji_{j_1} \ldots i_{j_p} i_{m_1} \ldots i_{m_{2t-p}}} A_{i_{j_1} \ldots i_{j_p} i_{j_{p+1}} i_{m_{2t-p+1}} \ldots i_{m_{2k-2p-1}}}$
h	$h = A_{i_{j_1} \ldots i_{j_p} i_{j_{p+1}} i_{m_1} \ldots i_{m_{2t-p+1}}} A_{iji_{j_1} \ldots i_{j_p} i_{m_{2t-p+2}} \ldots i_{m_{2k-2p-1}}}$
$H_A(s_1, s_2)$	Two-variable analog transfer function.
$H_D(z_1, z_2)$	Two-variable digital transfer function.
$\mid H_i(\omega_1, \omega_2) \mid$	Amplitude response of the ideal filter.
H_{2l}	$H_{2l} = \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_{2l} \le 2n \\ i_k \ne i, j}} \mu_{i_1}\mu_{i_2} \cdots \mu_{i_{2l}} A_{i_1i_2 \cdots i_{2l}}$
H_{2t}	$H_{2t} = \sum_{h=1}^{2t-p} \sum_{\substack{h'=1 \\ h' \ne h}}^{2t-p} \sum_{s_1} \alpha_{H_{2t}} a_{ii_{m_h}} a_{i_{j_{p+1}} i_{m_{h'}}} a_{js_1} \times$

$$\sqrt{A_{iji_{j_1}\ldots i_{j_{p+1}} i_{m_1}\ldots i_{m_{2t-p}} s_1} A_{iji_{j_1}\ldots i_{j_{p+1}} i_{m_{2t-p+1}}\ldots i_{m_{2t-2p-1}} i_{m_h} i_{m_{h'}}}}$$

h' $$h' = A_{iji_{j_1}\ldots i_{j_p} i_{j_{p+1}} i_{m_1}\ldots i_{m_{2t-p+1}}} A_{i_{j_1}\ldots i_{j_p} i_{m_{2t-p+2}}\ldots i_{m_{2t-2p-1}}}$$

I $$I \equiv |I_1|$$

I_1 $$I_1 = a_{56}$$

I_{2t} $$I_{2t} = \sum_{h=1}^{2t-p} \sum_{s_1} \sum_{s_2} \alpha_{I_{2t}} a_{ii_{m_h}} a_{js_1} a_{i_{j_{p+1}} s_2} \times$$

$$\sqrt{A_{iji_{j_1}\ldots i_{j_{p+1}} i_{m_1}\ldots i_{m_{2t-p}} s_2} A_{iji_{j_1}\ldots i_{j_{p+1}} i_{m_{2t-p+1}}\ldots i_{m_{2t-2p-1}} i_{m_h} s_1}}$$

IIR Infinite impulse response.

I sub P Set of all frequency points along ω_1 and ω_2 axis covering the passband of the filter only.

I sub PS Set of all frequency points along ω_1 and ω_2 axis within the passband and stopband of the filter.

i_{m_l} An integer in the range $l+1$ to $k-b+1$, where l, k, b, and m are given by the ascending inequality defined by $b \leq m_{l+1} < m_{l+2} < \ldots < m_{k-b+1} \leq k$.

i_{m_h} An integer in the range 1 to $l+1$, where l, k, b, and m are given by the ascending inequality defined by $b \leq m_1 < m_2 < \ldots < m_{l+1} \leq k$.

J $$J \equiv |J_1|$$

J_1 $$J_1 = a_{35} a_{36} + a_{45} a_{46}$$

J_{2t} $$J_{2t} = \sum_{h=1}^{2t-p} \sum_{\substack{h'=1 \\ h' \neq h}}^{2t-p} \sum_{s_1} \alpha_{J_{2t}} a_{ji_{m_h}} a_{i_{j_{p+1}} i_{m_{h'}}} a_{is_1} \times$$

$$\sqrt{A_{iji_{j_1}\ldots i_{j_{p+1}} i_{m_1}\ldots i_{m_{2t-p}} s_1} A_{iji_{j_1}\ldots i_{j_{p+1}} i_{m_{2t-p+1}}\ldots i_{m_{2t-2p-1}} i_{m_h} i_{m_{h'}}}}$$

K $$K \equiv |K_1|$$

K_1 $$K_1 = a_{34} \sqrt{A_{12}}$$

K_{2t}	$K_{2t}=\sum_{h=1}^{2t-p}\sum_{s_1}\sum_{s_2}\alpha_{K_{2t}}a_{ji_{m_h}^{r}}a_{is_1}a_{i_{j_{p+1}}s_2}\times$ $\sqrt{A_{iji_{j_1}\ldots i_{j_{p+1}}i_{m_1}\ldots i_{m_{2t-p}}s_2}A_{iji_{j_1}\ldots i_{j_{p+1}}i_{m_{2t-p+1}}\ldots i_{m_{2k-2p-1}}i_{m_h}s_1}}$
$2k$	The total number of variables considered in $M_{2k,p}$, where p of them are repeated (squares).
L, L'	Left side of the equation.
L_{2t}	$L_{2t}=\sum_{h=1}^{2t-p}\sum_{s_1}\alpha_{L_{2t}}a_{ii_{m_h}}a_{js_1}a_{i_{j_{p+1}}s_1}\times$ $\sqrt{A_{iji_{j_1}\ldots i_{j_{p+1}}i_{m_1}\ldots i_{m_{2t-p}}s_1}A_{iji_{j_1}\ldots i_{j_{p+1}}i_{m_{2t-p+1}}\ldots i_{m_{2k-2p-1}}i_{m_h}s_1}}$
M	Even part of the slope, $M=\left(\sum_{l=0}^{n-1}P_{2l}\right)^2$
M_{2n}	2n-variable polynomial.
$M_{2k,p}$	Product of 2k variables with p of them repeated twice. $M_{2k,p}=\mu_{i_{j_1}}^2\mu_{i_{j_2}}^2\cdots\mu_{i_{j_p}}^2\mu_{i_{j_{p+1}}}\cdots\mu_{i_{j_{2k-p}}}$
M_{2t}	$M_{2t}=\sum_{h=1}^{2t-p}\sum_{s_1}\alpha_{M_{2t}}a_{ji_{m_h}}a_{is_1}a_{i_{j_{p+1}}s_1}\times$ $\sqrt{A_{iji_{j_1}\ldots i_{j_{p+1}}i_{m_1}\ldots i_{m_{2t-p}}s_1}A_{iji_{j_1}\ldots i_{j_{p+1}}i_{m_{2t-p+1}}\ldots i_{m_{2k-2p-1}}i_{m_h}s_1}}$
M'	*2n*-variable Hurwitz Polynomial.
M'_{2n}	$M'_{2n}=\frac{\partial}{\partial\mu_{j_1}}\cdots\frac{\partial}{\partial\mu_{j_p}}M_{2n}$
N	Odd part of the slope, $N=\left(\sum_{l=1}^{n-1}P_{2l-1}\right)^2$
$N_{i,j}$	$N_{i,j}=M+N$
N_{2n+1}	The *2n+1*-variable odd polynomial.
N_{2t}	$N_{2t}=a_{ij}\,\alpha_{N_{2t}}\times\sqrt{A_{iji_{j_1}\ldots i_{j_p}i_{m_1}\ldots i_{m_{2t-p}}}A_{iji_{j_1}\ldots i_{j_{p+1}}i_{m_{2t-p+1}}\ldots i_{m_{2k-2p-1}}}}$

N'	$2n+1$-variable Hurwitz Polynomial.
$N'_{i,j}$	$N'_{i,j} = \left(\sum_{l=0}^{n-1} P'_{2l-p}\right)^2 - \left(\sum_{l=0}^{n-1} P'_{2l-p-1}\right)^2 \quad i \neq j$
$N'_{j,j}$	$N'_{j,j} = -\left(\sum_{l=1}^{n} P'_{2l-p-1}\right)^2$
$N''_{i,j}$	$N''_{i,j} = \left(\sum_{l=0}^{n-1} P_{2l+1}''\right)^2 - \left(\sum_{l=0}^{n-1} P_{2l}''\right)^2 \quad i \neq j$
$N''_{j,j}$	$N''_{j,j} = -\left(\sum_{l=0}^{n-1} P_{2l-2}''\right)^2$
p	$p = A_{ij_1 \ldots i_{j_p} i_{j_{p+1}} i_{m_1} \ldots i_{m_{2l-p}}} A_{ji_{j_1} \ldots i_{j_p} i_{m_{2l-p+1}} \ldots i_{m_{2k-2p-1}}}$
p'	$p' = A_{ji_{j_1} \ldots i_{j_p} i_{j_{p+1}} i_{m_1} \ldots i_{m_{2l-p}}} A_{ii_{j_1} \ldots i_{j_p} i_{m_{2l-p+1}} \ldots i_{m_{2k-2p-1}}}$
P_{2t}	$P_{2t} = \sum_{s_1} \sum_{s_1} \alpha_{P_{2l}} a_{is_1} a_{js_1} \times \sqrt{A_{iji_{j_1} \ldots i_{j_{p+1}} i_{m_1} \ldots i_{m_{2l-p}} s_1} A_{iji_{j_1} \ldots i_{j_p} i_{m_{2l-p+1}} \ldots i_{m_{2k-2p-1}} s_1}}$
P_{2l}	$P_{2l} = \sum_{\substack{1 \leq i_1 \cdots < i_{2l} \leq 2n \\ i_k \neq i,j}} \delta_{i_1 \cdots i_{2l}, ij} \mu_{i_1} \cdots \mu_{i_{2l}} \sqrt{A_{i_1 \cdots i_{2l}} A_{iji_1 \cdots i_{2l}}}$
P_{2l-1}	$P_{2l-1} = \sum_{\substack{1 \leq i_1 \cdots < i_{2l-1} \leq 2n \\ i_k \neq i,j}} \delta_{i_1 \cdots i_{2l-1}, ij} \mu_{i_1} \cdots \mu_{i_{2l-1}} \sqrt{A_{ii_1 \cdots i_{2l-1}} A_{ji_1 \ldots i_{2l-1}}}$
P'_{2l-p}	$P'_{2l-p} = \sum_{\substack{1 \leq i_1 < \ldots < i_{2l-p} \leq 2n \\ i_k \neq j_1 \ldots j_p, i, j}} \sqrt{A_{j j_1 \ldots j_p i_1 \ldots i_{2l-p}} A_{ijj_1 \ldots j_p i_1 \ldots i_{2l-p}}}$
P'_{2l-p-1}	$P'_{2l-p-1} = \sum_{\substack{1 \leq i_1 < \ldots < i_{2l-p-1} \leq 2n \\ i_k \neq j_1 \ldots j_p, i, j}} \sqrt{A_{ij_1 \ldots j_p i_1 \ldots i_{2l-p-1}} A_{jj_1 \ldots j_p i_1 \ldots i_{2l-p-1}}}$
P_{2l}''	$P_{2l}'' = \sum_{\substack{1 \leq i_1 \cdots < i_{2l} \leq 2n+1 \\ i_k \neq i,j}} \delta_{i_1 \cdots i_{2l}, ij} \mu_{i_1} \cdots \mu_{i_{2l}} \sqrt{A_{ii_1 \cdots i_{2l}} A_{ji_1 \cdots i_{2l}}}$
P_{2l+1}''	$P_{2l+1}'' = \sum_{\substack{1 \leq i_1 \cdots < i_{2l+1} \leq 2n+1 \\ i_k \neq i,j}} \delta_{i_1 \cdots i_{2l+1}, ij} \mu_{i_1} \cdots \mu_{i_{2l+1}} \sqrt{A_{i \cdots i_{2l+1}} A_{iji_1 \ldots i_{2l+1}}}$

q	$q = A_{ii_{j_1}\dots i_{j_p} i_{m_1}\dots i_{m_{2t+1-p}}} A_{ji_{j_1}\dots i_{j_p} i_{j_{p+1}} i_{m_{2t-p+2}}\dots i_{m_{2t-2p-1}}}$
q'	$q' = A_{ji_{j_1}\dots i_{j_p} i_{m_1}\dots i_{m_{2t+1-p}}} A_{i_{j_1}\dots i_{j_p} i_{j_{p+1}} i_{m_{2t-p+2}}\dots i_{m_{2t-2p-1}}}$
Q_{2t+1}	$Q_{2t+1} = \sum_{h=1}^{2t-p+1} \sum_{\substack{h'=1 \\ h' \neq h}}^{2t-p} \sum_{s_1} \alpha_{G_{2t+1}} a_{ii_{m_h}} a_{ji_{m_{h'}}} a_{i_{j_{p+1}} s_1} \times \sqrt{A_{iji_{j_1}\dots i_{j_{p+1}} i_{m_1}\dots i_{m_{2t-p+1}}} A_{iji_{j_1}\dots i_{j_{p+1}} i_{m_{2t-p+2}}\dots i_{m_{2t-2p-1}} i_{m_h} i_{m_{h'}} s_1}}$
r	For $k \geq n$, we define $k = n + r$, therefore, $0 \leq r \leq n-2$.
R, R'	Right side of the equation.
$R_{1,2t,2k}$	$R_{1,2t,2k} = X_{2t,2k} + X'_{2t,2k} - 2Y_{2t,2k} = \sum_J \sum_{\Lambda_{2t}} \sum_I M_{2k,p} (s^2 + s'^2 - 2\beta_p ss')$
$R_{2,2t+1,2k}$	$R_{2,2t+1,2k} = Z_{2t+1,2k} + Z'_{2t+1,2k} - 2Y_{2t+1,2k}$
R_{2t+1}	$R_{2t+1} = \sum_{h=1}^{2t-p+1} \sum_{\substack{h'=1 \\ h' \neq h}}^{2t-p+1} \sum_{s_1} \alpha_{H_{2t+1}} a_{ii_{m_h}} a_{i_{j_{p+1}} i_{m_{h'}}} a_{js_1} \times \sqrt{A_{iji_{j_1}\dots i_{j_{p+1}} i_{m_1}\dots i_{m_{2t-p+1}}} A_{iji_{j_1}\dots i_{j_{p+1}} i_{m_{2t-p+2}}\dots i_{m_{2t-2p-1}} i_{m_h} i_{m_{h'}} s_1}}$
s_1	any integer from 1 to $2n$ except the numbers in the set: $S = \{i, j, i_{j_1}, \dots, i_{j_{p+1}}, i_{m_1} \dots i_{m_{2t-2p-1}}\} = \{i, j, i_{j_1} \dots i_{j_{2t-2p}}\}$
s_2	s_2 can assume any integer from 1 to $2n$ except the numbers in the set: $S' = \{S, s_1\} = \{i, j, i_{j_1} \dots i_{j_{2t-2p}}, s_1\}$
S_a	$S_a = \sum_{s_1} \sum_{s_2} \alpha_{S_a} a_{is_1} a_{js_2} \times \sqrt{A_{iji_{j_1}\dots i_{j_p} i_{m_1}\dots i_{m_{2t-p}} s_1 s_2} A_{iji_{j_1}\dots i_{j_{p+1}} i_{m_{2t-p+1}}\dots i_{m_{2t-2p-1}}}}$
$S_{a'}$	$S_{a'} = \sum_{s_1} \sum_{s_2} \alpha_{S_{a'}} a_{is_1} a_{js_2} \times \sqrt{A_{iji_{j_1}\dots i_{j_p} i_{m_1}\dots i_{m_{2t-p}}} A_{iji_{j_1}\dots i_{j_{p+1}} i_{m_{2t-p+1}}\dots i_{m_{2t-2p-1}} s_1 s_2}}$

S_c	$S_c = \sum_{s_1}\sum_{s_2} \alpha_{S_c} a_{is_1} a_{js_2} \times \sqrt{A_{iji_{j_1}\cdots i_{j_p+1} i_{m_1}\cdots i_{m_{2t-p}} s_2} A_{iji_{j_1}\cdots i_{j_p} i_{m_{2t-p+1}}\cdots i_{m_{2k-2p-1}} s_1}}$
$S_{c'}$	$S_{c'} = \sum_{s_1}\sum_{s_2} \alpha_{S_{c'}} a_{is_1} a_{js_2} \times \sqrt{A_{iji_{j_1}\cdots i_{j_p+1} i_{m_1}\cdots i_{m_{2t-p}} s_1} A_{iji_{j_1}\cdots i_{j_p} i_{m_{2t-p+1}}\cdots i_{m_{2k-2p-1}} s_2}}$
SHP	Strictly Hurwitz Polynomial.
t	Defined as: $H_{2t} D_{2k-2t}$
T	Sampling period.
T_{2t+1}	$T_{2t+1} = \sum_{h=1}^{2t-p+1} \sum_{s_1}\sum_{s_2} \alpha_{I_{2t+1}} a_{ii_{m_h}} a_{js_1} a_{i_{j_p+1}s_2} \times \sqrt{A_{iji_{j_1}\cdots i_{j_p+1} i_{m_1}\cdots i_{m_{2t-p+1}} s_1 s_2} A_{iji_{j_1}\cdots i_{j_p+1} i_{m_{2t-p+2}}\cdots i_{m_{2k-2p-1}} i_{m_h}}}$
u	Universal set, $u = \left\{ b, b+1, b+2, \cdots, k \right\}$
U_{2t+1}	$U_{2t+1} = \sum_{h=1}^{2t-p+1} \sum_{\substack{h'=1 \\ h' \neq h}}^{2t-p+1} \sum_{s_1} \alpha_{J_{2t+1}} a_{ji_{m_h}} a_{i_{j_p+1} i_{m_{h'}}} a_{is_1} \times \sqrt{A_{iji_{j_1}\cdots i_{j_p+1} i_{m_1}\cdots i_{m_{2t-p+1}}} A_{iji_{j_1}\cdots i_{j_p+1} i_{m_{2t-p+2}}\cdots i_{m_{2k-2p-1}} i_{m_h} i_{m_{h'}} s_1}}$
v_i	Complement of the set w_i, i.e., $v_i \cup w_i = u$
v_i'	Complement of the set W prime back 100 nothing sub 1
v_β'	The set v_i' with β added to it.
V_{2t+1}	$V_{2t+1} = \sum_{h=1}^{2t-p+1} \sum_{s_1}\sum_{s_2} \alpha_{K_{2t+1}} a_{ji_{m_h}} a_{is_1} a_{i_{j_p+1}s_2} \times \sqrt{A_{iji_{j_1}\cdots i_{j_p+1} i_{m_1}\cdots i_{m_{2t-p+1}} s_1 s_2} A_{iji_{j_1}\cdots i_{j_p+1} i_{m_{2t-p+2}}\cdots i_{m_{2k-2p-1}} i_{m_h}}}$
VSHP	Very strictly Hurwitz polynomial.
W	The set $W = \left\{ w_1, w_2, \cdots, w_r \right\}$, where, $r = \binom{k-b+1}{l}$ and

each member of W is: $w_i = \left\{ m_1{}^i, m_2{}^i, \cdots, m_l{}^i \right\}$ where m_j's are given by: $b \leq m_1 < m_2 < \cdots < m_l \leq k$

w_i	The set, $w_i = \left\{ m_1{}^i, m_2{}^i, \cdots, m_l{}^i \right\}$, where: $w_1 = \left\{ 1,2, \cdots, l \right\}$ $w_2 = \left\{ 1,2, \cdots, l+1 \right\}$, and so on.
$\overline{w_i}$	$\overline{w_i} = v_i = \left\{ m_{l+1}{}^i, m_{l+2}{}^i, \cdots, m_{k-b+1}{}^i \right\}$
w_α	$w_\alpha = w_i \cup \alpha$
$X_{2t,2k}$	Terms in $H_{2t} D_{2k-2t}$ containing $M_{2k,p}$
$X'_{2t,2k}$	Terms in $H_{2k-2t} D_{2t}$ containing $M_{2k,p}$
$Y_{2t,2k}$	Terms in $P_{2t} P_{2k-2t}$ containing $M_{2k,p}$
Z_{LC}	Reactance function.
$Z_{2t+1,2k}$	Terms in $E_{2t+1} F_{2k-2t-1}$ containing $M_{2k,p}$
$Z'_{2t+1,2k}$	Terms in $E_{2k-2t-1} F_{2t+1}$ containing $M_{2k,p}$
α	Member of the set v_i, i.e., one of the integers belonging to the set $v_i = \overline{w_i} = \left\{ m_{l+1}{}^i, m_{l+2}{}^i, \cdots, m_{k-b+1}{}^i \right\}$
$\alpha_{B_{2t+1}}$	$\alpha_{B_{2t+1}} = \xi_1 \lambda_1 \beta_1 \eta_1$
$\alpha_{C_{2t}}$	$\alpha_{C_{2t}} = \xi_2 \lambda_2 \beta_2 \eta_2$
$\alpha_{D_{2t+1}}$	$\alpha_{D_{2t+1}} = \xi_1 \lambda_1 \beta_1 \eta_3$
$\alpha_{E_{2t}}$	$\alpha_{E_{2t}} = \xi_2 \lambda_2 \beta_2 \eta_4$
$\alpha_{F_{2t+3}}$	$\alpha_{F_{2t+3}} = \xi_3 \lambda_4 \beta_3 \eta_5$
$\alpha_{G_{2t}}$	$\alpha_{G_{2t}} = \xi_4 \lambda_4 \beta_4 \eta_6$

$\alpha_{H_{2t}}$	$\alpha_{H_{2t}} = \xi_4\lambda_4\beta_4\eta_7$
$\alpha_{I_{2t}}$	$\alpha_{I_{2t}} = \xi_5\lambda_6\beta_5\eta_8$
$\alpha_{J_{2t}}$	$\alpha_{J_{2t}} = \xi_4\lambda_4\beta_4\eta_9$
$\alpha_{L_{2t}}$	$\alpha_{L_{2t}} = \xi_6\lambda_8\beta_6\eta_{11}$
$\alpha_{M_{2t}}$	$\alpha_{M_{2t}} = \xi_6\lambda_8\beta_6\eta_{12}$
$\alpha_{P_{2t}}$	$\alpha_{P_{2t}} = \xi_7\lambda_{11}\beta_7\eta_{14}$
$\alpha_{Q_{2t+1}}$	$\alpha_{Q_{2t+1}} = \xi_4\lambda_5\beta_4\eta_6$
$\alpha_{R_{2t+1}}$	$\alpha_{R_{2t+1}} = \xi_4\lambda_5\beta_4\eta_7$
α_{S_s}	$\alpha_{S_s} = \xi_8\lambda_{13}\beta_8\eta_{15}$
$\alpha_{S_{s'}}$	$\alpha_{S_{s'}} = \xi_8\lambda_{14}\beta_8\eta_{15}$
α_{S_c}	$\alpha_{S_c} = \xi_8\lambda_{17}\beta_8\eta_{15}$
$\alpha_{S_{c'}}$	$\alpha_{S_{c'}} = \xi_8\lambda_{18}\beta_8\eta_{15}$
$\alpha_{T_{2t+1}}$	$\alpha_{T_{2t+1}} = \xi_5\lambda_7\beta_5\eta_8$
$\alpha_{U_{2t+1}}$	$\alpha_{U_{2t+1}} = \xi_4\lambda_5\beta_5\eta_9$
$\alpha_{V_{2t+1}}$	$\alpha_{V_{2t+1}} = \xi_5\lambda_8\beta_5\eta_{10}$
β	A member of the set w_i, i.e., an integer from the set, $w_i = \left\{ m_1{}^i, m_2{}^i, \cdots, m_l{}^i \right\}$
β_1	$\beta_1 = \delta_{i_{j_1} \ldots i_{j_p}, iji_{j_{p+1}} i_{m_k}}$
β_2	$\beta_2 = \delta_{i_{j_1} \ldots i_{j_p}, iji_{j_{p+1}} s_1}$
β_3	$\beta_3 = \delta_{i_{j_1} \ldots i_{j_p}, iji_{j_{p+1}} i_{m_k} i_{m_k'}, i_{m_k''}}$
β_4	$\beta_4 = \delta_{i_{j_1} \ldots i_{j_p}, iji_{j_{p+1}} i_{m_k} i_{m_k'}, s_1}$

β_5	$\beta_5 = \delta_{i_{j_1} \ldots i_{j_p}, i j i_{j_{p+1}} i_{m_k} s_1 s_2}$
β_6	$\beta_6 = \delta_{i_{j_1} \ldots i_{j_p}, i j i_{j_{p+1}} i_{m_k}}$
β_7	$\beta_7 = \delta_{i_{j_1} \ldots i_{j_p}, i j}$
β_8	$\beta_8 = \delta_{i_{j_1} \ldots i_{j_p}, i j s_1 s_2}$
γ_i	± 1, sign variable.
$\gamma(w)$	Sign function. With $w = \sqrt{A_{J\Lambda_1} A_{J\Lambda_2}}$ $\gamma(w) = \delta_{J\Lambda_1, J\Lambda_2} = \delta_{\Lambda_1, \Lambda_2}$
$\delta_{s_1 \ldots s_r, t_1 \ldots t_u}$	$\delta_{s_1 s_2 \ldots s_r, t_1 t_2 \ldots t_u} = \epsilon_{s_1 t_1} \cdots \epsilon_{s_r t_1} \epsilon_{s_1 t_2} \cdots \epsilon_{s_r t_2} \cdots \epsilon_{s_1 t_u} \cdots \epsilon_{s_r t_u}$
ϵ_{hk}	$\epsilon_{hk} = \begin{cases} +1 & \text{if } h < k \\ -1 & \text{if } h > k \end{cases}$
η_1	$\eta_1 = \epsilon_{i i_{j_{p+1}}} \epsilon_{j i_{m_k}} \delta_{i i_{j_{p+1}}, j i_{m_k}}$
η_2	$\eta_2 = \epsilon_{i i_{j_{p+1}}} \epsilon_{j s_1} \delta_{i i_{j_{p+1}}, j s_1}$
η_3	$\eta_3 = \epsilon_{j i_{j_{p+1}}} \epsilon_{j i_{m_k}} \delta_{j i_{j_{p+1}}, i i_{m_k}}$
η_4	$\eta_4 = \epsilon_{j i_{j_{p+1}}} \epsilon_{i s_1} \delta_{j i_{j_{p+1}}, i i_{m_k}}$
η_5	$\eta_5 = \epsilon_{i i_{m_k}} \epsilon_{j i_{m_{k'}}} \epsilon_{i_{j_{p+1}} i_{m_{k''}}} \delta_{i i_{m_k}, j i_{m_{k'}}} \delta_{i j i_{m_k} i_{m_{k'}}, i_{j_{p+1}} i_{m_{k''}}}$
η_6	$\eta_6 = \epsilon_{i i_{m_k}} \epsilon_{j i_{m_{k'}}} \epsilon_{i_{j_{p+1}} s_1} \delta_{i i_{m_k}, j i_{m_{k'}}} \delta_{i j i_{m_k} i_{m_{k'}}, i_{j_{p+1}} s_1}$
η_7	$\eta_7 = \epsilon_{i i_{m_k}} \epsilon_{i_{j_{p+1}} i_{m_{k'}}} \epsilon_{j s_1} \delta_{i i_{m_k}, i_{j_{p+1}} i_{m_{k'}}} \delta_{i i_{j_{p+1}} i_{m_k} i_{m_{k'}}, j s_1}$
η_8	$\eta_8 = \epsilon_{i i_{m_k}} \epsilon_{j s_1} \epsilon_{i_{j_{p+1}} s_2} \delta_{i i_{m_k}, j s_1} \delta_{i_{j_{p+1}}, i j i_{m_k} s_1}$
η_9	$\eta_9 = \epsilon_{j i_{m_k}} \epsilon_{i_{j_{p+1}} i_{m_{k'}}} \epsilon_{i s_1} \delta_{j i_{m_k}, i_{j_{p+1}} i_{m_{k'}}} \delta_{j i_{j_{p+1}} i_{m_k} i_{m_{k'}}, i s_1}$
η_{10}	$\eta_{10} = \epsilon_{j i_{m_k}} \epsilon_{i s_1} \epsilon_{i_{j_{p+1}} s_2} \delta_{j i_{m_k}, i s_1} \delta_{i_{j_{p+1}}, i j i_{m_k} s_1}$
η_{11}	$\eta_{11} = \epsilon_{i i_{m_k}} \epsilon_{j s_1} \epsilon_{i_{j_{p+1}} s_1} \epsilon_{i j} \epsilon_{i i_{j_{p+1}}} \epsilon_{i_{j_{p+1}} i_{m_k}} \delta_{i_{m_k}, j s_1} \delta_{j, i_{j_{p+1}} s_1}$
η_{12}	$\eta_{12} = \epsilon_{j i_{m_k}} \epsilon_{i s_1} \epsilon_{i_{j_{p+1}} s_1} \epsilon_{j i} \epsilon_{j i_{j_{p+1}}} \epsilon_{i_{j_{p+1}} i_{m_k}} \delta_{i_{m_k}, i s_1} \delta_{i, i_{j_{p+1}} s_1}$

η_{13}	$\eta_{13} = \epsilon_{ij}$
η_{14}	$\eta_{14} = \epsilon_{is_1}\epsilon_{js_1}\delta_{is_1,js_1}$
η_{15}	$\eta_{15} = \epsilon_{is_1}\epsilon_{js_2}\delta_{is_1,js_2}$
θ	Phase variable.
κ_{2t}	$\kappa_{2t} = B_{2t+1}+C_{2t}+D_{2t+1}+T_{2t+1}+J_{2t}+V_{2t+1}-$ $R_{2t+1}-I_{2t}-U_{2t+1}-K_{2t}-L_{2t}-M_{2t}$
λ_1	$\lambda_1 = \delta_{i_{m_k} i_{m_1} \dots i_{m_{2t-p+1}}, i_{m_{2t-p+2}} \dots i_{m_{2k-2p-1}}}$
λ_2	$\lambda_2 = \delta_{i_{m_1} \dots i_{m_{2t-p}}, s_1 i_{m_{2t-p+1}} \dots i_{m_{2k-2p-1}}}$
λ_3	$\lambda_3 = \delta_{i_{m_k} i_{m_k'} i_{m_1} \cdots i_{m_{2t-p}}, i_{m_{2t-p+1}} \dots i_{m_{2k-2p-1}}}$
λ_4	$\lambda_4 = \delta_{i_{m_k} i_{m_k'} i_{m_k''} i_{m_1} \cdots i_{m_{2t-p+3}}, i_{m_{2t-p+4}} \dots i_{m_{2k-2p-1}}}$
λ_5	$\lambda_5 = \delta_{i_{m_k} i_{m_k'} s_1 i_{m_1} \cdots i_{m_{2t-p+1}}, i_{m_{2t-p+1}} \dots i_{m_{2k-2p-1}}}$
λ_6	$\lambda_6 = \delta_{i_{m_k} i_{m_1} \cdots i_{m_{2t-p}}, s_1 i_{m_{2t-p+1}} \dots i_{m_{2k-2p-1}}} \delta_{ij i_{m_1} \cdots i_{m_{2t-p}}, s_2} \delta_{i_{m_{2t-p+1}} \dots i_{m_{2k-2p-1}}, s_1}$
λ_7	$\lambda_7 = \delta_{i_{m_k} i_{m_1} \cdots i_{m_{2t-p+1}}, s_1 s_2 i_{m_{2t-p+2}} \dots i_{m_{2k-2p-1}}}$
λ_8	$\lambda_8 = \delta_{i_{m_k} i_{m_1} \cdots i_{m_{2t-p}}, i_{m_{2t-p+1}} \dots i_{m_{2k-2p-1}}} \delta_{i_1 \dots i_{2k}, s_1}$
λ_9	$\lambda_9 = \delta_{i_{m_1} \cdots i_{m_{2t-p}}, i_{j_p+1} i_{m_{2t-p+1}} \dots i_{m_{2k-2p-1}}}$
λ_{11}	$\lambda_{11} = \delta_{s_1 i_{j_p+1} i_{m_1} \cdots i_{m_{2t-p}}, s_1 i_{m_{2t-p+1}} \dots i_{m_{2k-2p-1}}}$
λ_{13}	$\lambda_{13} = \delta_{i_{m_1} \cdots i_{m_{2t-p}}, s_1 s_2 i_{j_p+1} i_{m_{2t-p+1}} \dots i_{m_{2k-2p-1}}}$
λ_{14}	$\lambda_{14} = \delta_{s_1 s_2 i_{m_1} \cdots i_{m_{2t-p}}, i_{j_p+1} i_{m_{2t-p+1}} \dots i_{m_{2k-2p-1}}}$
λ_{17}	$\lambda_{17} = \delta_{s_2 i_{j_p+1} i_{m_1} \cdots i_{m_{2t-p}}, s_1 i_{m_{2t-p+1}} \dots i_{m_{2k-2p-1}}}$
λ_{18}	$\lambda_{18} = \delta_{s_1 i_{j_p+1} i_{m_1} \cdots i_{m_{2t-p}}, s_2 i_{m_{2t-p+1}} \dots i_{m_{2k-2p-1}}}$
μ_i	The i^{th} variable (axis).

ν_{2t}	$\nu_{2t} = S_a - S_{a'}$
ν'_{2t}	$\nu'_{2t} = S_c + S_{c'}$
ξ_1	$\xi_1 = (-1)^{i+j+i_{j_p+1}+i_{m_k}}$
ξ_2	$\xi_2 = (-1)^{i+j+i_{j_p+1}+s_1}$
ξ_3	$\xi_3 = (-1)^{i+j+i_{j_p+1}+i_{m_k}+i_{m_{k'}}+i_{m_{k''}}+3}$
ξ_4	$\xi_4 = (-1)^{i+j+i_{j_p+1}+i_{m_k}+i_{m_{k'}}+s_1+3}$
ξ_5	$\xi_5 = (-1)^{i+j+i_{j_p+1}+i_{m_k}+s_1+s_2+3}$
ξ_6	$\xi_6 = (-1)^{i+j+i_{j_p+1}+i_{m_k}}$
ξ_7	$\xi_7 = (-1)^{i+j}$
ξ_8	$\xi_8 = (-1)^{i+j+s_1+s_2}$
$\sum\limits_I$	$\sum\limits_I \equiv \sum\limits_{\substack{1\le i_1<i_2\cdots<i_{2k-p}\le 2n \\ i_l\ne i,j}}$
$\sum\limits_J$	$\sum\limits_J \equiv \sum\limits_{1\le j_1<j_2\cdots<j_p\le 2k-p}$
$\sum\limits_{\Lambda'}$	$\sum\limits_{\Lambda'} \equiv \sum\limits_{j_p+1\le m_1<m_2\cdots<m_{s-p}\le j_{2k-p}}$
$\sum\limits^{a}$	$\sum\limits^{a} \equiv \sum\limits_J \sum\limits_{\Lambda_s} \sum\limits_I$
$\sum\limits_{\Lambda_s}$	$\sum\limits_{\Lambda_s} \equiv \sum\limits_{j_p+2\le m_1\cdots<m_{s-p}\le j_{2k-p}}$
τ	Group delay
ω	digital frequency.
Ω	Analog frequency.
$\binom{n}{k}$	The number of different combinations of n different things k at a time without repetition.

Chapter 1
Introduction

1.1 General

Filtering is a process by which the frequency spectrum of a signal can be modified according to some desired specifications. The digital filter is a system that can be used to filter discrete-time signals.

One-dimensional (1-D) filtering finds diverse applications such as in biomedical engineering, acoustics, sonar, radar, seismology, speech communication, nuclear science, and many others. However 1-D has limitations in some applications which make two-dimensional (2-D) signal processing desirable. We can name picture processing of weather photos, air reconnaissance photos, medical x-rays, seismic records, etc. as examples. These signals are inherently 2-dimensional in nature.

2-D filtering is different from 1-D due to three factors: (1) 2-D problems generally involve considerably more data than 1-D ones; (2) the mathematics required for 2-D systems is less complete than the mathematics for 1-D systems; and (3) 2-D systems have many more degrees of freedom, which give the system designer a flexibility not encountered in the 1-D case.

By making components smaller and cheaper, a computer of the given size can handle more data nowadays than before and hence problem (1) given above will not be serious. However the other two problems require considerable attention.

1.2 Applications

Two dimensional (2-D) digital filters find numerous applications in image processing, geophysical prospecting, sonar, radar, radioastronomy, etc. [1,2].

Exploration geophysics has a dual scientific role, namely the research for new petroleum and mineral deposits and the search required to develop new methods and improve scientific understanding. Here among many other things we should mention analysis of seismic records.

Sonar is equivalent of sound propagation in the ocean environment. Examples of active sonar applications include target detection and localization, communication, navigation and mapping and charting.

Parameters that can be measured by a radar are: target's angular direction (azimuth and elevation), range, velocity and reflectivity (cross section).

In all of the above applications 2-D filters are the integral part of any processing scheme adopted.

1.3 FIR and IIR Filters

2-D filters are computational algorithms which can transform an input 2-D array of numbers to an output 2-D array of numbers according to a set of prespecified rules. These filters are divided into two classes, namely, Nonrecursive (Finite Impulse Response FIR) or Recursive (Infinite Impulse Response IIR).

An *FIR* (Nonrecursive) filter is one whose impulse response possesses only a finite number of nonzero samples. For such a filter, the impulse response is always absolutely summable and thus FIR filters are always stable. The difference equation of an FIR filter can be written as:

$$y(n_1,n_2) = \sum_{l_1=0}^{N_1-1} \sum_{l_2=0}^{N_2-1} h(l_1,l_2)\,x(n_1-l_1,n_2-l_2) \qquad (1.1)$$

where $\underline{x}$, $\underline{h}$, and $\underline{y}$ are the input array, the filter impulse response, and the output array respectively.

An *IIR* (Recursive) filter is one whose input and output satisfy a multidimensional difference equation of finite order. These filters may or may not be stable. IIR filters can be represented by their difference equation as follows:

$$\sum_{k_1=0}^{N_1}\sum_{k_2=0}^{N_2} b(k_1,k_2)y(n_1-k_1,n_2-k_2)=\sum_{l_1=0}^{L_1}\sum_{l_2=0}^{L_2} a(l_1,l_2)x(n_1-l_1,n_2-l_2) \qquad (1.2)$$

where $\underline{a}$ and $\underline{b}$ are the filters coefficients, while $\underline{x}$ and $\underline{y}$ are the input and output respectively.

As can be seen from Eq(1.2), present output of IIR filter is calculated using the present and past input as well as past outputs. Therefore the output can become arbitrary large independent of the size of the input signal, hence resulting in instability of the system.

In 2-D FIR (nonrecursive) filters, problems of stability do not exist and these filters are capable of providing linear phase. However, IIR (recursive) filters are preferred in practice because they provide a sharp cut-off response and they require less number of multipliers and memory than their FIR counterparts. But these filters suffer from the problem of stability and that exact linear phase characteristics are difficult to get with these filters. Also the order of FIR filter has to be very large compared with that of IIR. Thus the promise of IIR filters is a potential reduction in computation compared to FIR filters when performing comparable filtering operations.

Taking the z-transform of both sides of Eqs(1.1) and (1.2) yields the system function for FIR and IIR filters. For FIR it is:

$$H(z_1,z_2)=\sum_{l_1=0}^{L_1-1}\sum_{l_2=0}^{L_2-1} a(l_1,l_2)z_1^{-l_1}z_2^{-l_2} \qquad (1.3)$$

and for IIR filter is :

$$H(z_1,z_2) = \frac{A(z_1,z_2)}{B(z_1,z_2)}$$

$$= \frac{\sum_{l_1=0}^{L_1-1}\sum_{l_2=0}^{L_2-1} a(l_1,l_2) z_1^{-l_1} z_2^{-l_2}}{\sum_{k_1=0}^{K_1-1}\sum_{k_2=0}^{K_2-1} b(k_1,k_2) z_1^{-k_1} z_2^{-k_2}} \tag{1.4}$$

1.4 Design of IIR Filters

IIR filters can be designed in space domain or frequency domain. In space domain technique the aim in general is to find the coefficient arrays of 2-D IIR filter so that the filter response to a specified input signal, usually unit impulse, is a good approximation to some desired output signal. Usually the mean squared error norm is the most wildly used in conjunction with optimization.

One of the earliest methods of space domain designs is due to Shanks et al [3]. The major advantage of Shanks method is linearization of the cost function so that linear programming can be used for minimization of the cost function. Unfortunately this technique does not minimize the true mean squared error, and the resultant filter may not be stable.

There are other linear and nonlinear programming techniques for space domain design [1].

There are two general approaches for the frequency domain designs, namely, *spectral transformation* method and *direct (computer aided)* method. Below we represent a brief description of each method.

1.4.1 Spectral Transformation

In the spectral transformation technique some kind of transformation is applied to 1-D analog or digital filter to obtain 2-D filters, the transfer function of which is:

$$H(s_1,s_2) = \frac{A(s_1,s_2)}{B(s_1,s_2)}$$

$$= \frac{\sum_{l_1=0}^{L_1-1}\sum_{l_2=0}^{L_2-1} a(l_1,l_2)s_1^{l_1}s_2^{l_2}}{\sum_{k_1=0}^{K_1-1}\sum_{k_2=0}^{K_2-1} b(k_1,k_2)s_1^{k_1}s_2^{k_2}} \quad (1.5)$$

The first transformation proposed is due to [3]. In this paper the authors introduced the expression for rotation of axis as basis for transformation of 1-D analog filters to 2-D analog filters which is then discretized by the use of the double bilinear transformations. This technique, in spite of it's simplicity, suffers from the lack of producing a desired cutoff boundary and also guaranteeing stable 2-D filter.

Later this technique was modified [4] by finding suitable angles of rotation to ensure the stability of the designed filter. It was also shown that by cascading several rotated filters which have the angle of rotation between 270^o and 360^o, near circular symmetric filter is obtainable. This technique suffers from hardware complexity since several rotated filters are to be cascaded to obtain a desired circular cutoff boundary.

McClellan [5] has introduced a transformation for the design of 2-D FIR filters. Later this transformation has been used for the design of 2-D IIR filters [6]. The technique required is: (i) generation of 1-D zero phase IIR filter (ii) application of McClellan transformation to 1-D zero phase IIR filter and (iii) decomposition of the resultant 2-D zero phase filter to four simple quadrantal filters each recursing in different direction. This method cannot guarantee the stability of the designed filter.

Later a second order, two-variable reactance function has been applied to 1-D analog filters as a transformation [7]. This is followed by double bilinear transformation. Zero-phase 2-D filter is then obtained by cascading four single quadrant 2-D filters.

Higher order 2-variable reactance function has been used to arrive at approximately circularly symmetric 2-D filters [8].

For transformation from a 1-D digital IIR prototype filter $H(z)$ to a 2-D IIR filter $G(z_1,z_2)$ we can make the substitution

$$z^{-1} = F(z_1,z_2) \tag{1.6}$$

To ensure that G is a stable filter both H and F must be stable and F should be allpass [9]. It has been shown that [10] the only admissible transformation for mapping 1-D IIR filter into 2-D IIR filter has the form:

$$F(z_1,z_2) = z_1^{-p} z_2^{-q} \tag{1.7}$$

Most of these techniques produce circularly symmetric filters. The design of fan filters using the transformation as in [7] , is worked out by Ref.[11]. Later [12] proposed a complex transformation:

$$z_1 \rightarrow e^{j\phi} z_1^{\alpha_1/\beta_1} z_2^{\alpha_2/\beta_2} \tag{1.8}$$

to obtain stable fan filters.

1.4.2 Direct method of design of IIR filters

We group these filters according to their separability. Then three cases arise:

I) - Product Separable Transfer Functions [13,14]

These are given by:

$$H(z_1,z_2) = H_1(z_1)H_2(z_2) \tag{1.9}$$

In the special case that the unit-sample response or equivalently the system function is separable, the filter is stable if and only if the poles of $H_1(z_1)$ lie inside the unit circle in the z_1-plane and the poles of $H_2(z_2)$ lie inside the unit circle in the z_2-plane. Thus the stability test is indeed equivalent to two 1-D tests.

Considering that in the 1-D canonic realization of the filters the number of unit delays, which is almost the number of multipliers, is equal to the order of the filter, then if the order of $H_1(z_1)$ is n_1 and order of $H_2(z_2)$ is n_2, then n_1+n_2 multipliers are used to realize the separable transfer function of Eq(1.9). In the realization of the general non-separable filter as seen from Eq(1.4), $Max\,(L_1L_2,K_1K_2)$ of multipliers are required for canonic realization of the filter. Thus it is seen that number of multipliers are considerably less for separable filters as compared with general non-separable filters.

Linear phase is also obtainable with this scheme, since,

$$H(e^{j\omega_1},e^{j\omega_2}) = H_1(e^{j\omega_1})H_2(e^{j\omega_2})$$

the familiar optimization technique can now be formulated and implemented much easier than when H is non-separable.

However, this scheme has restriction that it gives rectangular cut off boundary. Let us assume that we want to design a low-pass filter. Then $H_1(e^{j\omega_1})$ is forced to be ideally flat up to cut-off frequency $\omega_{1,c}$ and similarly $H_2(e^{j\omega_2})$ will be flat up to cut-off frequency $\omega_{2,c}$. Therefore the cut-off boundary of $H=H_1H_2$ will be rectangular with sides at $\omega_{1,c}$ and $\omega_{2,c}$.

This rectangular boundary finds applications in image processing when the boundary of the spectrum of image has near rectangular shape.

II) - Denominator Separable Transfer Functions [15,16]

In this case, the transfer function is:

$$H(z_1,z_2) = \frac{A(z_1,z_2)}{B_1(z_1)B_2(z_2)} \tag{1.10}$$

Since the denominator is the determining stability factor and since the denominator is separable, the same argument as in I) above concerning the stability holds. The other advantage is that less number of multipliers are required as compared to the case of the nonseparable filters. Also nearly circularly symmetric filters are possible only with this scheme. The disadvantage is that if nonsymmetric cutoff boundary is required this method fails. It has been shown by [15] that quadrantal symmetry is only obtainable with separable denominator filters and in this case the numerator should be in the form

$$A(z_1,z_2) = P_1(z_1+z_1^{-1},z_2)P_2(z_1,z_2+z_2^{-1})z_1^m z_2^n \tag{1.11}$$

III) - General form

The transfer function is of the form:

$$H(z_1,z_2) = \frac{A(z_1,z_2)}{B(z_1,z_2)} \tag{1.12}$$

The advantage is that general class of 2-D filters can be designed with any given boundaries. But it has the disadvantage of stability problem, and it needs more multipliers as compared with the first two groups.

These filters are designed by linear [17-22] or nonlinear [23-27] programming.

1.5 Stability of IIR Filters

The stability requirement for Eq(1.4) is [28]:

$$B(z_1,z_2) \neq 0 \quad \text{for} \bigcap_{i=1}^{2} |z_i| \geq 1 \tag{1.13}$$

In the analog domain, referring to Eq(1.5), the requirement is,

$$B(s_1,s_2) \neq 0 \quad \text{for} \bigcap_{i=1}^{2} \mathrm{Re}\; s_i \geq 0 \tag{1.14}$$

Historically, in the direct design method, stability test was required after calculation of the parameters of the filter followed by one of the stabilization methods if the filter was found to be unstable. Note that stabilization is used in 1-D filters as well, but some techniques used for 1-D cannot be extended to 2-D.

As an example pole-zero cancellation technique in 1-D described below, cannot be extended to the case of 2-D.

In the design of 1-D recursive digital filter, it may happen that the filter's transfer function will have some of its poles located outside the unit circle in spite of a satisfactory magnitude response which is associated with the filter's transfer function. In this case it is desirable to stabilize the filter transfer function without affecting the magnitude characteristics of the filter. Let us assume that the designed filter which has a desired magnitude response is expressed as follows:

$$H(z) = \frac{\sum_{i=0}^{M} a_i z^i}{\prod_{i=1}^{K}\left(z - r_i e^{j\theta_i}\right) \prod_{i=K+1}^{M}\left(z - \frac{1}{r_i} e^{j\theta_i}\right)}, \quad |r_i| < 1 \tag{1.15}$$

This represents the transfer function of a filter with K of its poles inside the unit circle and the remaining *(M-K)* poles located outside the unit circle. If this transfer function is cascaded with the all pass function of the form below:

$$\prod_{i=k+1}^{M} \frac{\left(z - \frac{1}{r_i} e^{j\theta_i}\right)}{\left(z - r_i e^{j\theta_i}\right)} \tag{1.16}$$

the destabilizing effect will be removed without changing the magnitude response of the resultant filter. However this may not be possible in the case of 2-D, because 2-D polynomials may not be factorizable in general.

Another difficulty of 2-D which is not present in 1-D is the occurrence of non-essential singularity of the second kind and the role of the numerator polynomial in its stability. In [29], Goodman has considered three filters:

$$G_1(z_1,z_2) = \frac{(1-z_1)^8(1-z_2)^8}{2-z_1-z_2} \tag{1.17}$$

$$G_2(z_1,z_2) = \frac{(1-z_1)(1-z_2)}{2-z_1-z_2} \tag{1.18}$$

$$G_3(z_1,z_2) = \frac{2}{2-z_1-z_2} \tag{1.19}$$

In G_1 and G_2, both the numerator and the denominator become zero at $z_1 = z_2 = 1$. It is shown that G_1 is stable whereas G_2 and G_3 are unstable. It can be seen that the numerator of Eq(1.17) has played a role in the stability of the filter in spite of the fact that the poles and zeros of Eq(1.17) and (1.18) are identical.

This difficulty of stability test and stabilization has been overcome by *circuit analogy method* [30] as follows. In 1-D recursive digital filter design a bilinear transformation is often applied to analog filter transfer function with the desired characteristics to obtain digital filter transfer function. The rationale for this approach is that it is desirable to exploit considerable body of knowledge which has been built up in analog filter theory. Bearing in mind that it would be very desirable to have a class of network functions which are guaranteed to be stable, thus avoiding repeated stability tests, the following *two-dimensional circuit analogy* can be proposed: design a 2-D analog filter and apply double bilinear transformation to obtain the 2-D recursive digital filter. This method is analogous to 1-D case, but underlying motivation is different (guaranteed stability rather than previous experience).

Later, lossless frequency independent multiport network that was terminated at n_1 of its ports in unit capacitors s_1, n_2 of its ports in unit capacitors s_2, and n_r of its ports in unit resistors have been considered [31] and they produced Strictly Hurwitz Polynomial (SHP). Then by the application of double bilinear transformation 2-D stable filters were obtained.

However, it has been shown [32] that not all two variable analog functions with SHP denominators upon bilinear transformation yield stable 2-D digital transfer functions. There exist analog functions which on bilinear transformation result in digital functions possessing singularities of the second kind on the distinguished boundary of the unit circles in the z_1, z_2 plane. Here, we present several methods of generating 2-D SHP's free of non-essential singularities of the second kind. These polynomials are called *Very Strictly Hurwitz Polynomials (VSHP)* [33]. But first we define the singularities.

It is well known that a 2-variable rational function $H(s_1,s_2)=A(s_1,s_2)/B(s_1,s_2)$ may possess two types of singularities and they may be defined as follows:

i) $H(s_1,s_2)$ is said to possess non-essential singularity of the first kind at (s_1^*,s_2^*) if $B(s_1^*,s_2^*)=0$ and $A(s_1^*,s_2^*)\neq 0$.

ii) $H(s_1,s_2)$ is said to possess non-essential singularity of the second kind at (s_1^*,s_2^*) if $A(s_1^*,s_2^*)=B(s_1^*,s_2^*)=0$.

Some difficulties have been pointed out [32] with the generation of stable 2-D digital functions using double bilinear transformation. However, it may be verified that this maps the entire s_1 and s_2 planes on the entire z_1 and z_2 planes on a one-to-one basis and so the behavior of the function is not altered by the application of bilinear transformation. For instance, if $H(z_1,z_2)$ is the transformed function of $H(s_1,s_2)$ and $H(z_1,z_2)$ has a singularity at (z_1^*,z_2^*), then $H(s_1,s_2)$ ought to have the same type of singularity at (s_1^*,s_2^*) where (s_1^*,s_2^*) is the point corresponding to (z_1^*,z_2^*). This simply means that the analog function with strict Hurwitz polynomial denominators themselves have the second kind singularities. This is not surprising in view of the fact that a two variable polynomial may possess, in addition to zeros and poles, singularities of the second kind at infinite distant poles. In the definition of SHP these second kind singularities are not taken into account. Naturally, one obvious remedy for this problem is then to define a Hurwitz polynomial which avoids the non-essential singularities of the second kind as follows [33].

Definition 1: $D(s_1,s_2)$ is a strict Hurwitz polynomial (SHP) if $1/D(s_1,s_2)$ does not possess any singularities in the region:

$$(s_1,s_2) \mid Re(s_1) \geq 0,\ Re(s_2) \geq 0,\ |s_1| < \infty, \text{ and } |s_2| < \infty$$

Definition 2: $D(s_1,s_2)$ is a Very Strict Hurwitz Polynomial (VSHP) if $1/D(s_1,s_2)$ does not possess any singularities in the region:

$$(s_1,s_2) \mid Re(s_1) \geq 0 \text{ and } Re(s_2) \geq 0$$

The test to see if an SHP is also a VSHP consists of [33] verifying the following three conditions:

i) $\lim_{s_1 \to 0} D(1/s_1, s_2) \neq 0/0$ for $Re(s_2) = 0$

ii) $\lim_{s_2 \to 0} D(s_1, 1/s_2) \neq 0/0$ for $Re(s_1) = 0$

iii) $\lim_{\substack{s_1 \to 0 \\ s_2 \to 0}} D(1/s_1, 1/s_2) \neq 0/0$

1.6 Generation of VSHP

There already exist effective methods for generation of VSHP which are as follows:

1.6.1. Generation of VSHP using properties of the derivatives of even or odd parts of Hurwitz polynomials [25]

It was mentioned that multi-variable network theory has found application in the design of two dimensional recursive digital filters. It has been shown in [31] how a 2-D stable analog filter can be designed by using the properties of the immitance function of a lossless frequency independent N-port network. This, however, has failed to deal with the possibility of generating functions with nonessential singularities of the second kind which could result in an unstable discrete filter if a bilinear transformation is applied. In [26] the earlier technique has been modified to ensure that the polynomial in the denominator of the filter remains a VSHP at all times, hence avoiding any non essential singularities of the second kind. This can be done by using the constraints for the denominator of the filter to be VSHP as penalty function in the process of optimization. However, this assurance bears a heavy cost in computation. In [25] a method has been given for ensuring that the denominator of the filter is always VSHP. This avoids the uncertainity of the method in [31]. also it does not require any constraint optimization method [28], which results in considerable saving in computation time.

The method consist of the following steps:

(i) A suitable even or odd part of an n-variable Hurwitz polynomial is generated.

(ii) The odd or even part is obtained by the corresponding derivatives, associated with it.

(iii) The resulting n-variable Hurwitz polynomial is converted to a 2-variable VSHP.

Two cases arise.

Case A: The even part of a Hurwitz polynomial as the starting point:

Consider the polynomial M_{2n} given by

$$M_{2n} = \det \left| \boldsymbol{\mu}_{2n} + \mathbf{A}_{2n} \right| \tag{1.20}$$

where $\boldsymbol{\mu}_{2n}$ is a diagonal matrix of order $2n$ and $\mathbf{A}_{2n}$ is a skew symmetric matrix of order $2n$.

From diagonal expansion of determinant of a matrix [36], M_{2n} can be written as:

$$\begin{aligned} M_{2n} = \det A_{2n} &+ \sum_{1 \le i_1 < i_2 \le 2n} \mu_{i_1}\mu_{i_2}A_{i_1i_2} + \\ &\sum_{1 \le i_1 < i_2 < i_3 < i_4 \le 2n} \mu_{i_1}\mu_{i_2}\mu_{i_3}\mu_{i_4}A_{i_1i_2i_3i_4} + \cdots \\ &+ \mu_1\mu_2\mu_3 \cdots \mu_{2n} \end{aligned} \tag{1.21}$$

where $A_{i_1 i_2}$ is the determinant of the submatrix of $\mathbf{A}_{2n}$ obtained by deleting both the i_1^{th} and i_2^{th} rows and columns, and is of order $(2n-2)$, $A_{i_1 i_2 i_3 i_4}$ is the determinant of the submatrix $\mathbf{A}_{2n}$ obtained by deleting the i_1th, i_2th, i_3th, and i_4th rows and columns and is of order $(2n-4)$, and so on.

Since the matrix, $[\mu_{2n} + \mathbf{A}_{2n}]$, is always physically realizable, M_{2n} represents the even part of a 2n-variable Hurwitz polynomial. Therefore,

$$Z_{LC} = \sum_{i=1}^{2n} k_i \frac{\dfrac{\partial M_{2n}}{\partial \mu_i}}{M_{2n}} \qquad (1.22)$$

is a reactance function. As a consequence

$$M' = M_{2n} + \sum_{i=1}^{2n} k_i \frac{\partial M_{2n}}{\partial \mu_i}. \qquad (1.23)$$

is a 2n-variable Hurwitz polynomial.

From Eq(1.23), a 2-variable VSHP can be generated by putting some of the μ's equal to s_1 and the rest of μ's equal to s_2 and also ensuring that the conditions of 2-variable VSHP are satisfied.

Case B: The odd part of a Hurwitz polynomial as the starting point.

Consider the polynomial N_{2n+1} given by

$$N_{2n+1} = \det \mid \mu_{2n+1} + \mathbf{A}_{2n+1} \mid \qquad (1.24)$$

where μ_{2n+1} is a diagonal matrix of order $(2n+1)$ and $\mathbf{A}_{2n+1}$ is a skew-symmetric matrix of order $(2n+1)$. Similar development as in Case A produces a $(2n+1)$-variable Hurwitz polynomial

$$N' = N_{2n+1} + \sum_{i=1}^{2n+1} k_i \frac{\partial N_{2n+1}}{\partial \mu_i}. \qquad (1.25)$$

from which a 2-variable VSHP can be obtained by equating some of μ's to s_1 and the remaining μ's to s_2, and also making sure that the conditions of VSHP are satisfied.

1.6.2 Generation of 2-variable VSHP's using reactance matrices [27], [35]

It is well known that a symmetric positive definite (or positive semi-definite) matrix is always physically realizable. It is further known that any positive definite matrix $\mathbf{P}$ can always be decomposed as a product of two matrices $\mathbf{Q}$ and $\mathbf{Q}^t$, where $\mathbf{Q}$ is either an upper-triangular or a lower-triangular matrix. Define the matrix

$$\mathbf{D}=\mathbf{A}\Gamma\mathbf{A}^t s_1+\mathbf{B}\Delta\mathbf{B}^t s_2+\mathbf{G} \tag{1.26}$$

where $\mathbf{A}$ and $\mathbf{B}$ are upper-triangular matrices, Γ and Δ are arbitrary diagonal non-negative matrices, and $\mathbf{G}$ is a skew-symmetric matrix.

Matrix $\mathbf{D}$ is realizable as a two-variable reactance network. Therefore $\det\mathbf{D}$ constitutes either the even part or the odd part of a two-variable Hurwitz polynomial (HP), depending on whether the order is even or odd. It has to be noted that some of the elements of Γ and Δ can be zero. Since $\mathbf{D}$ is a physically realizable matrix, we derive a two-variable Hurwitz polynomial,

$$D(s_1,s_2)=\det\mathbf{D}+k_1\frac{\partial(det\,\mathbf{D})}{\partial s_1}+k_2\frac{\partial(det\,\mathbf{D})}{\partial s_2} \tag{1.27}$$

where k_1 and k_2 are non-negative constants. The various steps are given below assuming that $\det\mathbf{D}=M$

(i) Form

$$M_1=M+k_{11}\frac{\partial M}{\partial s_1}+k_{21}\frac{\partial M}{\partial s_2} \tag{1.28}$$

This is known to be a 2-variable HP.

(ii) Now, form

$$M_2=M_1+k_{12}\frac{\partial M_1}{\partial s_1}+k_{22}\frac{\partial M_1}{\partial s_2} \tag{1.29}$$

which involves higher order partial derivatives of M. This will also be a 2-variable HP.

(iii) This process can be continued until we form

$$M_n = M_{n-1} + K_{1n} \frac{\partial M_{n-1}}{\partial s_1} + k_{2n} \frac{\partial M_{n-1}}{\partial s_2} \tag{1.30}$$

which can be shown to be a VSHP. Other VSHP's can be formed using other derivatives of M, and hence there exist a number of choices.

1.7 Phase Consideration

To overcome the short coming of linear phase, (which is easily obtained with FIR filters), three iterative methods are thought for IIR filters: cascade, parallel, and simultaneous magnitude and phase minimization.

In cascade method a filter whose impulse response is $h(n_1,n_2)$ is cascaded with a filter whose impulse response is $h(-n_1,-n_2)$. The overall impulse response of the cascade is $h(n_1,n_2)**h(-n_1,-n_2)$ and the overall frequency response is the real nonnegative function:

$$C(\omega_1,\omega_2) = |H(\omega_1,\omega_2)|^2 \tag{1.31}$$

In the parallel approach the outputs of two nonsymmetric IIR filters are added to form the final output signal. As in the cascade case, the second filter is a space reversed version of the first, so the overall frequency response is given by:

$$P(\omega_1,\omega_2) = H(\omega_1,\omega_2) + H^*(\omega_1,\omega_2) = 2Re[H(\omega_1,\omega_2)] \tag{1-32}$$

However, both cascade and parallel methods produce noncausal filters. Another way of getting linear (or any other specified) phase is simultaneous minimization of a cost function involving both magnitude and phase of the 2-D filter. And this is the approach we have adopted. We consider a constant group delay associated with an ideal filter, against which the actual filter is compared and its error is optimized.

1.8 Scope of the Thesis

In this thesis, the design of 2-D IIR filters based on the property of the slope of the reactance functions (which is known to be positive along the imaginary axes) will be discussed.

In Chapter 2, a reactance function obtained by terminated multiport gyrator is considered. The slope of this reactance function is obtained. A VSHP is generated using those properties. Some theorems are proved to determine those properties and also to ensure that the generated polynomial is VSHP. The VSHP is assigned to the denominator of a rational function thereby assuring the stability of the filter.

In Chapter 3, designs of various digital filters are discussed. The transfer function of the digital filter is obtained by the double bilinear transformation from the corresponding 2-D analog filter. The digital filters which could be product separable or non-product separable have been designed to meet the symmetry requirements and also to satisfy a prescribed magnitude with or without constant group delay specifications. This technique requires minimization of the cost function which is the mean squared error between the desired and the designed magnitude and group delay responses of the filter by means of a non-linear program.

All these filters yield coefficients with infinite precision. A method is also discussed to design integer coefficient 2-D filter. It is shown that the error with this integer coefficients does not differ considerably from that obtained with infinite precision coefficients. The sensitivity properties of the various filters are further investigated.

Chapter 4 gives the summary and conclusions. Also the possible future investigations arising from this study are outlined.

Chapter 2
Generation of 2-Variable VSHP Using the Reactance Slope Property

2.1 Introduction

This chapter will deal with generation of 2-variable VSHP, i.e., as indicated in Chapter 1, transfer functions whose poles are in left half plane, and are free of non-essential singularity of the second kind.

In Sec. 1.6.1 we mentioned the generation of VSHP using properties of the derivative of even or odd parts of Hurwitz polynomial. Two cases were discussed.

Case A: The even part of a Hurwitz polynomial as the starting point:

Consider the polynomial M_{2n} given by

$$M_{2n} = \det \mid \mu_{2n} + \mathbf{A}_{2n} \mid \tag{2.1a}$$

where μ_{2n} is a diagonal matrix of order $2n$ and $\mathbf{A}_{2n}$ is a skew symmetric matrix of order $2n$, and these are given by:

$$\mu_{2n} = diag\,[\mu_1, \mu_2, \cdots, \mu_{2n}] \tag{2.1b}$$

and

$$\mathbf{A}_{2n} = \begin{bmatrix} 0 & a_{12} & a_{13} & & a_{1,2n} \\ -a_{12} & 0 & a_{23} & \cdots & a_{2,2n} \\ -a_{13} & -a_{23} & 0 & \cdots & a_{3,2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -a_{1,2n} & -a_{2,2n} & -a_{3,2n} & \cdots & 0 \end{bmatrix} \tag{2.1c}$$

From the diagonal expansion of determinant of a matrix [36], M_{2n} can be written as:

$$M_{2n} = \det A_{2n} + \sum_{1 \le i_1 < i_2 \le 2n} \mu_{i_1}\mu_{i_2} A_{i_1 i_2} + \sum_{1 \le i_1 < i_2 < i_3 < i_4 \le 2n} \mu_{i_1}\mu_{i_2}\mu_{i_3}\mu_{i_4} A_{i_1 i_2 i_3 i_4} + \cdots \tag{2.2}$$

$$+\mu_1\mu_2\mu_3 \cdots \mu_{2n}.$$

where $A_{i_1i_2}$ is the determinant of the submatrix of $\mathbf{A}_{2n}$ obtained by deleting both the i_1^{th} and i_2^{th} rows and columns, and is of order $(2n-2)$, $A_{i_1i_2i_3i_4}$ is the determinant of the submatrix of $\mathbf{A}_{2n}$ obtained by deleting the i_1^{th}, i_2^{th}, i_3^{th}, and i_4^{th} rows and columns and is of order $(2n-4)$, and so on.

Since the matrix $[\mu_{2n} + \mathbf{A}_{2n}]$ is always physically realizable, M_{2n} represents the even part of a 2n-variable Hurwitz polynomial. Therefore,

$$Z_{LC} = \sum_{i=1}^{2n} k_i \frac{\frac{\partial M_{2n}}{\partial \mu_i}}{M_{2n}} \tag{2.3}$$

is a reactance function. As a consequence

$$M' = M_{2n} + \sum_{i=1}^{2n} k_i \frac{\partial M_{2n}}{\partial \mu_i} \tag{2.4}$$

is a 2n-variable Hurwitz polynomial.

Case B: The odd part of a Hurwitz polynomial as the starting point.

Consider the polynomial N_{2n+1} given by

$$N_{2n+1} = \det \mid \mu_{2n+1} + \mathbf{A}_{2n+1} \mid \tag{2.5}$$

where μ_{2n+1} is a diagonal matrix of order *(2n+1)* and $\mathbf{A}_{2n+1}$ is a skew-symmetric matrix of order $(2n+1)$, and these are given by:

$$\mu_{2n+1} = diag\,[\mu_1, \mu_2, \cdots, \mu_{2n+1}] \tag{2.5b}$$

and

$$\mathbf{A}_{2n+1} = \begin{bmatrix} 0 & a_{12} & a_{13} & & a_{1,2n+1} \\ -a_{12} & 0 & a_{23} & \cdots & a_{2,2n+1} \\ -a_{13} & -a_{23} & 0 & \cdots & a_{3,2n+1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -a_{1,2n+1} & -a_{2,2n+1} & -a_{3,2n+1} & \cdots & 0 \end{bmatrix} \tag{2.5c}$$

Similar development as in Case A produces the following reactance function:

$$Z_{LC} = \sum_{i=1}^{2n+1} k_i \frac{\frac{\partial N_{2n+1}}{\partial \mu_i}}{N_{2n+1}} \tag{2.6}$$

and the following $2n+1$ variable Hurwitz Polynomial:

$$N' = N_{2n+1} + \sum_{i=1}^{2n+1} k_i \frac{\partial N_{2n+1}}{\partial \mu_i} \tag{2.7}$$

Note that [31] has come out with a general 2-D SHP, but it does not always give VSHP. Instead [25] has given two particular VSHP's, i.e., Eqs(2.3) and (2.6). Here we will derive the slope of the reactance functions Eqs(2.3) and (2.6), and come out with yet another VSHP.

Thus the main purpose of this chapter is to find the the slope of Z_{LC} of Eq(2.3) and Eq(2.6) and their properties. These slopes are given in Theorem 5 and Theorem 6 which is preceded by four other theorems needed in their proofs. Some applications of Theorem 5 follow in Section 2.5 and 2.6. In Section 2.7 a new method of generation of 2-variable VSHP using the reactance slope property of Theorem 5 is indicated which is used in subsequent development of the 2-D filter in Chapter 3. We proceed with the theorems as mathematical preliminaries for proofs of Theorems 5 and 6

2.2.1 Theorem 1 : Expansion of a Pfaffian

A Pfaffian $\sqrt{A_{s_1 s_2 \cdots s_{2r}}}$ can be expanded around a pivot i as a sum of lower order Pfaffians:

$$\sqrt{A_{s_1 s_2 \cdots s_{2r}}} = \sum_{\substack{j=1 \\ j \neq i, s_1, s_2, \ldots, s_{2r}}}^{2n} (-1)^{i+j+1} \epsilon_{ij} \delta_{s_1 s_2 \cdots s_{2r}, ij} \, a_{ij} \sqrt{A_{ij s_1 s_2 \cdots s_{2r}}} \tag{2.8}$$

where:

$$\delta_{s_1 s_2 \cdots s_{2r}, ij} = \epsilon_{s_1 i}\epsilon_{s_2 i} \cdots \epsilon_{s_{2r} i}\epsilon_{s_1 j}\epsilon_{s_2 j} \cdots \epsilon_{s_{2r} j} \tag{2.9}$$

with:

$$\epsilon_{hk} = \begin{cases} +1 & \text{if } h < k \\ -1 & \text{if } h > k \end{cases} \tag{2.10}$$

Proof :

It is well known that any skew-symmetrical determinant A of odd order is zero, and of even order is a perfect square. The square root of A is given the name *Pfaffian*. [37] has proved that the Pfaffian $\sqrt{A}$, is obtained by the following rule:

> "Write down all the arrangements in pair of numbers $1,2,3,\ldots,n$. Denote any such arrangements by $(hk),(lm),\ldots,(uv)$, then:
>
> $$\sqrt{A} = \sum_{h,k,l,m,\ \ ,u,v} \pm a_{hk}\, a_{lm} \cdots a_{uv}$$
>
> where the positive or negative sign is to be taken according as the arrangement $khlm\ldots uv$ is derived from $1234\ldots n$ by an even or by an odd number of transpositions. The term $a_{12}a_{34}\ldots a_{n-1,n}$ is to have the sign $+$."

Now, let's fix h as a number in the set $\{1,2,3,\ldots,n\}$ and vary k. Then k will take all the values $1,2,3,\ldots,h-1,h+1,\ldots,n$. Thus:

$$\sqrt{A} = \sum_{\substack{k=1 \\ k\neq h}}^{n} \left(\sum_{l,m,\ldots,u,v} \pm a_{hk}\, a_{lm} \cdots a_{uv} \right)$$

But $a_{lm} \cdots a_{uv}$ is another, lower order, Pfaffian obtained from A by deleting the h and k rows and columns:

$$\sqrt{A} = \sum_{\substack{k=1 \\ k\neq h}}^{n} \left(\pm a_{hk} \sqrt{A_{hk}} \right)$$

The new Pfaffian can be expanded around another pivot different from h and k, and so on.

The arrangement: $h,k,1,2,3,\ldots,h\text{-}1,h+1,\ldots,k\text{-}1,k+1,\ldots,n$ can be derived from $1,2,3,\ldots,n$ by $h+k\text{-}3$ transpositions , and there will be another transposition if $h<k$. Therefore the rule says that the sign is $(-1)^{h+k+1}\epsilon_{hk}$, where ϵ_{hk} is given in Eq(2.10). Therefore:

$$\sqrt{A} = \sum_{\substack{k=1 \\ k\neq h}}^{n} \epsilon_{hk}\,(-1)^{h+k+1}\sqrt{A_{hk}} \;.$$

Similarly:

$$\sqrt{A_{s_1 s_2 \cdots s_{2r}}} = \sum_{\substack{j=1 \\ j\neq i, s_1, s_2, \ldots, s_{2r}}}^{n} \pm a_{ij}\sqrt{A_{ijs_1s_2 \cdots s_{2r}}} \;.$$

To determine the sign from the rule, observe that the arrangement:

$$i\,,j\,,1,2,\ldots,s_1-1,s_1+1,\ldots,s_2-1,s_2+1,\ldots,s_{2r}-1,s_{2r}+1,\ldots,n$$

is obtained from the arrangement:

$$1,2,3,\ldots,s_1-1,s_1+1,\ldots,i\,,\ldots,j\,,\ldots,s_{2r}-1,s_{2r}+1,\ldots,n$$

according to position of i and j. i.e. if $s_1<s_2<\ldots<s_l<i$, and $s_1<s_2<\ldots<s_m<j$,then the sign would be $(-1)^{i+j+1}\times(-1)^{l+m}$.

We introduced the sign δ in Eq(2.9). Accordingly since $s_1<s_2<\ldots<s_l<i$, therefore, $\delta_{s_1s_2\ldots s_l,i}=(-1)^l$ and $\delta_{s_{l+1}\ldots s_{2r},i}=+1$ thus:

$$\delta_{s_1s_2\ldots s_{2r}\,,\,i} = (-1)^l \tag{2.11}$$

Similarly, since $s_1<s_2<\ldots<s_m<j$, then $\delta_{s_1s_2\ldots s_m\,,\,j}=(-1)^m$,and $\delta_{s_{m+1}\ldots s_{2r}\,,\,j}=+1$, thus:

$$\delta_{s_1s_2\ldots s_{2r}\,,\,j} = (-1)^m \tag{2.12}$$

From Eq(2.11) and Eq(2.12) we get:

$$\delta_{s_1 s_2 \ldots s_{2r}, ij} = (-1)^{l+m}$$

And therefore the theorem is proved.

Note that even though at the first sight δ sign seems complicated, but it simplifies considerably, since then we don't have to keep the track of the various s's to determine whether they are less or greater than i or j. We extend this notation to:

$$\delta_{s_1 s_2 \ldots s_r, t_1 t_2 \ldots t_u} = \epsilon_{s_1 t_1} \cdots \epsilon_{s_r t_1} \epsilon_{s_1 t_2} \cdots \epsilon_{s_r t_2} \cdots \epsilon_{s_1 t_u} \cdots \epsilon_{s_r t_u} \tag{2.13}$$

for future use.

Next we prove that the number of addends of $\sqrt{A}$, when A has dimension $2n \times 2n$ is $1 \times 3 \times 5 \times \ldots \times (2n-1)$. Notice that total of $(2n)!$, which is the permutation of $2n$ numbers, are possible, and observe that $n!$ of them are repeated since the order is immaterial, and also note that interchange of a_{ij} with a_{ji} is taken into account by the sign, i.e., $a_{ij} = -a_{ji}$ and therefore 2^n interchange of pairs are redundant. Therefore:

$$\begin{aligned}
no.\ of\ addends\ of\ \sqrt{A} &= \frac{(2n)!}{2^n \times n!} \\
&= 1 \times 3 \times 5 \times \ldots \times (2n-1) \times \frac{2 \times 4 \times 6 \times \ldots \times 2n}{2^n \times n!} \\
&= 1 \times 3 \times 5 \times \ldots \times (2n-1) \times \frac{n!}{n!} \\
&= 1 \times 3 \times 5 \times \ldots \times (2n-1)
\end{aligned}$$

2.2.2 Theorem 2 : Number of repetitions of any element in an ascending inequality

In an "*ascending inequality*" :

$$f \leq m_1 < m_2 < \cdots < m_l \leq k$$

the number of repetitions, M, of any element N, with:

$$f \leq N \leq k$$

is:

$$M = \binom{k-f}{l-1} \tag{2.14}$$

Proof :

We define $f \leq m_1 < m_2 < m_3 < \cdots < m_l \leq k$ as an "*ascending inequality*" and the numbers m_i as ***elements*** of such an inequality. We also define l as its *index*. Obviously an element N is in the range $f \leq N \leq k$. The full expansion of such an inequality ~~is shown in Table(2.1).~~

We notice that there are as many as permutation of k-f+1 chosen l of arrangements for the inequality, i.e.:

$$Number\ of\ arrangements = \binom{k-f+1}{l} \ .$$

For each N, we partition the inequality into all of smaller inequalities which contain N, e.g., for $N=f$ we will have f in all arrangements of:

$$m = f \text{ , and } f+1 \leq m_2 < m_3 \cdots < m_l \leq k$$

or $N=f$ is repeated as much as $f+1 \leq m_1 < m_2 < \ldots < m_{l-1} \leq k$ which according to Eq(2.14) has exactly $\binom{k-f}{l-1}$ arrangements. We can proceed in this manner for $N=f+1$, $N=f+2$, and so on.

This scheme of partitioning is shown in Table(2.2) which proceeds from left for $N=f$ and terminates at right for a general element N.

Table(2.1) : Full expansion of the ascending inequality

m_1	m_2	...	m_{l-1}	m_l
f	$f+1$	...	$f+l-1$	$f+l$
f	$f+1$	...	$f+l-1$	$f+l+1$
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
f	$f+1$	...	$f+l-1$	k
f	$f+1$	...	$f+l$	$f+l+1$
f	$f+1$	...	$f+l$	$f+l+2$
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
f	$f+1$	...	$f+l$	k
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
f	$f+2$	...	$f+l$	$f+l+1$
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
f	$k-l+2$	...	$k-1$	k
$f+1$	$f+2$	...	$f+l$	$f+l+1$
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
$f+1$	$k-l+2$	...	$k-1$	k
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
$k-l+1$	$k-l+2$	...	$k-1$	k

Table(2.2) : Partitioning the ascending inequality up to element N

$$
f+1\le m_1\ldots m_{l-1}\le k
\begin{cases}
f+2\le\ldots m_{l-2}\le k \\
f+2\le\ldots m_{l-1}\le k
\end{cases}
\cdots
\begin{array}{l}
\begin{cases}
N+f+1\le\ldots m_{l-N}\le k \\
N+f+1\le\ldots m_{l-N+1}\le k
\end{cases} \\
\quad\vdots \\
\begin{cases}
N+f+1\le\ldots m_{l-2}\le k \\
N+f+1\le\ldots m_{l-1}\le k
\end{cases}
\end{array}
$$

Table(2.3) : The result of repeated application of Eq(2.15)

$$\left.\begin{matrix}\binom{k-N-f}{l-N}\\ \binom{k-N-f}{l-N+1}\end{matrix}\right\}\binom{k-N-f+1}{l-N+1}$$

$$\vdots \qquad \cdots \quad \left.\begin{matrix}\binom{k-f-1}{l-2}\\ \binom{k-f-1}{l-1}\end{matrix}\right\}\binom{k-f-1}{l-1}$$

$$\left.\begin{matrix}\binom{k-N-f}{l-2}\\ \binom{k-N-f}{l-1}\end{matrix}\right\}\binom{k-N-f+1}{l-1}$$

Now we use the following identity to sum up each pair of inequalities inside the brackets in Table(2.2):

$$
\begin{aligned}
\binom{k}{l}+\binom{k}{l-1} &= \frac{k!}{l!(k-l)!}+\frac{k!}{(l-1)!(k-l+1)!} \\
&= \frac{k!}{(l-1)!(k-l)!}\left(\frac{1}{l}+\frac{1}{k-l+1}\right) \\
&= \frac{(k+1)!}{l!(k-l+1)!} \\
&= \binom{k+1}{l}
\end{aligned}
\qquad (2.15)
$$

To sum for all arrangements which happen for N in the rightmost column of Table(2.2), the sum is simplified as we apply Eq(2.15) repeatedly by pairing consecutive rows as shown in Table(2.3). The procedure is repeated for the next column until we reach the last column which has $\binom{k-f}{l-1}$ as the number of arrangements for N. Therefore the theorem is proved.

2.2.3 Theorem 3 : Incrementing the index of ascending inequality in a sum

To increment the index of ascending inequality in a sum from l to $l+1$ we use the following relationship:

$$
\sum_{b\le m_1<\ldots<m_l\le k}\ \sum_{f=l+1}^{k-b+1} \epsilon_{ii_{m_f}}(-1)^{i_{m_f}+i+1}\delta_{i_{m_1}\ldots i_{m_l},\, ii_{m_f}}\delta_{i_{m_1}\ldots i_{m_l},\, ii_{m_{l+1}}\ldots i_{m_{k-b+1}}}
a_{ii_{m_f}}\sqrt{A_{ii_{m_1}\ldots i_{m_l}i_{m_f}}}\sqrt{A_{ii_{m_{l+1}}\ldots i_{m_{k-b+1}}}}
$$

$$
= \sum_{b\le m_1<\ldots<m_{l+1}\le k}\ \sum_{h=1}^{l+1} \epsilon_{ii_{m_h}}(-1)^{i_{m_h}+i+1}\delta_{i_{m_{l+2}}\ldots i_{m_{k-b+1}},\, ii_{m_h}}\delta_{i_{m_1}\ldots i_{m_{l+1}},\, ii_{m_{l+2}}\ldots i_{m_{k-b+1}}}
a_{ii_{m_h}}\sqrt{A_{ii_{m_{l+2}}\ldots i_{m_{k-b+1}}}}\sqrt{A_{ii_{m_{l+2}}\ldots i_{m_{k-b+1}}i_{m_h}}}
\qquad (2.16)
$$

Proof :

I) - The ascending inequality can be represented as a set W with members w_i :

$$b \le m_1 < m_2 < \cdots < m_l \le k \; = \; W = \left\{ w_1, w_2, \cdots, w_r \right\} \tag{2.17}$$

where r according to Eq(2.13) is:

$$r = \binom{k-b+1}{l} \tag{2.18}$$

Each member of W is itself another set of real positive integers:

$$w_i = \left\{ m_1^{\ i}, m_2^{\ i}, \cdots, m_l^{\ i} \right\} \tag{2.19}$$

From definition of such inequality and Table(2.1) it follows that the necessary and sufficient conditions for W to be an ascending inequality are that:

$$m_{l_1}^{\ i} \neq m_{l_2}^{\ i} \text{ for } l_1 \neq l_2 \tag{2.20}$$

and,

$$w_i \neq w_j \text{ for } i \neq j \tag{2.21}$$

That is at least one member of w_i, say $m_t^{\ i}$ of Eq(2.19), should be different from one member of w_j, say $m_p^{\ j}$. Of course w_i and w_j could have equal members but they can't have all their members equal.

We denote the complement of w_i as v_i, such that:

$$v_i \cup w_i = u \tag{2.22}$$

where,

$$u = \left\{ b, b+1, b+2, \cdots, k \right\} \tag{2.23}$$

Therefore:

$$v_i = \overline{w_i} = \left\{ m_{l+1}^{\ i}, m_{l+2}^{\ i}, \cdots, m_{k-b+1}^{\ i} \right\} \tag{2.24}$$

and,

$$b \leq m_{l+1} < m_{l+2} < \ldots < m_{k-b+1} \leq k = \left\{ v_1, v_2, \ldots, v_r \right\} \tag{2.25}$$

where, r is given by Eq(2.18). Once w_i is specified v_i can be found from Eq(2.24), and vice-versa. Therefore:

$$\sum_{b \leq m_1 < \quad < m_l \leq k} g_{i_{m_1} \ldots i_{m_l}} g_{i_{m_{l+1}} \ldots i_{m_{k-b+1}}} = \sum_{b \leq m_{l+1} < \quad < m_{k-b+1} \leq k} g_{i_{m_1} \ldots i_{m_l}} g_{i_{m_{l+1}} \ldots i_{m_{k-b+1}}} \tag{2.26}$$

II) - Let,

$$L = \sum_{f=l+1}^{k-b+1} \sum_{b \leq m_{l+1} < \cdots < m_{k-b+1} \leq k} a_{i_{m_f}}$$

$$= \sum_{b \leq m_{l+1} < \cdots < m_{k-b+1} \leq k} \left(a_{i_{m_{l+1}}} + a_{i_{m_{l+2}}} + \cdots + a_{i_{m_{k-b+1}}} \right) \tag{2.27}$$

Therefore L is a summation of of all possible a_{i_m} 's defined by the ascending inequality , and range from a_{i_b} to a_{i_k} . According to Theorem 2 Eq(2.14), each a_{i_N} is repeated $\binom{k-b}{k-l-b}$ times. Therefore:

$$L = \binom{k-b}{k-l-b} \sum_{N=b}^{k} a_{i_N} \tag{2.28}$$

Similarly:

$$R = \sum_{h=1}^{l+1} \sum_{b \leq m_1 < \cdots < m_{l+1} \leq k} a_{i_{m_h}}$$

$$= \binom{k-b}{l} \sum_{N=b}^{k} a_{i_N} \tag{2.29}$$

but,

$$\binom{k-b}{k-b-l} = \frac{(k-b)!}{(k-b-l)!\,l!}$$
$$= \binom{k-b}{l} \tag{2.30}$$

Thus:

$$L = R \tag{2.31}$$

III) - Let:

$$L' = \sum_{f=l+1}^{k-b+1} \sum_{b \le m_{l+1} < \cdots < m_{k-b+1} \le k} d_{i_{m_f}} g_{i_{m_1} \cdots i_{m_l} i_{m_f}} q_{i_{m_{l+1}} \cdots i_{m_{k-b+1}}} \tag{2.32}$$

Since f ranges from $l+1$ to k-$b+1$ and since Eq(2.32) together with Eq(2.24) imply that $b \le m_1 < \cdots < m_l \le k$ and thus all m_1 to m_l are different, therefore the set:

$$w_i' = \left\{ m_1^i, m_2^i, \cdots, m_l^i, m_f^i \right\} \tag{2.33}$$

fulfills the first condition, Eq(2.20), for $l+1$ member ascending inequality:

$$W' = b \le m_1 < m_2 < \cdots < m_l < m_{l+1} \le k \tag{2.34}$$

However the second condition Eq(2.21) is not as such satisfied, and some of w_i''s are repeated. This happens because the set w_i in Eq(2.19) is an ordered set, i.e.,

$$m_1^i < m_2^i < \cdots < m_l^i$$

but even though the first l members of the set w' in Eq(2.33) are ordered but the last member m_f makes it unordered.

Consider the case when:

$$m_f = m_{l+1} > m_l$$

Then the set:

$$w_0' = \left\{ m_1, m_2, \cdots, m_l, m_{l+1} \right\}$$

is ordered. However the unordered set w' can also take:

$$w_1' = \left\{ m_2, m_3, \cdots, m_l, m_{l+1}, m_1 \right\}$$

as well as:

$$w_2' = \left\{ m_1, m_3, m_4, \cdots, m_l, m_{l+1}, m_2 \right\}$$

and so on up to:

$$w_l' = \left\{ m_1, m_2, m_3, \cdots, m_{l-1}, m_{l+1}, m_l \right\}$$

Therefore, we see that the set is repeated from w_0 to w_l, i.e., $l+1$ times. This makes the g part in Eq(2.32) to repeat $l+1$ times, q part k-b-l+1 times, and according to Eq(2.20) d part repeats $\binom{k-b}{l}$ times.

Note that:

$$v_i' \cup w_i' = u$$

with u defined in Eq(2.23), has k-b+1 members instead of k-b+1. Therefore, we find a match for L' in Eq(2.32) as R' :

$$R' = \sum_{h=1}^{l+1} \sum_{b \leq m_1 < \cdots < m_{l+1} \leq k} d_{i_{m_h}} g_{i_{m_1} \cdots i_{m_{l+1}}} q_{i_{m_{l+2}} \cdots i_{m_{k-b+1}} i_{m_h}} \tag{2.35}$$

With the same reasoning as above we see that:

$$v_i'' = \left\{ m_{l+2}^i, m_{l+3}^i, \cdots, m_{k-b+1}^i, m_h^i \right\}$$

is the complement of:

$$w'' = \left\{ m_1, m_2, \cdots, m_l \right\}$$

but v'' has k-b-l+1 repetitions.

Therefore g part in Eq(2.35) repeats $l+1$ times, q part repeats k-b-l+1 times, and d part $\binom{k-b}{l}$ times. Also

$$u'' = v_i'' \cup w_i''$$

has k-b+1 members.

IV) - To complete the proof that $L' = R'$ it remains to prove that for any dgq in L' there is a corresponding dgq in R' ,i.e.

$$(dgq)_{L'} \to (dgq)_{R'} \tag{2.36}$$

Let α be a member of the set v_i of Eq(2.24):

$$\alpha \in v_i \tag{2.37}$$

Let w_i be the complement of v_i as given in Eq(2.23), and let w_α be the set formed by adding α to the set w_i :

$$w_\alpha = w_i \cup \alpha \tag{2.38}$$

Then:

$$(dgq)_{L'} = d_\alpha g_{w_\alpha} q_{v_i}$$

similarly:

$$(dgq)_{R'} = d_\beta g_{w_i'} q_{v_\beta'} \tag{2.39}$$

where w_i' is as Eq(2.33), and β is a member of w_i and:

$$v_\beta' = v_i' \cup \beta \tag{2.40}$$

where v_i' is the complement of w_i' . By choosing $\beta = \alpha$,we can always get for some i, $w_i' = w_\alpha = w_i \cup \alpha = \overline{v_i'}$ and:

$$v_i = v_i' \cup \alpha = v_\beta'$$

Thus Eq(2.36) holds and $L' = R'$ .Using Eq(2.26) once more we get:

$$\sum_{f=l+1}^{k-b+1} \sum_{b \le m_1 < \cdots < m_l \le k} d_{i_{m_f}} g_{i_{m_1} \ldots i_{m_l} i_{m_f}} q_{i_{m_{l+1}} \ldots i_{m_{k-b+1}}} =$$

$$\sum_{h=1}^{l+1} \sum_{b \le m_1 < \cdots < m_{l+1} \le k} d_{i_{m_h}} g_{i_{m_1} \ldots i_{m_{l+1}}} q_{i_{m_{l+2}} \ldots i_{m_{k-b+1}} i_{m_h}} \tag{2.41}$$

Now by following substitutions in Eq(2.41):

$$d_{i_{m_f}} = a_{i i_{m_f}}$$

$$g_{i_{m_1}\cdots i_{m_l} i_{m_f}} = \sqrt{A_{ii_{m_1}\cdots i_{m_l} i_{m_f}}}$$

$$q_{i_{m_{l+1}}\cdots i_{m_{k-b+1}}} = \sqrt{A_{ii_{m_{l+1}}\cdots i_{m_{k-b+1}}}}$$

$$d_{i_{m_k}} = a_{ii_{m_k}}$$

$$g_{i_{m_1}\cdots i_{m_{l+1}}} = \sqrt{A_{ii_{m_1}\cdots i_{m_{l+1}}}}$$

$$q_{i_{m_{l+2}}\cdots i_{m_{k-b+1}} i_{m_k}} = \sqrt{A_{ii_{m_{l+2}}\cdots i_{m_{k-b+1}} i_{m_k}}}$$

and since $i_{m_f} = \alpha = i_{m_k}$ therefore:

$$(-1)^{i+i_{m_f}+1}\epsilon_{ii_{m_f}} = (-1)^{i+i_{m_k}+1}\epsilon_{ii_{m_k}} \tag{2.42}$$

$$\delta_{i_{m_1}\cdots i_{m_l},\, ii_{m_f}}\delta_{i_{m_1}\cdots i_{m_l},\, ii_{m_{l+1}}\cdots i_{m_{k-b+1}}} = \delta_{i_{m_1}\cdots i_{m_l},\, i_{m_f} i_{m_{l+1}}\cdots i_{m_{k-b+1}}}$$

$$= \delta_{w,v''} \tag{2.43}$$

where, $v'' \cup \alpha = v$ and we have used $\delta_{i_{m_1}\cdots i_{m_l},\, \alpha\alpha} = +1$

$$\delta_{i_{m_{l+2}}\cdots i_{m_{k-b+1}},\, ii_{m_k}}\delta_{ii_{m_1}\cdots i_{m_{l+1}},\, i_{m_{l+2}}\cdots i_{m_{k-b+1}}} = \delta_{i_{m_1}\cdots i_{m_{l+1}} i_{m_k},\, i_{m_{l+2}}\cdots i_{m_{k-b+1}}}$$

$$= \delta_{w,v''} \tag{2.44}$$

Therefore Eq(2.16) follows. Note that the index of ascending inequality in the left of Eq(2.16) is l whereas in the right is $l+1$.

2.2.4 Theorem 4 : Product of two sums

The product R=KL, with:

$$K = \sum_{1 \le i_1 < \cdots < i_k \le n} \mu_{i_1}\mu_{i_2}\ \ \mu_{i_k} A_{ii_1 i_2\ \cdots\ i_k}$$

and,

$$L = \sum_{1\leq i_1< \cdots < i_l \leq n} \mu_{i_1}\mu_{i_2} \; \mu_{i_l} A_{ji_1i_2 \cdots i_l}$$

when $l+k\leq n$ is given by:

$$R = \sum_{p=0}^{k} \sum_{1\leq j_1<\dots<j_p\leq k+l-p} \sum_{j_{p+1}\leq m_1<\dots<m_{k-p}\leq j_{l+k-p}} \sum_{1\leq i_1<\dots<i_{k+l-p}\leq n}$$

$$\mu_{i_{j_1}}^2 \mu_{i_{j_2}}^2 \cdots \mu_{i_{j_p}}^2 \mu_{i_{j_{p+1}}} \cdots \mu_{i_{j_{l+k-p}}}$$

$$A_{ii_{j_1}i_{j_2}\cdots i_{j_p} i_{m_1} i_{m_2}\dots i_{m_{k-p}}} A_{ji_{j_1}i_{j_2}\cdots i_{j_p} i_{m_{k-p+1}} i_{m_{k-p+2}}\dots i_{m_{k+l-2p}}} \tag{2.45}$$

Proof :

Without loss of generality we assume $k\leq l$. We can have μ's to the first or second power only. We first consider the case when k of μ's are squared, to this we add the component when k-1 of μ's are squared, and so on, until the last sum which has $k+l$ distinct μ's provided that $l+k\leq n$. Writing down in this order we get:

$$R = \sum_{1\leq j_1<j_2<\dots<j_k\leq l} \sum_{1\leq i_1<i_2<\dots<i_l\leq n}$$

$$\mu_{i_{j_1}}^2 \mu_{i_{j_2}}^2 \cdots \mu_{i_{j_k}}^2 \mu_{i_{j_{k+1}}} \cdots \mu_{i_{j_l}}$$

$$A_{ii_{j_1}i_{j_2}\cdots i_{j_k}} A_{ji_{j_1}i_{j_2}\cdots i_{j_l}}$$

$$+ \sum_{1\leq j_1<j_2<\dots<j_{k-1}\leq l} \sum_{m_1=j_k}^{j_{l+1}} \sum_{1\leq i_1<i_2<\dots<i_{l+1}\leq n}$$

$$\mu_{i_{j_1}}^2 \mu_{i_{j_2}}^2 \cdots \mu_{i_{j_{k-1}}}^2 \mu_{i_{j_k}} \cdots \mu_{i_{j_l}} \, .$$

$$A_{ii_{j_1}i_{j_2}\cdots i_{j_{k-1}} i_{m_1}} A_{ji_{j_1}i_{j_2}\cdots i_{j_{k-1}} i_{m_2} \cdots i_{m_{l-k+2}}}$$

$$+ \sum_{1 \le j_1 < j_2 < \; < j_p \le l} \; \sum_{j_{p+1} \le m_1 < m_2 < \ldots < m_{k-p} \le j_{l+k-p}} \; \sum_{1 \le i_1 < i_2 < \ldots < i_{l+k-p} \le n}$$

$$\mu_{i_{j_1}}^2 \mu_{i_{j_2}}^2 \cdots \mu_{i_{j_p}}^2 \mu_{i_{j_{p+1}}} \cdots \mu_{i_{j_{l+k-p}}}$$

$$A_{ii_{j_1} i_{j_2} \cdots i_{j_p} i_{m_1} \cdots i_{m_{k-p}}} A_{ji_{j_1} i_{j_2} \cdots i_{j_p} i_{m_{k-p+1}} \cdots i_{m_{l+k-2p}}}$$

$$+ \sum_{1 \le m_1 < m_2 < \ldots < m_k \le l+k} \; \sum_{1 \le i_1 < i_2 < \ldots < i_{l+k} \le n}$$

$$\mu_{i_1} \mu_{i_2} \cdots \mu_{i_{k+l}}$$

$$A_{ii_{m_1} i_{m_2} \cdots i_{m_k}} A_{ji_{m_{k+1}} i_{m_{k+2}} \cdots i_{m_{k+l}}} \tag{2.46}$$

Note that as explained in Theorem 3 the sets such as:

$$w = \left\{ m_1, m_2, \cdots, m_k \right\}$$

whose members appear in the ascending inequality:

$$A \le m_1 < m_2 < \cdots < m_k \le B$$

are the complements of the sets such as:

$$v = \left\{ m_{k+1}, m_{k+2}, \cdots, m_{B-A+1} \right\}$$

such that:

$$w \cup v = u$$

where,

$$u = \left\{ A, A+1, A+2, \cdots, B \right\}$$

Putting Eq(2.46) in the compact form we get Eq(2.45).

As a further check of Eq(2.45), note that k has $\binom{n}{k}$ and L has $\binom{n}{l}$ addends. Therefore:

$$no. \; of \; addends \; of \; R = \binom{n}{k} \binom{n}{l} \tag{2.47}$$

The total number of addends predicted by Eq(2.45) is:

$$Addends\ of\ R\ by\ Eq\,(2.45) = \sum_{p=0}^{k} \binom{k+l-p}{p} \binom{k-l-2p}{k-p} \binom{n}{l+k-p}$$

$$= \sum_{p=0}^{k} \frac{(k+l-p)!}{p!(k+l-2p)!} \frac{(k+l-2p)!}{(k-p)!(l-p)!} \times$$

$$\times \frac{n!}{(l+k-p)!(n-l-k+p)!}$$

$$= \sum_{p=0}^{k} \frac{n!}{p!(k-p)!(l-p)!(n-l-k+p)!}$$

$$= \frac{n!}{l!(n-l)!} \sum_{p=0}^{k} \frac{l!(n-l)!}{p!(k-p)!(l-p)!(n-l-k+p)!}$$

$$= \binom{n}{l} \sum_{p=0}^{k} \binom{l}{p} \binom{n-l}{k-p} \tag{2.48}$$

But it is well known in the theory of binomial coefficients that:

$$\sum_{p=0}^{k} \binom{l}{p} \binom{n-l}{k-p} = \binom{n}{k} \tag{2.49}$$

Therefore Eq(2.48) becomes:

$$Addends\ in\ the\ right\ of\ Eq\,(2.45) = \binom{n}{l} \binom{n}{k}$$

$$= Addends\ in\ the\ left\ of\ Eq\,(2.45)$$

Therefore the theorem is proved.

2.3 Theorem 5 : The slope of the reactance function

The derivative of:

$$Z_{LC} = \sum_{i=1}^{2n} k_i \frac{\dfrac{\partial M_{2n}}{\partial \mu_i}}{M_{2n}} \tag{2.50}$$

where:

$$M_{2n} = det \mid \mu_{2n} + \mathbf{A}_{2n} \mid \tag{2.51}$$

where $\mathbf{A}_{2n}$ is a $2n \times 2n$ skew-symmetric matrix, and μ_{2n} is a diagonal matrix of order $2n$ with diagonal elements μ_d, $d=1,2,\ldots,2n$ given by Eq(2.1) is:

$$\frac{\partial Z_{LC}}{\partial \mu_j} = \sum_{i=1}^{2n} k_i \frac{N_{i,j}}{M_{2n}^2} \tag{2.52}$$

Where:

$$N_{j,j} = -\left(\sum_{l=1}^{n} P_{2l-1} \right)^2, \quad i=j \tag{2.53}$$

and,

$$N_{i,j} = \left(\sum_{l=0}^{n-1} P_{2l} \right)^2 - \left(\sum_{l=1}^{n-1} P_{2l-1} \right)^2, \quad i \neq j \tag{2.54}$$

where:

$$P_{2l} = \sum_{\substack{1 \leq i_1 \cdots < i_{2l} \leq 2n \\ i_k \neq i,j.}} \delta_{i_1 \cdots i_{2l};ij} \mu_{i_1} \cdots \mu_{i_{2l}} \sqrt{A_{i_1 \cdots i_{2l}} A_{iji_1 \cdots i_{2l}}} \tag{2.55}$$

and,

$$P_{2l-1} = \sum_{\substack{1 \leq i_1 \cdots < i_{2l-1} \leq 2n \\ i_k \neq i,j}} \delta_{i_1 \cdots i_{2l-1},ij} \mu_{i_1} \cdots \mu_{i_{2l-1}} \sqrt{A_{ii_1 \cdots i_{2l-1}} A_{ji_1 \ldots i_{2l-1}}} \tag{2.56}$$

Proof :

The derivative of the reactance function in Eq(2.50), with M_{2n} given in Eq(1.21), with respect to j-axis is:

$$\frac{\partial Z_{LC}}{\partial \mu_j} = \sum_{i=1}^{2n} k_i \frac{M_{2n} \dfrac{\partial^2 M_{2n}}{\partial \mu_i \partial \mu_j} - \dfrac{\partial M_{2n}}{\partial \mu_i} \dfrac{\partial M_{2n}}{\partial \mu_j}}{M_{2n}^2}$$

$$= \frac{\sum_{i=1}^{2n} k_i N_{i,j}}{M_{2n}^2} \tag{2.57}$$

Two cases emerge:

1 - When the derivatives of M_{2n} and Z_{LC} are with respect to the same variables, i.e., $i=j$ in Eq(2.57).

2 - When the two derivatives are with respect to different variables, i.e, $i \neq j$.

Separating these two cases Eq(2.57) becomes:

$$\frac{\partial Z_{LC}}{\partial \mu_j} = \frac{k_j N_{j,j}}{M_{2n}^2} + \frac{\sum_{\substack{i=1 \\ i \neq j}}^{2n} k_i N_{i,j}}{M_{2n}^2} \tag{2.58}$$

2.3.1 : The component of slope when $i=j$

For $i=j$ in Eq(2.57) we get:

$$N_{j,j} = M_{2n} \frac{\partial^2 M_{2n}}{\partial^2 \mu_j} - \left(\frac{\partial M_{2n}}{\partial \mu_j} \right)^2 \tag{2.59}$$

But referring to Eq(1.21), M_{2n} can be written as:

$$M_{2n} = C_0 + C_2 + \cdots + C_{2n} \;, \tag{2.60}$$

and each C can be split into two parts: a part which contains μ_j which we show it as $\mu_j E_{2l-1}$,and a part missing μ_j which we call it H_{2l} .Therefore:

$$C_{2l} = \mu_j E_{2l-1} + H_{2l} \tag{2.61}$$

Then according to Eq(1.21):

$$E_{2l-1} = \sum_{\substack{1 \leq i_1 < i_2 < \cdots < i_{2l-1} \leq 2n \\ i_k \neq j}} \mu_{i_1} \mu_{i_2} \cdots \mu_{i_{2l-1}} A_{j i_1 i_2 \cdots i_{2l-1}} \tag{2.62}$$

$$H_{2l} = \sum_{\substack{1 \leq i_1 < i_2 < \cdots < i_{2l} \leq 2n \\ i_k \neq j}} \mu_{i_1} \mu_{i_2} \cdots \mu_{i_{2l}} A_{i_1 i_2 \cdots i_{2l}} \tag{2.63}$$

Then since E_{2l-1} and H_{2l} don't have any μ_j Eq(2.59) yields:

$$N_{j,j} = -\left(E_1 + E_3 + \ldots + E_{2l-1} + \ldots + E_{2n-1} \right)^2 \tag{2.64}$$

But since $\delta_{i_1 \dots i_{2l-1},\, jj} = +1$, by comparing Eq(2.62) and Eq(2.56) we see that:

$$P_{2l-1} = E_{2l-1} \quad i = j \tag{2.65}$$

and therefore Eq(2.53) is verified.

Note that $N_{j,j}$ in Eq(2.64) is always non-negative on imaginary axis, since as is evident from Eq(2.62) E_{2l-1} is an odd function of μ's, and thus on the imaginary axis it will have a common factor of j, and $(\mathrm{j})^2 = -1$.

2.3.2 : The component of the slope when $i \neq j$

I) - Simplification of $N_{i,j}$

We write Eq(2-2) again as:

$$M_{2n} = C_0 + C_2 + \cdots + C_{2n} \tag{2.66}$$

But now each C has the following form:

$$C_{2l} = \sum_{1 \le i_1 < i_2 < \cdots < i_{2l} \le 2n} \mu_{i_1}\mu_{i_2} \cdots \mu_{i_{2l}} A_{i_1 i_2 \cdots i_{2l}}$$

$$= \mu_i \mu_j D_{2l-2} + \mu_i E_{2l-1} + \mu_j F_{2l-1} + H_{2l} \tag{2.67}$$

where:

$$D_{2l-2} = \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_{2l-2} \le 2n \\ i_k \neq i,j}} \mu_{i_1}\mu_{i_2} \cdots \mu_{i_{2l-2}} A_{iji_1 i_2 \cdots i_{2l-2}} \tag{2.68}$$

$$E_{2l-1} = \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_{2l-1} \le 2n \\ i_k \neq i,j}} \mu_{i_1}\mu_{i_2} \cdots \mu_{i_{2l-1}} A_{ii_1 i_2 \cdots i_{2l-1}} \tag{2.69}$$

$$F_{2l-1} = \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_{2l-1} \le 2n \\ i_k \neq i,j}} \mu_{i_1}\mu_{i_2} \cdots \mu_{i_{2l-1}} A_{ji_1 i_2 \cdots i_{2l-1}} \tag{2.70}$$

$$H_{2l} = \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_{2l} \le 2n \\ i_k \neq i,j}} \mu_{i_1}\mu_{i_2} \cdots \mu_{i_{2l}} A_{i_1 i_2 \cdots i_{2l}} \tag{2.71}$$

Note that in Eq(2.66) $C_0 = A = H_0$,and $C_{2n} = \mu_1 \dots \mu_{2n} = \mu_i \mu_j D_{2n-2}$

Therefore, according to Eq(2.57):

$$N_{i,j}=M_{2n}\sum_{l=0}^{n}D_{2l-2}-\sum_{l=0}^{n}(D_{2l-2}\mu_j+E_{2l-1})\sum_{l=0}^{n}(D_{2l-2}\mu_i+F_{2l-1}) \qquad (2.72)$$

According to Eqs(2.66) and (2.67):

$$M_{2n}=\sum_{l=0}^{n}\mu_i\mu_j D_{2l-2}+\mu_i E_{2l-1}+\mu_j F_{2l-1}+H_{2l} \qquad (2.73)$$

substituting Eq(2.73) in Eq(2.72) we get, after some cancellations:

$$N_{i,j}=\sum_{l_1=0}^{n}D_{2l_1-2}\sum_{l_2=0}^{n}H_{2l_2}-\sum_{l_1=0}^{n}E_{2l_1-1}\sum_{l_2=0}^{n}F_{2l_2-1} \qquad (2.74)$$

Considering that $D_{-2}=D_{2n}=H_{2n}=E_{-1}=E_{2n-1}=F_{-1}=F_{2n-1}=0$, we get from Eq(2.74):

$$N_{i,j}=\sum_{l_1=0}^{n-1}D_{2l_1}\sum_{l_2=0}^{n-1}H_{2l_2}-\sum_{l_1=1}^{n-1}E_{2l_1-1}\sum_{l_2=1}^{n-1}F_{2l_2-1} \qquad (2.75)$$

An example is given below:

Example 1 :

Let n=2, then we should have:

$$(D_0+D_2)(H_0+H_2)-E_1F_1=(P_0+P_2)^2-P_1^2$$

And since:

$$D_0=A_{ij} \quad D_2=\mu_{i_1}\mu_{i_2}$$

$$H_0=A \quad H_2=\mu_{i_1}\mu_{i_2}A_{i_1i_2}$$

$$E_1=\mu_{i_1}A_{ii_1}+\mu_{i_2}A_{ii_2} \quad F_1=\mu_{i_1}A_{ji_1}+\mu_{i_2}A_{ji_2}$$

$$P_0=\sqrt{AA_{ij}} \quad P_1=\delta_{i_1,ij}\,\mu_{i_1}\sqrt{A_{ii_1}A_{ji_1}}+\delta_{i_2,ij}\,\mu_{i_2}\sqrt{A_{ii_2}A_{ji_2}}$$

$$P_2=\delta_{i_1i_2,ij}\,\mu_{i_1}\mu_{i_2}\sqrt{A_{i_1i_2}}$$

where,

$$1\le i_1<i_2\le 4$$
$$i_l\neq i,j$$

Therefore:

$$D_0H_0=AA_{ij}=P_0^2$$

Considering that:

$$F_1E_1=X_1+X_2$$

where,

$$X_1=\mu_{i_1}\mu_{i_2}A_{ji_1}A_{ii_2}+\mu_{i_1}\mu_{i_2}A_{ii_1}A_{ji_2}$$

$$X_2=\mu_{i_1}^2A_{ji_1}A_{ii_1}+\mu_{i_2}^2A_{ii_2}A_{ji_2}$$

And,

$$P_1^2=Y_1+Y_2$$

where

$$Y_1=2\delta_{i_1i_2,ij}\,\mu_{i_1}\mu_{i_2}\sqrt{A_{ii_1}A_{ji_1}A_{ii_2}A_{ji_2}}$$

$$Y_2=\mu_{i_1}^2A_{ii_1}A_{ji_1}+\mu_{i_2}^2A_{ii_2}A_{ji_2}$$

Equating the coefficients of $\mu_{i_1}\mu_{i_2}$, we get

$$D_0H_2+H_0D_2-X_1=2P_0P_2-Y_2$$

Thus, we have to verify that,

$$A_{ij}\,A_{i_1i_2}+A-A_{ji_1}A_{ii_2}-A_{ii_1}A_{ji_2}=\ldots$$

$$\ldots 2\sqrt{AA_{ij}}\,\delta_{i_1i_2,ij}\sqrt{A_{i_1i_2}}-2\delta_{i_1i_2,ij}\sqrt{A_{ii_1}A_{ji_1}A_{ii_2}A_{ji_2}}$$

We see that

$$A_{ji_1}A_{ii_2}+A_{ii_1}A_{ji_2}-2\delta_{i_1i_2,ij}\sqrt{A_{ii_1}A_{ji_1}A_{ii_2}A_{ji_2}}=(\gamma_1\sqrt{A_{ji_1}A_{ii_2}}+\gamma_2\sqrt{A_{ii_1}A_{ji_2}})^2$$

where:

$$\gamma_1=\delta_{ji_1,ii_2}\quad \gamma_2=\delta_{ii_1,ji_2}\quad \gamma_1\gamma_2=-\delta_{i_1i_2,ij}=\gamma_3$$

and,

$$A_{ij}\,A_{i_1i_2}+A-2\sqrt{AA_{ij}\,A_{i_1i_2}}\,\delta_{i_1i_2,ij}=\ldots$$

$$\ldots(\gamma_3\sqrt{A_{ij}\,A_{i_1i_2}}+\sqrt{A}\,)^2$$

Therefore, we have to prove that:

$$\gamma_1\sqrt{A_{ji_1}A_{ii_2}}+\gamma_2\sqrt{A_{ii_1}A_{ji_2}}=\gamma_3\sqrt{A_{ij}A_{i_1i_2}}+\sqrt{A} \tag{2.76}$$

But:

$$\sqrt{A}=\sum_{\substack{i_1=1\\ i_1\neq i}}^{4} a_{ii_1}(-1)^{i+i_1+1}\epsilon_{ii_1}\sqrt{A_{ii_1}}$$

$$=a_{ij}(-1)^{i+j+1}\epsilon_{ij}\sqrt{A_{ij}}$$

$$+a_{ii_1}(-1)^{i+i_1+1}\epsilon_{ii_1}\sqrt{A_{ii_1}}$$

$$+a_{ii_2}(-1)^{i+i_2+1}\epsilon_{ii_2}\sqrt{A_{ii_2}}$$

and,

$$\sqrt{A_{i_1i_2}}=(-1)^{i+j+1}\epsilon_{ij}\,a_{ij}\,\delta_{i_1i_2,ij}$$

$$\sqrt{A_{ji_1}}=(-1)^{i+i_2+1}\epsilon_{ii_2}a_{ii_2}\delta_{ji_1,ii_2}$$

$$\sqrt{A_{ji_2}}=(-1)^{i+i_1+1}\epsilon_{ii_1}a_{ii_1}\delta_{ji_2,ii_1}$$

Therefore Eq(2.76) follows.

For $\mu_{i_l}^2$ we clearly see that $X_2=Y_2$.

For $\mu_{i_1}^2\mu_{i_2}^2$ we have:

$$D_2H_2=\mu_{i_1}^2\mu_{i_2}^2A_{i_1i_2}=P_2^2$$

And, therefore, we have proved that for $n=2$ Eqs(2.54) and (2.75) are the same.

2.3.3) - Compact form of the equation.

To show that $N_{i,j}$, in Eq(2.75) is equivalent to that in Eq(2.54) we have to perform the product of the sums by use of Theorem 4. According to Theorem 4 the product KL, where K has k variables and L has l variables, has $l+k$ variables of which p of them are squares, where p is a number from zero to k. That is, KL will contain:

$$M_{l+k,p} = \mu_{i_{j_1}}^2 \mu_{i_{j_2}}^2 \cdots \mu_{i_{j_p}}^2 \mu_{i_{j_{p+1}}} \cdots \mu_{i_{j_{l+k-p}}}$$

Moreover it is seen from Eqs(2.68)-(2.71) and Eqs(2.55) and (2.56) that the subscript in $D,E,F,H,$ and P are the number of independent variables, μ_i's, in them and from Eqs(2.75) and (2.54) it is seen that we only encounter terms like $M_{2k,p}$, where:

$$M_{2k,p} = \mu_{i_{j_1}}^2 \mu_{i_{j_2}}^2 \cdots \mu_{i_{j_p}}^2 \mu_{i_{j_{p+1}}} \cdots \mu_{i_{j_{2k-p}}} \tag{2.77}$$

The terms containing $M_{2k,p}$ in Eq(2.75) and (2.54) are distinguished as:

$$\textit{Terms in } H_{2t} D_{2k-2t} \textit{ containing } M_{2k,p} = X_{2t,2k} \tag{2.78}$$
$$\textit{Terms in } H_{2k-2t} D_{2t} \textit{ containing } M_{2k,p} = X'_{2t,2k} \tag{2.79}$$
$$\textit{Terms in } P_{2t} P_{2k-2t} \textit{ containing } M_{2k,p} = Y_{2t,2k} \tag{2.80}$$
$$\textit{Terms in } E_{2t+1} F_{2k-2t-1} \textit{ containing } M_{2k,p} = Z_{2t+1,2k} \tag{2.81}$$
$$\textit{Terms in } E_{2k-2t-1} F_{2t+1} \textit{ containing } M_{2k,p} = Z'_{2t+1,2k} \tag{2.82}$$

Equating the terms with $M_{2k,p}$ in Eqs(2.75) and (2.54), we distinguish two cases for k even and odd. We first consider k even.

Notice that for $2t=k$:

$$H_{2t} D_{2k-2t} = H_{2k-2t} D_{2t} = H_k D_k \tag{2.83}$$

Or, in other words, $X_{k,2k} = X'_{k,2k}$, and therefore,

$$\begin{aligned} X_{k,2k} + X'_{k,2k} - 2Y_{k,2k} &= 2(X_{k,2k} - Y_{k,2k}) \\ &= R_{1,k,2k} \end{aligned} \tag{2.84}$$

Equating the coefficients of $M_{2k,p}$ in Eqs(2.75) and (2.54) we get:

$$\frac{1}{2} R_{1,k,2k} + \sum_{t=0}^{\frac{k-2}{2}} R_{1,2t,2k} - \sum_{t=0}^{\frac{k-2}{2}} R_{2,2t+1,2k} = 0 \quad k < n \tag{2.85}$$

where $R_{1,k,2k}$ is given by Eq(2.84) and

$$R_{1,2t,2k} = X_{2t,2k} + X'_{2t,2k} - 2Y_{2t,2k} \tag{2.86}$$

$$R_{2,2t+1,2k} = Z_{2t+1,2k} + Z'_{2t+1,2k} - 2Y_{2t+1,2k} \tag{2.87}$$

The largest value for k as is seen from Eq(2.75) is $k=2n-2$, and with this value of k we only get $\frac{1}{2}R_{1,k,2k} = 0$, with $M_{2k,k}$. Therefore:

$$0 \le k \le 2n-2 \tag{2.88}$$

For $k \ge n$ we can say:

$$k = n + r \qquad 0 \le r \le n-2 \tag{2.89}$$

Since in $H_{2t} D_{2k-2t}$ we should have $2k-2t \le 2n-2$, Therefore:

$$t \ge r+1 \tag{2.90}$$

Then we have:

$$\frac{1}{2}R_{1,k,2k} + \sum_{t=r+1}^{\frac{k-2}{2}} (R_{1,2t,2k} - R_{2,2t+1,2k}) = 0 \quad n \le r \le 2n-4 \quad 2n-4 \ge n \tag{2.91}$$

where, r is found from Eq(2.89): $r=k-n$ and,

$$R_{1,k,2k} = 0 \quad k=2n-2 \text{ , with } M_{2k,k} \tag{2.92}$$

meaning that :

$$H_{2n-2} D_{2n-2} = P^2_{2n-2}$$

which is trivial. To illustrate take $n=10$, then Eq(2.75) gives:

$$(H_0 + \cdots + H_{18})(D_0 + \cdots + D_{18}) \tag{2.93}$$

For $k=12$ we get $r=2$ and:

$$\frac{1}{2}R_{1,12,24} + \sum_{t=3}^{5} R_{1,2t,24} - R_{2,2t+1,24} = 0 \tag{2.94}$$

which is checked by considering $M_{12,p}$ in Eq(2.93) which is:

$$H_{12}D_{12} + (H_6 D_{18} + H_8 D_{16} + H_{10} D_{14}) \tag{2.95}$$

We first consider $k<n$. From Eq(2.45) of Theorem 4 and definition of $X_{2t,2k}$ in Eq(2.78) with H and D as given by Eqs(2.71) and (2.68) respectively we conclude that :

$$X_{2t,2k} = \sum_J \sum_{\Lambda_{2t}} \sum_I M_{2k,p}\, s^2 \tag{2.96}$$

where:

$$\sum_J \equiv \sum_{1 \le j_1 < j_2 \cdots < j_p \le 2k-p} \tag{2.97}$$

$$\sum_{\Lambda'} \equiv \sum_{j_{p+1} \le m_1 < m_2 \cdots < m_{s-p} \le j_{2k-p}} \tag{2.98}$$

$$\sum_I \equiv \sum_{\substack{1 \le i_1 < i_2 \cdots < i_{2k-p} \le 2n \\ i_l \ne i,j}} \tag{2.99}$$

$$s^2 \equiv A_{i_{j_1} \dots i_{j_p} i_{m_1} \dots i_{m_{2t-p}}} A_{iji_{j_1} \dots i_{j_p} i_{m_{2t-p+1}} \dots i_{m_{2k-2p}}} \tag{2.100}$$

Similarly:

$$X'_{2t,2k} = \sum_J \sum_{\Lambda_{2t}} \sum_I M_{2k,p}\, s'^2 \tag{2.101}$$

$$Z_{2t+1,2k} = \sum_J \sum_{\Lambda_{2t+1}} \sum_I M_{2k,p}\, u^2 \tag{2.102}$$

$$Z'_{2t+1,2k} = \sum_J \sum_{\Lambda_{2t+1}} \sum_I M_{2k,p}\, u'^2 \tag{2.103}$$

$$Y_{2t,2k} = \sum_J \sum_{\Lambda_{2t}} \sum_I M_{2k,p}\, \beta_p\, ss' \tag{2.104}$$

$$Y_{2t+1,2k} = \sum_J \sum_{\Lambda_{2t+1}} \sum_I M_{2k,p}\, \beta_p\, uu' \tag{2.105}$$

where:

$$s'^2 \equiv A_{iji_{j_1} \dots i_{j_p} i_{m_1} \dots i_{m_{2t-p}}} A_{i_{j_1} \dots i_{j_p} i_{m_{2t-p+1}} \dots i_{m_{2k-2p}}} \tag{2.106}$$

$$u^2 \equiv A_{ii_{j_1} \dots i_{j_p} i_{m_1} \dots i_{m_{2t-p}}} A_{ji_{j_1} \dots i_{j_p} i_{m_{2t-p+1}} \dots i_{m_{2k-2p}}} \tag{2.107}$$

$$u'^2 \equiv A_{ji_{j_1}\cdots i_{j_p} i_{m_1}\cdots i_{m_{2t-p}}} A_{ii_{j_1}\cdots i_{j_p} i_{m_{2t-p+1}}\cdots i_{m_{2t-2p}}} \qquad (2.108)$$

$$\beta_p \equiv \delta_{i_{m_1}\cdots i_{m_{2t-2p}},ij} = (\pm 1) \qquad (2.109)$$

From Eqs(2.86),(2.96), (2.101), and (2.104) we get:

$$R_{1,2t,2k} = X_{2t,2k} + X'_{2t,2k} - 2Y_{2t,2k}$$

$$= \sum_p \sum_J \sum_{\Lambda_{2t}} \sum_I M_{2k,p} (s^2 + s'^2 - 2\beta_p ss') \qquad (2.110)$$

Note that s, s', u, and u' are all products of two Pfaffians, i.e., they are of the form:

$$w = \sqrt{A_{J\Lambda_1} A_{J\Lambda_2}} \qquad (2.111)$$

where J, Λ_1, and Λ_2 represent all rows and columns deleted from the determinant A. Some simplifications arise by introducing the sign function:

$$\gamma(w) = \delta_{J\Lambda_1, J\Lambda_2}$$
$$= \delta_{\Lambda_1,\Lambda_2} \qquad (2.112)$$

where δ is defined by Eq(2.13). For instance, s in Eq(2.100) has:

$$J = i_{j_1} i_{j_2} \cdots i_{j_p}$$
$$\Lambda_1 = i_{m_1} i_{m_2} \cdots i_{m_{2t-p}} \qquad (2.113)$$
$$\Lambda_2 = iji_{m_{2t-p+1}} \cdots i_{m_{2t-2p}}$$

and, therefore:

$$\gamma(s) = \delta_{i_{m_1}\cdots i_{m_{2t-p}}, iji_{m_{2t-p+1}}\cdots i_{m_{2t-2p}}} \qquad (2.114)$$

With this notation Eq(2.110) takes the form:

$$R_{1,2t,2k} = \sum_J \sum_{\Lambda_{2t}} \sum_I M_{2k,p} (s\,\gamma(s) - s'\gamma(s'))^2 \qquad (2.115)$$

Following the same procedure we obtain:

$$R_{2,2t+1,2k} = \sum_J \sum_{\Lambda_{2t+1}} \sum_I M_{2k,p} (u\,\gamma(u) + u'\gamma(u'))^2 \qquad (2.116)$$

From Eq(2.98):

$$\Lambda_a = j_{p+1} \leq m_1' < \cdots < m_{a-p} \leq j_{2k-p}$$

and by Theorem 2 the ascending inequality Λ_a has $\binom{2k-2p-1}{a-p-1}$ of element j_{p+1}, and only m_1 can take value j_{p+1}. Grouping Λ_a as to those with element j_{p+1} and those without it, from Eqs(2.96)-(2.109) and Eqs(2.85) and (2.86) we get:

$$\sum_{t=0}^{\frac{k-2}{2}} R_{1,2t,2k} = \sum_{t=0}^{\frac{k-2}{2}} \sum^{2t} (a_{2t} - a_{2t}')^2 + \sum_{t=0}^{\frac{k-2}{2}} \sum^{2t} (b_{2t-1} - b_{2t-1}')^2 \qquad (2.117)$$

where:

$$\sum^{a} \equiv \sum_{J} \sum_{\Lambda_a} \sum_{I} \qquad (2.118)$$

with the sums the same as Eqs(2.97) and (2.99) except that now:

$$\sum_{\Lambda_a} \equiv \sum_{j_{p+2} \leq m_1 \ldots < m_{a-p} \leq j_{2t-p}} \qquad (2.119)$$

$$a_{2t} = \gamma(g)\sqrt{g} \qquad (2.120)$$

$$g = A_{i_{j_1} \ldots i_{j_p} i_{m_1} \ldots i_{m_{2t-p}}} A_{iji_{j_1} \ldots i_{j_p} i_{j_{p+1}} i_{m_{2t-p+1}} \ldots i_{m_{2k-2p-1}}} \qquad (2.121)$$

$$a_{2t}' = \gamma(g')\sqrt{g'} \qquad (2.122)$$

$$g' = A_{iji_{j_1} \ldots i_{j_p} i_{m_1} \ldots i_{m_{2t-p}}} A_{i_{j_1} \ldots i_{j_p} i_{j_{p+1}} i_{m_{2t-p+1}} \ldots i_{m_{2k-2p-1}}} \qquad (2.123)$$

$$b_{2t+1} = \gamma(h)\sqrt{h} \qquad (2.124)$$

$$h = A_{i_{j_1} \ldots i_{j_p} i_{j_{p+1}} i_{m_1} \ldots i_{m_{2t-p+1}}} A_{iji_{j_1} \ldots i_{j_p} i_{m_{2t-p+2}} \ldots i_{m_{2k-2p-1}}} \qquad (2.125)$$

$$b_{2t+1}' = \gamma(h')\sqrt{h'} \qquad (2.126)$$

$$h' = A_{iji_{j_1} \ldots i_{j_p} i_{j_{p+1}} i_{m_1} \ldots i_{m_{2t-p+1}}} A_{i_{j_1} \ldots i_{j_p} i_{m_{2t-p+2}} \ldots i_{m_{2k-2p-1}}} \qquad (2.127)$$

Since we have $2t$ rows and columns removed from A in $A_{i_{m_1}\cdots i_{m_{2t}}}$ of a_{2t} and $2k\text{-}2t$ in b'_{2t-1}, therefore:

$$\sum^{2t-1} b'_{2t-1} = \sum^{2k-2t} a_{2k-2t} \qquad t \geq 1 \tag{2.128}$$

which yields:

$$\sum^{2t} a_{2t} = \sum^{2k-2t-1} b'_{2k-2t-1}$$

therefore:

$$\sum_{t=0}^{\frac{k-2}{2}} \sum^{2t} a_{2t} = \sum_{t=0}^{\frac{k-2}{2}} \sum^{2k-2t-1} b'_{2k-2t-1} \tag{2.129}$$

But since:

$$\sum_{\alpha=\gamma_1}^{\gamma_2} b'_\alpha = \sum_{\beta=0}^{\gamma_2-\gamma_1} b'_{\beta+\gamma_1} \tag{2.130}$$

Therefore:

$$\sum_{t=0}^{\frac{k-2}{2}} \sum^{2t} a_{2t} = \sum_{t=0}^{\frac{k-2}{2}} \sum^{2k-2t-1} b'_{2k-2t-1}$$

$$= \sum_{t=k/2}^{k-1} \sum^{2t+1} b'_{2t+1} \tag{2.131}$$

Similarly:

$$\sum_{t=0}^{\frac{k-2}{2}} \sum^{2t+1} b'_{2t+1} = \sum^{2t} \sum_{t=k/2}^{k-1} a_{2t} \tag{2.132}$$

Using Eq(2.130) we get:

$$\sum_{t=1}^{\frac{k-2}{2}} \sum^{2t-1} (b_{2t-1} - b'_{2t-1})^2 = \sum_{t=0}^{\frac{k-4}{2}} \sum^{2t+1} (b_{2t+1} - b'_{2t+1})^2 \tag{2.133}$$

We establish:

$$1/2R_{1,k,2k} = \sum^{k-1} (b_{k-1} - b'_{k-1})^2 \tag{2.134}$$

Substituting Eqs(2.133) and (2.134) into Eq(2.117) we get:

$$1/2R_{1,k,2k} + \sum_{t=0}^{\frac{k-2}{2}} R_{1,2t,2k} = \sum_{t=0}^{\frac{k-2}{2}} \sum^{2t} (a_{2t} - a'_{2t})^2 + \sum_{t=0}^{\frac{k-2}{2}} \sum^{2t+1} (b_{2t+1} - b'_{2t+1})^2 \quad (2.135)$$

Now we substitute Eq(2.132) into Eq(2.135):

$$1/2R_{1,k,2k} + \sum_{t=0}^{\frac{k-2}{2}} R_{1,2t,2k} = \sum_{t=0}^{k-1} \sum^{2t} (a_{2t} - a'_{2t})^2 \quad (2.136)$$

Similarly:

$$R_{2,2t+1,2k} = \sum^{2t} (c_{2t} + c'_{2t})^2 + \sum^{2t+1} (d_{2t+1} + d'_{2t+1})^2 \quad (2.137)$$

where:

$$c_{2t} = \gamma(p)\sqrt{p} \quad (2.138)$$

$$p = A_{ii_{j_1} \ldots i_{j_p} i_{j_{p+1}} i_{m_1} \ldots i_{m_{2t-p}}} A_{ji_{j_1} \ldots i_{j_p} i_{m_{2t-p+1}} \ldots i_{m_{2k-2p-1}}} \quad (2.139)$$

$$c'_{2t} = \gamma(p')\sqrt{p'} \quad (2.140)$$

$$p' = A_{ji_{j_1} \ldots i_{j_p} i_{j_{p+1}} i_{m_1} \ldots i_{m_{2t-p}}} A_{ii_{j_1} \ldots i_{j_p} i_{m_{2t-p+1}} \ldots i_{m_{2k-2p-1}}} \quad (2.141)$$

$$d_{2t+1} = \gamma(q)\sqrt{q} \quad (2.142)$$

$$q = A_{ii_{j_1} \ldots i_{j_p} i_{m_1} \ldots i_{m_{2t+1-p}}} A_{ji_{j_1} \ldots i_{j_p} i_{j_{p+1}} i_{m_{2t-p+2}} \ldots i_{m_{2k-2p-1}}} \quad (2.143)$$

$$d'_{2t+1} = \gamma(q')\sqrt{q'} \quad (2.144)$$

$$q' = A_{ji_{j_1} \ldots i_{j_p} i_{m_1} \ldots i_{m_{2t+1-p}}} A_{ii_{j_1} \ldots i_{j_p} i_{j_{p+1}} i_{m_{2t-p+2}} \ldots i_{m_{2k-2p-1}}} \quad (2.145)$$

Proceeding as we did for Eqs(2.131) and (2.132), we get:

$$\sum^{2t-1} d'_{2t-1} = \sum^{2k-2t} c_{2k-2t} \qquad t \geq 1 \quad (2.146)$$

$$\sum_{t=0}^{\frac{k-2}{2}} \sum^{2t} c_{2t} = \sum_{t=k/2}^{k-1} \sum^{2t+1} d'_{2t+1} \quad (2.147)$$

$$\sum_{t=0}^{\frac{k-2}{2}} \sum^{2t+1} d'_{2t+1} = \sum_{t=k/2}^{k-1} \sum^{2t} c_{2t} \tag{2.148}$$

And, therefore, Eq(2.85) gives, for k even:

$$\sum_{t=0}^{k-1} \sum^{2t} (a_{2t} - a'_{2t})^2 = \sum_{t=0}^{k-1} \sum^{2t} (c_{2t} + c'_{2t})^2 \tag{2.149}$$

Now we consider k odd:

$$\sum_{t=0}^{\frac{k-1}{2}} R_{1,2t,2k} - \sum_{t=0}^{\frac{k-3}{2}} R_{2,2t+1,2k} - \frac{1}{2} R_{2,k,2k} = 0 \tag{2.150}$$

Similar to development leading to Eqs(2.133) and (2.134), we obtain:

$$\sum^{2t-1} b'_{2t-1} = \sum^{2k-2t} a_{2k-2t} \quad t \geq 1 \tag{2.151}$$

$$\sum_{t=0}^{\frac{k-3}{2}} \sum^{2t} a_{2t} = \sum_{t=0}^{\frac{k-3}{2}} \sum^{2k-2t-1} b'_{2k-2t-1}$$

$$= \sum_{t=\frac{k+1}{2}}^{k-1} \sum^{2t+1} b'_{2t+1} \tag{2.152}$$

$$1/2 R_{2,k,2k} = \sum^{k-1} (c_{k-1} + c'_{k-1})^2 \tag{2.153}$$

Therefore Eq(2.150) gives:

$$\sum_{t=0}^{\frac{k-1}{2}} \sum^{2t} (a_{2t} - a'_{2t})^2 + \sum_{t=0}^{\frac{k-1}{2}} \sum^{2t-1} (b_{2t-1} - b'_{2t-1})^2 -$$

$$\sum_{t=0}^{\frac{k-3}{2}} \sum^{2t} (c_{2t} + c'_{2t})^2 - \sum_{t=0}^{\frac{k-3}{2}} \sum^{2t+1} (d_{2t+1} + d'_{2t+1})^2 - \sum^{k-1} (c_{k-1} + c'_{k-1})^2 = 0 \tag{2.154}$$

or:

$$\sum_{t=0}^{\frac{k-1}{2}} \sum^{2t} (a_{2t} - a'_{2t})^2 + \sum_{t=0}^{\frac{k-3}{2}} \sum^{2t+1} (b_{2t+1} - b'_{2t+1})^2 -$$

$$\sum_{t=0}^{\frac{k-1}{2}} \sum^{2t} (c_{2t} + c'_{2t})^2 - \sum_{t=0}^{\frac{k-3}{2}} \sum^{2t+1} (d_{2t+1} + d'_{2t+1})^2 = 0 \tag{2.155}$$

Using variations of Eq(2.152) we get:

$$\sum_{t=0}^{k-1} \sum^{2t} (a_{2t} - a'_{2t})^2 = \sum_{t=0}^{k-1} \sum^{2t} (c_{2t} + c'_{2t})^2 \ . \tag{2.156}$$

Therefore Eq(2.149) is the same for k even or odd.

Similar development for $2n-4 \geq k \geq n$ leads to:

$$\sum_{t=r+1}^{k-r-2} \sum^{2t} (a_{2t} - a'_{2t})^2 = \sum_{t=r+1}^{k-r-2} \sum^{2t} (c_{2t} + c'_{2t})^2 \tag{2.157}$$

2.3.4) - Mathematical Induction.

The smallest dimension that the determinant A can take is $2n=2$. And with this dimension we have $p=0$, and from Eqs(2.120) and (2.121) we get:

$$a_{2t} = a_0 = \sqrt{A} = a_{ij}$$

and from Eqs(2.122) and (2.123) we get:

$$a'_{2t} = a'_0 = \sqrt{A} = a_{ij}$$

Similarly Eqs(2.138) and (2.139) yield:

$$c_{2t} = c_0 = 0$$

and Eqs(2.140) and (2.141) gives:

$$c'_{2t} = c'_0 = 0$$

Therefore for the smallest n, Eq(2.149) is satisfied.

Now we proceed to prove that if Eq(2.149) is valid for $(2n-2)\times(2n-2)$ determinant A_{ij} then it is also valid for $2n \times 2n$ determinant A.

We accomplish this by assuming that Eq(2.149) holds for the case when we remove i, and j rows and columns from A, and if i or j or both row(s) and column(s) are already taken away then we remove s_1, s_2, or both row(s), and column(s). That is, if we encounter subscript(s) i, or j, or both in A we add subscript(s) s_1 or s_2 or both, e.g.:

$$A \rightarrow A_{ij}$$

and:

$$A_{ij} \rightarrow A_{ijs_1s_2}$$

Therefore, by assuming that Eq(2.149) is valid for special case when Eqs(2.120) and (2.121) become:

$$a_{2t} \rightarrow S_a = \gamma(g_1)\sqrt{g_1} \tag{2.158}$$

$$g_1 = A_{iji_{j_1}\dots i_{j_p} i_{m_1}\dots i_{m_{2t-p}}} A_{iji_{j_1}\dots i_{j_{p+1}} i_{m_{2t-p+1}}\dots i_{m_{2k-2p-1}} s_1 s_2} \tag{2.159}$$

and Eqs(2.122) and (2.123) become:

$$a'_{2t} \rightarrow S_{a'} = \gamma(g'_1)\sqrt{g'_1} \tag{2.160}$$

$$g'_1 = A_{iji_{j_1}\dots i_{j_p} i_{m_1}\dots i_{m_{2t-p}} s_1 s_2} A_{iji_{j_1}\dots i_{j_{p+1}} i_{m_{2t-p+1}}\dots i_{m_{2k-2p-1}}}. \tag{2.161}$$

and Eqs(2.138) and (2.139) become:

$$c_{2t} \rightarrow S_c = \gamma(p_1)\sqrt{p_1} \tag{2.162}$$

$$p_1 = A_{ii_{j_1}\dots i_{j_{p+1}} i_{m_1}\dots i_{m_{2t-p}} s_1} A_{ji_{j_1}\dots i_{j_p} i_{m_{2t-p+1}}\dots i_{m_{2k-2p-1}} s_2} \tag{2.163}$$

and Eqs(2.140) and (2.141) become:

$$c'_{2t} \rightarrow S_{c'} = \gamma(p'_1)\sqrt{p'_1} \tag{2.164}$$

$$p'_1 = A_{ji_{j_1}\dots i_{j_{p+1}} i_{m_1}\dots i_{m_{2t-p}} s_2} A_{ii_{j_1}\dots i_{j_p} i_{m_{2t-p+1}}\dots i_{m_{2k-2p-1}} s_1} \tag{2.165}$$

where, the subscript s_1 in above equations can take any integer from 1 to $2n$ except the integers in the set:

$$S = \left\{ i, j, i_{j_1}, \ldots, i_{j_{p+1}}, i_{m_1} \ldots i_{m_{2k-2p-1}} \right\}$$

$$= \left\{ i, j, i_{j_1} \ldots i_{j_{2k-2p}} \right\} \tag{2.166}$$

and s_2 can assume any number from 1 to $2n$ except the numbers in the set:

$$S' = \left\{ S, s_1 \right\} = \left\{ i, j, i_{j_1} \ldots i_{j_{2k-2p}}, s_1 \right\} \tag{2.167}$$

That is to say, we are assuming:

$$\nu_{2t}^2 = \nu'^2_{2t} \quad or \quad \nu_{2t} = \nu'_{2t} \tag{2.168}$$

where:

$$\nu_{2t} = S_a - S_{a'} \quad \text{and} \quad \nu'_{2t} = S_c + S_{c'} \tag{2.169}$$

2.3.5) - Expansion of the terms in the equivalent form.

Now we use Eq(2.8) of Theorem 1 to expand the Pfaffians in Eqs(2.120),(2.122),(2.138), and (2.140) around the pivots $i, j,$ and $i_{j_{p+1}}$. For example to expand a_{2t} around the pivot i:

$$a_{2t} = B_{2t+1} + C_{2t} + D_{2t+1} + E_{2t} + X + Y \tag{2.170}$$

where, $B_{2t+1}, C_{2t}, D_{2t+1}$, and E_{2t} are given later in Eqs(2.187) to (2.190), and:

$$X = \sum_{f=2t-p+1}^{2k-2p-1} (-1)^{i+i_{m_f}+1} \epsilon_{ii_{m_f}} \beta_a \beta_x a_{ii_{m_f}} \times \sqrt{A_{ii_{j_1} \ldots i_{j_p} i_{m_1} \ldots i_{m_{2t-p}} i_{m_f}} A_{iji_{j_1} \ldots i_{j_{p+1}} i_{m_{2t-p+1}} \ldots i_{m_{2k-2p-1}}}} \tag{2.171}$$

and,

$$Y = \sum_{s_1} (-1)^{i+s_1+1} \epsilon_{is_1} \beta_a \beta_y a_{is_1} \times \sqrt{A_{ii_{j_1} \ldots i_{j_p} i_{m_1} \ldots i_{m_{2t-p}} i_{s_1}} A_{iji_{j_1} \ldots i_{j_{p+1}} i_{m_{2t-p+1}} \ldots i_{m_{2k-2p-1}}}} \tag{2.172}$$

where,

$$\beta_a = \delta_{i_{m_1}\dots i_{m_{2t-p}}, iji_{j_{p+1}} i_{m_{2t-p+1}} \dots i_{m_{2k-2p-1}}}$$

$$\beta_x = \delta_{i_{j_1}\dots i_{j_p} i_{m_1}\dots i_{m_{2t-p}}, ii_{m_f}}$$

$$\beta_y = \delta_{i_{j_1}\dots i_{j_p} i_{m_1}\dots i_{m_{2t-p}}, is_1} .$$

We use Theorem 3, Eq(2.16) to put Y in the form:

$$X = \sum_{h=1}^{2t-p+1} (-1)^{i+i_{m_h}+1} \epsilon_{ii_{m_h}} \beta_x' a_{ii_{m_h}} \times$$

$$\sqrt{A_{ii_{j_1}\dots i_{j_p} i_{m_1}\dots i_{m_{2t-p+1}}} A_{iji_{j_1}\dots i_{j_{p+1}} i_{m_{2t-p+2}} \dots i_{m_{2k-2p-1}} i_{m_h}}} \tag{2.173}$$

with:

$$\beta_x' = \delta_{i_{j_1}\dots i_{j_{p+1}} i_{m_{2t-p+2}} \dots i_{m_{2k-2p-1}}, ii_{m_h}} \times \delta_{ii_{m_h} i_{m_1}\dots i_{m_{2t-p+1}}, ji_{j_{p+1}} i_{m_{2t-p+2}} \dots i_{m_{2k-2p-1}}}$$

Now we expand X, in Eq(2.173) around the pivot j as:

$$X = X_1 + X_2$$

where,

$$X_1 = \sum_{f=2t-p+2}^{2k-2p-1} \sum_{h=1}^{2t-p+1} (-1)^{i+i_{m_h}+1} (-1)^{j+i_{m_f}+1} \epsilon_{ii_{m_h}} \epsilon_{ji_{m_f}} \beta_x' \beta_{x_1} a_{ii_{m_h}} a_{ji_{m_f}} Z_1 \tag{2.174}$$

with:

$$Z_1 = \sqrt{A_{iji_{j_1}\dots i_{j_p} i_{m_1}\dots i_{m_{2t-p+1}} i_{m_f}} A_{iji_{j_1}\dots i_{j_{p+1}} i_{m_{2t-p+2}} \dots i_{m_{2k-2p-1}} i_{m_h}}} \tag{2.175}$$

$$\beta_{x_1} = \delta_{ii_{j_1}\dots i_{j_p} i_{m_1}\dots i_{m_{2t-p+1}}, ji_{m_f}}$$

and,

$$X_2 = \sum_{h=1}^{2t-p+1} \sum_{s_1} (-1)^{j+s_1+1} \epsilon_{js_1} \epsilon_{ii_{m_h}} \beta_x' \beta_{x_2} a_{ii_{m_h}} a_{js_1} Z_2 \tag{2.176}$$

where,

$$Z_2 = \sqrt{A_{iji_{j_1}\dots i_{j_p} i_{m_1}\dots i_{m_{2t-p+1}} s_1} A_{iji_{j_1}\dots i_{j_{p+1}} i_{m_{2t-p+2}} \dots i_{m_{2k-2p-1}} i_{m_h}}} \tag{2.177}$$

$$\beta_{x_2} = \delta_{ii_{j_1}\dots i_{j_p} i_{m_1}\dots i_{m_{2t-p+1}}, js_1}$$

Again, we use Eq(2.16) to modify X_1 of Eq(2.174) to the following form:

$$X_1 = \sum_{h=1}^{2t-p+2} \sum_{\substack{h'=1 \\ h' \neq h}}^{2t-p+2} (-1)^{i+i_{m_h}+1} (-1)^{j+i_{m_{h'}}+1} \epsilon_{ii_{m_h}} \epsilon_{ji_{m_{h'}}} \beta'_{z1} a_{ii_{m_h}} a_{ji_{m_{h'}}} Z_3 \quad (2.178)$$

where,

$$Z_3 = \sqrt{A_{iji_{j_1} \ldots i_{j_p} i_{m_1} \ldots i_{m_{2t-p+2}}} A_{iji_{j_1} \ldots i_{j_p+1} i_{m_{2t-p+3}} \ldots i_{m_{2k-2p-1}} i_{m_h} i_{m_{h'}}}} \quad (2.179)$$

$$\beta'_{z1} = \delta_{iji_{m_1} \ldots i_{m_{2t-p+2}}, i_{j_p+1} i_{m_{2t-p+3}} \ldots i_{m_{2k-2p-1}}} \times \delta_{i_{j_1} \ldots i_{j_p+1} i_{m_{2t-p+3}} \ldots i_{m_{2k-2p-1}}, ji_{m_{h'}}} \times$$
$$\delta_{ji_{j_1} \ldots i_{j_p+1} i_{m_{2t-p+3}} \ldots i_{m_{2k-2p-1}}, ii_{m_h}}$$

Then, we expand X_1 around the pivot i_{j_p+1} to get:

$$X_1 = F_{2t+3} + G_{2t+2}$$

where F_{2t+3} and G_{2t+2} are given later in Eqs(2.191) and (2.192) respectively, and again Eq(2.16) is used in arriving at the final form of F_{2t+3}.

Next we expand X_2 in Eq(2.176) around i_{j_p+1} to obtain:

$$X_2 = H_{2t+2} + I_{2t+1} \quad (2.180)$$

where, Eq(2.16) is utilized to get the final form of H_{2t+2} which is given in Eq(2.193), and I_{2t+1} is given by Eq(2.194).

Then we repeat the same procedure to expand Y of Eq(2.172). We first expand it around the pivot j to obtain:

$$Y = Y_1 + S_a \quad (2.181)$$

with S_a as in Eq(2.203). Then we expand Y_1 around the pivot i_{j_p+1} to obtain:

$$Y_1 = J_{2t+2} + K_{2t+1} \quad (2.182)$$

with J_{2t+2}, and K_{2t+1} as in Eqs(2.195) and (2.196) respectively.

Therefore, for a_{2t} in Eq(2.120) we can write:

$$a_{2t} = B_{2t+1} + C_{2t} + D_{2t+1} + E_{2t} + F_{2t+3} + G_{2t+2} +$$
$$H_{2t+2} + T_{2t+1} + J_{2t+2} + V_{2t+1} + N_{2t} + S_a \quad (2.183)$$

Similarly:

$$a'_{2t} = F_{2t+1} + G_{2t} + R_{2t+1} + I_{2t} + U_{2t+1} + K_{2t} + L_{2t} + M_{2t} + N_{2t} + S_{a'} \qquad (2.184)$$

$$c_{2t} = B_{2t+1} + C_{2t} + F'_{2t+1} + Q_{2t+1} + H_{2t} - I_{2t} - U_{2t+1} + V_{2t+1} - L_{2t} + P_{2t} + S_c \qquad (2.185)$$

$$c'_{2t} = D_{2t+1} + E_{2t} - F'_{2t+1} - Q_{2t+1} - R_{2t+1} + J_{2t} + -K_{2t} - M_{2t} - P_{2t} + S_{c'} \qquad (2.186)$$

The relevant terms in Eqs(2.183) to (2.186) are listed below:

$$B_{2t+1} = \frac{a_{ii_{j_{p+1}}} \sum_{h=1}^{2t-p+1} \alpha_{B_{2t+1}} a_{ji_{m_h}} \times}{\sqrt{A_{iji_{j_1} \ldots i_{j_{p+1}} i_{m_1} \ldots i_{m_{2t-p}}} A_{iji_{j_1} \ldots i_{j_{p+1}} i_{m_{2t-p+2}} \ldots i_{m_{2t-2p-1}} i_{m_h}}}} \qquad (2.187)$$

$$C_{2t} = \frac{a_{ii_{j_{p+1}}} \sum_{s_1} \alpha_{C_{2t}} a_{js_1} \times}{\sqrt{A_{iji_{j_1} \ldots i_{j_{p+1}} i_{m_1} \ldots i_{m_{2t-p}} s_1} A_{iji_{j_1} \ldots i_{j_{p+1}} i_{m_{2t-p+1}} \ldots i_{m_{2t-2p-1}}}}} \qquad (2.188)$$

$$D_{2t+1} = \frac{a_{ji_{j_{p+1}}} \sum_{h=1}^{2t-p+1} \alpha_{D_{2t+1}} a_{ii_{m_h}} \times}{\sqrt{A_{iji_{j_1} \ldots i_{j_{p+1}} i_{m_1} \ldots i_{m_{2t-p+1}}} A_{iji_{j_1} \ldots i_{j_{p+1}} i_{m_{2t-p+2}} \ldots i_{m_{2t-2p-1}} i_{m_h}}}} \qquad (2.189)$$

$$E_{2t} = \frac{a_{ji_{j_{p+1}}} \sum_{s_1} \alpha_{E_{2t}} a_{is_1} \times}{\sqrt{A_{iji_{j_1} \ldots i_{j_{p+1}} i_{m_1} \ldots i_{m_{2t-p}} s_1} A_{iji_{j_1} \ldots i_{j_{p+1}} i_{m_{2t-p+1}} \ldots i_{m_{2t-2p-1}}}}} \qquad (2.190)$$

$$F_{2t+3} = \frac{\sum_{h=1}^{2t-p+3} \sum_{\substack{h'=1 \\ h' \neq h}}^{2t-p+3} \sum_{\substack{h''=1 \\ h'' \neq h,h'}}^{2t-p+3} \alpha_{F_{2t+3}} a_{ii_{m_h}} a_{ji_{m_{h'}}} a_{i_{j_{p+1}} i_{m_{h''}}} \times}{\sqrt{A_{iji_{j_1} \ldots i_{j_{p+1}} i_{m_1} \ldots i_{m_{2t-p+3}}} A_{iji_{j_1} \ldots i_{j_{p+1}} i_{m_{2t-p+4}} \ldots i_{m_{2t-2p-1}} i_{m_h} i_{m_{h'}} i_{m_{h''}}}}} \qquad (2.191)$$

$$F'_{2t+1} = \epsilon_{ij} . F_{2t+1}$$

$$G_{2t} = \frac{\sum_{h=1}^{2t-p} \sum_{\substack{h'=1 \\ h' \neq h}}^{2t-p} \sum_{s_1} \alpha_{G_{2t}} a_{ii_{m_h}} a_{ji_{m_{h'}}} a_{i_{j_{p+1}} s_1} \times}{\sqrt{A_{iji_{j_1} \ldots i_{j_{p+1}} i_{m_1} \ldots i_{m_{2t-p}} s_1} A_{iji_{j_1} \ldots i_{j_{p+1}} i_{m_{2t-p+1}} \ldots i_{m_{2t-2p-1}} i_{m_h} i_{m_{h'}}}}} \qquad (2.192)$$

$$H_{2t}=\sum_{h=1}^{2t-p}\sum_{\substack{h'=1\\h'\neq h}}^{2t-p}\sum_{s_1}\alpha_{H_{2t}}a_{ii_{m_h}}a_{i_{j_{p+1}}i_{m_{h'}}}a_{js_1}\times$$
$$\sqrt{A_{ijj_{j_1}\ldots i_{j_{p+1}}i_{m_1}\ldots i_{m_{2t-p}}s_1}A_{iji'_{j_1}\ldots i_{j_{p+1}}i_{m_{2t-p+1}}\ldots i_{m_{2k-2p-1}}i_{m_h}i_{m_{h'}}}} \tag{2.193}$$

$$I_{2t}=\sum_{h=1}^{2t-p}\sum_{s_1}\sum_{s_2}\alpha_{I_{2t}}a_{ii_{m_h}}a_{js_1}a_{i_{j_{p+1}}s_2}\times$$
$$\sqrt{A_{iji_{j_1}\ldots i_{j_{p+1}}i_{m_1}\ldots i_{m_{2t-p}}s_2}A_{iji_{j_1}\ldots i_{j_{p+1}}i_{m_{2t-p+1}}\ldots i_{m_{2k-2p-1}}i_{m_h}s_1}} \tag{2.194}$$

$$J_{2t}=\sum_{h=1}^{2t-p}\sum_{\substack{h'=1\\h'\neq h}}^{2t-p}\sum_{s_1}\alpha_{J_{2t}}a_{ji_{m_h}}a_{i_{j_{p+1}}i_{m_{h'}}}a_{is_1}\times$$
$$\sqrt{A_{iji_{j_1}\ldots i_{j_{p+1}}i_{m_1}\ldots i_{m_{2t-p}}s_1}A_{iji_{j_1}\ldots i_{j_{p+1}}i_{m_{2t-p+1}}\ldots i_{m_{2k-2p-1}}i_{m_h}i_{m_{h'}}}} \tag{2.195}$$

$$K_{2t}=\sum_{h=1}^{2t-p}\sum_{s_1}\sum_{s_2}\alpha_{K_{2t}}a_{ji_{m_h}}a_{is_1}a_{i_{j_{p+1}}s_2}\times$$
$$\sqrt{A_{iji_{j_1}\ldots i_{j_{p+1}}i_{m_1}\ldots i_{m_{2t-p}}s_2}A_{iji_{j_1}\ldots i_{j_{p+1}}i_{m_{2t-p+1}}\ldots i_{m_{2k-2p-1}}i_{m_h}s_1}} \tag{2.196}$$

$$L_{2t}=\sum_{h=1}^{2t-p}\sum_{s_1}\alpha_{L_{2t}}a_{ii_{m_h}}a_{js_1}a_{i_{j_{p+1}}s_1}\times$$
$$\sqrt{A_{iji_{j_1}\ldots i_{j_{p+1}}i_{m_1}\ldots i_{m_{2t-p}}s_1}A_{iji_{j_1}\ldots i_{j_{p+1}}i_{m_{2t-p+1}}\ldots i_{m_{2k-2p-1}}i_{m_h}s_1}} \tag{2.197}$$

$$M_{2t}=\sum_{h=1}^{2t-p}\sum_{s_1}\alpha_{M_{2t}}a_{ji_{m_h}}a_{is_1}a_{i_{j_{p+1}}s_1}\times$$
$$\sqrt{A_{iji_{j_1}\ldots i_{j_{p+1}}i_{m_1}\ldots i_{m_{2t-p}}s_1}A_{iji_{j_1}\ldots i_{j_{p+1}}i_{m_{2t-p+1}}\ldots i_{m_{2k-2p-1}}i_{m_h}s_1}} \tag{2.198}$$

$$N_{2t}=a_{ij}\,\alpha_{N_{2t}}\times$$
$$\sqrt{A_{iji_{j_1}\ldots i_{j_p}i_{m_1}\ldots i_{m_{2t-p}}}A_{iji_{j_1}\ldots i_{j_{p+1}}i_{m_{2t-p+1}}\ldots i_{m_{2k-2p-1}}}} \tag{2.199}$$

$$P_{2t}=\sum_{s_1}\sum_{s_1}\alpha_{P_{2t}}a_{is_1}a_{js_1}\times$$
$$\sqrt{A_{iji_{j_1}\ldots i_{j_{p+1}}i_{m_1}\ldots i_{m_{2t-p}}s_1}A_{iji_{j_1}\ldots i_{j_p}i_{m_{2t-p+1}}\ldots i_{m_{2k-2p-1}}s_1}} \tag{2.200}$$

$$Q_{2t+1}=\sum_{h=1}^{2t-p+1}\sum_{\substack{h'=1\\h'\neq h}}^{2t-p}\sum_{s_1}\alpha_{G_{2t+1}}a_{ii_{m_h}}a_{ji_{m_{h'}}}a_{i_{j_{p+1}}s_1}\times$$
$$\sqrt{A_{iji_{j_1}\ldots i_{j_{p+1}}i_{m_1}\ldots i_{m_{2t-p+1}}}A_{iji_{j_1}\ldots i_{j_{p+1}}i_{m_{2t-p+2}}\ldots i_{m_{2k-2p-1}}i_{m_h}i_{m_{h'}}s_1}} \tag{2.201}$$

$$R_{2t+1}=\sum_{h=1}^{2t-p+1}\sum_{\substack{h'=1\\h'\neq h}}^{2t-p+1}\sum_{s_1}\alpha_{H_{2t+1}}a_{ii_{m_h}}a_{i_{j_{p+1}}i_{m_{h'}}}a_{js_1}\times\sqrt{A_{iji_{j_1}\dots i_{j_{p+1}}i_{m_1}\dots i_{m_{2t-p+1}}}A_{iji_{j_1}\dots i_{j_{p+1}}i_{m_{2t-p+2}}\dots i_{m_{2t-2p-1}}i_{m_h}i_{m_{h'}}s_1}} \tag{2.202}$$

$$S_a=\sum_{s_1}\sum_{s_2}\alpha_{S_a}a_{is_1}a_{js_2}\times\sqrt{A_{iji_{j_1}\dots i_{j_p}i_{m_1}\dots i_{m_{2t-p}}s_1s_2}A_{iji_{j_1}\dots i_{j_{p+1}}i_{m_{2t-p+1}}\dots i_{m_{2t-2p-1}}}} \tag{2.203}$$

$$S_{a'}=\sum_{s_1}\sum_{s_2}\alpha_{S_{a'}}a_{is_1}a_{js_2}\times\sqrt{A_{iji_{j_1}\dots i_{j_p}i_{m_1}\dots i_{m_{2t-p}}}A_{iji_{j_1}\dots i_{j_{p+1}}i_{m_{2t-p+1}}\dots i_{m_{2t-2p-1}}s_1s_2}} \tag{2.204}$$

$$S_c=\sum_{s_1}\sum_{s_2}\alpha_{S_c}a_{is_1}a_{js_2}\times\sqrt{A_{iji_{j_1}\dots i_{j_{p+1}}i_{m_1}\dots i_{m_{2t-p}}s_2}A_{iji_{j_1}\dots i_{j_p}i_{m_{2t-p+1}}\dots i_{m_{2t-2p-1}}s_1}} \tag{2.205}$$

$$S_{c'}=\sum_{s_1}\sum_{s_2}\alpha_{S_{c'}}a_{is_1}a_{js_2}\times\sqrt{A_{iji_{j_1}\dots i_{j_{p+1}}i_{m_1}\dots i_{m_{2t-p}}s_1}A_{iji_{j_1}\dots i_{j_p}i_{m_{2t-p+1}}\dots i_{m_{2t-2p-1}}s_2}} \tag{2.206}$$

$$T_{2t+1}=\sum_{h=1}^{2t-p+1}\sum_{s_1}\sum_{s_2}\alpha_{I_{2t+1}}a_{ii_{m_h}}a_{js_1}a_{i_{j_{p+1}}s_2}\times\sqrt{A_{iji_{j_1}\dots i_{j_{p+1}}i_{m_1}\dots i_{m_{2t-p+1}}s_1s_2}A_{iji_{j_1}\dots i_{j_{p+1}}i_{m_{2t-p+2}}\dots i_{m_{2t-2p-1}}i_{m_h}}} \tag{2.207}$$

$$U_{2t+1}=\sum_{h=1}^{2t-p+1}\sum_{\substack{h'=1\\h'\neq h}}^{2t-p+1}\sum_{s_1}\alpha_{J_{2t+1}}a_{ji_{m_h}}a_{i_{j_{p+1}}i_{m_{h'}}}a_{is_1}\times\sqrt{A_{iji_{j_1}\dots i_{j_{p+1}}i_{m_1}\dots i_{m_{2t-p+1}}}A_{iji_{j_1}\dots i_{j_{p+1}}i_{m_{2t-p+2}}\dots i_{m_{2t-2p-1}}i_{m_h}i_{m_{h'}}s_1}} \tag{2.208}$$

$$V_{2t+1}=\sum_{h=1}^{2t-p+1}\sum_{s_1}\sum_{s_2}\alpha_{K_{2t+1}}a_{ji_{m_h}}a_{is_1}a_{i_{j_{p+1}}s_2}\times\sqrt{A_{iji_{j_1}\dots i_{j_{p+1}}i_{m_1}\dots i_{m_{2t-p+1}}s_1s_2}A_{iji_{j_1}\dots i_{j_{p+1}}i_{m_{2t-p+2}}\dots i_{m_{2t-2p-1}}i_{m_h}}} \tag{2.209}$$

The various α's, in Eqs(2.187) to (2.209), after simplifications, are:

$$\alpha_{B_{2t+1}}=\xi_1\lambda_1\beta_1\eta_1 \tag{2.210}$$

$$\xi_1 = (-1)^{i+j+i_{j_{p+1}}+i_{m_k}}$$

$$\lambda_1 = \delta_{i_{m_k} i_{m_1} \ldots i_{m_{2t-p+1}} i_{m_{2t-p+2}} \ldots i_{m_{2k-2p-1}}}$$

$$\beta_1 = \delta_{i_{j_1} \ldots i_{j_p}, i j i_{j_{p+1}} i_{m_k}}$$

$$\eta_1 = \epsilon_{i i_{j_{p+1}}} \epsilon_{j i_{m_k}} \delta_{i i_{j_{p+1}}, j i_{m_k}}$$

$$\alpha_{C_{2t}} = \xi_2 \lambda_2 \beta_2 \eta_2 \tag{2.211}$$

$$\xi_2 = (-1)^{i+j+i_{j_{p+1}}+s_1}$$

$$\lambda_2 = \delta_{i_{m_1} \ldots i_{m_{2t-p}}, s_1 i_{m_{2t-p+1}} \ldots i_{m_{2k-2p-1}}}$$

$$\beta_2 = \delta_{i_{j_1} \ldots i_{j_p}, i j i_{j_{p+1}} s_1}$$

$$\eta_2 = \epsilon_{i i_{j_{p+1}}} \epsilon_{j s_1} \delta_{i i_{j_{p+1}}, j s_1}$$

$$\alpha_{D_{2t+1}} = \xi_1 \lambda_1 \beta_1 \eta_3 \tag{2.212}$$

$$\eta_3 = \epsilon_{j i_{j_{p+1}}} \epsilon_{j i_{m_k}} \delta_{j i_{j_{p+1}}, i i_{m_k}}$$

$$\alpha_{E_{2t}} = \xi_2 \lambda_2 \beta_2 \eta_4 \tag{2.213}$$

$$\eta_4 = \epsilon_{j i_{j_{p+1}}} \epsilon_{i s_1} \delta_{j i_{j_{p+1}}, i i_{m_k}}$$

$$\alpha_{F_{2t+3}} = \xi_3 \lambda_4 \beta_3 \eta_5 \tag{2.214}$$

$$\xi_3 = (-1)^{i+j+i_{j_{p+1}}+i_{m_k}+i_{m_{k'}}+i_{m_{k''}}+3}$$

$$\lambda_4 = \delta_{i_{m_k} i_{m_{k'}} i_{m_{k''}} i_{m_1} \ldots i_{m_{2t-p+3}} i_{m_{2t-p+4}} \ldots i_{m_{2k-2p-1}}}$$

$$\beta_3 = \delta_{i_{j_1} \ldots i_{j_p}, i j i_{j_{p+1}} i_{m_k} i_{m_{k'}} i_{m_{k''}}}$$

$$\eta_5 = \epsilon_{ii_{m_k}} \epsilon_{ji_{m_{k'}}} \epsilon_{i_{j_{p+1}} i_{m_{k''}}} \delta_{ii_{m_k}, ji_{m_{k'}}} \delta_{iji_{m_k} i_{m_{k'}}, i_{j_{p+1}} i_{m_{k''}}}$$

$$\alpha_{G_{2t}} = \xi_4 \lambda_4 \beta_4 \eta_6 \tag{2.215}$$

$$\xi_4 = (-1)^{i+j+i_{j_{p+1}}+i_{m_k}+i_{m_{k'}}+s_1+3}$$

$$\beta_4 = \delta_{i_{j_1} \ldots i_{j_p}, iji_{j_{p+1}} i_{m_k} i_{m_{k'}} s_1}$$

$$\eta_6 = \epsilon_{ii_{m_k}} \epsilon_{ji_{m_{k'}}} \epsilon_{i_{j_{p+1}} s_1} \delta_{ii_{m_k}, ji_{m_{k'}}} \delta_{iji_{m_k} i_{m_{k'}}, i_{j_{p+1}} s_1}$$

$$\lambda_3 = \delta_{i_{m_k} i_{m_{k'}}, i_{m_1} \cdots i_{m_{2t-p}}, i_{m_{2t-p+1}} \ldots i_{m_{2k-2p-1}}}$$

$$\alpha_{H_{2t}} = \xi_4 \lambda_4 \beta_4 \eta_7 \tag{2.216}$$

$$\eta_7 = \epsilon_{ii_{m_k}} \epsilon_{i_{j_{p+1}} i_{m_{k'}}} \epsilon_{js_1} \delta_{ii_{m_k}, i_{j_{p+1}} i_{m_{k'}}} \delta_{ii_{j_{p+1}} i_{m_k} i_{m_{k'}}, js_1}$$

$$\alpha_{I_{2t}} = \xi_5 \lambda_6 \beta_5 \eta_8 \tag{2.217}$$

$$\xi_5 = (-1)^{i+j+i_{j_{p+1}}+i_{m_k}+s_1+s_2+3}$$

$$\lambda_6 = \delta_{i_{m_k} i_{m_1} \cdots i_{m_{2t-p}}, s_1 i_{m_{2t-p+1}} \ldots i_{m_{2k-2p-1}}} \delta_{iji_{m_1} \cdots i_{m_{2t-p}}, s_2} \delta_{i_{m_{2t-p+1}} \ldots i_{m_{2k-2p-1}}, s_1}$$

$$\beta_5 = \delta_{i_{j_1} \ldots i_{j_p}, iji_{j_{p+1}} i_{m_k} s_1 s_2}$$

$$\eta_8 = \epsilon_{ii_{m_k}} \epsilon_{js_1} \epsilon_{i_{j_{p+1}} s_2} \delta_{ii_{m_k}, js_1} \delta_{i_{j_{p+1}}, iji_{m_k} s_1}$$

$$\alpha_{J_{2t}} = \xi_4 \lambda_4 \beta_4 \eta_9 \tag{2.218}$$

$$\eta_9 = \epsilon_{ji_{m_k}} \epsilon_{i_{j_{p+1}} i_{m_{k'}}} \epsilon_{is_1} \delta_{ji_{m_k}, i_{j_{p+1}} i_{m_{k'}}} \delta_{ji_{j_{p+1}} i_{m_k} i_{m_{k'}}, is_1}$$

$$\alpha_{K_{2t}} = \xi_5 \lambda_6 \beta_5 \eta_{10} \tag{2.219}$$

$$\eta_{10} = \epsilon_{ji_{m_k}} \epsilon_{is_1} \epsilon_{i_{j_{p+1}} s_2} \delta_{ji_{m_k}, is_1} \delta_{i_{j_{p+1}}, iji_{m_k} s_1}$$

$$\alpha_{L_{2t}} = \xi_6\lambda_8\beta_6\eta_{11} \tag{2.220}$$

$$\xi_6 = (-1)^{i+j+i_{j_{p+1}}+i_{m_k}}$$

$$\lambda_8 = \delta_{i_{m_k} i_{m_1} \cdots i_{m_{2t-p}}, i_{m_{2t-p+1}} \cdots i_{m_{2k-2p-1}}} \delta_{i_1 \ldots i_{2k}, s_1}$$

$$\beta_6 = \delta_{i_{j_1} \ldots i_{j_p}, iji_{j_{p+1}} i_{m_k}}$$

$$\eta_{11} = \epsilon_{ii_{m_k}} \epsilon_{js_1} \epsilon_{i_{j_{p+1}} s_1} \epsilon_{ij} \epsilon_{ii_{j_{p+1}}} \epsilon_{i_{j_{p+1}} i_{m_k}} \delta_{i_{m_k}, js_1} \delta_{j, i_{j_{p+1}} s_1}$$

$$\alpha_{M_{2t}} = \xi_6\lambda_8\beta_6\eta_{12} \tag{2.221}$$

$$\eta_{12} = \epsilon_{ji_{m_k}} \epsilon_{is_1} \epsilon_{i_{j_{p+1}} s_1} \epsilon_{ji} \epsilon_{ji_{j_{p+1}}} \epsilon_{i_{j_{p+1}} i_{m_k}} \delta_{i_{m_k}, is_1} \delta_{i, i_{j_{p+1}} s_1}$$

$$\alpha_{N_{2t}} = \xi_7\lambda_9\beta_7\eta_{12} \tag{2.222}$$

$$\xi_7 = (-1)^{i+j}$$

$$\lambda_9 = \delta_{i_{m_1} \cdots i_{m_{2t-p}}, i_{j_{p+1}} i_{m_{2t-p+1}} \cdots i_{m_{2k-2p-1}}}$$

$$\beta_7 = \delta_{i_{j_1} \ldots i_{j_p}, ij}$$

$$\eta_{13} = \epsilon_{ij}$$

$$\alpha_{P_{2t}} = \xi_7\lambda_{11}\beta_7\eta_{14} \tag{2.223}$$

$$\lambda_{11} = \delta_{s_1 i_{j_{p+1}} i_{m_1} \cdots i_{m_{2t-p}}, s_1 i_{m_{2t-p+1}} \cdots i_{m_{2k-2p-1}}}$$

$$\eta 14 = \epsilon_{is_1} \epsilon_{js_1} \delta_{is_1, js_1}$$

$$\alpha_{Q_{2t+1}} = \xi_4\lambda_5\beta_4\eta_6 \tag{2.224}$$

$$\lambda_5 = \delta_{i_{m_k} i_{m_k}, s_1 i_{m_1} \cdots i_{m_{2t-p+1}}, i_{m_{2t-p+1}} \cdots i_{m_{2k-2p-1}}}$$

$$\alpha_{R_{2t+1}} = \xi_4\lambda_5\beta_4\eta_7 \tag{2.225}$$

$$\alpha_{S_s} = \xi_8\lambda_{13}\beta_8\eta_{15} \tag{2.226}$$

$$\xi_8 = (-1)^{i+j+s_1+s_2}$$

$$\lambda_{13} = \delta_{i_{m_1}\cdots i_{m_{2t-p}},s_1s_2i_{j_p+1}i_{m_{2t-p+1}}\cdots i_{m_{2k-2p-1}}}$$

$$\beta_8 = \delta_{i_{j_1}\ldots i_{j_p},ijs_1s_2}$$

$$\eta 15 = \epsilon_{is_1}\epsilon_{js_2}\delta_{is_1,js_2}$$

$$\alpha_{S_s'} = \xi_8\lambda_{14}\beta_8\eta_{15} \tag{2.227}$$

$$\lambda_{14} = \delta_{s_1s_2i_{m_1}\cdots i_{m_{2t-p}},i_{j_p+1}i_{m_{2t-p+1}}\cdots i_{m_{2k-2p-1}}}$$

$$\alpha_{S_t} = \xi_8\lambda_{17}\beta_8\eta_{15} \tag{2.228}$$

$$\lambda_{17} = \delta_{s_2i_{j_p+1}i_{m_1}\cdots i_{m_{2t-p}},s_1i_{m_{2t-p+1}}\cdots i_{m_{2k-2p-1}}}$$

$$\alpha_{S_t'} = \xi_8\lambda_{18}\beta_8\eta_{15} \tag{2.229}$$

$$\lambda_{18} = \delta_{s_1i_{j_p+1}i_{m_1}\cdots i_{m_{2t-p}},s_2i_{m_{2t-p+1}}\cdots i_{m_{2k-2p-1}}}$$

$$\alpha_{T_{2t+1}} = \xi_5\lambda_7\beta_5\eta_8 \tag{2.230}$$

$$\lambda_7 = \delta_{i_{m_k}i_{m_1}\cdots i_{m_{2t-p+1}},s_1s_2i_{m_{2t-p+2}}\cdots i_{m_{2k-2p-1}}}$$

$$\alpha_{U_{2t+1}} = \xi_4\lambda_5\beta_5\eta_9 \tag{2.231}$$

$$\alpha_{V_{2t+1}} = \xi_5\lambda_8\beta_5\eta_{10} \tag{2.232}$$

Considering that:

$$\sum_{t=0}^{k-1} F_{2t+3} = \sum_{t=1}^{k} F_{2t+1} \tag{2.233}$$

And since $F_{2k+1} = F_1 = 0$, we get:

$$\sum_{t=0}^{k-1} F_{2t+3} = \sum_{t=1}^{k-1} F_{2t+1} \tag{2.234}$$

Similarly, $G_{2k} = G_0 = H_{2k} = H_0 = 0$ and thus:

$$\sum_{t=0}^{k-1} G_{2t+2} = \sum_{t=1}^{k-1} G_{2t} \tag{2.235}$$

$$\sum_{t=0}^{k-1} H_{2t+2} = \sum_{t=1}^{k-1} H_{2t} \tag{2.236}$$

Then we get, after some cancellations;

$$\begin{aligned}\sum_{t=0}^{k-1}\sum^{2t}(a_{2t}-a'_{2t})^2 &= (\kappa_0+\nu_0)^2 + \sum_{t=1}^{k-1}\sum^{2t}(\kappa_{2t}+\nu_{2t})^2 \\ &= \sum_{t=0}^{k-1}\sum^{2t}(\kappa_{2t}+\nu_{2t})^2\end{aligned} \tag{2.237}$$

where:

$$\begin{aligned}\kappa_{2t} = {} & B_{2t+1}+C_{2t}+D_{2t+1}+T_{2t+1}+J_{2t}+V_{2t+1}- \\ & R_{2t+1}-I_{2t}-U_{2t+1}-K_{2t}-L_{2t}-M_{2t}\end{aligned} \tag{2.238}$$

and,

$$\nu_{2t} = S_a - S_{a'} \tag{2.239}$$

And, similarly:

$$\sum_{t=0}^{k-1}\sum^{2t}(c_{2t}-c'_{2t})^2 = \sum_{t=0}^{k-1}\sum^{2t}(\kappa_{2t}+\nu'_{2t})^2 \tag{2.240}$$

$$\nu'_{2t} = S_c - S_{c'} \tag{2.241}$$

But according to Eq(2.168), we have already assumed that:

$$\nu_{2t} = \nu'_{2t} \tag{2.242}$$

Therefore we see that with the assumption in Eq(2.168), Eqs(2.237) and (2.240) become identical, thus the mathematical induction is completed, and as the result Eq(2.149) is verified. Parallel reasoning holds for $2n-4 \geq k \geq n$. And this completes the proof of Theorem 5.

2.3.6 Lemma 1 :Higher Order Derivatives.

The derivative of:

$$Z_{LC}^{\prime} = \sum_{\substack{i=1 \\ i \neq j_1 \dots j_p}}^{2n} k_i \frac{\frac{\partial M_{2n}^{\prime}}{\partial \mu_i}}{M_{2n}^{\prime}} \tag{2.250}$$

where,

$$M_{2n}^{\prime} = \frac{\partial}{\partial \mu_{j_1}} \cdots \frac{\partial}{\partial \mu_{j_p}} M_{2n} \tag{2.251}$$

with respect to μ_j is:

$$\frac{\partial Z_{LC}^{\prime}}{\partial \mu_j} = \sum_{\substack{i=1 \\ i \neq j_1 \dots j_p}}^{2n} k_i \frac{N_{i,j}^{\prime}}{M_{2n}^{\prime\,2}} \tag{2.252}$$

where:

$$N_{j,j}^{\prime} = -\left(\sum_{l=1}^{n} P_{2l-p-1}^{\prime} \right)^2 \tag{2.253}$$

and,

$$N_{i,j}^{\prime} = \left(\sum_{l=0}^{n-1} P_{2l-p}^{\prime} \right)^2 - \left(\sum_{l=0}^{n-1} P_{2l-p-1}^{\prime} \right)^2 \quad i \neq j \tag{2.254}$$

with:

$$P_{2l-p}^{\prime} = \sum_{\substack{1 \leq i_1 < \dots < i_{2l-p} \leq 2n \\ i_k \neq j_1 \dots j_p, i, j}} \sqrt{A_{j_1 \dots j_p, i_1 \dots i_{2l-p}} A_{ijj_1 \dots j_p, i_1 \dots i_{2l-p}}} \tag{2.255}$$

$$P_{2l-p-1}^{\prime} = \sum_{\substack{1 \leq i_1 < \dots < i_{2l-p-1} \leq 2n \\ i_k \neq j_1 \dots j_p, i, j}} \sqrt{A_{ijj_1 \dots j_p, i_1 \dots i_{2l-p-1}} A_{jj_1 \dots j_p, i_1 \dots i_{2l-p-1}}} \tag{2.256}$$

Proof :

We again write Eq(1.21) as:

$$M_{2n} = \sum_{l=0}^{n} C_{2l} \tag{2.257}$$

We break C_{2l} into two components:

$$C_{2l} = \mu_{j_1} \dots \mu_{j_p} E_{2l-p} + D_{2l} \tag{2.258}$$

where:

$$E_{2l-p} = \sum_{\substack{1 \le i_1 < \dots < i_{2l-p} \le 2n \\ i_k \ne j_1 \dots j_p}} A_{j_1 \dots j_p i_1 \dots i_{2l-p}} \mu_{i_1} \cdots \mu_{i_{2l-p}} \tag{2.259}$$

$$D_{2l} = \sum_{\substack{1 \le i_1 < \dots < i_{2l-p} \le 2n \\ i_k \ne j_1 \dots j_p}} A_{i_1 \dots i_{2l}} \mu_{i_1} \cdots \mu_{i_{2l}} \tag{2.260}$$

Now we proceed as we did for Theorem 5 but with M_{2n} replaced with:

$$M'_{2n} = \sum_{l=0}^{n} E_{2l-p} \tag{2.261}$$

2.4 Theorem 6 : The Slope of The Alternative Reactance Function

The derivative of:

$$Z_{LC}'' = \sum_{i=1}^{2n+1} k_i \frac{\dfrac{\partial N_{2n+1}}{\partial \mu_i}}{N_{2n+1}}$$

where:

$$N_{2n+1} = det \mid \mu_{2n+1} I_{2n+1} + \mathbf{A}_{2n+1} \mid$$

where $\mathbf{A}_{2n+1}$ is a $2n+1 \times 2n+1$ skew-symmetric matrix and μ_{2n+1} a diagonal matrix of order $2n+1$ with diagonal elements μ_d $d=1,2,\dots,2n+1$ as given in Eq(2.5) is:

$$\frac{\partial Z_{LC}''}{\partial \mu_j} = \sum_{i=1}^{2n+1} k_i \frac{N''_{i,j}}{N^2_{2n+1}}$$

Where:

$$N''_{j,j} = -\left(\sum_{l=1}^{n+1} P_{2l}|'_{-2} \right)^2$$

and,

$$N''_{i,j} = \left(\sum_{l=0}^{n-1} P''_{2l+1}\right)^2 - \left(\sum_{l=0}^{n-1} P''_{2l}\right)^2$$

with:

$$P''_{2l} = \sum_{\substack{1 \le i_1 \cdots < i_{2l} \le 2n+1 \\ i_k \ne i,j}} \delta_{i_1 \cdots i_{2l}, ij}\, \mu_{i_1} \cdots \mu_{i_{2l}} \sqrt{A_{ii_1 \cdots i_{2l}} A_{ji_1 \cdots i_{2l}}}$$

and,

$$P''_{2l+1} = \sum_{\substack{1 \le i_1 \cdots < i_{2l+1} \le 2n+1 \\ i_k \ne i,j}} \delta_{i_1 \cdots i_{2l+1}, ij}\, \mu_{i_1} \cdots \mu_{i_{2l+1}} \sqrt{A_{ii_1 \cdots i_{2l+1}} A_{iji_1 \ldots i_{2l+1}}}$$

Proof :

Note that the diagonal expansion of determinant N_{2n+1} is now:

$$N_{2n+1} = \sum_{i_1=1}^{2n+1} \mu_{i_1} A_{i_1} + \sum_{1 \le i_1 < i_2 < i_3 \le 2n+1} \mu_{i_1}\mu_{i_2}\mu_{i_3} A_{i_1 i_2 i_3} + \cdots + \mu_1\mu_2\mu_3 \cdots \mu_{2n+1}$$

and that A is a skew symmetric determinant of order *2n+1*, and therefore it is zero. Where A_{i_1} is the determinant of the submatrix **A** obtained by deleting both the i_1th rows and columns, and is of order $(2n)$, and is a perfect square. Similar reasoning as for proof of Theorem 5 completes the proof of this theorem.

2.5 Further Properties.

Here we present two properties based on the reactance slope (Theorem 5):

I) - Property one

Z_{LC} in Eq(1.23) can be written in the following form:

$$Z_{LC} = \frac{A_1\mu_1+A_2}{B_1\mu_1+B_2} \tag{2.262}$$

Then,

$$A_1B_2-B_1A_2 = \sum_{i=1}^{2n} k_i N_{i,1} \tag{2.263}$$

where, $N_{i,1}$ is given by Eq(2.53) or (2.54). Since $N_{i,1}$ is non-negative on the imaginary axis, we conclude that $A_1B_2-B_1A_2$ is non-negative there.

Proof :

We have:

$$\frac{\partial Z_{LC}}{\partial \mu_1} = \frac{A_1(\mu_1B_1+B_2)-B_1(\mu_1A_1+A_2)}{M_{2n}^2}$$

$$= \frac{A_1B_2-B_1A_2}{M_{2n}^2}$$

$$= \frac{\sum_{i=1}^{2n} k_i N_{i,1}}{{M_{2n}}^2} \tag{2.264}$$

where, $N_{i,1}$ is given by Eq(2.53) or (2.54). Since $N_{i,1}$ is non-negative on the imaginary axis, we conclude that $A_1B_2-B_1A_2$ is non-negative there.

II) - Property two

Let M_{2n} in Eq(1.21) be written in the following form:

$$M_{2n} = \mu_1B_1+B_2 \tag{2.265}$$

Then the slope of:

$$Z = \frac{B_1}{B_2} \tag{2.266}$$

is non-negative on the imaginary axis.

Proof :

The slope of the reactance function:

$$Z_{LC} = \frac{\dfrac{\partial M_{2n}}{\partial \mu_i}}{M_{2n}} \quad i=1,2,\ldots,2n \tag{2.267}$$

along μ_1-axis is:

$$\frac{\partial Z_{LC}}{\partial \mu_1} = \frac{M_{2n}\dfrac{\partial}{\partial \mu_1}\left(\dfrac{\partial M_{2n}}{\partial \mu_i}\right) - B_1\dfrac{dM_{2n}}{\partial \mu_i}}{M_{2n}^{\ 2}}$$

$$= \frac{(\mu_1 B_1 + B_2)\dfrac{\partial B_1}{\partial \mu_i} - B_1\left(\mu_1\dfrac{\partial B_1}{\partial \mu_i} + \dfrac{\partial B_2}{\partial \mu_i}\right)}{M_{2n}^{\ 2}}$$

$$= \frac{B_2\dfrac{\partial B_1}{\partial \mu_i} - B_1\dfrac{\partial B_2}{\partial \mu_i}}{M_{2n}^{\ 2}}$$

$$= \frac{1}{B_2^{\ 2}M_{2n}^{\ 2}}\frac{\partial}{\partial \mu_i}\left(\frac{B_1}{B_2}\right) \tag{2.268}$$

And since, we have already proved that $\dfrac{\partial Z_{LC}}{\partial \mu_1}$ is non-negative on the imaginary axis, and since $B_2^{\ 2}M_{2n}^{\ 2}$ is non-negative there, therefore the slope of $\dfrac{B_1}{B_2}$

is non-negative on the imaginary axis.

2.6 Application in Group Delay Approximation.

The group delay of the analog all pass filter H:

$$H(\mu_1,\mu_2,\ldots,\mu_{2n}) = \frac{M_{2n} - \dfrac{\partial M_{2n}}{\partial \mu_i}}{M_{2n} + \dfrac{\partial M_{2n}}{\partial \mu_i}} \tag{2.269}$$

is:

$$\tau_a = 2\frac{M^2-N^2}{A^2-B^2} \tag{2.270}$$

where, M^2-N^2 is calculated as numerator of $\dfrac{\partial Z_{LC}}{\partial \mu_j}$ as in Eq(2.53) or (2.54) on the imaginary axis and, A^2-B^2 is the magnitude square of the denominator of Eq(2.269) on the imaginary axis, and M is the even part and N is the odd part of the polynomial.

The group delay for the derived digital filter, assuming bilinear transformation, is:

$$\tau_d = \sec^2(\frac{T}{2}\omega_i)\tau_a \tag{2.271}$$

Proof :

I) - Analog Filter

In Eq(2.269), M_{2n} is an even function, therefore, on the imaginary axis ($\mu_i = j\Omega_i$) is $A(j\Omega_1,\ldots,j\Omega_{2n})$, and $\dfrac{\partial M_{2n}}{\partial \mu_i}$ is an odd function which we denote it as $jB(j\Omega_1,\ldots,j\Omega_{2n})$. Therefore, on the imaginary axis:

$$H = \frac{A - jB}{A + jB} \tag{2.272}$$

The phase angle of H is, therefore:

$$\theta = -2\tan^{-1}\frac{B}{A} \tag{2.273}$$

The group delay with respect to μ_j axis is:

$$\tau_d = -\frac{\partial\theta}{\partial\Omega_j} = 2\frac{\partial}{\partial\Omega_j}(\tan^{-1}\frac{B}{A})$$

$$= \frac{2}{1+\left(\frac{B}{A}\right)^2}\frac{\partial}{\partial\Omega_j}\left(\frac{B}{A}\right) \tag{2.274}$$

But we have already deduced the value of $\frac{\partial}{\partial\Omega_j}\left(\frac{B}{A}\right)$, in Theorem 5, Eq(2.53) or (2.54), as:

$$\frac{\partial}{\partial\Omega_j}\left(\frac{B}{A}\right) = \frac{M^2-N^2}{A^2} \tag{2.275}$$

Therefore,

$$\tau_a = 2\frac{M^2-N^2}{A^2+B^2} \tag{2.276}$$

Since H is a stable circuit, and since we have analytically determined the group delay, then we can optimize this group delay with respect to the coefficients in Eq(2.53) or (2.54). Also notice that the coefficients in $M, N, A,$ and B are the elements of the $2n \times 2n$ skew-symmetrical determinant and thus they are $\binom{2n}{2}$ in number.

II) - Digital Filter.

In this case some form of transformation should be used. Denoting it by:

$$\Omega_i = f\ (\omega_i). \tag{2.277}$$

where Ω is analog, and ω is digital frequencies. Then

$$\tau_d = -\frac{\partial\theta}{\partial\Omega_j}\frac{\partial\Omega_j}{\partial\omega_j}$$

$$= -\frac{\partial\theta}{\partial\Omega_j}\frac{\partial f\ (\omega_j)}{\partial\omega_j} \tag{2.278}$$

And following the same derivation as in I), we get:

$$\tau_d = \tau_a \frac{\partial f\ (\omega_j)}{\partial\omega_j} \tag{2.279}$$

Assuming bilinear transformation, then:

$$\Omega_i = \frac{2}{T} tan\,(\frac{T}{2}\omega_i) \tag{2.280}$$

and, thus:

$$\frac{\partial\Omega_i}{\partial\omega_i} = \sec^2(\frac{T}{2}\omega_i)\ . \tag{2.281}$$

2.7 Generation of New VSHP

In this section we exploit the result of Theorem 5 to obtain a new reactance function, and thus a Very Strictly Hurwitz Polynomial (VSHP) and to use it in generating stable 2-D digital filters. Because of the complexity involved, this will be illustrated for a special case.

2.7.1 A special case of Theorem 5.

Here we consider a special case of Eq(2.54) when $2n=6$, $i=5$, and $j=6$. Then:

$$N_{5,6} = (P_0+P_2+P_4)^2-(P_1+P_3)^2 \tag{2.282}$$

where, according to Eq(2.55):

$$P_0 = \sqrt{AA_{56}} \tag{2.283}$$

$$P_2 = \mu_1\mu_2\sqrt{A_{12}A_{1256}}+\mu_1\mu_3\sqrt{A_{13}A_{1356}}+\mu_1\mu_4\sqrt{A_{14}A_{1456}}+$$
$$\mu_2\mu_3\sqrt{A_{23}A_{2356}}+\mu_2\mu_4\sqrt{A_{24}A_{2456}}+\mu_3\mu_4\sqrt{A_{34}A_{3456}} \tag{2.284}$$

$$P_4 = \mu_1\mu_2\mu_3\mu_4\sqrt{A_{1234}A_{123456}} \tag{2.285}$$

And according to Eq(2.56):

$$P_1=\mu_1\sqrt{A_{15}A_{16}}+\mu_2\sqrt{A_{25}A_{26}}+\mu_3\sqrt{A_{35}A_{36}}+\mu_4\sqrt{A_{45}A_{46}} \tag{2.286}$$

$$P_3 = \mu_1\mu_2\mu_3\sqrt{A_{1235}A_{1236}}+\mu_1\mu_2\mu_4\sqrt{A_{1245}A_{1246}}+$$
$$\mu_1\mu_3\mu_4\sqrt{A_{1345}A_{1346}}+\mu_2\mu_3\mu_4\sqrt{A_{2345}A_{2346}} \tag{2.287}$$

where A is a 6×6 determinant:

$$A = \begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ -a_{12} & 0 & a_{23} & a_{24} & a_{25} & a_{26} \\ -a_{13} & -a_{23} & 0 & a_{34} & a_{35} & a_{36} \\ -a_{14} & -a_{24} & -a_{34} & 0 & a_{45} & a_{46} \\ -a_{15} & -a_{25} & -a_{35} & -a_{45} & 0 & a_{56} \\ -a_{16} & -a_{26} & -a_{36} & -a_{46} & -a_{56} & 0 \end{vmatrix} \tag{2.288}$$

which according to Theorem 1 can be expressed as:

$$A = \left(a_{12}\sqrt{A_{12}}-a_{13}\sqrt{A_{13}}+a_{14}\sqrt{A_{14}}-a_{15}\sqrt{A_{15}}+a_{16}\sqrt{A_{16}}\right)^2$$

The variables contained in determinant A are the elements of the set A', where:

$$A' = \left\{a_{12}, a_{13}, \ldots, a_{16}, \ldots, a_{23}, \ldots, a_{26}, \ldots, a_{56}\right\} \tag{2.289}$$

and are $\binom{6}{2}$ in number.

The lower order Pfaffians can be similarly found by further application of Theorem 1. They are listed below for future reference:

$$\sqrt{A_{12}} = a_{34}a_{56} - a_{35}a_{46} + a_{36}a_{45} \tag{2.290}$$

$$\sqrt{A_{13}} = a_{24}a_{56} - a_{25}a_{46} + a_{26}a_{45} \tag{2.291}$$

$$\sqrt{A_{14}} = a_{23}a_{56} - a_{25}a_{36} + a_{26}a_{35} \tag{2.292}$$

$$\sqrt{A_{15}} = a_{23}a_{46} - a_{24}a_{36} + a_{26}a_{34} \tag{2.293}$$

$$\sqrt{A_{16}} = a_{23}a_{45} - a_{24}a_{35} + a_{25}a_{34} \tag{2.294}$$

$$\sqrt{A_{23}} = a_{14}a_{56} - a_{15}a_{46} + a_{16}a_{45} \tag{2.295}$$

$$\sqrt{A_{24}} = a_{13}a_{56} - a_{15}a_{36} + a_{16}a_{35} \tag{2.296}$$

$$\sqrt{A_{25}} = a_{13}a_{46} - a_{14}a_{36} + a_{16}a_{34} \tag{2.297}$$

$$\sqrt{A_{26}} = a_{13}a_{45} - a_{14}a_{35} + a_{15}a_{34} \tag{2.298}$$

$$\sqrt{A_{34}} = a_{12}a_{56} - a_{15}a_{26} + a_{16}a_{25} \tag{2.299}$$

$$\sqrt{A_{35}} = a_{12}a_{46} - a_{14}a_{26} + a_{16}a_{24} \tag{2.300}$$

$$\sqrt{A_{36}} = a_{12}a_{45} - a_{14}a_{25} + a_{15}a_{24} \tag{2.301}$$

$$\sqrt{A_{45}} = a_{12}a_{36} - a_{13}a_{26} + a_{16}a_{23} \tag{2.302}$$

$$\sqrt{A_{46}} = a_{12}a_{35} - a_{13}a_{25} + a_{15}a_{23} \tag{2.303}$$

$$\sqrt{A_{56}} = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} \tag{2.304}$$

We see that $N_{5,6}$ is of the form:

$$N_{5,6} = M^2 - N^2 = (M+N)(M-N) \tag{2.305}$$

where, M is an even, and N is an odd polynomial. In order to retain a Strictly Hurwitz Polynomial (SHP), we keep $M+N$, and by letting $\mu_1 = \mu_2 = s_1$ and

$\mu_3=\mu_4=s_2$ we obtain:

$$H(s_1,s_2) = M+N \tag{2.306}$$

with:

$$M = F_1+K_1 s_1^2+G_1 s_2^2+B_1 s_1 s_2+I_1 s_1^2 s_2^2 \tag{2.307}$$

$$N = C_1 s_1+D_1 s_2+J_1 s_1^2 s_2+E_1 s_1 s_2^2 \tag{2.308}$$

where:

$$B_1 = a_{24}\sqrt{A_{13}}+a_{23}\sqrt{A_{14}}+a_{14}\sqrt{A_{23}}+a_{13}\sqrt{A_{24}} \tag{2.309}$$

$$C_1 = \sqrt{A_{15}A_{16}}+\sqrt{A_{25}A_{26}} \tag{2.310}$$

$$D_1 = \sqrt{A_{35}A_{36}}+\sqrt{A_{45}A_{46}} \tag{2.311}$$

$$E_1 = a_{25}a_{26}+a_{15}a_{16} \tag{2.312}$$

$$F_1 = \sqrt{AA_{56}} \tag{2.313}$$

$$G_1 = a_{12}\sqrt{A_{34}} \tag{2.314}$$

$$I_1 = \hat{a}_{56} \tag{2.315}$$

$$J_1 = a_{45}a_{46}+a_{35}a_{36} \tag{2.316}$$

$$K_1 = a_{34}\sqrt{A_{12}} \tag{2.317}$$

2.7.2 The New VSHP.

Equations (2.290) to (2.304) are 15 highly nonlinear and difficult to manipulate equations. To ease the mathematics considerably, and yet not to obtain a degenerate solution, that is starting from even part Eq(2.1) and ending with the same even part, the following five sufficiency conditions

$$a_{26}a_{34} = a_{24}a_{36} \tag{2.318}$$
$$a_{25}a_{34} = a_{24}a_{35} \tag{2.319}$$
$$a_{16}a_{35} = a_{15}a_{36} \tag{2.320}$$

$$a_{16}a_{34} = a_{14}a_{36} \qquad (2.321)$$

$$a_{16}a_{23} = a_{13}a_{26} \qquad (2.322)$$

are used to obtain a new transfer function.

We notice that the fifteen Pfaffians in Eqs(2.290) to (2.304) each has three addends, but with the new conditions twelve of them become single term each. Now they are:

$$\sqrt{A_{12}} = a_{34}a_{56} - a_{35}a_{46} + a_{36}a_{45} \qquad (2.323)$$

$$\sqrt{A_{13}} = \frac{a_{25}}{a_{35}}\sqrt{A_{12}} \qquad (2.324)$$

$$\sqrt{A_{23}} = \frac{a_{16}}{a_{36}}\sqrt{A_{12}} \qquad (2.325)$$

$$\sqrt{A_{14}} = a_{23}a_{56} \qquad (2.326)$$

$$\sqrt{A_{15}} = a_{23}a_{46} \qquad (2.327)$$

$$\sqrt{A_{16}} = a_{23}a_{45} \qquad (2.328)$$

$$\sqrt{A_{24}} = a_{13}a_{56} \qquad (2.329)$$

$$\sqrt{A_{25}} = a_{13}a_{46} \qquad (2.330)$$

$$\sqrt{A_{26}} = a_{13}a_{45} \qquad (2.331)$$

$$\sqrt{A_{34}} = a_{12}a_{56} \qquad (2.332)$$

$$\sqrt{A_{35}} = a_{12}a_{46} \qquad (2.333)$$

$$\sqrt{A_{36}} = a_{12}a_{45} \qquad (2.334)$$

$$\sqrt{A_{45}} = a_{12}a_{36} \qquad (2.335)$$

$$\sqrt{A_{46}} = a_{12}a_{35} \tag{2.336}$$

$$\sqrt{A_{56}} = a_{12}a_{34} \tag{2.337}$$

Further, we see that we have 10 independent variables, instead of 15 as in A' of Eq(2.289), these which we call it the set A'' are:

$$A'' = \left\{ a_{12}, a_{16}, a_{23}, a_{26}, a_{34}, a_{35}, a_{36}, a_{45}, a_{46}, a_{56} \right\} \tag{2.338}$$

The other 5 variables in the set A' can be expressed in terms of these 10 variables as:

$$a_{13} = \frac{a_{16}a_{23}}{a_{26}}, \quad a_{14} = \frac{a_{16}a_{34}}{a_{36}}, \quad a_{15} = \frac{a_{16}a_{35}}{a_{36}}$$

$$a_{24} = \frac{a_{26}a_{34}}{a_{36}}, \quad a_{25} = \frac{a_{24}a_{35}}{a_{34}} = \frac{a_{26}a_{35}}{a_{36}} \tag{2.339}$$

With these variables Eqs(2.309) to (2.317) becomes:

$$B_1 = \sqrt{A_{12}}a_{34}\left(a_{16}^2 + a_{26}^2\right)\frac{1}{a_{36}^2} + a_{56}\left(a_{16}^2 + a_{26}^2\right)\frac{a_{23}^2}{a_{26}^2} \tag{2.340}$$

$$C_1 = \left(a_{16}^2 + a_{26}^2\right)\frac{a_{23}^2}{a_{26}^2} \tag{2.341}$$

$$D_1 = \left(a_{35}a_{36} + a_{45}a_{46}\right)a_{12}^2 \tag{2.342}$$

$$E_1 = \frac{a_{35}}{a_{36}}\left(a_{16}^2 + a_{26}^2\right) \tag{2.343}$$

$$F_1 = a_{12}^2\sqrt{A_{12}}a_{34} \tag{2.344}$$

$$G_1 = a_{12}^2a_{56} \tag{2.345}$$

$$I_1 = a_{56} \tag{2.346}$$

$$J_1 = a_{35}a_{36}+a_{45}a_{46}$$

$$K_1 = a_{34}\sqrt{A_{12}} \tag{2.348}$$

Moreover these coefficients, B_1 to K_1 need to be positive, which can be achieved by setting:

$$\left|a_{34}\sqrt{A_{12}}\right| = Z_1 \quad , \quad \left|a_{35}a_{36}\right| = Z_2 \quad , \quad \left|a_{45}a_{46}\right| = Z_3$$

$$\left|a_{56}\right| = Z_4 \quad , \quad a_{16}^2+a_{26}^2 = Z_5 \tag{2.349}$$

and then we let

$$B = Z_1Z_5\frac{1}{a_{36}^2}+\frac{a_{23}^2}{a_{26}^2}Z_4Z_5>0$$

$$C = \frac{a_{23}^2}{a_{26}^2}Z_3Z_5>0$$

$$D = \left(Z_2+Z_3\right)a_{12}^2>0$$

$$E = \frac{Z_2}{a_{36}^2}Z_5>0 \tag{2.350}$$

$$F = a_{12}^2Z_1>0$$

$$G = a_{12}^2Z_4>0$$

$$I = Z_4>0$$

$$J = Z_2+Z_3>0$$

$$K = Z_1>0$$

And from now on we take B, $\cdots$, K, which are positive coefficients instead of B_1, $\cdots$, K_1

It is of interest to see if the resulting second order 2-variable function is VSHP. We first check the SHP nature of the function by applying Ansel's test [38].

To this end we have to show that $T(s_1,s_2)=\frac{M}{N}$ is a reactance function, that is it obeys the following three conditions:

Condition 1) -

$$T(-s_1,-s_2) = -T(s_1,s_2) \tag{2.347}$$

which is fulfilled, since M is an even and and N is an odd polynomial. Therefore $T=\frac{M}{N}$ is odd.

Condition 2) - The polynomial in s_1, $H(s_1,1)$ must not have zeros in the closed right half plane, that is

$$H(s_1,1) = (D_1+F_1+G_1)+(B_1+C_1+E_1)s_1+(I_1+J_1+K_1)s_1^2 \tag{2.351}$$

should be SHP, which is always the case since coefficients of s_1 are positive. where: *Condition 3)* - We form the nonzero polynomial:

$$H(j\omega,j\Omega) = b_0\Omega^2+b_1\Omega+b_2+j(a_0\Omega^2+a_1\Omega+a_2) \tag{2.352}$$

$$b_0 = I\omega^2-G \tag{2.353}$$

$$b_1 = -B\omega \tag{2.354}$$

$$b_2 = -K\omega^2+F \tag{2.355}$$

$$a_0 = -E\omega \tag{2.356}$$

$$a_1 = -J\omega^2+D \tag{2.357}$$

$$a_2 = C\omega \tag{2.358}$$

Let $c_{r,s}(\omega)$ denote the polynomial defined by:

$$c_{r,s} = a_r\, b_s - a_s\, b_r \tag{2.359}$$

for:

$$0 \leq r, s \leq 2 \tag{2.360}$$

Let $R(\omega)$ denote the 2×2 symmetric polynomial matrix whose typical element $R_{ij}(\omega)$ $(1 \leq i,j \leq 2)$ is the sum of all those polynomials $c_{r,s}(\omega)$ for which both

$$s + r = i + j - 1 \tag{2.361}$$

and

$$s - r > |\, i - j \,| \tag{2.362}$$

are satisfied. Then the two successive principal minors of $R(\omega)$, that is $M_1 = R_{11}$ and

$$M_2 = \begin{vmatrix} R_{11} & R_{12} \\ R_{12} & R_{22} \end{vmatrix}$$

must be non-negative for all real ω.

Here we have:

$$M_1 = JI\,\omega^4 + \omega^2(EB - DI - JG) + DG \tag{2.363}$$

$$M_2 = R_{11}R_{22} - R_{12}^2$$

$$= \left[JI\,\omega^4 + (EB - DI - JG)\omega^2 + DG\right] \times \left[JK\,\omega^4 + (BC - JF - DH)\omega^2 + DF\right] - \left[(EK - CI)\omega^3 + (CG - EF)\omega\right]^2 \tag{2.364}$$

Here we have:

$$M_1 = S_1 + S_2 \tag{2.365}$$

where,

$$S_1 = JI\,\omega^4 - \omega^2(DI + JG) + DG \tag{2.366}$$

$$S_2 = EB\,\omega^2 \tag{2.367}$$

S_2 is non-negative since E and B are absolute values. We can make S_1 non-negative if we take $DI = JG$. Then:

$$S_1 = JI\omega^4 - 2JG\omega^2 + \frac{JG^2}{I}$$
$$= \frac{J}{I}(I^2\omega 4 - \omega^2 IG + G^2)$$
$$= \frac{A}{I}(I\omega^2 - G)^2 . \qquad (2.368)$$

which is non-negative. Since M_1 is the sum of two non-negative quantities, it is non-negative.

The second principle minor is:

$$M_2 = M_1 Q_1 - \omega^2 Q_2^2 \qquad (2.369)$$

where:

$$Q_1 = S_3 + S_4 \qquad (2.370)$$

with:

$$S_3 = JK\omega^4 - (JF + DK)\omega^2 + DF \qquad (2.371)$$

$$S_4 = BC\omega^2 \qquad (2.372)$$

and,

$$Q_2 = \omega^2(EK - CI) + (CG - EF) \qquad (2.373)$$

If we assume that:

$$DI = JG \text{ and } DK = JF \qquad (2.374)$$

so that:

$$\frac{G}{F} = \frac{I}{K} \text{ or } \frac{I}{G} = \frac{K}{F} \qquad (2.375)$$

since,

$$S_1 = \frac{JG^2}{I}\left(\frac{I}{G}\omega^2 - 1\right)^2 \qquad (2.376)$$

we substitute the value of $\frac{I}{G}$ from Eq(2.375), then:

$$S_1 = \frac{JG^2}{I}\left(\frac{K}{F}\omega^2-1\right)^2$$

$$= \frac{JG^2}{IF^2}(K\omega^2-F)^2$$

$$= \frac{JG^2}{IF^2}Y^2 \tag{2.377}$$

where,

$$Y = K\omega^2-F \tag{2.378}$$

and from Eq(2.375):

$$I^2F^2 = K^2G^2 \tag{2.379}$$

Substituting Eq(2.379) in Eq(2.377):

$$S_1 = \frac{JI}{K^2}Y^2 \tag{2.380}$$

Similarly:

$$S_3 = \frac{J}{K}Y^2 \tag{2.381}$$

We write Eq(2.373) in slightly different form:

$$Q_2 = E(\omega^2K-F)-C(\omega^2I-G) \tag{2.382}$$

since from Eq(2.374) $I = \frac{JG}{D} = \frac{KG}{F}$, therefore;

$$Q_2 = EY-C\left[\omega^2\frac{KG}{F}-G\right]$$

$$= EY-\frac{CG}{F}Y$$

$$= Y\left[E-\frac{CG}{F}\right] \tag{2.383}$$

Therefore,

$$-\omega^2Q_2^2 = -S_5-S_6+S_7 \tag{2.384}$$

where,

$$S_5 = E^2\omega^2Y^2 \tag{2.385}$$

$$S_6 = \frac{C^2G^2}{F^2}\omega^2Y^2$$

$$= \frac{C^2 I^2}{K^2} \omega^2 Y^2 \tag{2.386}$$

$$S_7 = 2\frac{ECI}{K} \tag{2.387}$$

Where S_1 to S_7 are all non-negative.

Substituting Eqs(2.365),(2.370), and (2.384) in Eq(2.369) we get:

$$M_2 = S_1S_3+S_1S_4+S_2S_3+S_2S_4-S_5-S_6+S_7 \tag{2.388}$$

Since:

$$S_1S_4-S_6 = \frac{\omega^2 Y^2 CI}{K^2}(JB-IC) \tag{2.389}$$

If we let:

$$B = B_2+B_3 \quad J=J_2+J_3 \tag{2.390}$$

Then by taking:

$$J_2B_2-IC = 0 \tag{2.391}$$

we get:

$$S_1S_4-S_6 \geq 0 \tag{2.392}$$

Similarly:

$$S_2S_3-S_5 = E\,\omega^2 Y^2 \left(\frac{BJ}{K}-E\right) \tag{2.393}$$

If we take

$$B_3J_3 = EK \tag{2.394}$$

then we get:

$$S_2S_3-S_5 \geq 0 \tag{2.395}$$

Therefore, with the assumptions:

$$DI=JG \quad DK=JF \quad J_2B_2=IC \quad J_3B_3=EK \tag{2.396}$$

Note that the four conditions in Eq(2.396) are simplification of Ansel's test, and by yet we have not specialized it for our case. That is they are sufficient conditions for stability of any second order polynomial as in Eqs(2.306) to (2.308), we can be sure that M_1 and M_2 are always non-negative for all real values of ω in our case.

For the first condition in Eq(2.396) we have

$$D_1 I_1 = J_1 a_{12}^2 a_{56} = J_1 G_1 \tag{2.398}$$

Similarly, with:

$$B_2 = \frac{a_{23}^2}{a_{26}^2} Z_4 Z_5 \tag{2.399}$$

$$B_3 = \frac{Z_1 Z_5}{a_{36}^2} \tag{2.400}$$

$$J_2 = Z_3 \tag{2.401}$$

$$J_3 = Z_2 \tag{2.402}$$

we easily check other conditions in Eq(2.396).

Now that we have an SHP, we would like to apply double bilinear transformation to get a digital filter. However, it is shown [32] that not all SHP's are candidates for double bilinear transformation. The remedy for this is given by VSHP [33]. Applying the VSHP test we get:

$$H\left(\frac{1}{s_1}, j\omega_2\right) = \tag{2.403}$$

$$\frac{Fs_1^2 + K - G\omega_2^2 s_1^2 + j\omega_2 s_1 - I\omega_2^2 + Cs_1 + jD\omega_2 s_1^2 + jJ\omega_2 - E\omega_2^2 s_1}{s_1^2}$$

Therefore:

$$\lim_{s_1 \to 0} H\left(\frac{1}{s_1}; j\omega_2\right) = \frac{K - I\omega_2^2 + jJ\omega_2}{0} \tag{2.404}$$

which is not $\frac{0}{0}$ if

$$K - I\omega_2^2 + jJ\omega_2 \neq 0 \tag{2.405}$$

Similarly:

$$\lim_{\substack{s_1 \to 0 \\ s_2 \to 0}} H\left(\frac{1}{s_1},\frac{1}{s_2}\right) = \frac{I}{0} \tag{2.406}$$

which is not $\frac{0}{0}$ if $I \neq 0$

In all we should have $I \neq 0$, $K \neq 0$, and $G \neq 0$ which leads to:

$$a_{12} \neq 0 \quad a_{56} \neq 0 \quad A_{12} \neq 0 \tag{2.407}$$

In short our 2-D digital filter is a bilinear transformation:

$$s_i = \frac{2}{T}\frac{z_i - 1}{z_i + 1} \qquad i = 1,2 \tag{2.408}$$

to the 2-D continuous transfer function:

$$T(s_1,s_2) = \frac{\sum_{i=0}^{2}\sum_{j=0}^{2} c_{ij}\, s_1^i s_2^j}{H(s_1,s_2)} \tag{2.409}$$

where:

$$\begin{aligned} H(s_1,s_2) = {} & F + Ks_1^2 + Gs_2^2 + Bs_1 s_2 + Is_1^2 s_2^2 \\ & Cs_1 + Ds_2 + Js_1^2 s_2 + Es_1 s_2^2 \end{aligned} \tag{2.410}$$

with:

$$\begin{aligned} B &= Z_1 Z_5 \frac{1}{a_{36}^2} + \frac{a_{23}^2}{a_{26}^2} Z_4 Z_5 > 0 \\ C &= \frac{a_{23}^2}{a_{26}^2} Z_3 Z_5 > 0 \\ D &= \left(Z_2 + Z_3\right) a_{12}^2 > 0 \\ E &= \frac{Z_2}{a_{36}^2} Z_5 > 0 \\ F &= a_{12}^2 Z_1 > 0 \end{aligned} \tag{2.411}$$

$$G = a_{12}^2 Z_4 > 0$$

$$I = Z_4 > 0$$

$$J = Z_2 + Z_3 > 0$$

$$K = Z_1 > 0$$

and:

$$\left| a_{34}\sqrt{A_{12}} \right| = Z_1 \quad , \quad \left| a_{35} a_{36} \right| = Z_2 \quad , \quad \left| a_{45} a_{46} \right| = Z_3$$

$$\left| a_{56} \right| = Z_4 \quad , \quad a_{16}^2 + a_{26}^2 = Z_5$$

$$\sqrt{A_{12}} = a_{34} a_{56} - a_{35} a_{46} + a_{36} a_{45}$$

2.7.3 Comparison

Since in the reactance function of Eq(2.3), which we started with, the generating polynomial was its even part M_{2n} of Eq(2.2). Therefore, it is fair to compare M_4 with the even part of its derivative $N_{5,6}$ of Eq(2.282), which we call it M_6':

$$M_6' = P_0 + P_2 + P_4 \tag{2.412}$$

where P_0, P_2, and P_4 are given in Eqs(2.283) to (2.285).

For this we indicate the elements of the determinant A in Eq(2.2) by b_{uv} and those of A_6' in Eq(2.411) as a_{uv}.

Equating the coefficients of like variables we get:

$$a_{56} = 1 \tag{2.413}$$

$$b_{34}^2 = a_{34}\sqrt{A_{12}} \tag{2.414}$$

$$b_{24}^2 = a_{24}\sqrt{A_{13}} \tag{2.415}$$

$$b_{23}^2 = a_{23}^2 a_{56} \tag{2.416}$$

$$b_{14}^2 = a_{14}\sqrt{A_{23}} \tag{2.417}$$

$$b_{13}^2 = a_{13}^2 a_{56} \tag{2.418}$$

$$b_{12}^2 = a_{12}^2 a_{56} \tag{2.419}$$

$$B = \left(b_{12}b_{34} - b_{13}b_{24} + b_{14}b_{23}\right)^2 = a_{12}^2 a_{34}\sqrt{A_{12}} \tag{2.420}$$

The two sides are equal, if and only if:

$$a_{35}a_{46} = a_{36}a_{45}$$

which makes:

$$\sqrt{A_{12}} = a_{34}a_{56}$$

in which case we find a degenerate case, that we have started from one form and end up with the same case.

2.8 Summary and Discussion

The derivative of the reactance functions, Z_{LC}, for two cases of Eqs(2.3) and (2.6) are established in Theorems 5 and 6 respectively. Some properties of these derivatives are indicated in Sections 2.5 and 2.6.

A special case of Theorem 5 is used to generate a new VSHP in Section 2.7. It should be noted that higher order VSHP can be formed if the order of the generating skew-symmetrical determinant is increased. This new VSHP will be used in Chapter 3 in making of various filters.

Chapter 3
Design of 2-D Stable Recursive Digital filters

3.1 Introduction

In this chapter, based on the results of previous chapter, designs are given for stable 2-D recursive filters (a)-having separable denominator transfer function, (b)-having non-separable numerator and denominator transfer functions. These are suitable for the design of 2-D filters satisfying (i) a prescribed magnitude specification only (ii) a prescribed magnitude and group delay specifications.

These design approaches yield a transfer function for the desired 2-D filter having coefficients with high precision. Implementation of this transfer function on a finite register machine would alter the filter specification, and some times the stability of the filter, which is undesirable. To overcome this problem a method will be presented here enabling us to design an integer coefficient 2-D filter so that (a)-fixed point arithmetic is employed in the implementation of such filter rather than floating point arithmetics, (b)-the effect of round off error in the implementation of such filter on the finite register machine is kept to minimum. Examples will also be given to illustrate the usefulness of these proposed techniques.

3.2 Design of Separable Denominator 2-D Recursive Filter

In this approach the transfer function of the filter is of the form:

$$H_D(z_1,z_2) = \frac{\sum_{i=0}^{M}\sum_{j=0}^{N} a_{ij} z_1^{-i} z_2^{-j}}{\left(\sum_{i=0}^{M} b_{1i} z_1^{-i}\right)\left(\sum_{j=0}^{N} b_{2i} z_1^{-j}\right)}$$

$$= \frac{N(z_1,z_2)}{D(z_1,z_2)} = \frac{N(z_1,z_2)}{D_1(z_1)D_2(z_2)} \tag{3.1}$$

Using the method in previous chapter for generating 2-variable VSHP it can easily be shown as an example that if in Eq(2.289) all members of the set A' are equal, and their absolute value is c_1, then Eq(2.410), apart from a constant multiplier c_1, reduces to:

$$Q(s_1,s_2) = (s_1+c_1)^2(s_2+c_1)^2$$
$$= Q_1(s_1)Q_2(s_2) \tag{3.2}$$

and is portraying a stable separable 2-variable polynomial in the variables s_1 and s_2. Other examples can similarly be found. It should be noted that the matrix chosen in Eq(2.288) is a 6x6 matrix which has resulted in a cascade of the second order polynomial in s_1 and s_2 as in Eq(3.2). Higher order 2-variable polynomials are obtainable by increasing the order of the determinant A in Eq(2.288).

Now $Q(s_1,s_2)$ of Eq(3.2) is assigned to the denominator of a 2-variable separable denominator analog transfer function of the form:

$$H_A(s_1,s_2) = \frac{\sum_{i=0}^{2}\sum_{j=0}^{2} p_{ij}\, s_1^i s_2^j}{Q(s_1,s_2)}$$
$$= \frac{P(s_1,s_2)}{Q(s_1,s_2)} \tag{3.3}$$

The numerator is chosen as in Eq(1.11) so that the filter can have quadrantal symmetry.

To discretize this 2-D analog transfer function, bilinear transformations are applied to Eq(3.3) as follows:

$$H_D(z_1,z_2) = H_A(s_1,s_2)\Big|_{s_i=\frac{2}{T}\frac{z_i-1}{z_i+1}} \quad i=1,2 \tag{3.5}$$

Now the error between the magnitude response of the ideal 2-D filter and the desired one is calculated using the relationship:

$$E_{Mag}(\omega_1,\omega_2,\underline{\Psi}) = |H_d(e^{j\omega_1 T},e^{j\omega_2 T})| - |H_i(e^{j\omega_1 T},e^{j\omega_2 T})| \tag{3.6}$$

where, $\underline{\Psi}$ is the coefficient vector, $|H_d|$ and $|H_i|$ are the magnitude responses of designed and ideal filter respectively.

The error between the group-delay responses of the designed and ideal 2-D filter is calculated using the relationship:

$$E_{\tau_i} = \tau_I \cdot T - \tau_i (e^{j\omega_1 T}, e^{j\omega_2 T}) \quad i=1,2 \tag{3.7}$$

where τ_I is the ideal group-delay response and τ_i is the designed group-delay with respect to ω_i, and it is proven [18] that τ_I is equal to the order of the filter ± 1.

The gradients of the absolute values in Eq(3.6) are calculated analytically by:

$$\frac{\partial |H|}{\partial x} = \frac{1}{|H|} \mathrm{Re} \left[\overline{H} \frac{\partial H}{\partial x} \right] \tag{3.8}$$

and the group-delays in Eq(3-7) are calculated by [40]:

$$\tau_i = -Re \left[z_i \frac{\dfrac{\partial H(z_i)}{\partial z_i}}{H(z_i)} \right] \quad i=1,2 \tag{3.9}$$

with $z_i = e^{j\omega T}$

Now two cases will be considered.

(i) - Formulation of the Design Problem with Magnitude Specification Only with Quadrantal Symmetry

In this approach the least mean square error of the magnitude response is calculated using Eq(3.6) in the following relationship:

$$E_{l_2}(j\omega_1, j\omega_2, \underline{\Psi}) = \sum_{m,n \in I_{PS}}\sum E_{Mag}^2(\omega_1, \omega_2, \underline{\Psi}) \tag{3.10}$$

where I_{PS} is a set of all frequency points along ω_1 and ω_2 axis within passband and stopband of the filter. Coefficient vector $\underline{\Psi}$ can now be calculated by minimizing E_{l_2} in Eq(3.10). This is a nonlinear programming problem and can be solved by using the conjugate gradient method of Fletcher and Powell [41]. All programs were run on VAX 11/780 with double precision. The continuous time frequency were sampled at 21 points.

The flow chart of the programs is shown in Fig(3.1). Note that this flow chart is valid for Example 3.1 through 3.4, the only block which changes is the Block A.

Example 3.1

To show how this technique works we design a 2-D low pass filter with H_i in Eq(3.6) expressing the following specification:

$$| H_i(\omega_1,\omega_2) | = \begin{cases} 1 & \text{for} \quad \sqrt{\omega_1^2+\omega_2^2} \leq 1 \\ 0 & \text{for} \quad \sqrt{\omega_1^2+\omega_2^2} > 2 \end{cases}$$

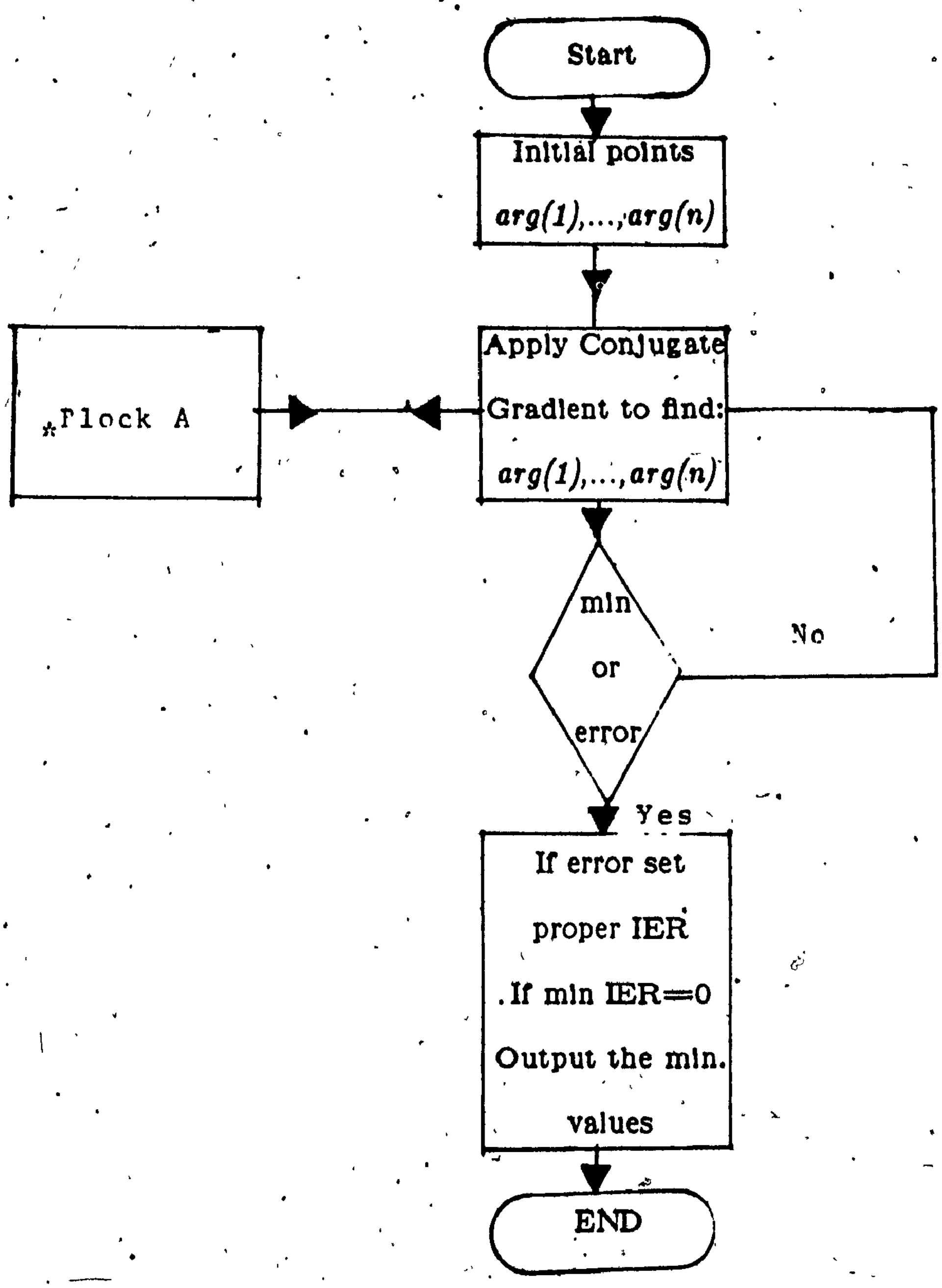
and the transfer function $H_d = N/D$ in Eq(3.6) is obtained with numerator in the form of Eq(1.11) as:

$$N = (c_2(z_1+1./z_1)+c_3 z_2)(c_4(z_2+1/z_2)+c_5 z_1) \Big|_{z_i = e^{j\omega_i T}} \quad i=1,2$$

and the denominator as Eq(3.2) repeated here:

$$D = c_1(c_1+s_1)^2(c_1+s_2)^2 \Big|_{\frac{2}{T}\frac{z_i-1}{z_i+1}} \quad i=1,2$$

Fig(3.2) shows the particular part of the flow chart for this Example, the other parts of the flow chart are as in Fig(3.1). The minimized error in Eq(3.6) was 2.70, and the Central Processor Unit (CPU) time was 31 seconds. Table(3.1) indicates the coefficients, whereas Fig(3.3) shows the magnitude response of this filter.



*Block A:

For Example 3.1 see fig(3.2)

For Example 3.2 see fig(3.4)

For Example 3.3 see fig(3.8)

For Example 3.4 see fig(3.12)

Fig(3.1) - Part of the flow chart for Examples 3.1 through 3.4

$$z_1 = e^{jTk\Delta} \qquad z_2 = e^{jTl\Delta}$$
$$s_m = \frac{2}{T}\frac{z_m - 1}{z_m + 1} \qquad m = 1,2$$

$$\downarrow$$

$$N1_{k,l} = z_1 z_2$$
$$N2_{k,l} = arg(2)(z_1 + z_1^{-1}) + arg(3) * z_2$$
$$N3_{k,l} = arg(4)(z_2 + z_2^{-1}) + arg(5) * z_1$$
$$N_{k,l} = N1_{k,l}\, N2_{k,l}\, N3_{k,l}$$

$$\downarrow$$

$$F = |\, arg(1)\, |$$
$$D_{k,l} = F(s_1 + F)^2 (s_2 + F)^2$$

$$\downarrow$$

$$HMN_{k,l} = \frac{\left| \dfrac{N_{k,l}}{D_{k,l}} \right|}{|H_{0,0}|} \; *$$

$$\downarrow$$

$$G_{k,l} = \begin{cases} 1 & \omega \le 1 \\ e^{-2(\omega - 1)} & 1 \le \omega \le 2 \\ 0 & otherwise \end{cases}$$

$$\downarrow$$

$$val = \sum_{k=-10}^{10} \sum_{l=-10}^{10} \left(HMN_{k,l} - G_{k,l} \right)^2$$

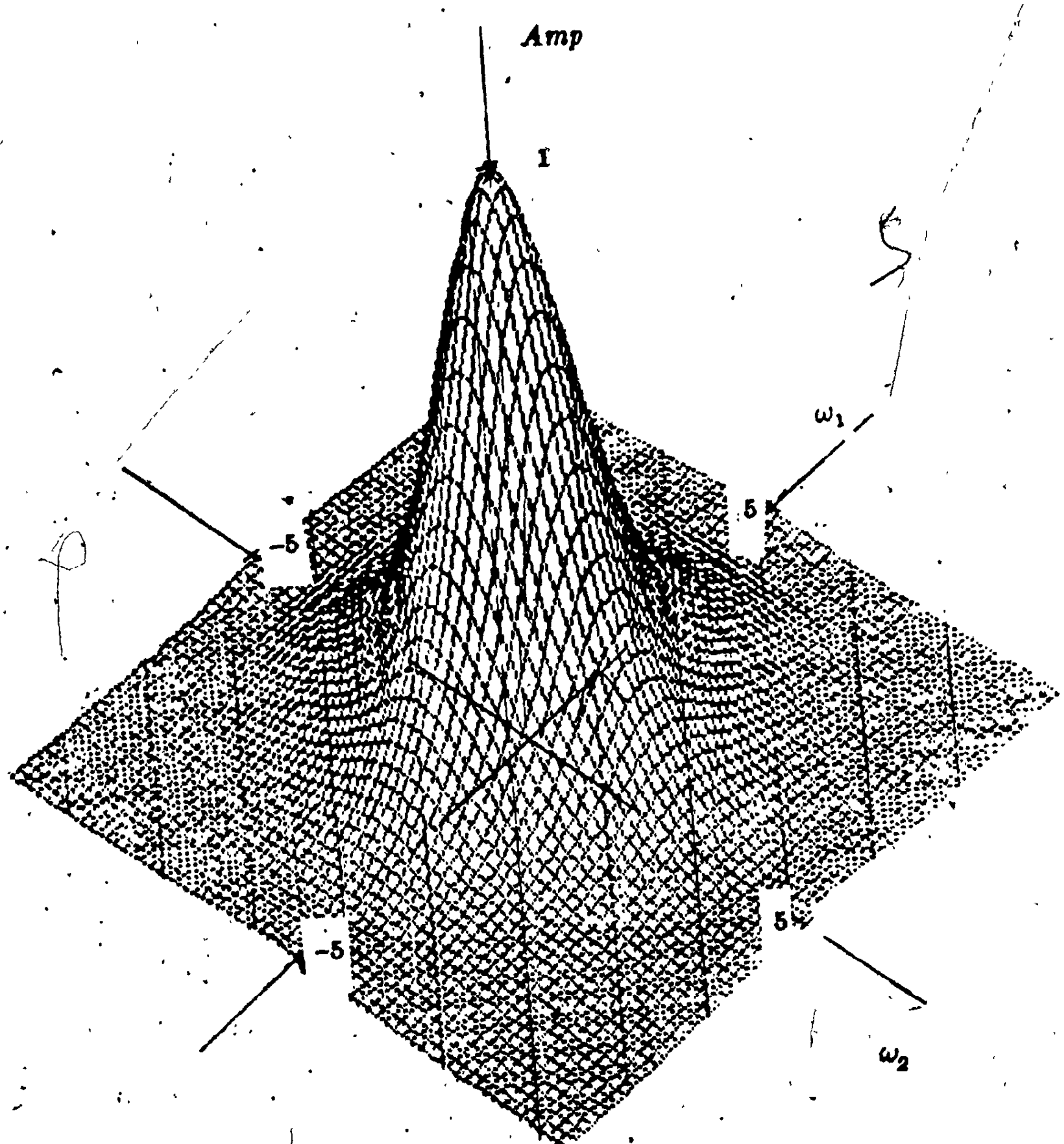
$$grad(m) = \partial val / \partial arg(m) \qquad m = 1, \ldots, n$$

$*H_{0,0}$ is N/D at freq. (0,0)

Fig(3.2) - Block A in Fig(3.1) for Example 3.1

Table(3.1) :Coefficients of the filter in Example 3.1

c_1	0.1333186633E+01	c_4	0.2559358544E+01
c_2	0.2559358544E+01	c_5	-0.138610581E+01
c_3	-0.1386105815E+01		



Fig(3.3) - Amplitude response of the filter in Example 3,1

(ii) - Formulation of the Design Problem with Magnitude and Group-Delay Specifications with Quadrantal Symmetry

In many image processing applications it has been proven that the the 2-D filters must have constant group-delay characteristics [42] to be useful in any preprocessing or postprocessing in the image processing application. For this reason a modification to the above algorithm will be given here to include the design of 2-D filters with constant group-delay specification. In this formulation the general least mean square error is calculated using the following relationship:

$$E_G(j\omega_1,j\omega_2,\underline{\Psi}) = \sum_{m,n \in I_{PS}}\sum E^2_{Mag}(\omega_1;\omega_2,\underline{\Psi}) + \sum_{m,n \in I_P}\sum E^2_{\tau_1}(\omega_1,\omega_2,\underline{\Psi}) + \sum_{m,n \in I_P}\sum E^2_{\tau_2}(\omega_1,\omega_2,\underline{\Psi}) \tag{3.11}$$

where E_{Mag} and E_{τ_j} $j=1,2$ are given as in Eqs(3.6) and (3.7), I_{PS} is a set of all discrete frequency pairs along ω_1 and ω_2 axis covering the passband and stopband of the filter while I_P is a set of all desired frequency points along ω_1, ω_2 axis covering passband of the filter only. Now the coefficient vector $\underline{\Psi}$ can be calculated by minimizing E_G in Eq(3.11) using any suitable nonlinear optimization technique.

Example 3.2

To show how this technique works we design a 2-D low pass filter with H_i in Eq(3.6) expressing the following specification:

$$|H_i(\omega_1,\omega_2)| = \begin{cases} 1 & \text{for } \sqrt{\omega_1^2+\omega_2^2} \le 1 \\ 0 & \text{for } \sqrt{\omega_1^2+\omega_2^2} > 2 \end{cases}$$

and the transfer function $H_d = N/D$ in Eq(3.6) is obtained with numerator in the form of Eq(3.11) as:

$$N = (c_2(z_1+1./z_1)+c_3z_2)(c_4(z_2+1/z_2)+c_5z_1)\Big|_{z_i=e^{j\omega_i T}} \quad i=1,2$$

and the denominator as Eq(3.2) repeated here:

$$D = c_1(c_1+s_1)^2(c_1+s_2)^2 \Big|_{\frac{2}{T}\frac{z_i-1}{z_i+1}} \quad i=1,2$$

The error between the group-delay responses of the designed and ideal 2-D filter is calculated using the relationship:

$$E_{\tau_i} = \tau_I\, T - \tau_i\,(e^{j\omega_1 T}, e^{j\omega_2 T}) \quad i=1,2$$

where τ_I is the ideal group-delay response and τ_i is the designed group-delay with respect to ω_i, and τ_I is equal to the order of the filter ± 1. The corresponding flow chart to fill-up the block A in Fig(3.1) is shown in Fig(3.4), and the coefficients were as given in Table(3.2).

The 3-D plot of amplitude and group delays are shown in Fig(3.5) to Fig(3.7). The minimized error in Eq(3.11) was 6.31, and the CPU time was 3.6 seconds.

It is interesting to note that Fig(3.3) and Figs(3.5) to (3.7) indicate good responses and all these have been obtained with only five coefficients, c_1 to c_5.

Block A

in Fig(3.2)

$$\tau_m = -Real\left[z_m \frac{\dfrac{\partial H(z_m)}{\partial z_m}}{H(z_m)} \right] \qquad m = 1,2$$

tt=order of filter ± 1

For $\omega < 2$

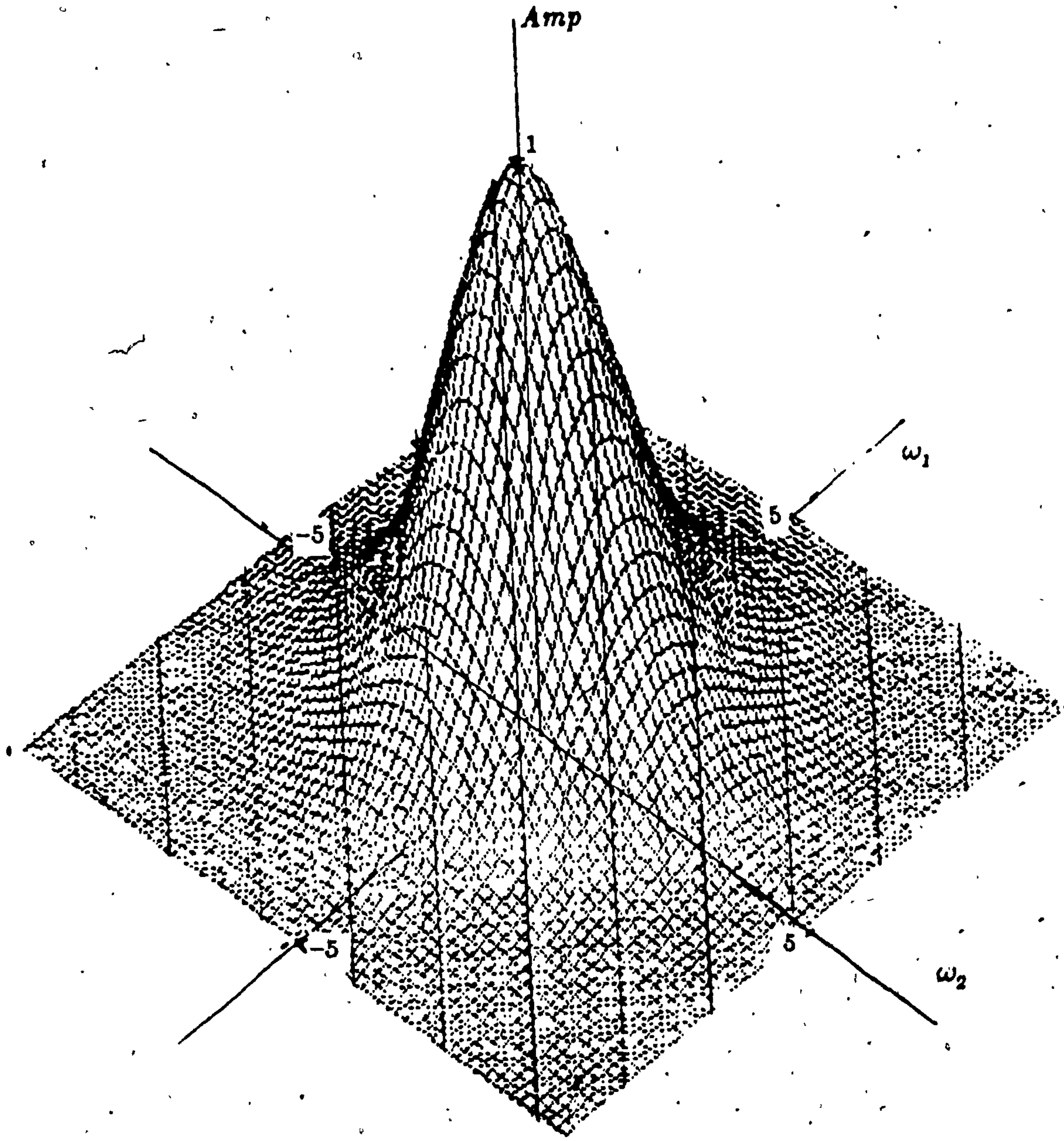
$$val' = val + \sum_{k=-10}^{10} \sum_{l=-10}^{10} (\tau_1 - tt)^2 + (\tau_2 - tt)^2$$

$$grad(m) = \partial val' / \partial arg(m) \qquad m = 1,\dots,n$$

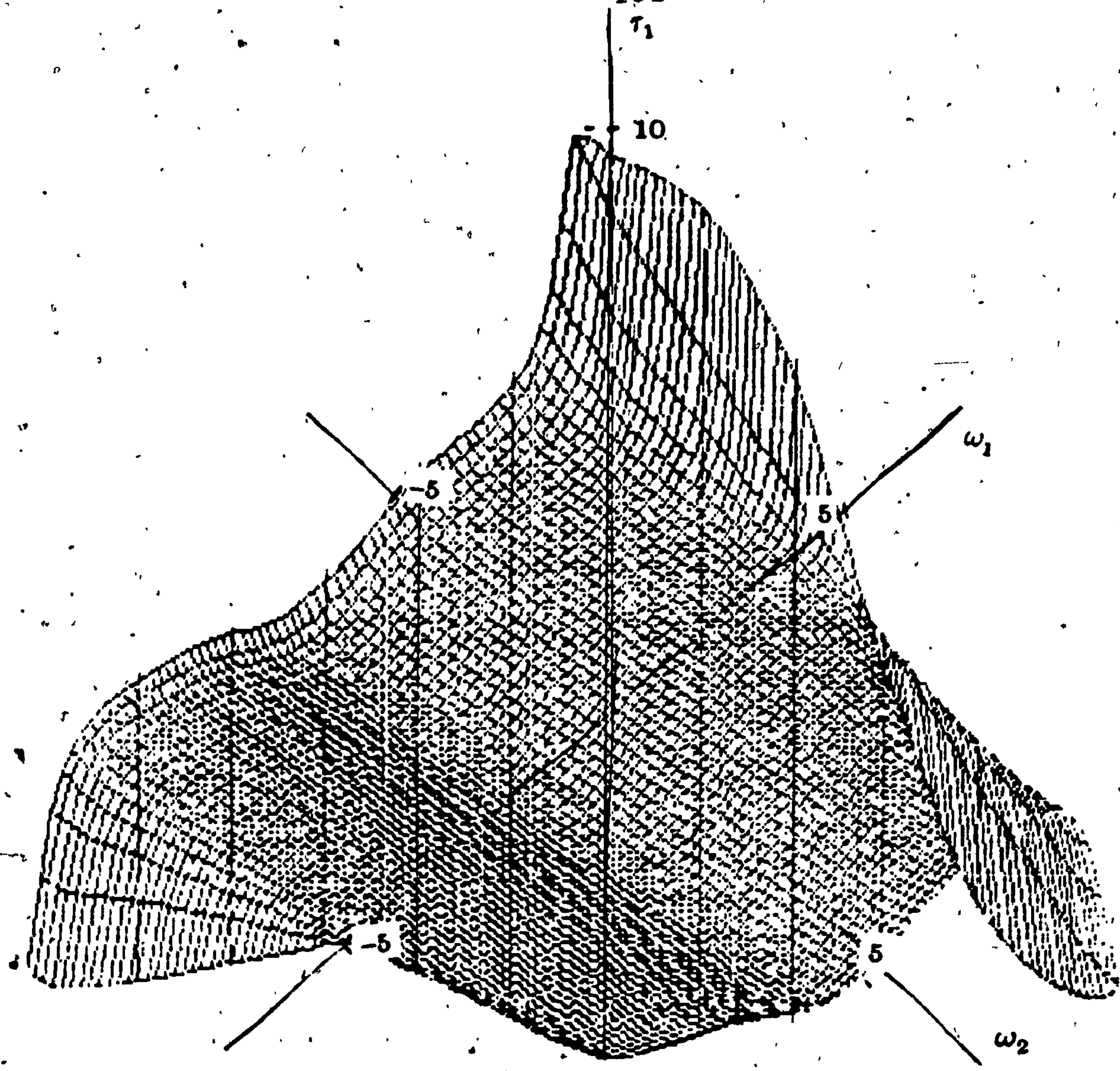
Fig(3.4) - Block A in Fig(3.1) for Example 3.2.

Table(3.2) :Coefficients of the filter in Example 3.2

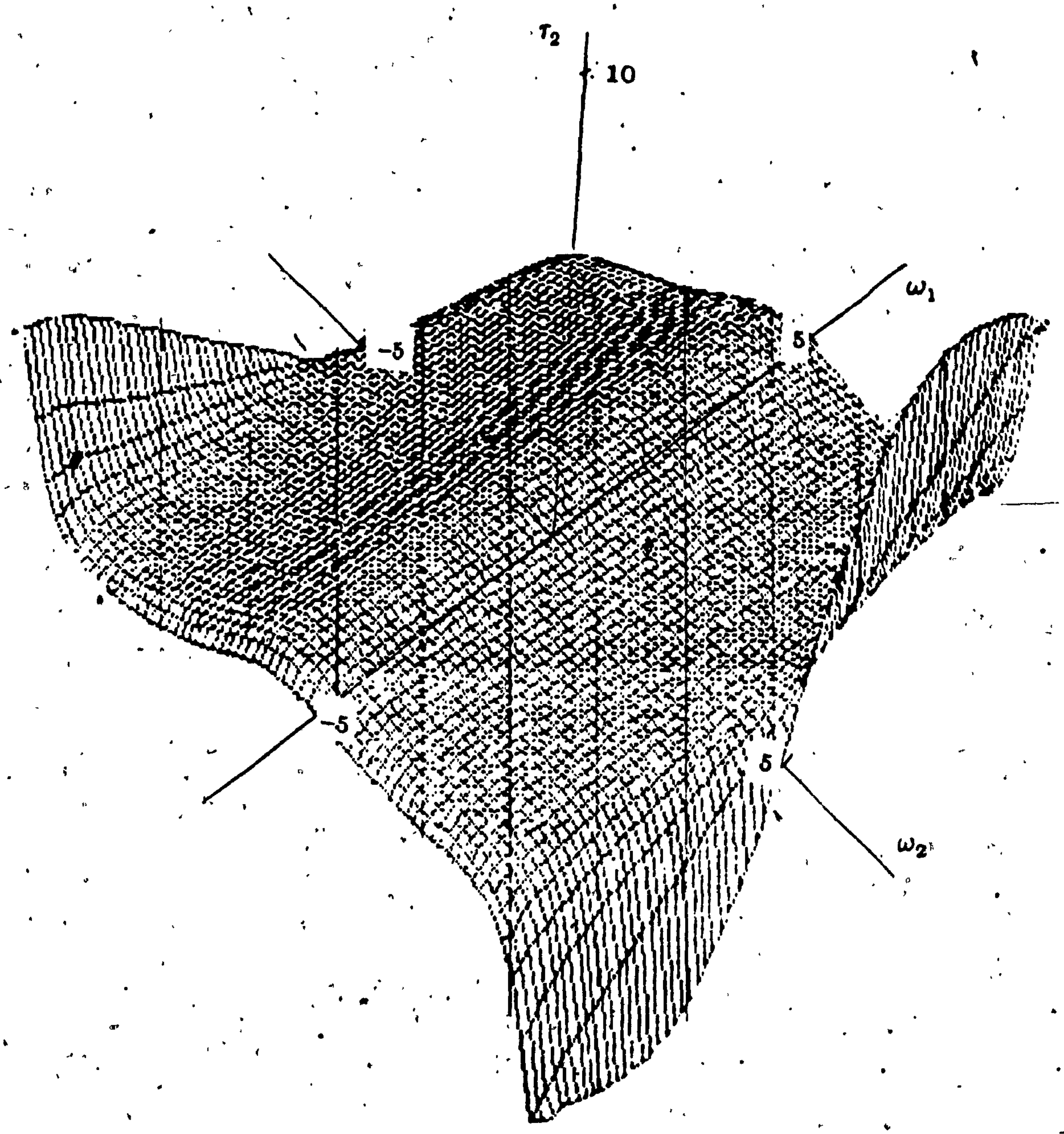
c_1	0.1773225247E+01	c_4	0.4460828576E+01
c_2	0.4460828576E+01	c_5	0.7796479967E+01
c_3	0.7796479967E+01		



Fig(3.5) - Amplitude response of the filter in Example 3.2



Fig(3.6) - Group delay 1 of the filter in Example 3.2



Fig(3.7) - Group delay 2 of the filter in Example 3.2

3.3 Design of General Class of 2-D Filters with Nonseparable Numerator and Denominator Transfer Functions

In the method to be presented, here the transfer function of the filter is of the form of Eq(1.5) and is rewritten as:

$$H_A(s_1,s_2) = \frac{A(s_1,s_2)}{B(s_1,s_2)}$$

$$= \frac{\sum_{l_1=0}^{L_1-1}\sum_{l_2=0}^{L_2-1} a(l_1,l_2)s_1^{l_1}s_2^{l_2}}{\sum_{k_1=0}^{K_1-1}\sum_{k_2=0}^{K_2-1} b(k_1,k_2)s_1^{k_1}s_2^{k_2}} \tag{3.12}$$

This is discretized by the application of bilinear transformation to obtain the digital transfer function of the form Eq(1.4) rewritten as:

$$H_D(z_1,z_2) = \frac{A(z_1,z_2)}{B(z_1,z_2)}$$

$$= \frac{\sum_{l_1=0}^{L_1-1}\sum_{l_2=0}^{L_2-1} a(l_1,l_2)z_1^{-l_1}z_2^{-l_2}}{\sum_{k_1=0}^{K_1-1}\sum_{k_2=0}^{K_2-1} b(k_1,k_2)z_1^{-k_1}z_2^{-k_2}} \tag{3.13}$$

To ensure the stability of the designed filter the denominator polynomial in the analog transfer function of the Eq(3.12) is chosen to have VSHP property. The technique discussed in Chapter 2 is used to generate such a polynomial and then assigned to the denominator of the analog transfer function Eq(3.12), i.e., $B(s_1,s_2)$ in Eq(3.12) is as given by Eq(2.410) with parameters as Eq(2.411). Then we follow the discretization as mentioned earlier. Two cases will be considered.

(i) - Design of 2-D Recursive Filter with Magnitude Specification Only

Example 3.3

Here we follow the technique already discussed in (i) of the previous section, and as an example we design a fan filter. The frequency response of the ideal filter was:

$$G = 1 \quad \text{for} \quad |\omega_{d_1}| \leq |\omega_{d_2}| \tan\frac{\pi}{3}$$

$$G = 0 \quad \textit{otherwise}$$

Block A part of flow chart for this Example, to fill-up Fig(3.[illegible]) is shown in Fig(3.8). The coefficients of the filter:

$$H_A\left(s_1,s_2\right) = \frac{\sum_{i=0}^{2}\sum_{j=0}^{2} c_{ij}\, s_1^i s_2^j}{D\left(s_1,s_2\right)} \tag{3.14}$$

is tabulated in Table(3.3), where $D(s_1,s_2)$ in Eq(3.14) is as given by H in Eq(2.410) with its parameters as indicated by Eq(2.411). Figs(3.9) to (3.11) shows the characteristics of the designed filter. The minimized error was 19.17 and the CPU time was 82.2 seconds.

$$D_{k,l}' = F + Ks_1^2 + Gs_2^2 + Bs_1s_2 + Is_1^2s_2^2 + Cs_1 + Ds_2 + Js_1^2s_2 + Es_1s_2^2$$

with $B,C,D,E,F,G,I,J,,K$ as in Eq(2.411)

$$H_{k,l} = \frac{\sum_{i=0}^{2} \sum_{j=0}^{2} c_{ij} s_1^i s_2^j}{D_{k,l}'}$$

$$HMN_{k,l} = H_{k,l} / H_{0,0}$$

$$G_{k,l} = \begin{cases} 1 & \left|\omega_{d_1}\right| \le \left|\omega_{d_2}\right| \tan\pi/3 \\ 0 & otherwise \end{cases}$$

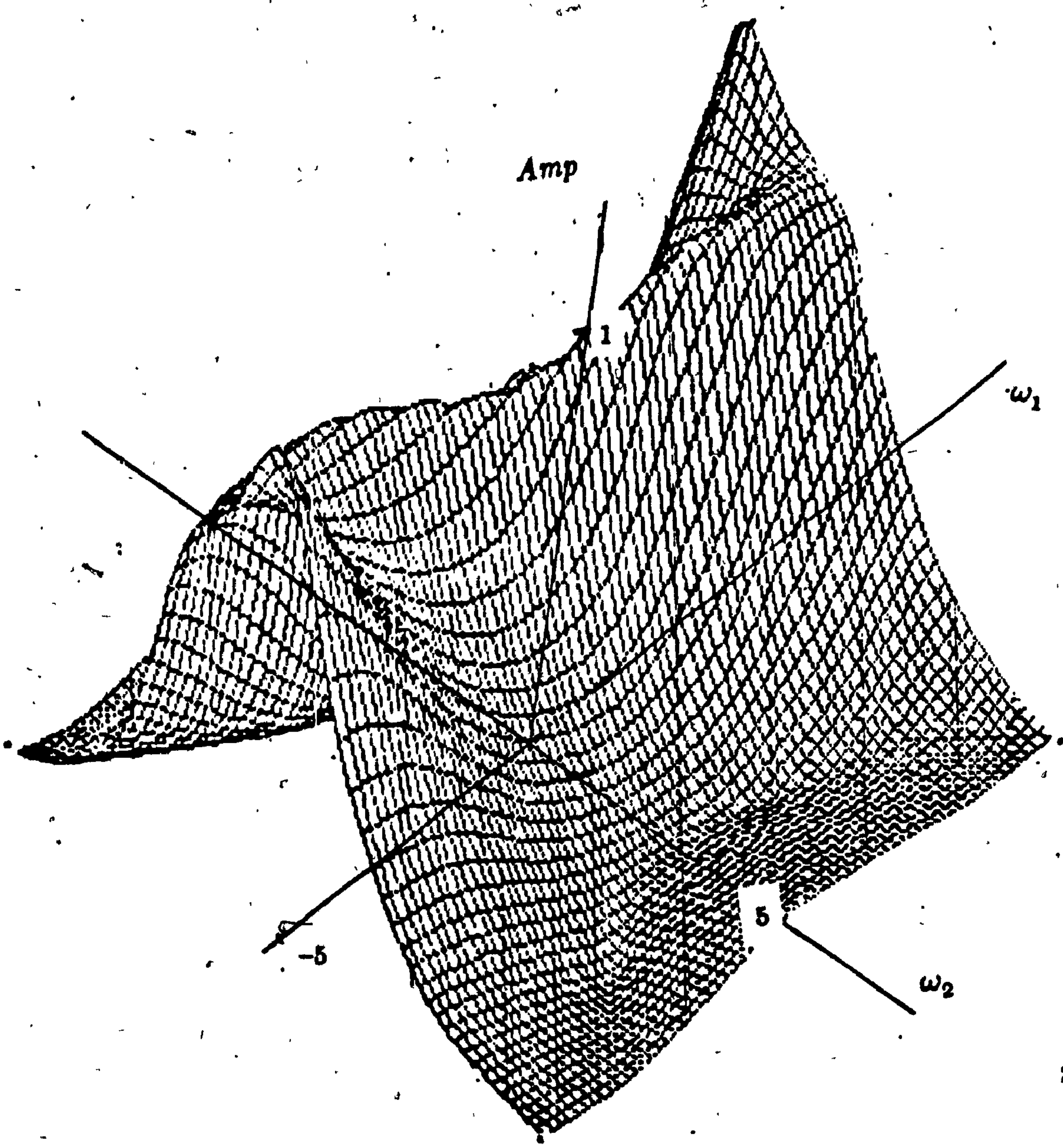
$$val = \sum_{k=-10}^{10} \sum_{l=-10}^{10} \left(HMN_{k,l} - G_{k,l} \right)$$

$$grad(m) = \partial val / \partial arg(m) \quad m = 1,\ldots,n$$

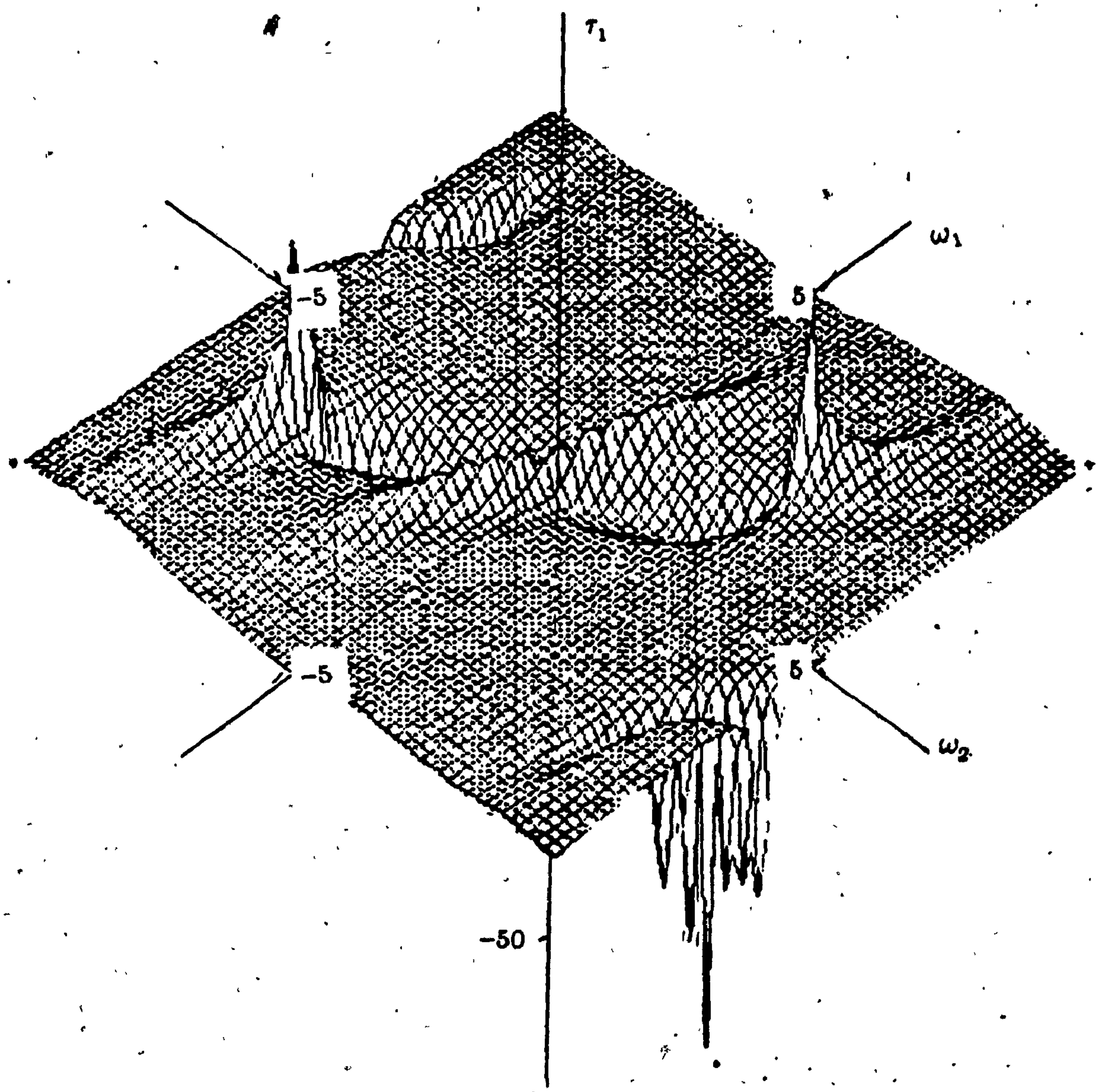
Fig(3.8) - Block A in Fig(3.1) for Example 3.3.

Table(3.3) :Coefficients of the filter in Example 3.3

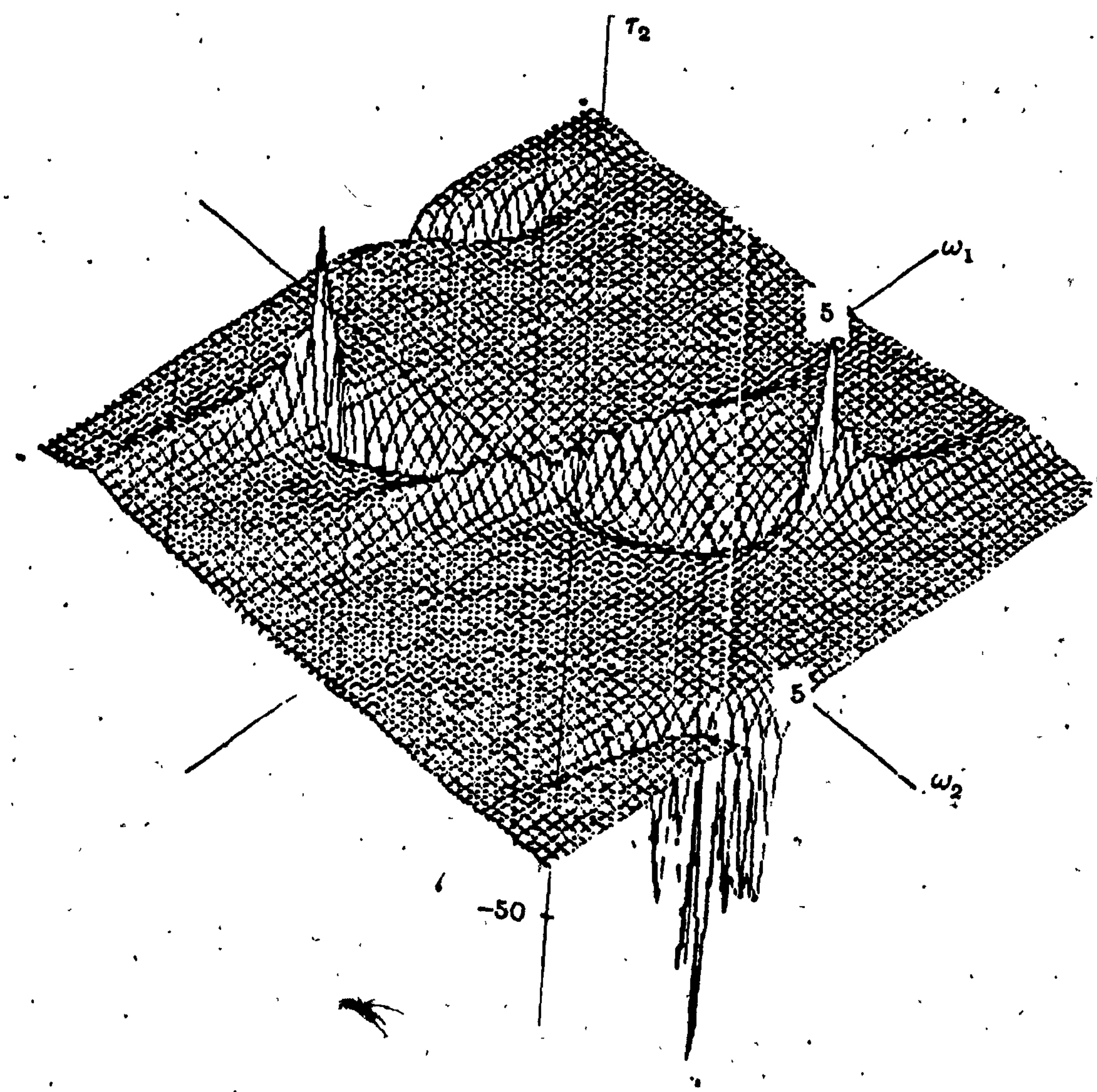
a_{12}	0.2804555728E+01	c_{00}	0.1774678234E+01
a_{16}	-0.8086411362E-01	c_{01}	0.1871468762E+01
a_{23}	0.2476525028E+01	c_{02}	0.2652894352E+00
a_{26}	0.3317034010E+01	c_{10}	0.5896821107E+01
a_{34}	0.1019347210E+01	c_{11}	0.8333040815E-01
a_{35}	0.1599320433E+01	c_{12}	0.1[illegible]65731560E+01
a_{36}	0.1273504768E+01	c_{20}	0.2491684446E+01
a_{45}	0.9483519026E+00	c_{21}	0.1417640774E+00
a_{46}	0.1795200209E+01	c_{22}	0.3701634931E+00
a_{56}	0.1917506653E+01		



Fig(3.9) - Amplitude response of the filter in Example 3.3



Fig(3.10) - Group delay 1 of the filter in Example 3.3



Fig(3.11) - Group delay 2 of the filter in Example 3.3

(ii) - Design of 2-D Recursive Filter with Magnitude and Phase Specification

Example 3.4

Following the technique (ii) of the section 3.2, we design a fan filter with ideal response:

$$G = 1 \quad \text{for} \quad |\omega_{d_1}| \leq |\omega_{d_2}| \tan\frac{\pi}{3}$$

$$G = 0 \quad \textit{otherwise}$$

with group-delay equal to the order of the filter ± 1. Part of flow chart for this example which completes the flow chart in Fig(3.1) is shown in Fig(3.12). The coefficients of the filter are as shown in Table(3.4).

The responses of the filter are shown in Figs(3.13) to (3.15). The minimized error was 25.99 and the CPU time was 92.0 seconds.

Block A

In Fig(3.8)

$$\tau_m = -Real\left[z_m \frac{\frac{\partial H(z_m)}{\partial z_m}}{H(z_m)} \right] \quad m=1,2$$

tt=order of filter ± 1

For $\omega < 2$

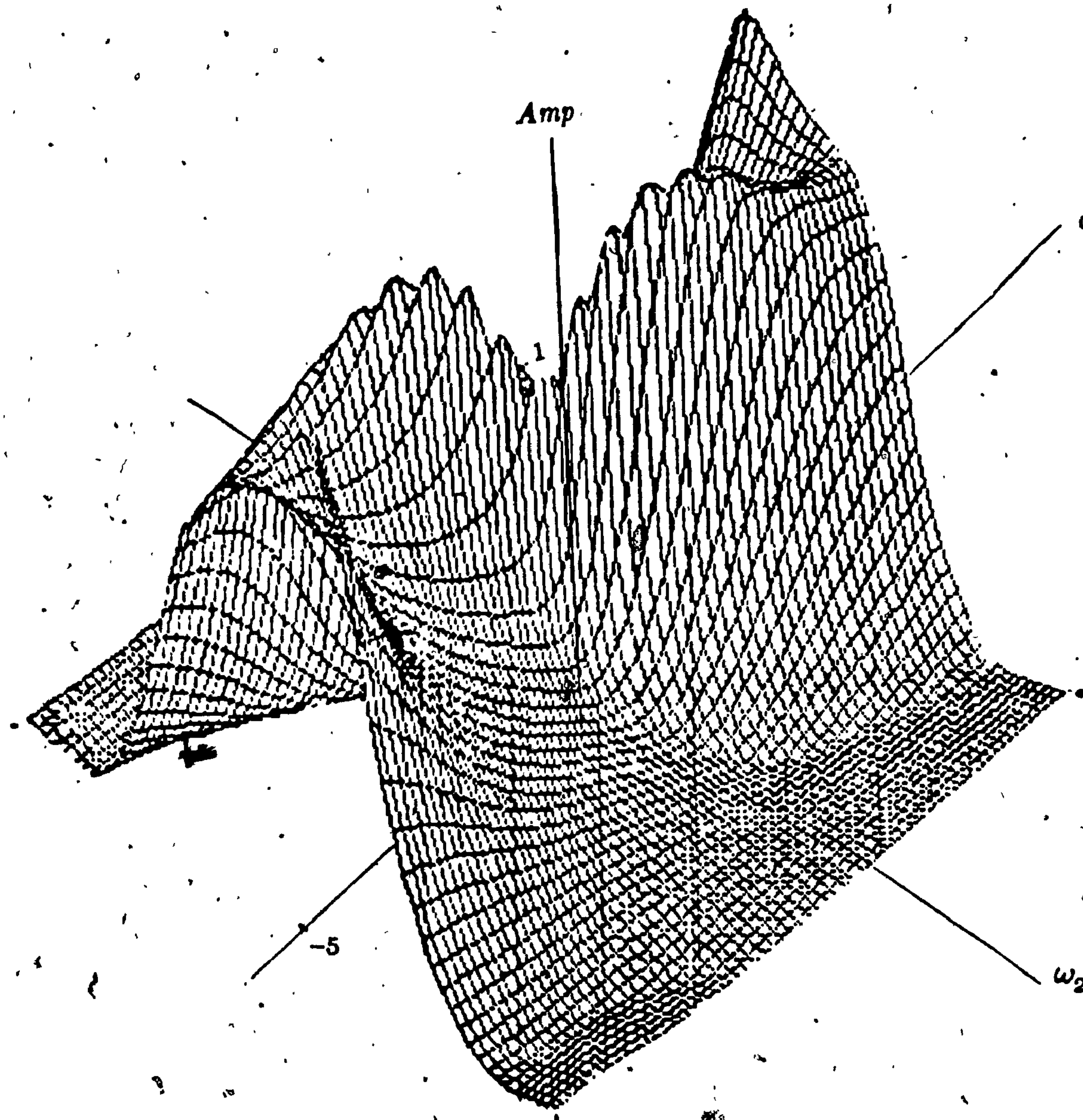
$$val' = val + \sum_{k=-10}^{10} \sum_{l=-10}^{10} (\tau_1 - tt)^2 + (\tau_2 - tt)^2$$

$$grad(m) = \partial val' / \partial arg(m) \quad m=1,\ldots,n$$

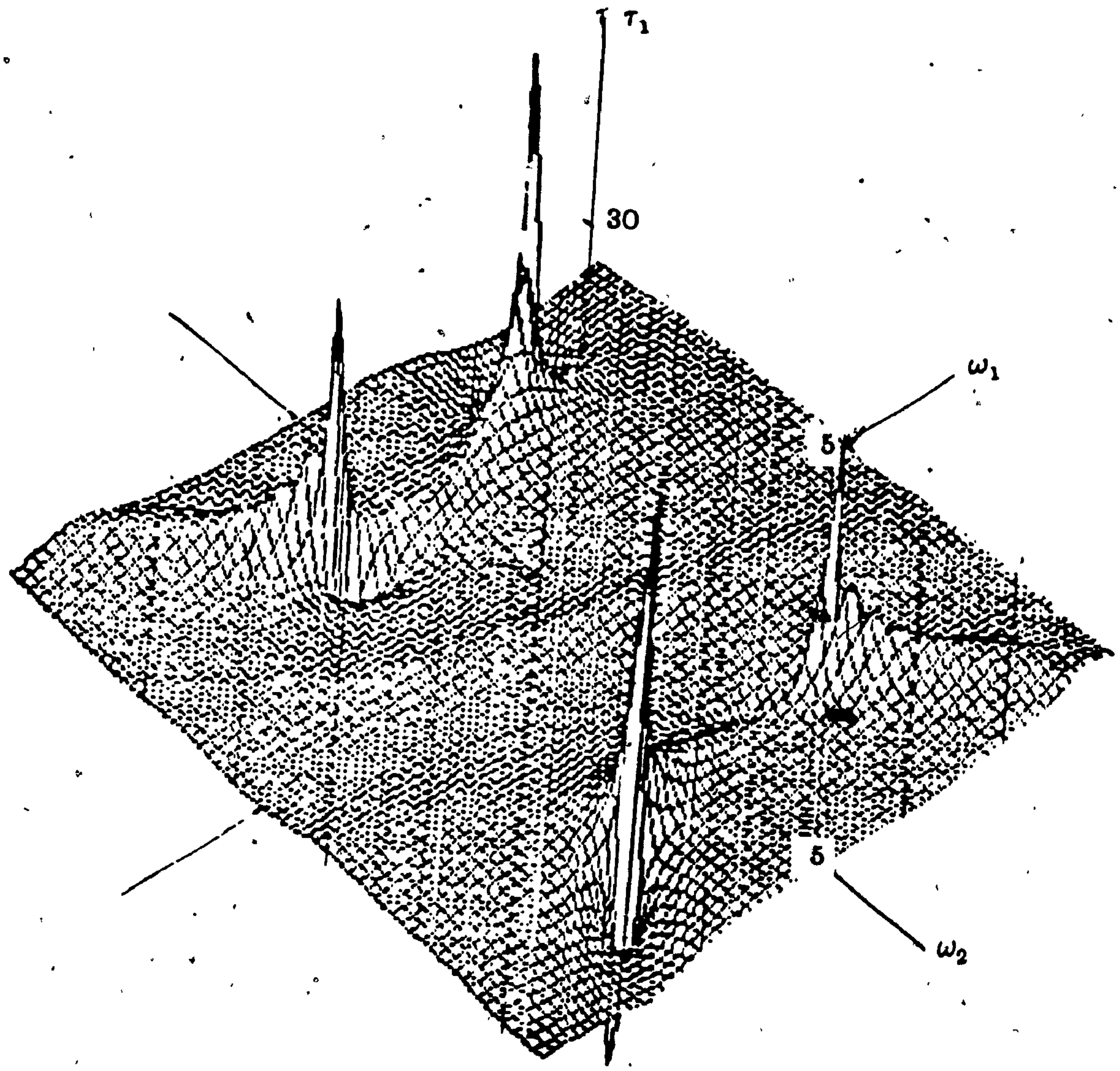
Fig(3.12) - Block A in Fig(3.1) for Example 3.4.

Table(3.4) :Coefficients of the filter in Example 4

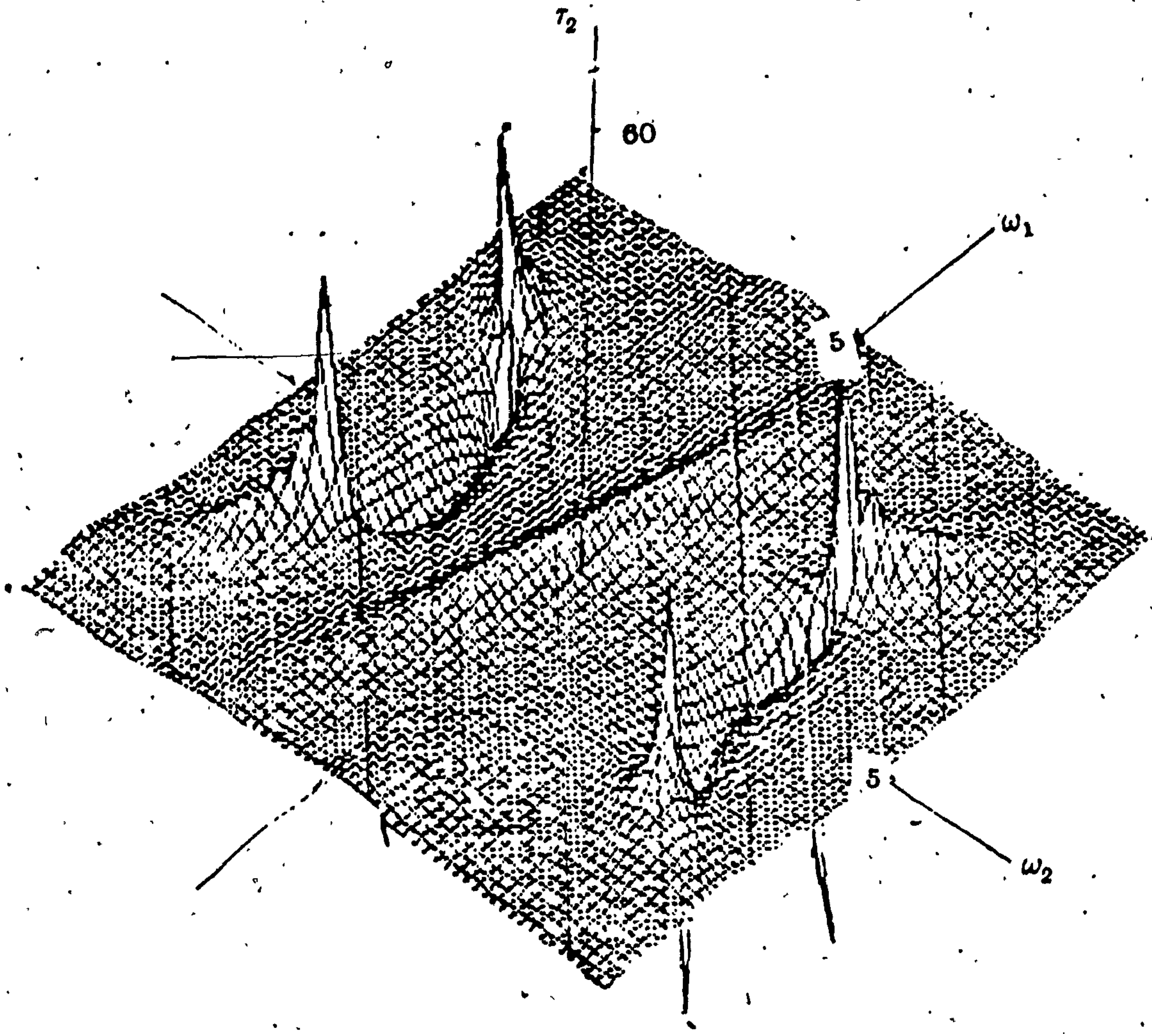
a_{12}	0.2505666089E+01	c_{00}	0.1656491286E+01
a_{16}	0.5702157329E-01	c_{01}	0.3820012793E+00
a_{23}	0.1807669756E+01	c_{02}	0.8337748514E+00
a_{26}	0.2616831065E+01	c_{10}	0.6568883071E+00
a_{34}	0.1441730470E+01	c_{11}	-0.2056726373E+00
a_{35}	0.2540574868E+01	c_{12}	0.1460924178E+00
a_{36}	0.9950267311E+00	c_{20}	-0.6451703889E+00
a_{45}	0.1576072479E+01	c_{21}	0.4084838108E+00
a_{46}	0.1745085460E+01	c_{22}	0.3072355539E-01
a_{56}	0.2512277407E+01		



Fig(3.13) - Amplitude response of the filter in Example 3.4



Fig(3.14) - Group delay 1 of the filter in Example 3.4



Fig(3.15) - Group delay 2 of the filter in Example 3.4

3.4 Design of Integer Coefficients 2-D Recursive Digital Filter

All filters designed using the method discussed in this chapter yield coefficients with infinite precision. A method will be presented in this section for the design of integer coefficient 2-D filter.

In this procedure a 2-D integer coefficient recursive filter is obtained using a technique called discretization and reoptimization [43]. The starting value of this algorithm is the coefficients of the designed filter with infinite precision using any of the technique mentioned above. Then the following steps have to be implemented to obtain a 2-D integer coefficient filter.

step (1) - Choose the parameter which has the largest value greater than unity (if no parameter is found with value greater than unity an integer scalar will be used to create parameters value greater than unity).

step (2) - The value of this parameter will be rounded up and down to the closest integer value.

step (3) - The least mean square error for the two values of the chosen parameter is calculated.

step (4) - Choose the value of the parameter for which the least mean square error is minimum, assign this value to the chosen parameter.

step (5) - Go through the optimization process by having the chosen parameter as a constant [$(n-1)$ variable].

GO TO STEP (1).

This procedure is repeated until all values of the filter's coefficients are discretized.

Example 3.5

Following these steps for the filter in Example 3.3, we obtain:

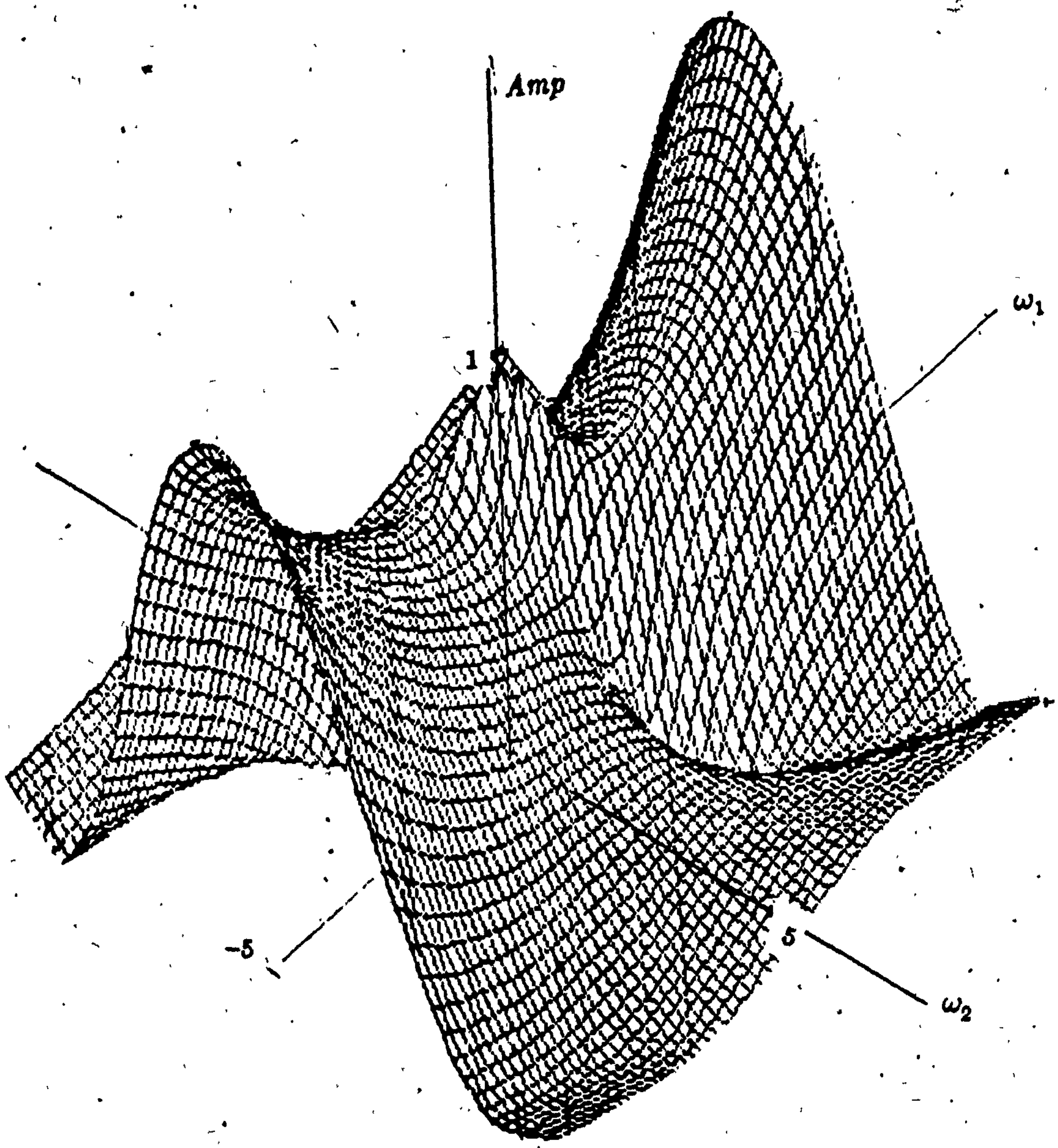
Table(3.5) :Coefficients of the filter in Example 3.5

a_{12}	3	c_{00}	3
a_{16}	4	c_{01}	6
a_{23}	2	c_{02}	1
a_{26}	3	c_{10}	12
a_{34}	1	c_{11}	1
a_{35}	3	c_{12}	3
a_{36}	1	c_{20}	6
a_{45}	3	c_{21}	1
a_{46}	3	c_{22}	1
a_{56}	8		

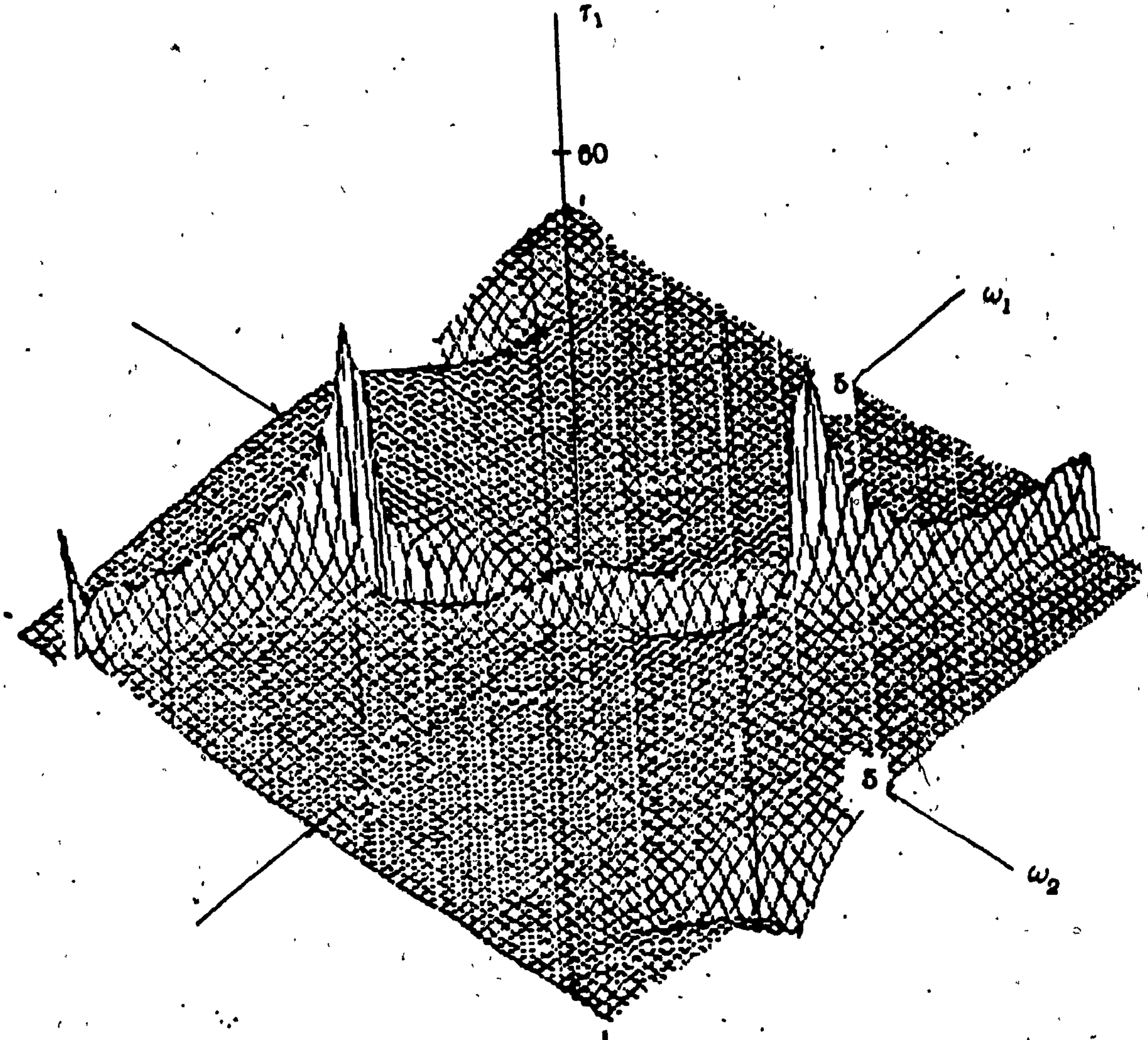
The coefficients of the filter, i.e., the parameters in Eq(2.420)were:

$$B = 1250 \quad C = 900 \quad D = 972 \quad E = 162$$
$$F = 162 \quad G = 648 \quad I = 72 \quad J = 108$$
$$K = 18$$

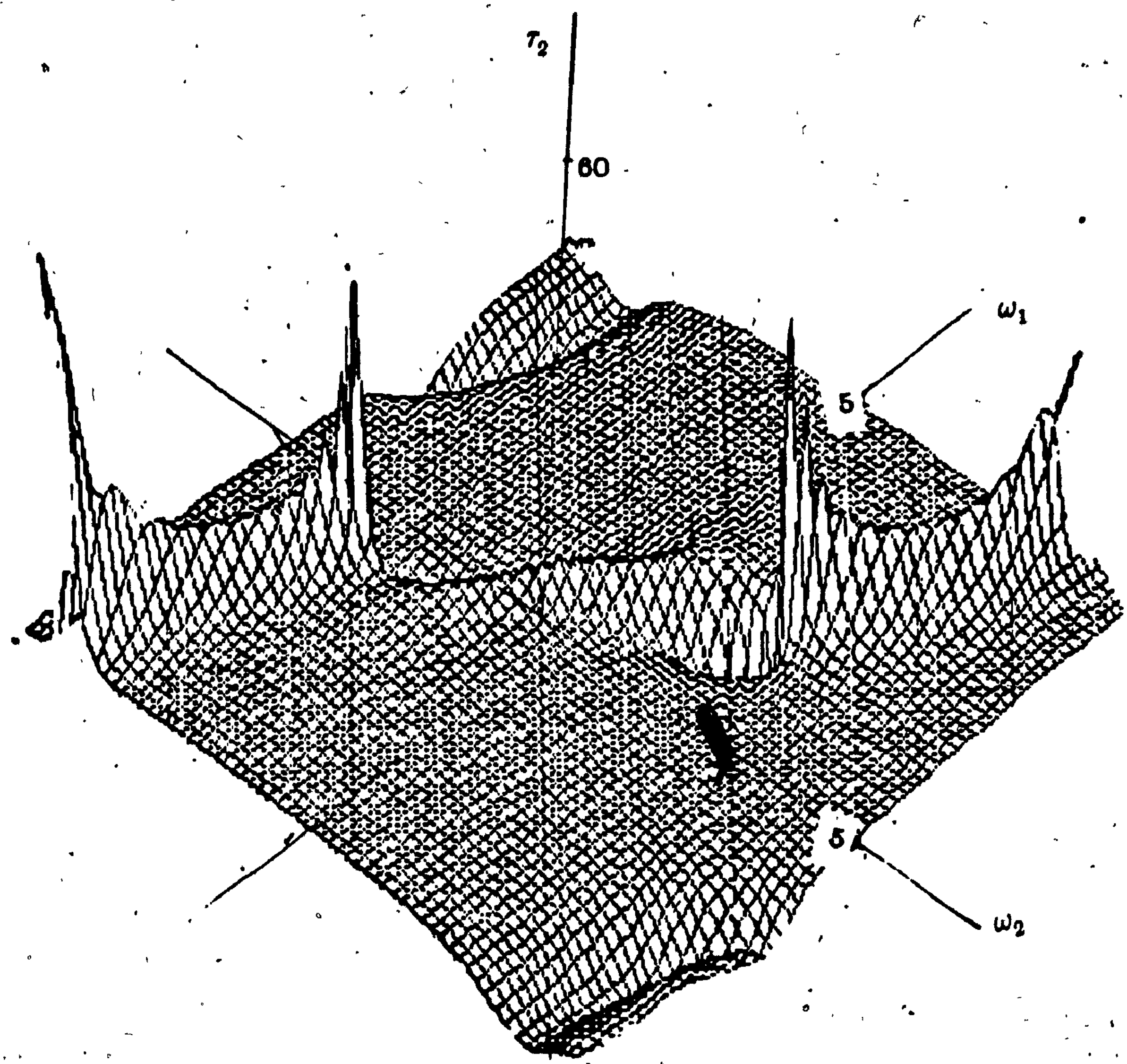
The error which was 19.17 in Example 3.3, now becomes 30.84. Figs(3.16) to (3.18) show the results. It is seen that the filter can be implemented even on a personal computer.



Fig(3.16) - Amplitude response of the filter in Example 3.5



Fig(3.17) - Group delay 1 of the filter in Example 3.5



Fig(3.18) - Group delay 2 of the filter in Example 3.5

3.5 Coefficient Sensitivity and Effects of Finite Word Length.

The results of the integer coefficient filter in Example 3.4 indicates low coefficient sensitivity. This encourages us to enquire more about sensitivity and effects of finite word length.

There are several ways to define quantities for the characterization of the parameter sensitivity of a system. Of these we choose relative (logarithmic) sensitivity function, which is:

$$S_j^T = \left.\frac{\partial \ln T}{\partial \ln \alpha_j}\right|_{\alpha_0} \quad j=1,2,\ldots,N$$

$$= \left.\frac{\dfrac{\partial T}{T}}{\dfrac{\partial \alpha_j}{\alpha_j}}\right|_{\alpha_0} \tag{3.15}$$

Since we are dealing with sensitivity functions in the frequency or z-domain it is seen that these functions depend on the frequencies as well as parameters, which makes it impossible to show it graphically, and huge tables are necessary to represent them.

Thus, we choose sensitivity measures which are defined on the entirety of the sensitivity functions and allow for a global characterization of the sensitivity by a single number [44]. To this end we form:

$$S'_{Max} = \sum_{j=1}^{N} \left| S_{j,Max}^T \right| \tag{3.16}$$

$$S'_{\min} = \sum_{j=1}^{N} \left| S_{j,min}^T \right| \tag{3.17}$$

where $S_{j,Max}^T$ ($S_{j,min}^T$) is formed by changing frequencies over their entire ranges and sort out for the maximum (minimum) sensitivity.

Note that:

$$T = \frac{\sum_{i=0}^{2} \sum_{j=0}^{2} c_{ij} s_1^i s_2^j}{\sum_{i=0}^{2} \sum_{j=0}^{2} d_{ij} s_1^i s_2^j} \tag{3.18}$$

followed by bilinear transformation. Therefore, $N=18$, and furthermore T is complex.

In order not run into complication we tabulate these measures for amplitude, group delay one and two.

Since, eventually, the digital filter is realized by some finite digits, it becomes of interest to see the performance of the filter with finite number of digits.

One easy way to show and compare different filters with respect to finite world length is to consider rounding the mantissa of the floating point arithmetics.

Table 3.6 shows the sensitivity performance of the filters in Examples 3.1 to 3.4. The entries of Table 3.6 were formed by rounding the unsigned mantissa of the filter coefficients, which are all expressed by floating point arithmetics, to the indicated precision. The coefficients in Tables (3.1)to (3.4) were rounded to 10, 6, 5, 4, 3, 2, and 1 decimal digit.

The sensitivity of the transfer function were calculated with respect to variables c's and d's according to Eq(3.15) and it was sorted out to find the minimum and maximum sensitivities with respect to each variable, and then they were added according to Eqs(3.16) and (3.17).

From Table 3.6 we conclude that the overall sensitivities are low. In addition rounding effects as far as single digit mantissa is concerned doesn't change the low sensitivity measure considerably. The following example further shows the effect of low sensitivity.

Example 3.6

In this example we truncate the coefficients of Example 3.1, which are tabulated in Table(3.1), with a single byte. In this one byte 2 bits are dedicated to the signs of mantissa and the exponents, 3 bits for mantissa, and 3 bits for exponent.

The binary coefficients are shown in Table(3.8), and the amplitude response is shown in Fig(3.19), which when compared with Fig(3.1) shows that they are exact replica of each other.

3.6 Summary and Discussion :

In this chapter, designs of 2-D filters have been carried out. The designs are based on the concept of generating VSHPs using the slope of a 2-variable or n-variable reactance function, thereby guaranteeing stability. It is further shown that it is possible to design such 2-D filters which are less sensitive to finite word-length effects.

Table 3.6 - Sensitivity Performance.

decimal points		ten		six		five		four	
		min	Max	min	Max	min	Max	min	Max
Ex.3.1	Amp	1.7E-7	30.7	2.1E-7	30.7	1.6E-7	30.7	3.0E-8	30.5
	GD1	0.6	1.5	0.6	1.5	.59	1.5	0.59	1.5
	GD2	0.6	1.5	0.6	1.5	0.6	1.5	0.6	1.5
Ex.3.2	Amp	9.7E-8	7.3	1.3E-8	7.4	1.9E-7	7.4	1.6E-7	7.4
	GD1	0.9	1.5	0.9	1.5	0.9	1.5	0.9	1.5
	GD2	0.9	1.5	0.9	1.5	0.9	1.5	0.9	1.5
Ex.3.3	Amp	3.0E-4	77.2	3.0E-7	77.2	3.0E-4	77.2	4.0E-4	77.2
	GD1	0.1	182.0	0.1	182.0	0.1	181.4	0.1	177.0
	GD2	8.1E-3	12.0	8.1E-3	12.0	8.1E-3	12.0	8.2E-3	12.0
Ex.3.4	Amp	5.0E-3	50.7	5.8E-3	50.0	5.8E-3	50.7	5.8E-3	50.7
	GD1	8.0E-2	14.0	8.0E-2	14.7	8.0E-2	14.7	8.0E-2	14.7
	GD2	6.0E-2	10.0	6.0E-2	10.0	6.0E-2	10.0	6.0E-2	10.3

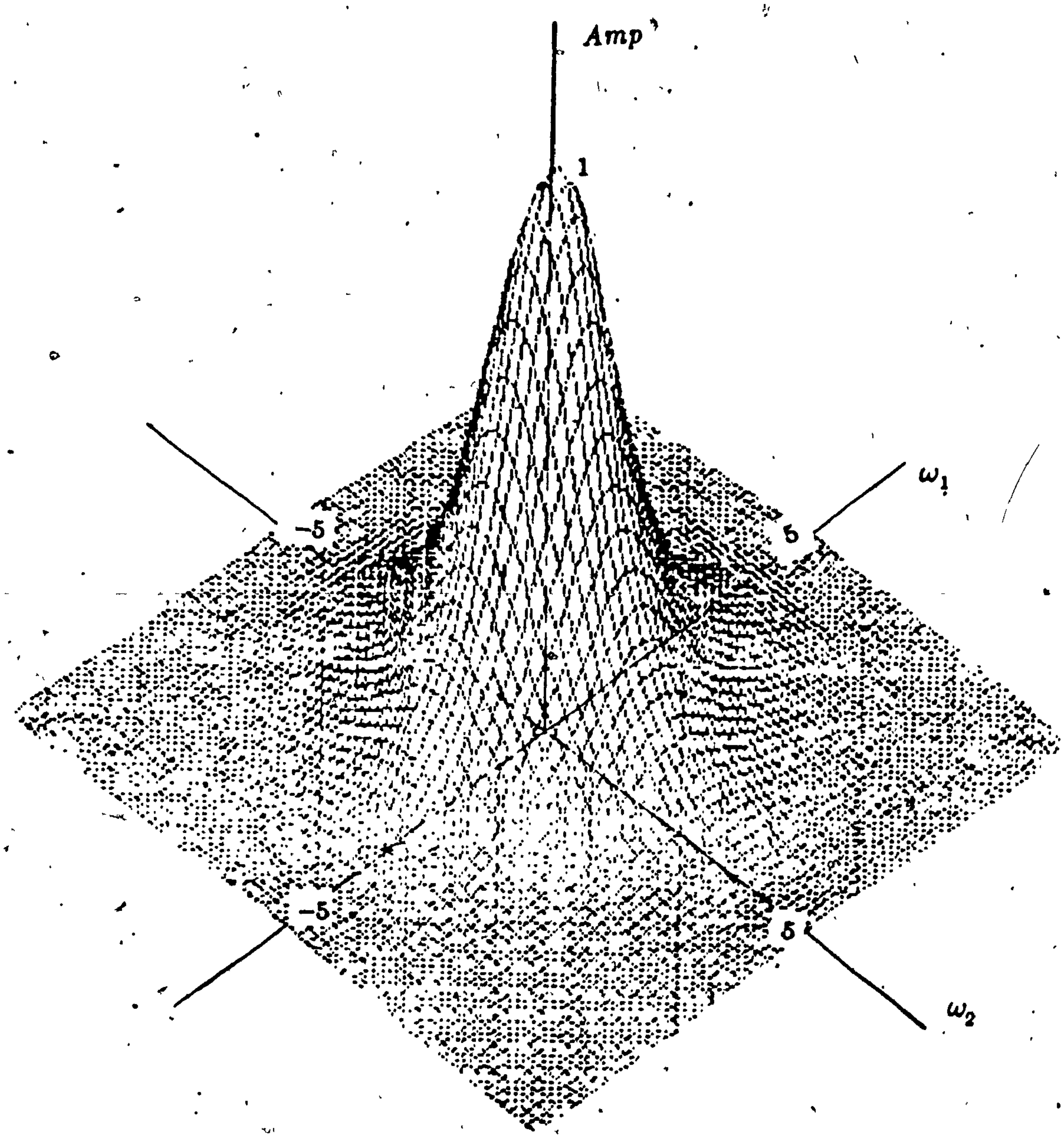
Table 3.6 (continued)

decimal points		three		two		one	
		min	Max	min	Max	min	Max
Ex.3.1	Amp	1.6E-7	31.3	9.9E-8	29.4	2.2E-16	69.8
	GD1	0.6	1.5	0.6	1.49	0.1	1.3
	GD2	0.6	1.5	0.55	1.5	0.1	1.3
Ex.3.2	Amp	1.7E-7	7.4	1.3E-7	7.4	1.4E-7	7.4
	GD1	0.9	1.5	1.0	1.5	1.0	1.6
	GD2	0.9	1.5	1.0	1.5	1.0	1.6
Ex.3.3	Amp	4.7E-4	75.0	3.0E-4	66.0	3.1E-3	43.0
	GD1	0.14	229.0	0.13	4595.0	0.1	2692.0
	GD2	8.4E-3	12.0	6.5E-3	12.8	1.8E-2	14.9
Ex.3.4	Amp	4.5E-3	50.9	3.0E-3	58.6	2.0E-2	54.0
	GD1	0.12	14.8	1.0E-2	16.2	0.1	16.8
	GD2	5.6E-2	10.4	4.9E-2	11.5	3.9E-2	11.0

Table (3.7): One byte floating point representation

of entries of Table (3.1)

c_1	0.101×2^1	c_4	0.101×2^2
c_2	0.101×2^2	c_5	-0.101×2^1
c_3	-0.101×2^1		



Fig(3.19) - Amplitude response of the filter in Example 3.5

Chapter 4
Summary and Conclusions

The purpose of this chapter is to provide a summary of the contributions made in this thesis and to indicate some future investigations.

The aim of this thesis is to design stable 2-D recursive digital filters satisfying prescribed magnitude and/or group delay specifications. The stability is ensured by taking care to see that the transfer function does not contain any non-essential singularities of the second kind. This has been accomplished by considering the properties of the slope of a 2-variable or a n-variable reactance function on the imaginary axis. Since it is difficult to generate a general n-variable reactance function, a class of reactance functions is considered in Chapter Two. The reactance function is obtained by a n-port gyrator terminated by n-variable reactances, n being either even or odd. The slope of these functions on the imaginary axes is known to be positive. It is shown that the numerator of the slope can always be written in a compact form as the difference of the squares of two polynomials, each coefficient of the polynomial being obtained by the gyrator matrix. This enables one to generate VSHP.

In Chapter Three designs of stable 2-D recursive digital filters are considered. The VSHPs generated earlier are assigned to the denominator of the transfer functions and the numerators are suitably obtained. The 2-D recursive filters considered are: (i) having separable denominator transfer functions and (ii) having non-separable numerator and denominator transfer functions. These are chosen to give required symmetry in the response. These filters approximate magnitude only and magnitude and group delay specifications also. In all these cases, an unconstrained optimization has been used.

However, the coefficients of these filters were based on infinite precision. A method is given where, through discretization and reoptimization, filters having integer coefficients have been obtained.

The coefficient sensitivities of the filters designed have been studied. They show that sensitivities are indeed low and rounding of the coefficients does not change the filter performance appreciably and that they can be implemented on a personal computer.

4.3 Future Research Suggestions.

The following problems seem to be promising research topics in connection with this thesis:

1) - In Chapter 2 we considered a special case of Theorem 5, where 2n=6, and we formed a second ordered section and use it for different filters. one can cascade many of these second order sections and then apply non-linear programming to find the appropriate coefficients.

Also one can take 2n>6 and form direct form of realization and apply optimization to that.

2) - In the optimization routine we usually force the gradient of the objective function to vanish. The objective function in examples of Chapter 3 were all mean square error between an ideal and actual responses. For example:

$$f = \sum_{s_1}\sum_{s_2}\left(| T | - G \right)^2$$

Then:

$$\nabla f = \sum_{s_1}\sum_{s_2} 2\left(| T | - G \right) \frac{\partial | T |}{\partial x_i}$$

But we have the sensitivity

$$S^{|T|}_{x_i} = \frac{\partial | T |}{\partial x_i}$$

Therefore, if we need a very low sensitivity we can use some other optimization criteria such that the gradient is a constant times the sensitivity. To this end mini-max appears to have superiority over the mean square error criteria.

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