UPPER BOUNDS FOR THE NUMBER OF ABSOLUTELY CONTINUOUS MEASURES INVARIANT UNDER TRANSFORMATIONS

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A Thesis

in

The Department

of

Mathematics

Presented in Partial Fulfillment of the Requirements
for the degree of Master of Science at
Concordia University
Montreal, Quebec, Canada

March, 1980

ABSTRACT

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Given n linearly independent functions invariant under a non-singular transformation, there exists a collection of n non-negative invariant functions with disjoint supports. This fact is fundamental in establishing an upper bound for the number of absolutely continuous measures invariant under a piecewise monotonic transformation. Improved upper bounds are obtained for special subclasses of these transformations. In particular, for piecewise linear Markov maps, the number of absolutely continuous invariant measures is equal to the dimension of an eigenspace of a certain matrix.

ACKNOWLEDGEMENTS

The author is greatly indebted to his thesis supervisor, Dr. Abraham Boyarsky of Concordia University, for his guidance, encouragement and valuable suggestions. The numerous conversations the author had with Dr. Boyarsky were most helpful and stimulating.

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CHAPTER I

INTRODUCTION

An important question in ergodic theory is to find conditions on a transformation which will guarantee the existence of absolutely continuous invariant measures. Although of mathematical interest by itself, this problem has many applications in other areas, namely in the physical and biological sciences. This question has been the subject of intensive research by many authors and several verifiable conditions on transformations have been discovered [1, 10, 11, 12, 13].

Once the existence problem is settled, the next question which arises naturally is to determine the number of such invariant measures. In this thesis, we will primarily be interested in finding upper bounds for the number of absolutely continuous measures invariant under a transformation. We will see that the supports of the densities of these measures play a vital role in establishing this bound. The bound itself is computed easily from the transformation.

In Chapter II, we fix notations, introduce standard definitions and state without proof two existence theorems which will be used throughout the text. We also prove a property about the structure of the support of a function of bounded variation.

In Chapter III we introduce invariant sets and explore some properties of invariant functions under non-singular transformations. It will be seen that we can always assume that independent invariant functions are non-negative and have disjoint supports. We present this fundamental result in a separate chapter to stress the fact that it requires only the non-singularity of the transformation under consideration.

Chapter IV contains the main results of this dissertation. We state and prove three theorems, each of them giving an upper bound for the number of absolutely continuous invariant measures. In the last section we show that the number of these measures is invariant under topological conjugacy.

In Chapter V we focus our attention on two classes of transformations: Renyi transformations and Markov maps. Under appropriate hypothesis, these transformations and all their iterates are shown to have a unique invariant function.

Chapter VI deals with piecewise linear Markov maps. Following [7] we show the existence of invariant step functions regardless of slope conditions on the transformation. The method developed in this chapter allows us to find at least one invariant function and in the case where the map is uniformly expanding, we can obtain explicitly all the invariant functions simply by solving a system of linear equations.

Finally we mention some of the contributions to this thesis that are original. All of section 4.2 is new as well as Theorem 5.4 and Proposition 6.3. There were a number of erroneous statements in the paper of Li and Yorke [2] which we corrected in Chapter 3.

CHAPTER II

PRELIMINARIES

2.1. INTRODUCTION AND DEFINITIONS

Denote by $(L^1, \|\cdot\|)$ the space of all integrable functions on the interval $J = [a,b]$. Let m denote Lebesgue measure on J, and M the class of all measurable subsets of J. We say $\tau: J \to J$ is a measurable transformation if the set $\tau^{-1}(A) = \{x \in J : \tau(x) \in A\}$ is in M for each $A \in M$, and non-singular if $m(\tau^{-1}(A)) = 0$ whenever $A \in M$ and $m(A) = 0$. A measure μ is said to be invariant under τ if $\mu(A) = \mu(\tau^{-1}(A))$ for all $A \in M$. Also, μ is absolutely continuous with respect to m, in notation $\mu \ll m$, if there exists an $f \in L^1$ such that $\mu(A) = \int_A f\,dm$ for all $A \in M$. We refer to this f as the density of μ, and it is unique a.e.. Notice when μ is absolutely continuous and invariant under τ, its density function f satisfies

$$\int_A f\,dm = \int_{\tau^{-1}(A)} f\,dm$$

for all $A \in M$. With this in mind, we define a function $f \in L^1$ to be invariant (under τ) if the above equality holds for every $A \in M$.

We now introduce the Frobenious-Perron operator, a very useful tool in the study of absolutely continuous invariant measures. Let $\tau: J \to J$ be a measurable non-singular transformation, and $f \in L^1$. Defining the measure μ_f by

$$\mu_f(A) = \int_{\tau^{-1}(A)} f\, dm ,$$

we see that $m(A) = 0 \Rightarrow m(\tau^{-1}(A)) = 0 \Rightarrow \mu_f(A) = 0$, that is $\mu_f \ll m$. By the Radon-Nikodym theorem, there exists a function $g \in L^1$ such that

$$\mu_f(A) = \int_A g\, dm ,$$

and g is unique a.e. . We define the Frobenious-Perron operator P_τ by setting $P_\tau f = g$. Thus, P_τ maps L^1 into L^1, and

$$\int_A P_\tau f\, dm = \int_{\tau^{-1}(A)} f\, dm \qquad (2.1)$$

for all $A \in \mathcal{M}$ and $f \in L^1$. Clearly, f is invariant under τ if and only if $P_\tau f = f$ a.e., i.e. f is a fixed point of the Frobenious-Perron operator. Letting $A = [0,x]$ and differentiating both sides of (2.1), we obtain

$$P_\tau f(x) = \frac{d}{dx} \int_{\tau^{-1}[a,x]} f(t)\, dt . \qquad (2.2)$$

It can be shown that P_τ as defined by (2.2) is equivalent to the definition given by (2.1).

We now list, without proof, some well-known properties of the operator P_τ:

(1) Linearity: $P_\tau(f+g) = P_\tau f + P_\tau g$

$P_\tau(cf) = cP_\tau f$ for real c.

(2) Continuity: $\| P_\tau f \| \le \| f \|$.

(3) P_τ is positive: $f \ge 0 \Rightarrow P_\tau f \ge 0$.

(4) P_τ preserves integrals:

$\int_J f\,dm = \int_J P_\tau f\,dm$ for every $f \in L^1$.

(5) $P_{\tau^n} = P_\tau^n$ where $\tau^n = \tau \circ \tau^{n-1}$ is the nth iterate of τ.

(6) $P_\tau f = f$ a.e. $\Leftrightarrow$ the measure $d\mu = f dm$ is invariant under τ.

If we denote by F the set of all functions invariant under τ, then property (1) combined with property (6) imply that F is a linear subspace of L^1. However, when we say that τ has n invariant functions, we will always mean n linearly independent functions. A set of functions $\{f_1, \ldots, f_n\} \subset L^1$ is said to be linearly independent if $\sum_{i=1}^{n} c_i f_i = 0$ a.e. implies $c_1 = \ldots = c_n = 0$. We shall say that the absolutely continuous measures $\mu_1, \ldots, \mu_n$

are independent if their density functions are linearly independent.

2.2. EXISTENCE THEOREMS

In this section, we state without proof two theorems which guarantee the existence of absolutely continuous invariant measures for a class of transformations. See [1]. First we need a definition.

A transformation $\tau: [a,b] \to [a,b]$ will be called <u>piecewise C^2</u> if there exists a partition

$$a = a_0 < a_1 < \ldots < a_p = b$$

of $[a,b]$ such that for each integer i, $1 \le i \le p$, the restriction of τ to the open interval (a_{i-1}, a_i) is a C^2 function which can be extended to the closed interval $[a_{i-1}, a_i]$ as a C^2 function.

<u>Theorem 2.1</u> [1]

Let $\tau: [0,1] \to [0,1]$ be a piecewise C^2 function such that $\inf|\tau'| > 1$, where $\tau'(x)$ is defined. Then for any $f \in L^1$, the sequence $\frac{1}{n} \sum_{k=0}^{n-1} P_\tau^k f$ is convergent in norm to a function $f^* \in L^1$ having the following properties:

(1) $f \geq 0 \Rightarrow f^* \geq 0$

(2) $\int_0^1 f^* dm = \int_0^1 f\, dm$

(3) $P_\tau f^* = f^*$ and consequently the measure $d\mu^* = f^* dm$ is invariant under τ.

(4) the function f^* is of bounded variation; moreover, there exists a constant c independant of the choice of initial f such that the the variation of the limiting f^* satisfies the inequality

$$\overset{1}{\underset{0}{V}} f^* \leq c\|f\| .$$

Theorem 2.2 [1]

Let $\tau: [0,1] \to [0,1]$ be a piecewise C^2 function such that $\inf\left|\frac{d\tau^N}{dx}\right| > 1$ for a positive integer N. Then for any $f \in L^1$ the sequence $\frac{1}{n}\sum_{k=0}^{n-1} P_\tau^k f$ is convergent in norm to a function f^* which satisfies conditions (1), (2) and (3) of Theorem 2.1. If, in addition, $\inf|\tau'| > 0$ then condition (4) is also satisfied.

2.3. FUNCTIONS OF BOUNDED VARIATION IN L^1

We say $f \in L^1$ is a function of bounded variation in L^1 if f equals almost everywhere some function of bounded variation. When $\tau: [0,1] \to [0,1]$ is piecewise C^2 with $\inf|\tau'| > 1$, Theorem 2.1 asserts that every function invariant under τ is a function of bounded variation in L^1. The structure of the support of a function of bounded variation will be crucial in the sequel. By the support of any real-valued function f, we mean the set on which f is non-zero. The notation $\text{spt}\, f$ for this set will be used throughout this text. Notice that $\text{spt}\, f$ need not be closed in our definition. The following proposition is partially proved in [2]:

Proposition 2.1

If $f \in BV[a,b]$ then

$$\text{spt}\, f = \left(\bigcup_{n=0}^{p} K_n \right) \cup M , \qquad 0 \le p \le \infty$$

where the K_n are open disjoint intervals, M is a countable set and

$$M \cap \left(\bigcup_{n=0}^{p} K_n \right) = \phi$$

Proof:

First recall that every open set in $[a,b]$ is a countable (or finite) union of disjoint intervals, each of them being open relative to the topology of $[a,b]$. Thus, if $(\operatorname{spt} f)^\circ$ denotes the interior of $\operatorname{spt} f$, we may write

$$(\operatorname{spt} f)^\circ = \bigcup_{n=0}^{p} K_n \,, \quad 0 \le p \le \infty$$

where the K_n are open disjoint intervals in $[a,b]$. Let

$$M = (\operatorname{spt} f) - (\operatorname{spt} f)^\circ$$

and

$$A_n = M \cap \left\{x: \ |f(x)| \ge \frac{1}{n}\right\} .$$

Clearly $M = \bigcup_{n=1}^{\infty} A_n$. We claim that A_n is a finite set for every n. Suppose this is not true for some n. Then choose points

$$a < a_1 < a_2 < \ldots < a_N < b \,,$$

where $a_i \in A_n$ for each $1 \le i \le N$ and N arbitrary, and consider a partition

$$a = b_0 < b_1 < \ldots . < b_N = b$$

such that $a_i \in (b_{i-1}, b_i)$. Since a_i is not in the interior of $\operatorname{spt} f$, there exists a point $c_i \in (b_{i-1}, b_i)$ such that $f(c_i) = 0$. Thus $|f(a_i)| \geq \frac{1}{n}$ implies $\overset{b_i}{\underset{b_{i-1}}{V}} f \geq \frac{1}{n}$ for each i, and

$$\overset{b}{\underset{a}{V}} f = \sum_{i=1}^{N} \overset{b_i}{\underset{b_{i-1}}{V}} f \geq \frac{N}{n} .$$

Since N is arbitrary, this implies that $\overset{b}{\underset{a}{V}} f = \infty$. Contradiction. Hence A_n is finite for each n, and M is countable. The conclusion of the proposition follows.

Q.E.D.

We conclude this chapter with a discussion. For τ a piecewise C^2 transformation with $\inf|\tau'| > 1$, let F be the space of functions invariant under τ. If $f \in L^1$, denote by $[f]$ the class of all functions which are equal a.e. to f. By what we have mentioned earlier, for each $f \in F$ there exists a function of bounded variation $g \in [f]$. By the preceding proposition, $\operatorname{spt} g = \left(\bigcup_{n\geq 0} K_n\right) \cup M$ where the K_n are disjoint intervals and M is a countable set. Letting $f_1 = g$ on $\bigcup_{n\geq 0} K_n$ and $f_1 = 0$ elsewhere, we get $f_1 \in [f]$ with $\operatorname{spt} f_1 = \bigcup_{n\geq 0} K_n$. Changing the values of f_1 on the end points of each K_n, if necessary, we obtain a function f_2 with support equal to a countable union of disjoint closed intervals. Notice that f_2 is not necessarily of bounded variation, but since $f_2 \in [f] = [g]$, it is of bounded variation in L^1. Summarizing, given $f \in F$, there exists a function f_2 equal a.e. to f such that $\operatorname{spt} f_2$ is a countable (or finite) union of closed disjoint intervals. Hence we may assume, without loss of generality, that each function invariant under τ is of bounded variation in L^1 and its support consists of a union of disjoint closed intervals.

We will make a further assumption about the support of an invariant function: any two closed intervals in $\operatorname{spt} f = \bigcup_{\ell \geq 0} I_\ell$ will have to be separated by a set of positive measure. We explain how this is possible. Let $S = \operatorname{spt} f = \bigcup_{\ell \geq 0} I_\ell$ where the I_ℓ's are closed disjoint intervals. If $S \approx [a,b]$ then $m(S^c) = 0$ and by changing the values of f on S^c we can extend the support to the whole interval $[a,b]$. If $m(S^c) > 0$, we define for $x \in S$ the following functions:

$$\rho(x) = \sup\{h : m([x,x+h] \cap S) = h\}$$

$$\lambda(x) = \sup\{h : m([x-h,x] \cap S) = h\} .$$

Now for $I_\ell = [a_\ell, b_\ell]$, let $K_\ell = [a_\ell - \lambda(a_\ell), b_\ell + \rho(b_\ell)]$ and $T = \bigcup_{\ell \geq 0} K_\ell$. What we have done is to extend the intervals I_ℓ's as far as we can, ignoring sets of measure zero in S^c, in such a way that any pair of intervals are separated by a set of positive measure in S^c. The K_ℓ's are closed and disjoint. Furthermore, if an interval K is contained a.e. in $\bigcup_{\ell \geq 0} K_\ell$, then K is completely contained in one of the K_ℓ's. If this were not true, then $K \cap K_m \neq \phi$ and $K \cap K_n \neq \phi$ with $n \neq m$. Suppose then that $K_n < K_m$, i.e. $\sup K_n < \inf K_m$. Take any interval $[x,y]$ in K with $x \in K_n$ and $y \in K_m$. Then $m([x,y] \cap T) < m[x,y]$ since K_m and K_n are disjoint and are separated by a set of positive measure. In particular, $m(K \cap T) < m(K)$ and consequently $K \not\subseteq T$. This is a contradiction.

Thus, with this representation of the support, we can affirm that if an internal K is contained a.e. in $\bigcup_{\ell \geq 0} K_\ell$, then it is contained in one of the K_ℓ's.

CHAPTER III

INVARIANT FUNCTIONS

Throughout this chapter, τ is any measurable non-singular transformation from a compact interval I into itself. We assume that $\dim F \geq 1$, where F is the space of functions invariant under τ.

3.1 INVARIANT SETS

Let A and B be measurable subsets of I. We say that A is included almost everywhere in B, in notation $A \underset{\sim}{\subseteq} B$, if almost every element of A is in B, that is $m(A-B) = 0$. Also we write $A \approx B \leftrightarrow A \underset{\sim}{\subseteq} B$ and $B \underset{\sim}{\subseteq} A$. Clearly $A \approx B$ if and only if $m(A \Delta B) = 0$, where $A \Delta B$ is the symmetric difference of these two sets.

Definition 3.1: The set A is said to be invariant (under τ) if A is measurable and $\tau(A) \approx A$. Notice that this definition does not imply $\tau^{-1}(A) \approx A$ but only $A \underset{\sim}{\subseteq} \tau^{-1}(A)$.

We now list some obvious consequences of these definitions which will be used freely in the sequel:

(a) $A \underset{\sim}{\subseteq} B \Rightarrow m(A) = m(A \cap B) \leq m(B)$.

(b) $A \underset{\sim}{\subseteq} B \Rightarrow \int_A f\, dm = \int_{A \cap B} f\, dm$ for every $f \in L^1(I)$.

(c) If A and B are invariant under τ, so is $A \cup B$ since $\tau(A \cup B) = \tau(A) \cup \tau(B) \approx A \cup B$.

In general, contrary to what is affirmed in [2], the sets $A \cap B$ and $A - B$ will not be invariant when A and B are invariant, as can be seen from the following example:

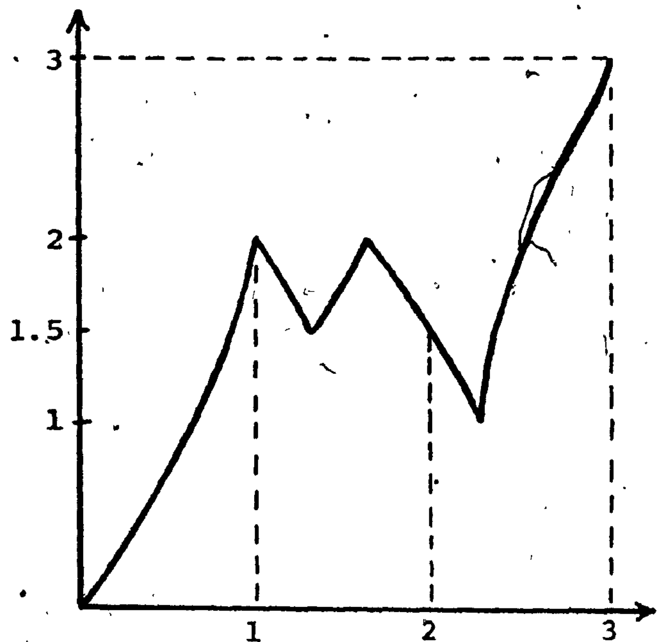

Figure 3.1

Let $\tau: [0,3] \to [0,3]$ have the above graph. If we let $A = [0,2]$ and $B = [1,3]$, then A and B are invariant, but

$$\tau(A \cap B) = [1.5,2] \not\approx A \cap B$$

and

$$\tau(A - B) = [0,2) \not\approx A - B\ .$$

However, in Lemma 3.3, we show that if A is the support of some invariant function, then $A \cap B$ and $A - B$ will be invariant for every invariant set B.

Before establishing some important properties of invariant functions, we need two preliminary results.

<u>Lemma 3.1</u>: Suppose A is a measurable subset of I satisfying $\tau(A) \subseteqq A$. Then, for every invariant function f,

$$\int_{\tau^{-1}(A)-A} f \, dm = 0$$

<u>Proof</u>: We note that $\tau(A) \subseteqq A$ implies $A \subseteqq \tau^{-1}(A)$. Since f is invariant, we have:

$$\int_A f \, dm = \int_{\tau^{-1}(A)} f \, dm$$

$$= \int_{\tau^{-1}(A) \cap A} f \, dm + \int_{\tau^{-1}(A)-A} f \, dm$$

$$= \int_A f \, dm + \int_{\tau^{-1}(A)-A} f \, dm$$

and the conclusion follows.

Q.E.D.

<u>Lemma 3.2</u>: Let f be a non-negative invariant function. If $A \subset \operatorname{spt} f$ and $\tau(A) \subseteqq A$, then A is an invariant set.

<u>Proof</u>: $\tau(A) \subsetneqq A$ implies $A \subset \tau^{-1}(\tau(A)) \subsetneqq \tau^{-1}(A)$. Let $B = \tau(A)$. Then we get the following chain of inclusions:

$$B \subsetneqq A \subset \tau^{-1}(B) \subsetneqq \tau^{-1}(A).$$

Since $\tau(B) \subsetneqq B$, we obtain in view of the preceding lemma

$$\int_{\tau^{-1}(B)-B} f\,dm = 0 .$$

But

$$0 = \int_{\tau^{-1}(B)-B} f\,dm = \int_{\tau^{-1}(B)-A} f\,dm + \int_{A-B} f\,dm .$$

Therefore,

$$\int_{A-B} f\,dm = 0 ,$$

and since $f > 0$ on A, it follows that $m(A-B) = 0$, i.e. $A \subsetneqq B$ and A is invariant.

Q.E.D.

3.2. PROPERTIES OF INVARIANT FUNCTIONS

Our objective in this section is to prove two important properties of invariant functions.

(1) If f is invariant, then $f^+ = \max(f,0)$ and $f^- = -\min(f,0)$ are also invariant, and $\operatorname{spt} f$ is an invariant set.

(2) Given any finite collection of linearly independent invariant functions, it is possible to "transform" it into a collection of non-negative invariant functions with disjoint supports.

For f any real-valued function defined on I, we let $P(f)$, $N(f)$ and $Z(f)$ denote the sets where f is positive, negative and zero respectively. Notice that $\operatorname{spt} f = P(f) \cup N(f)$. We will often write P, N and Z for these sets when no ambiguity can arise. Also, if A is any subset of I, then χ_A will denote the characteristic function of A, i.e., $\chi_A(x) = 1$ if $x \in A$ and 0 otherwise.

<u>Proposition 3.I</u>: Let f be invariant under τ. Then

(1) the sets P, N and $\operatorname{spt} f$ are invariant.

(2) f^+ and f^- are invariant functions.

<u>Proof</u>:

(1) Since f is invariant, we may write:

$$\int_P f\,dm = \int_{\tau^{-1}(P)} f\,dm = \int_{\tau^{-1}(P)\cap P} f\,dm + \int_{\tau^{-1}(P)\cap N} f\,dm\ .$$

If $m(\tau^{-1}(P)\cap N) > 0$, then

$$\int_{\tau^{-1}(P)\cap N} f\,dm < 0$$

and we get

$$\int_P f\,dm < \int_{\tau^{-1}(P)\cap P} f\,dm\ .$$

which is impossible. Therefore we must have $m(\tau^{-1}(P)\cap N) = 0$ and $m(\tau^{-1}(P)\cap P) = m(P)$,

i.e. $\tau^{-1}(P) \underset{\sim}{\subseteq} N^c = P \cup Z$ and $P \underset{\sim}{\subseteq} \tau^{-1}(P)$.

Hence $P \underset{\sim}{\subseteq} \tau^{-1}(P) \underset{\sim}{\subseteq} P \cup Z$.

(Notice this implies that f is a.e. zero on $\tau^{-1}(P)-P$.) Consequently, we get the following inclusions:

$$\tau(P) \underset{\sim}{\subseteq} P \subseteq \tau^{-1}(\tau(P)) \underset{\sim}{\subseteq} \tau^{-1}(P)\,.$$

Letting $A = \tau(P)$, we then have

$$\tau(A) \subseteq A \subseteq P \subset \tau^{-1}(A) \subseteq \tau^{-1}(P).$$

Now, by Lemma 3.1,

$$\int_{\tau^{-1}(A)-A} f\,dm = 0 ,$$

and so

$$0 \le \int_{P-A} f\,dm \le \int_{\tau^{-1}(A)-A} f\,dm = 0.$$

Therefore,

$$\int_{P-A} f\,dm = 0 ,$$

and since $f > 0$ on P, we must have $m(P-A) = 0$. Thus, $P \subseteq A = \tau(P)$ and P is an invariant set.

A similar argument can be used to prove that N is invariant. Finally, $\operatorname{spt} f$ is also invariant since $\operatorname{spt} f = P \cup N$, a union of two invariant sets.

(2) Let B be any measurable subset of I. Noticing that $f^+ = f\chi_P$, we have

$$\int_B f^+\,dm = \int_B f\chi_P\,dm = \int_{B\cap P} f\,dm$$

$$= \int_{\tau^{-1}(B\cap P)} f\,dm = \int_{\tau^{-1}(B)\cap\tau^{-1}(P)} f\,dm$$

But, as shown in the first part of this lemma,

$$P \subseteq \tau^{-1}(P) \subseteq P \cup Z .$$

Therefore,

$$\int_{\tau^{-1}(B)\cap\tau^{-1}(P)} f\,dm = \int_{\tau^{-1}(B)\cap P} f\,dm$$

$$= \int_{\tau^{-1}(B)} f\chi_P\,dm$$

$$= \int_{\tau^{-1}(B)} f^+\,dm .$$

Thus f^+ is invariant under τ. The proof that f^- is also invariant follows from the relation $f^- = f^+ - f$ and the fact that invariant functions form a subspace of $L^1(I)$.

Q.E.D.

Lemma 3.3: If f is invariant and A any invariant set, then

(1) $f\chi_A$ is an invariant function.

(2) the sets $(\operatorname{spt} f)\cap A$ and $(\operatorname{spt} f)-A$ are invariant.

Proof:

(1) It suffices to show that $f^+\chi_A$ and $f^-\chi_A$ are invariant. Let B be any measurable subset of I. Since $A \underset{\sim}{\subseteq} \tau^{-1}(A)$ and f^+ is invariant, we get

$$\int_B f^+\chi_A \, dm = \int_{B\cap A} f^+ \, dm = \int_{\tau^{-1}(B\cap A)} f^+ \, dm$$

$$= \int_{\tau^{-1}(B)\cap\tau^{-1}(A)} f^+ \, dm$$

$$= \int_{\tau^{-1}(B)\cap A} f^+ \, dm + \int_{\tau^{-1}(B)\cap[\tau^{-1}(A)-A]} f^+ \, dm$$

$$= \int_{\tau^{-1}(B)} f^+\chi_A \, dm + 0$$

(by Lemma 3.1).

This proves the invariance of $f^+\chi_A$. A similar argument holds for $f^-\chi_A$.

(2) Since $\operatorname{spt} f = P \cup N$, it suffices to show that $P \cap A$ and $N \cap A$ are both invariant. Notice that $P = \operatorname{spt} f^+$ and $N = \operatorname{spt} f^-$, and that these sets are invariant by Proposition 3.1.

Now,

$$\tau(P \cap A) \subset \tau(P) \cap \tau(A) \approx P \cap A .$$

Applying Lemma 3.2 to f^+ and $P \cap A$, we obtain the invariance of $P \cap A$. Similarly, $N \cap A$ is an invariant set. Thus $(\text{spt } f) \cap A$ is invariant.

To prove the invariance of $(\text{spt } f) - A$, it suffices to show that $P - A$ and $N - A$ are both invariant sets. If we could show that $\tau(P-A) \underset{\sim}{\subset} P-A$, then Lemma 3.2 applied to f^+ and $(P-A)$ will establish the invariance of $P-A$. To prove that $\tau(P-A) \underset{\sim}{\subset} P-A$, first notice that

$$\int_{\tau^{-1}(A)-A} f^+ \, dm = 0$$

by virtue of Lemma 3.1. Therefore,

$$\int_{P \cap (\tau^{-1}(A)-A)} f^+ \, dm = 0$$

and $m[P \cap (\tau^{-1}(A)-A)] = 0$. But

$$(P-A) - \tau^{-1}(P-A) = [(P-\tau^{-1}(P))-A] \cup [P \cap (\tau^{-1}(A)-A)]$$

and since $P \underset{\sim}{\subset} \tau^{-1}(P)$, we get $m[(P-A)-\tau^{-1}(P-A)] = 0$.

i.e. $\quad (P-A) \underset{\sim}{\subset} \tau^{-1}(P-A)$

or $\quad \tau(P-A) \underset{\sim}{\subset} P-A$.

Hence P-A is invariant. It can be similarly shown that N-A is invariant.

Q.E.D.

Lemma 3.4: If f_1 and f_2 are linearly independent functions in F, then there exist g_1 and g_2 in F such that

(1) $g_1 \geq 0$, $g_2 \geq 0$ and $\| g_1 \| = \| g_2 \| = 1$

(2) $\operatorname{spt} g_1$ and $\operatorname{spt} g_2$ are disjoint.

(3) For each $i = 1,2$, $\operatorname{spt} g_i$ is contained in $(\operatorname{spt} f_1) \cup (\operatorname{spt} f_2)$.

Proof: Dividing by their L^1-norm if necessary, we may assume that $\| f_1 \| = \| f_2 \| = 1$. If for $i = 1$ or 2, we have $m(P(f_i)) > 0$ and $m(N(f_i)) > 0$, we may take

$$g_1 = \frac{f_i^+}{\| f_i^+ \|} \quad \text{and} \quad g_2 = \frac{f_i^-}{\| f_i^- \|}$$

and the lemma is proved. We have the remaining cases when both f_1 and f_2 do not change sign. We assume $f_i \geq 0$ for each i, replacing f_i by $-f_i$ if necessary. Now, if $f_1 \geq f_2$ a.e., then $\| f_1 - f_2 \| = \| f_1 \| - \| f_2 \| = 0$ and so $f_1 = f_2$ a.e.

which contradicts their linear independence.

Similarly, we can't have $f_1 \le f_2$ a.e.. Therefore neither $f_1 - f_2 \ge 0$ a.e. nor $f_1 - f_2 \le 0$ a.e. is true. Consequently $(f_1-f_2)^+$ and $(f_1-f_2)^-$ are not zero a.e.. Let

$$g_1 = \frac{(f_1-f_2)^+}{\|(f_1-f_2)^+\|} \quad \text{and} \quad g_2 = \frac{(f_1-f_2)^-}{\|(f_1-f_2)^-\|}$$

Clearly, these two functions satisfy the conclusions of the lemma.

Q.E.D.

<u>Lemma 3.5</u>: Let $\{f_1, f_2, \ldots, f_m\}$ be a subset of F with disjoint supports, $\|f_i\| = 1$ and $f_i \ge 0$ for all $1 \le i \le m$. If $f \in F$ is independent of $\{f_1, \ldots, f_m\}$, there exists a set of non-negative functions $\{g_1, \ldots, g_m, g_{m+1}\} \subset F$ with disjoint supports and $\|g_i\| = 1$ for $1 \le i \le m+1$.

<u>Proof</u>: Without loss of generality, we may suppose $f \ge 0$ a.e.. For if both $P(f)$ and $N(f)$ have positive measure, then we claim either f^+ or f^- is linearly independent of the f_i's. Otherwise,

$$f^+ = \sum_{i=1}^{m} a_i f_i \quad \text{and} \quad f^- = \sum_{i=1}^{m} b_i f_i$$

imply

$$f = f^+ - f^- = \sum_{i=1}^{m} (a_i - b_i) f_i$$

and f is dependent on $\{f_1, \ldots, f_m\}$. Hence we replace f by f^+ or f^- and we obtain a non-negative function in F independent of the f_i's.

Now, let $S_i = \operatorname{spt} f_i$, $S = \bigcup_{i=1}^{m} S_i$, $A = \operatorname{spt} f$ and consider the following cases:

(1) $S \cap A \approx \phi$: The lemma is obvious if we let $g_i = f_i$ for $1 \le i \le m$ and $g_{m+1} = \frac{f}{\| f \|}$.

(2) $S \cap A \not\approx \phi$ and $A \not\subseteq S$: By Lemma 3.3, the set $A - S$ is invariant, and therefore the function $f^* = f\chi_{A-S}$ is invariant. Let $g_i = f_i$ for $1 \le i \le m$ and

$$g_{m+1} = \frac{f^*}{\| f^* \|} .$$

(3) $S \approx A$: Suppose for every $1 \le i \le m$ there exists α_i such that $f\chi_{S_i} = \alpha_i f_i$. Then

$$f = f\chi_S = \sum_{i=1}^{m} f\chi_{S_i} = \sum_{i=1}^{m} \alpha_i f_i$$

and f is dependent on the f_i's. Hence there must exist an index j, $1 \le j \le m$, such that $f\chi_{S_j}$ is independent of f_j. By applying

Lemma 3.4 to these two functions, we get g_j and g_{m+1} with disjoint supports, each contained in S_j, hence disjoint from $\bigcup_{\substack{i=1 \\ i \neq j}}^{m} S_i$. Take $g_i = f_i$ for $i \neq j$, and the lemma is proved.

(4) $m(A \cap S) = m(A) < m(S)$: Here we have to consider two possibilities:

(a) If A is a union of some S_i's, say $A = \bigcup_{i=1}^{k} S_i$, $1 \leq k < m$, then for some $1 \leq j \leq k$, f has to be independent of f_j on S_j (otherwise f will be dependent on all of the f_i's). Apply Lemma 3.4 to f_j and $f\chi_{S_j}$ to obtain g_j and g_{m+1}. Now let $g_i = f_i$ for $i \neq j$.

(b) There exists an index k such that $\phi \neq$ $\phi \neq A \cap S_k \subsetneq S_k$.

Let $f_k^* = f_k \chi_{S_k - A}$ and $f^* = f\chi_{S_k \cap A}$.

These two functions are invariant and have disjoint supports included in S_k. Let

$$g_k = \frac{f_k^*}{\| f_k^* \|} \quad , \quad g_{m+1} = \frac{f^*}{\| f^* \|}$$

and $g_i = f_i$ for $1 \le i \le m$, $i \ne k$. This completes the proof of the lemma.

Q.E.D.

We now come to the main result of this chapter.

Proposition 3.2

Let $\{f_1,\ldots,f_n\}$ be any independent set in F, where $n \ge 2$. Then there exists a set of non-negative functions $\{g_1,\ldots,g_n\}$ in F, with disjoint supports and $\| g_i \| = 1$ for each i.

Proof:

The proof is by induction on n. For $n = 2$, this is just Lemma 3.4. Suppose the proposition is true for $n \ge 2$ and let $\{f_1,\ldots,f_{n+1}\}$ be a linearly independent set in F, assuming that such a set exists. By the induction hypothesis applied to $\{f_1,\ldots,f_n\}$ there exist functions $h_1,\ldots,h_n$ in F which are non-negative, having disjoint supports and L^1-norm equal to one. Since $\dim F \ge n+1$, let h_{n+1} be any function in F independent of $\{h_1,\ldots,h_n\}$. Applying Lemma 3.5 to the h_i's , we get $\{g_1,\ldots,g_{n+1}\}$ which satisfies the conclusions of the proposition.

Q.E.D.

We close this chapter with a remark. If $\dim F = n < \infty$, then F has a basis consisting of non-negative functions with norm one, and having disjoint supports. If $\{f_1, \ldots, f_n\}$ is such a basis, then, for each i, the measure $d\mu_i = f_i \, dm$ is not only invariant, but also ergodic. For if μ_i were not ergodic for some i, there would exist a set $A \subset \operatorname{spt} f_i$ such that $\tau^{-1}(A) = A$ but $0 < \mu_i(A) < 1$. Define two new measures by $\nu = \mu_i|_A$ and $\eta = \mu_i|_{(\operatorname{spt} f_i) - A}$. It is easy to see that these measures are invariant and independent. Thus we increased the number of absolutely continuous independent invariant measures by one, which yields a contradiction.

Also, it is worth noticing that such a basis is unique. To see this, suppose $\{f_1, \ldots, f_n\}$ and $\{g_1, \ldots, g_n\}$ are two bases, each of them consisting of non-negative invariant functions with norm one and having disjoint supports. If one of the g_i's, say g_1, is not equal a.e. to any of the f_i's, then we must have $g_1 = a_1 f_1 + \ldots + a_n f_n$ with at least two non-zero scalars, say a_i and a_k. Let $\hat{g}_1 = g_1 \chi_{\operatorname{spt} f_i}$ and $\tilde{g}_1 = g_1 \chi_{\operatorname{spt} f_k}$. By Lemma 3.3, these two functions are invariant and consequently the set of functions $\{\hat{g}_1, \tilde{g}_1, g_2, \ldots, g_n\}$ have disjoint supports and thus are independent. This is a contradiction since $\dim F = n$.

CHAPTER IV

UPPER BOUNDS FOR THE NUMBER OF INVARIANT FUNCTIONS

4.1. THE LI AND YORKE THEOREM

Throughout this section, τ is a piecewise C^2 transformation mapping the interval into itself, with $\inf|\tau'| > 1$ where the derivative exists. We also denote by F the space of functions invariant under τ. As mentioned at the end of Chapter II, we will assume that the support of each $f \in F$ consists of a countable union of disjoint closed intervals. Let $\{x_1, \ldots, x_k\}$ be those points in (0,1) where the derivative τ' does not exist. We will refer to these points as discontinuities of τ. Our objective is to show that the number of independent invariant functions is bounded above by the number of discontinuities of τ, that is $\dim F \le k$. This is the main result in [2]. First we need an important lemma.

Lemma 4.1

Let $f \in F$ be non-negative with $\operatorname{spt} f = \bigcup_{k=0}^{p} I_k$, $1 \le p \le \infty$, where the I_k's are disjoint closed intervals. Then

(1) There exists an index ℓ such that I_ℓ contains at least one discontinuity x_j in its interior.

(2) $p < \infty$.

Proof:

(1) Suppose for each ℓ, I_ℓ does not contain any discontinuity x_j in its interior. This means that τ is strictly monotonic and continuous on the interior of each I_ℓ, and since $\inf|\tau'| > 1$, τ is uniformly expanding. Therefore for each $0 \le \ell \le p$, $\tau(I_\ell)$ is an interval with length greater than $m(I_\ell)$. Recalling that spt f is invariant under τ, we have

$$\tau(\operatorname{spt} f) = \tau\left(\bigcup_{\ell=0}^{p} I_\ell\right) = \bigcup_{\ell=0}^{p} \tau(I_\ell) \approx \bigcup_{\ell=0}^{p} I_\ell .$$

Now let k_1 be any index; $\tau(I_{k_1})$ is an interval contained a.e. in $\bigcup_{\ell=0}^{p} I_\ell$ and since these are disjoint and $m(\tau(I_{k_1})) > m(I_{k_1})$, there must exist $k_2 \neq k_1$ such that $\tau(I_{k_1}) \subsetneq I_{k_2}$ and $m(I_{k_2}) > m(I_{k_1})$. Repeating the same argument, we may construct a sequence of intervals $\{I_{k_i}\}_{i=0}^{\infty}$ with strictly increasing measures which are bounded below by $m(I_{k_1})$. This is a contradiction since the I_{k_i}'s are disjoint and contained in a finite interval. Therefore, at least one of the I_k's must contain an x_j in its interior.

(2) Let $D = \{0 \le k \le p : I_k$ contains a discontinuity of τ in its interior$\}$. By the first part of this lemma, D is not empty and is finite since there are only finitely many discontinuities. Notice that when $k \in D$, $\tau(I_k)$

consists of a finite union of intervals. Let J be the shortest interval in the collection of intervals

$$\{I_k\}_{k\in D} \cup \{\tau(I_k)\}_{k\in D}$$

and let S be the union of those intervals I_ℓ such that $m(I_\ell) \geq m(J)$. Clearly S is a finite union of closed disjoint intervals and $I_k \subset S$ when $k \in D$.

We claim that $\tau(S) \subseteq S$. To see this, let $I_k \subset S$. If $k \notin D$, then $\tau(I_k)$ is an interval contained in I_{k_1} for some $k_1 \neq k$ and $m(I_{k_1}) \geq m(\tau(I_k)) > m(I_k) \geq m(J)$, which implies that $\tau(I_k) \subseteq I_{k_1} \subset S$. If $k \in D$, $\tau(I_k)$ consists of a finite union of intervals, say $\bigcup_{i=1}^{m} J_i$, with $m(J_i) \geq m(J)$ for each $1 \leq i \leq m$. Also each J_i is contained in some I_{k_i}. Therefore $m(I_{k_i}) \geq m(J_i)$ and $I_{k_i} \subset S$ for each i, and

$$\tau(I_k) = \bigcup_{i=1}^{m} J_i \subseteq \bigcup_{i=1}^{m} I_{k_i} \subset S .$$

This proves our claim.

Now if $\operatorname{spt} f = S$, then $p < \infty$ and the lemma is proved. Otherwise, $(\operatorname{spt} f) - S$ is a union of disjoint intervals and if we let K denote the largest interval in this collection, $\tau(K)$ is an interval with length greater than $m(K)$. Hence $\tau(K) \not\subset (\operatorname{spt} f) - S$, thus $\tau(K) \subset S$ and $K \subset \tau^{-1}(S)$. By Lemma 3.1,

$$\int_{\tau^{-1}(S)-S} f\,dm = 0$$

and since $K \subset \tau^{-1}(S) - S$, we get

$$\int_K f\,dm = 0 .$$

This is a contradiction since f is non-negative and $K \subset \operatorname{spt} f$. Thus $\operatorname{spt} f = S$ and $p < \infty$.

Q.E.D.

Theorem 4.1

With the above assumptions on τ, there exists a finite collection of sets $M_1, \ldots, M_n$ and a set of non-negative functions $\{f_1, \ldots, f_n\} \subset F$ such that

(1) Each M_i is a finite union of closed intervals.

(2) $M_i \cap M_j$ contains at most a finite number of points when $i \neq j$.

(3) Each M_i contains at least one discontinuity x_j in its interior, and hence $n \leq k$.

(4) $f_i(x) = 0$ for $x \notin M_i$ and $f_i(x) > 0$ for almost all x in M_i.

(5) $\int_{M_i} f_i\,dm = 1$ for each $1 \leq i \leq n$.

(6) If $g \in F$ satisfies (4) and (5) for some $1 \leq i \leq n$, then $g = f_i$ a.e. .

(7) Every $f \in F$ can be written as $f = \sum_{i=1}^{n} a_i f_i$ with suitably chosen $\{a_i\}$.

<u>Proof</u>:

Most of the work has already been done. We know that $\dim\mathcal{F} \geq 1$ by Theorem 2.1. Let $\{g_1,\ldots g_j\}$ be any independent set in $\mathcal{F}$. By Proposition 3.2, there exist $\{f_1,\ldots f_j\}$ in $\mathcal{F}$ with disjoint supports and, in view of the preceding lemma, the support of each f_i has to contain at least one discontinuity x_j. Thus $j \leq k$, i.e. each independent set in $\mathcal{F}$ contains at most k elements. Hence $\mathcal{F}$ is finite-dimensional with dimension $n \leq k$.

Let $\{f_1,\ldots f_n\}$ be a basis for $\mathcal{F}$ consisting of non-negative functions with norm one, and having disjoint supports. If we let $M_i = \operatorname{spt} f_i$, then conclusions (1) to (5) and conclusion (7) follow.

It remains to prove (6). If $g \in \mathcal{F}$ satisfies (4) and (5) for some i and g is not equal a.e. to f_i, then the functions $(g-f_i)^+$ and $(g-f_i)^-$ are invariant with disjoint supports. Also both are not zero a.e. (see a similar argument in the proof of Lemma 3.4). Therefore, we get $n+1$ linearly independent functions in $\mathcal{F}$, which is impossible. Thus $g = f_i$ a.e.

Q.E.D.

<u>Definition 4.1</u>: By a <u>maximal set of disjoint (probability) density functions</u> for τ we mean a set of non-negative functions $\{f_1,\ldots f_n\}$ which satisfy the conclusions of Theorem 4.1.

We close this section by proving an interesting result. For $x \in [0,1]$, consider the orbit $\{x_n\}_{n=0}^{\infty}$ where $x_n = \tau^n(x)$, $x_o = x$, and denote by $\Lambda(x)$ the set of its limit points, that is

$$\Lambda(x) = \bigcap_{N=1}^{\infty} \overline{\{\tau^n(x)\}_{n=N}^{\infty}} .$$

We will show that for almost all x in $[0,1]$, $\Lambda(x)$ is one of M_i's. Notice that if $y \in \Lambda(x)$ and τ is continuous at this point, then for some subsequence $\{x_{n_k}\}_{k\geq 0}$ converging to y, we have

$$\tau(y) = \lim_{k\to\infty} \tau(x_{n_k}) = \lim_{k\to\infty} x_{1+n_k} .$$

Therefore $\tau(y) \in \Lambda(x)$ and we conclude that $\tau(\Lambda(x)) \subseteq \Lambda(x)$.

<u>Proposition 4.1</u>

For almost every x in $[0,1]$, $\Lambda(x) \approx M_i$ for some $1 \leq i \leq n$.

<u>Proof</u>:

Let $L_i = \bigcup_{k=0}^{\infty} \tau^{-k}(M_i)$ for $1 \leq i \leq n$, where $\tau^{-0}(M_i) \equiv M_i$. We first prove that $\bigcup_{i=1}^{n} L_i \approx [0,1]$. Suppose this is not the case. Then there exists a set B with $m(B) > 0$ in $[0,1] - \bigcup_{i=1}^{n} L_i$. Let $f = \chi_B$. By Theorem 2.1, the function $\frac{1}{m}\sum_{k=0}^{m-1} P_\tau^k f$ converges to a function $g \neq 0$ in the L^1-norm and g is invariant under τ. Let $L_o = \operatorname{spt} g$. Without loss of generality we may suppose $g > 0$ in L_o. We claim that $m(L_o \cap M_i) = 0$ for each i. To see this, let $A \subset M_i$ for some $i \in \{1,\ldots,n\}$. Then $\tau^{-k}(A) \subset L_i$ for all k. Hence, since $L_i \cap \operatorname{spt} f = \phi$,

$$\int_A P_\tau^k f\, dm = \int_A P_{\tau^k} f\, dm = \int_{\tau^{-k}(A)} f\, dm = 0.$$

for all k. Therefore $\int_A g\, dm = 0$ and $m(L_o \cap M_i) = 0$. This contradicts conclusion (7) of Theorem 4.1. Thus, $[0,1] \approx \bigcup_{i=1}^{n} L_i$.

Now for almost all x in M_i, by applying the Birkhoff Ergodic Theorem [3], we have

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} \chi_{M_i}(\tau^k(x)) = \int_{M_i} f_i\, dm = 1 .$$

Hence $\Lambda(x) \subsetneq M_i = \operatorname{spt} f_i$. Since $\tau(\Lambda(x)) \subsetneq \Lambda(x)$, Lemma 3.2 implies the invariance of $\Lambda(x)$. We claim that $\Lambda(x) \approx M_i$. If this were not true, then by Lemma 3.3, f_i restricted to $\Lambda(x)$ would be an invariant function which could not be written as a linear combination of $\{f_1,\ldots,f_n\}$, and this contradicts conclusion (7) of Theorem 4.1.

Q.E.D.

4.2. IMPROVED UPPER BOUNDS (New results)

Again, let τ be a piecewise C^2 function with $\inf|\tau'| > 1$. In the preceding section, we considered the points $\{x_1,\ldots x_k\}$ in $(0,1)$ where τ' did not exist and we found that k constituted an upper bound for n, the dimension of F. It is actually possible in some cases to improve this bound.

In this section, we will consider the partition

$$0 = b_0 < b_1 < \ldots < b_m < b_{m+1} = 1$$

where τ is continuous and monotonic on each interval (b_{i-1}, b_i). Clearly $m \le k$.

Theorem 4.2: With the above notations, $\dim F \le m$.

Proof:

Let $\{f_1,\ldots,f_n\}$ and $M_1,\ldots M_n$ be as in Theorem 4.1. We claim that for each $i = 1,2,\ldots,n$, M_i contains some b_j, $1 \le j \le m$, in its interior. Suppose this is not true for some i, and let $[a,b]$ be the largest interval in M_i. Then τ is monotonic and continuous on (a,b) and since $\inf|\tau'| > 1$, $\tau(a,b)$ is an interval with length strictly greater than $[a,b]$. But M_i is invariant under τ. Thus $\tau(a,b) \subset \tau(M_i) \approx M_i$ and M_i contains an interval larger than $[a,b]$. This contradicts our choice of the interval $[a,b]$, and the claim is proved. Since the M_i's have disjoint interiors, we see that n cannot be greater than m.

Q.E.D.

Remark: Roughly speaking, this theorem says that the number of independent invariant functions (under τ) is at most one less than the number of continuous monotonic pieces in the graph of τ. In the special case where τ is continuous on $[0,1]$, the total number of peaks and valleys in the graph of τ constitutes an upper bound for $\dim F$.

In section 3 of [4] an upper bound for the number of absolutely continuous invariant measures is given in terms of the number of "independent pairs." With the same partition as above, let $u_k = \tau(b_k^-)$ and $v_k = \tau(b_k^+)$ for each $1 \le k \le m$. The pair $< u_k, v_k >$ will then denote the open interval (u_k, v_k) or (v_k, u_k). If $u_k = v_k$, then $< u_k, v_k > = \{u_k\}$. Two pairs $< u_i, v_i >, < u_j, v_j >$ are said to be independent if the corresponding intervals have no end points in common, that is either they are completely disjoint or one lies strictly inside the other. If N_τ denotes the maximal number of independent pairs, then Theorem 2 [4] affirms that the number of independent invariant functions is bounded above by N_τ. This is actually not correct and we furnish a counter-example:

Consider the map τ whose graph is given in Figure 4.1.

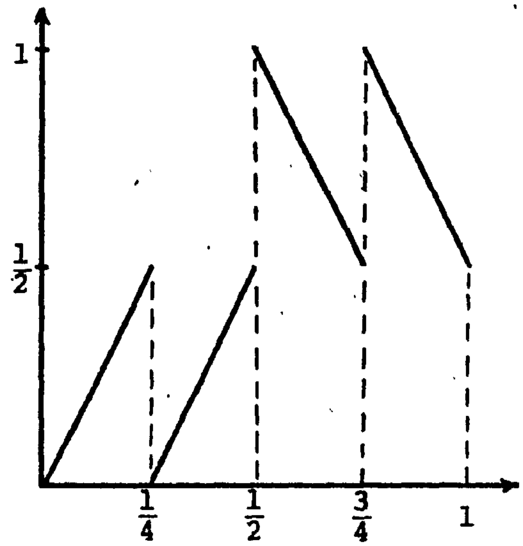

Figure 4.1

There are three pairs $<\frac{1}{2}, 0>$, $<\frac{1}{2}, 1>$ and $<\frac{1}{2}, 1>$ and, according to the above definition, they are all dependent. Hence $N_\tau = 1$ and consequently there exists a unique invariant function. But if we let

$$f_1 = \chi_{[0,1/2]} \ , \quad f_2 = \chi_{[1/2,1]}$$

and $d\mu_i = f_i \, dm$, then clearly μ_1 and μ_2 are both invariant and independent. Hence N_τ is not an upper bound.

We suggest an alternative bound based upon a modified definition of "dependence". With the above assumptions on τ and its partition, let $\mathcal{D} = \{b_1, b_2, \ldots, b_m\}$. We shall say that b_i and b_j are dependent if

$$\tau(b_i-\varepsilon, b_i+\varepsilon) \cap \tau(b_j-\varepsilon, b_j+\varepsilon)$$

has positive measure for every $\varepsilon > 0$. This implies, but is not equivalent to

$$\{\tau(b_i^-), \tau(b_i^+)\} \cap \{\tau(b_j^-), \tau(b_j^+)\} \neq \phi.$$

This definition of dependence for a pair of discontinuities in $\mathcal{D}$ is reflexive, symmetric but not transitive. A collection $S \subset \mathcal{D}$ is said to be dependent if every pair of points in this collection is dependent, and maximal if S is not a proper subset of any dependent collection. Notice that two distinct maximal dependent collections may have non-empty intersection, and such a collection may consist of a single point. Thus given $b_j \in \mathcal{D}$ there exists at least one and at most two distinct maximal dependent collections containing b_j. In particular, when τ is continuous at b_j, there exists only one maximal dependent collection containing this point.

Let N_τ be the number of distinct maximal dependent collections. We have the following result:

Theorem 4.3

The number of independent invariant functions for τ is bounded above by N_τ.

Proof:

We show first that if f_1 and f_2 are invariant with disjoint supports, then to each f_i corresponds one maximal dependent collection S_i and $S_1 \neq S_2$. Letting $M_i = \text{spt}\, f_i$, we know from the proof of Theorem 4.2 that $\text{int}\, M_i$ has to contain at least one point of $\mathcal{D}$, say b_i'. Let S_1 and S_2 be any maximal dependent collections containing b_1' and b_2' respectively, and suppose $S_1 = S_2$. Then b_1' and b_2' are dependent and since $\tau(M_i) \subseteq M_i$ and $(b_i'-\varepsilon, b_i'+\varepsilon) \subset M_i$ for some $\varepsilon > 0$, their dependence implies

$$m(M_1 \cap M_2) \geq m[\tau(b_1'-\varepsilon\, ,\, b_1'+\varepsilon) \cap \tau(b_2'-\varepsilon\, ,\, b_2'+\varepsilon)] > 0.$$

This is a contradiction. Therefore S_1 and S_2 must be distinct.

Now let $\{f_1, \ldots, f_n\}$ be a maximal set of disjoint density functions for τ. By the preceding argument, we see that there exists a one-to-one mapping from $\{f_1, \ldots f_n\}$ into $\{S_1, \ldots, S_{N_\tau}\}$. Thus $n \leq N_\tau$

Q.E.D.

Example 1:

Consider the following transformation:

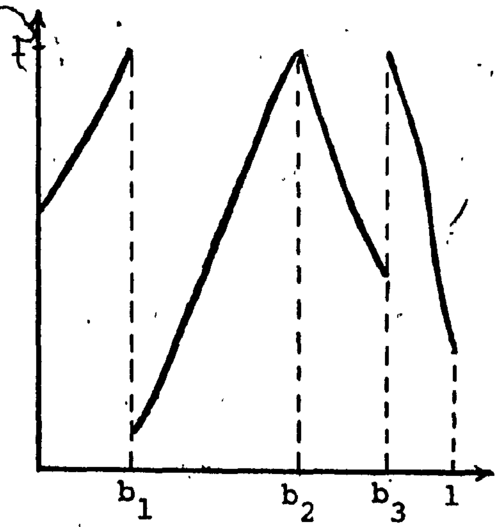

Figure 4.2

We see that $\{b_1, b_2, b_3\}$ is the unique collection which is dependent and maximal. Thus $N_\tau = 1$ and there exists a unique absolutely continuous invariant measure.

Example 2:

Let τ have the following graph:

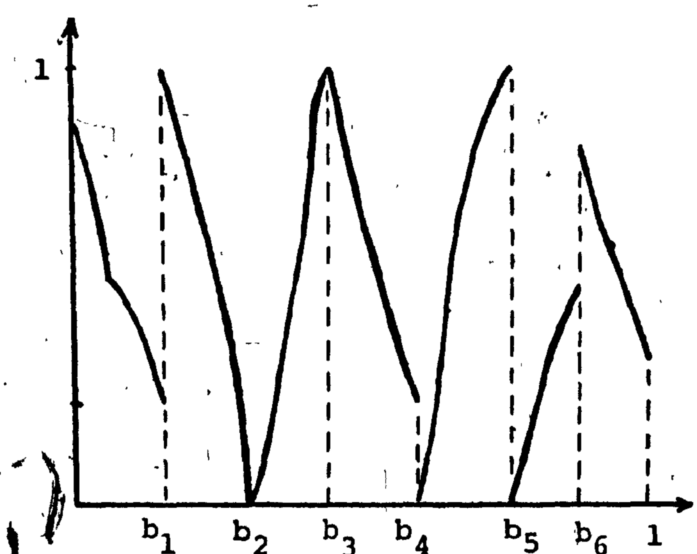

Figure 4.3

For each discontinuity, we give the corresponding maximal dependent collection(s):

b_1: $\{b_1, b_3, b_5\}$ and $\{b_1, b_4\}$

b_2: $\{b_2, b_4, b_5\}$

b_3: $\{b_1, b_3, b_5\}$

b_4: $\{b_1, b_4\}$ and $\{b_2, b_4, b_5\}$

b_5: $\{b_1, b_3, b_5\}$ and $\{b_2, b_4, b_5\}$

b_6: $\{b_6\}$.

We see that $N_\tau = 4$ and therefore there exist at most four invariant functions. Notice that the bounds given by Theorems 4.1 and 4.2 are 7 and 6 respectively.

4.3. TOPOLOGICALLY CONJUGATE TRANSFORMATIONS

Definition:

Let τ be a piecewise monotonic transformation mapping the interval $J = [a,b]$ into itself. If $h : J \to J$ is a homeomorphism then $\sigma = h^{-1} \circ \tau \circ h$ is a transformation from J into J, and τ and σ are said to be topologically conjugate.

Theorem 4.4

Let τ be a piecewise monotonic transformation having invariant functions. Assume that the homeomorphism $h: J \to J$ is differentiable. Let $\sigma = h^{-1} \circ \tau \circ h$. Then

(i) If f is invariant under τ, then the function $f(h(x))\frac{dh}{dx}$ is invariant under σ.

(ii) τ and σ have the same number of absolutely continuous invariant measures.

Proof:

(i) If f is invariant under τ, then for any measurable $A \subset J$

$$\int_{\tau^{-1}(A)} f\,dm = \int_A f\,dm \ .$$

Without loss of generality we shall assume that h is strictly increasing. It follows that for every x,

$$\int_{h[a,x]} f\,dm = \int_a^{h(x)} f\,dm = \int_a^x f(h(u))\frac{dh(u)}{du}\,dm$$

by a change of variable. Define $\bar{f}(u) = f(h(u))\frac{dh(u)}{du}$.

Then for every interval $B = [a,x]$ we have

$$\int_{h(B)} f\,dm = \int_B \bar{f}\,dm \ . \qquad (1)$$

Equation (1) is also valid for open sets B and, from that, for all measurable sets B of $[a,b]$. In particular for $B=\sigma^{-1}[a,x]$ we get

$$\int_{h(\sigma^{-1}[a,x])} f\,dm = \int_{\sigma^{-1}[a,x]} \bar{f}\,dm .$$

Since $h(\sigma^{-1}(A))=\tau^{-1}(h(A))$ for any set A, it follows that

$$\int_{h(\sigma^{-1}[a,x])} f\,dm = \int_{\tau^{-1}(h[a,x])} f\,dm = \int_{h[a,x]} f\,dm$$

$$= \int_a^x \bar{f}\,dm$$

where the last equality follows from (1). Therefore, for every x,

$$\int_{\sigma^{-1}[a,x]} \bar{f}\,dm = \int_a^x \bar{f}\,dm$$

and $\bar{f}$ is invariant under σ.

(ii) Suppose τ has exactly n independent invariant functions. Let $\{f_1,\ldots,f_n\}$ be a set of invariant functions (under τ) with disjoint supports. Then, for each i, the function $\bar{f}_i(x)=f_i(h(x))\frac{dh}{dx}$ is invariant under σ. If $\bar{f}_i(x)\neq 0$ then $f_i(h(x))\neq 0$ and $x\in h^{-1}(\operatorname{spt} f_i)$. Thus $\operatorname{spt}\bar{f}_i\subset h^{-1}(\operatorname{spt} f_i)$ and the $\bar{f}_i$'s will also have disjoint supports. Therefore σ has at least n independent invariant functions. If we apply the same argument to σ and its topologically conjugate $\tau = h \circ \sigma \circ h^{-1}$, we see that both must have the same number of invariant functions.

Q.E.D.

CHAPTER V

TWO CLASSES OF TRANSFORMATIONS WITH UNIQUE INVARIANT MEASURE

5.1. RENYI TRANSFORMATIONS

In [5] Renyi has shown that the transformation $\tau(x) = \lambda x \pmod 1$ from the unit interval into itself has a unique non-negative invariant function with norm one for $\lambda > 1$. In this section, we generalize this result by replacing λx by any C^2 function $p(x)$ with slope greater than one. The discontinuities of τ will then be all those x's in $(0,1)$ where $p(x)$ is an integer. Thus τ is a piecewise increasing (or decreasing) function with only a finite number of jump discontinuities, each of these jumps having magnitude equal to one.

Theorem 5.1

Let $p(x)$ be a C^2 function with $|p'(x)| > 1$ for $x \in [0,1]$. Then the map $\tau(x) = p(x) \pmod 1$ has a unique non-negative invariant function f with $\|f\| = 1$.

Proof:

Notice that the continuity of $p'(x)$ over the compact set $[0,1]$ implies $\inf|p'(x)| > 1$, so τ is a piecewise C^2 function with $\inf|\tau'| > 1$ where the derivative exists. Existence of an invariant function for τ is guaranteed by Theorem 2.1. If $a_1, \ldots, a_k$ denote the discontinuities of τ, it is easy to see that $\{a_1, \ldots, a_k\}$ is the unique maximal dependent collection of discontinuities. Therefore, by Theorem 4.3, there exists a unique absolutely continuous invariant measure for $\overset{\circ}{\tau}$.

Q.E.D.

Next we show that if $|p'(x)| > 2$, then every iterate of τ will have a unique invariant function. Before proceeding with the proof, we need a few lemmas. First, we recall two definitions pertinent to square matrices: An $n \times n$ matrix $A = (a_{ij})$ is stochastic if $a_{ij} \geq 0$ and $\sum_{j=1}^{n} a_{ij} = 1$ for each $1 \leq i \leq n$. It is well known that A^k will also be stochastic for every $k \geq 1$. We say that a matrix B is a permutation matrix if B is obtained from the identity matrix by permutations of rows.

<u>Lemma 5.1</u>

Let A be an $n \times n$ stochastic matrix. If $A^N = I$ (the identity matrix) for some $N > 1$, then A^k is a permutation matrix for each $1 \leq k \leq N$.

<u>Proof:</u>

We claim that for any matrix $B = (b_{ij})$, the inequality

$$\operatorname*{Max}_{1 \leq i \leq n} (AB)_{ij} \leq \operatorname*{Max}_{1 \leq i \leq n} b_{ij}$$

holds for each $1 \leq j \leq n$. To see this, let $M_j = \operatorname*{Max}_{1 \leq k \leq n} b_{kj}$.

Then

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$\leq \sum_{k=1}^{n} a_{ik} M_j = M_j \left(\sum_{k=1}^{n} a_{ik} \right) = M_j .$$

Therefore $\underset{1 \leq i \leq n}{\text{Max}} (AB)_{ij} \leq M_j$ for each fixed j, and the claim is proved. As a consequence, we have that for each j and any k,

$$\underset{1 \leq i \leq n}{\text{Max}} (A^{k+1})_{ij} = \underset{1 \leq i \leq n}{\text{Max}} (A.A^k)_{ij} \leq \underset{1 \leq i \leq n}{\text{Max}} (A^k)_{ij} .$$

Now the assumption that $A^N = I$ combined with the last inequality imply that A^{N-1} has an entry equal to one in each column. Since A^{N-1} is also stochastic, it must be a permutation matrix. Repeating the same argument, we see that $A^{N-2}, \ldots, A^2$ and A are all permutation matrices.

Q.E.D.

For the next lemma, we let P denote the Frobenious-Perron operator corresponding to τ. Notice that if f is invariant under τ^N, then so is $P^k f$ for each $k \geq 1$ since P^N is the operator corresponding to τ^N and

$$P^N(P^k f) = P^k(P^N f) = P^k f .$$

<u>Lemma 5.2</u>

Let $f_1, \ldots f_n$ be a maximal set of disjoint density functions for τ^N. Then for every $1 \leq i \leq n$, $Pf_i = f_j$ for

for some $1 \le j \le n$.

Proof:

Invoking Theorem 4.1 and the fact that Pf_i is invariant under τ^N, we may write

$$Pf_i = \sum_{j=1}^{n} a_{ij} f_j$$

for some matrix $A = (a_{ij})$. The Frobenious-Perron operator is known to be positive and to preserve integrals. Therefore, for each i,

$$1 = \int_0^1 f_i \, dm = \int_0^1 Pf_i \, dm = \sum_{j=1}^{n} \left(a_{ij} \int_0^1 f_j \, dm \right) = \sum_{j=1}^{n} a_{ij}$$

and $a_{ij} \ge 0$ for each j (If $a_{ij} < 0$ for some j, then Pf_i would be negative on $\operatorname{spt} f_j$, which contradicts the positivity of P). Hence A is a stochastic matrix.

Since $P^N f_i = f_i$ for all i, it is easy to see that $A^N = I$, and Lemma 5.1 implies that A is a permutation matrix. Thus $P:\{f_1,\dots,f_n\} \to \{f_1,\dots,f_n\}$ is a permutation.

Q.E.D.

Lemma 5.3

For $\{f_1,\dots f_n\}$ as in Lemma 5.2, let $M_i = \operatorname{spt} f_i$ and $[a,b]$ be the largest interval in all of the M_i's. Then

(1) $[a,b]$ contains at least two discontinuities of τ in its interior.

(2) There exists an interval $(x,y) \subset (a,b)$ such that $\tau^N(x,y) = (0,1)$ and τ is continuous on (x,y).

Proof:

(1) We have $(a,b) \subset M_i = \operatorname{spt} f_i$ for some i. By the preceding lemma, $Pf_i = f_j$ for some $1 \le j \le n$, and consequently

$$1 = \int_{M_j} f_j \, dm = \int_{M_j} Pf_i \, dm = \int_{\tau^{-1}(M_j)} f_i \, dm .$$

This implies that $M_i \subseteq \tau^{-1}(M_j)$, i.e. $\tau(M_i) \subset M_j$. Therefore $\tau(a,b) \subset \operatorname{spt} f_j = M_j$. Notice that (a,b) has to contain at least one discontinuity of τ, otherwise $\tau(a,b)$ would be an interval in M_j with length greater than $[a,b]$ and this contradicts our choice of $[a,b]$. Suppose that (a,b) contains exactly one discontinuity of τ, say z. There is no loss of generality in assuming that $z-a \ge b-z$. Since τ is continuous on (a,z) and $\inf|\tau'| > 2$, we see that $\tau(a,z)$ is an interval in M_j with length strictly greater than $2(z-a)$, i.e. greater than $b-a$. This is a contradiction. Hence, $[a,b]$ has to contain at least two discontinuities in its interior.

(2) Let x and y be two consecutive discontinuities in (a,b). Clearly, $\tau(x)$ and $\tau(y)$ are integers

(zero or one) and since τ is continuous and monotonic on (x,y), it must be that $\tau(x,y) = (0,1)$. Thus $\tau^N(x,y) = (0,1)$ and the lemma is proved.

Q.E.D.

Theorem 5.2

Let $p(x)$ be a C^2 function with $|p'(x)| > 2$ on the interval $[0,1]$. Let $\tau(x) = p(x) \pmod 1$. Then for any positive integer N, τ^N has a unique invariant function.

Proof:

For any fixed N, the existence of an invariant function is ensured by Theorem 2.1, since τ^N is a piecewise C^2 map with $\left|\frac{d\tau^N}{dx}\right| \geq 2$ where the derivative exists.

Let $\{f_1,\ldots,f_n\}$ be as in Lemma 5.2. By the preceding lemma, the support of one of these functions, say f_1, contains an interval (x,y) with the property that $\tau^N(x,y) = (0,1)$. Since $\operatorname{spt} f_1$ is invariant under τ^N, we have

$$(0,1) = \tau^N(x,y) \subset \tau^N(\operatorname{spt} f_1) \approx \operatorname{spt} f_1 .$$

The f_i's having disjoint supports and $\operatorname{spt} f_1$ being equal almost everywhere to $[0,1]$, we conclude that $n = 1$, i.e. τ^N has a unique invariant function.

Q.E.D.

5.2 MARKOV MAPS

Let J be any compact interval of the real line. We say that $\tau : J \to J$ takes partition points into partition points if there exists a partition $P = \{(a_0,a_1),(a_1,a_2),\ldots,(a_{N-1},a_N)\}$ of J such that $\tau(Q) \subset Q$ where Q are the partition points of P. If τ is discontinuous at some $a_i \in Q$, we shall require that both $\tau(a_i^-)$ and $\tau(a_i^+)$ be in Q. Without loss of generality, we shall always assume that τ is either left or right continuous at each point of Q. In the case where τ is piecewise C^2 with respect to P, this means that each interval of P is mapped onto a finite number of adjoining or contiguous intervals of P. Notice that $\tau(Q) \subset Q$ is equivalent to the statement that partition points are eventually periodic. The point $x \in J$ is an eventually periodic point of τ if there exists an $n = n(x)$ such that $\tau^n(x)$ is periodic.

A map which takes partition points into partition points is often called a Markov map. We will study these maps under an additional condition: the partition P must have the communication property under τ. This means given any $I_i, I_j \in P$ there exist integers n and m such that $I_j \subset \tau^n(I_i)$ and $I_i \subset \tau^m(I_j)$.

<u>Definition</u>:

A point transformation $\tau : J \to J$ is in <u>class C</u> if there exists a partition $\mathcal{P}$ such that:

(1) τ is piecewise C^2 with respect to $\mathcal{P}$ and $\inf|\tau'| > 1$.

(2) $\tau(Q) \subset Q$ where Q are the partition points of $\mathcal{P}$

(3) $\mathcal{P}$ has the communication property under τ.

We will show that each τ in class C has a unique absolutely continuous invariant measure. Our first objective will be to prove the existence of a dense orbit in J. Using symbolic dynamics, we associate with each of the intervals $(a_0,a_1),(a_1,a_2),\ldots,(a_{N-1},a_N)$ of $\mathcal{P}$ a symbol such as $\alpha,\beta,\gamma,\ldots.$ and code the orbit of x by an infinite sequence

$$< x > = .\alpha\beta\gamma\ldots$$

to mean that $x \in I(\alpha)$, $\tau(x) \in I(\beta)$, $\tau^2(x) \in I(\gamma),\ldots$ where $I(\alpha)$ is the interval in $\mathcal{P}$ whose symbol is α. Note that this coding is uniquely defined except for possibly the points eventually entering the partition points Q. To avoid this difficulty, we will code the orbit of only those x's which never enter in Q, i.e. all x in

$\bar{J} = J - \bigcup_{k=0}^{\infty} \tau^{-k}(Q)$ where $\tau^0(Q) \equiv Q$. Notice that $m(\bar{J}) = m(J)$.

<u>Lemma 5.4</u>

Let τ satisfies condition (1) defining class C. If $x, y \in \bar{J}$ are such that $\langle x \rangle = \langle y \rangle$ then $x = y$.

<u>Proof:</u>

Suppose $\langle x \rangle = \langle y \rangle$ but $|x-y| > 0$, and let $d = \inf|\tau'| > 1$. By hypothesis, $\tau^n(x)$ and $\tau^n(y)$ belong to the same open interval of P for each $n \geq 0$. Thus

$$\begin{aligned} |\tau^n(x) - \tau^n(y)| &= |\tau(\tau^{n-1}(x)) - \tau(\tau^{n-1}(y))| \\ &\geq d|\tau^{n-1}(x) - \tau^{n-1}(y)| \\ &\vdots \\ &\geq d^n|x-y| \to \infty \text{ as } n \to \infty . \end{aligned}$$

Contradiction. Hence $x = y$.

Q.E.D.

<u>Lemma 5.5</u>

Let τ be as in Lemma 5.4. If $\sigma = .\alpha_1\alpha_2\alpha_3\ldots$ is a sequence with the property that $\tau(I(\alpha_k)) \supset I(\alpha_{k+1})$ for each $k \geq 1$, then there exists a unique $x \in \bar{J}$ with $\langle x \rangle = \sigma$.

<u>Proof:</u> For $n > 1$, let

$$\begin{aligned} J_n &= \{x \in J : x \in I(\alpha_1), \tau(x) \in I(\alpha_2), \ldots, \tau^{n-1}(x) \in I(\alpha_n)\} \\ &= I(\alpha_1) \cap \tau^{-1}(I(\alpha_2)) \cap \ldots \cap \tau^{-n+1}(I(\alpha_n)). \end{aligned}$$

We claim that J_n is a non-empty closed interval for each

n. To see this, notice that τ is monotonic and continuous on each interval of the partition and $\tau(I(\alpha_{n-1})) \supset I(\alpha_n)$. This implies that $\tau^{-1}(I(\alpha_n)) \cap I(\alpha_{n-1})$ is a non-empty closed interval in $I(\alpha_{n-1})$, call it B_{n-1}. Similarly $\tau(I(\alpha_{n-2})) \supset I(\alpha_{n-1})$ implies $\tau^{-1}(B_{n-1}) \cap I(\alpha_{n-2})$ is a closed interval in $I(\alpha_{n-2})$, say B_{n-2}. Continuing this way, we obtain a sequence of non-empty closed intervals $B_{n-1}, B_{n-2}, \ldots, B_2$ with $B_k \subset I(\alpha_k)$ and $B_{k-1} = \tau^{-1}(B_k) \cap I(\alpha_{k-1})$. In particular, $\tau^{-1}(B_2) \cap I(\alpha_1)$ is an interval in $I(\alpha_1)$. But

$$\begin{aligned}\tau^{-1}(B_2) \cap I(\alpha_1) &= \tau^{-2}(B_3) \cap \tau^{-1}(I(\alpha_2)) \cap I(\alpha_1) \\ &= \tau^{-3}(B_4) \cap \tau^{-2}(I(\alpha_3)) \cap \tau^{-1}(I(\alpha_2)) \cap I(\alpha_1) \\ &\;\vdots \\ &= J_n\end{aligned}$$

and the claim is proved.

Now $J_n \supset J_{n+1} \Rightarrow \bigcap_{n=1}^{\infty} J_n \neq \phi$. If x is in this intersection, then $\langle x \rangle = \sigma$ and Lemma 5.4 implies that this x is unique in $\bar{J}$.

Q.E.D.

<u>Lemma 5.6</u>

Let τ satisfy conditions (1) and (2) defining class C. Let $\xi \subset P$ be a collection of intervals satisfying the communication property: if $I_1, I_2 \in \xi$, there exist n and m such that $I_1 \subset \tau^m(I_2)$ and $I_2 \subset \tau^n(I_1)$. Assume ξ contains at least two intervals and let $V = \bigcup_{I \in \xi} I$. Then there exists an $x \in V$ such that $\{\tau^i(x)\}$ is dense in V. (Notice that if $\tau \in C$, there exists a dense orbit in all of J).

Proof:

Consider the set of all possible finite sequences $.\alpha_1\alpha_2\ldots\alpha_k$ where $I(\alpha_1)$ and $I(\alpha_k) \in \xi$, and $\tau(I(\alpha_j)) \supset I(\alpha_{j+1})$, $1 \le j \le k-1$. Such sequences exist by condition (2), and the set of all such sequences is countable. Let $S_1, S_2, S_3, \ldots$ be an enumeration, and form the sequence

$$< x > = .S_1T_1S_2T_2\ldots$$

where the T_i's are finite sequences joining the last symbol of S_i to the first symbol of S_{i+1}. That this can be done follows from the communication property of intervals in ξ. Thus, by the preceding lemma, a real x exists corresponding to the coding $< x >$.

Now given $y \in V$ and $\varepsilon > 0$, we claim there exists an integer n such that $|\tau^n(x) - y| < \varepsilon$. Choose m such that $2M/d^m < \varepsilon$ where $M = \max_{x \in J} \tau(x)$ and $d = \inf|\tau'| > 1$. Consider the orbit $< y > = .\beta_1\beta_2\beta_3\ldots$ and let $S = .\beta_1\beta_2\ldots\beta_{m+1}$. This S occurs in the coding of x, therefore for some n, $\tau^{n+k}(x)$ and $\tau^k(y)$ belong to the same intervals for

$0 \le k \le m$. But

$$\begin{aligned} |\tau^n(x)-y| &\le \frac{1}{d}|\tau^{n+1}(x) - \tau(y)| \\ &\le \frac{1}{d^2}|\tau^{n+2}(x) - \tau^2(y)| \\ &\vdots \\ &\le \frac{1}{d^m}|\tau^{n+m}(x) - \tau^m(y)| \\ &\le \frac{2M}{d^m} < \varepsilon. \end{aligned}$$

Thus the orbit of x is dense in V.

Q.E.D.

Theorem 5.3

Let $\tau \in C$. Then τ has a unique absolutely continuous invariant measure.

Proof:

From Theorem 2.1 we know there exists an absolutely continuous measure invariant under τ. Suppose there exist two such measures with densities f_1 and f_2. We may assume that these densities are non-negative with disjoint supports S_1 and S_2, and $\|f_1\| = \|f_2\| = 1$. Also each S_i is a finite union of closed intervals.

Let $x \in J$ be a point which has a dense orbit in J, such a point exists by the preceding lemma. If we let $x_n = \tau^n(x)$, then $x_n \notin Q$ for any $n > 0$ (otherwise x would be an eventually periodic point of τ), where Q

are the partition points of P. Thus, corresponding to n there exists an open ball O_n centered at x_n such that τ is continuous and monotonic on O_n. Clearly, $\tau(O_n)$ will contain an open ball centered at x_{n+1} and consequently $\tau^k(O_n)$ contains also some open interval around x_{n+k} for any $k \geq 1$.

Now the denseness of $\{x_n\}$ implies the existence of points x_k and x_ℓ such that $x_k \in \text{int}\, S_1$, $x_\ell \in \text{int}\, S_2$ and $\ell > k$, where int denotes interior. By the preceding argument, we can find an open interval O_k such that $x_k \in O_k \subset \text{int}\, S_1$ and $\tau^{\ell-k}(O_k)$ contains some open interval O_ℓ around x_ℓ included in $\text{int}\, S_2$. Thus

$$m(\tau^{\ell-k}(O_k) \cap \text{int}\, S_2) > 0.$$

But S_1 is invariant under τ, therefore

$$\tau^{\ell-k}(O_k) \subset \tau^{\ell-k}(S_1) \approx S_1.$$

Consequently $m(S_1 \cap S_2) > 0$ which is a contradiction. Hence there exists only one absolutely continuous invariant measure.

Q.E.D.

<u>Corollary 5.1</u>

Let τ be a piecewise C^2 map with $\inf|\tau'| > 0$. If τ^ℓ is in class C for some integer ℓ, then τ has a unique absolutely continuous invariant measure.

Proof:

The existence of an invariant function for τ follows from Theorem 2.2. Since τ^{ℓ} has a unique invariant function and every function invariant under τ is also invariant under τ^{ℓ}, the conclusion follows.

Q.E.D.

We would like to generalize the result of Theorem 5.3 to the iterates of τ. If τ is in class C then, for every n, τ^n will satisfy conditions (1) and (2) defining this class (see next theorem). However, in general, τ^n will fail to satisfy condition (3) and thus we cannot conclude that $\tau^n \in C$. Here is an example:

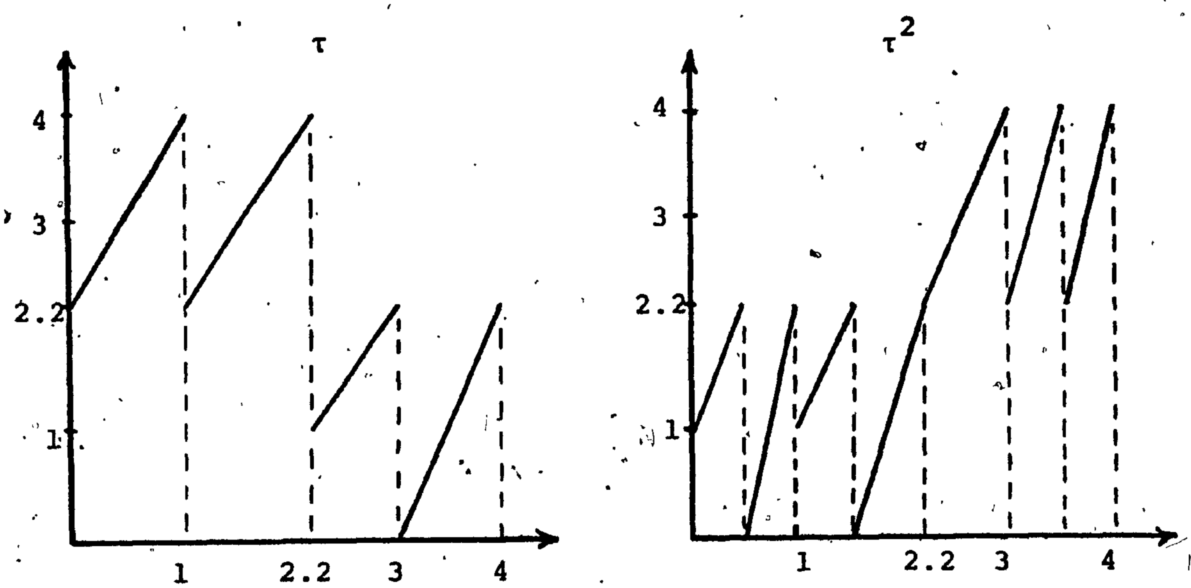

Figure 5.1

Let τ have the above graph. Clearly this is a Markov map, and it is easy to see that starting with the interval (0,1) we can go to any other interval and come back to (0,1). Thus the partition {0,1, 2.2, 3, 4} has the communication property under τ. However τ^2 does not share this property with respect to its partition: this is immediate since the intervals (0, 2.2) and (2.2, 4) are both invariant under τ^2. Actually, using the method developed in Chapter VI, it can be seen that τ^2 has two independent invariant functions. Thus, in general, the conclusion of Theorem 5.3 is not valid for the iterates of a $\tau \in C$. However, if we replace condition (3) of this class by the stronger condition:

(3′) For every $I_i \in P$ there exists an integer n_i such that $\tau^{n_i}(I_i) = J$

then every iterate of τ will have a unique invariant function. This is proved in the next theorem.

<u>Theorem 5.4</u> (New result)

Let τ be piecewise C^2 with respect to partition P_1. If

(1) $\inf|\tau'| > 0$ and $\inf|\frac{d\tau^{\ell}}{dx}| > 1$ for some integer ℓ,

(2) τ is Markov with respect to P_1,

(3) For every $I_i \in P_1$ there exists n_i such that

$$\tau^{n_i}(I_i) = J,$$

then τ^n has a unique invariant function for each $n \geq 1$.

Proof:

We claim first that for any $n \geqslant 1$, τ^n will satisfy conditions (2) and (3′) with respect to its partition P_n. If Q_n are the partition points of P_n, then

$$Q_n = Q_1 \cup \tau^{-1}(Q_1) \cup \tau^{-2}(Q_1) \cup \ldots \cup \tau^{-n+1}(Q_1).$$

This implies

$$\tau(Q_n) \subset Q_1 \cup \tau^{-1}(Q_1) \cup \ldots \cup \tau^{-n+2}(Q_1)$$
$$= Q_{n-1}.$$

Therefore

$$\tau^n(Q_n) \subset \tau^{n-1}(Q_{n-1})$$

for every n, and so

$$\tau^n(Q_n) \subset \tau^{n-1}(Q_{n-1}) \subset \ldots \subset \tau(Q_1) \subset Q_1 \subset Q_2 \subset \ldots \subset Q_n.$$

This proves that τ^n is a Markov map. Notice that $\tau^n(Q_n) \subset Q_1$ means that τ^n maps each interval of P_n onto contiguous intervals of P_1. Let $\rho = \tau^n$. If I is any interval of P_n, $\rho(I)$ contains some interval $I' \in P_1$. By condition (3′) there exists an integer m such that $\tau^m(I') = J$. Therefore

$$\rho^{m+1}(I) = \rho^m(\rho(I)) \supset \rho^m(I') = J$$

and the claim is proved.

In particular, for each $n \geq 1$, the map $\tau^{n\ell}$ is in class C and, by Theorem 5.3, it has a unique invariant function. If for some k, τ^k has more than one invariant function, so does $\tau^{\ell k}$ and this leads to a contradiction. Therefore τ^n has a unique invariant function for each $n \geq 1$.

Q.E.D.

CHAPTER VI

PIECEWISE LINEAR MARKOV MAPS

Even if it is known that a transformation has a unique invariant function, finding it may be a formidable task. However, there is a class of transformations for which it is relatively easy to exhibit invariant functions simply by solving a system of linear equations. This is the class of piecewise linear Markov maps.

Let $I = [a,b]$ and $\tau : I \to I$ be a (non-singular) piecewise linear Markov map with respect to the partition

$$P = \{a = a_0 < a_1 < \ldots < a_N = b\}.$$

For each i, let $I_i = (a_{i-1}, a_i)$ and denote by τ_i the restriction of τ to the interval I_i. Then τ_i is a homeomorphism from I_i onto some interval $(a_{j(i)}, a_{k(i)})$, having τ_i^{-1} as inverse. Let S be the class of all functions which are piecewise constant on the above partition, that is,

$$f \in S \iff f = \sum_{i=1}^{N} c_i \chi_{I_i}$$

for some constants $c_1, \ldots, c_N$. Such an f will also be represented by the column vector $(c_1, \ldots, c_N)^t$ where t denotes transpose.

Proposition 6.1

With the foregoing assumptions, there exists an $N \times N$ matrix M_τ such that $P_\tau f = M_\tau f$ for every $f \in S$.

Proof:

A simple computation shows that the Frobenious-Perron operator for τ is given by [1]:

$$P_\tau f(x) = \sum_{i=1}^{N} f(\tau_i^{-1}(x)) \left| \frac{d\tau_i^{-1}(x)}{dx} \right| \chi_{\tau_i(I_i)}(x)$$

$$= \sum_{i=1}^{N} f(\tau_i^{-1}(x)) \left| \tau_i' \right|^{-1} \chi_{\tau_i(I_i)}(x) \, .$$

Suppose first that $f = \chi_{I_k}$ for some $1 \le k \le N$. Then

$$P_\tau f(x) = \sum_{i=1}^{N} \chi_{I_k}(\tau_i^{-1}(x)) \left| \tau_i' \right|^{-1} \chi_{\tau_i(I_i)}(x) \, ,$$

and since τ_i^{-1} has range I_i, $\chi_{I_k}(\tau_i^{-1}(x)$ will be zero for all $i \neq k$. Thus

$$P_\tau f(x) = \left| \tau_k' \right|^{-1} \chi_{\tau_k(I_k)}(x) \, .$$

Now let $f \in S$, i.e. $f = \sum_{k=1}^{N} c_k \chi_{I_k} = (c_1, \ldots, c_N)^t$. Since P_τ is a linear operator, we have

$$P_\tau f = \sum_{k=1}^{N} c_k P_\tau(\chi_{I_k})$$

$$= \sum_{k=1}^{N} c_k \left|\tau_k'\right|^{-1} \chi_{\tau_k(I_k)} \qquad (6.1)$$

This proves that $P_\tau f \in S$. Let us write $P_\tau f = (d_1,\ldots d_N)^t$. When $x \in I_j$, $P_\tau f(x) = d_j$. Now, the kth term in the right hand side of (6.1) equals $c_k|\tau_k'|^{-1}$ iff $x \in \tau_k(I_k)$, that is $I_j \subset \tau_k(I_k)$. Let $\Delta_{jk} = 1$ if $I_j \subset \tau_k(I_k)$ and zero otherwise, and define the matrix $M_\tau = (m_{jk}) = \Delta_{jk}|\tau_k'|^{-1}$. Then

$$d_j = \sum_{k=1}^{N} c_k m_{jk}$$

and

$$M_\tau \begin{pmatrix} c_1 \\ \vdots \\ c_N \end{pmatrix} = \begin{pmatrix} d_1 \\ \vdots \\ d_N \end{pmatrix} = P_\tau f .$$

Q.E.D.

The matrix M_τ defined in the above proposition is called the <u>matrix induced by τ</u>. This matrix is non-negative and, for each $j \in \{1,2,\ldots,N\}$, the non-zero entries in the jth column are contiguous and equal to $|\tau_j'|^{-1}$. Notice that τ is not the only map which induces M_τ. For, on any segment I_i, the function τ_i can be replaced by a linear function with the same domain and range, with slope equal to $-\tau_i'$, and the matrix induced by this new map will also be M_τ . Thus, there exists 2^N piecewise linear Markov maps which induce the same matrix.

<u>Proposition 6.2</u>

The matrix $M = M_\tau$ has 1 as the eigenvalue of maximum modulus.

<u>Proof:</u>

We recall that the eigenvalues of a matrix are invariant under similarity transformations and under transposition. Let us define

$$\delta = \prod_{j=1}^{N} (a_j - a_{j-1})$$

and

$$\delta_i = \frac{\delta}{a_i - a_{i-1}} = \prod_{\substack{j=1 \\ j \neq i}}^{N} (a_j - a_{j-1}) .$$

Define the diagonal matrix D to have entries $d_{ii} = \delta_i$, $i = 1,2,\ldots,N$. Then $E = D^{-1}$ is a diagonal matrix with entries $e_{ii} = \delta_i^{-1}$ for each i.

Suppose τ maps I_i onto $I_j \cup I_{j+1} \cup \ldots \cup I_{j+k}$. Then $|\tau_i'| = (a_{j+k} - a_{j-1})/(a_i - a_{i-1})$. It follows that the ith column of M has entries $(a_i - a_{i-1})/(a_{j+k} - a_{j-1})$ in rows j to $j+k$, and zero in all the remaining rows. Let $B = D^{-1}MD$. Then $b_{rs} = \delta_r^{-1} m_{rs} \delta_s$. We claim that B is column stochastic. Consider the column sum of the ith column for B:

$$\sum_{r=1}^{N} b_{ri} = \sum_{r=1}^{N} \delta_r^{-1} m_{ri} \delta_i$$

$$= \sum_{r=j}^{j+k} \delta_r^{-1} \frac{a_i - a_{i-1}}{a_{j+k} - a_{j-1}} \delta_i$$

$$= \frac{\delta}{a_{j+k} - a_{j-1}} \left[\frac{1}{\delta_j} + \frac{1}{\delta_{j+1}} + \ldots + \frac{1}{\delta_{j+k}} \right]$$

$$= \frac{\delta}{a_{j+k} - a_{j-1}} \left[\frac{a_j - a_{j-1}}{\delta} + \ldots + \frac{a_{j+k} - a_{j+k-1}}{\delta} \right]$$

$$= 1 .$$

Thus B^t is row stochastic. Invoking Theorem 9.5.1 in [8], the matrix B^t has one as the eigenvalue of maximum modulus. The conclusion of the proposition now follows.

Q.E.D.

It follows from Proposition 6.2 that the system of linear equations $M_\tau \pi = \pi$ always has a non-trivial solution, and this is equivalent to the statement that there always exists a step function invariant under τ. Notice that we have tacitly proved the existence of invariant functions for any piecewise linear Markov map τ with $\inf|\tau'| > 0$. If τ is known to have a unique invariant function, this function has to be piecewise constant on the same partition for τ. Also, the dimension of the (right) eigenspace of the eigenvalue 1 of the matrix M_τ constitutes a lower bound for the number of functions invariant under τ, i.e.

the fixed points of M_τ are fixed points of P_τ. In the special case where $|\tau'| > 1$, the next proposition ensure us that all invariant functions are in S and hence the space of invariant functions is precisely the eigenspace of eigenvalue 1 of the matrix M_τ.

Proposition 6.3 (New result)

If $\inf|\tau'| = \alpha > 1$ then every invariant function is piecewise constant on the partition defined by τ.

Proof:

Let f be invariant under τ. By Theorem 2.1 we know that f is of bounded variation on $[a,b]$. Moreover, we have

$$P_\tau f(x) = \sum_{i=1}^{N} f(\tau_i^{-1}(x)) \frac{1}{|\tau_i'|} \chi_{\tau_i(I_i)}(x) = f(x) .$$

Notice that f has to be identically zero outside the range of τ. Let $I_k \subset \tau(I)$ be any interval of the partition and let $x, y \in I_k$ be distinct and fixed. Then $\chi_{\tau_i(I_i)}(x) = \chi_{\tau_i(I_i)}(y)$ for all i. Thus

$$\begin{aligned} f(x) - f(y) &= P_\tau f(x) - P_\tau f(y) \\ &= \sum_{i=1}^{N} \frac{1}{|\tau_i'|}\left[f(\tau_i^{-1}(x)) - f(\tau_i^{-1}(y))\right]\chi_{\tau_i(I_i)}(x) \\ &= \sum_{i_1} \frac{1}{|\tau_{i_1}'|}\left[f(\tau_{i_1}^{-1}(x)) - f(\tau_{i_1}^{-1}(y))\right] \end{aligned}$$

where, to avoid heavy notations, the index i_1 vary over some appropriate non-empty subset of $\{1,2,\ldots,N\}$.

Similarly, for each i_1,

$$f(\tau_{i_1}^{-1}(x)) - f(\tau_{i_1}^{-1}(y)) = \sum_{i_2} \frac{1}{|\tau'_{i_2}|}\left[f(\tau_{i_2}^{-1}\tau_{i_1}^{-1}(x)) - f(\tau_{i_2}^{-1}\tau_{i_1}^{-1}(y))\right]$$

and so on. Therefore

$$\begin{aligned} |f(x)-f(y)| &\le \frac{1}{\alpha}\sum_{i_1}\left|f(\tau_{i_1}^{-1}(x)) - f(\tau_{i_1}^{-1}(y))\right| \\ &\le \frac{1}{\alpha^2}\sum_{i_1}\sum_{i_2}\left|f(\tau_{i_2}^{-1}\tau_{i_1}^{-1}(x)) - f(\tau_{i_2}^{-1}\tau_{i_1}^{-1}(y))\right| \\ &\ \vdots \\ &\le \frac{1}{\alpha^n}\sum_{i_1}\cdots\sum_{i_n}\left|f(\tau_{i_n}^{-1}\cdots\tau_{i_1}^{-1}(x)) - f(\tau_{i_n}^{-1}\cdots\tau_{i_1}^{-1}(y))\right| . \end{aligned} \tag{1}$$

Now it is easy to see that

$$\{ (\tau_{i_n}^{-1}\cdots\tau_{i_2}^{-1}\tau_{i_1}^{-1}(x) \ , \ \tau_{i_n}^{-1}\cdots\tau_{i_2}^{-1}\tau_{i_1}^{-1}(y)) \}_{i_i,i_2,\ldots,i_n}$$

is a finite collection of at most N^n non-overlapping intervals. Consequently, the summation in (1) is bounded above by the total variation of f and hence

$$|f(x) - f(y)| \le \frac{1}{\alpha^n} \overset{b}{\underset{a}{V}} f < \varepsilon$$

for large n. Therefore $f(x) = f(y)$ and f is constant on I_k.

Q.E.D.

It is worth noting that the slope condition in Proposition 6.3 is essential. For if $\inf|\tau'| \le 1$, there may exist invariant functions in $BV[a,b]$ which are not piecewise constant on the partition of τ. Consider for instance the map $\tau : [0,1] \to [0,1]$ defined by

$$\tau(x) = \begin{cases} 2x , & 0 \le x \le \frac{1}{2} \\ -x + \frac{3}{2} , & \frac{1}{2} < x \le 1 \end{cases}$$

Then the corresponding Frobenious-Perron operator is given by

$$P_\tau f(x) = \begin{cases} \frac{1}{2} f(\frac{x}{2}) , & 0 \le x < \frac{1}{2} \\ \frac{1}{2} f(\frac{x}{2}) + f(\frac{3}{2} - x) , & \frac{1}{2} \le x \le 1 \end{cases}$$

Let f be any function of bounded variation which is zero on $(0,\frac{1}{2})$ and symmetric with respect to the line $x = \frac{3}{4}$ on the interval $(\frac{1}{2},1)$. Clearly, this function will satisfy $P_\tau f = f$ and hence will be invariant under τ. Thus, invariant functions need not be piecewise constant.

We summarize all these results in a theorem.

<u>Theorem 6.1</u>

If τ is a non-singular piecewise linear Markov map with respect to a partition P, then there exists a piecewise constant function on P which is invariant under τ. If, in addition, $\inf|\tau'| > 1$, then every invariant function is piecewise constant on P and the space of invariant functions is precisely the eigenspace of eigenvalue one of the matrix M_τ.

Example 1

Let $\tau : [0,1] \to [0,1]$ be defined by

$$\tau(x) = \begin{cases} 2x + \frac{1}{2}, & x \in I_1 = [0, \frac{1}{4}] \\ -x + \frac{5}{4}, & x \in I_2 = [\frac{1}{4}, \frac{1}{2}] \\ -2x + \frac{7}{4}, & x \in I_3 = [\frac{1}{2}, \frac{3}{4}] \\ -x + 1, & x \in I_4 = [\frac{3}{4}, 1] \end{cases}$$

We see that τ is Markov with respect to $\{I_1, I_2, I_3, I_4\}$ and $\tau^4(I_1) = \tau^6(I_2) = \tau^3(I_3) = \tau^5(I_4) = [0,1]$. The line segments in the graph of τ have slopes -1, ± 2; however, the third iterate of τ has slopes >1 in absolute value for all segments. Thus, by Theorem 5.4, τ and all its iterates have a unique invariant function.

Now the matrix induced by τ is given by

$$M_\tau = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 & 0 \end{pmatrix}$$

and the vector $\pi = (2,1,2,2)^t$ is an eigenvector of eigenvalue one. Thus the unique invariant density for τ (and all τ^n) is

$$f(x) = \begin{cases} 2 & \text{on } (0,\frac{1}{4}) \cup (\frac{1}{2},1) \\ 1 & \text{on } (\frac{1}{4},\frac{1}{2}) \ . \end{cases}$$

Example 2

Let $h : [0,1] \to [0,1]$ be the homeomorphism defined by $h(x) = \sqrt{x}$. For τ as in Example 1, let $\tau_1 = h^{-1} \circ \tau \circ h$. Then τ and τ_1 are topologically conjugate transformations and

$$\tau_1(x) = \begin{cases} (2\sqrt{x} + \frac{1}{2})^2 , & x \in [0,\frac{1}{16}] \\ (-\sqrt{x} + \frac{5}{4})^2 , & x \in [\frac{1}{16},\frac{1}{4}] \\ (-2\sqrt{x} + \frac{7}{4})^2 , & x \in [\frac{1}{4},\frac{9}{16}] \\ (-\sqrt{x} + 1)^2 , & x \in [\frac{9}{16},1] \end{cases}$$

By the results of section 4.3, τ_1 has a unique invariant function f_1 given by $f_1 = (f \circ h)h'$ where f is the unique invariant density for τ. Explicitly, we have

$$f_1(x) = \begin{cases} \frac{1}{\sqrt{x}} , & x \in (0,\frac{1}{16}) \cup (\frac{1}{4},1) \\ \frac{1}{2\sqrt{x}} , & x \in (\frac{1}{16},\frac{1}{4}) \ . \end{cases}$$

To close this chapter we will discuss briefly an application to functional equations. Suppose we are given a functional equation on some interval, to be solved in L^1

and, somewhow, we are able to recognize a map τ such that the original equation reduces to $P_\tau f = f$, where P_τ is the Frobenious-Perron operator corresponding to the transformation, then, using the results of Chapter IV, we can get an upper bound for the number of independent solutions. If it happens that τ is piecewise linear and Markov with slope greater than one, then we know that all solutions of $P_\tau f = f$ are piecewise constant on some fixed partition and they can all be obtained by solving a system of linear equations, namely $M_\tau \pi = \pi$. We illustrate this method by some examples.

Example 3

Let a be in $[0,1/2)$ and consider the functional equation

$$f(x) = \begin{cases} \frac{1}{2} f(\frac{x}{2}) & 0 \le x < a \\ \frac{1}{2} f(\frac{x}{2}) + cf(\frac{1}{2} + c(1-x)) & a \le x \le 1 \end{cases} \qquad (6.1)$$

where $c = 1/2(1-a)$. If we define $\tau : [0,1] \to [0,1]$ by

$$\tau(x) = \begin{cases} 2x & 0 \le x \le \frac{1}{2} \\ (2-a)-2(1-a)x & \frac{1}{2} < x \le 1 \end{cases},$$

then a simple computation shows that the Frobenious-Perron operator of τ is given by the right hand side of (6.1). Thus the original problem reduces to finding fixed points for P_τ. Invoking Theorems 2.1 and 4.1, we know that a solution exists and is unique. Therefore, the functional equation (6.1) must have a unique solution in L^1.

In the special case where $a = 1/2^n$ for some integer $n \geq 2$, we see that τ is a piecewise linear Markov map with respect to the partition

$$P = \{0 < \frac{1}{2^n} < \frac{1}{2^{n-1}} < \ldots < \frac{1}{2} < 1\}.$$

Using the results of this chapter, the unique solution is piecewise constant on P and the solution of the equation $M_\tau \pi = \pi$, where M_τ is the matrix induced by τ. Simple computations will show that the unique solution (up to constant multiples) is given by

$$f(x) = \sum_{k=2}^{n+1} \left(2^{n+1} - 2^{n-k+2}\right) \chi_{I_k}(x)$$

where $I_k = \left(1/2^{n-k+2}, 1/2^{n-k+1}\right)$ for $2 \leq k \leq n+1$.

Example 4

On the interval $[0,1]$ consider the functional equation

$$f(x) = \frac{1}{n}\left[f(\frac{x}{n}) + f(\frac{x+1}{n}) + \ldots + f(\frac{x+n-1}{n})\right]$$

where $n \geq 2$ is a fixed integer. For $1 \leq k \leq n$, let $I_k = ((k-1)/n, k/n)$ and define $\tau_k : I_k \to [0,1]$ by

$\tau_k(x) = nx + 1 - k$. Finally let $\tau : [0,1] \to [0,1]$ be such that $\tau|_{I_k} = \tau_k$. Clearly τ is a piecewise linear Markov map and also a Renyi transformation. Hence there exists a unique function invariant under τ . Now for every $f \in L^1$ we have

$$P_\tau f(x) = \frac{1}{n} \sum_{k=0}^{n-1} f\left(\frac{x+k}{n}\right)$$

and hence f is invariant under τ iff it is a solution of the functional equation. Also the matrix induced by τ has all entries equal to $1/n$ and the vector $(1,1,\ldots,1)^t$ is the unique fixed point of this matrix. Consequently every solution to the original equation has to be a constant function on $[0,1]$.

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