A Study of the Problem solving behavior of Twenty-four Students

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ABSTRACT

A STUDY OF THE PROBLEM SOLVING BEHAVIOR OF TWENTY-FOUR STUDENTS

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Chapter I describes the research project into problem solving on which this paper is based: -both the wider Concordia project which began in 1978 and the author's own systematic investigation of the problem solving behavior of twenty-four selected students interviewed in the Spring of 1980. Chapters II to IV describe and analyse student behavior for each of the three problems posed. This is followed by a reflection on individual student problem solving behavior across the three problems (i.e. behavior constants). In the concluding chapter an attempt is made to confront problem solving theory as formulated by authors such as Polya, Bell, Polanyi and Wertheimer, Newell and Simon with what actually takes place in the seventy-two protocols examined here. This is followed by a discussion of some hypotheses on problem solving behavior that have grown out of this study as well as some directions for future research.

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Chapter I: The research project

1:1 Introduction

In September 1978 a group of professors and graduate students in the M.T.M. program at Concordia University began work on a research project entitled: "Aspects de la résolution de problèmes". During the first year of research the main preoccupation was to develop a research technique that maximized flexibility and uniformity. The aim was to study mathematical problem solving and the method chosen was the clinical interview. By the end of the first year of the research project there was a fairly strong consensus on the technical recording method, the format for writing protocols, and the low-intervention interview style. During the second year of research, although there was some experimentation with variations in interview situations, more accent was put into the analysis of the protocols themselves. It was toward the end of the second research year that this study was undertaken.

The project had reached a point where the need to systematically study the behavior of a circumscribed group on a limited number of problems was beginning to be felt. The major obstacle to any large scale research in this area is the time involved in obtaining, transcribing, and analysing a single protocol. Access to groups of students is not easy and involves fitting in to a school schedule where two or three interviews are often the most that can be hoped for in a day. Interview sessions last about an hour each. It is, however, the time required for transcription of the recorded interviews that is the biggest obstacle to research. Transcribing a forty-five minute tape can take up to seven or eight hours. The subsequent review and analysis can take equally long. And so some sort of compromise has to be made in terms of numbers.

This study grew out of the Concordia research project and the hope is to contribute to it through a systematic study of protocols of twenty-four students solving three problems each under as similar conditions as possible.

1.2 Contents of this paper

The clinical interview method, the protocols, the problems used, the interviews, and the interviewees, will be described and discussed in this chapter.

In chapters II to IV student behavior on the three problems will be studied while chapter V looks at individual student behavior across the three problems. In chapter VI an attempt is made to confront the problem solving behavior found in these protocols with the major theories in this area. Some hypotheses about problem solving behavior are proposed and a résumé of the major conclusions of this study is attempted. The paper closes with a few suggestions for further research.

1.3 The problems

The three problems chosen for this study had been used frequently in the Concordia research with very interesting results. The first is called the "parking lot" problem. The pros and cons of this problem are discussed in Chapter II. It is a problem that can be solved algebraically or by a trial and adjustment technique. The second, called the "square cutting" problem, is discussed in Chapter III. Its demands are more geometric and visual. The third problem, discussed in Chapter IV, is a simple crypto-arithmetic problem where the demands are arithmetical. The three problems together are short enough to be given in a 3/4 hour interview. They are not age specific and in fact are both accessible and challenging to anyone who has mastered the four basic arithmetic operations.

English versions of the problems:

1. There are 40 vehicles in a parking lot, cars and motorcycles. All together they have 100 wheels. How many of each kind of vehicle are there in the lot?

2. You are given a square and you want to cut in into a number of pieces so that each piece is also a square. Can you cut it into 9 squares? 7 squares?

3. In the following problem A, B, and C are different digits. Find A, B, and C.

$$\begin{array}{r} AB \\ \underline{\times\ 4} \\ CA \end{array}$$

1.4 The interviews

The interview method adopted for this study was developped during the problem solving project at Concordia. Essentially it is a clinical interview where there is one interviewer and one interviewee. Characteristics are:

- A small Sony tape recorder and microphone are placed on the table at which the interviewer and student are seated side by side. The quality of the taped interview was sufficiently good and it was felt that a heavier equipment might be more intimidating to the student. The aim was to put the student in a setting where he was as at ease as possible and would hopefully forget the recording machinery.

- Students are asked to think aloud.

- Students are asked to write on plain paper with a black pen. The work sheets provide the visual recording of the student's work during the interview. The black pen was proposed first of all because it can be photocopied more easily than another color but mainly because the student could not subsequently erase his written efforts (which students are fond of doing).

- The interviewer speaks as little as possible and comes in after long silences or to try to put the student at ease or to get him to verbalize. The main role of the interviewer is to direct the student's attention to the problem and the need to verbalize.

- A short conversation generally takes place before the taping begins. Its aim is to put the student at ease and explain what is about to happen. If time permits, the session is concluded with a brief discussion of the interview experience. Students are asked how they felt, whether they enjoyed the session and so on. Although these are often taped they are not transcribed.

- Sessions last about 3/4 hr. This seems to be an optimal period for a clinical interview: —long enough for the student to relax and concentrate and not long enough to tire him.

- No school records, family history, psychological data, etc are sought on the interviewee. It was felt that these might distort the interviewer's behavior during the interview and the subsequent analysis of the protocols.

The general idea is to create as simple an interview situation as possible where the student has little more stress than that produced by the problem itself and works at the problem in the closest possible way to how he would do it if left alone.

Some interview ideas which were explored by the Concordia research group and rejected for various reasons should be mentionned here because they help to understand why this particular format was chosen.

They are:

- leaving the student alone with the problems and recorder. This was rejected because it would require the training of interviewees in verbalizing. Many students would never get started: —for example, those who begin the session by saying "It can't be done". Others would quit before completely understanding the problem. The advantage of course, would be that there would be no interviewer distortion of the session.

. video taping the interviews. This technique of interview recording was used by the Montpellier research group. It requires a studio setting with overhead and floor cameras. It was rejected because it is neither portable nor discreet. It was also felt that the masses of visual as well as audio information would be difficult if not impossible to study.

. group solving sessions. Individuals solve problems in groups very differently than they do when alone or with an interviewer. The information coming out of group protocols is richer in information about group dynamics than in providing answers to the question as to what happens when an individual attempts to solve a problem. All sorts of social inhibitors can prevent individuals from verbalizing or contributing to various stages of the problem's solution.

For researchers in this area an "interesting" interview is one in which the student verbalizes well, works through the problem with high motivation and perserverence, and provides the interviewer with a rich portrait of behavior. A decision had to be made early in this project as to how to collect protocols for study: —to do a large number of interviews and select the best or to do the desired number of interviews and retain them all. Since access to students was difficult the second option, that of retaining all interviews for the study, was taken. The interview style had to be adapted to this decision.

In the Concordia project (where all interviewees were volunteers), the interviewer was to intervene as little as possible. When every interview must be retained for analysis it is not always possible to follow the low profile style. With highly motivated verbal students it was possible. With others it became, at the worst, a situation of trying to wring something mathematical out of a student in the available interview time. And so subquestions, reformulations of questions, coaxing, hinting, joking ... anything that would keep things going was used.

It was considered better to have the student take a few assisted steps and a few on his own than to get absolutely nothing out of the interview.

Making every interview count is much more demanding on the interviewer. To judge when to intervene, to avoid over intervention and yet to keep the interview going is difficult. It is particularly hard when the student is a perfect stranger. Some rapport has to be established fairly quickly. The student has to be put in a relaxed and cooperative mood and from there on every gesture must be studied to know when and how to intervene. Silences must be interpreted: —Is the student thinking? Is it about the problem? Has he quit somewhere along the way? Throughout the interview decisions have to be made: —Is this a student I should sit close to? Or should I keep a certain distance? Is there something about me or the situation distracting him? And, in some cases, —Does he know I'm here?

In these protocols it sometimes appears that the interviewer intervened more often than was needed. This is particularly noticeable in the second and third problem protocols in those cases where the student had considerable difficulty with the first. The interventions were not always helpful, either, and in some cases may have hindered progress. For example in a parking lot problem protocol the interviewer tries to explain to the student, Chantal, the meaning of the word "véhicules" and to get her to see that it means both cars and motorcycles. There are a total of forty vehicles in the parking lot and yet Chantal maintains that there are forty cars. The interviewer asks "If there are 40 cars in the parking lot that means there are how many motorcycles?" The interviewer has to reformulate and repeat the question many times and by the time Chantal finally produces the desired response, "zero motorcycles", she seems totally muddled. The interviewer's insistance on the question leads Chantal to conclude that this is the final answer. It is likely that Chantal, who was aware of the wheel constraint in the problem, could have worked her way through the problem without this long harangue about forty cars and how many motorcycles.

If research in this area is to continue some work will need to be done on the problem of interviewer behavior. A bank of useful questions for the interviewer should be developped and interview techniques strengthened. If there is to be any uniformity in the research done by different interviewers, some rules of conduct which will serve as guidelines for interviewing the unmotivated and the shy as well as the very verbal student, will have to be agreed upon.

This problem was not too acute in the Concordia project. There, the students who were interviewed were generally volunteers and highly motivated. The unsuccessful or uninteresting interviews were discarded. If a student was uncommunicative or unproductive no effort was made to squeeze something out of the situation. However, if we wish to study problem solving behavior for a cross-section of students such as are found in the normal classroom situation, then the problem of the nature and extent of the interviewer's input will have to be studied.

1.5 The interviewees

Twenty-four students were interviewed for the purpose of this study. Twelve were cegep students aged eighteen and over, and twelve were grade six students about twelve years of age. Equal numbers of boys and girls were studied in each age group.

1.5.1 Cegep students:

All twelve cegep students studied were enrolled in the "Cours d'appoint" in the winter term of 1980 at Cegep Saint-Jean-sur-Richelieu. With one exception all were in the same group. One girl, Martine, was selected from a second group because of the insufficient number of girls in the chosen group.

For the selected group the clinical interview was an integral part of the course and students were required to choose one of the taped interview questions, transcribe, and comment on it. Where available, student transcriptions are included in dossiers. These are generally inaccurate and incomplete but they do give a good idea of the writing abilities of students. Classroom discussion of some of the protocols followed after work had been graded. Students seemed to enjoy the whole exercise and some had some interesting insights into their mathematical behavior.

The cegep "Cours d'appoint" is essentially a review of high school algebra. Interviews began at the end of the second week of classes. The classroom approach taken was one of problem solving. Students were presented with a series of problems which could be solved algebraicly using a linear equation with one unknown, two unknowns, ... Technical manipulations were considered separately in short drill sessions where the accent was on acquiring speed as well as accuracy. During the first week of class no algebra was used in solving problems, which were chosen mainly for their motivational qualities (i.e. fun or funny). In the second week simple algebraic solutions in one unknown were imposed on a number of problems which had been previously solved without algebra. Several times during the interviews there are references made to the reasoning or "logique" approach of the first week and the algebraic approach that had just begun. It should be noted that the winter 1980 term began at the beginning of March due to strike action in the Cegeps that year.

Students in the algebra course (Cours d'appoint) tend for the most part to be weak in mathematics. They do not have the required math. to enter cegep level courses. Many have failed to attain 60% or more in high school math. Some have repeated math. failures and some are taking the algebra course for the third time. The failure rate tends to be around 50%. It is a course no one wants to teach because of the lack of motivation of students.

An entire class of students was interviewed and tapes are available for all twenty-five students. All five female interviews were retained for this study and the six males were chosen on the basis of the most complete dossiers. Some students, for example, lost work sheets or didn't write protocols or did not do the three problems for various reasons. Some interesting material is to be found in their dossiers as for example an algebraic solution attempt at the car-motorcycle problem by Pierre which is a masterpiece in confused algebra.

Interviews were done in a sound proofed fully equiped taping laboratory in the Cegep.

1.5.2 Grade six students:

The second group is composed of twelve grade six students from Bélanger School in the parish of Notre-Dame-Auxiliatrice in Saint-Jean. The school is in a working class area and is known as a medium standard school. Many handicapped and immigrant children attend. The school's principal, vice principal, and grade six teachers were very open and helpful about the project. Medium to slow in math. children were requested on the hypothesis that these produce richer and more interesting protocols. The children were aware that they had problems in mathematics and seemed to think the interviewer was there to help them. They came willingly and seemed fairly relaxed and chatty considering the unusualness of the taping experience. It was after reading the first problem involving cars and motorcycles that the children seemed to jam up and be unable to do or say anything. At that point the decision was made to retell the problem story situating it in the school yard outside the window. The aim was not so much to clarify the problem as to relax the student and allow him time to pull himself together and to get used to the interviewer, the equipment, the entire situation.

As for the students' math background it seemed to be standard for the end of grade six. The students were about to write provincial exams for entrance into high school. Some of the students had spent the previous year with a male teacher who is very interested in problem solving and a great fan of Polya. This accounts for the occasional reference to problem solving techniques and the way a few students wrote down information.

Interviews were held in a sort of storage area which was bright and looked as though it would be one of the quieter spots in a somewhat noisy school. However it turned out that there was not only a fair bit of traffic through the room but also on the street outside. For one session a power mower moved back and forth under the window. In spite of all this the quality of the tapes - this time recorded on a small portable Sony - is relatively good.

1.5.3 Some questions:

Why these particular age or grade groups?

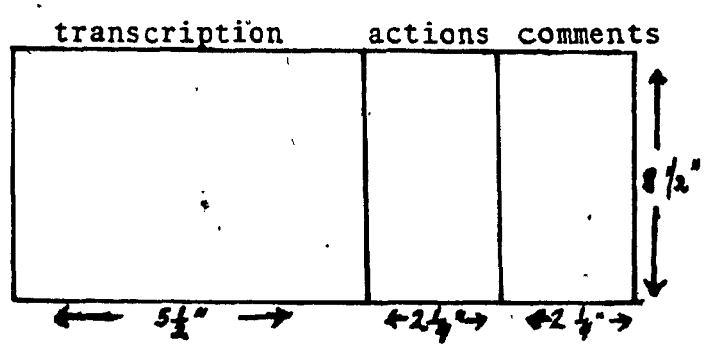
The main reason for choosing these particular groups was their availability: —the cegep students being in the interviewer's class and the grade six students being accessible because of several contacts made over the years with personnel in Bélanger school. It was hoped that students at the end of primary school and end of high school levels in math. might provide interesting material for comparison. The problems seemed to be adequately interesting and challenging for both groups.

Why an equal number of males and females in both groups?

Naturally no definite conclusions can be reached on sex differences in problem solving because of the very small sample size. However the intention was to examine the protocols to see if there were any indications of qualitative differences on the basis of sex.

1.6 The protocols

Protocols are written on 8½" x 11" paper. divided into three columns:



A. Notes on the transcriptions:

Every effort was made to have word accuracy however 100% accuracy is impossible. Some sentences or words are incomprehensible even after playing and replaying tapes at every speed and volume possible. Students had a tendency to whisper or mumble particularly when they were unsure of themselves. Many non-words were used. For example a grunt could often be understood as "pis" or "ben" or "mais". During periods of accelerated thinking students tended to telescope words so that you get the beginning of one word with the ending of another or simply a piece missing out of the middle of a word. There was a very heavy use of joual which is hard to transcribe except for some more common words such as "ouai" for "oui", "tsi" for "tu sais", and "icitte" for "ici". In some spots external noises particularly knocking of the microphone block out the words on the tape. In cases where it is impossible to know what was said a space with a question mark appears in the transcription.

The timing of silences is very difficult on a tape because the time that the tape is silent does not necessarily correspond exactly to clock time and the tape progress on the counter is even more inaccurate since it counts rotations and varies from recorder to recorder. In a few protocols tape countings were used to measure silences. For example – (213 - 227) – means there was silence from 213 to 227 on the counter of the transcribing machine used. However the length of the majority of silences were measured on the clock. A long dash (—) represents five seconds of silence. If periods of silence were longer than about 20 seconds the silence time was recorded between two dashes: – i.e. —(45 sec.)— for 45 seconds of silence.

Often while one person was talking the other made short comments that do not really interrupt the flow of the speaker. In these cases the interjection is included with brackets around. For example:

P: J'ai divisé par 9. (Um - huh) Pis j'ai ...

Here L. said Um huh while "P:" was talking. Note that L. represents the interviewer and the interviewee is represented by the initial of his christian name.

At the end of some transcriptions appears a short résumé of a conversation not involved in the problem solution and not worth transcribing in whole. For example, student responses to questions about how they liked doing the problems would be found in résumé in the transcription column.

B. Notes on the actions column:

During the interview students were asked to do any paper work on a sheet using a black pen or pencil. These work sheets were retained and are included in the dossiers. While students were working the interviewer tried to keep track of when various calculations were put on paper as well as significant gestures made by students. These gestures and written work appear in the actions column to the right of what was being said at the time. Occasionally in the square drawing problem it was possible to note the order in which each line of a square division was drawn. In these cases tiny encircled numbers indicate that order.

C. Notes on the comments column:

This column contains notes made on the student's behavior during the interview, during the transcription, and later during a study of the protocol. The main lines of the problem solution attempt are indicated in red ink and numbered. It is possible to quickly review the student's solution path by reading through those steps written in red. In black ink there are comments on student's language, personality, gestures, and any number of factors found interesting in the protocol.

On protocols:

There is a basic assumption on the part of researchers studying problem solving protocols that what a student says he is thinking (or doing) is what he believes he is thinking. In other words, students are essentially honest in the clinical interview situation.

There a number of factors which contribute to the reliability of the protocols. First of all, the interview situation is such that the student would have very little motivation for deceit. A second factor that contributes to honesty is the pace of the interviews. It would be very difficult to fabricate a fictional story of what is going on - complete with written indications - and at the same time solve the problem.

A short example taken at random will serve to illustrate.[1]

> D: Ça veut dire qu'ils sont pas très éloignés l'un de l'autre. Ça veut dire que 60 ... 50 ... 2 ... ça arrivera pas encore. 44? pis 56 ,.. 14. Ce qui donne 36. J'ai un de plus ... pas ben, ben. 48 divisé par 2, 24. 100 moins 48 — 52, divisé par 4 — 13. Faire 37, j'approche.

At this clip, (there are no pauses), it would be impossible for anyone to be doing something else entirely. I think it can be clearly seen that D. is sincerely trying to verbalize what he is up to.

[1] Note that extracts from protocols are taken verbatim. No attempt is made to correct the students' French.

On the other hand, and with the same reasoning we can treat with a little more caution those parts of the protocol in which the student responds to questions such as "What are you thinking about?" or "How did you get that?" In spite of the opportunity for students to tidy up their thoughts before responding to these questions, in the majority of cases their reviews of what they were doing correspond exactly to the data in the protocol (the written work and verbalized portions). For example in Daniel's protocol the above intervention is followed by the question:

L: Qu'est-ce que t'avais là?

and the response systematically reviews the work done.

D: Là, là j'avais 56 divisé par 4, égale 14 véhicules à 4 roues. Là j'ai 44 divisé par 2. Là, j'ai mis ... j'ai décidé ... de raccourçir encore plus. 50 divisé par 2, 25. Non. 48, oh là là. — — Ca va donner 24. Bon.

Frequently students are questionned at the end of the protocol on how they got the answer. The responses are more inadequate than dishonest. In most cases the student simply doesn't know how he got the solution. In some cases he simply says so. Daniel's crypto-arithmetic puzzle is an example of this:

L: Je peux te laisser avec la question là à moins que tu penses que t'es proche.

D: — Uh, uh; 4 fois 3, 12. 4 fois 2, 8, 9. Je l'ai trouvé.

L: Comment tu l'as trouvé?

D: Je sais pas. Là, j'ai dit, aye, comment ça se fait ... ah oui c'est, parce que ... j'ai fait 4 fois 23.

D. then goes on to explain how he multiplied 23 by 4. L. was ready to move on to another interview since D. was going nowhere with the problem; when suddenly D. produces the answer. When questionned as to how he got it D. simply says "I don't know". He then does what most students do in such a case which is to review in detail the answer. It is very rare in these interviews that any insight into how the student got the answer can be gained from directly asking the question. This however is not a case of the student trying to deceive the interviewer but more a case of the fact that he simply does not know.

Once it is established that the students are trying as earnestly and honestly as possible to verbalize what is going on in the solution process we then are led to the question of the relationship between what is verbalized and what is going on in the head. Are we really getting a complete message? Or are we, as Alan Bell suggests in his paper on problem solving, getting only "the tip of the iceburg"?[1]

It is obvious from a quick review of these interviews that the majority of students have considerable trouble with reading and expression in French at least in this area. The fairly frequent confusion over the meaning of words such as "véhicules" and "catégories", the fuzziness around the meaning of simple mathematical terms such as "impair", "égale", "droite", the absence in many cases of words to express even simple mathematical notions, all indicate that students are struggling not only with the problems themselves but with the language in which they are expressed and with their own limited abilities in expression. In the case of the Cegep group, an examination of their own attempts at protocol writing indicates that abilities in written French are also seriously low. One example will serve to illustrate here. It is taken from Carole's own protocol for the square cutting problem.

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[1]Alan Bell, The Problem Solving Process (Unpublished paper, 1981).

L:	très bien. Comment t'as trouver ça?
C:	J'y avais pas pensé, j'ai dis ... parce que moi j'ai dis ... la j'en ai 4 pis si j'en fait d'autre ici je comptais un autre fois culà.

Carole makes at least twelve errors in these few lines. Furthermore her explanation is not at all clear to someone who doesn't already know that she thought she would get eight squares when she subdivided a four square. Without reviewing the long and ongoing debate over the degree of correlation between the qualities of language and thought, and keeping in mind recent research that tends to indicate that abilities in problem solving have low correlation with language abilities, it seems reasonable that in a talking aloud situation the completeness of the portrait of what is going on mentally (as well as emotionally) depends to a large extent on the student's verbal abilities. It is also very clear in the protocols that confusion over the meaning of a word like "véhicules" can be a great handicap to the student who is trying to organize a plan of attack. In response to the question concerning the completeness of the message, we could say here that one important limitation on this is the language ability of the student: -in reading, writing, and speaking French. Even the most sincere effort to verbalize everything that is going on will be hampered by lack of vocabulary and other constraints of language.

A second limitation on the completeness of the verbal message is due to the complexity of thinking itself. Thinking takes place at various levels of complexity from that which is close to the verbal, (talking to one's self), to that which is extremely abstract and symbolic, (performing geometrical contortions in the mind's eye). It seems logical to conclude that verbal thought will take precedence over the more abstract and difficult to express thought processes in a situation where there is a fair bit of pressure on the student to verbalize. In other words, he says that which comes most easily - that which has been wholly or partially verbalized "in his head". For most students the learning of the four basic operations and most especially multiplication were very verbal experiences. Multiplication tables were rattled off through primary school to the point that most students still hear in their heads the sound of those tables.

Looking through the protocols one can see page after page of performance of operations out loud. This does not necessarily mean that nothing else is going on at the time but rather that these operations are the most natural and easy to express. The message we get will be incomplete to the extent that more abstract thought will be drowned out by the more accessible verbal thinking.

It is also possible that thought can take place at essentially equivalent levels of complexity but on different themes at the same time as for instance when students appear to be exploring two (or more) solution paths at a time. This possibility is explored and discussed in chapter VI. It is obvious that if two thought processes are taking place simultaneously, the verbal message will be inadequate in that it can express only one of these.

In the case of very rapid thinking, it is reasonable to expect that verbal expression which is limited in its speed will not be adequate or complete. It is possible that the demand to verbalize may even have hampered the thought process by slowing it down considerably. For one reason or another there are not a great deal of examples of rapid thought here, although we can suspect that silences before solutions might be possible areas of unexpressed thought. In general, all long silences are holes in the message. We cannot believe as one student said that when he is not talking it's because there is "nothing, absolutely nothing, going on".

In conclusion we might say that there are a certain number of limitations on the completeness of the verbal message due to language constraints, to the complexity of what has to be expressed, and the impossibility of verbalizing more than one thought process at a time. Nevertheless we are getting a sufficiently significant portrait of the thought processes and should not use the tip-of-the-iceburg analogy here.

A number of bonus factors in the interviews compensate for the incompleteness of the portrait of the thinking process. When the student verbalizes he tells us so much more than what is going on in his head. A lot of information as to the emotional ups and downs of the student during problem solving, his attitude towards mathematics, his dominant personality traits, his interpersonal skills, his self image, and so much more, can be gleaned from a careful study of these interviews. These factors are as important to the teacher of mathematics as are the thought processes themselves and cannot be entirely dissociated from them. The problem solver is a complete human being who addresses the problem, the interviewer, and the entire contextual situation with his entire self. The interviewer, the solver, and the problem are modified by the experience. The next ten years could be spent exploring these very complex areas of these twenty-four protocols and still not exhaust all the possibilities. So many questions are raised by the information provided - questions about thought itself. For instance, a problem solution is often accompanied by an inner assurance that this is it, by a relaxation, laughter, and a sort of explosion of joy. Are the mental processes themselves the source of all this? And what about the agony, the sighs, groans, writhing? The very simple context of the interview situation does not somehow seem an adequate source for the emotional range that is expressed in many interviews. The questions are endless and very exciting. Since the thought processes cannot be entirely untangled from the rest (except by information processors) they will of necessity have to be included in discussions of the protocols. The problem here is not so much the inadequacy or incompleteness of the message but rather the vastness of the messages. Wertheimer expresses the same problem in Productive Thinking.

> "But if we want really to understand how the performance comes about (or does not), a much broader field must be confronted. The question then is one of the organization of the whole field in which the actual happening is only a part - the personal, the social, the historical field." [1]

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[1] Max Wertheimer, Productive Thinking, Enlarged edition, (New York: Harper and Brothers, 1959), p. 64.

Chapter II: The "parking lot" problem

"Dans un parking sont placées 40 véhicules, des automobiles et des motocyclettes. On compte en tout 100 roues. Combien y a-t-il de véhicules de chaque catégorie?"

There are 40 vehicles in a parking lot, cars and motorcycles. All together they have 100 wheels. How many of each kind of vehicle are there in the lot?"

(English version)

2.1 Problem presentation

The parking lot problem was hand written on a filing card by a colleague who had done the translation of the problem statement from English. The card was presented to the student with the request that he read it aloud and then attempt to solve it. It was the first problem presented to each student.

For grade six students there was a second presentation of the problem in the form of a verbal retelling of the parking story by the interviewer for reasons mentionned earlier (see Ch. I: 1.5.2).

In reading the problem aloud a number of students exhibit some reading problems. Several stumble over the words "véhicules" and "catégories". Others read the last sentence as if it were not a question. These two reading flaws correspond precisely to major difficulties experienced by the students in understanding the problem: -confusion over the meaning of the words "vehicle" and "categories" and difficulty in grasping what exactly is being asked of them.

The meaning of the word "véhicule" is given in the first sentence of the problem statement. That sentence is structurally complicated by the fact that "des automobiles et des motocyclettes" appears after a comma. For students who are very uncertain about the use of a comma and who tend to avoid them in their own writing (see cegep students' own protocols), the indication that vehicles corresponds to both cars and motorcycles could easily be missed. For the majority of students their identification of "véhicules" with "automobiles" and in some cases their conclusion that there are 40 cars and some motorcycles seems to indicate a reading that might be punctuated as follows:

"40 véhicules, des automobiles, et des motocyclettes". Many problems might have been avoided by replacing the last sentence by: "Combien sont des automobiles et combien sont des motocyclettes?" In fact, at the grade six level, that is how the question is asked in the "retelling of the problem story". It is surprising that cegep students also had trouble understanding the problem statement.

Example from the beginning of Mario's protocol:

M: (reads problem) Faut je ... essaie de le résoudre?

L: Um huh. Tu peux prendre un papier si ça t'aide.

M: Bon dans un parking il y a 40 véhicules, les automobiles ont 4 roues, 40 fois 4 ... 40 fois 4 roues. Pis, il y a des motocyclettes ... Il y a 40 véhicules. Bon, c'est pas ça. Il y a des automobiles pis des motocyclettes. Il y a 100 roues en tout. 40 véhicules avec 100 roues.

— Oh, mais là j'ai ... j'ai juste le nombre de roues, j'ai le nombre de véhicules — (30 sec.) — Si on va avoir des véhicules à 4 roues ... $x + 4$ ... j'ai des véhicules à deux roues ... ça m'en prend 40. Faut que ça égale à 100. — Il y a 2 inconnues là dedans? — (25 sec.) — Attend, là je suis bloqué là ... Faut que je trouve le nombre de véhicules — (30 sec.) — Ah je pense c'est ... je peux dire x ...

M. begins by asking if he must solve the problem. That he is unsure what the question is soon becomes clearer. He begins rereading the problem and stops after the word automobiles: "dans un parking il y a 40 véhicules, les automobiles". Here he introduces the needed fact that cars have four wheels. His equating of vehicles with cars leads him to multiply 40 by 4. He then continues by saying that "then there are some motorcyles". At this point he seems to conclude that there are 40 cars and some motorcycles. He realizes that this is not correct and a period of silences and confusion ensues. When he mentions that he has 4 - wheeled and 2 - wheeled vehicles, he appears to have resolved the car - vehicle confusion. But something is still confusing him and after another period of silence he says that he must find the number of vehicles. This conclusion may have come from a partial reading of the last sentence: "Combien y a-t-il de véhicules ...." Unclear about the word vehicle and possibly "catégorie" as well, M. embarks on an algebraic solution attempt.

Louise, a middle-aged cegep student, equates vehicles and cars at the beginning of her protocol: -"Des motos ça a deux roues; des véhicules ça en a quatre" and "Si je mets une trentaine de ... de véhicules ... par 4, ça fait 120 roues".

Vocabulary problems and other evidence of difficulties in understanding the problem statement are followed by a confusion over the meaning assigned to x in the case of cegep students or, more generally, a confusion over what exactly the student is supposed to find. Students who set x equal to the number of vehicles or the number of "véhicules de chaque catégorie" generally have experienced some degree of difficulty with reading and understanding the problem statement.

2.1.1 Pros and cons of the parking lot problem:

The discussion in the preceding section of the problem presentation centered on the problem created for the students by the choice of vocabulary and sentence structure. Beyond the actual wording of the problem there are several other aspects which might be seen as negative.

The answer to the problem (30 motorcycles and 10 cars) can quickly be hit upon almost by chance. The student who begins with 20 of each very naturally moves to a 10 - 30 division on the second try. Had the answer been say 13 - 27, all students would have been required to do a certain amount of trial and adjustment to arrive at the solution and we would not be left wondering whether a quick solution is not just a bit of luck.

Another unfortunate aspect of the answer is that it does not correspond to North American reality where there are always more cars than motorcycles in a parking lot. The student must therefore work against his intuition.

Nearly all first attempts in this problem involve more cars than motorcycles or an equal number of each. When Jean-Pierre and Michel are asked directly whether they think there will be more cars or motorcycles in the lot they both reply "cars". Jean-Pierre answers affirmatively when asked if it is because in most parking lots there are more cars.

L: ... penses-tu qu'il y a plus d'autos ou plus de motos?
Jean-Pierre: Plus de voitures.
L: D'autos? (oui). Pourquoi?
J: — — — —
L: Tu sais pas. C'est juste une intuition, là? Tu te

	dis "Mais la majorité des stationnements c'est plutôt des autos"?
J:	Oui.

The lack of superfluous numerical information and the need to add relevant information on car and moto wheels can be seen as either advantageous or disadvantageous. For the researcher it is interesting to have some superfluous information because much can be learned about the degree of the student's understanding of the problem through his treatment of this information. On the other hand, the problem does require the students to bring into play two figures that are not given.

On the positive side, the problem requires the coordination of two variables which is manageable but makes definite demands on students. The problem can be done by anyone who has learned multiplication and addition. It is not a familiar problem for students being very different from school textbook questions. It requires, therefore, a certain inventiveness and thought.

The problem might be considered as a classic. Krutetskii makes great use of it in his enquiry into mathematical abilities in schoolchildren. In Krutetskii's work the problem appears in various forms although in Series III it is given in almost identical form as here:

> "A2. There are 40 vehicles in a parking lot - cars and motorcycles. All together they have 100 wheels and 40 steering devices. How many of each kind of vehicle are in the lot?" (page 110)[1]

Krutetskii's addition of the unnecessary information about the "40 steering devices" permitted him to examine the pupil's ability to handle surplus information. Another common form of this type of problem involves two and four-legged animals. It appears in Series VI in Krutetskii:

> "Chickens and rabbits are running around outdoors. Together they have 35 heads and 94 feet. How many chickens and rabbits are there?"[2]

And finally, the interest and effort generated by the problem as demonstrated by the very lengthy and rich protocols elicited in this experiment is perhaps evidence in itself that the definite advantages of this particular problem far outweigh its disadvantages.

This is not to say that the problem statement could not be improved. Figures and wording could and should be adjusted for future use.

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[1] V.A. Krutetskii, The Psychology of Mathematical Abilities in Schoolchildren, ed. Jeremy Kilpatrick and Izaak Wirszup, trans. Joan Teller (Chicago: University of Chicago Press, 1976), p. 110.

[2] Ibid., p. 121.

2.2 Algebraic solution attempts:

After reading the problem statement cegep students embarked upon an algebraic solution attempt whereas grade six students began some form of trial-and adjustment approach to the problem. With the exception of two students who successfully solved the problem with algebra, all students ended up sooner or later using a non-algebraic approach. In this section eleven algebraic efforts will be studied and in the following section the twenty-two non-algebraic attempts.

2.2.1

With the exception of Louise all cegep students embarked on an algebraic solution for the parking lot problem. The most obvious reason for this is that the interviewer was also their algebra teacher and it would therefore be assumed that an algebraic solution would be the desired response. The strength of this teacher-algebra association is so strong that in some cases students approach the square cutting and crypto-arithmetic problems either immediately looking for an algebraic solution or apologizing afterward for not having found one.

A second reason for algebra attempts may be that students realize that there is in algebra an algorithm which will more or less solve the problem for them. This involves, first of all, the recognition that there exists an algebraic algorithm for this type of problem and secondly, that if properly applied an economy of thinking results. (Although with only two exceptionsalgebraic attempts did not lead to a solution of the problem, certain important elements are present: - recognizing that algebra is applicable, realizing that the unknowns must be represented by x or expressions in x, recognizing that equations must be established and solved.)

2.2.2 Can the parking lot problem be considered a problem for cegep students?

Kantowski defines a problem as a task for which no algorithm is available to the solver and Alan Bell says:

> "... a problem exists when a situation demands the coordination of a number of items of data to produce a new unit of knowledge and this cannot be dealt with by the subject by the recognition of the relevance of a known procedure and the direct application of it to give the required result."[1]

After examination of the cegep protocols, it would be hard to say that the parking lot problem is not a problem for these students. And yet for all of these students the algebra algorithm is available to the extent

[1]Alan Bell, The Problem Solving Process (Unpublished paper, 1981).

that they have done at least three years of highschool algebra. It is available to them in that they recognize its' appropriateness for one reason or another, they possess varying degrees of proficiency in establishing expressions for the unknowns, and they realize equations must be established and solved. In Bell's definition we would say that the subject recognizes "the relevance of a known procedure" but cannot directly apply this known procedure to give the required result. Given Kantowski's definition of a problem the algebra portions of the protocols lead one to ask exactly what she means by "availabity".

There are two cases (Daniel R. and Hélène) in which students successfully solve the problem with algebra and yet not even here, where the algorithm was both available and correctly applied could we say that the parking lot problem was not a problem for these students. Both students recognized that a problem exists and worked very hard at solving it with all the relevant states of perplexity, planning, solving, and tension release.

I think that the existence (degree of availability) of an algorithm greatly modifies student problem solving behavior but I would hesitate, on the basis of what happens in these interviews, to say that what is taking place is not problem solving because of the available algorithm.

Nor would I say that the problem immediately becomes less of a problem (i.e. easier) because of the existing algorithm. Notice, for instance, the number of students who, when released from algebra, express the simplicity of the non-algebraic problem and solve it almost immediately (see Richard, Pierre G., Daniel B., Mario). Several students oppose logic to algebra saying that the use of algebra in this problem is far more difficult than the use of "logic". Algebra is seen as a handicap to the student and, in effect, it is.

2.2.3 Algebraic behavior

The eleven students who used algebra displayed a certain degree of knowledge about how to solve problems with algebra. The basic steps or strategies of an algebraic solution are present and may be summarized as a series of three reactions which are found in the majority of protocols:

. Immediate reaction: <u>This is a job for *x* .</u>

Often *x* is introduced without any definition immediately after reading the problem statement. "Soit *x* ..." It is the first symbol to go down on the work sheet in many cases.

. Second reaction: <u>*x* has got to be something, represent something</u>

Few students have the idea that x represents a number and many students assign several meanings to x. Several students study the sentence with the question mark and try to assign x to words in it. In one case, (Carole), x represents the unknown meaning of the words "véhicules" and "catégories". As the solution attempt progresses, the meaning given to x also evolves.

. Third reaction: <u>An equation must be established and solved</u>.

Students realize that they must establish an equation with x and as many given numbers as possible. Most feel there must be a 100 and a 40 in the equation and for some the 4 and 2 corresponding to the number of wheels on cars and motorcycles as well.

Beyond these basic algebraic reflexes, the majority of students do not possess any useful tools with which to tackle this problem algebraically. An examination of the evolution of x in a couple of protocols will serve to show how students flounder in spite of the three essentially sound reactions described above.

2.2.4 Evolution of x in two protocols:

A. Daniel D. on reading the problem statement immediately introduces x without giving it any meaning: "Soit x, soit x ..." His solution attempt then evolves as follows:

. x, number of vehicles

. "I'm looking for the x ..."

. x, the number of vehicles in each category

. $40x$, the number of vehicles

. x, the number of wheels for the cars and wheels for the motorcycles

. x, the number of wheels

. x, the number of car wheels

. $4x$, the number of wheels for the cars

. $2x$, the number of wheels for the motorcycles

. $4x + 2x = 100$ $6x = 100$ "x is the wheels here"

. $4x$, the number of wheels per car

. "x, by itself is the number of wheels of a car"

. "y, the number of wheels per motorcycle"

In commenting on his solution later Daniel calls the above attempts his "tentatives d'algébrications". "Algébrications", a word of his own invention, might well be coined to describe what takes place in Carole's protocol as well as in many others.

B. Carole, on reading the problem statement, writes

$x =$ on her work sheet saying "x est, uh, ...".

x then evolves as follows:

. "x is the number of categories? of automobiles? of vehicles?"

. "A car has 4 wheels. Therefore, $4x$."

. "A moto has 2 wheels. Therefore, plus $2x$."

. $x = 4x + 2x$

. "x, number of categ ...
x, number of vehicles"

. "$x = \frac{4x + 2x}{100}$ but where can I stick the 40 vehicles?"

. Solves above equation and gets $x = 94$ as follows:

$100 = 4x + 2x$

$100 - 4x - 2x$ (sends rt. hand terms to left side and lets = sign drop)

$x = 94$ (simplifies above but instead of getting $94x$ she brings back the equal sign)

In spite of the fact that she has never settled on a meaning for x, Carole realizes that $x = 94$ doesn't work and accepts the invitation to try a non-algebraic solution.

2.2.5 Algebraic flaws:

The faulty algebraic behavior demonstrated in the two protocols just reviewed is repeated over and over in the others. Major flaws seem to be:

- <u>not realizing x represents a number and only one number</u>.

. For example Sylvie begins her solution saying "Bon, j'ai 40 véhicules, des automobiles ... x, des motocyclettes ... x".

As she speaks she writes:

$$\boxed{\begin{array}{c}\underline{40\ v} \\ a = x \\ m = x\end{array}}$$

In fact very many students never say or write that x represents the <u>number</u> of automobiles or whatever.

. Another example is Edes who asks "Faudrais-tu que je donne mettons un x automobiles pis un autre x, à motocyclettes?"

She writes automobile $\longrightarrow x$

moto $\longrightarrow x$

but later introduces the idea of number and writes

nbre de véhic $\longrightarrow x$ and

nbre de auto $\longrightarrow x$

This practice of allowing x to represent more than one number can be seen in the majority of protocols. Often x evolves in meaning as it does in the two examples given in the preceding section. In such cases it is possible that as x takes on a new meaning the old one is abandoned. However, in many cases such as in this example from Edes' protocol, it is quite clear that the student intends x to have two different number values at the same time.

. <u>inability to form simple expressions in x.</u>

For instance, a student who defines x as the number of cars, states that the number of car wheels is $x + 4$ (see Martine). For Mario x represents the two-wheeled vehicles and $x + 4$ the four-wheeled.

. <u>inability to establish equations</u>

This is obviously a consequence of students' inability to form simple expressions in x as described above. Most students are so lost at this stage that they simply throw down on either side of an equal sign as many expressions as possible involving x, 100, 40, and often 4 and 2. Carole's dissatisfaction with her equation $\left\{ x = \frac{4x + 2x}{100} \right\}$ is obvious in her question "Où est-ce qu'on mettrait les 40 véhicules?" 40 is the only number missing in her equation and she feels she ought to include it. Mario, who gets as far as $x + 4$ four-wheeled vehicles, reminds himself: -"Faut que ça égale à 100" - and establishes the equation $(x + 4)x = 100$. Richard and Martine do not introduce 4 and 2 into their first attempts at equation formulation: $x + 40 = 100$ (Richard) and $x - 40 = 100$ (Martine).

A brief look at the beginning of Martine's algebra protocol will serve to demonstrate the above mentioned inabilities as well as to conclude the study of students algebraic attempts in the parking lot problem.

2.2.6 Review of a partial algebra protocol:

	Work sheet
M: (reads question aloud slowly) Bon, 40 véhicules ... t'as en ça des automobiles et des motocyclettes, um ... ce serait bien facile s'il y en avait 20 de chacun. On compte en tout 100 roues. Les, uh, automobiles ont 4 roues; les motocyclettes ont 2 roues. Bon ... Qu'est-ce que je vais faire? (laughs) Combien y a-t-il de véhicules dans chaque catégorie? ... pour qu'on arrive à ....100 roues. — Bon ... "Sont placées 40 véhicules des automobiles et des motocyclettes. On compte en tout 100 roues. Combien ..."	
x, on veut savoir combien qu'il y a de véhicules dans chaque catégorie. Um, bon ... 40 véhicules.	x

M. shows an immediate understanding of the problem when she says that it would be easy if there were twenty of each.

She then reviews several times the problem statement and introduces the needed 4 and 2 wheels per car and motorcycle respectively. She seems to be off to a good start for a non-algebraic solution when on a rereading of the problem she responds to the word "combien" by introducing an x. At this point x seems to represent the entire question sentence.

L:	x, c'est quoi?	
M:	x, c'est le nombre de véhicules ... véhicules ... chaque catégorie, ça	$x \to$ nbre de véhicules
	veut dire que ... ce serait x pour les automobiles vue que c'est pour chaque	$x \to$ automobiles
	catégorie, pis x pour les motocyclettes	$x \to$ motocyclette
	— Bon, uh, ... comment je vais faire ... pour faire une équation avec x? "Dans un parking sont placé 40 véhicules ... des automobiles et des motocyclettes. On compte en tout 100 roues" — — — (sighs). Ca ferait ... ben, ... les automobiles s'ils ont 4 roues — ce ferait x ... moins, moins ou plus quelque	
	chose qui va égaler 100, égale 100. Ben	$x-40=100$
	non, ça marche pas ... Alors ... un peu ... les 4 roues viennent faire quelque chose là dedans sûrement (sighs) — 100 roues ... ça va être peut-être ...,	
	peut-être aussi ... x plus 4 est égale à	$x+4=$

— — . Mais non, le x ça n'a pas de rapport pantoute avec les roues. — ? — — — Ça va être x égale 100. Quelle équation qu'il faut? — — — x ou peut-être fois ... fois 4 roues égale 100. Pis l'autre x fois 2 égale 100. — $4x$ égale 100, $2x$ ... ça a pas d'allure pantoute. (Pourquoi?) Je sais pas. J'ai multiplié, uh, 4 roues pour que ça égale 100. Ça arrive pas, je ... c'est parce que je comprends pas le problème, sûrement.	$x = 100$ $x(4) = 100$ $x(2) = 100$ $4x = 100$ $2x = 100$

Having made the initial "this-is-a-job-for-x" reaction, M. proceeds to give some meaning to x in response to L's question. First of all x is the number of vehicles for each category. She explains that this means x is for the automobiles and the motorcycles "since it is for each category". It is the question sentence in the problem statement that seems to cause her difficulty. She still identifies x with the entire sentence and can't seem to break it down. The third reaction, that of establishing an equation, now takes over with all the disastrous effects of an undefined x. Bringing together x, 40, and 100 she produces $x - 40 = 100$ and then reminding herself that the 4 wheels/car should appear in the equation she begins a second attempt $x + 4 =$ which she rejects on the grounds that x has no relationship to the wheels and presumably the 4 wheels/car. After a relatively long silence she states that, whatever it is, her equation must be equal to 100 and she writes $x \quad = 100$. She then tries to complete the left hand side introducing 4 and then 2 as multipliers of x in the blank space. She rejects this saying it doesn't make sense but cannot say why. She concludes that it is not working out because she really must not understand the problem. Martine seems to understand the problem and her use of the word "sûrement" indicates that she too feels she understands it. L. picks her up on this

L: Tu comprends pas le problème? M: Je sais, en tout cas, qu'il faut que j'aie une équation qui égale à cent. Pis je pense aussi que c'est pas deux équations séparées, que ça va être une équation qui égale à 100. Pis je pense aussi que c'est pas deux équations séparées, que ça va être une équation qui elle, ... pas séparé comme ça. — Au moins que ça soit ... $8x$ égale 100. Je penserais pas. — — — . Au moins qu'on divise — — — $4x$ divisé par 40 — — — 4000 divisé par 40 ... Ca se peut pas. — — —	$\frac{4x}{40} = 100$ $4000 \lfloor 4$ 1000

M. now reviews everything she knows about the equation she is looking for: that it is a single equation with an expression in x equal to 100. She then tries to combine her two previous equations ($4x = 100$ and $2x = 100$) into a single equation: $8x = 100$. This she rejects and after a period of silence she seems to be off on a new formulation of the equation which brings in $4x$, 40, and 100: $\frac{4x}{40} = 100$ which she correctly solves for $x = 1000$ and rejects.

Martine's algebraic protocol goes on much longer. It becomes very clear that she fully understands the problem. The rest of her algebraic efforts involve trying to establish the single equation with 100 on the right hand side.

2.2.7 Student views of algebra:

Through student interjections one gets a portrait of what algebra is for the student. For most it is a ritual they have never understood – a bag of tricks. Sylvie on reading the question sentence says: "Ça doit être une affaire niaiseux, encore ça" (Note: "niaiseux" is a popular form for the adjective "niais" which means according to the dictionnary Robert "dont la simplicité va jusqu'à la bêtise"). For Richard "... l'algèbre c'est ... je sais pas moé. J'imagine que c'est plus compliqué que d'autre chose." Louise would give anything to be able to do algebra: -"Je ne sais pas qu'est-ce que je donnerais pour être capable de penser algébriquement mais ça ne me rentre pas dans la tête pantoute."

Daniel B. is the most eloquent on algebra which he sees as the opposite of logic. He explains that he generally uses a method of successive approximations in problem solving. This seems to be logic. In algebra he can never find an equation with x's.

> "J'aime mieux l'affaire logique parce que l'algèbre, je sais pas, j'arrive pas. Je pars avec un tout, c'est tout le temps en soustrayant. Des fois c'est ça, je pars en plus bas pis je monte. Comme ça j'arrive plus. Je sais pas je ... des fois je dis, la manière ... ça ça doit être avec les x. C'est ça, je dois comprendre des variables ... je sais pas quoi. J'arrive pas à les placer."

Like the majority of students, Daniel flounders in algebra and expresses relief when released of it. Students do not appear to even understand the problem and seem to stop thinking while involved in an algebraic solution and yet when released of algebra they seem to know exactly what the problem is demanding and have a ready and effective plan of attack. In his algebra attempt Daniel flounders because he cannot establish an expression for the number of wheels. It is only when he switches to "logique" that he brings into play or even seems to think of the fact that there are 4 wheels per car and 2 wheels per motorcycle.

The following section will examine the "logique" of the cegep students as well as the solution attempts of the grade six group.

2.3 Non-algebraic behavior:

Five solution types appear in the twenty-two non-algebraic solution attempts of the cegep and grade six students. They are listed below and then each is dealt with separately.

I Keeping the total number of vehicles at 40, the student tries various combinations of cars and motorcycles until wheels come out right.

II Keeping wheels fixed at 100, the student chooses the number of cars and calculates the number of car wheels and hence the possible number of motorcycles until the total number of vehicles comes out right.

III Keeping neither the total number of wheels nor vehicles fixed, the student chooses a number of cars and motorcycles and then checks for the total number of wheels and, if he remembers, the total number of vehicles.

IV Keeping the total number of wheels at 100, the student tries various combinations of car wheels and moto wheels until the number of vehicles comes out right.

V The student randomly combines numbers in the problem.

It is important to note that while these five types cover all non-algebraic solution attempts, the individual student uses more than one solution type in the course of his attempts to solve the parking ~~lot~~ problem.

2.3.1 I Keeping the total number of vehicles at 40, the student tries various combinations of cars and motorcycles until wheels come out right.

Chooses # of cars (or motos) C	⇒	Calculates # of motos (or cars) 40 - C	⇒	Calculates # of wheels 4C+(40-C)2	⇒	Decides whether to increase or decrease C

Daniel B. uses this method very efficiently. He succeeds, as follows, in solving the problem in three circuits:

20 cars ⇛ 20 motos ⇛ 120 wheels ⇛ decrease C by 5

15 cars ⇛ 25 motos ⇛ 110 wheels ⇛ decrease C by 5

10 cars ⇛ 30 motos ⇛ 100 wheels.

An example of a less efficient use of this method is found in François' protocol:

22 cars ⇛ 18 motos ⇛ 88 + 36 wheels (too many) ⇛ switch

~~18~~ cars ⇛ 22 motos ⇛ 116 wheels .

15 cars ⇛ 25 motos ⇛ 60 + 50 wheels (too many) ⇛ switch

25 cars ⇛ 15 motos ⇛ 100 + 30 wheels. Doesn't work.

10 cars ⇛ 30 motos ⇛ 100 wheels ⇛ Switch should also work.

30 cars ⇛ 10 motos ⇛ too many wheels.

Francois' added strategy of inverting numbers of cars and motos and his lack of attention to the progress of the total number of wheels makes his use of this method much slower. He has combined two good strategies but does not fully profit from either. Although he tends more to reduce the number of cars than increase them he does not fully realize the link between reducing the wheels and reducing the number of cars.

Lucie's protocol provides an example of a student who begins her solution attempt with this method:

20 cars $\Longrightarrow$ 20 motos $\Longrightarrow$ 120 wheels

30 cars $\Longrightarrow$ 10 motos $\Longrightarrow$ more than 120 wheels

A lack of decision on whether C should be increased or decreased leads Lucie further away from the desired number of wheels and is probably a major factor in her abandoning this method. In her next step she cuts numbers in half and tries 15 cars and 5 motos thereby controlling neither vehicles nor wheels.

2.3.2 <u>II</u> Keeping wheels fixed at 100, the student chooses the number of cars and calculates the number of car wheels and hence the possible number of motorcycles until the total number of vehicles comes out right.

$$\begin{tikzcd}
\begin{array}{c}\text{Chooses \# of}\\ \text{cars}\\ C\end{array} \arrow[r, Rightarrow] & \begin{array}{c}\text{Calculates \# of}\\ \text{car wheels}\\ 4C\end{array} \arrow[r, Rightarrow] & \begin{array}{c}\text{Calculates \# of}\\ \text{moto wheels}\\ 100 - 4C\end{array} \arrow[d, Rightarrow] \\
\begin{array}{c}\text{Decides whether}\\ \text{or not to increase}\\ C\end{array} \arrow[u, Rightarrow] & \begin{array}{c}\text{Calculates \# of}\\ \text{vehicles}\\ C + \frac{100 - 4C}{2}\end{array} \arrow[l, Rightarrow] & \begin{array}{c}\text{Calculates \# of}\\ \text{motos}\\ \frac{100 - 4C}{2}\end{array} \arrow[l, Rightarrow]
\end{tikzcd}$$

Strangely enough, the other version of this method which puts the accent on the number of motorcycles rather than cars never occurs except momentarily in one protocol. There are three possible reasons for this: -the word car appears before motorcycles in the problem, the cars control more wheels and therefore make bigger changes when varied, and finally the interviewer's accent on cars (i.e. always asking "How many cars?").

As with I, the fewer times the student must go through the circuit, the more efficient his use of the method might be considered to be. Richard, for example, makes very efficient use of the technique:

15 cars ⇛ 60 car wheels ⇛ 40 moto wheels ⇛ 20 motos ⇛ 35 vehicles
⇛ need 5 more vehicles

10 cars ⇛ 40 car wheels ⇛ 60 moto wheels ⇛ 30 motos ⇛ 40 vehicles

He demonstrates how quickly the circuits can be run through as he solves the problem in about a minute.

This method (II) tends however to be slightly more cumbersome for the majority of students than the first (I).

The length of each loop and the possibility of calculation errors along the way contribute to protocol lengths. Decisions as to whether to increase or decrease C and by how much also retard progress.

Louise's protocol provides a good example of how forgetting the vehicle constraint and how small variations in C slow down an otherwise successful use of this method:

30 cars ⇒ too many car wheels. Reduce cars.

20 cars ⇒ 80 car wheels ⇒ 10 motos. Thinks she has it.
(misses vehicle constraint)

25 cars ⇒ 100 car wheels ⇒ 0 moto

15 cars ⇒ 60 car wheels ⇒ 40 moto wheels ⇒ 20 motos.
Thinks she has it (misses vehicle constraint)

12 cars ⇒ 48 car wheels ⇒ 52 moto wheels ⇒ 26 motos

11 cars ⇒ 44 car wheels ⇒ 56 moto wheels ⇒ 28 motos

10 cars ⇒ 40 car wheels ⇒ 60 moto wheels ⇒ 30 motos

Actually Louise's method is less direct than appears above because of the use of a strategy all her own between the 12 and 11 car attempts as well as between the 11 and 10 attempts. She decreases cars by 1 and increases motos by 1 on these two occasions, calculating wheels and finding she no longer has 100. Fortunately she returns to her method (II) each time.

2.3.3 III Keeping neither the total number of wheels nor vehicles fixed, the student chooses a number of cars and motorcycles and then checks for the total number of wheels and, if he remembers, the total number of vehicles.

Chooses # of →	Calculates total →	Calculates total →	Decides whether
cars & motos	# of wheels	# of vehicles	to increase or
C, M	4C + 2M	C + M	decrease C & M

This method is the most unwieldy of all. Decisions must be made on increases or decreases of both C (cars) and M (motorcycles) at every loop or circuit taking into account the total number of wheels and vehicles simultaneously.

The method appears more often in the grade six protocols. It is never a major strategy except in the case of Pierre-Paul whose only concrete attempt at a trial solution involves 5 cars and 10 motorcycles. It is significant that Pierre-Paul felt he had to deal with the figures 100, 40, 4, 2 and later, 25 and 50 simultaneously. This is characteristic of the type III frame of mind: -everything has to be dealt with at once. Needless to say, the majority of students cannot maintain so many variables at once.

Method III appears in protocols generally as a consequence of losing track in methods I or II of one of the variables being held constant. Lucie's protocol provides a good example of a method I which deteriorates into III as she loses track first, of the total number of vehicles she was holding fixed (I) and later, of the number of wheels being held fixed at 100. The steps in her solution can be summarized as follows:

20 cars $\Rightarrow$ 20 motos $\Rightarrow$ 120 wheels Method I

30 cars $\Rightarrow$ 10 motos $\Rightarrow$ 120 wheels Method I

She then decides to cut the number of cars and motos in half in order to drastically reduce the number of wheels.

15 cars and 5 motos $\Rightarrow$ 60 car wheels + 10 moto wheels $\Rightarrow$ 70 wheels Method III

She now has insufficient wheels and decides to add 5 more of each.

20 cars and 10 motos $\Rightarrow$ 80 car wheels + 20 moto wheels $\Rightarrow$ 100 wheels Method III

Having forgotten the total number of vehicles constraint she now thinks she has arrived at the answer.
Set on the right track by the interviewer (i.e. reminded she must have a total of 40 vehicles), Lucie now embarks on a type II method which eventually leads to a solution.

We see in Chantal's protocol another example of a deterioration of a type I method into type III through the forgetting of the 40 vehicle constraint when another strategy, —that of finding numbers divisible by four and two — comes into play. However in Nathalie V.'s protocol the introduction of method III represents an improvement in method in the sense that choosing a certain number of cars and motorcycles is a more structured or ordered approach than simply throwing numbers together haphazardly as she does in the beginning:

. divides 100 by 40 getting 20 and 20 remainder

. multiplies 50 (half of 100) by 20 (half of 40)

These efforts will later be classified as method V.

. 18 cars (forgetsmotos) $\Longrightarrow$ 72 wheels   Method III

. 18 cars and 40 moto wheels $\Rightarrow$ 112 wheels

. 16 cars and 40 moto wheels $\Rightarrow$ 104 wheels

. 15 cars and 40 moto wheels $\Rightarrow$ 100 wheels

and later,

. 14 cars and 40 moto wheels $\Rightarrow$ 96 wheels

In the hands of a very competent problem solver method III might eventually lead to a solution, given of course that the solver does not forget the 40 vehicle constraint. It is for this reason that it has been called a method and not simply a I or II method gone astray. The good problem solver would not however, choose method III because of the obvious difficulty of attending to two variables (wheels and vehicles) at once. The method has some merits in that it often leads the solver to clarify the requirements of the question and to eventually adopt a more efficient method. It is also better than doing nothing at all and does indicate the student has some hold on the problem and is applying a sometimes very organized form of trial and adjustment. Lucie, for instance, very systematically reduces the number of cars.

2.3.4	IV Keeping the total number of wheels at 100, the student tries various combinations of car wheels and moto wheels until the number of vehicles comes out right.

$$
\begin{tikzcd}
\begin{array}{c}\text{Chooses \# of}\\ \text{car wheels}\\ W\end{array} \arrow[r, Rightarrow] & \begin{array}{c}\text{Calculates \#}\\ \text{of moto}\\ \text{wheels}\\ 100 - W\end{array} \arrow[r, Rightarrow] & \begin{array}{c}\text{Calculates \#}\\ \text{of cars}\\ \frac{W}{4}\end{array} \arrow[d, Rightarrow] \\
\begin{array}{c}\text{Increase of \# of}\\ \text{car wheels?}\end{array} \arrow[u, Rightarrow] & \begin{array}{c}\text{Calculates}\\ \text{total \# of}\\ \text{vehiciles}\\ \frac{W}{4} + \frac{100 - W}{2}\end{array} \arrow[l, Rightarrow] & \begin{array}{c}\text{Calculates \#}\\ \text{of motos}\\ \frac{100 - W}{2}\end{array} \arrow[l, Rightarrow]
\end{tikzcd}
$$

This method is structurally equivalent to method I. Here total wheels rather than vehicles are being kept constant.

And yet, in terms of difficulty in application, the method is much more demanding in this context. First of all, the method demands division rather than multiplication and looking at these protocols we can conclude that division is a much more difficult and less successful operation for students. Secondly, dividing a 100 wheels up into possible combinations provides many more possibilities than does 40 vehicles. One can have anywhere from 0 to 100 car wheels whereas one can only have from 0 to 40 cars.

Method IV appears in a somewhat modified form in Chantal's protocol near the beginning where she takes the total number of wheels and divides by 4 to get the number of cars. Edes begins her solution with method IV when she divides the 100 wheels up into two groups of fifty. She does not, however, pursue this and after an algebra attempt she embarks on a method I solution.

It is Daniel D. who successfully pushes through with method IV to a solution. The steps are worth listing here because they illustrate the length and complexity of the method.

D. begins with a method II approach:

20 cars $\Longrightarrow$ 80 car wheels $\Longrightarrow$ 10 motos

which he thinks is one of many answers until he is reminded of the vehicle constraint.

His second attempt is a beginning of method IV in that he concentrates on the total number of wheels and assigns 20 to the cars. This however gives him five cars which he considers too few. He continues:

40 car wheels $\Longrightarrow$ 10 cars (which he considers too few)

80 car wheels & 20 moto wheels $\Rightarrow$ 20 cars & 10 motos $\Rightarrow$ 30 veh.

70 car wheels & 30 moto wheels $\Rightarrow$ 15 motos & $\frac{70}{4}$ cars

(Rejects because 70 not divisible by 4)

60 car wheels & 40 moto wheels $\Rightarrow$ 15 cars & 20 motos $\Rightarrow$ 35 veh.

50 car wheels & 50 moto wheels $\Rightarrow$ $\frac{50}{4}$ cars Rejects

Gets turned around here.

44 moto wheels & 56 car wheels $\Rightarrow$ 22 motos & 14 cars $\Rightarrow$ 36 veh.

48 moto wheels & 52 car wheels $\Rightarrow$ 24 motos & 13 cars $\Rightarrow$ 37 veh.

Once again he gets turned around intending to leap to 64 moto wheels and instead taking 64 car wheels.

64 car wheels & 36 moto wheels $\Rightarrow$ 16 cars & 18 motos $\Rightarrow$ 34 veh.

Confusion ensues because he expected to be nearer 40 veh. He decides to return to "le centre".

52 car wheels & 48 moto wheels $\Rightarrow$ 13 cars & 24 motos $\Rightarrow$ 37 veh.

After a review of the 4 multiplication table he decides to increase number of car wheels by 4:

56 car wheels & 44 moto wheels $\Rightarrow$ 14 cars & 22 motos $\Rightarrow$ 36 veh.

More confusion ensues. He decides to switch car & moto wheels:

44 car wheels & 56 moto wheels ⟹ 11 cars & 28 motos ⟹ 39 veh.
40 car wheels & 60 moto wheels ⟹ 10 cars & 30 motos ⟹ 40 veh.

At a certain point Daniel has realized that the choice of the number of car wheels must be limited to numbers divisible by 4 which cuts the possibilities down considerably but still leaves him with twenty-five choices and at each step he must ask the question: Is my choice divisible by 4?

2.3.5 V The student randomly combines numbers in the problem.

What are the numbers to combine? ⟹ Combination through one or more operations ⟹ Interpretation of result in terms of demands of problem

This method is one of genuine trial and error, with the accent on error. Often it serves the role of getting the student started on the problem and increases understanding of the problem statement by throwing the solver back on the statement in order to interpret his results. The most common manifestation of the method is the division of 100 by 40 as a first step in the solution process. Maryse, for instance, divides 100 by 40 and somehow gets 25 and decides there are 25 cars. Nathalie L. divides 100 by 40 correctly but doesn't know what to do with it. Nathalie V. divides 100 by 40 getting 20 and 20 remainder. She concludes there are 20 cars and 20 motorcycles. François begins by subtracting 40 from 100 getting 60 which he suggests might be the number of motorcycles.

It is Maryse's protocol which is entirely situated within method V. Having, as mentionned above, divided 100 by 40 to get 25 cars she proceeds as follows:

> L. asks, —"If 25 cars how many motos?" and
> M. replies — " 4 motos and therefore 29 vehicules."
> When the question is repeated she subtracts her 29 vehicles from 40 and gets 11 motos. When asked how many wheels on 29 vehicles, she replies 29 X 4 = 116. Realizing she has too many she divides 116 by 40. Seeing that that does not come out evenly she focuses on the 16 wheels in excess and decides to divide 116 by 16 instead.

This is an extreme form of method V where the student applies no intuition or judgement in her selection and juggling of numbers. The student does little or no learning from her errors. For these reasons method V should be situated at the bottom of the strategy scale. It has its value in helping the student get into the problem but as a solution method it is disastrous, as Maryse's protocol demonstrates. It can happen, however, in a problem this simple, that a method V juggling of numbers can, with a bit of luck and intuition, lead to a solution. Nathalie L., for instance, with method V reasoning all the way, ends up with 30 motos and 40 cars to get a total of 70 vehicles. Reminded that there are only 40 vehicles and retaining her 30 motos she concludes there must be only 10 cars. She checks this out and realizes she has it.

2.3.6 Solutions to the parking lot problem:

The majority of students use more than one of the five solution methods studied. Sometimes the switch from one to another is due to a certain bogging down with an unwieldy method. Often it is a consequence of losing track of one's method as for example the student who is keeping wheels fixed and varying the number of cars and who forgets this and begins fixing vehicles and calculating wheels. Some students seem to be able to fix on a method and in spite of interruptions or confusion maintain their original method or strategy. Flexibility in changing methods is of course important. For instance, the student using method III who begins to realize how unwieldy it is and who switches as does Lucie to method II is showing good judgement. Chantal's protocol contains examples of some form of all five methods and terminates in a solution to the problem. The variety of methods used by individual solvers can be seen in the following table.

Table of solution methods:

NOMS NAMES	METHOD I	II	III	IV	V
Lucie	✓	✓	✓		
Maryse					✓
Chantal	✓	✓	✓	✓	✓
Pierre-Paul			✓		
Jean-François	✓	✓			
Marc	✓	✓			
Jean-Pierre	✓	✓			✓
François	✓				✓
Michel	✓				
France	✓				
Nathalie L.					✓
Nathalie V.			✓		✓
Sylvie	✓	✓			
Carole					✓
Hélène		(successful algebra solution)			
Edes	✓			✓	
Louise		✓	✓		
Martine	✓	✓			
Daniel B.	✓				✓
Pierre G.	✓	✓			
Benoit		(successful algebra solution)			
Dani[illegible]		✓		✓	
Mario		✓			
Richard		✓			

2.3.7 Trial and adjustment in the parking lot problem:

Methods I to V are all trial and adjustment methods. Method V could be called trial and error. Establishing a hierarchy in the methods as to their efficiency in solving this problem would place them in the following order: I, II, III, V with a question about IV which theoretically should be classed with I but which for this particular problem would probably have to come after II. Within any particular method there is a wide spectrum of use of trial and adjustment. The most efficient form of trial and adjustment might be considered to be "bracketing" where the student comes at the answer from both sides. For example, in method I he might produce too many wheels, then too few, too many but nearer to 100, and so on. In these protocols there is no evidence of this level of trial and adjustment. Often the interviewer starts students off in this way by asking "If all the vehicles were cars, how many wheels would there be?" and then "If all the vehicles were motorcycles, how many wheels?" The student responses of 160 and 80 wheels respectively and the nearness of 80 to 100 does not however lead students to continue with the method taking much fewer cars than motorcycles.

The least efficient form would be one which could be called random trial and error. The student learns nothing from an attempt except that it does not work. If we look again at method I, the student who tries, say, 20 cars and 20 motos and concludes that there are 120 wheels and who, on the subsequent attempt, tries 25 cars and 15 motos and then perhaps 30 cars and 10 motos, is not making very profitable use of his efforts and could be said to be proceeding at the level of trial and error.

Some students systematically use trial and adjustment but take extremely small steps in adjusting. Pierre G's protocol is an example of this in the context of method I. After 20 cars he tries 15 cars and 25 motos, then 14 cars, then 12 cars, then 13 cars, and finally 10 cars. A more efficient use of the method would have involved seeing the effect on the wheels of reducing the cars from 20 to 15 and then trying another leap of 5 to 10 cars. In general, one could say that the fewer the number of trials needed the more efficient is the use of the trial and adjustment strategy.

A wide variety of activities in problem solving come under the "trial-and-adjustment" or "trial-and-error" umbrella: -indeed, all non-algebraic solution attempts made on the parking lot problem. It will be seen to be the major strategy in the square cutting and crypto-arithmetic problems as well. For this reason it needs some analysis as a strategy. Here the five solution types represent very different forms of trial and adjustment and in terms of efficiency for this problem a certain hierarchy has been suggested. Within a particular solution type varying degrees of efficiency in the use of trial and adjustment can also be seen to go from what is called trial and error here to the most efficient use of trial and adjustment where the student passes through the solution circuit a minimal number of times.

Chapter III: The square cutting problem

"Vous souhaitez diviser un carré donné de telle sorte que chaque partie obtenue soit aussi un carré.
Pouvez-vous le faire pour obtenir 9 carrés? 7 carrés?"

"You are given a square and you want to cut it into a number of pieces so that each piece is also a square.
Can you cut it into 9 squares? 7 squares?"

(English version)

3.1 Problem presentation:

The problem was presented on a filing card as shown on the previous page. Like the parking lot problem, it was translated from the English version by a colleague who also sketched in a small square at the top of the card.

Two reading problems were common. The first appeared with the word "donné" which students tended to read in this way: "Vous souhaitez diviser un carré . . . donné de telle sorte" instead of associating the word with "carré": un carré donné . . . de telle sorte. The second problem appeared at the end of the problem statement with "9 carrés? 7 carrés?" It did not seem to be clear that there were two questions being asked. One student asked whether "7 carrés" was the answer. Others wondered if they were supposed to achieve 7 and 9 in the same division.

It is hard to say whether the sketch of a square on the filing card threw students off or not.

Although most students' immediate reaction to reading the problem statement is to draw the 9 division there are a number of reactions which are essentially dealing with two questions: "What kind of problem is this?" and "What are the rules of the game?" Many students expected an algebra problem and in one way or another asked whether or not it should be done with algebra. Benoit who successfully solved the first problem with algebra thrashes about for several minutes trying to come up with a "formula" to solve this problem and goes so far as to reject his solution on the grounds that it is not algebraic. Edes asks in the middle of her protocol: "Tu peux-tu le faire par uh . . . comme des *x* pis des *y*?" Others read the problem and ask if this is geometry. Sylvie's comment on the problem statement is: "Ca, là, c'est des affaires de géométrie ça, eh? C'est-tu ça? Parce que j'ai ben de la misère. Non? C'est une question d'habilité je suppose ou . . . ?" Questions as to the rules of square dividing come up throughout the problem. It is not clear in the question how squares are to be counted. For some a four division would be counted as five squares by including the original square. Similarly, it is not clear that implanting  is not allowed. Triangles which could be regrouped to make squares also appear in several protocols. This possibility is not excluded in the problem statement.

3.1.1 Pros and cons of the problem:

The question sentence seems to be demanding a yes or no answer and not how to do it. Students spend a lot of time evaluating whether or not 7 can be done and try to invent theories to support their decision (which is generally "no"). The accent is not on action here and even though the interviewer tells students to draw on the work sheet, they seem to see their main job as a contemplative one.

The 9 request produces to some extent a blockage in the search for 7 which grows out of the 4 division. It does however provide interesting protocol evidence as to how students deal with blocks and the degree of flexibility of each.

The problem has very definite advantages. It is accessible to both age groups. In fact one of the quickest solutions is that of a grade six student, Marc. The diversity of solutions and square associations are revealed as students deal with the notion of square in a way they have never done before. Students are generally highly motivated, succeed with at least part of the problem, and express their enjoyment at the end. Sylvie, whose protocol goes on for 17 pages, when she returns to the problem after her crypto-arithmetic effort says this about it:

> "Bon, ben je vais revenir à mon 9 carrés. Lui il m'intrigue. (Oui?) Ah oui, ça m'intrigue ça. Je va revenir à lui là. Bon que lui là, faut que je l'aie, eh."[1]

---

[1]Much use is made of this problem in research at the IREM in Montpellier. There the problem is presented in a group situation as a more open ended problem:

On a une feuille de papier de forme carrée. Peut-on la découper en neuf morceaux carrés? Peut-on la découper en dix morceaux carrés? Peut-on la découper en onze morceaux carrés? De combien de façons peut-on la découper en douze morceaux carrés?

3.2 Solution attempts:

With one exception all students succeed in the 9 division. The majority see it as a three row-column division or as some put it, "tic-tac-toe". A few build it up square by square or overshoot in terms of rows and columns and scratch out the excess. However, about half the students draw the 9 division right after reading the problem statement or within a few seconds after. Ten of the twenty-two students were able to do the 7 division.

3.2.1 Solution behavior:

Supporters of the associationist theory of thought would delight in the reactions of students to the square cutting problem. The word "square" seems to trigger a series of associations in students' minds which both help and, more often, hinder progress in this problem. It is through a discussion of these very potent associations that the student behavior in this problem can best be summarized.

<u>I</u> square-equal

This is one of the strongest associations and by far the most frequent in these protocols. Students associate some form of equality with squares which although it should be that sides of squares are of equal length, appears here more often as a notion that squares must be of equal size.

This is so strong that even after many interjections on the part of the interviewer to say that squares need not be of equal size or that "you may have big squares and little squares", some students persist in their search for a seven division with squares of equal size. This has, of course, been reinforced by the nine division where this kind of equality exists. Have cube models used as teaching aides in the classroom reinforced this notion also?

• When L. points out to Edes that she has been trying to draw squares of equal size and asks whether the question requires this, Edes replies that "a square is a square" and goes on with her search for equal sized squares.

L: Oui, il faut que . . . il faut que ça prend tout le carré mais, uh, . . . Mais t'as fait tous les carrés de la même grandeur. Est-ce que la question exige ça?

E: Ben, un carré c'est un carré.

• François never escaped from the equal-square association. When asked to draw the 9 division he asks: "Mais faut-il soient des parties égales?" He goes on to the end of the protocol trying to construct divisions with squares of equal size. His only interpretation of L's insistence on unequal sizes is to produce for the seven division a drawing which shows a small square tacked on to the original outline in which he had just attempted a six-division. 

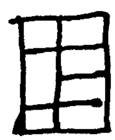
For Lucie (C) a real breakthrough occurs when she finally understands that squares need not be the same size. It is L's exaggerated drawing of a small and large square within the outline that really unblocks her.

L: En 7 carrés.

C: Egaux?

L: Non, il peut avoir des . . . différentes grandeurs.

C: ______ Comme ça?

draws 

L: Est-ce que ce sont tous des carrés, là?

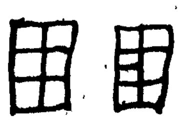
(Non) Pourquoi?

C: ___ ___

L: Qu'est-ce qui va pas?

C: Ben, il y en a plus de ce côté là que celui là.

(points to right then left sides of drawing)

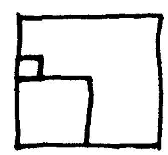
Later in the protocol she comes back to this type of situation. She has drawn what she believes is a 6 division and then adds a line  and says "Je peux en faire 7 mais ils seraient pas égaux, égaux." Once again

L. points out that squares don't have to be the same size. Later L. confronts her.

L: Tu peux avoir des petits carrés et des gros carrés. Ca, à date, tu essaies de les faire à peu près la même grandeur, tu sais.

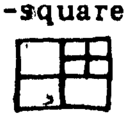
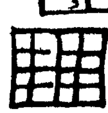
C: ___ ___ Je comprends pas.

L: Bon, ce que je veux dire . . . Tu comprends pas le gros pis le petit là? . . Ca veut dire, par exemple, si t'as un carré ici, tu peux avoir peut-être un gros carré là-dedans pis peut-être un petit aussi longtemps qu'ils soient des carrés.



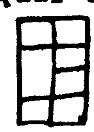
C: O . . oh!

Here C. begins a series of attempts which finally lead to the solution.

Michel who gets the seven division asks. "Veux-tu que je fasse les autres comme lui?" Here he points to the subdivision in one of the squares and is asking if he should subdivide the other three squares. L. asks if it is because he thinks that all squares should be equal and he replies "Yes". Here the equal-squares notion makes this seven solution unacceptable  and makes him want to continue subdividing  in order to have equal sized squares.

II square-even

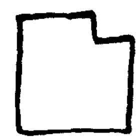
Almost as persistant as the equal squares association and to some extent linked to it, is the notion that one can only cut a square into an even number of pieces. This is a particularly astounding association considering that the majority of students have just solved the 9 division when they begin to elaborate their even number theory.

- For Lucie her even number constraint grows out of her equal-squares idea. She has explained that her division  is incorrect because there are more squares on one side than the other. Here she is thinking more of the fact that the pieces are not of equal size. She declares that 7 doesn't work. When L asks why, she says: "Ben, 7 c'est un nombre impair, pis uh, ben tu peux pas rentrer dedans, ça marche pas. Je sais pas." Later she explains that there is a square missing in her division: "Non, parce qu'il y en manque un là. Mais parce que, regarde. Il y a un carré plus, uh, parce que s'il y en a 4 là, il faut qu'il y en a 4 là."

- François has the most elaborate even number theory. So convinced is he that a square can only be divided into an even number of pieces, that he never gets the 9 division. He tries to come at 9 through an 18 division which he would later divide in two. He suggests that "toutes" les nombres pairs ils y rentrera dedans" and produces a four division and what he thinks is a 6 division. L. asks him for 16 which he gets after much effort and yet he still cannot do 9. Asked if 7 would be easier he replies it wouldn't.

F: Ça prend tout le temps des nombres pairs pour faire ça. Parce que si . . . si on prend des nombres impairs c'est sûr ça arrivera pas. Ça va faire comme un . . . un carré mais il va manquer une partie.

Draws



Louise's protocol provides an interesting modification of the even number theory. In response to a request to draw a 25 division she responds that it cannot be done because it is not divisible by two. But then she goes on to say the number must also be divisible by 4 because there are four sides to a square.

L.L: 25? Non. Ça divise pas . . . par deux.
L. : Divise pas par . . .?
L.L: Ben, il y a 4 côtés, faut que ça soit au moins divisible . . . par 4. On a 4 côtés . . . Faut que ça soit divisible par 4.

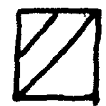
III square-diagonal

Another curious association that shows up in a number of protocols is that of a diagonal in the square. Nothing in the question elicits the presence of the diagonal. Often students draw it as a first gesture in their square outline and then proceed to ignore it when making their divisions. Of Daniel B's eight division efforts on his work sheet, four contain a single diagonal from the lower left to upper right corners, and one, a four division, has two diagonals. In each case these were drawn in immediately after drawing the square outline. His 9 division and a 7 attempt based on the 9 division contain no diagonals. Nor does his final solution. When asked about this he doesn't know why he made diagonals and says that they really did not help. Some vague motivations are however expressed:

D: Ben. Je me ... parce que ... passer une diagonale ... mais je sais pas ... je me dis qu'avec un ....? avec la moitié ... un triangle ... avec un triangle ... avec un, je sais pas ... avec un autre ça me ... donner un carré dans un ... je sais pas pourquoi ... je me suis donné de faire ça ... une diagonale.

Diagonals cut squares in half and make triangles and triangles can be put together to make squares.

• Richard exploits the diagonal idea to produce triangles and squares. His division,  gives him 4 squares and 8 triangles which produce 2 more squares. After this, in a 4 division he includes a diagonal although he appears to have given up on his triangle idea.

• Martine begins her 7 attempt with diagonals, , which she then removes because "lines like that make triangles".

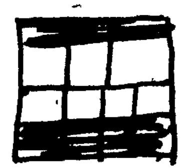
None of the grade six students draw diagonals which indicates that for them the diagonal has not yet become strongly associated with squares.

<u>IV</u> square-small rectangles

When challenged on whether certain divisions are really squares, a number of students reply that they look like rectangles but that if they were made small enough they would be squares.

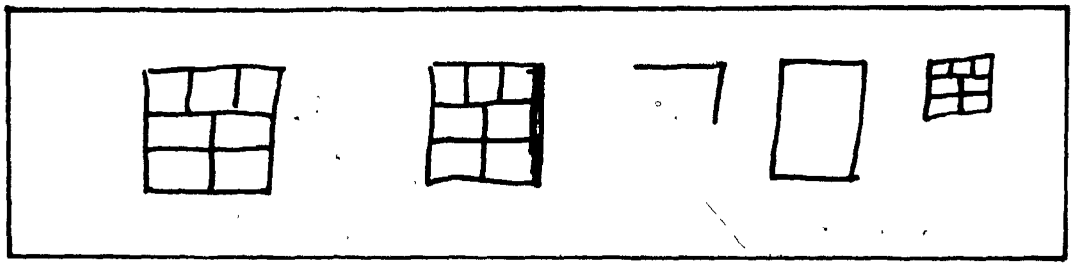
• France has drawn what she thinks is a successful 7 division:  When asked about this she replies that making the drawing smaller by chopping off the top and bottom would correct it. She then proceeds to redo the division in successively smaller outline squares. When asked if making a rectangle smaller turns it into a square she responds affirmatively.

L: Est-ce que c'est des carrés?
F: Il y en a qui ressemble un peu à des rectangles dans le bas.
L: Est-ce que tu peux l'arranger pour que ça soit des carrés partout?
F: Oui, il faudrait que je rapetisse mon carré. Le bas, là ... Comme ça. Pis du haut aussi un peu.



L: Est-ce qu'il y a une différence entre ça pis ça?
F: Oui. (Quoi?) Mais celle là c'est parce qu'on a rapetissé les carrés. Ca fait ...
L: Si tu rapetisses un rectangle, est-ce que ça va devenir un carré?
F: Oui. C'est si je fais ça comme ça ... ça devient ...

Nathalie L. also responds to a challenge over rectangles in her seven division by redrawing it very small:



François when told that all divisions must be squares immediately responds "all little squares" and then says it's lucky his original square is not too big, otherwise it would take him all day (presumably, to fill it in with little squares).

L: Il faut qu'ils soient des carrés.
F: Oh, tous des petits carrés! Ah! Whew! Sacre, whew! Une chance c'est pas un gros carré parce qu'on aurait pour la journée.

Chantal is the most articulate on the square-small rectangle association. In successive attempts to get an 8 division she makes her squares smaller and smaller and when asked about a particular division she reproduces it in increasingly smaller versions. Asked if her final minute version would be composed of squares if examined under a microscope she says: "No, but without a microscope they are."

L: Est-ce que ce sont tous des carrés? Hein?

C: _ _ Uh (laughs). Non.

L: Des rectangles?

C: Je vais faire un plus petit _ _

L: Ça, c'est des carrés maintenant?

C: Non _ _

L: Penses-tu que si tu pouvais le faire assez petit, ça deviendrait des carrés?

C: Oui.

L: Ouais? Veux-tu essayer encore?

C: _ _ _

L: Là, ils sont des carrés?

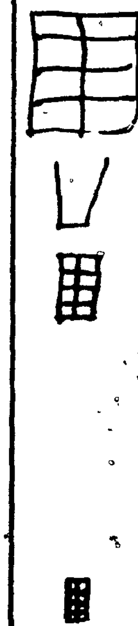
C: Oui.

L: Si je prends un microscope et je regarde ça est-ce que ça va être des carrés?

C: (laughs) Non, je pense pas.

L: Comme ça, ils sont des carrés?

C: Oui. (both laugh)



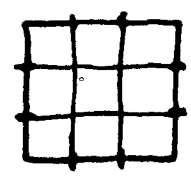
V square-dimensions

In contrast to those students who work with an idealized "perfect" square (which for some is the limiting case of a shrinking rectangle), a few students demand dimensions on their squares. They feel that not just any square can be subdivided and one suspects that different dimensions would imply different divisions for them.

Benoit, when asked to draw a 9 division, immediately begins working on a 3 cm by 3 cm square which he later explains is a perfect square.

L: Montre-moi neuf carrés.

B: Neuf? Ca donnerait un carré qui a 3 cm chaque côté. On le divise en 1 cm. On fait ... on trace une ligne à 1 cm de .... à chaque endroit. Ça fait 3 lignes sur ... 2 lignes ... Ça fait 3 carrés séparés sur un côté, pis 3 carrés sur l'autre. Si on les multiplie ensemble ça fait 9 carrés.



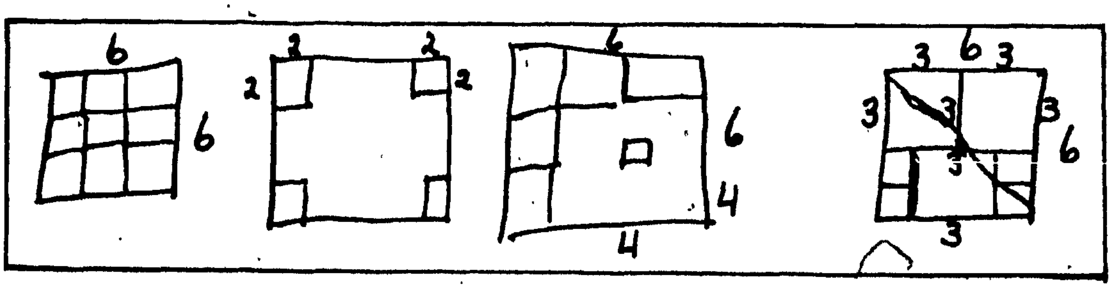
L: O.K. Est-ce qu'il faut que ça soit 3 cm?

B: C'est parce qu'il faut que ça soit ... faut que ça soit un cube parfait ... un carré parfait ... pour que ... ça puisse donner des carrés parfaits dans le milieu.

He then comes to realize that it is the 9 and not the dimensions of the square which has to be a perfect square for this division.

France, still trying to defend her seven division  says she would need a ruler to really draw it correctly. Since no ruler is available L. asks her to explain how she would do it with a ruler. She explains she would take a 2 cm square and make three squares of ½ cm each across the top. Across the bottom she would draw ¼ cm squares. She feels certain that by measuring it will turn out.

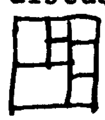
Daniel D. whose first reaction to the problem statement is: "Mais ils disent pas les dimensions du carré. Qu'est-ce qu'on fait dans ce temps-là?", works consistently with a 6 by 6 square. To get the 9 division he draws what he specifies are 2 by 2 squares one at a time. All drawings on his work sheet are accompanied by dimension notation.



His answer is expressed in dimensions as well: "Je pense que je l'ai, 6 par 6, je veux dire 3, 3, 3, 3, 3, 1½, 1½. C'est-tu ça? Ça marches-tu?"

VI square-precision

Many students when asked if they have produced squares will admit they haven't but that if the division were tidied up or done with more care, the rectangles would become squares. This is another form of the idealized square evident in IV. For some a real square is impossible to draw and the best that can be done is to try to get the rectangle as neat as is humanly possible in the situation in order that it will be as close to a square as possible. Some ask for a ruler in order to make lines as straight as possible. This is the case of France mentionned in V.

• In a long discussion with Martine over the validity of her 7 attempt, , she expresses again and again, the need to draw and measure it with a ruler in order that it be correct. She eventually repeats it with a ruler measuring off a 4 cm by 4 cm square (unfortunately here).

• Nathalie V. having produced a 4 column 3 row division indicates that it is very hard to draw squares:

"Parce que ceux-là dans le milieu - ils sont presque tout - mais ceux-là ils ressemblent à des rectangles. C'est dur d'avoir tous des carrés."

Later after a successful 25 division she explains:

"Ça c'est mes lignes. J'essaie tout le temps droit

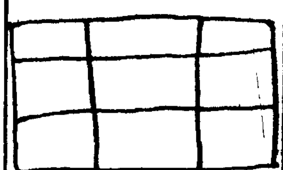
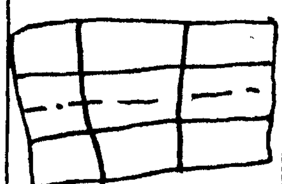
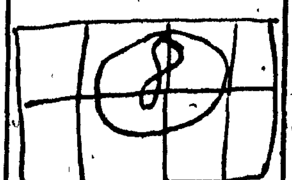
... fait que ça fait, ça fait, ça fait pas des càrrés. Si j'aurais une règle j'y fais, j'y fais droit."

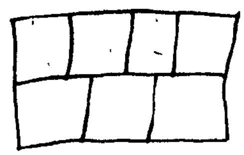
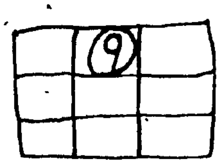
In both cases there is nothing wrong for Nathalie that good straight lines would not fix.

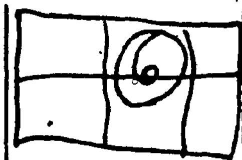
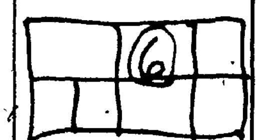
VII square-perimeter, area

Benoit is the only student who expresses the area association. On reading the problem statement he writes $C^2 = A^1$ and says "La formule d'un carré c'est – la longueur d'un des côtés au carré." Daniel D. introduces the notion of perimeter for his 6 by 6 square saying: "OK. j'ai un carré de – 6 par 6 ce qui donne un périmètre de 4 fois 6, 36." He does not, however, use the notion of perimeter in any way. These associations are rare but have been included here because the spontaneity of their appearance was similar to that of the other associations mentionned above. They are the weakest not only because of their rarety but because their influence is very short-lived.

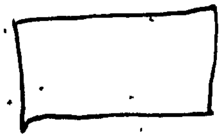
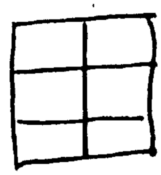
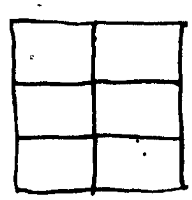
In order to demonstrate some of these associations within the context of a complete protocol, I have chosen one of the shorter interviews - that of Pierre G. - to examine.

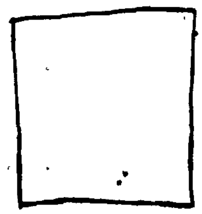
TEXT	WORK SHEET	COMMENTS
P: (reads problem) 9 carrés? 1, 2, 3 ... 1, 2, 3 ... (rereads first sentence). Un vrai, vrai carré là ... hum ...		P. seems to have the 9 division in his head but doesn't realize he should draw it.
L: Oh! mais tu vas faire un dessin ...		L. asks him to draw.
P: OK. Mais comme ça, ça fait 9. (laughs)		P. has 9 division and his laugh indicates it is a trivial question.
L: C'est ça.		
P: Sept ... Oh! ... (laughs). Uh, ah, sept? Ah! OK . Si je fais deux de même ... — Non ça fait 9 pareil. Uh, hum ... une seconde là ... hum ... sept ... Oh! des	 	7 division is obviously not so trivial for P. He intends to try something new and ends up with 9 division. His 8 attempt is probably what he intended.

	carrés ... Oh! — C'est pas sept carrés ... pour qu'il soit un carré ... supposer être égales? Sont pas obligés d'être égales les carrés? (Non) . Bon, je fais 7 carrés n'importe comment? Ben, rien qu'à faire des carrés, uh — — qui donnent 1 ... 9. Je vais avoir trop. 1, 2, 3, 4, 5, 6, 7 ... Non? Faut-il qu'ils soient égales? Oui?		(Two lines makes 3, not 2 divisions) "square-equal" I association is expressed here.
L:	Non, non il faut pas ... mais il faut qu'ils soient des carrés. (Oui)		"square-even" II is introduced here to explain impossibility of 7 division.
P:	Oh! Dans un carré, 7 carrés ... Ca se		Here he reviews his work:

peut pas ... (laughs) ... parce que c'est un chiffre impair.		9 can be done and he redoes it
Ben, ça se peut pas. Je sais pas là ... De même, de même ... Ça fait 9 ... Ça fait 8.		8 can be done and goes over his 8 division
Six ... uh, j'ai 6, 8, 9 (laughs). Il me manque juste 7 _ _ _.		6 and he draws what he feels is a 6 division
Je sais plus là, là. Dans ce sens là ... Ah! ben oui comme ça. _		P. subdivides a rectangle in the 6 division to get 7.
L: Est-ce que ce sont des carrés?		
P: (laughs) Non. Ben, ah! ... on pourrait tout le temps s'arranger pour que ça fasse des carrés.		P. says we can always arrange the drawing so that rectangles become squares. This seems to be a "square-precision" <u>VI</u> association.
L: Tu es sûr?		

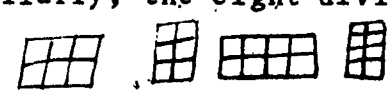
P:	Ben, oui, mais ils sont pas égales. Je veux dire, ils sont ...	What troubles P. more about his 7 division is that the squares are not equal: "square-equal" I
L:	-- Ça, ça fait rien mais ... moi je m'en doute que ce sont des carrés.	
P:	Mais, pour moi à sept en tout cas on sera jamais capable là ... pour que ça ... que ça fasse 7 carrés, des vrais carrés. Je me doute. Je sais pas là mais, hum, ...	P. is convinced 7 can't be done. It appears he has rejected his last attempt on the bases of I.
L:	Pourquoi tu dessines toujours un rectangle. Ça, c'est pas très carré pour commencer. Ça peut causer des problèmes.	L. points out that his outline square is very rectangular and that this won't help.

P:	Supposé être carré ça? Bon, je l'aurais pas plus anyway. 1, 2, 3, 4, 5, 6. C'est 6, 8.		P. says the outline doesn't really matter since he won't get 7 anyway.
L:	Montre-moi 6.		Because he says 6 can be done and it fits in with his II association, L. requests it.
P:	6 je l'ai, uh. 6 c'est pas dur. 1, 2, 3, 4, 5, 6. 6 là.		P. says it's easy.
L:	Ca a l'air des rectangles pour moi.		L. says it looks like rectangles so P. redraws more carefully (VI association). He realizes himself that they still look like rectangles but he doesn't know why.
P:	(laughs) - Là, es-tu mieux là? ... Oui ça, a l'air des rectangles ça encore. (laughs) Je sais pas là.		
L:	Veux-tu changer de page?		

P: Aye yaie yaie ... OK. Carrés --- Y: Faut que ... que tout le carré soit utilisé, oui? (hum huh) Hum huh, OK. — Icitte là ... 7 ça arrivera pas jamais en tout cas. Je pense pas d'être capable de faire 7.		Here he changes work sheet, draws a square outline which he stares at for quite a while and decides 7 can't be done.

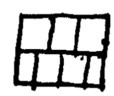
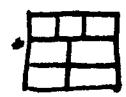
Pierre exhibits a number of behavior patterns here which are found across the protocols and which to a large extent are the result of the strong associations mentionned here.

the six division which is generally felt to be successful (it looks like squares, 6 is an even number, squares are of equal size) and similarly, the eight division.



From the 6 division students subdivide one of the rectangles to make 7 but are generally not pleased with this. (for the same reasons considered in acceptance of the 6 division).

- the modelling of many attempts on the successful 9 division and so starting each drawing with either three horizontal or vertical lines. In some cases students try to achieve 7 by somehow removing two squares from the successful 9 division. For example, Chantal draws a 9 division leaving out two squares which she says "have been eaten".

- 7 as a 4 plus 3 combination produces many drawings of the type  or  which are then distorted, adjusted, or reduced to make rectangles look like squares.

Pierre sees the question as: "Can it be done?" The meditative rather than active response to the question brought about by the formulation of the problem statement was mentionned earlier (3.1.1).
Pierre complies to the interviewer's request for paper work but never digresses from the search for a Yes or No answer to the question.

- He indicates 9 can be done but sees no need to draw it.
- "Ça se peut pas ... parce que c'est un chiffre impair."* He responds that the answer to the

7 question is "No" and then adds a supportive theory for his answer.

- "Ben, ça se peut pas". Once again he expresses that 7 cannot be done.
- "Ah, ben, oui comme ça". Here he thinks it can be done.
- "Mais pour moi à sept en tout cas, on sera jamais capable. Pour que ça ... que ça fasse 7 carrés, des vrais carrés. Je me doute". Here his formulation of the response that 7 cannot be done is beginning to have a ring of finality to it.

He indicates that it doesn't really matter what sort of outline he draws on the paper because 7 cannot be done anyway.

- "7 ça arrivera pas jamais en tout cas. Je pense pas d'être capable de faire 7." The question "Can you cut it into 7 squares?" has been answered in the negative and Pierre goes on to the next problem.

It is, however, the force of three square associations that is the most determinant and remarkable element in Pierre's behavior here. A glance at the comments column in the protocol shows how the first association, "square-equal", persists throughout the interview. Each time Pierre indicates that squares must be equal he is told that they need not be. Yet right to the end he tries to achieve a division with squares of equal size. The second association, "square-even", appears as a first justification of why 7 cannot be done. And yet immediately afterwards he repeats his 9 division. The sixth association, "square-precision", appears first in defense of an unsuccessful 7 division attempt which P. feels could be "fixed up" so that the rectangles become squares. Later when L. questions his 6 division he tries to carefully redraw the division but realizes himself that his precision has not converted rectangles into squares.

Chapter IV: The crypto-arithmetic problem

"Dans le problème suivant A, B, et C sont des chiffres différents. Trouver A, B, et C"

$$\begin{array}{r} AB \\ \times\ \underline{4} \\ CA \end{array}$$

"In the following problem, A, B, and C are different digits. Find A, B, and C."

$$\begin{array}{r} AB \\ \times\ \underline{4} \\ CA \end{array}$$

(English version)

4.1 Problem presentation

The third and final problem involving crypto-arithmetic was presented, as were the first two, on a small card with instructions to read it aloud. Although there are no difficult words or sentence structures, some students experienced difficulty reading the problem. Michel reads "A virgule, B virgule" for A,B,. The major difficulty appears however in reading $\begin{array}{r} AB \\ \underline{X\ 4} \\ CA \end{array}$.

Maryse and France have to be told to read it. Edes asks "C'est quoi ça?" as she points to the multiplication. Richard asks "Ça, AB fois 4, là, c'est égale ... égale CA c'est — — —. Qu'est-ce que ça veut dire ça CA? C'est le problème ça, quoi?" Benoit reads the problem with great difficulty and reads AB as "A fois B". This reading of AB as a product persists in spite of several correction attempts.

Difficulties in reading do not appear to be syntactical. They are probably due to the newness of the problem type: - students do not appear to have seen a crypto-arithmetic problem of any variety before. For the cegep students there are the added difficulties of associating the letters with algebra (where AB is a product) and of a certain distance from simple multiplication problems.

Pros and cons of the problem situation.

It is the newness of the problem situation (crypto-arithmetic) that really takes the spotlight here. Student behavior in this particular problem would have been very different had students already worked through or at least seen an example of problems of this type before. And so it is not the student's reaction to "AB times 4 equals CA" as much as his reaction to an entirely new problem situation that is of importance here. How does a student behave when confronted with crypto-arithmetic for the first time?

This unexpected bonus factor in the choice of this particular problem can certainly be seen as a positive element which far outweighs the problems of reading and understanding occasioned by the problem presentation. It would be difficult to imagine any other way of presenting the problem which would be clearer and not take away from the richness of the portrait of student behavior in a new situation. All the necessary information is given in the problem statement. To add more would be to take away from the number and variety of elements the student must bring to bear on the problem.

4.1.1 Solution behavior:

Before examining the three major methods (or strategies) used by students in this problem, it is necessary to examine the elements of understanding involved in the problem and, to a large degree, in even being able to arrive at a strategy.

U1 "chiffres"

"chiffres" means digits, that is {0, 1, 2, ... 9} . It must be understood that A, B, and C are single digits and not just any numbers at all. Without this the student cannot see AB as a two digit number. France, when blocked over the value of B (which she feels has to be smaller than 1), takes B as a two digit number and tries

B = 12: $\begin{array}{r} A\ 12 \\ 4 \\ \hline 8 \end{array}$ and B = 10: $\begin{array}{r} A\ 10 \\ 4 \\ \hline 0 \end{array}$

U2 "différents"

The student is told that A, B, and C are different digits. That is A≠B, B≠C, and A≠C. Futile efforts are made because students have either not noticed or understood this piece of information. Students who are trying to produce some symmetry in order that the A's will come out equal will often try AB = 11 or AB = 22 (see, for example, Edes, Sylvie, Mario).

U3 linking "A, B, et C" and $\begin{array}{r} AB \\ X\ 4 \\ \hline CA \end{array}$

Here various degrees of understanding are involved. When Louise, for instance, sees absolutely no connection between the written sentences and the multiplication (which she ignores), she demonstrates that simply putting the multiplication $\left(\begin{array}{r}AB\\ \underline{X4}\\ CA\end{array}\right)$ under the problem statement does not suffice in all cases to tell the students that these are the same A, B, and C as given in the problem statement and that they have certain restrictions on them as defined by the multiplication. As mentionned in the previous section, a number of other students do not make this connection and wonder what the multiplication is doing on the question card.

At another level, students must see A, B, and C as a code. They have never multiplied letters and so must interpret the problem as saying that there are three unknown digits which when arranged and multiplied by 4 in a certain way give a certain arrangement of these digits. There are varying degrees of freedom students feel with respect to the A, B, C link. Some can jump right into the solution attempts writing only numbers whereas others, like Nathalie L., persist throughout in writing first $\begin{array}{r}AB\\ \underline{X4}\\ CA\end{array}$ and then numbers beside the letters as if they are afraid to lose track of which is A, B, or C. Sylvie writes her numbers and then AB and CA above and below respectively

(from Sylvie's work sheet)			(from Nathalie's work sheet)		
AB 32 x 4 128 CA	AB 20 x4 80	AB 32 x3 96 CA	4AB1 x4 164	0AB1 x4 CA	4AB4 x4 164

(from Sylvie's work sheet) (from Nathalie's work sheet)

The student who understands both that A, B, and C are a code and their link to the multiplication, is not always able to respond to the question: "What is A? What is B? and What is C?" once he has performed the multiplication. This involves pulling the values out of the multiplication arrangement. For example, Nathalie L. who, as mentionned above, writes her numbers beside the letters, when asked after her "AB = 41" attempt to identify A, B, and C cannot do it.

N: — — D'abord je ferais icitte, ben — je ferais 4 fois, que ça serait ici en place 1, là. 4 fois 1, 4. Pis uh — 4 fois 4, 16. L: Um huh. Ca ferait 41 fois 4 égale ...? N: 164	4AB1 x4 164

L:	À ce moment là, A c'est égale à quoi?	
N:	À lui.	points to upper 4
L:	À 4?	
N:	Pis à lui.	points to lower 4
L:	Et B est égale à ...?	
N:	À lui.	points to 16
L:	B. B. Regardez la question.	
N:	— — —	
L:	T'as pris B égale quoi ici dans ta question? T'as placé B égale à ... ?	L. points to her mult'n effort
N:	Um	
L:	Um huh. Et C serait égale à ... ?	
N:	À 6.	

Lucie can identify A and B correctly but has trouble with C. Since Lucie (C.) seems fairly certain of her $\begin{array}{r} 21 \\ \times\ 4 \\ \hline 84 \end{array}$ solution, L. questions her about it.

L:	... si c'est ça ce serait quoi A, ce serait quoi B, ce serait quoi C?
C:	Ben, A ce serait 2, B ce serait 1, pis C ce serait la réponse?
L:	Mais regarde, il y a CA. Alors C est égale à quoi?
C:	84?

$\boxed{U4}$ AB is a two digit number

It is one thing to understand that A and B are single digits and another to see that AB is a two digit number. For cegep students who understand AB as "A times B" this is not obvious. Most students, once the non-algebraic nature of the problem is cleared up, seem to understand this. No one chooses more than two digit numbers for AB. Some, however, choose a single digit but often this is based on the idea that A can be zero. (For example, $\begin{array}{r} 05 \\ \underline{X\ 4} \\ 20 \end{array}$)

Sylvie who fixes A = B = 4 and should therefore multiply 44 by 4 instead writes $\begin{array}{r} 16 \\ \underline{X\ 4} \\ 64 \end{array}$ because for her, if A = 4 and B = 4, then AB = 4 X 4 or 16.

$\boxed{U5}$ CA is a two digit number

Students who consistently use two digit numbers for AB do not always balk at three digit numbers for CA. Many students believe that they have the answer when they try AB = 41 which gives CA = 164. (See Richard, Pierre G., Mario, and Daniel D.) In many cases the interviewer has to point out that CA must be a two digit number or that C must be a single digit. Richard's protocol is typical.

R:	Ouai, ouai, OK — (1 min. 10 sec.) — 41 fois 4 ... je sais pas si c'est ça ... CA —	$\begin{array}{r}41\\ \underline{4}\\ 164\end{array}$
L:	41 fois 4?	
R:	4 fois B, ben 4 fois 1, ça donne 4, ça a la même valeur que A. Pis là, C. 4 fois, 4, 16. Y ...	
L:	C sera quoi? (Pardon?) C sera égale à quoi?	
R:	C serait égale à 16.	

A full understanding of CA as a two digit number would lead to the conclusion that $CA < 100$ and possibly that $AB < 25$ and therefore that $A \leq 2$.

$\boxed{U6}$ the two A's represent the same digit

The appearance of two A's in $\begin{array}{r}AB\\ \underline{X\ 4}\\ CA\end{array}$ does not always lead to the conclusion that the digits in the A positions must be the same. Some students simply do not notice the two A's and treat the problem as if it were: $\begin{array}{r}AB\\ \underline{X\ 4}\\ CD\end{array}$. Others have obviously noticed the A's but drawn no conclusion about equality. A few seem to think A can represent two different numbers. Michel, who is quite happy with his $\begin{array}{r}12\\ \underline{X\ 4}\\ 48\end{array}$ multiplication because he has two digits for AB and two for CA, appears at first not to have noticed the two A's but later indicates that A=1 above and A=8 below does not bother him.

L:	Est-ce que ça marche ça?
M:	— Oui .... Il y a deux chiffres en haut, il y a deux en bas.
L:	Um huh, A serait quoi dans ce cas-là?
M:	A ce serait 1. B serait 2.
L:	C ?
M:	4
L:	Et le A en bas?
M:	Uh, 8.

After an explanation that A must be the same in both places he asks with a tone of surprise "Faut que ça soit la même affaire?" Very shortly after this clarification he gets the solution.

U7 "carrying" in multiplication.

Although all students are able to multiply a two digit number by four, when confronted with this problem many, if not all, seem to temporarily forget about the possibility of carrying in multiplication. They conclude that 4B=A and tend to limit choices of B to 1 and 2 for that reason. Sometimes they are unblocked by an example that they or the interviewer produce where carrying is necessary.

(For example AB =36). Bringing to mind the "carrying" possibility is essential for the resolution of this problem.

Jean-François concludes from the problem statement that 4B equals A and all his solution attempts until the very last involve a 1 or 2 in the B position and hence no carrying.

J:	Oui, c'est ça. Il y a A qui peut aider, mais l'autre — 4 fois B ça donnera A. — C'est-à-dire que A est plus fort que B.
L:	A c'est quoi?
J:	A est plus fort que B pour que 4 fois B ça donne A. Ben non, c'est vrai ...

France experiences a similar block. She realizes A<4 and therefore since 4B =A concludes that B must be less than 1 and therefore that the problem cannot be done.

Pierre G., having tried B=1 and B=2, and feeling for the same reasons as France that B cannot be any bigger, momentarily envisages making B a negative number. He is so stuck on this A=4B notion that he is ready to quit. At this point L. suggests that he multiply 26 by 3 and before he has done the multiplication P. realizes that the "carrying" possibility is what he needed. He then jumps into the problem with great zest and eventually gets the answer.

As mentionned above, all students know how to multiply a two digit number by 4. There are however, a surprising number of multiplication errors made particularly by cegep students. They seem to be of two types. First, there are errors due to weakness in *their* multiplication tables. For instance, France multiplies 4 by 3 saying and writing that 4 times 3 is 16. The second type of error involves various forms of confusion in multiplying such as adding rather than multyplying and inversing the digit to be written and the digit to be carried. Pierre G. multiplies 28 by 4 and gets 92 and naturally believes he has the solution. He seems to have multiplied 4 by 8, put down his 2 and carried the 3 correctly. Then he multiplies 4 times 2 and gets 6 (adding) plus the carried 3 which gives him 9. In a subsequent multiplication of 26 by 4 he once again gets 92. This time he multiplies 4 times 6 and either gets 24 and puts down the 2 instead of 4 or he has multiplied 4 by 6 to get 12. When asked to correct his error he gets 26 times 4 equal to 94. He has corrected the 4 but not the 9. It is only on the third attempt that he corrects it to 104. Edes almost misses the answer when she multiplies 23 times 4 to get 112. She appears to have carried a 3 rather than a 1. Sylvie multiplies 56 by 4 to get 220. She may have multiplied the 4 by 5 twice ignoring the 6. Mario multiplies 23 by 4 and gets 52.

Not all of the above mentioned elements of understanding need be present for the student to successfully solve the problem. Nor does the student need to be capable of flawless multiplication although errors do hinder progress. The three basic strategies used in this problem put different accent on those elements of understanding that are required. The choice of a strategy is generally based on the elements of understanding the student possesses. All three solution types (or strategies) are a form of trial and adjustment. In descending order of efficiency for this problem they are:

<u>I</u> Fixing A and finding B
<u>II</u> Fixing B and finding A
<u>III</u> Fixing AB and checking for CA.

I <u>Fixing A and finding B</u>

With the exception of the second point of understanding, namely that A, B, and C are different, all other elements are necessary for this solution approach. Concluding from the fact that CA is a two digit number that AB is a number less than 25 and therefore that A is either 1 or 2, makes this a particularly efficient method. Sometimes the understanding about "carrying" arrives later and students block on the 4B=1 or 2 problem. Once unblocked, the problem becomes one of finding a number which when multiplied by 4 gives 1 (which is quickly seen as impossible) or 2 (for which there are only two possibilities, namely, 3 and 8).

Martine's protocol shows how efficient this method can be. She focuses on the lower A and decides it is probably a 2 since 4 times many numbers end in a two. She then chooses B=3 and has solved the problem.

M:	(reads problem aloud) Faut que je trouve A, B, et C en termes des chiffres.	
L:	Um huh. C'est des chiffres de 1 à 9.	
M:	1à 9. . . Mais en premier est-ce que je fais la multiplication? Ben non, elle est faite la multiplication. Faut que je remplace par des chiffres d'abord. — Tu veux dire que ces deux A, A ça va être le même chiffre. Uh, — bon — — (whispers) 4 fois — 40 — — — — deux. Parce qu'il y a beaucoup de multiplications par 4 qui finissent par un deux. Pour que A sera le même chiffre — — — A égalerait 2, B trois, C neuf.	23 4 92 A=2 B=3 C=9

Asked if other values are possible, she chooses for B the other value which would produce a 2 in the lower A position namely, 8. But on checking this out concludes that her solution is unique.

Mario also produces a very rapid solution once he has realized that CA is only two digits. He rejects A=1, tries A=2 and very quickly concludes AB=23.

Daniel makes slower progress with Method I. Focusing on A he tries A=4 which he rejects because C is too big. Next he tries A=3 and realizes C is still too big. He tries A=2 briefly but also decides B should be 2. Here he loses track of the method and moves in to what might best be classified as method III.

It is when Jean-François switches to a method I approach that he solves the problem. He has realized that A cannot be bigger than 2 but he believed that A≠2 because 4B will not give 2. Suddenly he realizes (without ever doing a carrying example) that B=3 will do the trick.

II. Fixing B and finding A

Here the student tries different values for B, multiplies by 4 to get the lower A and then places the same digit in the upper A position, completes the multiplication and checks that C is a single digit.

A common first choice for B is 1.

The student proceeds as follows:

$\begin{array}{r} 1 \\ \underline{4} \\ 4 \end{array}$ then places 4 in upper A position $\begin{array}{r} 41 \\ \underline{4} \\ 4 \end{array}$

and completes the multiplication $\begin{array}{r} 41 \\ \underline{4} \\ 164 \end{array}$

Here he may or may not conclude that C is too big. In some cases he has to be told. The next natural choice for B would be B=2. Here some students hesitate because B=1 produced a CA that was too big and so fear that values of B cannot be any bigger than 1. And, of course, if they try B=2 do get CA even bigger than when B=1. Pierre G. having tried B=1 and B=2 wonders if B can be negative which is the only way to make it smaller. Once unblocked over the carrying possibility he tries B=8. There follows a sequence of type III attempts before he returns to method II and tries B=5, B=4, and finally B=3. Edes indicates quite a breakthrough when she suddenly embarks on a type II solution with B=9 and then B=4 and B=3. She misses the solution because of a multiplication error but she sticks to her method trying B=6, 5, 8. She rereads the question, rechecks B=8 then tries B=7 and B=6. Once again she makes a multiplication error in verifying B=3 but finally coming at it again she settles on B=3 and gets the solution. France tries B=1 and concludes that the problem cannot be done because B would have to be less than 1. She tries to change the rules placing B=12 and then 10. Then B=6, B=8, and finally B=3. She too almost misses the solution because of a multiplication error when B=3. In general, the use of Method II is not as efficient as it might be due to the two obstacles evident in these examples namely, multiplication errors and a carrying block which leads students to conclude that B is less than 1.

III Fixing AB and checking for CA

This approach is feasible if the student realizes or comes to realize through experience that AB is less than 25. If the student chooses AB≥25 he gets a three digit number for CA. Some students have to be told that CA is a two digit number and not all conclude that AB is therefore less than 25. More often, they conclude that AB should be smaller than their previous choice. Very often students feel that both A and B cannot be bigger than 2 because 4A and 4B must be less than 9. Not all possible values for AB are tried by students. Many try 11, 21, 12, 22, 20, 10 because of the A<2 B<2 block. Hélène tries AB=19 just after L. has said that A and B are digits between 1 and 9. Sylvie having tried AB=20, 18, 56, 30 and 22 concludes that if AB=30 then CA is too big and if AB=22 then CA is too small and therefore AB should be between 22 and 30 and closer to 22. She then tries AB=21, 22, and 23. Daniel D. tries to use the divisibility of CA by 4 to fix his choices of AB. He wrestles with the inverse expression of AB X 4 =CA trying AB ÷4 =CA, CA X 4 =AB, and 4 ÷ AB =CA. Louise uses this inverse process successfully. Having tried A, B, C equal to 3, 6, 9 she realizes that CA should be 93. Dividing 93 by 4 she gets 23 with 1 remainder. Her logic is that to get 93 for CA she should have used AB=24 or AB=23.

L.P.	Ah, faut j'essaie de réaliser ça. (Um huh). AB, ça fait ... ça fait 36, multiplié par 4 ... ça fait 4 fois 6, 24. 4 fois 3, 12 et 2, 14. CA. C, qu'est-ce que ça veut dire ça, c'est ça que je comprends ... Ah oui, CA. Ah, faut que ça donne, ah bon, là, je comprends. Je viens de comprendre là. Faut que A et B multiplié par 4 donne le chiffre C et A. (Um huh) C'est ça? Ah, bon, bon, bon, bon. Là, je suis ... là je viens de comprendre. (laughs) Ça prend du temps. Je suis dure de comprenure. Alors si je mets A et B multiplié par 4 ... là ça marche peu. Faudrait que ça donne un 9 icitte (Um huh). Pis faudrait que ça donne un 3 là. Ben, à ce moment-là si je fais ... si je mets	36 x4 144 Writes 93 over 144: 36 x4 193 | 4 8 | 23 13

un 9 pis un 3 ici ... faudrait je diviserais par 4 ... ça me donnerait mon chiffre. — 2 fois pour 8, 3 fois pour 12. Non ça ne marche pas. Il reste des ... des retenus. Ça marche pas. — Uh, j'essaierais deux ... un ... deux — fait 6, 24, non ça marche pas ça.	

Louise has not only fixed A and B but also C in her 3, 6, 9 attempt. A more frequent choice of A, B, C is 1, 2, 3. The rhyming of C and 3 seems to have something to do with this choice as does the coding nature of ABC for which the most natural number relation is 1, 2, 3. Pre-algebra students often think that this is what algebra is all about: —a replacing of the numbers by their corresponding letters in the alphabet. Marc, a grade six student suggests that A, B, C could be 1, 2, 3 and explains his position.

M:	Regarde c'est comme l'alphabet. La première lettre c'est 1. Je sais pas si ...	
L:	Mais. Essaie-le voir si ça marche. A égale 1, B égale?	Writes A=1 B=2 C=3
M:	Deux	
L:	Pis C égale?	

M:	Trois (laughs). Je sais pas si ça marche.	
L:	Bon, essayer de voir si ça marche.	$\begin{array}{r}AB\\ \times 4\\ \hline CA\end{array}$ $\begin{array}{r}12\\ \times 4\\ \hline 31\end{array}$
M:	— — —	

At this point M. suddenly switches to another effort. He writes $\begin{array}{r}10\\ \times\ 4\\ \hline 40\end{array}$ and explains that he chose the 40 for its sound. Here the CA was chosen first on the basis of the fact that it is pronounced like the first syllable in "quarante".

L:	Est-ce que ça marche, 10 fois 4, 40?	
M:	M'a essayer...?..fois 4. CA commence "ça" ... c'est pareil comme 40, CA.	
L:	Pourquoi c'est pareil comme quarante?	
M:	Ben, c'est ... c'est semblable. CA, quarante. Tu sais. CA ben "qua..." pis quarante. Je sais pas.	
L:	Tu veux dire?	
M:	CA là. Ça commence pareil comme 40.	
L:	Oh, le son!	
M:	Oui.	

Lucie and Michel also start out with A, B, C, 1, 2, 3. Nathalie L. is disappointed when she tries AB=12 and discovers that the C is 4 instead of 3. What is a little surprising is to find two cegep students who also try this 1, 2, 3 tactic. Pierre G. begins the problem: —"OK. faut que je donne des valeurs à A, à B, pis à C ... Commencer à 1, 2, 3". Edes first attempt involves A=1 and B=2.

Method III is the most commonly used method in this problem. It sometimes appears, as in the case of Pierre G., as a temporary losing track of the fixed variable in methods I or II. Pierre, for example, has just tried B=8 and hence AB=28 and realizing this is too big, cuts AB in half, and tries AB=14. He returns later to his Method II approach. More often Method III is supplanted by Method I or II as the student gains more understanding of the problem's constraints.

In order to portray the various understandings and methods at work in a single protocol, Mario's will be examined. It has been chosen for its brevity. The narrow right hand column will indicate the method being used and the particular understanding being demonstrated:

M:	(reads problem) — Ça, c'est-tu A fois B ou ...?		
L:	... Non, ça c'est ...		
M:	... ou mettons 47.		U4
L:	... un chiffre comme 36 ou 47.		

	C'est pas une multiplication.		
M:	On peut dire ... trouver ... bon,		
	prend n'importe quel chiffre,		U3
	multipliez par 4, pis ça donne		
	une réponse.		
L:	Mais, essaie-le.		
M:	— Moi, je dis 24 ... Ah, les		III
	deux A sont pareils. Ah, ah oui	24 4	U6
	— — — Non, ça marche pas —	96	U7
	(25 sec.), Ça, faut le faire		
	jusqu'à temps que ça arrive.		
	... 10 fois 4, 40. — C' ...?	10 4 20 4 22 4 88	III
	là. — (35 sec.) — Je l'ai.		
	41 fois 4. A, B, ... Icitte les	41 4	II
	deux A sont pareils. C'est le	164	
	même A.		
L:	C c'est égale à quoi?		
M:	Seize?		
L:	Oui, mais, c'est juste supposé		
	d'être un chiffre. Un seul chiffre.		U1
M:	Juste un chiffre?		U5
L:	OK.? Entre 1 et 9.		
M:	— Oh, ça marchera peu là si		
	j'enlève mon 4. Si je mets un onze	11	III
	— — — — Avec 1 ça marche pas.	4 L	
	2 — (45 sec.) ça me semble 22,	23	I
	23 fois 4.	4 52	

L: 23 fois 4.		
M: A, B, C.		
L: Veux-tu vérifier ta multiplication là.		

M. corrects his multiplication.

A discussion of his solution which he credits to "tatonnement" or trial and error follows. He explains why he rejected B=0. He says that he tried A=1, then 2 and would have gone on with 3, 4, 5 except that he had tried 80 and knew it was too big. He also alludes to the AB=41 effort which gave a CA which was too big.

Mario never demonstrates any realization of U2 (that A, B, and C are different digits).

Chapter V: Student behavior across the problems

Introduction

Chapter II examined the parking lot problem and the solution behavior of the twenty-four students. Chapters III and IV dealt with the square cutting and crypto-arithmetic problems respectively. The accent has been on the problem and the varied solution behavior it gives rise to.

Although the problem has a lot to do with what happens in solving, some problem behavior is maintained by the student across all three problems. This chapter will look, first of all, at individual behavior constants across the problems. This will be followed by a discussion of grade and sex differences in solution behavior.

5.1 Each student comes into the problem solving situation with his own personality, style of social interaction, and relationship to mathematics and problem solving. That relationship to mathematics and problems has a history of which this session will become a part.

5.1.1 State of stress:

Although the majority of students arrive at the interview in a fairly relaxed state and seem to be looking forward to the experience, or at least not dreading it, a few students appear stressed and apprehensive on arrival. Jean-François visibly shakes and before leaving explains: -"Je suis toujours un peu nerveux. Ça dépend ce que je fais aussi." Carole and Hélène, both cegep students, arrive looking perfectly miserable and never overcome their nervousness sufficiently to be able to perform well on the problems. Hélène is very conscious of this.

H:	(after 1 min, 20 sec. of silence) Faut que je me concentre. (follows another 50 seconds of silence).
L:	Peux-tu expliquer un peu ce que t'as pensé dans ta tête ... ? ... problème surtout — (1 min.) — Es-tu bloqué un peu?
H:	— (20 sec.) — Je suis un peu nerveuse.
L:	Oui? Pourquoi?
H:	J'ai peur.

For some students this initial tension is positive in the sense that they are able to channel it into a heightened performance. Richard, for instance, solves all three problems in record time whereas in the classroom situation he does nothing. However in the cases of Carole and Hélène the stress seems to paralyse them.

Somehow related to this initial state of tension, are the opening self-deragatory remarks made by several students, before they have even sat down in some cases. Edes comes through the door explaining how she cannot do problems and will need help. François announces "Moi je suis pas bon pour le vrai" as if to say that the others may have strung you a line about not being good in math., but I am the genuine disaster. Such confessions of weakness are often beneficial. Perhaps the student feels stressed about the level of performance he asks of himself and projects his expectations on the interviewer. By lowering the interviewer's expectations and hence his own, he can relax and get on with the problem solving.

5.1.2 Interaction with interviewer:

From the very beginning of the interview session each student establishes some way of relating to the interviewer. About half the students show very little interest in the interviewer. She is initially on about a par with the recording equipment and in many cases disappears altogether from consciousness once the student is immersed in the problems.

. Martine, for example, never asks for help and obviously doesn't want any interference from L. in spite of having great difficulty with the problems. She gets so involved in the problems themselves that she takes the unresolved square cutting one home and comes back with what she believes is a solution the following day. L's questions to Martine are mainly of the "What-are-you-doing-now?" type.

- Louise also gets very engrossed in her problems and her protocols are almost monologues. She arrived at the session wanting to talk about everything and anything except mathematics. She spoke of her achievements in handicrafts ("There's nothing I can't make") and offered addresses for bargains in wool and other supplies. Once confronted with the first problem however, she addresses it entirely and interviewer, macramé, and bargains are forgotten.

- Daniel elicits no help from L. other than the occasional "Am I on the right track?". L's questions are answered patiently but do not seem to influence his solution.

- Mario, who produces three very rapid solutions to the problems, says that if left alone he would not have solved one of these problems. In his own protocol writing he begins with a short reflection on his performance. (Note that the written errors are Mario's)

"Je me suis appercu que si j'avais été seul je n'aurait pas résolue les problèmes j'aurait abandone. Mais là j'était obligé de les resoudres mais dans ma tête j'avais aucun espoire au premier embuche." (sic)

And yet if the protocols are examined it can be seen that there is very little interaction between Mario and the interviewer. It is probable that if left alone with these problems Mario would have, as he said, "abandoned" them. It was not the interviewer in as much as the whole interview situation that forced Mario to persist.

The relationship of the grade six students to the interviewer is influenced by the fact that she is a stranger and by their need for explanations of the problem statements. Once the problems are clear and they have settled in to the interview situation, these students tend to more or less forget the interviewer and immerse themselves in the problems in about the same proportion as cegep students. Jean-Pierre ignores L. to the point that in order to get a response L. must repeat a question several times. Michel waits for a long time before responding to L's questions. Compared to the cegep students, grade six students are generally less communicative. The majority are however problem centered and relatively disinterested in the interviewer.

At the other extremity are those students for whom the interviewer is the central problem to be attended to. Her every sound and gesture is interpreted and used as guidance in the problem. These students check every step in their solution with L. usually through questions. Some never make a statement without raising their voice at the end and turning it into a question. Even when they arrive at a solution they are not convinced by verification or a sense of rightness about it but rather by a positive reaction on the part of the interviewer.

. Richard on solving the crypto-arithmetic problem asks three times if he has the right answer.

R:	Ça se pourrais-tu?
L:	Ouai.
R:	— Des grosses chances?
L:	Oui, ça marche là.
R:	Ça marche certain?

. Sylvie, during her second attempt at the square cutting problem produces an incorrect nine-division attempt, misinterprets L's reaction, and experiences all the elation of success.

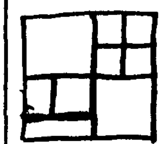
S: ... Bon ... comme ça ... voyons, quand on fait ça ... comme ça ... tudk! ... comme ça. Ça doit pas faire. Je suis persuadé ça marche pas. Pis çà, ça donne ... pas un carré. C'est ça, eh? C'est-tu ça?

L: C'est ça le problème, oui.

S: Bon! 1, 2, 3, 4, 5, 6, 7, 8, 9. Fallait y penser, eh? Je savais c'était une niaiserie de même. Bon. O.K., on a deux de réussies. Il manque mon espèce de petit dernier, là. Faut je l'aie, lui.

L: — Oh mais, ça tu ... non ... je me suis mal ... Ça ici, est-ce que c'est un carré?

S: Oh! ... Ça marche pas d'abord! (Non) Oh, O.K! Je pensais que ça marchait.



Sylvie recognizes, when questionned about this attempt, that the bottom left divisions are rectangles. External feedback takes precedence over her own sense that the division is incorrect.

It is Syvie who maintains the most intense interaction with the interviewer through a very elaborate and unique interview style. She establishes a form of Socratic dialogue with herself in which the interviewer becomes the audience. She then feels her way through the solution process by reading the interviewer's reactions to her drama. The following example is taken from her dialogue just preceding the above example.

S: ... Bon ici, bon, mon sept celle-là est trouvé. Mon neuf carrés là. Oh Jupiter, faut je le trouve. Neuf carrés. Ici, j'ai sept carrés. — Me semble, crime, me va essayer de .. ? ... d'abord. O.K. Comme ça je peux pas diviser sans enfin ... Ah, attend un petit peu. — Tu peux pas diviser comme ça. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. Ah non. Ça donne pas un carré. Ah je pensais ben je l'avais. Non ça donne un ... Attend un petit peu que je refais le carré parce que je me comprends peu ...

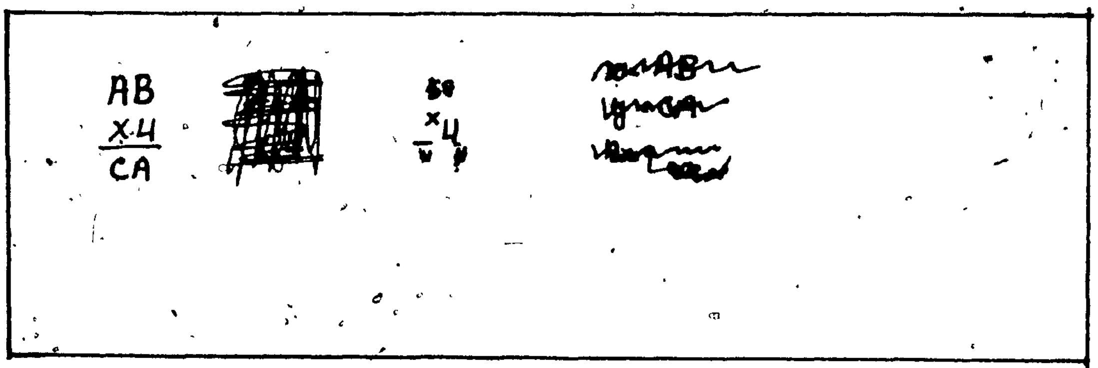
Another example of her unique style is drawn randomly from the parking problem.

S: ... Bon là mes 100 roues était correcte mais il y a une affaire, j'ai 40 véhicules. Bon c'est que je

> vais en rajouter ... Ah, ben non. Heh! ... Attends un petit peu là – Eh, ça peut pas. Ben ça me donne hien, trente véhicules. Il m'en manque dix pis j'ai mis 100 roues. — Attend un petit p... —— Attend un peu — 30 véhicules. Ça, ça doit être un attrape là que j'ai pas pantoute, uh? Je suppose que c'est rien qu'un petite affaire là? —— Mais je peux pas. Ecoute donc, t'as une automobile, ça a 4 roues, ...

Students are dispersed along the spectrum of low to high interaction with the interviewer. Low interaction students are problem centred and field independent. High interaction students are interviewer centred and field dependent. The majority of students are somewhere in between the two extremities. Some evolution from dependence to independence can be seen in a few interviews. The majority, however, maintain the same interactional style throughout.

The students who are most difficult to classify are those who never lose their initial nervousness in the interview situation. They react neither with the problem nor with the interviewer. Hélène for instance, is so nervous she can hardly think. Her tape is almost total silence and in the eight pages of protocol there are very few clues as to what she is doing. On her crypto work sheet she scratches out everything as she goes along. She is verbally uncommunicative and even hides her written work. She recognizes her problem and tries to overcome her nervousness but is unsuccessful. Hélène cannot be situated anywhere between problem centred and interviewer centred. She is dealing with neither the problem nor the interviewer but with her own extreme state of stress.



(contents of crypto work sheet)

Some students are interviewer centred until the first problem is given. They become problem centred until all three problems are completed and occasionally revert to an interest in the interviewer during the closing discussion. Marc, who obviously wants to impress L. at the beginning, gets very involved in the problems and only returns to L. once he has finished.

5.1.3 Solver openness:

Highly related to the degree of stress and the relationship established with the interviewer is the degree of openness the student maintains during the interview situation. This too evolves slightly in the direction of a greater openness as the student relaxes and feels a certain level of trust in the interviewer. The range of openness is however very wide. Hélène is typical of one end of the spectrum. She is closed to the entire situation and turned inside herself. Other students exhibit some of this closed behavior. Carole bites her fingernails and plays with her bangs during long periods of silence and when she speaks it is in whispers. Her ten pages of protocol contain little information other than "it-can't-be-done" which she expresses in one way or another eleven times. Jean-Pierre also hides his work, scratches it out, and writes minutely small. Others, like Michel, only write when they are told to and speak when spoken to. At the other extremity are students like Edes, Louise, François, and Pierre who fill their work sheets with large very visible calculations and whose protocols are rich in information about their progress through the problems.

5.1.4 Student self-image:

The protocols provide occasional glimpses of the student's self-image and the evolution of that image.

during the interview session. Although the majority who have a negative self-image with respect to mathematics tend to maintain it in spite of success on these problems, Edes shows how a negative self-image can evolve into something more positive as she succeeds in these problems.

Edes began the interview asking for help since she could not do written problems: "Des problèmes écrits, je peux pas." She said it was the fourth time she'd taken algebra and that she was always in the weak group. All this is said before she even sits down. Once the microphone is installed we hear:

E: Ça parce que les problèmes comme les ... comme tu fais au tableau, ah ça, j'ai tout le temps de la misère avec ça.

She reads the first problem statement and says: -"J'espère que t'as longue parce que je pense ça va me prendre du temps". She begins work on paper but when L. asks her what she's doing she says it's nothing and no good.

E: J'essaie même pas rien. Je ... je veux tout faire en même temps, pis uh, ...

L: (points to the work on the work sheet) 50 - 50, qu'est-ce que c'est?

E: Me semble ça arrive pas ça. Ça c'est pas bon ça.

Negative comments about herself continue:

"Je veux tout faire en même temps, c'est ça l'affaire."
"J'arriverai pas, je suis sûre ...
" ... je sais pas, là."
" ... je pourrais pas dire pas plus qu'avant ..."
"je ne sais pas pantoute ..."

At this point she abandons the parking lot problem. In the following question, (square cutting), her negative comments - of which there are eleven - are mainly of the "I-don't-know" or "I-can't-do-it" type. She begins swearing towards the end of this protocol both at herself and at the problem: -"mausus", "merde", "collick". This is important because it indicates a certain impatience with herself and a self-image which has improved to the point that she can engage in a genuine battle with the problem rather than her original beaten-before-she-begins attitude. In fact "collick" accompanies her successful 7 division. She laughs when L. says that the solution seems to have fallen from the sky. The protocol ends on a high with Edes declaring: " Ah, ben, je suis pas pire." The third protocol is completely free of self-derogatory or negative comments. As it proceeds Edes seems more and more confident and asks L. fewer and fewer questions. Returning to the parking problem she is completely in command and does all the talking, asks no questions, and moves confidently toward the solution. Twice here she says to herself "Voyons!" which seems to indicate a certain impatience and you-can-do-better attitude. The entire session ends on a high and Edes expresses her enjoyment.

> E: Ben, celles-là, c'est le fun. J'aime ça faire des problèmes comme ça... J'aime ... (end of tape).

5.1.5 Emotional involvement:

From Sylvie who is completely immersed emotionally in the problems to Jean-Pierre who remains distant and uninvolved, a whole range of emotional involvement is evident in the interviews. For the individual student this involvement varies only slightly from question to question. Sylvie personalizes the problems. They become her problems, her x's her numbers, ...

> S: Alors, attend un peu. Ici mettons je dis AB fois 4, c'est égale à CA. AB fois 4 va me donner 4A, 4B O.K. Alors toute suite là mon chiffre de A ... vue que j'ai mon A, ce serait donc 4. Va essayer, on va

> le voir. Oh, ça ce serait 4. — Pis mon chiffre de B, ça peut pas ... Ce serait 4 aussi? Oh ben, peut-être parce que quand je les multipliais ça donnait 4A, 4B, Mon — ...

This level of personal involvement in the problem is accompanied throughout by a high emotional level. She expresses her nervousness: -"C'est à cause j'ai peur de ne pas réussir ... c'est pour ça je suis tout énervée", her surprise: -"Oh! ça m'en fait 30. Ah, c'est vrai!", her excitement: -"Ah! je pense que je l'ai!", and her disappointment: -"Oh ... Ca marche pas d'abord! Oh, O.K. Je pensais que ça marchait", and her despair: -

> S: Um ... m. — — Ouai — Et franchement là. Je ... je le sais pas! Je te dis là, je suis là, j'ai essayé ... ben ... je dois pas surement avoir essayé toutes les manières mais ... Il me manque deux. Hé franche ...

Jean-Pierre never gets involved emotionally in the problems. He does not even appear to want to solve them and is always ready to quit. He is neither intrigued nor curious. His basic emotion seems to be his desire to leave the interview particularly when he hears the recess bell. Asked if he understands or if he has any ideas about the problems he always responds in the negative. He writes little and spends most of his time sitting back in the chair with his arms folded.

5.1.6 Solver persistance:

Another solver trait that varies greatly here is the degree of determination or persistance or what one author calls courage span.[1] Some students, even highly motivated, are quickly discouraged and ready to abandon. Others stick to the problems like a dog at a bone and are disappointed when L. has to terminate the interview. In the previous examples Sylvie refuses to quit until she has solved every problem. She returns to the square cutting determined to solve it and when that is accomplished moves back to the crypto-problem with the same determination.

[1]Richard Werthime, "Students, Problems and 'Courage Spans'", Cognitive Process Instruction, ed. Lockhead and Clement (Philadelphia: The Franklin Institute Press, 1979).

S: (returns to square cutting) Bon je vais revenir à mon neuf carrés. Lui, il m'intrigue. (Oui?) Ah oui, ça m'intrigue ça. Je va revenir à lui là. Bon, lui là, faut que je l'aie eh.

Lucie expresses a similar persistance: "Un jour je vais l'avoir" ... "Je vais tout les passer".

Jean-Pierre, on the other hand, is ready to quit at every minute. In the square cutting for instance he is discouraged after three attempts at the seven division. There is a long silence after which he says "Pas capable", puts his pencil down, and sits back in his chair folding his arms. He never tries anything else.

5.1.7 Enjoyment of the session:

Often students are asked if they like doing this kind of problem, or if they enjoyed the session, and their responses vary from very negative or apathetic to enthusiastic and positive. Enjoyment does not necessarily mean success although those who succeeded tend more often to express enjoyment. François who succeeds with the first problem, misses the second, and has no time for the third, expresses his enjoyment and asks for more problems to take home. Yet he is the student who sighs "whew!" throughout the interview and who insists that he really is bad in math. Pierre enjoys himself, laughing and joking throughout the interview, and saying he finds the problems fun. He doesn't have an easy time of it though.

Nathalie L., on the other hand, who succeeds with all except the square cutting, says she does not like these problems.

L:	Aimes-tu les problèmes écrits comme ça?
N:	Pas ben ben.
L:	Non? Pourquoi?
N:	Ah, j'sais pas. Je suis tout le temps mélangé là-dedans pis ...
L:	Um huh. Pis la division du carré?
N:	Oh non ... (Non?) ... tu sais quand on comprend pas (laughs) ....

On the whole, though, students seem to experience some degree of enjoyment of the session.

5.1.8 Technical abilities:

Each student brings to the problem solving session his own level of abilities in arithmetic and language. In arithmetic the four basic operations are the most important. In language, the ability to read the problems, understand key words, and verbalize are necessary.

In the discussion of the three problems in the preceding chapters, language and arithmetic factors have been shown to be deterrents to performance. Looking now from the point of view of the individual students, it is obvious that a difficulty in reading the problem or performing a basic operation will show up in all three problem attempts. Some students have difficulty reading the problems aloud. Some are unclear about word-meanings in all three problems. Performance on the basic operations does not change from problem to problem either.

In arithmetic Maryse does her operations technically well (with the exception of a division) and very neatly but they seem to have no meaning for her. This is reflected in her success with the crypto-arithmetic problem and her failure in the parking problem where she divides, multiplies, adds and subtracts without giving any meaning to these operations or drawing any from her results. When asked for the number of cars Maryse divides 100 by 40, gets 25 and responds 25 cars. In response to the question "How many motos if there are 25 cars?" Maryse responds 4 which gives her a total of 29 vehicles. When asked again she subtracts: 40 - 29 = 11 and responds 11 motos.

L. tries to draw her attention to the wheels:

L: 40. Il y a 40 véhicules. Il y en a parmi les véhicules qui ont 4 roues pis il y en a qui ont 2 roues. Ca dépend s'ils sont des autos ou des motocyclettes.

M: Moi, je dis 29 fois 4.

L: Um huh. Ca donne quoi?

M: — — 116?

L: Oui, ça fait trop de roues eh?

M: Oui.

L: Um. Qu'est-ce que tu ferais avec ça?

M: — — — Je le diviserais par 40.

L: Tu diviserais quoi?

M: 116.

L: Pis ça va te donner quoi?

M: — — — Non, ça marche pas.

Maryse's decision not to divide by 40 appears to have to do with the fact that 116 is not divisible by 40. She then decides to divide 116 by 16 instead. Her reasoning is that she has 16 wheels too many. When the division gets a little difficult she tries multiplication instead: 16 X 5, 16 X 6, and 16 X 8.

Nathalie does addition well and seems to understand it. When confronted with the need for subtraction she replaces it by an addition problem. When asked how she decided there were 60 car wheels given that she had 40 moto wheels she replies "Um, ben, j'ai pris 40 plus quoi qui donne 100". Later when she would be expected to subtract 15 from 40 she asks herself "il y a 15, um, véhicules ... des uh, pis uh, plus quoi qui donnerait 40 en tout. Fait que là j'ai pris 25 pour faire le 10. J'ai mis un retenu pis ça m'a donné 40." On her work sheet she writes 15 and then

$\overline{40}$

fills in the 25 under the 15.

Her division is wrong and has no meaning for her. Although multiplication does not mean much either she is able to perform and succeeds with the crypto-arithmetic problem.

Pierre, a cegep student, experiences great technical difficulties. Addition and multiplication errors slow up his progress in both the parking and crypto-arithmetic problems.

5.1.9 Problem representation:

There seems to be general agreement among researchers in problem solving that the way a student sees the problem or represents it in his mind greatly influences the way he will attack it and the eventual success or failure of the enterprise. Where researchers differ is on the degree of importance they attach to this initial problem representation. Gestaltists see it as the most dominant element in the solution process. Newell and Simon attach much importance to it while Krutetskii[1] recognizes it but attaches more importance to the solution processes.

In these protocols two aspects of problem representation are of particular interest. First, there appears to be a gradual evolution of the problem representation as understanding of the problem grows. In other words, the problem reconstruction is not a static element appearing near the beginning of the protocol but a dynamic mental representation that follows closely or lags behind the student's evolving understanding of the problem statement. The crypto-arithmetic problem provides many examples of this evolving problem representation from seeing the problem as one of finding any three or four digits to one of searching for three very constrained digits. Since considerable discussion of the evolutionary nature of understanding can be

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[1] Krutetskii is, in fact, quite critical of the Gestalt psychologists on this score: -"We in the USSR have subjected Gestalt psychology to just criticism for reducing thought to a personal construction of the problem situation, leaving out any activity by the thinking subject, and for ignoring, in point of fact, the influence of a person's past experience."
(The Psychology of Mathematical Abilities in School children, p. 41)

found in the following chapter and since problem representation changes with understanding no further discussion of this seems necessary here.

Secondly, although the problems themselves are very determinant in the student's reconstruction or representation, in some cases there is evidence that the individual's style of constructing the problem space seems to be fairly consistent across the three problems. This individual style of problem representation is very difficult to study because of the predominance of the specific problem in that representation and because of its evolutionary nature within a particular problem. Hints of an individual style appear in the work of Benoit, Sylvie, and Daniel D. for instance.

- Benoit sees each problem as an algebraic one and the central problem is initially seen as one of finding an algebraic formula.

- Sylvie, once she has sufficient understanding to reconstruct her initial problem space tends to see the problem as one of adding a missing number of elements. In the parking lot problem she is missing ten vehicles and her problem is how to add them without disturbing the number of wheels.

S: ... Ben ça me donne, hien, trente véhicules. Il m'en manque 10 pis j'ai mis 100 roues. — Attend un petit p ... — —

In the square cutting she has the seven division and the problem is how to add two more squares to make 9.

S: ... Bon, O.K. Ca, ça m'en ... ça fait mes sept. Pis, il manque mes deux autres ... Mais écou ... donc 1, 2, 3, 4, 5, 6, 7 ... il m'en manque deux ...

Sylvie's progress in these problems is blocked for a time by these constructions of the problem space.

. Daniel D. sees all three problems as division situations. He divides up the 100 wheels between the cars and motorcycles. He gives dimensions to his squares (6 cm) and then divides these up into 3 cm, $1\frac{1}{2}$ cm, 2 cm divisions for smaller squares. In the crypto-arithmetic he tries to construct and solve the inverse problem: CA + 4 = AB.

These protocols seems to indicate that the idea of a problem space as something static and initial needs some revision and that the area of individual style of problem reconstruction merits some further exploration.

5.1.10 Solution style

Individual styles in problem solving can be identified. Some students are remarkable for their use of mental calculation throughout, others for their use of trial and adjustment, others for their leap-frog thinking, and still others for their flexibility, originality, or eureka solutions.

- Jean-Pierre, Michel, Jean-François, and Marc are remarkable for their resistance to using paper and pencil. Even for the square cutting problem they try to handle the divisions in their heads.

- Richard and Mario are remarkable for the rapidity of their solutions and along with Edes for their "eureka" nature.

- Edes is also interesting for the blocking and unblocking progression through each of the problems and by a fairly early near-solution in each.

. Louise is slow to understand each problem but remarkably persistant. In fact, although a few students seem to understand each problem as soon as they've read it, others add slowly to their understanding throughout the entire solution process. Pierre-Paul is interesting in this respect. He seems to fully understand each problem right at the beginning and yet he can never get started. He appears to lack the trial and error reflex and remains frozen in front of the entire problem situation.

. Proficiency at trial and adjustment techniques varies from individual to individual. Students like Marc and Mario make very efficient use of trial and adjustment whereas other students tend to choose a slower or less efficient method or lose track.

5.1.11 Conclusion:

There is ample evidence in these protocols that although the problems themselves are highly determinative in directing behavior, the students bring to the problems a certain personal style which orients behavior across all three problems. In order to be able to fully explore the personal style of students it would be necessary to examine their behavior across a much greater number and wider range of problems.

In this section some of the factors making up that individual style have been suggested and supported by a few examples. For purposes of individual help to students experiencing difficulties in the area of problem solving it would be important to develop a much richer portrait of each student.

In the case of the cegep students, an examination of the limited self-portraits provided by the tape recordings proved helpful to some in overcoming certain detrimental behavior in future problem solving. One student, for example, decided it was time he learned his multiplication tables. Many were impressed with their own abilities and approached future work with greater confidence and less stress. More complex characteristics such as the efficiency of the trial and adjustment method were not however picked up by the students in any conscious way.

Much work has been done in the area of how to make students better problem solvers. Various authors have drawn up lists of helpful hints such as read and reread the problem, retell the problem in your own words, think of similar problems, have lots of paper and pencils handy, and so on ... This examination of some elements of individual problem solving behavior seems to point to a very wide range of personality traits, technical abilities, and solution styles which are very determinative in what happens when the individual confronts the problem. For older students, making that individual style conscious could prove to be more helpful than the general heuristic check lists. It appears here to be at least an area worth exploring.

5.2

The following brief look at sex and age differences in problem solving performance could well have been placed in annexe or omitted completely. The temptation to include this very brief and inconclusive reflection was too great to resist particularly when the sample included equal numbers of either sex and both age groups and at a time when sex differences are the subject of hot debate. It was included in this chapter because it deals with behavior constants across all three problems looked at not on an individual basis but collectively according to the parameters of sex and age. No statistical significance is possible with such a small sample size. The hope is simply to raise a few questions.

5.2.1 Sex differences:

"We should say with all certainty that our

> research, as well as the studies by Dubrovina and Shapiro, did not reveal any qualitative, specific characteristics of the mathematical thinking of boys and girls. The teachers we questioned did not note these differences either." [1]

This is Krutetskii's conclusion after an exhaustive study of mathematical abilities in schoolchildren which he explored through use of problem solving in the clinical interview situation. Quantitatively, Krutetskii does note a difference.

> "Of course, boys actually show mathematical abilities (as well as mechanical ones) more often. This is almost unnoticeable in the primary grades; in the upper grades it becomes quite marked." [2]

Krutetskii explains the quantitative difference by "a difference in tradition, in the upbringing of boys and girls, and the widespread view of professions as masculine or feminine".

Current research into problem solving abilities tends to conclude either that there are no sex differences or that boys outperform girls on certain specific problem

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[1]V.A. Krutetskii, The Psychology of Mathematical Abilities in Schoolchildren, ed. Jeremy Kilpatrick and Izaak Wirszup, trans. Joan Teller (Chicago: University of Chicago Press, 1976), p. 343.

[2]Ibid.

tasks. Fennema and Sherman, who maintain the equality of the sexes here, explain any quantitative differences in a similar way to Krutetskii: i.e. through affective and social variables such as confidence in learning mathematics and mathematics as a male domain. In this way they account for the sex differences that begin to appear in early adolescence. (Fennema and Sherman 1978).

Very recently a media event has taken place over the Stanley and Benbow assertion that boys outperform girls in mathematics because of their genetic superiority. When an examination is made of the very varied problem solving performance of any two boys in Krutetskii's study, in the classroom situation, or in the present research, it never occurs to anyone to explain the difference biologically. Krutetskii never speaks of sex when he enumerates the characteristics of a good problem solver.

An examination of Krutetskii's problems indicates some sexual bias towards boys interests and interest is, according to Krutetskii, the key factor in the development of mathematical ability.

Which brings us to the three problems examined here. The parking lot is definitely a male situation in that boys have a greater interest in cars and motorcycles and at an earlier age. Marc, for instance, bought a mobilette only a few weeks later. Among cegep students the owners or drivers of cars or motorcycles to class are mainly boys. All six cegep boys succeeded on this problem as compared with four of the girls. At the grade six level, performance in terms of success was the same: - four girls and four boys.

The square cutting is hard to classify in terms of male or female biais. At the grade six level one girl and two boys succeed in solving it. Four boys and three girls succeeded at the cegep level.

Those who would call the square cutting a male question because of its geometric nature would probably classify the crypto-arithmetic problem as female. Five of the grade six girls as compared to 2 of the boys get the answer. However all six cegep boys and four cegep girls solve it successfully.

In fact, nothing conclusive can be said about sex differences here from a quantitative point of view. The samples are too small and any statistical differences are insignificant. Looking at the overall portrait of the number of problems solved the boys solved 24 and the girls 21 making an average of 2 out of 3 problems for the boys and 1.75 for the girls. Problem success is a particularly shakey variable here considering that some students were helped and prodded all the way while others worked almost entirely on their own. The interviews were not designed with any statistical comparisons in mind.

Keeping in mind the statistical weakness of the comparisons, and yielding to the temptation to make a few more quantitative tests, a comparison of the length of protocols can be examined in order to see if one group verbalizes more than the other. The suspicion is, of course, that girls talk more than boys. The boys contribute 186 pages of protocol or 15.5 pages each on the average. The standard deviation is 5.37. The girls fill 227 pages or 18.9 each with a standard deviation of 6.43. The differences in the averages prove to be statistically insignificant.

Still searching for a quantitative sex difference, a comparison of the numbers of work sheets needed by each group can be examined in order to see whether one group writes more than the other. Alas, the results are once again insignificant. The boys produce 19.34 work sheets or 1.61 each with a standard deviation of 1.5 and the girls 26.75, or 2.23 each, but with a standard deviation of only .89.

Abandoning the quantitative, are there any significant qualitative sex differences evident in these protocols?

- An examination of the methods used in the parking lot and crypto-arithmetic problems does not indicate that the girls choose any particular method more than the boys. As many boys as girls experience square-association blocks in the second problem.

- Nervousness is apparent in as many boys as girls. In general, affective elements do not appear to be sex specific. The girls may verbalize their emotions slightly more at the cegep level.

- Technical abilities or inabilities here appear as frequently in boys as in girls. More grade six boys show their inability at mental arithmetic quite simply because no girls try it.

In conclusion, although there is a wide variety of student behavior evident in these protocols none of it provides any indication of either a quantitative or qualitative difference in problem solving behavior between the sexes.

5.2.2 Age differences

There seems to be a general consensus among researchers and educators that problem solving abilities increase with age and instruction. Krutetskii studies the evolution of the components of mathematical abilities in primary through secondary schoolchildren as demonstrated in problem solving and concludes: "Research has shown the presence of regular quantitative and qualitative changes in the manifestations of these components according to age."[1] There is some disagreement among researchers as to the ages and stages of mathematical development however all indicators would lead us to suspect a significant difference in performance between sixth grade students and first year cegep students. Studies in developmental changes in problem solving behavior (Weir, Piaget, Stevenson, Lester, Krutetskii) would lead us to expect from cegep students a greater flexibility, a more rapid solution, fewer errors, less reliance on trial and error, a more efficient use of trial and error, less attention to distractors, a greater tendency to be concept rather than data driven, greater field independence, and so on.

And yet, in the study of student behavior within each of the three problems in Chapters II to IV as well as in the study of individual behavior across the three problems, little mention was made of the grade level of the

[1] The components of mathematical abilities according to Krutetskii are: formalized perception of mathematical material, the generalization of mathematical material, the curtailed quality of mathematical thinking, the flexibility of mental processes, the striving for elegant solutions, and mathematical memory. (see Krutetskii, p. 340)

students. It might be helpful (and surprising) at this point to point out who's who.

Grade six students: France, Marc, Nathalie L., Maryse, Nathalie V., Chantal, Lucie, Pierre-Paul, Jean-François, Jean-Pierre, François, Michel.

Cegep students: Pierre G., Louise, Sylvie, Edes, Daniel B., Richard, Benoit, Daniel D., Carole, Martine, Mario, Hélène.

Where no grade distinctions are made it is because they did not seem necessary. Examples of most solution behavior can be found as easily in one group as the other. This is not to say that on examining the protocols of grade six students no differences can be found with those of cegep students.

An examination of the quantitative elements (pages of protocol and work sheets, numbers of problems solved) does show more statistically significant age differences than sex differences. While the cegep students provide on the average 14.92 pages of protocol each, grade six students provide 19.5. We might conclude with some reservations that the grade six students tend to verbalize more than the cegep students. A look at the protocols shows that in the grade six protocols L. does about half the talking and so a more acceptable conclusion would be that L. talks more with the grade six students.

It was very difficult, in fact, to get the grade six students to verbalize. Although the grade six protocols go on longer than the cegep ones, the latter say much more in the interview situation. The difference in the number of work sheets used by either group is statistically insignificant. The cegep students solve 27 problems or an average of 2.25 each whereas the grade six students solve 18 or 1.5 problems each. One might conclude, with some reservations, that cegep students have greater success in problem solving than grade six students. In general, because of the small sample size, the very different interview situations, and the absence in the research design of any elements which would permit statistical comparison of the two groups, no quantitative conclusions can be made. Comparisons of numbers of problems solved and pages of protocol do at least show some statistical significance here whereas when comparisons on a sex basis were made none of the parameters showed any significance.

Qualitatively, the differences between grade six and cegep protocols are much clearer and conclusive. Solution behavior is particularly interesting. Although both groups make exclusive use of trial and adjustment techniques in all three problems, the efficiency of the technique differs. On the parking lot problem methods III and V appear slightly more often in the grade six protocols than in the cegep ones. Since method III (where both wheels and vehicles are varied at the same time) and method V (where numbers are thrown together in a more or less haphazard fashion) are the least efficient of the trial and adjustment methods we might conclude, with reservations, that grade six students use a less efficient trial and adjustment technique. However, more grade six students use method I in their solutions and it is considered here to be the most efficient method for this problem. Comparisons are complicated by the fact that two of the cegep students solved the problem with algebra and so did not use any solution type. It is hard to say that within a solution type one group shows more efficiency than the other or that one group learns more from their errors. If, for instance, François' (grade six) solution is compared with that of Daniel B. (cegep) on the basis of the efficiency of method I, the conclusion would be that the cegep student makes more efficient use of trial and adjustment. François gets the solution on the sixth trial whereas Daniel B. succeeds on his third trial. Yet if we compare François with another cegep student who uses method I, namely Pierre G., François appears more efficient here. Grade six students do jump around slightly more in their use of the five methods. One could interpret this as a possible sign of the greater flexibility of grade six students. Of course another possibility is that grade six students are slower to understand the problem and so change methods as their problem representation evolves.

It is on this latter point, understanding the problem, that cegep students seem to have a slight edge over grade six students. Certainly, in the parking lot problem cegep students do not require the same retelling of the story and get into their algebraic attempts almost immediately. The words "véhicules" and "catégories" appear however to give equal difficulty to both groups. In the square cutting problem neither group has great difficulty in understanding the problem. More cegep students actually solve the square cutting, however, in looking at the protocols it would be very difficult to classify most of them according to age groups. In the crypto-arithmetic problem grade six students seem to form their first problem representation earlier than cegep students perhaps because multiplication is closer to them and they haven't the algebra analogy to deal with. Yet grade six students are slower to come to a full understanding of the problem, that is, to recognize the carrying possibility, the limitations on the size of A, the two A's the same, and so on.

Age differences in this problem solving experiment were surprisingly subtle and few. Although on some scores the overall portrait of cegep students shows a slight advance over grade six students, there is always at least one cegep student whose performance on a particular problem is comparable to that of the poorest grade six solver and often a grade six student who ranks among the best of the cegep solvers. From this research no conclusions can be drawn about greater flexibility, speed, accuracy, field independence and less use of trial and error or less attention to distractors of cegep students. Cegep students appear to make slightly more efficient use of trial and error although this is debatable. The significant age differences in these protocols are to be found in the areas of understanding the problem and facility at verbalization where cegep students outperform grade six students. The most remarkable conclusion to be drawn from age comparisons here is that there is so little evidence of qualitative or quantitative differences in problem solving behavior. Individual differences within an age range appear to be much more significant than differences between the two age groups.

Chapter VI: Problem solving theory and problem solving behavior

6. Introduction

In the preceding chapters the problem solving behavior of twenty-four students has been examined from the perspective of the three problems, and then in Chapter V from the point of view of the individual solver. Many authors have written about problem solving behavior in a theoretical framework and in this chapter an attempt will be made to confront the major authors' theories with what actually takes place in these protocols.

This will be followed by a discussion of some hypotheses on problem solving behavior that have grown out of this study as well as some directions for future research.

In conclusion, an attempt will be made to bring together the various discussions into a response to the initial question as to what occurs when twenty-four students attempt to solve three specific problems.

6.1 Polya

The first name that comes to mind when one thinks of authors on problem solving is that of George Polya. Although Polya's concern in How to Solve It is directed toward helping students become better problem solvers, his "four phases of problem solving" have become the most widely known model for problem solving behavior.

> "First, we have to understand the problem; we have to see clearly what is required. Second, we have to see how the various items are connected, how the unknown is linked to the data, in order to obtain the idea of the solution, to make a plan. Third, we carry out our plan. Fourth, we look back at the completed solution, we review and discuss it."[1]

In a discussion of these four phases Polya emphasizes that successful problem solving involves strict and ordered adherence to these four steps: -understanding the problem, making a plan, carrying out the plan, and reviewing or checking the solution.

To what extent can these four phases of problem solving be seen in these protocols? Do students solving the parking lot problem, for instance, tend to adhere to these steps?

6.1.1 Polya's model and the parking lot problem:

i) **Understanding the problem:**

The length of this stage varies from protocol to

[1]George Polya, How to Solve it, 2nd edition (Princeton: Princeton University Press, 1957), p. 5.

protocol. In a few cases it is very brief as in the case of the majority of cegep students. In general, students took a few minutes to understand the problem and its demands, the main hurdles being the words "catégories" and "véhicules". In the case of Maryse the problem was never fully understood. For some students understanding the problem's demands was the major problem. For example, it is not until the eighth page of her protocol (#228 on tape) that Chantal indicates she has had the first glimmer of understanding and not until the eleventh page that she really seems to have fully understood the problem.

Understanding the problem is not, however, a cut and dried stage in the problem solving process here. As was the case with Chantal, there is often a point at which one can say that the student seems to fully understand the problem. That point is very often well on into the protocol and follows several unsuccessful trial and adjustment attempts. François, for example, tries a type V solution of subtracting 40 from 100 to get 60 motos, and then a type I 20-20 division before he finally arrives at a full understanding of the problem statement on page six of his protocol. Like François, some students are well on their way to a solution before fully understanding the problem. In fact, some students successfully solve the problem without seeming to have fully understood it. Nathalie L., who arrives at the answer through an erroneous juggling of numbers, leaves us wondering whether she has understood the problem and how she would go about solving a very similar one. At no point in her protocol does she indicate an understanding of the problem. Her first reaction is to divide 100 (wheels) by 40 (vehicles) to get $2\frac{20}{40}$. She then suggests the answer is 40 multiplied by 2. In response to L's question as to how many wheels on 40 cars she gets 160 correctly but then subtracts 100 wheels to get 60 wheels, the surplus, which she divides by 2 and declares there are 40 cars, 30 motorcycles, and therefore 70 vehicles. Told that there are only 40 vehicles in all she subtracts 30 motorcycles from 40 vehicles and arrives at the answer of 10 cars. She checks this out and realizes it works and it is only at this point that she shows she might now understand the question. Her decision that the solution is unique also indicates understanding but one is left wondering.

In conclusion, although there is much evidence of students trying to come to grips with the problem's requirements particularly at the beginning of protocols, "understanding the problem" is not a first stage preceding the choice of a strategy but rather an ongoing process that may or may not be terminated even at the end of a successful problem solving attempt. In general, understanding deepens as the student works on the solution.

ii) Devising a plan or strategy:

There is very little evidence of students devising a plan for the solution of the problem. There is some evidence of partial plans, as for instance when students are involved in one of the forms of trial and adjustment and decide to stick to it. Here it seems more a case of trying something and deciding to continue with it in a somewhat ordered fashion. Daniel B's post-algebraic protocol is very brief and yet indicates a form of partial planning or decision to stick to method I that is typical in these protocols.

	40
D: Ah ... S'il y a 40 véhicules —	100 $\quad \frac{140}{2}$
On divise par, uh, je veux dire	
... 100 roues divisé par 40	automobile = 4 roues
véhicules qui donne — Un	motocyclette=2 roues
automobile égale 4 roues,	
motocyclette 2 roues, — cent,	20 auto $\quad$ 20 moto
40 véhicules ... Je vais prendre	
un chiffre au total ... 20, 20	80 roues $\quad$ 40 roues
automobiles ... va donner 80	
roues. Avec 20 motocyclettes qui	
va donner 40 roues ... Ça arrive	
pas encore. On va soustraire ...	
on va mettre ... soit ... 15	
automobiles? ... ça donne 15	

fois 4, 60, 15 fois 4, c'est ça 60. 60. Avec 25 qui donne 50 ...? ... encore.	15 automobiles 25 60 50
5 ça lui donne 10 automobiles à 40 roues ... à 30 motocyclettes qui fait 60 roues.	10 autos 30 moto 40 roues 60 roues
Le tout il y a 100 roues. 100 roues. Bon. C'est à quoi je veux arriver.	100 roues rép.

Daniel first tries dividing 100 by 40 but as he is performing this division he begins to formulate a plan. "Je vais prendre un chiffre au total ... 20" announces his conscious plan which has been called type I. He tries the 20 - 20 division and when it does not work he expresses his intention to reduce the number of cars: -"On va soustraire". When the 15 - 25 division does not work he decides to reduce the number of cars again by five and when his 10 - 30 division succeeds he announces that his plan has succeeded: -"C'est à quoi je veux arriver". Few students appear to be as conscious of what they are doing and as able to verbalize their intentions as is Daniel. Daniel appears to have a conscious plan of reducing by five the number of cars on each attempt. This plan is very short-lived as are most plans in these protocols. In Daniel's case it is short-lived because it leads him quickly to the answer. Yet most plans that appear in these protocols do not last much longer. Students tend to try one thing and then another and plans seem to be formulated as the student goes along and abandoned as quickly. Such short-lived strategies do not merit the title of a plan which in Polya's sense is much more global and carries the solver through a large part if not the entire solution process.

iii) Carrying out the plan:

It is difficult to separate the divising and carrying out of plans in these protocols. Plans evolve as they are being carried out. Frequently students seem to forget their original plans, which indicates that perhaps the plans were never very conscious or present in the first place. Devising and carrying out a plan are generally too short-lived to be called stages here. Exceptions would be the two successful algebraic attempts where the students planned to use algebra and successfully carried it out.

iv) Verification:

Most students are fairly confident when they reach the answer, and seem to feel no need to check it against the problem statement or even to check over their last calculations. The general feeling of elation at getting the answer is often accompanied by a loss of interest in formal verification or any further discussion of the problem. Typically, the student has been moving closer and closer to the solution with a trial and adjustment strategy. He knows when he has got it. He is elated, he may want to celebrate his success, but he has lost all interest in the problem itself. Pierre G. is very confused at one point in his solution attempt when a multiplication error produces an anomaly which he cannot untangle. Shortly after, when he has successfully solved the problem, L. tries to have him look again at this point of confusion which remains unresolved. Pierre has lost all interest.

P: ... Ah, c'est ça, 10 autos, pis ah, 30 motocyclettes.

L: Um, huh.

P: C'est ça?

L: Qu'est-ce qui t'a fait essayer 10 et 30?

P: Oh, je sais pas. (rire)

L: Parce que avec 12 et 13 ça marchait pas.

P: Oui. J'ai essayé 15, 14, 12, 13. Et j'ai essayé 10, pis ça marche. Ça fait 40.

The discussion continues but Pierre never tries to resolve the anomaly and continues to come back to his successful solution. 10 works and that's good enough for Pierre.

Conclusion:

There is little evidence of Polya's four stages of problem solving in these protocols. When they do appear they are short lived and incomplete. Nor are the stages chronological. There appears to be a more spiral or circular movement through these stages.

It must be remembered that Polya was describing what ought to happen, and not what actually happens. Given that many students successfully solve the parking lot problem, and given that there is little evidence of the four stages as defined by Polya, the question arises as to the reliability of the model in looking at problem solving behavior. This is not to say that as an instructional aid the steps are of no use. Undoubtedly most of these students could benefit from Polya's instruction in the four steps of problem solving. As a theoretical model though, it does not provide a helpful basis for examination of what is going on in these protocols.

6.1.2 Polya's model and the square cutting problem:

The conclusion reached above concerning the inappropriateness of Polya's four stages to the parking lot problem are not specific to that particular problem. In the square cutting problem the stages are even harder to identify.

Although some students ask a question at the beginning of the problem as to whether they are dealing with algebra, geometry, or something else, and although a few have trouble reading the question statement, the majority of students have very little trouble understanding the problem. It is later when they come up against certain blocks, that questions of understanding appear, particularly ones dealing with the unexplicit rules of the game. Once again, understanding the problem is not a cut and dried stage nor is it very important or lengthy here. In the case of two students who solve the problem correctly and reject their solutions, it appears that the solution has been reached before a full understanding of the problem.

The other three stages of Polya could be said to be totally absent here.

6.2 Bell

In an unpublished paper, <u>The Problem Solving Process</u>, Alan Bell suggests a model for problem solving behavior which has grown out of a reflection on a number of the Concordia research protocols and a close association with that project particularly in its initial stages.

> "The problem solving process normally begins with a recognition that this is so (i.e. the situation cannot be dealt with by application of a known procedure) and that a "problem" exists. The process adopted then consists primarily of selecting a subset of the data small enough to be processed, leading to the production of a new piece of knowledge integrating that subset; the end of such a phase is often marked by an act of verification, a check. Attention then moves to a new item, and this new subset is similarly processed. In this way the number of items to be coordinated is progressively reduced until they can be processed simultaneously to resolve the problem."[1]

A schematic presentation or listing of the steps in problem solving behavior suggested by Bell could be as follows:

1. Recognition that a probem exists
2. Selection of a subset of data (small enough to be processed)
3. Processing the subset of data to get a new piece of knowledge (which integrates the subset).
4. (Verification)

[1]Alan Bell, The Problem Solving Process (Unpublished paper, 1981).

This description is appealing because of its cyclic nature which seems closer to what happens in these protocols than do Polya's well defined stages.

i) Recognition that a problem exists:

This is obviously a prerequisite to any problem solving activity. Cegep students were aware that what was on the problem card was indeed a problem because they had been told so. Grade six students who were unclear why the interviewer was there tended to read the problem statement and look up as if to say "Now what are we going to do?" The main task with these students was to make them see that there was a job to be done here: -that a problem had indeed been set before them.

ii) Selection of a subset of data:

One student, Pierre-Paul, could not move on the parking lot problem because he did not select a subset of the data and tried to handle it all at once. Otherwise, most students showed signs of making some selection in the parking lot problem. Some concentrated mainly on vehicles and others mainly on wheels and major errors occured as a result of ignoring or completely forgetting the unselected data. Many students, for instance, thought that their 20 - 20 attempt was correct since it gave a total of 40 vehicles while others thought that the 20 - 10 response was correct because it produced 100 wheels. Yet in the trial and adjustment methods of this first problem it is hard to say that the fixing of either the number of vehicles or the number of wheels is a selection of data in Bell's sense. Bell later refers to this as "chunking", and appears to be speaking more of a process which involves selecting and solving a sub-problem.

In the square cutting problem there is, to some extent, what might be called a selection of a subset of the data in the frequent use of a column of the 9 division with a 4 division for instance. This never leads to a solution though. The problem seems to be one that must be dealt with in its entirety in order to be solved.

iii) Processing the subset of data to get a new piece of knowledge which integrates the subset:

Looking at a type I solution in the parking lot problem where the student fixes the number of vehicles at 40 and tries various combinations of cars and motorcycles until the wheels come out right, it is difficult to see which part of this trial and adjustment method corresponds to Bell's third step. If we suppose that the wheels are the subset of data being processed, vehicles being fixed, the processing would consist of multiplying the number of cars by 4, motorcycles by 2, and adding. For instance in the frequent 20 - 20 attempt in these protocols, the processing would involve the calculation of the 120 wheels involved. It would be hard to say that this new piece of knowledge, 120 wheels, integrates the subset.

iv) Verification:

As mentionned in the discussion of Polya, the verification is almost totally absent as a last stage in the solution process. If the example just mentionned is continued, the step of comparing the 120 wheels to the required 100 wheels, and thus realizing that the next attempt must involve a reduction of the wheels, might be called a verification attempt.

For the parking lot problem Bell's model seems a bit strained, although to some extent, the cyclic movement through the steps does correspond to the process described in the various solution types. In the square cutting problem it is completely inappropriate as a model for solution behavior. Here the cyclic nature of the solution process is completely absent. The crypto-arithmetic problem, being a trial and adjustment type, fits the Bell model about as well as does the parking lot problem. A model more consistent with these solutions might be:

1. Reading the problem (several times)
2. Trying something (varying degrees of relevance)
3. Evaluation of effort (worth pursuing? needs adjusting?)

6.3 Polanyi and Wertheimer

Polanyi chracterizes the problem solving process as a state of perplexity followed by a relaxation of tension and a purposive action.

Wertheimer often mentions the emotions of the problem solver: -the joyous outcome, the sense of rightness about the final solution, a satisfying of some basic human urge. And at the beginning of the problem solving session are the sense of uneasiness, disturbance, the sense that something is amiss or unbalanced, and the accompanying desire to set it right.

> "In human terms there is at bottom the desire, the craving to face the true issue, the structural core, the radix of the situation; to go on from an unclear, inadequate relation to a clear, transparent, direct confrontation - straight from heart of the thinker to the heart of his object, of his problem."[1]

Polanyi and Wertheimer have begun to formulate what could be called an affective model for problem solving behavior. While Polya and Bell describe problem solving in terms of a series of actions, these authors see it as a series of affective or feeling states. So far, very little attention has been paid to affective elements in human problem solving. The gestaltists with their accent on the wholeness of the problem solving experience come the closest

[1]Max Wertheimer, Productive Thinking, Enlarged edition, (New York: Harper and Brothers, 1959), p.p. 64-65.

to dealing with this aspect although Wertheimer recognizes that this is an area to which he has not paid sufficient attention.[1] The model deals with the student's feelings when confronted with the problem, when he comes to terms with the problem and begins to tackle it, and when he arrives at the solution. Does the student behavior in this experiment correspond to this model?

i) When confronted with the problem:

Here Polanyi speaks of a state of perplexity and Wertheimer of a sense of uneasiness and the emergence of a desire to set things right (which vary in intensity according to the student's life experience).

A look at protocols and listening to tapes of the beginnings of each problem solving situation bear out these descriptions of the problem-confrontation experience. Student voices are tense, higher pitched, nervous laughter in some, and in several cases a confession of nervousness and attempts to relax. This state is not entirely generated by the problem itself. Some students come into the problem solving session in a nervous state which Wertheimer would say is due to the student's "life experience" particularly in the problem solving area. That tension or disturbance

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[1]Ibid.

which is more directly due to the particular problem posed can be observed more at the beginning of the second and third problem solving efforts where the student is in a more or less relaxed and happy state having resolved the previous problem. These protocols bear out the claims that when confronted with a problem, the student experiences a certain "malaise" which varies in intensity from student to student.

ii) Coming to terms with the problem and beginning to tackle it :

Polanyi says the state of perplexity is followed by a relaxation of tension and a purposive action. We would expect that having confronted the problem the student relaxes as he sets about solving it. Some degree of relaxation is both necessary and observable in these interviews. Hélène, for instance, realizes she must relax in order to be able to get started on the problem. However the degree of relaxation is very varied and shortlived. In many cases the student returns to the initial state of perplexity and tension, particularly when his solution attempt comes up against a dead end and he must return to the original problem and, to some extent, start again.

Wertheimer would ascribe the purposive action to the "desire, the craving, to face the true issue". It is amazing that the majority of students interviewed, in spite of a past history of difficulty in mathematics and problem solving, jump right into the solution of the problem with a fairly elevated degree of motivation. Nothing in the social or environmental context of the situation could provide sufficient motivation, and it can only be assumed that the problem itself provides the impetus. A student such as Jean-Pierre who has to be pushed and prodded all the way, and who very obviously does not want to do these problems, is the exception that brings to light the high degree of motivation of the majority of students. In many cases motivation is so high that it is hard to stop the student when time has run out for the session.

iii) Arriving at the solution:

Polanyi does not go into this stage in the problem solving protess. In these protocols it is upon arriving at the solution that the most dramatic release of tension takes place. Wertheimer's "joyous outcome" and "sense of rightness about the final solution" are very evident. It is perhaps this certainty about the solution that is the major reason why students do not tend to verify solutions. Relaxed and happy and filled with a sense of achievement, the student is also reluctant to talk about the problem solving stage. Many seem impatient to get on to the next problem. This seems to be evidence that problem solving is a pleasurable experience. Some students are disappointed when the session is over and express their enjoyment of the experience. Some even request other problems to take home. This was particularly true for the grade six students who knew they would never see the interviewer again. A couple of grade six students explained that, in school, problems were not like these. France says this about school problems:

> L: (after parking lot problem) Fais-tu des questions comme ça en classe?
>
> F: Oui ... mais c'est pas, c'est pas pareil. (Non?) Non, c'est comme, uh, — Mettons ils diraient "Martine a 125 autos pis uh, ... moins, uh, ... Pis uh, elle s'en va ... Son amie elle reste chez eux et elle y emprunte, mettons, 40. Faut trouver qu'est-ce qui reste. (Ah oui.) Mais plus dure que ça, là.

While there is great variation in the intensity of the affective states of problem solvers as well as very different abilities and willingness to express affective elements, the affective model of problem solving behavior has a certain degree of relevance to solution behavior in these protocols. Examining affective states was not the intention of this experiment, and no request was made to students to express what they were feeling. The resulting portrait is therefore incomplete. There is, however, sufficient material in these protocols to suggest that problem solving is a very emotional activity. Sheila Tobias has studied what she calls "math. anxiety", but the pleasurable experiences evident in these protocols are, as yet, relatively unexplored. The intense pleasure found in mathematical activity by Bertrand Russell does not surprise anyone. But the students in this study are all relatively weak, and have experienced some, if not repeated, failure in mathematics. And yet there is no denying the intense satisfaction and elation of students such as Chantal, Sylvie, and Pierre, when they succeed in solving any one of the problems.

Exploration and development of the affective model could provide great dividends for the teaching of mathematics. Sheila Tobias has succeeded in helping thousands of students through an unblocking of negative attitudes and emotions. Making conscious the pleasurable experience of problem solving might provide even more dramatic breakthroughs.

6.4 Newell and Simon; Information Processing.

While Newell and Simon and other information-processing adherents have never attempted to formulate a clearly defined problem solving model, the implicit assumption in most of their work is that humans solve problems or process information in a way that is very similar to a computer. Like cameras and electrical circuitry in the past, the computer provides a very mechanistic model for human thinking. One of the most appealing aspects of the computer analogy is the language that accompanies it. Input, output, feedback, bits, memory bank, scanning, ... are all welcome and useful words in an area where the dearth of vocabulary is one of the major problems. Organigrams, and other visual aids from computer theory, are also helpful in plotting certain aspects of student behavior. The consensus that has arisen around computer language and symbolism is perhaps the most attractive aspect of the information processing package.

As a model for student behavior in these protocols, the information processing theory is not particularly helpful. Students exhibit no behavior that is comparable to computer performance. As an illustration of this, an information processing analysis of a crypto-arithmetic problem can be compared with the crypto-arithmetic performance in these protocols.

In a chapter entitled "<u>Analysis of human problem solving protocols</u>" in Johnson-Laird and Wason's book <u>Thinking</u>, the author, Newell, analyzes a few lines of a crypto-arithmetic protocol phrase by phrase: -defining the problem space and drawing a "problem behavior graph" for the partial, and then the entire, protocol.[1] The protocol examined is that of a very competent solver attacking the DONALD plus GERALD problem. No unnecessary remarks are made by the student. He does not stutter or stammer nor do there appear to be any long pauses. He proceeds with efficiency through the problem, never expressing an emotion. That portion of the protocol quoted by Newell is presented below. The student, S, has presumably been asked to think aloud and has read the problem.

S: Each letter has one and only one numerical value?

E: One numerical value.

S: There are ten different letters and each of them has one numerical value. Therefore, I can, looking at the two D's ... each D is 5, therefore T is zero. So I think I'll start by writing that problem here. I'll write 5, 5 is zero. Now, do I have any other T's? No. But I have another D. That means I have a 5 over the other side. No I have two A's and two L's that are each ... somewhere ... and this R, three R's ... Two L's equal an R. Of course I'm carrying a 1.

[1] Johnson-Laird and Wason, Thinking, Readings in Cognitive Science (Cambridge: Cambridge University Press, 1977), p.p. 50-57.

Which will mean that R has to be an odd number. Because the two L's ... any two numbers added together has to be an even number – and 1 will be an odd number. So R can be 1, 3, not 5 or 7 or 9.

With this degree of solver competence our complete crypto-arithmetic protocol should have looked like this:

S: (reads problem) Chaque lettre a une et seulement une valeur numérique?

L: Une valeur numérique.

S: Il y a trois lettres différentes et chacune a une valeur numérique. J'ai deux A's. CA est de deux chiffres, alors AB ne peut pas dépasser 24. Ça veut dire que A peut être 1 ou 2. A ne peut pas être égale à 1 parce que 4 fois B est toujours un nombre pair. Alors j'ai 2B en haut et C2 en bas. 4 fois quoi donne un nombre qui finit en 2. 4 fois 3. Essaie 23 X 4 = 92. Ca marche. A = 2 B = 3 et C = 9.

In reality our protocols go on for pages. Yet this is essentially the stuff of the protocol that interests the information processor. From this he would produce a problem behavior graph[1] similar to that for the previous

[1]Ibid., p.p. 52-53.

crypto-problem (Donald & Gerald) and then what more would we know about human problem solving?

The crypto-arithmetic problem is examined here because it is a favorite of information processors who seem to have a very small repertoire of problems. The parking lot and square cutting problems, particularly the latter, would be hard, if not impossible, to deal with through an information processing analysis. Not only do information processors seem to deal with a very restrained set of problems, they seem to use a very select group of problem solvers. It is for these reasons that their contribution to understanding what is going on in these twenty-four protocols is almost nil.

The information processing analysis of problem solving behavior seems impoverished when compared to other studies. The criticism Wertheimer leveled at the Associationists and Logicians could be addressed today to adherents to information processing:

> "In their aim to get at the elements of thinking they cut to pieces living thinking processes, deal with them blind to structure, assuming that the process is an aggregate, a sum of those elements. In dealing with processes of our type they can do nothing but dissect them, and thus show a dead picture stripped of all that is alive in them."[1]

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[1]Max Wertheimer, Productive Thinking, Enlarged edition, (New York: Harper and Brothers, 1959), p. 237.

To analyse these protocols using the information processing model is to take away all their richness (affective elements, creativity, leaps of thought, ...), their humanness, and to look at the dead, and distorted, story of the steps taken.

6.5 A problem solving model for these protocols:

Polya, Bell, Polanyi and Wertheimer, and the information processing group were discussed here because they offer four fairly complete and yet very different models of problem solving behavior. All four models were found to be ~~inadequate~~ in describing behavior or helping to understand what went on in these protocols.

In order to correspond to problem solving behavior in these protocols the model would have to contain affective as well as transactional and cognitive elements. The cyclic movement through the problem's solution (both intellectually and emotionally) would also need to be a part of the model. And finally the model would need to encompass problem solving behavior for a very wide range of problems and an equally wide range of solver ability.

Ginsberg, in a recent article, indicates that we are still in the "discovery stage" in our study of human problem solving.[1] This stage is characterized by the formulation of hypotheses for which he feels the clinical interview and protocol analysis are privileged tools. Lester provides us with an important "state of the art" look at problem solving research in an NCTM article. Many authors, while they do not attempt to offer a complete model of problem solving behavior do have very important contributions to make. Lester provides us with a rather lengthy biography of major works in this area.[2]

An eventual problem solving paradigm will have to wait for a certain level of consensus in the area of cognitive theory. Johnson-Laird and Wason see a convergence of points of view and feel that we will soon see the emergence of a cognitive science.[3] Since much investigation in this area has been conducted through problem solving experiments, it may be hoped that the emerging consensus will include a problem solving paradigm.

In the following section, leap-frog thinking, a cognitive element which appeared in these protocols and

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[1]Ginsburg et al., "The Clinical Interview: Its Effectiveness and Use in Diagnosis" (Unpublished article, 1981).

[2]Frank K. Lester Jr., "Research on Mathematical Problem Solving", Research in Mathematics Education, ed. Richard Shumway (Reston, Virginia: NCTM)

[3]Johnson-Laird and Wason, Thinking, Readings in Cognitive Science (Cambridge: Cambridge University Press, 1977).

which does not appear to have captured the attention of researchers in this area, will be discussed. Other cognitive factors which are present in these protocols will follow in an attempt to bring together those elements which might contribute to an eventual cognitive theory.

6.6.1 Leap-frog solutions:

Student solutions in these protocols do not flow smoothly but give the impression of jumping from one train of thought or solution path to another. At times the student seems to have leaped backward in his work on the problem and be picking up a solution attempt that had been abandoned. At other times the leap seems to be in the other direction and an answer or partial solution seems to have been pulled out of the air. Often the protocol can be analysed in terms of two or three major solution strategies or trains of thought which appear and reappear in alternance. For example, Nathalie V. in the crypto-arithmetic problem, tries AB = 25 and scratches it out because it is "too big". She then writes AB = 12 and believes she has the solution. Once it has been pointed out that the two A's must be the same, she tries AB = 14 and possibly AB = 15 since she writes 60 on the paper. Then suddenly she writes AB = 23. When asked where this sudden solution came from she responds that it came from the AB = 25 attempt.

L:	Mais tu travaillais sur différentes choses. (Oui) Comment t'as pigé ça de l'air comme ça?
N:	Comme ça. Ben j'ai ess ... j'ai essayé vingt ... vingt-cinq. Ça marchait pas. Fait que là, ben, j'ai descendu un peu pis j'ai pris 23 pis ça marchait.

Nathalie ignores the 12, 14, and possibly 15 attempts and says her solution idea came from her first solution effort.

In the parking lot problem, right in the middle of an algebraic attempt, Richard appears to have tested the 20 car possibility. When he abandons the algebraic method he does not try the usual 20 - 20 division but starts right in with a 15 and then a 10 car test.

R: ... x égale 100 roues - (30 sec.) — Bon, ben, uh ... Il y a 20 autos pis ... Non, non, non, ça marche pas. (Pourquoi?) Non; parce que moi, ici, ... j'ai ben du trouble avec ça, tsi.	x = 100 Roues x - 40 = $\frac{20\ 20}{\frac{4}{80}}$

It appears that when Richard begins his non-algebraic solution he picks up from his 20 car attempt, reduces the number of cars by 5 twice, and gets the answer. Edes also tries a non-algebraic solution, a 50 - 50 wheel division, right in the middle of her work in algebra.

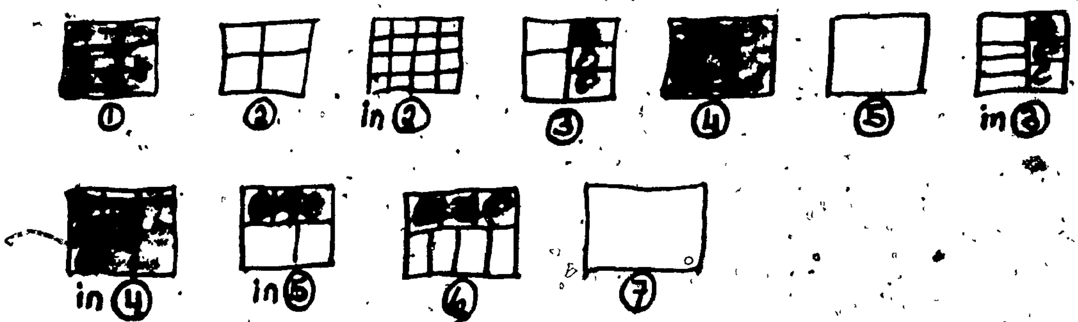
The leap-frog solutions appear at first to be the result of leap-frog thinking. They give the impression that students actually leap about mentally from one train of thought to another and at times with great rapidity. For example Nathalie tried 25, put it aside, tried the increasing 12, 14, 15 sequence, and then mentally leaped back to the 25 solution and decided to decrease it slightly and try 23.

A second explanation is that students are actually pursuing two, or more, lines of thought simultaneously. They only appear to be leap-frogging because of the linear and chronological nature of the written protocol and the limitations of language. The student can only verbalize one thought at a time and so no protocol evidence for two or more simultaneous thought flows is possible. What is verbalized, and hence appears in the protocols, is the most conscious or dominant line of thought at the time. The other lines of thought could, however, be progressing, or at least remaining present, at another less verbally accessible, or less conscious, level.

In the case of Nathalie, then, there would be two major thought flows: that involving the 25 and 23 attempts and the other the 12, 14, and 15 efforts. While she works on the 12 to 15 figures, which are verbalized, the first thought flow progresses from 25 to 23 reasoning that something a little smaller than 25 should be tried. What appears as an "eureka" solution here is then simply the logical consequence of one line of thought which has progressed while the other is being verbalized.

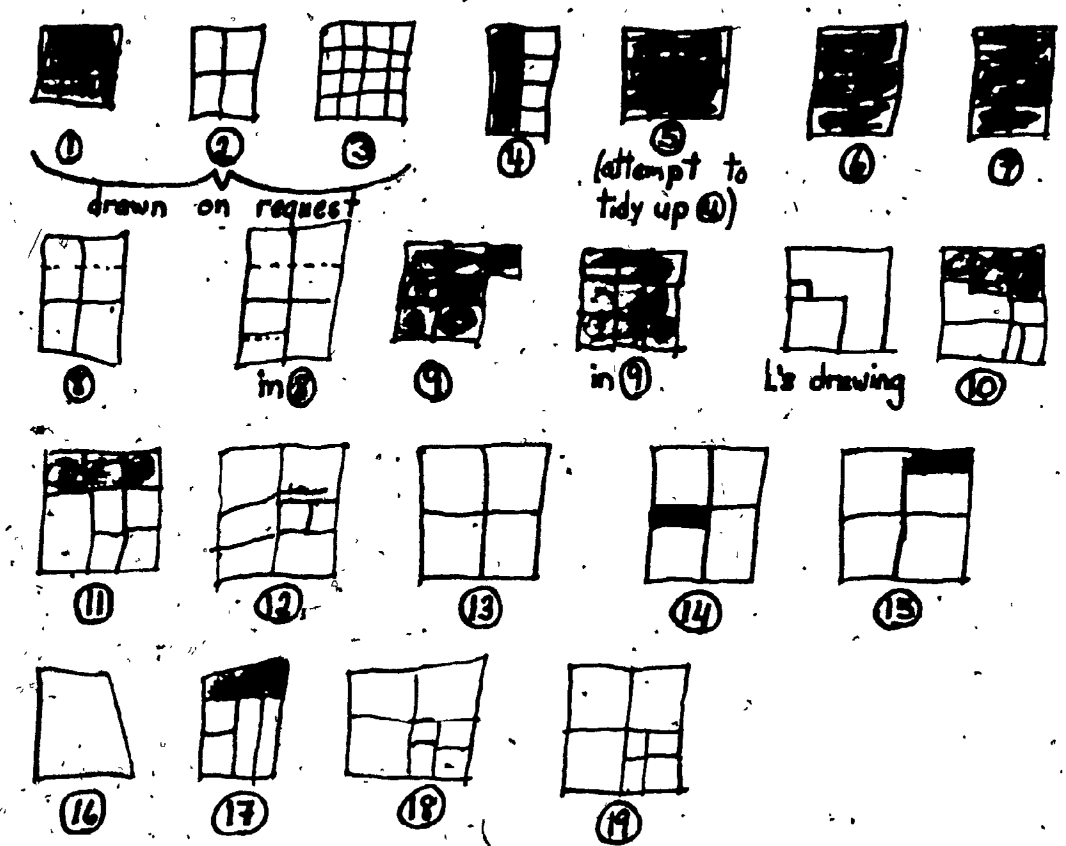
In Richard's case, when he begins the non-algebraic solution, which is very rapid, he seems to have it almost worked out. One suspects that while floundering about in algebra he has made a great deal of progress on the non-algebraic solution, and the appearance in the middle of the algebra work of a very sudden 20 - 20 attempt supports this hypothesis. For Richard, two parallel trains of thought appear to be functioning. One is algebraic and floundering, and the other is an efficient type II reasoning, involving 20, 15, and 10 car attempts.

The square cutting protocols provide excellent examples of leap-frogging where strong visual evidence supports the verbal indications. In the seven-square problem the majority of students tend to alternate between solution attempts based on the 9 and 4 divisions. If student drawing attempts are regrouped according to whether they are inspired by the 9 or 4 division, two complete developments of thought can be seen. Look, for example at Pierre-Paul's drawings in the order they appeared on the work sheet.

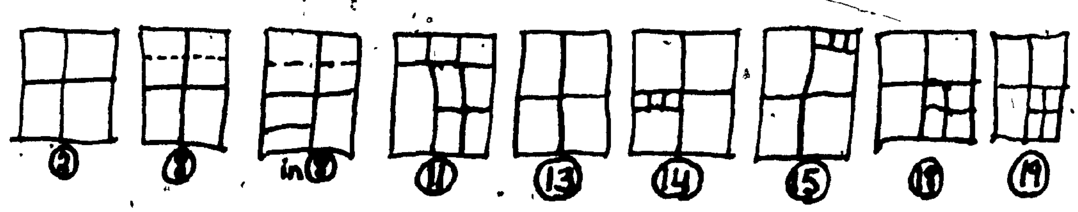


Pierre-Paul not only leap-frogs in his actual drawing, starting one, completing another, going back, etc ... but the divisions of 9 inspiration (in yellow) alternate with variations of the 3 plus 4 inspiration. Version 6 may be seen as a merger of his 9 and 16 division: - a row of each coming together vertically in the final form of 3 and horizontally in 6. But in between he is working on a 6 plus 1 effort which appears in the final form of 4.

In Lucie's work the 4 division in 2 reappears in 8 in 11 and then in 13, 14, 15 and once again in 18 and 19.

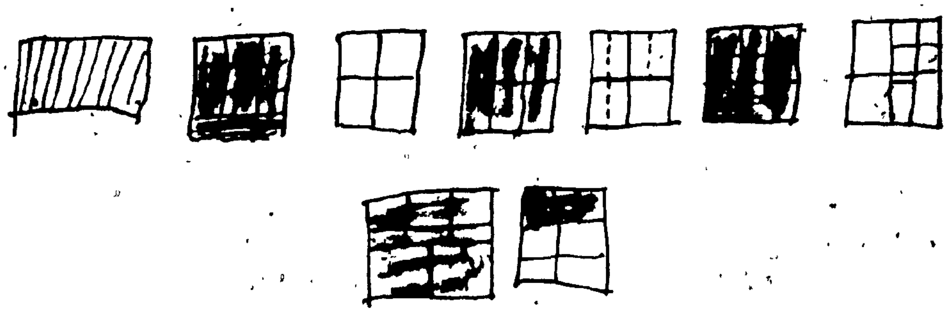


There is a logical evolution to the final successful drawing through these eight drawings. Placing them side by side we have:



In 8 Lucie tries to subdivide a four-division with horizontal lines. In 11 she creates a four-division putting in squares one at a time. The four and three clusters in 11 may be what inspired 14 and 15 where she tries to add three small squares to one square of the four-division. Then in 18 she tries drawing in the small squares to fill one of the squares of the four-division and in 19 she just straightens this into a cross cut and has it. This evolution toward the answer is interspersed with divisions inspired by the nine and, possibly, the sixteen divisions.

Nathalie Lamothe rigorously alternates between drawings inspired by the nine and four divisions finally merging the two in her last effort.



That there are two, and occasionally more, lines of thought taking place in these examples seems evident. That the student is actually pursuing both at once is still debatable but seems a plausible hypothesis here. W.K. Estes in an article entitled "The Information-Processing Approach to Cognition: A Confluence of Metaphors and Methods" says:

> "There are good reasons to believe that the serial-processing conception is oversimplified and that actually much parallel processing goes on in the human problem solver in contrast to the digital computer." [1]

The implications of the parallel processing hypothesis are many. As mentionned by Estes, parallel processing undoes the information processing model for problem solving. It also puts into question the other models where discrete stages are defined as, for example, in Polya.

> "Just as the conception of a serial concatenation of elementary operations, the notion of a succession of discrete stages of processing is doubtless oversimplified ..." [2]

Stages are possible if the mind performs successive operations sequentially. If, however, the mind can be at different points in two or more thought directions at the same time, then defining the cognitive stages becomes problematic.

"Eureka" solutions lose their mystery with the parallel processing hypothesis. Appearing as great leaps forward in thought, they are simply the conclusion of one of the thought processes which has not surfaced or been

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[1]W.K. Estes, Handbook of Learning and Cognitive Processes, Vol V, Human Information Processing (Hillsdale, N.J.: Lawrence, Erlbaum Associates, 1978), p. 10.

[2]Ibid., p. 11.

expressed for a time but has nevertheless progressed to a solution. This seems a more satisfactory explanation than the one offered by information processing theory where because of the necessarily serial nature of thinking "eureka" solutions make their sudden appearance through very accelerated thinking.

The parallel-thinking hypothesis, while it provides answers to many questions, gives rise to others. In these protocols only one train of thought is verbalized at a time. The question that arises is: How does the selection take place as to which thought process is verbalized or translated into an action such as in the square cutting problem? Does the parallel processing take place at different levels of consciousness? The first investigators of problem solving at the end of the 19th century believed that much of thinking goes on below the level of consciousness. The fact that people do solve problems while sleeping or while wholly engaged in a totally unrelated activity lends support to the subconscious or unconscious thought possibility. Often in these interviews when students were having a difficult time verbalizing, it seemed that they were pulling something out of an area of the mind that was hard to get at. The language used and the style of verbal expression were similar to someone's telling of a dream. Is it possible that parallel thought processing occurs at different levels of consciousness and that on some occasions even progresses at the unconscious level? If much, or even some, of human problem solving takes place at an unconscious level, the implications for protocol analysis, not to mention classroom teaching, would be tremendous.

The leap-frog thinking that appears in these protocols is an important cognitive element because it gives rise to a number of hypotheses and questions while demonstrating the inadequacy of existing cognitive theories.

6.6.2 Other cognitive elements:

The presence of mental blocks in problem solving has been documented by most researchers. The mental blocks that occured in these interviews seemed to be associated with a successful partial solution at some point in the problem. For this reason they will be called "success blocks". The question then arose: -Are all blocks success blocks?

Since the square cutting problem has just been discussed in the previous section it will serve here to illustrate "success blocks". In the square cutting problem the student generally succeeds with the nine-square division and then blocks over the seven-division. Over and over again students try to use some part of the successful nine-division in order to obtain the seven-division. As mentionned earlier, they try taking two squares away from the nine-division, using one of its rows or columns, or using the three column or three row division. Many students can never put aside the successful nine-division in order to be able to solve the seven. The majority of protocols give evidence of what could be called a nine-division success block.

By far the most unusual and strong success block is Sylvie's, who in trying to get the nine-division, stumbles on the seven-division. Having succeeded on the seven-division she thereafter puts all her efforts into trying to add two more squares to it to get nine. "Il m'en manque deux", she repeats over and over. In spite of several attempts to get her to forget the seven-division she keeps coming back to it. She goes on to solve the crypto-arithmetic problem and when she comes back to the nine-division immediately draws another seven-division on her work sheet and says "Bon, ici, mon sept celle-là est trouvé. Mon neuf carrés là ... Oh Jupiter, faut je le trouve. Neuf carrés. Ici j'ai sept carrés, —" Later when L. insists she take a new sheet of paper and forget the seven-division, she takes the new sheet and immediately draws a seven-division on it.

Clinging to a successful partial solution the student is blocked from further progress. A line of thinking or a strategy that produced a partial success no longer is helpful and the mind blocks rather than trying a different approach. In these protocols the majority of strong mental blocks seemed to be of this nature.

Another well documented cognitive aspect of problem solving is forgetting. Students often forget achievements and efforts that have taken place only minutes earlier. Ample evidence of the tendency to forget can be found in these as well as most problem solving protocols.

6.7 Conclusion:

In this chapter existing problem solving theories have been to some extent challenged by the problem solving behavior of the twenty-four students studied here. Three cognitive elements present in the protocols and important considerations in the development of an eventual problem solving paradigm were discussed. In order to pull together these discussions and others from preceding chapters, it might be helpful to conclude this paper with a very rapid review of the highlights of the problem solving behavior studied.

. Trial and adjustment is the major solution strategy for all three problems. Great variation, however, can be found in the style and efficiency of its use.

- Affective elements are a major factor in problem solving activity.

- No evidence of sex differences and remarkably few age differences are found.

- Students have a personal style which directs their problem solving behavior. Persistance, openess, self-image, social interaction, technical abilities, and problem representation are discussed here as elements in this personal style.

- Problem solving behavior here cannot be adequately described by existing models or theories.

- Thinking in these problem solving efforts appears to be characterized by success blocks, forgetting, and leap-frogging.

and some directions for future study.

- exploration of leap-frogging and the parallel and subconscious thinking hypotheses

- exploration of the positive affective elements in problem solving and their possible use in the classroom

- building up an adequate interviewer intervention model as, for instance, a list of questions to ask, responses to make, ...

- linguistic analysis of these protocols:

After a brief study of a sampling of interview tapes, Suzanne Beauchemin, a linguist at Collège Saint-Jean-sur-Richelieu, indicates that a study of the language used by students could be a very rewarding area of research. Among other observations she mentions what might be a first sex difference — a linguistic one.

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