A New Approach to the Approximation of Constant Group Delay Low-pass One Dimensional Analog and Digital Transfer Functions

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ABSTRACT

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In this thesis, a new procedure for constant group delay approximation of low-pass, analog and discrete domain filters is developed. Starting with the phase function as an odd infinite series, a new set of parameters are defined. A linear matrix equation is formed in terms of these new variables and the coefficients of the transfer function. Some properties of the elements of the matrix lead to some further interesting properties with respect to stability, generation of transfer functions by recurrence relation, and the solutions to the coefficients of the transfer functions. These properties and the structural properties of the matrix are exploited in order to enhance the computational ability of the approximation procedure, the criteria being the least mean square. The procedure incorporates a minimization algorithm which requires an initial guess values for the variables. A method to obtain these initial values for a given set of coefficients of a transfer function, is described. The stability constraints, solutions to some of the coefficients, and in the case of analog domain, the elemental values of a ladder realization, are obtained simultaneously.

In the analog domain, the method of generating the denominator polynomial of the all-pole transfer function is developed. Some properties of the generating matrix with respect to differentiability and integrability of the elements and of the determinants of the matrix

are presented. An important property of this matrix with respect to the stability of the transfer function is established. The existence of the recurrence relation for the generation of the denominator polynomial are shown.

The structural properties of the matrix increases the computational ability in evaluating its determinants by 68.75%(when compared with that of the Gaussian method). Also, these properties lead to the evaluation of the determinant in terms of lower order determinants and coefficients of the polynomials of lower orders. It is shown that in each iteration the elemental values of a doubly terminated ladder network can be evaluated simultaneously during the minimization process. These are illustrated with examples.

Starting in the t-domain (Richard's variable), similar properties are established for the generation of discrete transfer functions. The difference and anti-difference properties in the discrete domain are analogous to the derivative and the integral properties in the analog domain. Also, the structural properties are interestingly similar.

Similar to the analog case, an algorithm is developed for the approximation of constant group delay in the least mean square sense. The minimization algorithm incorporated in the procedure requires a set of initial values for the variables. A method to obtain these initial values is described. The approximation procedure is illustrated with examples.

Scope for further research is discussed.

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LIST OF IMPORTANT ABBREVATIONS AND SYMBOLS

$[A_{a,n}]$	Generating Matrix of order n in the p-domain
$[C_{a,i}]$	i^{th} column of the matrix $[A_{a,n}]$ replaced with the vector $[B_{an}]$
$[A_{d,n}]$	Generating Matrix of order n in the t-domain
$[C_{d,i}]$	i^{th} column in the matrix $[A_{d,n}]$ replaced with the vector $[B_{dn}]$
Bw	The bandwidth
$[B_{an}]$	Column vector of length n in the p-domain
$[B_{dn}]$	Column vector of length n in the t-domain
c_{an}	A positive real constant number in the p-domain
c_{dn}	A positive real constant number in the t-domain
c_{un}	A positive real constant number in the z-domain
$E_{a,n}$	Even part of the polynomial $P_{a,n}(p)$ in the p-domain
$E_{d,n}$	Even part of the polynomial $P_{d,n}(p)$ in the t-domain
$F_0, F_1, .., F_k$	n^{th} column elements of the matrix $[A_{a,n}]$
$F_{0,d}, F1_{,d}, ..., F_{k,d}$	n^{th} column elements of the matrix $[A_{d,n}]$
$G(\alpha_a)$	The n^{th} element of the vector $[B_{an}]$
$G(\alpha_a)^{(i)}$	The i^{th} partial derivative of $G(\alpha_a)$ with respect to α_a
$G_d(\alpha_d)$	The n^{th} element of the vector $[B_{dn}]$ in the t-domain
$G_d(\mathring{\alpha}_d)^{(i)}$	The i^{th} partial forward difference with respect to α_d
$[H_i]$	Hurwitz matrix of order i with respect the polynomial $P_{a,n}(p)$
$[H_{i,d}]$	Hurwitz matrix of order i with respect to the polynomial $P_{d,n}(t)$

I_i's A Function of the coefficients $a_{k,n}$'s of the polynomial $P_{a,n}(p)$ and the phase slope α_a

$\bar{I}_{i,d}$'s A Function of I_i's and the new set of variables b_i's

$I_{i,d}$'s A Function of the coefficients $d_{k,n}$'s of the $P_{d,n}(t)$ and the factorial polynomials in α_d, the phase slope in the t-domain

$\bar{I}_{i,d}$'s A Function of $I_{i,d}$'s and the new variables $b_{i,d}$'s

$J_0, J_1, .., J_k$ The n^{th} column elements in terms of $G(\alpha_a)$ in matrix $[A_{a,n}]$

$J_{0,d}, J_{1,d}, .., J_{k,d}$ The n^{th} column elements in terms of $G_d(\alpha_d)$ in the matrix $[A_{d,n}]$

$L_0, L_1, ..., L_k$ The $(n-1)^{th}$ column elements in terms of $G(\alpha_a)$ in matrix $[C_{a,n}]$

$L_{0,d}, L_{1,d}, .., L_{k,d}$ The $(n-1)^{th}$ column elements in terms of $G_d(\alpha_d)$ in matrix $[C_{d,n}]$

$O_{a,n}(p)$ Odd part of polynomial $P_{a,n}(p)$

$O_{d,n}(t)$ Odd part of the polynomial $P_{d,n}(t)$

$P_{a,n}(p)$ Polynomial of order n in the p-domain

$P_{d,n}(t)$ Polynomial of order n in the t-domain

$P_{u,n}(z)$ Polynomial of order n in the z-domain

$|Q_1|$ First non vanishing determinant in the partially differentiated $|C_{a,n-1}|$ with respect to α_a

$|Q_{1,d}|$ First non vanishing determinant in the partially differed $|C_{d,n-1}|$ with respect to α_d

$|Q_2|$ Second non vanishing determinant in the partially differentiated $|C_{a,n-1}|$ with respect to α_d

$|Q_{2,d}|$ Second non vanishing determinant in the partially differed $|C_{d,n-1}|$ with respect to α_d

$R_{a,n}(p)$ A second order homogeneous polynomial in p

$R_{d,n}(t)$ A second order homogeneous polynomial in t

$(R-H)_i$ i^{th} element in the first column of Routh-Hurwitz array

$S_{1,1}$, $S_{1,2}$, $S_{2,1}$ and $S_{2,2}$

Submatrices of the matrix $[A_{a,n}]$ after rearranging the columns

T The sampling period in seconds

$T_{a,n}(p)$ Analog transfer function of order n

$T_{d,n}(t)$ Transfer function of order n in the t-domain

$T_{u,n}(z)$ Transfer function of order n in the z-domain

$U_0, U_1, .., U_k$ $(n-2)^{th}$ column elements of $|Q_1|$

$U_{0,d}, U_{1,d}, .., U_{k,d}$ $(n-2)^{th}$ column elements of $|Q_{1,d}|$

$W_0, W_1, .., W_k$ $(n-1)^{th}$ column elements of $|Q_1|$

$W_{0,d}, W_{1,d}, .., W_{k,d}$ $(n-1)^{th}$ column elements of $|Q_{1,d}|$

$W_0', W_1', .., W_k'$ $(n-1)^{th}$ column elements of $|Q_2|$

$[X_{an}]$ Unknown vector whose elements are coefficients $a_{k,n}$'s

$[X_{dn}]$ Unknown vector whose elements are coefficients $d_{k,n}$'s

$[\overline{X}_{an}]$ Unknown vector whose elements are the variables b_i's

$[\overline{X}_{dn}]$ Unknown vector whose elements are the variables $b_{i,d}$'s

$a_{k,n}$'s Coefficients of the polynomial $P_{a,n}(p)$

$d_{k,n}$'s Coefficients of the polynomial $P_{d,n}(t)$

$u_{k,n}$'s Coefficients of the polynomial $P_{u,n}(z)$

$b_1, b_3, .., b_{2n-3}$ New variables in the p-domain

$b_{1,d}, b_{3,d}, .., b_{2n-3,d}$ New variables in the t-domain

c_1 Constant of integration

c_2 Constant of integration

$c_{0,d}$ Constant of anti-difference

$c_{1,d}$ Constant of anti-difference

$c_{2,d}$ Constant of anti-difference

c_b Constant of multiplication of the bilinear transform of p

$c_{m,k}$ (m is 2i or 2i-1) Coefficient of the k^{th} power term in the expansion of $\{\delta_a(p)\}^m$

$\nu_{m,k}$ (m is 2i or 2i-1) Coefficient of k^{th} power term in the expansion of $\{\delta_d(t)\}^m$

h_{2i}'s Denominator coefficients of the analog group delay function

$\eta_{2k+1,1}$'s Coefficients of the error phase function in the t-domain

$f_{i,j}$ Elements of the matrix $[A_{a,n}]$, and vector $[B_{an}]$

f(.) A function of t and represents pT/2 as an infinite series

f_k's Factorial polynomial of order k

f Frequency in Hz

g_{2i}'s Numerator coefficients of the analog group delay function

m Specified number of points in the bandwidth Bw

p Analog complex frequency variable(or discrete)

$s_{i,k}$ Elements of the matrices S's

t Richard's variable

V Vector whose elements are coefficients $u_{k,n}$'s

$x_{i,k}$ Elements of the matrix $[A_{d,n}]$

z Independent variable in the z-domain

$\Delta^{( )}_{\alpha_d}$	Difference operator representation
$\Delta^{(-)}_{\alpha_d}$	Anti-difference operator representation
α_a	Phase slope in the p-domain
α_d	Phase slope in the t-domain
β_i's	Coefficients in the continued fraction expansion of even and odd parts of $P_{a,n}(p)$
γ_i's	Coefficients in the continued fraction expansion of even and odd parts of $P_{d,n}(t)$
δ_a	Error phase function in the p-domain
δ_d	Error phase function in the discrete domain
$\epsilon_{2i+1,1}$'s	Coefficients of the error phase function $\delta_a(p)$
$\zeta_{2i+1,1}$'s	Coefficients of the error phase function $\delta_d(j\Omega T)$
σ	Real part of the complex frequency variable p
τ_{an}	Group delay function in the p-domain
τ_{dn}	Group delay function in the t-domain
τ_{un}	Group delay function in z-domain
τ_{sp}	Specified constant group delay in p-domain
τ_{sd}	Specified constant group delay in the z-domain
ϕ_a	Phase function in the p-domain
ϕ_d	Phase function in the t-domain
ϕ_u	Phase function in the z-domain
ω	Imaginary part of the complex frequency variable p in the analog domain
Σ	Real part of the complex frequency variable p

in the discrete domain

Ω Imaginary part of the complex frequency variable in the discrete domain

CHAPTER ONE

INTRODUCTION

1.1 GENERAL:

Constant group delay(or linear phase) filters find extensive applications in signal processing. It is known [1] that a number of important features of a signal are well preserved only when the phase is also considered. Further, it has been shown [1] that a signal can be reconstructed in a large number of cases when the phase component of the spectrum of the signal is known.

In addition, distortionless transmission is required in order to preserve the signal wave shape. This necessitates synthesis of filters having constant group delay characteristics in the entire passband. In practice, it is not possible to realize such filters. Therefore, design procedures are needed to realize filters which approximate constant group delay characteristics over a required bandwidth.

In what follows, the various contributions made in order to approximate constant group delay over a band of frequencies will be briefly reviewed.

1.2 CONSTANT GROUP DELAY APPROXIMATION:

Constant group delay low-pass ladder networks having transfer functions with Bessel polynomials as the denominators are well known [2]. The method employed is to obtain the continued fraction expansion of e^{-p} ($p = \sigma + j\omega$ is a complex frequency variable) around the origin.

Since the coefficients of the resulting continued fraction expansion are positive, the corresponding transfer function is always stable, and this also provides the elemental values of the resulting passive ladder network.

In [3,4], approximations of arbitrary phase characteristics have been considered, and from the solutions, maximally flat delay filter can be deduced as a particular case. As the group delay response is maximally flat about the origin, the error of approximation is more towards the edge of the passband.

Explicit solution to the approximation of constant group delay at equidistant points in the specified band has also been obtained [5]. In this, the deviation of the delay at the points other than the equidistance points are not considered. All the above methods with explicit solutions do not have the best error norm.

Design techniques considering error norms such as equi-ripple and least mean square sense have been developed [6-8]. These techniques depend heavily on optimization algorithms involving numerical computations. In [6], the coefficients of the all-pole transfer function are obtained such that the phase approximates a quadratic function in the Chebyshev sense. In [7,8], the results are obtained as a product of second order factors with a first-order factor when needed, thereby ensuring the Hurwitz nature of the denominator polynomial. The error norms in [7] and [8] are respectively equi-ripple and least mean square.

In the discrete domain, similar work has been carried out. In [9], explicit solution for the coefficients of a filter possessing maximally flat group delay characteristics has been obtained in terms of the variable z ($z = e^{-j\Omega T}$, T is the sampling period). This same solution is obtained in a simpler way by making use of the bilinear transform of the variable z [10]. Also, in [11], by considering Taylor series expansion of the group delay function, the same solution as in [9,10] but in a different form has been reported. There are other contributions [12,13], describing techniques of approximating constant group delay in the equi-ripple sense. In [12], equi-ripple conditions are formed by a set of non-linear equations and are solved numerically. The same problem is solved using a different approach in [13].

As the solution to the approximation problem is the same in the digital and the distributed domains, the techniques developed in the latter can be used to obtain digital filters also [14].

In the above methods, emphasis is only on constant group delay approximation. This does not mean that the magnitude response is not important. Next, we briefly discuss some of the techniques that consider improving magnitude response while retaining the group delay response. Also, there are some other techniques that consider improving magnitude response, but at the expense of some constant group delay characteristics. These are also discussed.

1.3 CONSTANT GROUP DELAY AND MAGNITUDE APPROXIMATIONS:

There are several contributions, where different conditions on the amplitude and group delay characteristics have been considered and solutions to the approximation problem have been obtained[15-31]. In [15], a rational filter is obtained as a ratio of two Bessel polynomials with frequency scale difference of unity. As the denominator and the numerator possess maximally flat delay characteristics, the resulting rational filter also possesses maximally delay characteristics about the origin. By varying the scaling factor, a variety of amplitude-group delay characteristics are obtained.

In [16], the above problem is considered and, expressions for time delay and frequency response characteristics in explicit form are given.

In [17], the same approach is followed as in [15], except that the Bessel polynomial is replaced by Generalized Bessel polynomials.

In [18], by adding an additional parameter to the coefficients of the continued fraction expansion of a Bessel filter, flexibility of approximation is enhanced while retaining the generation of transfer function by recurrence relations. By varying this parameter in the region of stability, a variety of responses can be obtained.

In [19], with two additional parameters, the Bessel polynomial is generalized. In [20], by introducing a transformation, the exponential function is approximated as a summation. In this summation, the function is a product of Bessel polynomial and a binomial expansion of $(1- r/2)^{i}$ $(1<i<n)$, where r is the new parameter. For different values of this parameter in the stability region, various frequency response characteristics can be obtained. By varying certain parameters (poles and zeros), various combinations of amplitude and group delay characteritics are obtained [21-22].

In [23], closed form solutions for the transfer function possessing linear phase in the passband and steep amplitude selectivity have been obtained. In this, the passband region over which the phase is linear is determined by amplitude selectivity. For the approximation of phase response of a filter, the number of available conditions are (n-1), where n is the degree of the denominator polynomial of the transfer function. By assigning (n-2) conditions to approximate maximally flat group delay and the remaining one condition to approximate prescribed magnitude response, explicit solutions have been obtained [24]. Also, it is shown that by adjusting only two coefficients in the truncated continued fraction expansion of the complex phase angle of an ideal delay function, a large variety of passband loss specifications can be fulfilled, while retaining phase linearity at lower frequencies. In the above method, the Bessel filter can be recovered specifically. A family of low-pass filters possessing maximally flat group delay as well as maximally flat amplitude characteristics has been obtained in [25].

Another approach to the problem of approximating linear phase and frequency selective magnitude characteristics is to have equal-ripple attenuation in the stopband and flat attenuation in the passband, where the numerator is iteratively determined while the denominator is a Bessel polynomial [26].

In the discrete domain, similar work has been carried out [27-31]. The method described in [27] can be considered as an extension of the analog case [15] to the discrete domain. Also, the method described in [28] is an extended version of the analog method [23] to the discrete domain. In [29], the method is based on the Laurent expansion of the filter transfer function with no zeros or poles on the unit circle. The closed form solutions for the coefficients of filter's transfer function possessing both maximally flat group delay and amplitude characteristics, have been obtained [30]. In this, the analysis is carried out in the domain of the variable which is the bilinear transform of the discrete variable z. Also, the corresponding analog transfer function can be recovered specifically. In [31], the method determines explicitly the coefficients of the transfer function possessing Chebyshev characteristics as its attenuation in the stopband and linear phase or constant group delay specifiable independently of the attenuation.

In all the above methods the mean square error is not considered.

Design methods for microwave filters can be used to obtain corresponding discrete filters [32-35]. Of course, a number of design techniques based on iterative and numerical computational methods for microwave as well as digital filters can be found in the literature [36-42].

1.4 SCOPE OF THE THESIS:

From the above, it can be seen that the problem of approximating constant group delay minimizing the mean square error in a bandwidth has not attracted much attention. [8] discusses this problem, however, the resulting denominator Hurwitz polynomial of the transfer function is a function obtained as a product of second-order factors and a first-order factor where needed. As a consequence, realization of such transfer functions by terminated LC-ladder networks could result in certain problems like the variation of the response characteristics with respect to the elements of the ladder network. Also, in the discrete domain, such an approximation does not appear to exist.

Therefore, in this thesis, an approximation technique is given incorporating the following:

(a) The group delay response approximates the specified constant group delay in the least mean square criterion and in a specified bandwidth.

(b) The transfer function (both in the analog and the discrete domains) satisfies the stability conditions

(c) In the analog domain, the elements of the terminated low-pass LC-ladder network is simultaneously obtained.

(d) The approximaton procedure is simple and reduces the number of computations (multiplications) as much as possible.

Only all-pole low-pass transfer functions are considered. In the analog domain, a filter transfer function is represented as

$$T_{a,n}(p) = \frac{c_{an}}{\sum_{i=0}^{n} a_{i,n}\, p^{i}} \tag{1.1}$$

In the discrete domain, the transfer function is represented as

$$T_{u,n}(z) = \left. \frac{c_{un}}{\sum_{i=0}^{n} d_{i,n}\, t^{i}} \right|_{t = c_b \frac{z-1}{z-1}} \tag{1.2}$$

where the variable "t" is the Richard's variable

$$t = c_b \tanh(pT/2)$$

with T as the sampling period and c_b is a constant.

It suffices to consider an all-pole transfer function only, because the denominator polynomial can be determined such that it approximates the desired group delay and / or magnitude characteristics and then a suitable even polynomial in the analog domain or a mirror-image polynomial in the discrete domain can be introduced as the numerator in order to improve the magnitude characteristics.

In Chapter Two, the generation technique for the denominator polynomial of an analog transfer function is considered. This is done by formulating a linear matrix equation. Some interesting properties of the generating matrix with respect to differentiability , and integrability of the elements and the determinants of the matrix, and the matrix structural properties facilitating the computational effort are presented. It is shown that the principal minors of the generating matrix are related to the Hurwitz determinants. Some properties with respect to the recurrence relation of the polynomials are discussed. An approximation algorithm is developed in order to obtain the elemental values of a network realizable as a LC-ladder structure terminated in resistances and such that the transfer function approximates a specified constant group delay in the least mean square sense and in a specified bandwidth. Some examples illustrate the technique.

In Chapter Three, a similar method to generate denominator polynomial of the all-pole transfer function in the variable t is presented. Some interesting properties of the generating matrix with respect to forward difference and anti-difference of the elements and the determinats of the matrix are discussed. Structural properties of the matrix similar to that of the generating matrix in the analog domain are obtained. It is shown how a stable discrete transfer function can be obtained. An approximation algorithm is developed in order to obtain the coefficients and the poles of a digital transfer function such that it approximates a specified constant group delay in the least mean square sense and in a specified bandwidth. Some examples are worked out and a number of responses are obtained.

In Chapter Four, the conclusions are drawn and the scope for further research is discussed.

CHAPTER TWO

CONSTANT GROUP DELAY APPROXIMATION OF 1-D ANALOG LOW-PASS TRANSFER FUNCTIONS

2.1 INTRODUCTION:

In this chapter, we shall consider the generation of the analog transfer function approximating a specified constant group delay. The various analytical properties are first developed and these are effectively used to minimize the error between the constant group delay and the actual group delay.

2.2 FORMULATION OF THE GENERATING MATRIX:

The analog all-pole transfer function of order n can be represented as

$$T_{an}(p) = \frac{c_{an}}{P_{a,n}(p)} \qquad (2.2.1)$$

where c_{an} is a real positive constant and the denominator polynomial is given as

$$P_{a,n}(p) = \sum_{k=0}^{n} a_{k,n}\, p^k \quad \text{with } a_{0,n} = 1, \qquad (2.2.2)$$

and p is the complex frequency variable = $\sigma + j\omega$

Therefore,

$$T_{an}(j\omega) = \frac{c_{an}}{P_{a,n}(j\omega)} \tag{2.2.3}$$

$$= \frac{c_{an}}{|P_{a,n}(j\omega)|} e^{\Phi_a(j\omega)} \tag{2.2.4}$$

where $\Phi_a(j\omega)$ is the phase function. The denominator polynomial $P_{a,n}(p)$ gives

$$P_{a,n}(j\omega) = |P_{a,n}(j\omega)| e^{\Phi_p(j\omega)} \tag{2.2.5}$$

where $\Phi_p(j\omega)$ is the phase function of the polynomial. Expressing the phase function as a sum of two odd polynomials with α_a as the slope of the phase, we have

$$\Phi_a(j\omega) = -\alpha_a j\omega + j\delta_a(\omega) \tag{2.2.6}$$

and $\Phi_p(j\omega) = -\Phi_a(j\omega)$

$$= \alpha_a j\omega - j\delta_a(\omega) \tag{2.2.7}$$

The odd polynomial $\delta_a(j\omega)$ is termed as the error phase polynomial containing odd terms only and is represented as

$$\delta_a(j\omega) = \sum_{i=1}^{\infty} \varepsilon_{2i+1,1}\, p^{2i+1}\Big|_{p=j\omega} \tag{2.2.8}$$

($\varepsilon_{2i+1,1}$'s are the real coefficients and the second right hand suffix 1 indicates the power of $\delta_a(p)$.)

We have

$$\Phi_a(p) = -\Phi_p(p)$$

$$= -\alpha_a p + \delta_a(p) \qquad (2.2.9)$$

We can now consider the all pass function

$$\frac{P_{a,n}(p)}{P_{a,n}(-p)} = e^{2\Phi_p(p)} \qquad (2.2.10)$$

Eq.(2.2.10) can be written as

$$P_{a,n}(p)\, e^{-\alpha_a p} - P_{a,n}(-p)\, e^{\alpha_a p}$$

$$= P_{a,n}(-p)\, e^{\alpha_a p}\{e^{-2\delta_a(p)} - 1\} \qquad (2.2.11)$$

By the infinite series expansion of the exponential function the left hand side of Eq.(2.2.11) can be expressed as

$$P_{a,n}(p)\, e^{-\alpha_a p} - P_{a,n}(-p)\, e^{\alpha_a p}$$

$$= 2 \sum_{i=0}^{\infty} \sum_{k=0}^{2i+1} a_{2i+1-k,n} \frac{\alpha_a^{\,k}}{k!} p^{2i+1} \qquad (2.2.12)$$

and the right hand side can be expressed as

$$P_{a,n}(p)\, e^{-\alpha_a p} \{e^{-2\delta_a(p)} - 1\}$$

$$= \{2\sum_{i=0}^{\infty} \sum_{k=0}^{i} a_{i-k,n} \frac{\alpha_a^k}{k!} (-p)^i \}.$$

$$\{b_1\, p^3 + b_3\, p^5 + \sum_{k=4}^{\infty} b_k\, p^{k+2}\} \qquad (2.2.13)$$

The coefficients α_a and b_i's, in Eq.(2.2.13), are considered as new parameters. These new parameters and the coefficients $\varepsilon_{2i+1,1}$'s in Eq.(2.2.8) are related by

$$b_{k-2} = -\sum_{i=1}^{j/2} \frac{2^{(2i-2)}}{(2i-1)!} \, o_{2i-1,k} \text{ for } (k-2) \text{ odd and} \qquad (2.2.14)$$

$$b_{k-2} = \sum_{i=1}^{j/2} \frac{2^{(2i-1)}}{2i!} \, o_{2i,k} \text{ for } (k-2) \text{ even} \qquad (2.2.15)$$

where j is the largest integer such that $3j \leq k$ (if j is found to be odd then j is j+1) and $o_{m,k}$ (m is 2i-1 or 2i) is the coefficient of the k^{th} degree term in the expression

$$\{\delta_a(p)\}^m = \{\sum_{i=1}^{\infty} \varepsilon_{2i+1,1}\, p^{2i+1}\}^m \quad \text{for } m > 1 \qquad (2.2.16)$$

$$= \sum_{i=0}^{\infty} \varepsilon_{2i+3m,m}\, p^{2i+3m} \qquad (2.2.17)$$

where

$$\varepsilon_{2i+3m,m} = \sum_{j=1}^{i+1} \varepsilon_{2j+1,1}\, \varepsilon_{2i+3m-2j-1,m-1} \qquad (2.2.18)$$

Using Eqs.(2.2.16) to (2.2.18), the coefficients $c_{m,k}$'s(m is 2i-1 or 2i) are obtained. It is required that $a_{k,n}$'s shall be obtained as a solution for a given set of values of the new variables α_a and b_i's. A set of linear equations can be formed from which the coefficients $a_{k,n}$'s are obtained as a solution.

Substituting Eqs.(2.2.12) and (2.2.13) in Eq.(2.2.11) and equating corresponding odd powered terms on both sides, infinite number of linear equations can be deduced. Of these, only the first n equations shall be considered, because of the following reason: When we obtain the coefficient $a_{k,n}$ from these equations in terms of new variables all the remaining variables with even suffixes can be evaluated.

Now the set of first n linear equations can be expressed as a matrix equation as follows:

$$[A_{a,n}]\,[X_{an}] = [B_{an}] \tag{2.2.19}$$

where

$[A_{a,n}]$ is a square matrix of order n,

$[X_{an}]$ is a column vector of length n,

and $[B_{an}]$ is also a column vector of length n.

The elements of the matrix $[A_{a,n}]$, the vectors $[X_{an}]$ and $[B_{an}]$ are respectively as follows:

$$[A_{a,n}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ f_{2,1} & -f_{2,2} & 1 & 0 & 0 & \cdots & 0 \\ f_{3,1} & -f_{3,2} & f_{2,1} & -f_{2,2} & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & F_0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & F_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ f_{n,1} & -f_{n,2} & f_{n-1,1} & -f_{n-1,2} & \cdot & \cdots & F_n \end{bmatrix} \qquad (2.2.20)$$

where

$$f_{i,k} = \left(\frac{\alpha_a^{2i-1-k}}{(2i-1-k)!} + \frac{\alpha_a^{2i-4-k}}{(2i-4-k)!} b_1 + \sum_{j=1}^{2i-5-k} \frac{\alpha_a^{2i-5-k-j}}{(2i-5-k-j)!} b_{j+2}\right)^*$$

for $k = 1,2$, with $f_{1,1} = 1$, (2.2.21)

for n odd: $F_0 = 1$, $F_1 = f_{2,1}$,......, $F_n = f_{(n+1)/2,1}$,

for n even: $F_0 = 0$, $F_1 = -f_{2,2}$,....., $F_n = -f_{(n/2)+1,2}$ (2.2.22)

$$[X_{an}] = (a_{1,n}\ a_{2,n}\ \cdots\cdots\cdots\ a_{n,n})' \qquad (2.2.23)$$

$$[B_{an}] = (f_{2,2}\ \ f_{3,2}\ \ \cdots\cdots\cdots f_{n+1,2})' \qquad (2.2.24)$$

(Prime indicates transpose.)

---

*when power suffix of α_a is less than zero, then α_a with that suffix is zero.

The elements of the matrix $[A_{a,n}]$ and the vector $[B_{an}]$ in Eqs.(2.2.19) and (2.2.24) respectively are because of the consideration of the first set of n consecutive linear independent equations. The n^{th} element of the vector $[B_{an}]$ is a function of (2n-3)-variables. Among these, n-variables (α_a, b_1, b_3, b_5,...., b_{2n-3}) with odd suffixes are independent. The remaining (n-3) variables (b_2, b_4, b_6,...., b_{2n-4}) with even suffixes can be obtained as a function of the independent variables. The vector $[X_{an}]$ is obtained as a solution to Eq.(2.2.19) for a set of values for the new parameters (α_a, b_1, b_3, b_5,...., b_{2n-3}). Thus the vector $[X_{an}]$ containing the coefficients $a_{k,n}$'s as its elements is generated. For all b_i's equal to zero, the generated polynomial is the Bessel polynomial of order n. Before attempting to establish the strictly Hurwitz nature of the polynomial $P_{a,n}(p)$, we wish to discuss some properties of the elements of the matrix $[A_{a,n}]$. Henceforth this matrix $[A_{a,n}]$ is called the generating matrix.

2.3 SOME PROPERTIES OF THE ELEMENTS OF THE GENERATING MATRIX:

It is evident from the structure of the matrix $[A_{a,n}]$ that there are only (2n-1) different elements which are to be evaluated. Any $(i,j)^{th}$ element of the matrix can be evaluated from the following expression.

$$f_{i,j} = (-1)^{j+1} \left\{ \frac{\alpha_a^{2i-1-j}}{(2i-1-j)!} + \frac{\alpha_a^{2i-4-j}}{(2i-4-j)!} b_1 + \sum_{k=1}^{2i-5-j} \frac{\alpha_a^{2i-5-j-k}}{(2i-5-j-k)!} b_{k+2} \right\} \qquad (2.3.1)$$

When $f_{i,j}$'s are evaluated according to Eq.(2.3.1), we get the elements of the matrix $[A_{a,n}]$ along with the required sign.

Several theorems are proved below.

Theorem 2.3.1

$$f_{i+1,j+2} = f_{i,j} \tag{2.3.2}$$

Proof:

Substituting (i+1) for i and (j+2) for j in Eq.(2.3.1), the power suffixes in Eq.(2.3.1) will remain the same. Hence, the $(i,j)^{th}$ element is exactly equal to $(i+1,j+2)^{th}$ element*.

Hence the result follows.

The elements corresponding to two consecutive columns(in the ascending order) and a row are related through partial derivatives with respect to the phase slope α_a and is given by the following theorem.

Theorem 2.3.2

$$f_{i,j+1} = - \frac{\partial f_{i,j}}{\partial \alpha_a} \tag{2.3.3}$$

Proof:

Differentiating Eq.(2.3.1) with respect to α_a and multiplying by (-1) we have

* $(-1)^{j+1} = (-1)^{j+3}$

$$\frac{\partial f_{i,j}}{\partial \alpha_a} = (-1)^{j+1}\{\frac{\alpha_a^{2i-j}}{(2i-j)!} + \frac{\alpha_a^{2i-5-j}}{(2i-5-j)!} b_1$$

$$+ \sum_{k=1}^{2i-6-j} \frac{\alpha_a^{2i-6-j-k}}{(2i-6-j-k)!} b_{k+2} \} \qquad (2.3.4)$$

Hence the result follows.

The elements corresponding to two consecutive rows (in the descending order) and a column are related through second order partial derivatives with respect to the phase slope α_a and is given by the following theorem.

Theorem 2.3.3

$$f_{i-1,j} = \frac{\partial^2 f_{i,j}}{\partial \alpha_a^2} \qquad (2.3.5)$$

Proof:

Differentiating Eq.(2.3.1) twice with respect to the phase slope α_a, we get

$$\frac{\partial^2 f_{i,j}}{\partial \alpha_a^2} = (-1)^{j+1} \{\frac{\alpha_a^{2i-3-j}}{(2i-3-j)!} + \frac{\alpha_a^{2i-6-j}}{(2i-6-j)!} b_1$$

$$+ \sum_{k=1}^{2i-7-j} \frac{\alpha_a^{2i-7-j-k}}{(2i-7-jk)!} b_{k+2}\} \qquad (2.3.6)$$

It can be easily verified that

$$f_{i-1,j} = \frac{\partial^2 f_{i,j}}{\partial \alpha_a^2}$$

Hence the result follows.

The elements of the generating matrix $[A_{a,n}]$ are related through integral relationships. These are presented as below.

Theorem 2.3.4

$$f_{i,j} = -\int f_{i,j+1}\, d\alpha_a + (-1)^{(j+1)}\, b_{2i-j-3} \tag{2.3.7}$$

Proof:

Let

$$f_{i,j} = -\int f_{i,j+1}\, d\alpha_a + c_1 \tag{2.3.8}$$

The first term in the right hand side is true from Theorem 2.3.2. It remains to be established that the constant of integration c_1 is $(-1)^{j+1}\, b_{2i-j-3}$.

From Eq.(2.3.8), we have

$$c_1 = f_{i,j} + \int f_{i,j+1}\, d\alpha_a \tag{2.3.9}$$

Substituting expressions for $f_{i,j}$ by using Eq.(2.3.1) and Theorem 2.3.2, Eq.(2.3.9) results as

$$c_1 = (-1)^{j+1}\, b_{2i-j-3} \tag{2.3.10}$$

Hence the result follows.

Theorem 2.3.5

$$f_{i,j} = \iint f_{i-1,j}\, d\alpha_a d\alpha_a + (-1)^{j+1}\, b_{2i-j-4} \tag{2.3.11}$$

Proof:

Let

$$f_{i,j} = \iint f_{i-1,j}\, d\alpha_a d\alpha_a + c_2 \tag{2.3.12}$$

The first term in the right hand side of the above equation is true due to Theorem 2.3.3. It remains to be established that the constant of integration c_2 is $(-1)^{j+1}\, b_{2i-j-4}$.

From Eq.(2.3.13) we have

$$c_2 = f_{i,j} - \iint f_{i-1,j}\, d\alpha_a d\alpha_a \qquad (2.3.13)$$

Substituting expressions for $f_{i,j}$, by using Eq.(2.3.1) and Theorem 2.3.3, Eq.(2.3.13) results as

$$c_2 = (-1)^{j+1}\, b_{2i-j-4} \qquad (2.3.14)$$

Hence the result follows.

In the next section, we shall discuss some properties of the determinants resulting from the, generating matrix $[A_{a,n}]$.

2.4 PROPERTIES OF THE DETERMINANTS OF THE GENERATING MATRIX:

By Cramers's rule, we have

$$a_{k,n} = \frac{|C_{a,k}|}{|A_{a,n}|} \qquad (2.4.1)$$

where $|C_{a,k}|$ is the determinant of the generating matrix $[A_{a,n}]$ with its k^{th} column replaced by the vector $[B_{an}]$ and $|A_{a,n}|$ is the determinant of the matrix $[A_{a,n}]$. There exist some relationships among these determinants, and hence among the coefficients $a_{k,n}$'s of the polynominal $P_{a,n}(p)$. In this section these relations are discussed.

Theorem 2.4.1

$$|C_{a,n}| = (-1)^{(n)} \quad |A_{a,n+1}| \tag{2.4.2}$$

Proof:

The first element in the matrix $[A_{a,n+1}]$ is always unity. Therefore, its determinant is that of the submatrix of order n obtained by deleting the first row and the first column of the matrix $[A_{a,n+1}]$. In this submatrix if the first, the second, the third,... and the n^{th} columns are replaced respectively with the second, the third, the fourth,...$(n-1)^{th}$ and the first columns, the matrix $[C_{a,n}]$ is obtained where the n^{th} column will have the elements of the vector $-[B_{an}]$. When n is odd or even, there will be (n-1) number of column changes. Taking into account the negative sign of the vector $[B_{an}]$, there are n number sign changes. This can be put as in Eq.(2.4.2).

Hence the result follows.

Theorem 2.4.2

$$a_{n,n} = (-1)^{(n)} \frac{|A_{a,n+1}|}{|A_{a,n}|} \tag{2.4.3}$$

Proof:

From Cramer's rule, we have

$$a_{n,n} = \frac{|C_{a,n}|}{|A_{a,n}|} \tag{2.4.4}$$

From Theorem 2.4.1, substituting for $|C_{a,n}|$, we get

$$a_{n,n} = (-1)^{(n)} \frac{|A_{a,n+1}|}{|A_{a,n}|}$$

Hence the results follows.

Using Theorems 2.3.2 and 2.3.3, the elements of the generating matrix $[A_{a,n}]$ can be shown to be sequentially related to the partial derivatives of the n^{th} element of the vector $[B_{an}]$ with respect to the phase slope α_a. This n^{th} element is designated as $G(\alpha_a)$ which is equal to $f_{n+1,2}$.

In order to prove some further properties of the generating matrix $[A_{a,n}]$, the matrix $[C_{a,k}]$, and the relations among the coefficients $a_{k,n}$'s of the polynomial $P_{a,n}(p)$, the elements of the generating matrix $[A_{a,n}]$ are expressed as the respective partial derivatives of the function $G(\alpha_a)$ which is the n^{th} element of the column vector $[B_{an}]$ given by $G(\alpha_a)$ as

$$G(\alpha_a) = \{\frac{\alpha_a^{2n-1}}{(2n-1)!} + \frac{\alpha_a^{2n-4}}{(2n-4)!} b_1 + \sum_{k=1}^{2n-5} \frac{\alpha_a^{2n-5-k}}{(2n-5-k)!} b_{k+2}\} \quad (2.4.5)$$

$$[A_{a,n}] = \begin{bmatrix} G^{(2n-1)} & 0 & 0 & \cdots\cdots & .0 \\ G^{(2n-3)} & -G^{(2n-2)} & G^{(2n-1)} & \cdots\cdots & .0 \\ G^{(2n-5)} & -G^{(2n-4)} & G^{(2n-3)} & \cdots\cdots & .0 \\ \cdot & \cdot & \cdot & \cdots\cdots & .0 \\ \cdot & \cdot & \cdot & \cdots\cdots & .0 \\ \cdot & \cdot & \cdot & \cdots\cdots & .J_0 \\ \cdot & \cdot & \cdot & \cdots\cdots & .J_1 \\ \cdot & \cdot & \cdot & \cdots\cdots & .. \\ G^{(3)} & -G^{(4)} & G^{(5)} & \cdots\cdots & .. \\ G^{(1)} & -G^{(2)} & G^{(3)} & \cdots\cdots & .J_n \end{bmatrix} \quad (2.4.6)$$

where

for n odd: $J_0 = G^{(2n-1)}$, $J_1 = G^{(2n-3)}$, .., $J_n = G^{(n)}$, (2.4.7)

for n even: $J_0 = -G^{(2n-2)}$, $J_1 = -G^{(2n-4)}$,.., $J_n = -G^{(n)}$ (2.4.8)

and $G^{(2n-1)}$ is $(2n-1)^{th}$ partial derivative with repect to α_a.

The elements of the matrix $[C_{a,n}]$ are also expressed as partial derivatives of the function $G(\alpha_a)$. The matrix $[C_{a,n}]$ is

$$[C_{a,n}] = \begin{bmatrix} G^{(2n-1)} & 0 & 0 & \cdot\cdot & & G^{(2n-2)} \\ G^{(2n-3)} & -G^{(2n-2)} & G^{(2n-1)} & \cdot\cdot & & G^{(2n-4)} \\ \cdot & \cdot & \cdot & \cdot\cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot\cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot\cdot & L_0 & \cdot \\ \cdot & \cdot & \cdot & \cdot\cdot & L_1 & \cdot \\ \cdot & \cdot & \cdot & \cdot\cdot & \cdot & \cdot \\ G^{(3)} & -G^{(4)} & G^{(5)} & \cdot\cdot & L_{k-1} & G^{(2)} \\ G^{(1)} & -G^{(2)} & G^{(3)} & \cdot\cdot & L_k & G \end{bmatrix} \quad (2.4.9)$$

where

for odd n: $L_0 = -G^{(2n-2)}$, $L_1 = -G^{(2n-4)}, \ldots, L_k = -G^{(n-1)}$ (2.4.10)

and for even n: $L_0 = G^{(2n-1)}$, $L_1 = G^{(2n-3)}, \ldots, L_k = G^{(n-1)}$ (2.4.11)

Next, the relationship between the determinant $|C_{a,n-1}|$ and the partial derivative of the determinant $|C_{a,n}|$ with respect to α_a is established.

Theorem 2.4.3

The determinant $|C_{a,n-1}|$ is equal to the partial derivative of the determinant $|C_{a,n}|$ with respect to phase slope α_a.

$$|C_{a,n-1}| = \frac{\partial\,|C_{a,n}|}{\partial\alpha_a} \quad (2.4.12)$$

Proof:

The partial derivative of the determinant $|C_{a,n}|$ with respect to α_a is the sum of n partially differentiated determinants. The first determinant is the determinant of the matrix $[C_{a,n}]$ with its first column elements replaced by the partial derivatives of the elements in the first column of the original matrix $[C_{a,n}]$; the second partially differentiated determinant is the determinant of the matrix $[C_{a,n}]$ with its second column elements replaced by the partial derivatives of the elements in the second column of the original matrix $[C_{a,n}]$;...; and the n^{th} partially differentiated determinant is the determinant of the matrix $[C_{a,n}]$ with its n^{th} column elements replaced by the partial derivatives of the n^{th} column elements of the original matrix $[C_{a,n}]$. In the k^{th} determinant, the k^{th} column elements (partial derivatives of the original k^{th} column elements with respect to α_a are the same as the corresponding elements of the $(k+1)^{th}$ column, except when k is (n-1). And when k is n, k+1 is 1. As the two columns have the same elememnts in order, the value of the determinant is zero. Hence, in the summation of n determinants, (n-1) determinants vanish. The non zero valued determinant is the one whose $(n-1)^{th}$ column elements are the respective partial derivatives of the $(n-1)^{th}$ column elements of the original matrix $[C_{a,n}]$. Therefore we have

$$\frac{\partial |C_{a,n}|}{\partial \alpha_a} = \begin{vmatrix} G^{(2n-1)} & 0 & 0 & 0. & G^{(2n-2)} \\ G^{(2n-3)} & -G^{(2n-2)} & G^{(2n-1)} & 0. & G^{(2n-4)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & W_0 & \cdot \\ \cdot & \cdot & \cdot & W_1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ G^{(3)} & -G^{(4)} & G^{(5)} & W_{k-1} & G^{(2)} \\ G^{(1)} & -G^{(2)} & G^{(3)} & W_k & G \end{vmatrix} \tag{2.4.13}$$

where

for odd n

$$W_0 = -G^{(2n-1)},\ W_1 = -G^{(2n-3)}, \ldots ,\ W_{k-1} = -G^{(n+2)},\ W_k = -G^{(n)}, \tag{2.4.14}$$

and for even n

$$W_0 = 0,\ W_1 = G^{(2n-2)}, \ldots,\ W_{k-1} = G^{(n+2)},\ W_k = G^{(n)}. \tag{2.4.15}$$

Interchanging the $(n-1)^{th}$ and n^{th} columns in $|C_{a,n}|$ we get the determinant $|C_{a,n-1}|$.

Hence the result follows.

Next, we shall establish a relation between the n^{th} and the $(n-1)^{th}$ coefficients of $P_{a,n}(p)$.

Theorem 2.4.4

The coefficient $a_{n,n}$ and the coefficient $a_{n-1,n}$ are related by

$$\frac{a_{n-1,n}}{a_{n,n}} = (-1)^{(n)} \frac{\frac{\partial |A_{a,n+1}|}{\partial \alpha_a}}{|A_{a,n+1}|} \tag{2.4.16}$$

Proof:

From Theorem 2.4.1 we have

$$|C_{a,n}| = (-1)^{(n)} |A_{a,n+1}| \tag{2.4.17a}$$

and from Theorem 2.4.3 we have

$$|C_{a,n-1}| = \frac{\partial |A_{a,n+1}|}{\partial \alpha_a} \tag{2.4.17b}$$

From Cramer's rule, we have

$$a_{n,n} = \frac{|C_{a,n}|}{|A_{a,n}|} \tag{2.4.18}$$

and

$$a_{n-1,n} = \frac{|C_{a,n-1}|}{|A_{a,n}|} \tag{2.4.19}$$

Forming the ratio of Eq.(2.4.18) and Eq.(2.4.19) and substituting for $|C_{a,n}|$ and $|C_{a,n-1}|$, we obtain the required result as Eq.(2.4.16). Hence the result follows.

The determinant of the matrix $[C_{a,n-2}]$ can be expressed as function of the partial derivatives of the determinant of the matrix $[C_{a,n-1}]$ or

$[C_{a,n}]$ or $[A_{a,n+1}]$.

Theorem 2.4.5

$$|C_{a,n-2}| = \frac{1}{2}\frac{\partial\,|C_{a,n-1}|}{\partial\alpha_a}\,, \tag{2.4.20a}$$

$$= \frac{1}{2}\frac{\partial^2\,|C_{a,n}|}{\partial\alpha_a^2} \tag{2.4.20b}$$

$$= \frac{1}{2}(-1)^{(n)}\frac{\partial^2\,|A_{a,n+1}|}{\partial\alpha_a^2} \tag{2.4.20c}$$

Proof:

The same procedure is followed as in the previous theorem. That is, the determinant $|C_{a,n-1}|$ is partially differentiated column wise n times and their summation is considered. The partially differentiated k^{th} column in the determinant $|C_{a,n-1}|$ is the same as the elements in $(k+1)^{th}$ column except when k is (n-1) and (n-2). Hence, in the summation of the determinants all the detrminants except two, vanish as they have same respective elements in two of their columns. Therefore we have

$$\frac{\partial\,|C_{a,n-1}|}{\partial\alpha_a} = |Q_1| + |Q_2| \tag{2.4.21}$$

where

$$|Q_1| = \begin{vmatrix} G^{(2n-1)} & 0 & 0 & \cdots & 0 & G^{(2n-2)} \\ G^{(2n-3)} & -G^{(2n-2)} & G^{(2n-1)} & \cdots & 0 & G^{(2n-4)} \\ G^{(2n-5)} & -G^{(2n-4)} & G^{(2n-3)} & \cdots & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdots & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot\ U_0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot\ U_1 & W_0 & \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot \\ G^{(3)} & -G^{(4)} & G^{(5)} & \cdot\ U_{k-1} & W_{k-1} & G^{(2)} \\ G^{(1)} & -G^{(2)} & G^{(3)} & \cdot\ U_k & W_k & G \end{vmatrix} \qquad (2.4.22)$$

and

$$|Q_2| = \begin{vmatrix} G^{(2n-1)} & 0 & 0 & \cdot\cdot\ 0 & 0 & G^{(2n-2)} \\ G^{(2n-3)} & -G^{(2n-2)} & G^{(2n-1)} & \cdot\cdot\ 0 & 0 & G^{(2n-4)} \\ \cdot & \cdot & \cdot & \cdot\cdot \int U_0\, d\alpha_a & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot\cdot \int U_1\, d\alpha_a & W'_0 & \cdot \\ \cdot & \cdot & \cdot & \cdot\cdot\ \cdot & W'_1 & \cdot \\ \cdot & \cdot & \cdot & \cdot\cdot\ \cdot & \cdot & \cdot \\ G^{(3)} & -G^{(4)} & G^{(5)} & \cdot\cdot \int U_{k-1}\, d\alpha_a & W'_{k-1} & G^{(2)} \\ G^{(1)} & -G^{(2)} & G^{(3)} & \cdot\cdot \int U_k\, d\alpha_a & W'_k & G \end{vmatrix} \qquad (2.4.23)$$

(Prime indicates partial derivative with respect to α_a.)

where

for odd n

$$
\begin{array}{llll}
U_0 = 0, & W_0 = -G^{(2n-1)}, & \int U_0 \, d\alpha_a = G^{(2n-1)}, & W'_0 = 0 \\
U_1 = G^{(2n-2)}, & W_1 = -G^{(2n-3)}, & \int U_1 \, d\alpha_a = G^{(2n-3)}, & W'_1 = -G^{(2n-2)} \\
\vdots & \vdots & \vdots & \vdots \\
U_{k-1} = G^{(n-1)}, & W_{k-1} = -G^{(n+2)}, & U_{k-1} \, d\alpha_a = G^{(n)}, & W'_{k-1} = -G^{(n+3)} \\
U_k = G^{(n-3)}, & W_k = -G^{(n)}, & \int U_k \, d\alpha_a = G^{(n-3)}, & W'_k = -G^{(n+1)}
\end{array}
\quad (2.4.24)
$$

and for n even

$$
\begin{array}{llll}
U_0 = -G^{(2n-1)}, & W_0 = 0, & \int U_0 \, d\alpha_a = -G^{(2n-2)}, & W'_0 = 0 \\
U_1 = -G^{(2n-3)}, & W_1 = G^{(2n-2)}, & \int U_1 \, d\alpha_a = -G^{(2n-4)}, & W'_1 = G^{(2n-1)} \\
\vdots & \vdots & \vdots & \vdots \\
U_{k-1} = -G^{(n-1)}, & W_{k-1} = G^{(n)}, & \int U_{k-1} \, d\alpha_a = -G^{(n)}, & W'_{k-1} = G^{(n+1)} \\
U_k = -G^{(n-3)}, & W_k = G^{(n-2)}, & \int U_k \, d\alpha_a = -G^{(n-2)}, & W'_k = G^{(n-1)}
\end{array}
\quad (2.4.25)
$$

The first determinant $|Q_1|$ in Eq.(2.4.21) is the detetminant $|C_{a,n-2}|$. It remains to establish that the second determinant $|Q_2|$ is also the same as $|C_{a,n-2}|$.

By partially differentiating $(n-2)^{th}$ column and partially integrating $(n-1)^{th}$ column of the second determinant $|Q_2|$ and further applying the Theorems 2.3.2 and 2.3.4, it is clearly seen that the determinants $|Q_1|$ and $|Q_2|$ are the same. The other relationships follow from Theorem 2.4.4.

Hence the results follows.

We shall now establish the relationships of $a_{n-2,n}$ with respect to the coefficients $a_{n-1,n}$ and $a_{n,n}$.

Theorem 2.4.6

$$\frac{a_{n-2,n}}{a_{n,n}} = \frac{|C_{a,n-2}|}{|C_{a,n}|} \tag{2.4.26a}$$

$$= \frac{1}{2}\,\frac{\dfrac{\partial\,|C_{a,n-1}|}{\partial\alpha_a}}{|A_{a,n+1}|(-1)^{(n)}} \tag{2.4.26b}$$

$$= \frac{1}{2}\,\frac{\dfrac{\partial^2\,|A_{a,n+1}|}{\partial\alpha_a^2}}{|A_{a,n+1}|(-1)^{(n)}} \tag{2.4.26c}$$

Proof:

By Cramer's rule we have

$$a_{n-2,n} = \frac{|C_{a,n-2}|}{|A_{a,n}|} \tag{2.4.27}$$

and $$a_{n,n} = \frac{|C_{a,n}|}{|A_{a,n}|} \tag{2.4.28}$$

Taking the ratio of Eq.(2.4.27) and Eq.(2.4.28) and using Theorems 2.4.3 and 2.4.5, we obtain the required result as Eqs.(2.4.26a to 2.4.26c).

Hence the results follows.

Thus Theorems 2.4.4 and 2.4.6 establish the relation among the three coefficients.

Theorem 2.4.7

The coefficient $a_{1,n}$ for any order n is always the phase slope α_a. That is for all n

$$a_{1,n} = \alpha_a \tag{2.4.29}$$

Proof:

The vector $[X_{an}]$ is

$$[X_{an}] = [A_{a,n}]^{-1}[B_{an}] \tag{2.4.30}$$

where

$$[A_{a,n}]^{-1} = \text{Adjoint of } [A_{a,n}] / |A_{a,n}| \tag{2.4.31}$$

In the generating matrix $[A_{a,n}]$, the first element is the element corresponding to the first row and first column which is always unity and the rest of the elements in the first row are all zeros. Hence

$$|A_{a,n}| = Z_{11} \tag{2.4.32}$$

where Z_{11} is the cofactor of $[A_{a,n}]$.

$$\text{Adjoint of } [A_{a,n}] = |C_{a,1}| = f_{2,2}\, Z_{11} \tag{2.4.33}$$

Therefore $$a_{1,n} = f_{2,2} = \alpha_a \tag{2.4.34}$$

Hence the result follows.

We shall next discuss the Hurwitz nature of the polynomial $P_{a,n}(p)$.

2.5 HURWITZ PROPERTIES OF THE POLYNOMIAL $P_{a,n}(p)$:

Two of the important requirements of a transfer function are the stability and the realizability. For the all-pole analog transfer function, if the denominator polynomial $P_{a,n}(p)$ is strictly Hurwitz, the transfer function is realizable as a reactance function terminated in a resistance [43].

Whenever it is not easy to generate the polynomial $P_{a,n}(p)$ in the closed or analytical form, generation of the same by numerical or other techniques is unavoidable. In such a technique, it is advantageous to incorporate stability constraints which are to be obtained as a function of a set of parameters. The advantages are the reduction in the computational effort and time.

In this section, a method to generate the stability constraints as a function of the new variables (α_a, b_1, b_3, b_5,.....) is explained. It will be shown that these stability constraints are equivalent to the Hurwitz stability criteria.

The denominator polynomial $P_{a,n}(p)$ is expressed as sum of its odd and even parts, that is,

$$P_{a,n}(p) = O_{a,n}(p) + E_{a,n}(p) \tag{2.5.1}$$

where $O_{a,n}(p)$ and $E_{a,n}(p)$ are odd and even parts of $P_{a,n}(p)$ respectively.

The stability constraints depend on the requirements that the principal minors of the generating matrix should satisfy

$$(-1)^{(i(i-1)/2)} \; |A_{a,i}| > 0 \quad \text{for } (1 \leq i \leq n+1) \tag{2.5.2}$$

or the coefficients in the continued fraction expansion of the even part $E_{a,n}(p)$ to the odd part $O_{a,n}(p)$ shall be positive. First, the continued fraction expansion of $E_{a,n}(p)$ by $O_{a,n}(p)$ about the origin is expressed in the form of Hurwitz determinants. Then, it will be shown that these are equivalent to the determinants of the generating matrix $[A_{a,i}](1 \leq i \leq n+1)$ with respect to their absolute value.

The continued fraction expansion of $E_{a,n}(p)$ and $O_{a,n}(p)$ is

$$\frac{E_{a,n}(p)}{O_{a,n}(p)} = \frac{\beta_1}{p} + \cfrac{1}{\cfrac{\beta_2}{p} + \cfrac{1}{+ \cdots + \cfrac{\beta_{n-1}}{p} + \cfrac{1}{\cfrac{\beta_n}{p}}}} \tag{2.5.3}$$

where β_i is a function of determinants known as Hurwitz determinants [44]. The coefficient β_i is

$$\beta_i = \frac{|H_i|^2}{|H_{i+1}| \; |H_{i-1}|} \quad \text{for } (1 \leq i \leq n) \tag{2.5.4}$$

In Eq.(2.5.4), $|H_i|$ is the principal minor of order i in the Hurwitz matrix $[H_{n+1}]$ of order (n+1). This matrix $[H_{n+1}]$ is obtained by arranging the coefficients of the denominator polynomial $P_{a,n}(p)$ as follows.

Eq.(2.2.02) for $P_{a,n}(p)$ is rewritten as

$$P_{a,n}(p) = 1 + \sum_{k=1}^{n} a_{k,n}\, p^k \qquad (2.5.5)$$

The Hurwitz matrix for odd n is

$$[H_{n+1}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{1,n} & a_{3,n} & a_{5,n} & a_{7,n} & a_{n,n} & 0 & 0 & 0 \\ 0 & 1 & a_{2,n} & a_{4,n} & a_{6,n} & a_{n-1,n} & 0 & 0 & 0 \\ 0 & 0 & a_{1,n} & a_{3,n} & a_{5,n} & a_{n-2,n} & a_{n,n} & 0 & 0 \\ 0 & 0 & 1 & a_{2,n} & a_{4,n} & \cdot & \vdots & a_{n-1,n} & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & 0 & a_{1,n} & a_{3,n} & \cdots & a_{n,n} \\ 0 & 0 & 0 & & 0 & 1 & a_{2,n} & \cdots & a_{n-1,n} \end{bmatrix} \qquad (2.5.6)$$

Similarly, it is simple to construct the Hurwitz matrix $[H_{n+1}]$ where n is even. This is shown below.

$$[H_{n+1}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & & \cdot & 0 \\ 0 & a_{1,n} & a_{3,n} & a_{5,n} & a_{7,n} & \cdot & a_{n-1,n} & & & \cdot & 0 \\ 0 & 1 & a_{2,n} & a_{4,n} & a_{6,n} & \cdot & a_{n-2,n} & a_{n,n} & \cdot & \cdot & 0 \\ 0 & 0 & a_{1,n} & a_{3,n} & a_{5,n} & \cdot & a_{n-3,n} & a_{n-1,n} & & \cdot & 0 \\ \cdot & \cdot & \cdot & & & & & & & \cdot & \\ 0 & 0 & 0 & \cdot & \cdot & & \cdot & \cdot & & \cdot & 0 \\ 0 & 0 & 0 & & 0 & a_{1,n} & a_{3,n} & \cdot & & a_{n-1,n} & 0 \\ 0 & \cdot & \cdot\cdot & & 1 & a_{2,n} & \cdot & \cdot\cdot & & a_{n-2,n} & a_{n,n} \end{bmatrix} \qquad (2.5.7)$$

It is well known that when all the principal minors are greater than zero, then the polynomial $P_{a,n}(p)$ is strictly Hurwitz [44]. These principal minors are

$$|H_0| = 1 \text{ (an assumption)}$$

$$|H_1| = 1$$

$$|H_2| = a_{1,n}$$

$$|H_3| = a_{1,n}\ a_{2,n} - a_{3,n}$$

$$|H_4| = a_{3,n}\ (a_{1,n}\ a_{2,n} - a_{3,n}) - a_{1,n}\ (a_{1,n}\ a_{1,n} - a_{5,n})$$

$$\vdots$$

$$\text{etc,} \qquad (2.5.8)$$

The coefficients in the continued fraction expansion of $E_{a,n}(p)$ and $O_{a,n}(p)$ of $P_{a,n}(p)$ are the β_i's ($1 \leq i \leq n$). For the polynomial $P_{a,n}(p)$, the Routh-Hurwitz array(R-H array) can be constructed. The elements in the first column of Routh-Hurwitz array can be expressed as a ratio of the principal minors of the Hurwitz matrix $[H_{n+1}]$. That is, the first column elements of R-H array are

$$(R\text{-}H)_{i,1} = \frac{|H_{i+1}|}{|H_i|} \qquad (2.5.9)$$

It can also be shown that the coefficients β_i's can be expressed in terms of the these determinants which are the principal minors of the Hurwitz matrix $[H_{n+1}]$. That is,

$$\beta_i = \frac{|H_i|^2}{|H_{i-1}|\ |H_{i+1}|} \quad \text{for } (1 \leq i \leq n) \qquad (2.5.10)$$

Our aim is to obtain β_i as a function of the principal minors of the generating matrix $[A_{a,n}]$. We shall now show that there exists a relation between principal minors of the Hurwitz matrix and the generating matrix $[A_{a,n}]$.

Theorem 2.5.1

The determinant of the generating matrix $[A_{a,i}]$ of order i is related to the determinant of the Hurwitz matrix $[H_i]$ of the same order i as

$$(-1)^{(i(i-1)/2)}\ |A_{a,i}| = |H_i| \qquad (2.5.11)$$

(Proof is given in Appendix A.)

Theorem 2.5.2

The coefficient β_i in terms of the determinants of the generating matrix $[A_{a,i}]$ is given by

$$\beta_i = \frac{|A_{a,i}|^2}{-|A_{a,i+1}|\,|A_{a,i-1}|} \quad \text{for } (1 \leq i \leq n) \tag{2.5.12}$$

This result can be proved as a consequence of Theorem 2.5.1 in view of the foregoing discussion.

Theorem 2.5.3

The necessary and sufficient condition for the polynomial $P_{a,n}(p)$ to be strictly Hurwitz, is

$$(-1)^{(i(i-1)/2)}\,|A_{a,i}| > 0 \text{ for } (1 \leq i \leq n+1) \tag{2.5.13}$$

Where $|A_{a,i}|$ $(1 \leq i \leq n+1)$ is the pricipal minor(of order i) of the generating matrix $[A_{a,n+1}]$.

Proof:

From Theorem 2.5.1, for β_i to be greater than zero, it is required that

$$-|A_{a,i+1}|\cdot|A_{a,i-1}| > 0 \text{ for } (1 \leq i \leq n) \tag{2.5.14}$$

The sign of $|A_{a,i}|$ is determined by the order i as given in Eq.(2.5.13). The $|A_{a,i}|$ will have the same negative (or positive) sign for any two consecutive orders(i,i+1), when (i(i-1)/2) is odd(or even). In the above expression the difference between the orders of the two matrices is two. Therefore, one of the determinants either $|A_{a,i+1}|$ or $|A_{a,i-1}|$ will have a negative sign. The conclusion is, if $|A_{a,i+1}|$ is less than zero (or greater than zero), then $|A_{a,i-1}|$ is greater than zero (or less than zero). Therefore, the condition Eq.(2.5.13) is necessary and sufficient.

Hence the result follows.

2.6 GENERATION OF $P_{a,n+1}(p)$ BY RECURRENCE RELATION:

The intention is to show that in general, a polynomial of degree (n+1) can be generated from the recurrence relation

$$P_{a,n+1}(p) = P_{a,n}(p) + R_{a,n}(p) \quad P_{a,n-1}(p) \tag{2.6.1}$$

with $P_{a,0}(p) = 1$, $P_{a,1}(p) = 1 + \alpha_a p$ and

$$R_{a,n}(p) = p^2 \frac{|A_{a,n-1}| \; |A_{a,n+2}|}{|A_{a,n}| \; |A_{a,n+1}|} \tag{2.6.2}$$

$$= \frac{P_{a,n+1}(p) - P_{a,n}(p)}{P_{a,n-1}(p)} \tag{2.6.3}$$

Several authors have developed methods to generate transfer functions by recurrence relations [3], [4], [5]. A variety of phase(group delay) responses such as arbitrary phase, equidistant linear phase, maximally flat linear phase, etc, can be satisfied with these transfer functions. From the recurrence relation, stability and realizability criteria can be established [5]. In our case, these two criteria depend on $R_{a,n}(p)$ in Eq.(2.6.2). In particular, these criteria depend on the

$$\frac{|A_{a,n-1}|\ |A_{a,n+2}|}{|A_{a,n}|\ |A_{a,n+1}|} \tag{2.6.4}$$

Eq.(2.6.4) is a function of the coefficients of the polynomials $P_{a,n+1}(p)$, $P_{a,n}(p)$, and $P_{a,n-1}(p)$. This function is

$$\frac{a_{n+1-i,n+1} - a_{n+1-i,n}}{a_{n-i-1,n-1}} \tag{2.6.5}$$

for $i = 0, 1, 2, \ldots, n-1$ and $a_{n+1-i,n} = 0$ for $i = 0$.

<u>Theorem 2.6.1</u>

The polynomial $P_{a,n}(p)$ can be generated from the recurrence relation

$$P_{a,n+1}(p) = P_{a,n}(p) + R_{a,n}(p)\ \ P_{a,n-1}(p)$$

where

$$R_{a,n}(p) = p^2\ \frac{|A_{a,n+2}|\ |A_{a,n-1}|}{|A_{a,n}|\ |A_{a,n+1}|} \tag{2.6.6}$$

(Proof is given in Appendix B.)

This leads to a relationship among the coefficients of the polynomial $P_{a,n-1}(p)$, $P_{a,n}(p)$ and $P_{a,n+1}(p)$ which is given by the following theorem.

Theorem 2.6.2

$$\frac{a_{n-\ell,n-1}}{a_{n-k,n-1}} = \frac{a_{n+1-j,n+1} - a_{n+1-j,n}}{a_{n+1-i,n+1} - a_{n+1-i,n}} \tag{2.6.7}$$

with $\ell \neq k$, $n-\ell \leq n-1$ and $n-k \leq n-1$

$j \neq i$, $n+1-j \leq n+1$, if $n+1-j > n$, then $a_{n+1-j,n+1} = 0$.

$n+1-i \leq n+1$, if $n+1-i > n$, then $a_{n+1-i,n+1} = 0$.

Proof:

In Eq.(2.6.3), if Eq.(2.6.5) and p^2 are taken out as common factors, the numerator and the denominator will be identical. By equating the respective coefficients the relation by Eq.(2.6.7) is established. As $R_{a,n}(p)$ is unique, so is the Eq.(2.6.7).

Hence the result follows.

In the next section, we shall discuss the structural properties of the generating matrix which will enable us to reduce the computational complexity.

2.7 STRUCTURAL PROPERTIES OF THE GENERATING MATRIX:

We have seen in earlier sections, that the generating matrix $[A_{a,n}]$ is important as the properties of the denominator polynomial $P_{a,n}(p)$ depends on the matrix $[A_{a,n}]$. The vector $[X_{an}]$ in Eq.(2.2.19) can be obtained numerically as well as analytically. Analytical solutions are simple to obtain for low order matrix equations. But for higher orders, mathematical complexity increases. Hence evaluation of the determinants by numerical methods is unavoidable. These methods involve number of multiplications, divisions, additions and subtractions. This number is also known as complexity involved in computing the determinant.

In this section, we present some structural properties of the generating matrix $[A_{a,n}]$ which lead to a reduction in the complexity (number of additions and subtractions are not considered in our case) in computing the determinants. Using these determinants in the recurrence relation Eqs.(2.6.1) and (2.6.2), the polynomial $P_{a,n}(p)$ can be obtained. The columns of the generating matrix are rearranged and partitioned into four submatrices as follows.

$$[A_{a,n}] = \left[\begin{array}{c|c} S_{11} & S_{12} \\ \hline S_{21} & S_{22} \end{array}\right] \qquad (2.7.1)$$

where $[S_{11}]$ is a lower triangular Toeplitz matrix, $[S_{12}]$ is a lower triangular Toeplitz matrix with all its diagonal elements zero, and $[S_{21}]$ and $[S_{22}]$ are Toeplitz matrices. These matrices are given below:

For even order

$$[S_{11}] = \begin{bmatrix} 1 & 0 & 0 & 0 & . & 0 \\ f_{2,1} & 1 & . & & & . \\ f_{3,1} & f_{2,1} & 1 & & & \\ . & . & f_{2,1} & . & & . \\ . & . & . & & & \\ . & . & . & . & . & 0 \\ f_{(n/2),n} & f_{(n/2)-1,n} & f_{(n/2)-2,n} & . & f_{2,1} & 1 \end{bmatrix} \qquad (2.7.2)$$

$$[S_{12}] = \begin{bmatrix} 0 & & & 0 & & 0 \\ f_{2,2} & 0 & & \cdot & & \\ f_{3,2} & f_{2,2} & 0 & 0 & & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & 0 & & 0 \\ f_{(n/2),2} & f_{(n/2)-1,n} & \cdot & f_{3,2} & f_{2,2} & 0 \end{bmatrix} \qquad (2.7.3)$$

$$[S_{21}] = \begin{bmatrix} f_{(n/2)+1,1} & f_{(n/2),1} & \cdot & f_{3,1} & f_{2,1} \\ f_{(n/2)+2,1} & f_{(n/2)+1,1} & \cdot & f_{4,1} & f_{3,1} \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ f_{n-1,1} & \cdot & \cdot & \cdot & \\ f_{n,1} & f_{n-1,1} & \cdot & \cdot & f_{(n/2)+1,1} \end{bmatrix} \qquad (2.7.4)$$

and

$$[S_{22}] = \begin{bmatrix} f_{(n/2)+1,2} & f_{(n/2),2} & \cdot f_{3,2} & f_{2,2} \\ f_{(n/2)+2,2} & f_{(n/2)+1,2} & \cdot f_{4,2} & f_{3,2} \\ \cdot & \cdot & \cdot\cdot & \cdot \\ \cdot & \cdot & \cdot\cdot & \cdot \\ \cdot & \cdot & \cdot\cdot & \cdot \\ f_{n-1,2} & f_{n-2,2} & \cdot\cdot & \cdot \\ f_{n,2} & f_{n-1,2} & \cdot f_{(n/2)+2,1} & f_{(n/2)+2,2} \end{bmatrix} \qquad (2.7.5)$$

Similarly for odd orders, these matrices are as follows:

$$[S_{11}] = \begin{bmatrix} 1 & 0 & 0 & & 0 \\ f_{2,1} & 1 & & & 0 \\ f_{3,1} & f_{2,1} & 1 & & 0 \\ \cdot & \cdot & \cdot & \cdot & \\ & \cdot & \cdot & \cdot & \\ f_{(n-1)/2,1} & f_{(n-3)/2,} & \cdot & 1 & 0 \\ f_{(n+1)/2,1} & f_{(n-1)/2,1} & & & 1 \end{bmatrix} \quad (2.7.6)$$

$$[S_{12}] = \begin{bmatrix} 0 & 0 & \cdot & \cdot & 0 \\ f_{2,2} & 0 & 0 & \cdot & 0 \\ f_{3,2} & f_{2,2} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f_{(n+1)/2,2} & f_{(n-1)/2,2} & & f_{2,2} & 0 \end{bmatrix} \quad (2.7.8)$$

$$[S_{21}] = \begin{bmatrix} f_{(n+3)/2,1} & f_{(n+1)/2,1} & \cdot & f_{2,1} \\ f_{(n+5)/2,1} & f_{(n+3)/2,1} & \cdot & f_{3,1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ f_{n,1} & f_{n-1,2} & \cdot & f_{(n+3)/2,1} \end{bmatrix} \quad (2.7.9)$$

$$[S_{22}] = \begin{bmatrix} f_{(n+3)/2,2} & f_{(n+1)/2,2} & \cdots & f_{3,2} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ f_{n,2} & f_{n-1,2} & \cdots & f_{(n+3)/2,2} \end{bmatrix} \quad (2.7.10)$$

It is noticed that

$[S_{11}]$ is a square matrix of order $(n+1)/2$,

$[S_{12}]$ is a rectangular matrix of order $(n+1)/2$ by $(n-1)/2$

$[S_{21}]$ is a rectangular matrix of order $(n-1)/2$ by $(n+1)/2$

and $[S_{22}]$ is a square matrix of order $(n-1)/2$

Now, we shall evaluate the determinant of the matrix $[A_{a,n}]$. The order of the matrix $[A_{a,n}]$ is directly constrained to the degree of the polynomial $P_{a,n}(p)$ being generated. Memory space, computational effort and time could be minimized by reducing the order n of the determinant of the generating matrix which is possible in this case due to the nature of the submatrices. Specifically, the order of $[A_{a,n}]$ is reduced from n to n/2 for n even, and $(n-1)/2$ for n odd. This is shown as follows.

The generating matrix $[A_{an}]$ is of even order n and its determinant in terms of the submatrices $[S_{11}]$, $[S_{12}]$, $[S_{21}]$ and $[S_{22}]$ is

$$|A_{a,n}| = |S_{11}| \; |[S_{22}] - [S_{21}] \, [S_{11}]^{-1}[S_{12}]| \qquad (2.7.11)$$

As $[S_{11}]$ is a lower triangular matrix with unity as its diagonal elements its determinant is always unity. That is,

$$|S_{11}| = 1$$

Also, as $[S_{11}]$ is a lower triangular Toeplitz matrix, so is its inverse. The elements of $[S_{11}]^{-1}$ are

$$s_{1,1} = 1$$

$$s_{i,1} = -\sum_{k=1}^{i-1} s_{i-k,1}\, f_{k+1,1} \quad \text{for } (2 \le i \le n)$$

$$s_{i+1,j+1} = s_{i,j} \quad \text{for } (2 \le (i,j) \le n) \qquad (2.7.12)$$

Let

$$|[S_{22}] - [S_{21}]\,[S_{11}]^{-1}[S_{12}]| = |Y_{a,n/2}| \qquad (2.7.13)$$

Then

$$|A_{a,n}| = |Y_{a,n/2}| \qquad (2.7.14)$$

$$y_{i,n/2} = f_{i+1,2} \quad \text{for } (1 \le i \le n/2) \qquad (2.7.15)$$

and $y_{i,j}$ is the $(i,j)^{th}$ element of the matrix $[Y_{a,n/2}]$ which is given as

$$y_{i,m-j} = -\sum_{k=1}^{j} y_{1,1-k+m}\, f_{i+j-k,1} + f_{i+j,2} \qquad (2.7.16)$$

for $(i,j) = 1, 2, 3, \ldots, m$ and $m = n/2$

An odd order matrix can similarly be reduced to a matrix $[Y_{a,(n-1)/2}]$ of order $(n-1)/2$ and its elemnts are

$$y_{i,m-j} = f_{i+j+1,2} - f_{3,2}\, f_{i+j-1,1} - \sum_{k=1}^{j} y_{1,1-k+m}\, f_{i+j-k,1} \qquad (2.7.17)$$

for (i, j) = 1, 2, 3, ... ,m and m = (n+1)/2

Since $[S_{11}]^{-1}$ is a lower triangular matrix, $[Y_{a,n/2}]$ or $[Y_{a,(n-1)/2}]$ is easier to construct. The order of the determinant is thereby reduced.

Now, we shall discuss the computational complexity reduction of the determinants of the generating matrix $[A_{a,n}]$. It is known that the number of multiplications and divisions (known as complexity) in evaluating the determinant by Gaussian method is

$$(n-1)\,(n^2 + n + 3)/3 \qquad (2.7.18)$$

In our case, the n^{th} order determinant can be treated as an $(n-1)^{th}$ order determinant due to the fact that the elements in the first row are zero, except for the first element which is unity. Then, the total number of multiplications and divisions will be

$$(n^3 - 3n^2 + 5n - 6)/3 \qquad (2.7.19)$$

Case 1: Even order n

The order of the reduced matrix $[Y_{a,n/2}]$ is n/2. The number of multiplications required to reduce the order is

$$n^2 (n - 2)/16 \tag{2.7.20}$$

and the complexity to evaluate its determinant is

$$(n - 2) (n^2 + 2n + 12)/24 \tag{2.7.21}$$

Total complexity involved is

$$(5n^3 - 6n^2 + 16n - 48)/48 \tag{2.7.22}$$

Due to the structure of the generating matrix, the complexity is

$$(n^3 + 3n^2 - n)/6 \tag{2.7.23}$$

That is, compared to Eq.(2.7.19), the reduction in complexity is

$$(n^3 - 9n^2 + 11n - 12)/6 \tag{2.7.24}$$

Further, because of the reduction in the order of the matrix, the reduction in complexity is

$$(11 n^3 - 42 n^2 + 64 n - 48)/48 \tag{2.7.25}$$

Therefore, the overall reduction in complexity is approximately 68.75%.

Case 2: Odd order n

The order of the reduced matrix $[Y_{a,(n-1)/2}]$ is (n-1)/2. The number of multiplications required to reduce the order is

$$(n - 1)^2 (n + 1)/16 \tag{2.7.26}$$

and the complexity to evaluate its determinant is

$$(n - 3)(n^2 + 11)/24 \qquad (2.7.27)$$

Due to the structure of the generating matrix, the complexity is

$$(5n^3 - 9n^2 + 19n - 63)/48 \qquad (2.7.28)$$

That is, compared to Eq.(2.7.19) the reduction in complexity is

$$(n - 1)(4n^2 + 7n - 9)/24 \qquad (2.7.29)$$

Further, because of the reduction in the order of the matrix, the reduction in complexity is

$$(11 n^3 - 39 n^2 + 61 n - 33)/48 \qquad (2.7.30)$$

Therefore, the overall reduction in complexity is approximately 68.75%.

The determinants of the generating matrix $[A_{a,n}]$ are also generated as a function of the elements of the matrix and the coefficients of lower order polynomial. It can be shown that the determinants of even order can be expressed as

$$|A_{a,n}| = |A_{a,n-2}| \{ - f_{n,2} + \sum_{i=1}^{(n/2)-1} (f_{n-i,1}\, a_{2i-1,n-2} - f_{n-i,2}\, a_{2i,n-2}) \} \qquad (2.7.31)$$

and odd order determinant can be expressed as

$$|A_{a,n}| = |A_{a,n-2}| \sum_{i=1}^{\frac{n-1}{2}} \{f_{n+1-i,2}\, a_{2i-2,n-2} - f_{n-i,1}\, a_{2i-1,n-2}\} \qquad (2.7.32)$$

with $a_{0,n-2} = 1$

2.8 GENERATION OF A PHASE FUNCTION FROM THE COEFFICIENTS OF THE TRANSFER FUNCTION:

In this section we present a method for obtaining the coefficients of an odd infinite series representing the phase function of a network, given its parameters (the coefficients of the transfer function representing the network).

As a transfer function is represented by a ratio of two polynomials, it suffices to obtain the coefficients of the odd, phase function, given the coefficients of a polynomial $P_{a,n}(p)$.

Our objective is to obtain b_i's given $a_{k,n}$'s of the polynomial $P_{a,n}(p)$. This will enable us to obtain a new set of variables b_i's during the optimization procedure.

The matrix equation Eq.(2.2.19) can be rearranged such that the unknown column vector $[X_{an}]$ has the new variables b_{2i-1}'s as its elements and the elements of the matrix $[A_{a,n}]$ and $[B_{an}]$ are functions of the coefficients $a_{k,n}$'s of the polynomial $P_{a,n}(p)$. The matrix equation becomes

$$[\bar{A}_{a,n-1}]\, [\bar{X}_{an-1}] = [\bar{B}_{an-1}] \qquad (2.8.1)$$

where

$$[\bar{X}_{an-1}] = (b_1 \; b_3 \; . \; . \; . b_{2i-1}) \quad (2.8.2)$$

$$[\bar{B}_{an-1}] = (\bar{I}_3 \; \bar{I}_5 \; . \; . \; . \; \bar{I}_{2i+1}) \quad (2.8.3)$$

with $\bar{I}_{2i+1}$'s given as

$$\bar{I}_{2i+1} = I_{2i+1} + \sum_{k=1}^{i-3} I_{2k+1} \cdot b_{2i+2-2k} \quad (2.8.4)$$

and the elements I_i's are given as

$$I_i = \sum_{k=0}^{i} a_{i-k} \frac{\alpha}{\overline{k!}} a(-1)^k \quad (2.8.5)$$

The matrix $[\bar{A}_{a,n-1}]$ is given as

$$[\bar{A}_{a,n-1}] = \begin{bmatrix} 1 & 0 & 0 & 0.. & 0 \\ I_2 & 1 & 0 & 0.. & 0 \\ I_4 & I_2 & 1 & 0 & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ I_{2n-2} & I_{2n-4} & .. & I_2 & 1 \end{bmatrix} \quad (2.8.6)$$

The lower suffix in Eq.(2.8.1) does not indicate the order of the column or the matrix except for the vector $[\bar{B}_{an-1}]$.

The solution can be obtained in a recursive manner as shown below.

$$b_1 = I_3$$

$$b_3 = I_5 - I_2 \cdot b_1$$

$$b_5 = I_7 - I_4 \cdot b_1 - I_2 \cdot b_3$$

$$b_7 = I_9 - I_6 \cdot b_1 - I_4 \cdot b_3$$

$$+ I_3 \cdot b_4 - I_2 \cdot b_5$$

.

.

.

$$b_{2i+1} = I_{2i+3} - I_{2i} \cdot b_1 - \sum_{k=1}^{2i-3} I_{2i-1-k} \cdot b_{k+2} \qquad (2.8.7)$$

The variables with even numbers as suffixes are related to the variables with odd numbers as suffixes as can be seen below:

$$b_2 = 0$$

$$b_4 = (b_1)^2$$

$$b_6 = 2\ b_1\ b_3$$

$$b_8 = 2\ b_1\ b_5 + (b_3)^2$$

$$b_{10} = 2\ b_1\ b_7 + 2\ b_3\ b_5 + (b_1)^4/3 \qquad (2.8.8)$$

So when an odd numbered variable is determined from the recurrence relation, the next even numbered variable is determined. The next odd numbered variable will have term consisting of the previous lower even numbered variable. Likewise any number of variables with odd numbers as suffixes can be generated for a given n^{th} degree polynomial $P_{a,n}(p)$.

Alternatively instead of treating b_i's as the new variables, the coefficients $\varepsilon_{2i+1,1}$'s of the defined phase function as an infinite series, can be be considered as new parameters. We will now show how any desired number of coefficients $\varepsilon_{2i+1,1}$'s could be generated and hence the phase function as an a truncated series could be obtained. The relation between the odd numbered variable b_i's and the coefficients

$\varepsilon_{2i+1,1}$'s of the error phase polynomial $\delta_a(p)$ is given by Eqs.(2.2.14) and (2.2.15) which are rewritten as below

$$b_{k-2} = -\sum_{i=1}^{j/2} \frac{2^{(2i-2)}}{(2i-1)!} \, c_{2i-1,k} \text{ for k odd} \qquad (2.8.9)$$

$$b_{k-2} = \sum_{i=1}^{(j/2)} \frac{2^{(2i-1)}}{(2i)!} \, c_{2i,k} \text{ for even} \qquad (2.8.10)$$

where j is the largest odd integer such that $3j \leq k$ (if j is found to be odd then j is j+1) and $c_{m,k}$ is the coefficient of the k^{th} degree term in(for m=2i-1 or 2i and m>1)

$$\{\delta_a(p)\}^m = \{ \sum_{i=1}^{\infty} \varepsilon_{2i+1,1} \; p^{2i+1} \}^m \qquad (2.8.11)$$

In Eq.(2.8.9), $c_{m,k}$ represents the coefficient of the k^{th} term of p in the expansion of the error phase polynomial $\delta_a(p)$ raised to the power m($\{\delta_a(p)\}^m$). As described earlier, using Eqs.(2.8.7) and (2.8.8) the coefficients b_i's can be determined. Then using Eqs.(2.8.9), (2.8.10) and (2.8.11) the coefficients $\varepsilon_{2i+1,1}$ of the error phase function can be determined in a recursive manner. Thus any number of coefficients $\varepsilon_{2i+1,1}$ can be determined.

With respect to the phase function which is an infinite series several observations can be made. Firstly, for a given n^{th} order polynomial, there exists only n independent coefficients in the phase function. The rest of the coefficients are dependent on the first n coefficients. Secondly, these can be generated recursively. Depending on the accuracy requirement this infinite series can be truncated to the required number of terms.

In the next section we shall show that an approximation technique can be developed by considering the new variables b_i's as new parameters.

2.9 AN APPROXIMATION PROCEDURE:

From the foregoing discussion the existence of the direct relation between the coefficients of the phase function(represented as an infinite odd series) and the coefficients of the polynomial $P_{a,n}(p)$ has been established. Using the above results and properties, it now remains to develop an approximation procedure for obtaining a all-pole analog transfer function approximating a specified constant group delay over a specified band of frequencies(or bandwidth Bw) which includes the stability constraints. The approximation shall be carried out using the least mean square error criterion.

The objective is to obtain a stable low-pass, all-pole, n^{th} order filter such that the group delay of the filter approximates a desired constant group delay in a specified bandwidth Bw. The parameters of the filter shall be the new variables namely, α_a, b_1, $b_3, \ldots,$ and b_{2i-3} which will define the component or the elemental values of the realization of the filter(which is a LC low-pass network terminated in resistances) in terms of the determinants of the generating matrix as well as stability constraints. It is noted that the coefficients $a_{k,n}$'s are generated indirectly. This is in direct contrast to the method contrast to the method adopted in [8] where the denominator of the transfer function is obtained, either as a product of factors or as a polynomial. In such cases the objective function parameters are the coefficients of the transfer function. When the transfer function is realizable, it has been observed that the small variations of the coefficients of the transfer function would cause large variations in the value of the elements of the realized network. On the other hand, when β_i's(functions of parameters α_a,and b_i's), the cofficients in the continued fraction expansion of even by odd parts representing the elemental values are varied, large variations in the values of the coefficients $d_{k,n}$'s of the transfer function do not occur [47]. The present method incorporates this property.

The objective function in this approximation is defined as

$$Ob(\alpha_a,\bar{X}_{an-1}) = \sum_{i=1}^{m}(\tau_{an}(\omega_i,\alpha_a,\bar{X}_{an-1}) - \tau_{sp})^2 \qquad (2.9.1)$$

where α_a, $\bar{X}_{an-1}$ are the parmaters

$\tau_{an}(\omega_i, \alpha_a, \bar{X}_{an-1})$ is the group delay function,

n is the order of the filter,

m is the number of points considerd in the specified band width Bw,

ω_i is the frequency interval equal to Bw/(m-1) and

τ_{sp} is the specified group delay. This is minimized subject to the stability constraints as

$$(-1)^{(i(i-1)/2)} |A_{a,i}| > 0 \qquad for(1 \le i \le n+1)$$

(The actual evaluation of the determinants is discussed in Section 2.7.)

It is seen that the elemental values of the ladder network is obtained directly.

Example 2.9.1:

It is required to design a fourth order all-pole analog filter such that its group delay response approximates a specified group delay τ_{sp} in a bandwidth of 3.5 radians per seconds.

The transfer function is

$$T_{a,4}(p) = \frac{c_{a4}}{P_{a,4}(p)} \qquad (2.9.2)$$

where

$$P_{a,4}(p) = 1 + a_{1,4}p + a_{2,4}p^2 + a_{3,4}p^3 + a_{4,4}p^4 \qquad (2.9.3)$$

and $c_{a4} = 1$

The group delay function is

$$\tau_{a,4}(\omega, a_{k,4}\text{'s}) = \frac{g_0 + g_2\omega^2 + g_4\omega^4 + g_6\omega^6}{h_0 + h_2\omega^2 + h_4\omega^4 + h_6\omega^6 + h_8\omega^8} \qquad (2.9.4)$$

where

$$g_0 = a_{1,4}$$

$$g_2 = a_{2,4} - 3\,a_{3,4}$$

$$g_4 = a_{2,4}\,a_{3,4} - 3\,a_{3,4}\,a_{1,4}$$

$$g_6 = a_{3,4}\,a_{4,4}$$

and

$$h_0 = 1$$

$$h_2 = a_{1,4}^2 - 2\, a_{2,4}$$

$$h_4 = a_{2,4}^2 - 2\, a_{1,4}\, a_{4,4} + 2\, a_{4,4}$$

$$h_6 = a_{3,4}^2 - 2\, a_{2,2}\, a_{4,4}$$

$$h_8 = a_{4,4}^2$$

The group delay function Eq.(2.9.4) is a function of and the coefficients $a_{k,4}$'s. The next step required is to transform the coefficients parameters $a_{k,4}$'s to the new variables α_a, b_1, b_3 and b_5. This is done as follows.

The matrix equation Eq.(2.2.19) for order four is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ f_{2,1} & -f_{2,2} & 1 & 0 \\ f_{3,1} & -f_{3,2} & f_{2,1} & -f_{2,2} \\ f_{4,1} & -f_{4,2} & f_{3,1} & -f_{3,2} \end{bmatrix} \begin{bmatrix} a_{1,4} \\ a_{2,4} \\ a_{3,4} \\ a_{4,4} \end{bmatrix} = \begin{bmatrix} f_{2,2} \\ f_{3,2} \\ f_{4,2} \\ f_{5,2} \end{bmatrix}$$

where

$$f_{2,1} = \frac{\alpha_a^2}{2!}$$

$$f_{3,1} = \frac{\alpha_a^4}{4!} + \alpha_a\, b_1$$

$$f_{4,1} = \frac{\alpha_a^6}{6!} + \frac{\alpha_a^3}{3!}\, b_1 + \alpha_a\, b_3 + b_4$$

$$f_{2,2} = \alpha_a$$

$$f_{3,2} = \frac{\alpha_a^3}{3!} + b_1$$

$$f_{4,2} = \frac{\alpha_a^{5}}{5!} + \frac{\alpha_a^{2}}{2!} b_1 + b_3$$

$$f_{5,2} = \frac{\alpha_a^{7}}{7!} + \frac{\alpha_a^{4}}{4!} b_1 + \frac{\alpha_a^{2}}{2!} b_3 + \alpha_a b_4 + b_5$$

The above matrix equation is solved analytically and the solution is

$$a_{1,4} = \alpha_a$$

$$a_{2,4} = \frac{\dfrac{\alpha_a^{8}}{105} - \dfrac{\alpha_a^{5}}{5} b_1 + \dfrac{2}{3}\alpha_a^{3} b_3 - \alpha_a b_5 + b_1 b_5}{|A_{a,4}|}$$

$$a_{3,4} = \frac{\dfrac{2}{945}\alpha_a^{9} - \dfrac{\alpha_a^{6}}{15} b_1 + \dfrac{\alpha_a^{4}}{3} b_3 + 2\alpha_a b_1 b_3 - \alpha_a^{2} b_5 - b_1^{3}}{|A_{a,4}|}$$

$$a_{4,4} = \frac{|A_{a,5}|}{|A_{a,4}|}$$

where

$$|A_{a,4}| = \frac{\alpha_a^{6}}{45} - \frac{\alpha_a^{3}}{3} b_1 + \alpha_a b_3 - b_1^{2}$$

$$|A_{a,5}| = \frac{\alpha_a^{10}}{4725} - \frac{\alpha_a^{7}}{105} b_1 + \frac{\alpha_a^{5}}{15} b_3 - \frac{\alpha_a^{3}}{3} b_5 + \alpha_a^{2} b_1 b_3 + b_1 b_5 - b_3^{2} - \alpha_a b_1^{3}$$

The coefficients $a_{k,4}$'s are thus obtained as a function of a set of variables α_a, b_1, b_3 and b_5. This gives $\tau_{a4}(\omega, a_{k,4}\text{'s})$ as a function of

α_a and b_i's. The objective function is

$$Ob(\omega_a, \overline{X}_{a,3}) = \sum_{i=1}^{m} (\tau_{a4}(\omega_i, \alpha_a, \overline{X}_{a,3}) - \tau_{sp})^2$$

and the stability constraints are

1) $|A_{a,1}| > 0$

2) $-|A_{a,2}| > 0$

3) $-|A_{a,3}| > 0$

4) $|A_{a,4}| > 0$

5) $|A_{a,5}| > 0$

The stability constraints can be further simplified as follows.

The first constraint is always unity as the determinant of the generating matrix $[A_{a,1}]$ is unity. Hence, there are actually four constraints. These are

1) $\alpha_a > 0$

2) $\frac{\alpha_a^3}{3} - b_1 > 0$

3) $|A_{a,4}| > 0$

4) $|A_{a,5}| > 0$

The above problem is a non-linear least squares data fitting problem. The algorithm described by Fletcher is used to minimize the objective function [48]. This minimization algorithm requires an initial approximation to the variables α_a, b_1, b_3 and b_5. These are determined as follows.

The relation between the variables and the coefficient $a_{k,4}$'s are established in Section 2.8. From equation Eq.(2.8.9) the variables α_a, b_1, b_3 and b_5 are obtained as a function of the coefficients $a_{k,4}$'s of the polynomial $P_{a,4}(p)$. They are

$$\alpha_a = \frac{a_{1,4}}{a_{0,4}}$$

$$b_1 = a_{3,4} - a_{1,4}\, a_{2,4} + \frac{a_{1,4}^2}{2!}\, a_{1,4} - \frac{a_{1,4}^3}{3!}$$

$$b_3 = I_5 - I_2\, b_1$$

$$b_5 = I_7 - I_4\, b_1 - I_2\, b_3$$

where

$$I_2 = a_{2,4} - a_{1,4} \cdot a_{1,4} + \frac{a_{1,4}^2}{2!}$$

$$I_4 = a_{4,4} - a_{3,4}\, a_{1,4} + \frac{a_{1,4}^2}{2!}\, a_{2,4} - \frac{a_{1,4}^3}{3!}\, a_{1,4}\, \frac{a_{1,4}^4}{4!}$$

$$I_5 = -a_{4,4}\, a_{1,4} - a_{3,4} \frac{a_{1,4}^2}{2!} - a_{2,4} \frac{a_{1,4}^3}{3!} + a_{1,4} \frac{a_{1,4}^4}{4!} - \frac{a_{1,4}^5}{5!}$$

and

$$I_7 = -a_{4,4} \frac{a_{1,4}^3}{3!} + a_{3,4} \frac{a_{1,4}^4}{4!} - a_{2,4} \frac{a_{1,4}^5}{5!} + a_{1,4} \frac{a_{1,4}^6}{6!} - \frac{a_{1,4}^7}{7!}$$

The flowchart for the optimization is given in Figure 2.9.1. It is seen that this flowchart is for any order n. A known analog all-pole low-pass filter such as Bessel, Butterworth, Chebyshev, ... etc, are chosen to determine the corresponding new variables α_a, b_1, b_3 and b_5. These variable values are used as initial guess values as required by the optmization algorithm. Figure 2.9.3 shows the various group delay responses. In the Table 2.9.2, the elemental values, the deviation of the group delay from the specified group delay τ_{sp} (which is normalized to unity), the percentage root means square error, and the sum of the like kind elements are tabulated. By adopting similar procedures, the results for the cases n = 3 and n = 5 are obtained and these are shown in Tables 2.9.1 and 2.9.3 respectively. Figures 2.9.2 and 2.9.4 show the responses corresponding to these cases. Figure 2.9.5 is a low-pass LC-ladder network terminated in resistances at both ends. Figures 2.9.6, 2.9.7 and 2.9.8 show the magnitude responses corresponding to these cases.

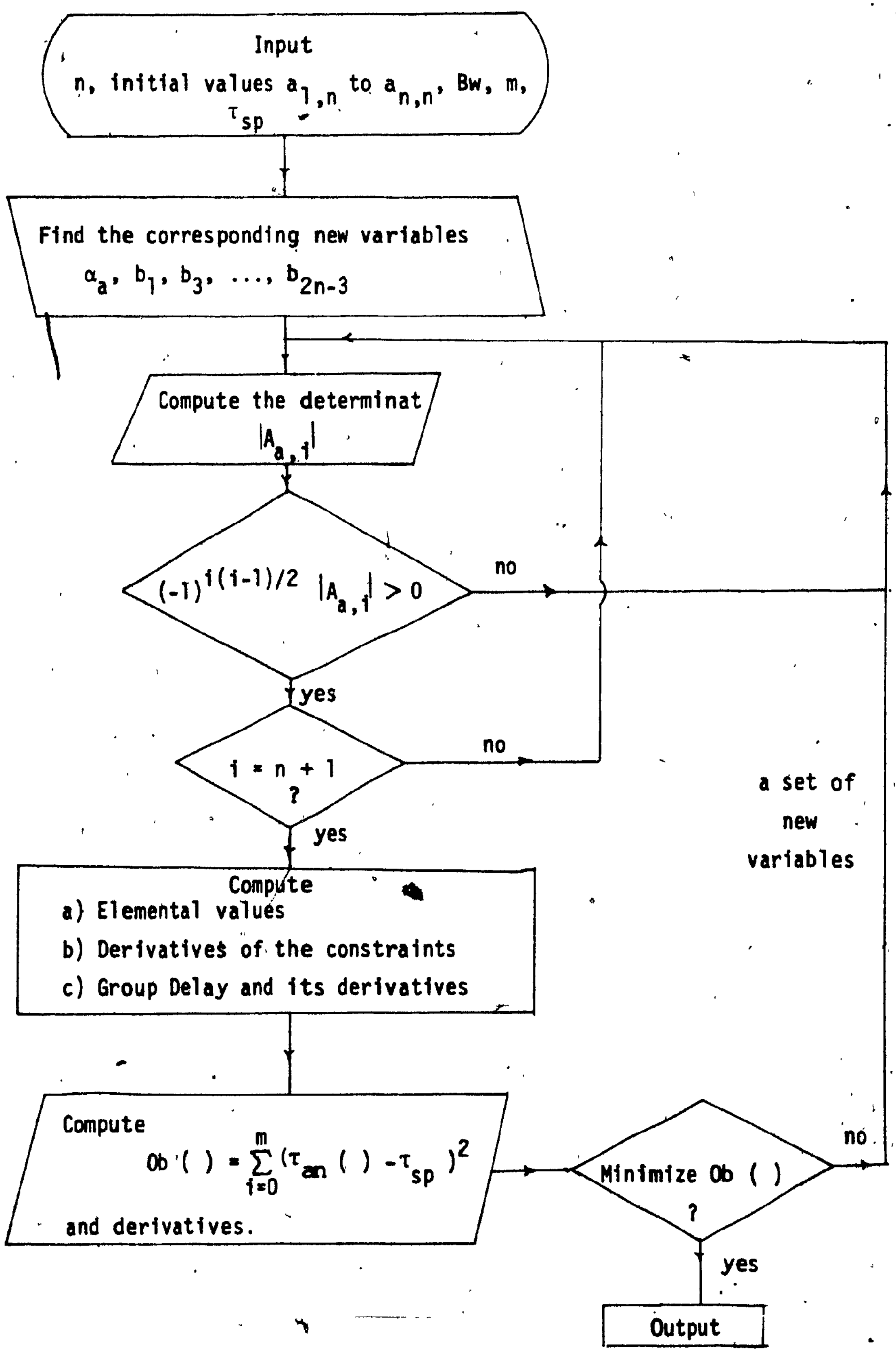


Fig.2.9.1. Flowchart for the Approximation Procedure: Analog Domain.

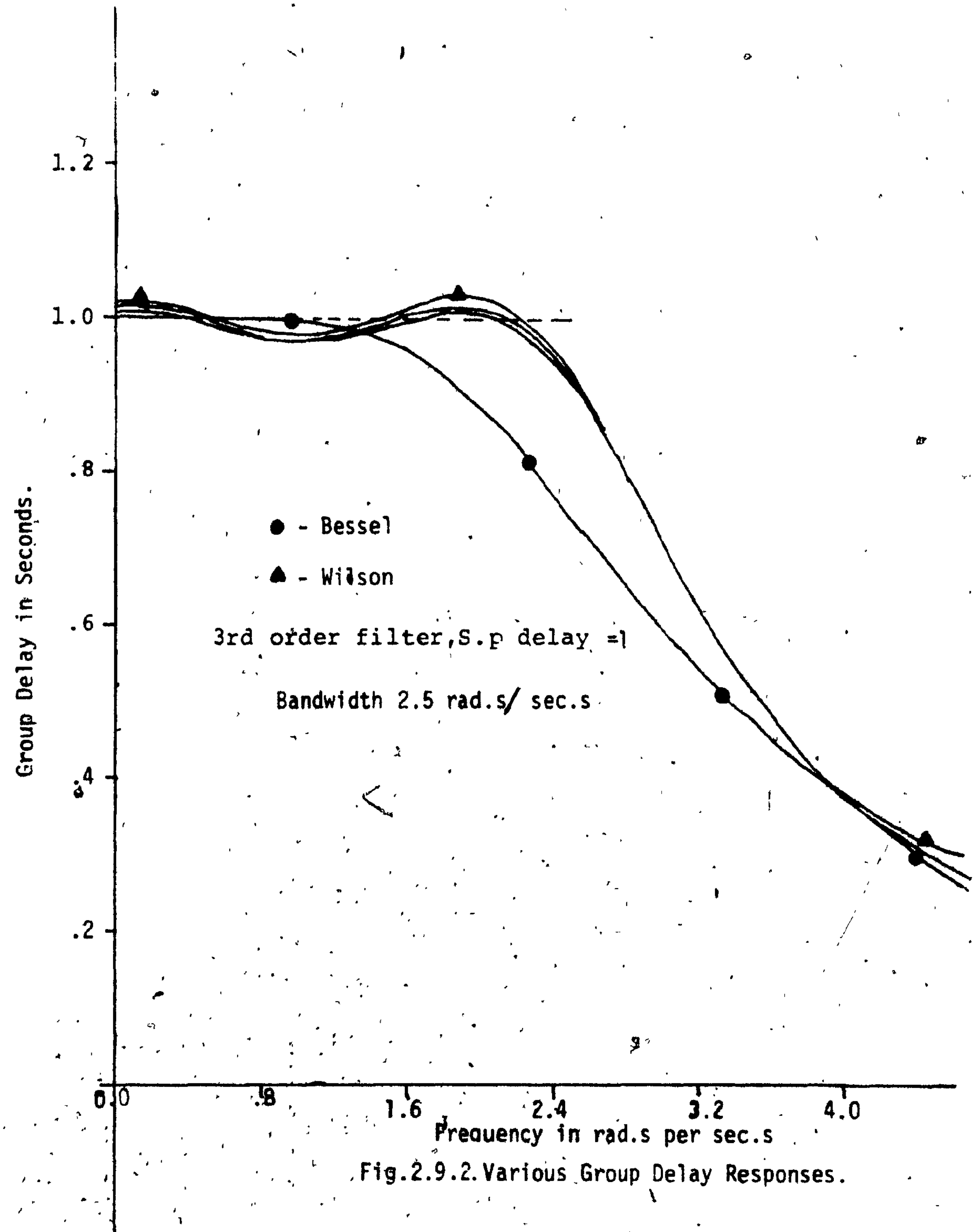


Fig.2.9.2. Various Group Delay Responses.

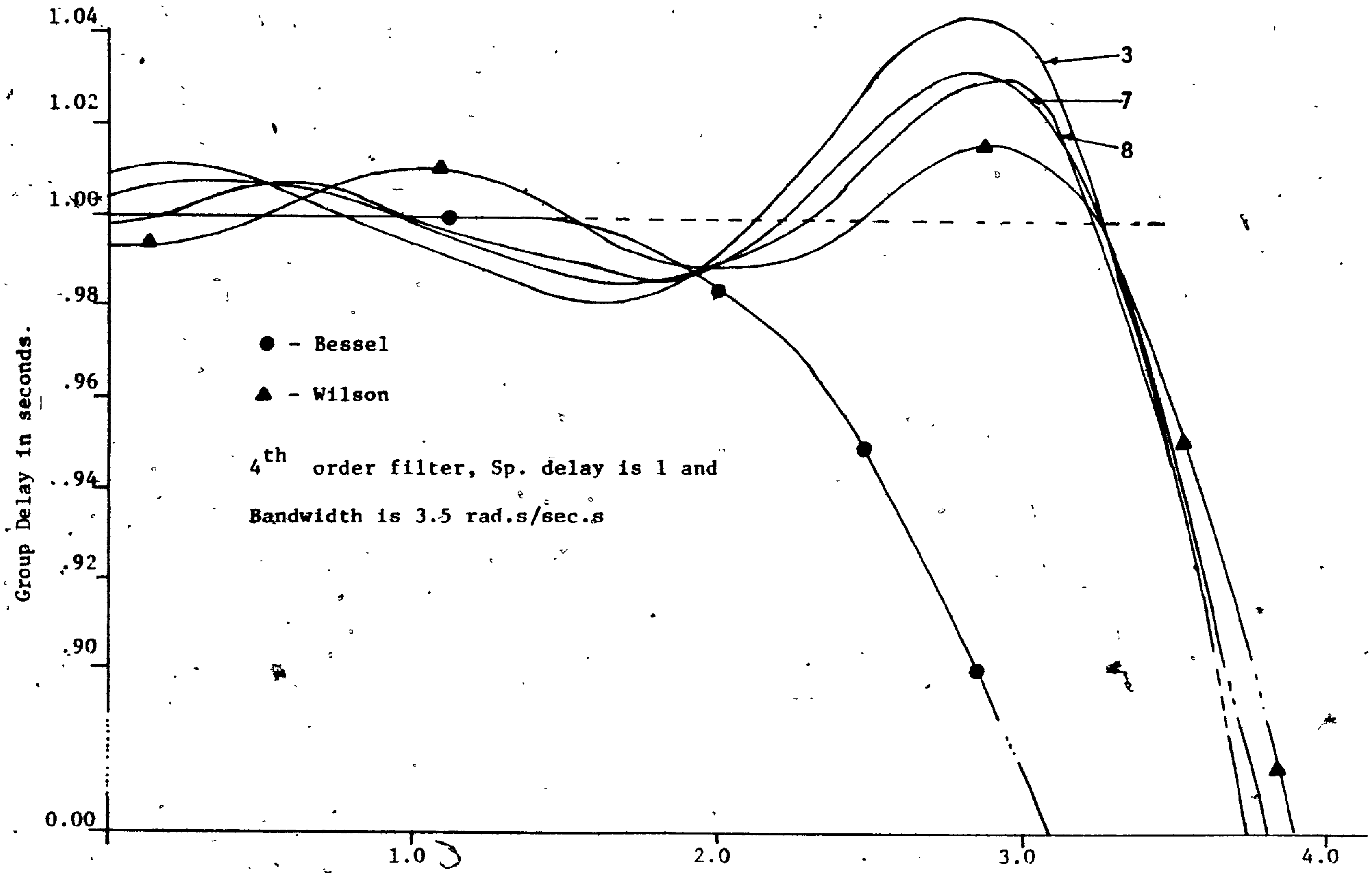


Fig.2.9.3. Various group delay response of a 4th order analog filter.

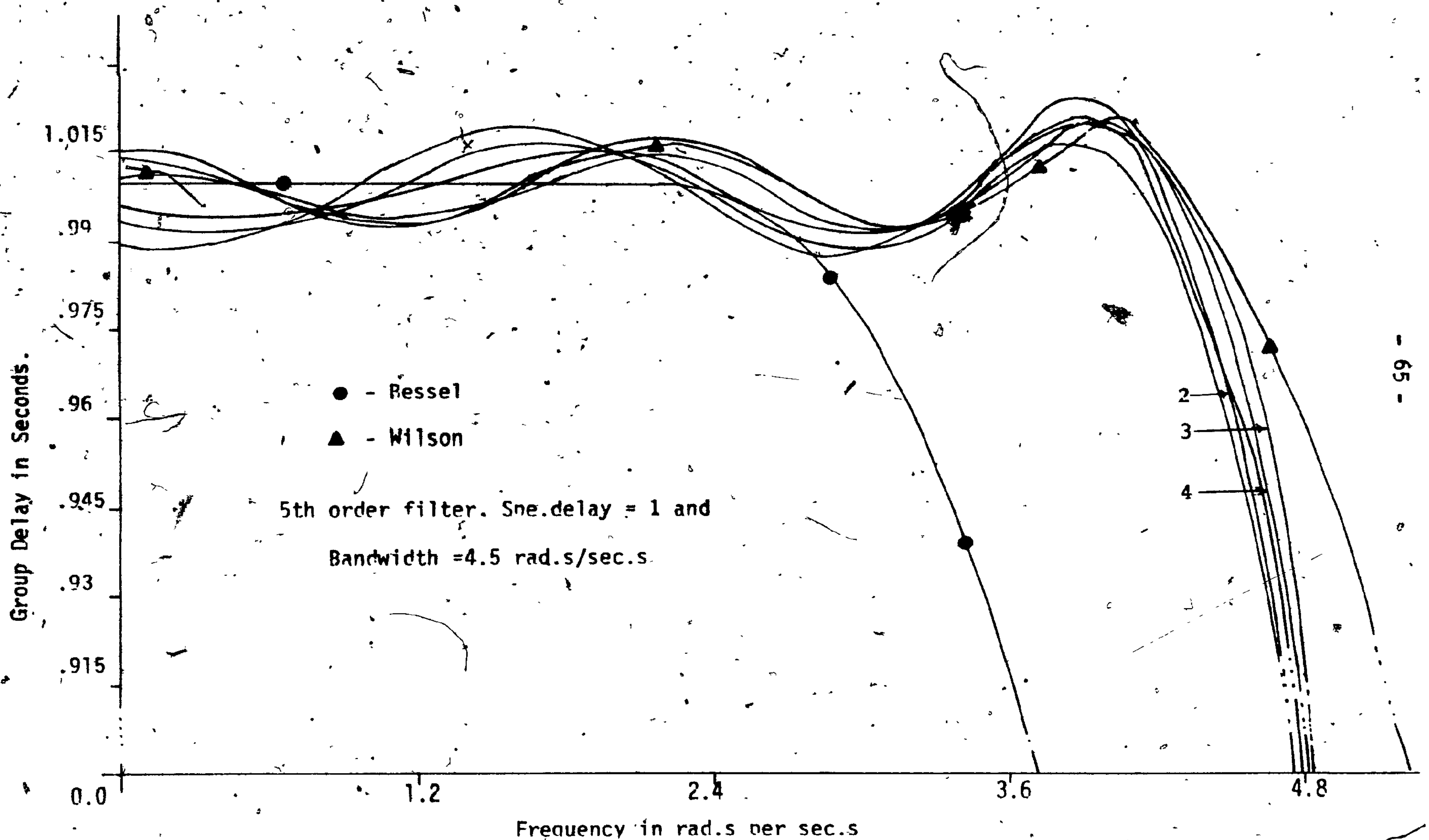
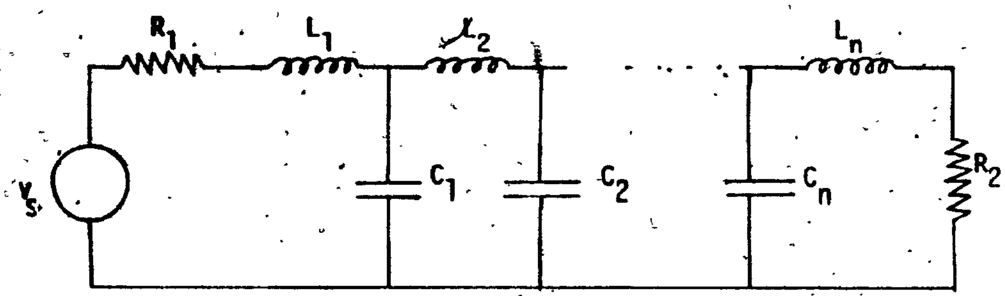


Fig.2.9.4 Various Group Delay Responses: Analog Domain.



For n odd: $C_n = 0$, and $L_n \neq 0$

For n even: $L_n = 0$, and $C_n \neq 0$

Fig.2.9.5. LC-ladder network terminated in resistances at both ends.

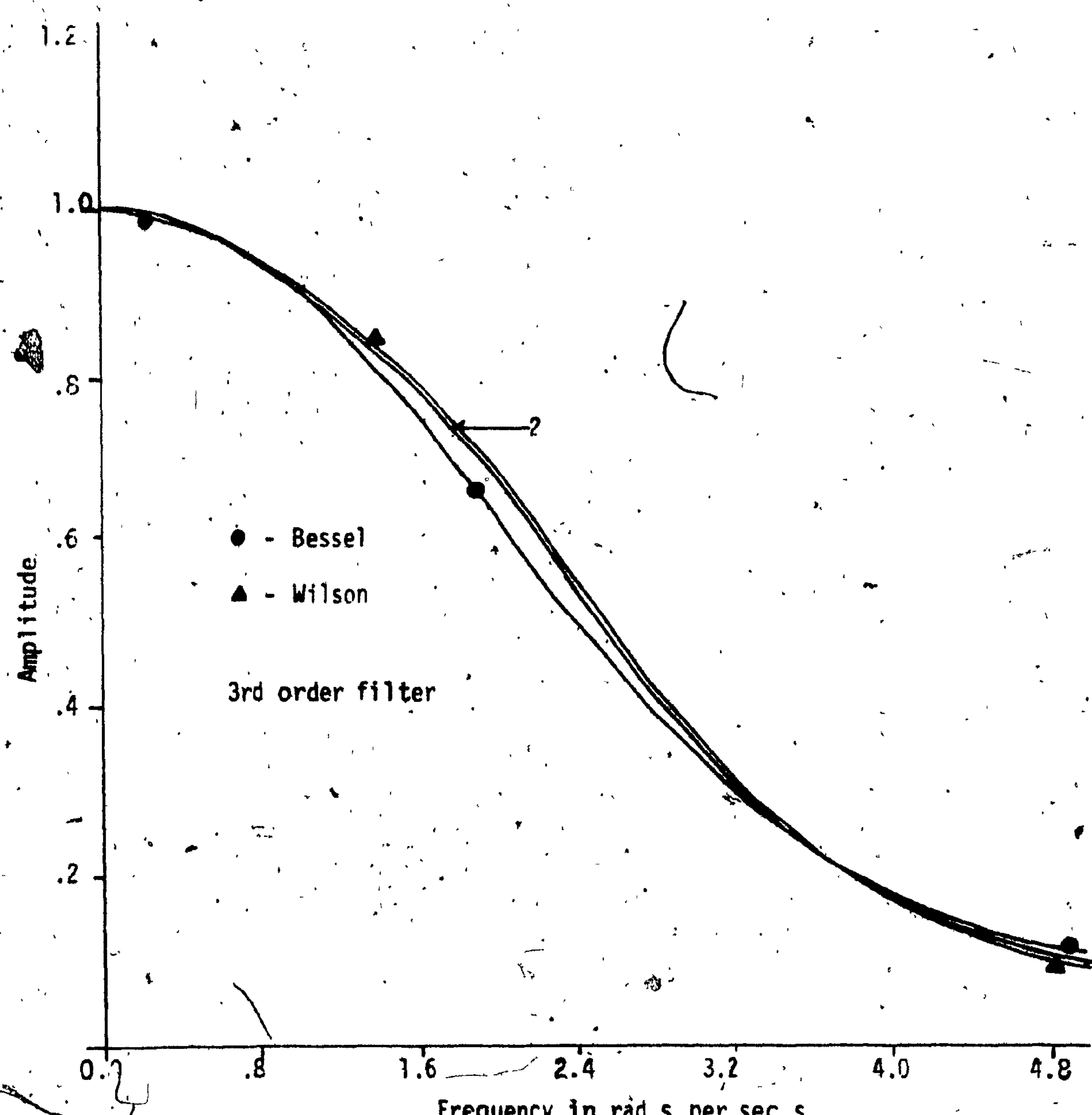


Fig.2.9.6. Various Amplitude Response

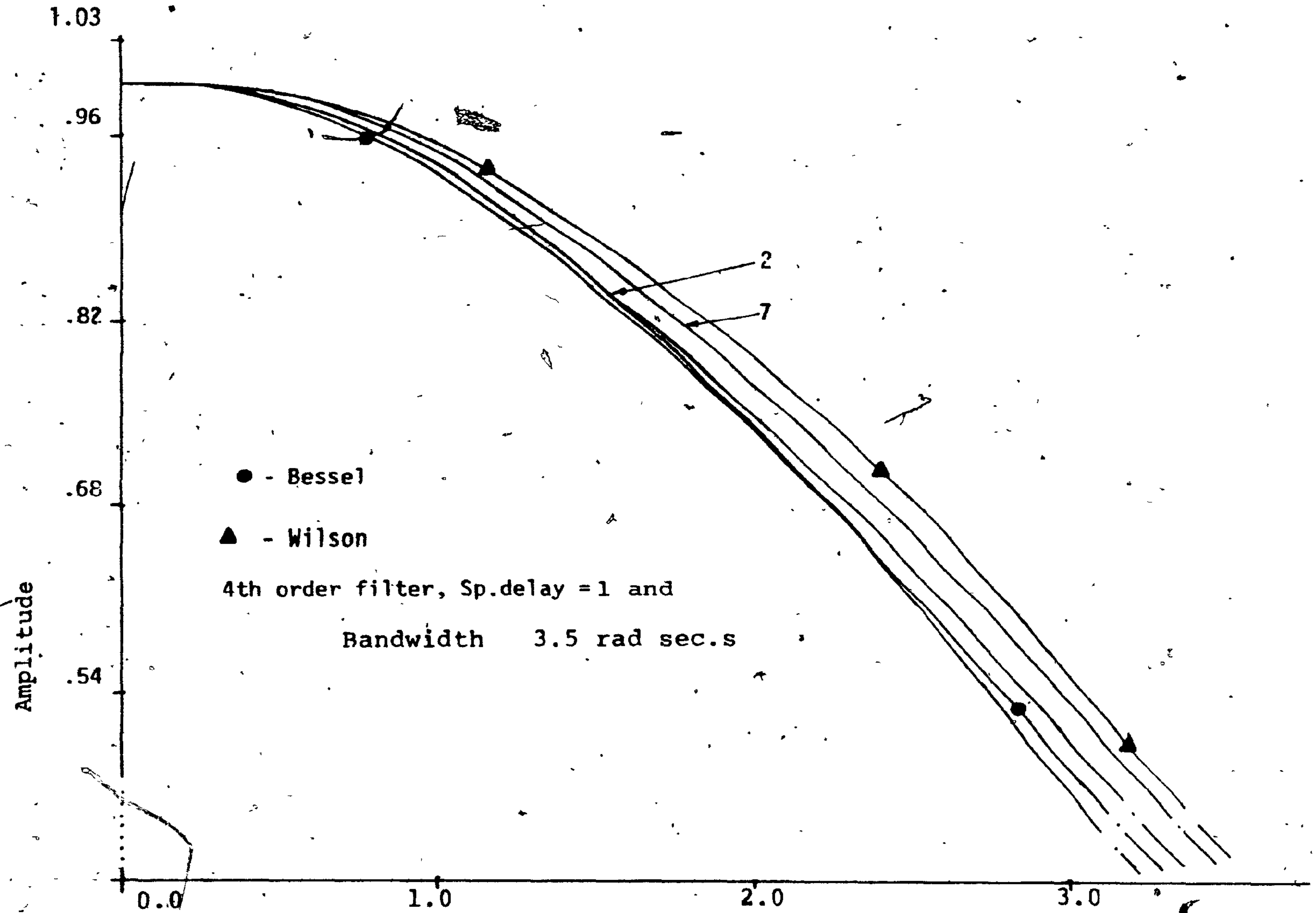


Fig.2.9.7. Various Amplitude Responses.

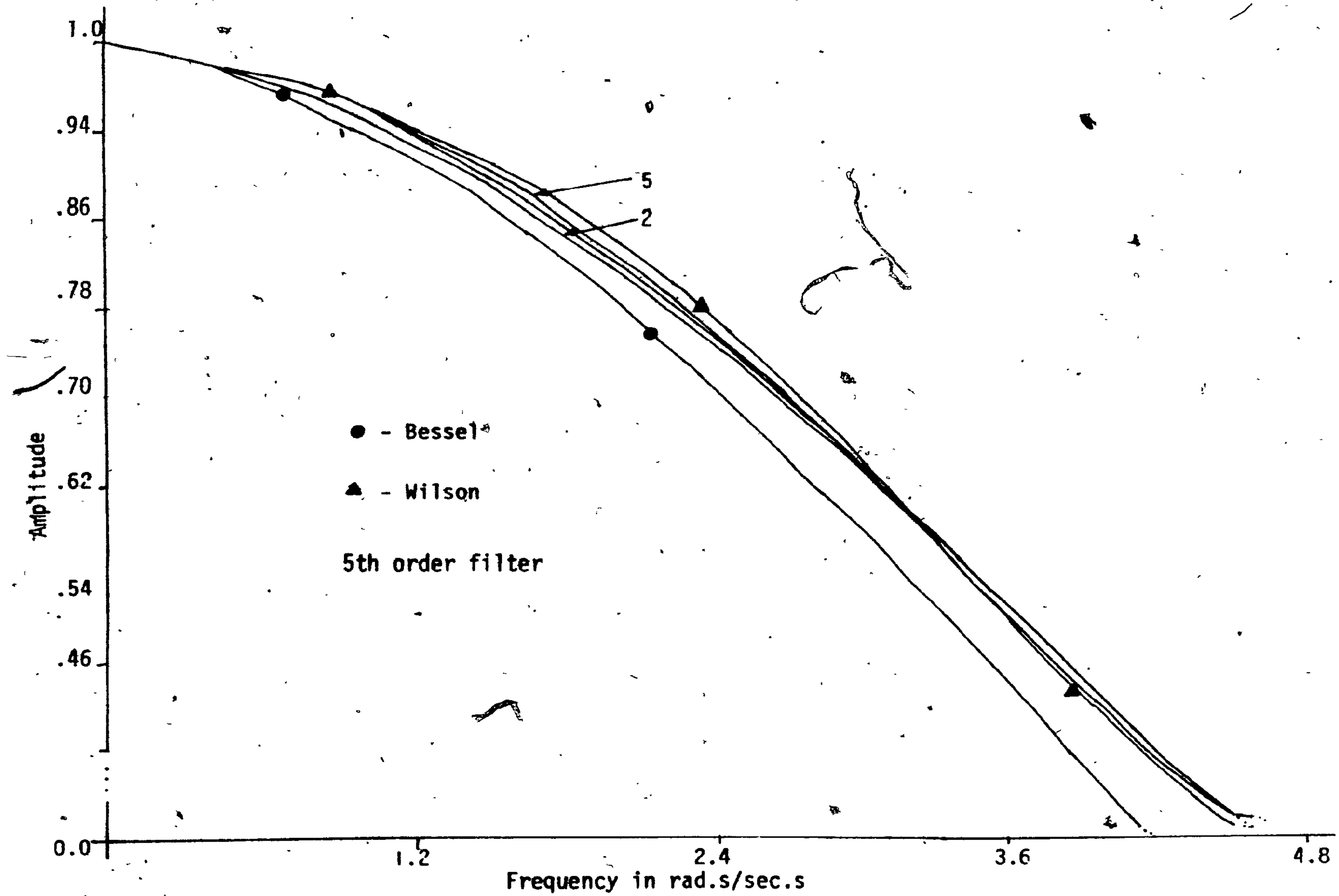


Fig.2.9.8 Varicus Amplitude Responses: Analog domain.

Table 2.9.1

Ite. No.	Elemental values			Sum of like kind elements		% r.m.s. error
	L_1	C_1	L_2	L_1+L_2	C_2	
1	1.0	3.0	5.0	6.0	3.0	8.704172065
2	.980041	3.2734	3.50956	4.4896	3.2734	2.195313879
3	.981874	3.26525	3.46505	4.44692	3.26525	2.073872900
4	.981776	3.26601	3.46459	4.44636	3.26601	2.073826789
5	.981778	3.26603	3.464547	4.446325	3.26603	2.073826744

Table 2.9.2

Iteration	Elemental Values				L_1+L_2	C_1+C_2	% r.m.s error
	L_1	C_2	L_2	C_2			
1	1.0	3.0	5.0	7.0	6.0	10.0	7.97024
2	1.01041	2.89087	6.12237	2.67956	7.13278	5.57043	19.6710
3	.991111	3.00636	5.41266	4.51842	6.40377	7.52478	1.93367
4	.998936	2.96370	5.59472	4.17694	6.59366	7.14065	2.18001
5	.99521	2.98385	5.50067	4.44825	6.49588	7.43211	1.50594
6	1.00149	2.95277	5.63203	4.21094	6.63351	7.16370	1.50837
7	.99933	2.96468	5.57742	4.34947	6.57675	7.31415	1.30315
8	1.00430	2.93906	5.68451	4.17761	6.68881	7.11667	1.23454
9	1.00734	2.92293	5.74672	4.13810	6.75407	7.06103	1.01901
10	1.00940	2.91192	5.79093	4.09452	6.80033	7.00644	.976848
11	1.01035	2.90683	5.81121	4.07770	6.82155	6.98453	.968853
12	1.01037	2.90687	5.82095	4.08025	6.82132	6.98714	.968609

Table 2.9.3

Ite. No.	Elemental Values					$L_1+L_2+L_3$	C_1+C_2	% r.m.s error
	L_1	C_1	L_2	C_2	L_3			
1	1.0	3.0	5.0	7.0	9.0	15.0	10.0	7.4935878
2	1.00731	3.00649	4.88792	8.17971	5.37062	11.2659	11.1862	.8722958
3	1.00738	3.00657	4.87750	8.29719	4.88258	10.7675	11.3038	.7597358
4	1.00435	3.01862	4.83109	8.47984	4.74456	10.5800	11.4985	.6431743
5	1.00331	3.02102	4.81586	8.55804	4.63886	10.4580	11.5791	.5730598
6	1.00094	3.02892	4.78508	8.68597	4.53915	10.3252	11.7149	.5107661
7	.998823	3.03503	4.75969	8.79764	4.44426	10.2028	11.8327	.4863268
8	.996871	3.04064	4.73745	8.89438	4.37145	10.1058	11.9350	.4629005
9	.995975	3.04229	4.72786	8.94256	4.32832	10.0522	11.9848	.4608977
10	.995632	3.04285	4.72430	8.96941	4.31491	10.0348	12.0033	.4607420

2.10. SUMMARY AND DISCUSSIONS:

In this chapter, a method is developed which will generate the denominator polynomial $P_{a,n}(p)$ of an all-pole analog transfer function in order to approximate constant group delay. Starting from an all-pass function(obtained from the transfer function considered), a set of new variables (α_a and b_i's) is obtained which determines the error between the actual and the constant group delays. This results in the generating matrix $[A_{a,n}]$ whose elements are functions of α_a and b_i's. Some important properties of the elements of $[A_{a,n}]$ and the principal minors of $[A_{a,n}]$ are discussed. In particular, it is shown that the principal minors of the generating matrix are equivalent to the corresponding Hurwitz determinants formulated by the coefficients $a_{k,n}$'s of $P_{a,n}(p)$. This will enable us to obtain the various stability constraints in terms α_a and b_i's. In addition, a recurrence relationship is obtained which will permit us to obtain higher order polynomials starting from lower-order ones and incorporating the stability constraints.

Using the above properties, an optimization procedure is formulated in order to approximate a constant group dealy. It is shown that the structure of $[A_{a,n}]$ considerably reduces the computational complexity in that the order of any determinant is approximately reduced by 68.75% as compared with that of Gaussian method.

In this procedure, it is shown that the objective function and the stability constraints are obtained as functions of the parameters that are related to the phase function expressed as an infinite series. It is shown that the Bessel polynomial is one particular case(all b_i's are equal to zero) of this procedure and a large number of responses can be obtained depending on the extent of minimization of the objective function. In addition, the elemental values of the realized LC-ladder network terminated in resistances is also obtained simultaneously.

CHAPTER III

CONSTANT GROUP DELAY APPROXIMATION OF 1-D LOW-PASS DIGITAL FILTERS

3.1 INTRODUCTION:

In this chapter, we shall consider the generation of a digital transfer function approximating a specified constant group delay. The various analytical properties are first developed and these are effectively used to minimize the error between the constant group delay and the actual group delay.

3.2 FORMULATION OF THE GENERATING MATRIX:

In this approximation, we consider the Richard's variable

$$t = c_b \operatorname{Tanh}(pT/2) \tag{3.2.1}$$

where T is the the sampling period, c_b is a positive constant, and p is the complex frequency variable defined as $p = \Sigma + j\Omega$.

By making use of the relationship

$$z = e^{pT} \tag{3.2.2}$$

we have

$$t = c_b \frac{z-1}{z+1} \tag{3.2.3}$$

where z is the variable in the digital domain.

This means that starting from a strictly Hurwitz polynomial in the variable t, one can get a polynomial in z which has all its zeros within the unit circle. Also, as the solution to the approximation problem in the t-domain and the digital z-domain is the same except for realization, we have

$$\Phi_d(t)\Big|_{t = c_b \tanh(pT/2)} = \Phi_u(z)\Big|_{z = e^{pT}} \tag{3.2.4}$$

where $\Phi_d(t)$ and $\Phi_u(z)$ are respective phase functions in the t-domain and the z-domain.

In terms of the Richard's variable t, the all-pole transfer function in the t-domain can be represented as

$$T_{d,n}(t) = \frac{c_{dn}}{P_{d,n}(t)} \tag{3.2.5}$$

where c_{dn} is a positive constant and

$$P_{d,n}(t) = \sum_{i=0}^{n} d_{i,n}\, t^i, \text{ with } d_{0,n} = 1 \tag{3.2.6}$$

The variable p can be expressed as a function of the Richard's variable t as

$$pT/2 = \sum_{i=0}^{\infty} \frac{\left(\frac{t}{c_b}\right)^{2i+1}}{2i+1}$$

$$= f(t) \qquad (3.2.7)$$

The polynomial $P_{d,n}(t)$ as a function of $j\Omega T/2$ is expressed as

$$P_{d,n}(j\Omega T/2) = |P_{d,n}(j\Omega T/2)|\ e^{\Phi_d(j\Omega T/2)} \qquad (3.2.8)$$

where phase function $\Phi_d(j\Omega T/2)$ is

$$\Phi_d(j\Omega T/2) = j\Omega T/2\ \alpha_d - \delta_d(j\Omega T/2) \qquad (3.2.9)$$

In Eq.(3.2.9) α_d represents the phase slope and $\delta_d(j\Omega T/2)$ is the error phase polynomial in the discrete domain given as

$$\delta_d(pT/2) = \sum_{i=1}^{\infty} \zeta_{2i+1,1}(pT/2)^{2i+1} \qquad (3.2.10)$$

where $\zeta_{2i+1,1}$'s are real coefficients and the second right hand suffix 1 indicates the power of $\delta_d(pT/2)$. This error phase polynomial is an odd infinite series.

We can now consider the all pass transfer function in the t-domain with t as the independent variable.

$$\frac{P_{d,n}(t)}{P_{d,n}(-t)} = e^{2\,\Phi_d(t)}$$

$$= e^{2\alpha_d \, pT/2 - 2\delta_d(pT/2)} \Big|_{pT/2 = f(t)} \tag{3.2.11}$$

we have

$$e^{2\alpha_d \, pT/2} = \left[\frac{[1 + t/c_b]}{[1 - t/c_b]}\right]^{\alpha_d} \tag{3.2.12}$$

Substituting Eq.(3.2.12) in Eq.(3.2.11) and rearranging terms we get

$$P_{d,n}(t)\,[1 - t/c_b]^{\alpha_d} - P_{d,n}(-t)\,[1 + t/c_b]^{\alpha_d}$$

$$= P_{d,n}(-t)\,[1 + t/c_b]^{\alpha_d}\,\{e^{-2\delta_d(pT/2)} - 1\} \tag{3.2.13}$$

where f(t) (Eq.(3.2.7)) is substituted for pT/2.

The left hand side of the above expression can be expressed as

$$P_{d,n}(t)\,[1 - t/c_b]^{\alpha_d} - P_{d,n}(-t)\,[1 + t/c_b]^{\alpha_d}$$

$$= 2\sum_{i=0}^{\infty}\{\sum_{k=0}^{i}[\frac{\alpha_d}{k!}\prod_{\ell=1}^{k-1}(\alpha_d - \ell)]\,d_{k,n}\,(-1)^k\}\,t^{2i+1}$$

$$= 2\sum_{i=0}^{\infty}\{\sum_{k=0}^{2i+1} f_k\,d_{i-k,n}(-1)^k\}\,t^{2i+1} \tag{3.2.14}$$

where f_k's are factorial polynomials given as

$$f_k = \frac{\alpha_d}{k!}\prod_{\ell=1}^{k-1}(\alpha_d - \ell) \tag{3.2.15}$$

The Eq.(3.2.14) is obtained by the infinite series expansion of the term $[1+t/c_b]^{\alpha_d}$ by the binomial series. The right hand side of Eq.(3.2.13) can be expressed as

$$P_{d,n}(-t)\,[1 + t/c_b]^{\alpha_d}\,\{e^{-2\delta_d(pT/2)} - 1\}\Big|_{pT/2 = f(t)}$$

$$= \{b_{1,d}(t/c_b)^3 + \sum_{i=1}^{\infty} b_{i,d}(t/c_b)^{i+2}\} \qquad (3.2.16)$$

The coefficients $b_{i,d}$'s are obtained by the expansion of the infinite series due to the exponential function $e^{-2\delta_d(pT/2)}$ and the binomial series expansion of $[1+t/c_b]^{\alpha_d}$. These coefficients along with the coefficient α_d are considered as new parameters. $b_{i,d}$'s in Eq.(3.2.16) and the coefficients $\zeta_{2i+1,1}$'s in Eq.(3.2.10) are related as follows:

For k odd

$$b_{k,d} = -(c_b)^{k+2} \sum_{k_1=1}^{j/2} \frac{2^{2k_1 - 1}}{(2k_1 - 1)!}\,\nu_{m,k} \qquad (3.2.17)$$

and for k even

$$b_{k,d} = (c_b)^{k+2} \sum_{k_1=1}^{j/2} \frac{2^{2k_1}}{(2k_1)!}\,\nu_{m,k} \qquad (3.2.18)$$

where j is the largest integer such that $3j \leq k$ (if j is odd then j=j+1) and $\nu_{m,k}$ (m is $2k_1$-1 or $2k_1$) is the coefficient of the k^{th} powered term of the variable t in the expansion of

$$\{\delta_d(t)\}^m = \{\sum_{k=0}^{\infty} \eta_{2k+1,1}\, t^{2k+1}\}^m \qquad (3.2.19)$$

and

$$\eta_{2k+1,1} = \sum_{j=1}^{k} \zeta_{2j+1,1} \xi_{2j+1,2k+1} \qquad (3.2.20)$$

where $\xi_{2j+1,2k+1}$ is the coefficient of $(2k+1)^{th}$ powered term of t in the expansion of

$$\{\sum_{i=0}^{\infty} \rho_{2i+1,1} \, t^{2i+1}\}^{2j+1} \qquad (3.2.21)$$

with $$\rho_{2i+1,1} = \frac{2}{2j+1}(T/c_b)^{2i+1} \qquad (3.2.22)$$

Using the above equations the coefficients $\nu_{m,k}$'s(m is 2i-1 or 2i) are obtained.

Eq.(3.2.13) is to be solved, that is, $d_{k,n}$'s, the coefficients of the polynomial $P_{d,n}(t)$ are to be obtained as a solution for a given set of values of the variables α_d and $b_{i,d}$'s. A set of linear equations can be formed from which the coefficients $d_{k,n}$'s of the polynomial $P_{d,n}(t)$ are obtained as a solution. Using the relationship Eq.(3.2.3) the desired denominator polynomial of a digital transfer function can be obtained.

The solution $d_{k,n}$'s is a function of $b_{i,d}$'s and α_d. Substituing Eqs.(3.2.14) and (3.2.16) in Eq.(3.2.13) and equating corresponding odd powered terms on both sides, a set of infinite number of linear independent equations can be deduced. Of these, only the first n equations shall be considered, because of the following reason: When we obtain the coefficients $d_{k,n}$'s from these equations in terms of the variables α_d and $b_{i,d}$'s, all the remaining coefficients of the phase function can be evaluated.

Now the set of n linear equations can be expressed as

$$[A_{d,n}]\ [X_{dn}] = [B_{dn}] \qquad (3.2.23)$$

where $[A_{d,n}]$ is square matrix of order n, $[X_{dn}]$ is column vector of length n, and $[B_{dn}]$ is also a column vector of length n.

The elements of the matrix $[A_{d,n}]$, the vectors $[X_{dn}]$ and $[B_{dn}]$ are respectively as follows.

$$[A_{d,n}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ x_{2,1} & -x_{2,2} & 1 & 0 & 0 & \cdots & 0 \\ x_{3,1} & -x_{3,2} & x_{2,1} & -x_{2,2} & 1 & \cdots & 0 \\ . & . & . & . & . & \cdots & 0 \\ . & . & . & . & . & \cdots & 0 \\ . & . & . & . & . & \cdots & 0 \\ . & . & . & . & . & \cdots & 0 \\ . & . & . & . & . & \cdots & F_{0,d} \\ . & . & . & . & . & \cdots & F_{1,d} \\ . & . & . & . & . & \cdots & . \\ . & . & . & . & . & \cdots & . \\ x_{n,1} & -x_{n,2} & x_{n-1,1} & -x_{n-1,2} & . & \cdots & F_{n,d} \end{bmatrix} \qquad (3.2.24)$$

where

$$x_{i,k} = \{f_{2i-1-k} + f_{2i-4-k}\, b''_{1,d} + \sum_{j=1}^{2i-k-5} f_{2i-1-k-j}\, b_{j+2,d}\}^{*}$$

---

* when suffix is less than zero, that particular f is zero.

for $k = 1,2$ and $x_{1,1} = 1$ (3.2.25)

where for n odd: $F_{0,d} = 1,\ F_{1,d} = x_{2,1}, \ldots\ldots, F_{n,d} = x_{(n+1)/2,1}$, and

for n even: $F_{0,d} = 0, F_{1,d} = -x_{2,2}, \ldots\ldots, F_{n,d} = x_{(n/2)+1,2}$. (3.2.25)

$$[X_{dn}] = (d_{1,n}\ d_{2,n}\ \ldots\ldots\ldots\ d_{n,n})' \tag{3.2.27}$$

$$[B_{dn}] = (x_{2,2}\ \ x_{3,2}\ \ \ldots\ \ x_{n+1,2})' \tag{3.2.28}$$

(Prime indicates transpose.)

Eqs.(3.2.23) and (3.2.24) are because of consideration of the set of first n consecutive linear indpendent equations. The n^{th} element of the vector $[B_{dn}]$ is a function of (2n-3)-variables. Among these, n-variables (α_d, $b_{1,d}$, $b_{3,d}$, $b_{5,d}, \ldots, b_{2n-3,d}$) with odd suffixes are independent. The remaining (n-3)-variables ($b_{2,d}$, $b_{4,d}, \ldots, b_{2n-4,d}$) with even suffixes can be obtained as a function of independent variables. The vector $[X_{dn}]$ is obtained as a solution to Eq.(3.2.23) for a set of values for the new parameters (α_d, $b_{1,d}$, $b_{3,d}$, b_4, $b_5, \ldots,$ $b_{2n-3,d}$). Thus the vector $[X_{dn}]$ containing the coefficients $d_{k,n}$'s of the polynomial $P_{d,n}(t)$ as its elements is generated. For all $b_{i,d}$'s equal to zero, the generated polynomial and the resulting filter will have constant group delay in the maximally flat sense about the origin. Before attempting to establish the strictly Hurwitz nature of the polynomial $P_{d,n}(t)$, we wish to discuss some properties of the elements of the matrix $[A_{d,n}]$. Henceforth, this matrix $[A_{d,n}]$ is called the generating matrix of the discrete domain.

3.3 SOME PROPERTIES OF THE ELEMENTS OF THE GENERATING MATRIX:

- It is evident from the elements of the matrix $[A_{d,n}]$ that there are only (2n-1) different elements which are to be evaluated. Any $(i,j)^{th}$ element of the matrix can be evaluated from the following expression.

$$x_{i,j} = (-1)^{j+1} \{f_{2i-1-j} + f_{2i-4-j}\, b_{1,d} + \sum_{j=1}^{2i-5-k} f_{2i-5-k-j}\, b_{j+2,d}\} \quad (3.3.1)$$

Several theorems are proved below.

Theorem 3.3.1

$$x_{i+1,j+2} = x_{i,j} \quad (3.3.2)$$

Proof:

Substituting (i+1) for i and (j+2) for j in Eq.(3.3.1), the power suffixes in Eq.(3.3.1) will remain the same. Hence, the $(i,j)^{th}$ element is exactly equal to $(i+1,j+2)^{th}$ element**.

Hence the result follows.

The elements corresponding to two consecutive columns(in the ascending order) and a row are related through partial forward differences with respect to the phase slope α_d. Before we present these properties as theorems, we wish to present the two well known difference and anti-difference properties of the factorial polynomial Eq.(3.2.15)

**: $(-1)^{(j+1)} = (-1)^{(j+3)}$.

as Lemmás.

<u>Lemma 3.3.1</u>

If $f_k(\alpha_d)$ is the k^{th} order factorial polynomial then its first forward partial difference with respect to the phase slope α_d is $f_{k-1}(\alpha_d)$ of order k-1; that is,

$$\Delta^{(1)*}_{\alpha_d} f_k(\alpha_d) = f_{k-1}(\alpha_d) \tag{3.3.3}$$

<u>Proof:</u>

We have

$$f_k(\alpha_d) = \frac{\alpha_d}{k!} \prod_{i=0}^{k-1} (\alpha_d - i) \tag{3.3.4}$$

$$f_k(\alpha_d) = f_k(\alpha_d+1) - f_k(\alpha_d) \tag{3.3.5}$$

$$= \frac{\alpha_d+1}{k!} \prod_{i=1}^{k-1} (\alpha_d - i) - \frac{\alpha_d}{k!} \prod_{i=1}^{k-1} (\alpha_d - i) \tag{3.3.6}$$

which simplifies to

* $\Delta^{(1)}_{\alpha_d}$ is the symbolic representation for partial difference operator, which means the difference shall be taken with respect to the variable α_d, the other variables remaining invariant [49].

$$= \frac{\alpha_d}{k!} \prod_{i=1}^{k-2} (\alpha_d - i) \qquad (3.3.7)$$

$$= f_{k-1} \qquad (3.3.8)$$

Hence the result follows.

<u>Lemma 3.3.2</u>

If $f_k(\alpha_d)$ is the k^{th} order factorial, then its first anti-difference with respect to phase solpe α_d is $f_{k+1}(\alpha_d)$ of order k+1.

$$\Delta_{\alpha_d}^{(-1)**} f_k(\alpha_d) = f_{k+1}(\alpha_d) \qquad (3.3.9)$$

<u>Proof:</u>

We have the expression for $f_k(\alpha_d)$ as Eq.(3.3.4). Then from Lemma 3.3.1, the first partial difference of $f_{k+1}(\alpha_d)$ is $f_k(\alpha_d)$. Therefore, the first partial anti-difference of $f_k(\alpha_d)$ is

$$\Delta_{\alpha_d}^{(-1)} f_k(\alpha_d) = f_{k+1}(\alpha_d) + c_{0,d} \qquad (3.3.10)$$

where $c_{0,d}$ is a constant of anti-difference. As $f_{k+1}(\alpha_d)$ has to satisfy Eq.(3.3.4) it cannot have a non zero value. Therefore, $c_{0,d}$ has to be zero.

Hence the result follows.

** $\Delta_{\alpha_d}^{(-1)}$ is the symbolic representaion for the partial anti-difference operator [49].

Theorem 3.3.2

The element $x_{i,j+1}$ is the negative of the first partial forward difference of the element $x_{i,j}$ with respect to the phase slope α_d.

$$x_{i,j+1} = -\Delta^{(1)}_{\alpha_d} x_{i,j} \tag{3.3.11}$$

Proof:

Taking the first partial forward difference of the Eq (3.3.1) with respect to α_d and multiplying by (-1) and applying Lemma 3.3.1, we have

$$\Delta^{(1)}_{\alpha_d} (x_{i,j}) = (-1)^{j+1} \{f_{2i-j} + f_{2i-5-j}\, b_{1,d} + \sum_{k=1}^{2i-6-j} f_{2i-6-j-k}\, b_{k+2,d}\}$$

$$= x_{i,j+1} \tag{3.3.12}$$

Hence, the result follows.

The elements corresponding to two consecutive rows (in the descending order) and a column are related through second order partial forward differences with respect to the phase slope α_d and is given by the following theorem.

Theorem 3.3.3

$$x_{i-1,j} = \Delta^{(2)}_{\alpha_d} x_{i,j} \tag{3.3.13}$$

Proof:

Taking the second partial difference of Eq.(3.3.1) with respect to the phase slope α_d, and using Lemma 3.3.1, we get

$$\Delta^{(2)}_{\alpha_d} (x_{i,j}) = (-1)^{j+1} \{f_{2i-2-j} + f_{2i-6-j}\, b_{1,d} + \sum_{k=1}^{2i-7-j} f_{2i-7-j-k}\, b_{k+2,d}\} \qquad (3.3.14)$$

It can be easily verified that

$$x_{i,j} = \Delta^{(2)}_{\alpha_d} x_{i-1,j}$$

Hence the result follows.

The elements of the generating matrix $[A_{d,n}]$ are related through anti-difference relationships. These are presented as follows.

<u>Theorem 3.3.4</u>

$$x_{i,j} = -\Delta^{(-1)}_{\alpha_d} x_{i,j+1} + (-1)^{j+1}\, b_{2i-j-3} \qquad (3.3.15)$$

<u>Proof:</u>

$$\text{Let } x_{i,j} = -\Delta^{(-1)}_{\alpha_d} x_{i,j+1} + c_{1,d} \qquad (3.3.16)$$

The first term in the right hand side is true from Theorem 3.3.2. It remains to establish that the constant of anti-difference $c_{1,d}$ is $(-1)^{j+1}\, b_{2i-j-3}$.

From Eq.(3.3.16), we have

$$c_{1,d} = x_{i,j} + \Delta^{(-1)}_{\alpha_d} x_{i,j+1} \tag{3.3.17}$$

Substituting expressions for $x_{i,j}$ by using Eq.(3.3.1) and Theorem 3.3.2, Eq.(3.3.16) results as

$$c_{1,d} = (-1)^{j+1} b_{2i-j-3}, \tag{3.3.18}$$

Hence the result follows.

Theorem 3.3.5

$$x_{i,j} = \Delta^{(-2)}_{\alpha_d} x_{i-1,j} + (-1)^{j+1} b_{2i-j-4} \tag{3.3.19}$$

Proof:

$$\text{Let } x_{i,j} = \Delta^{(-2)}_{\alpha_d} x_{i-1,j} + c_2 \tag{3.3.20}$$

The first term in the right hand side of the above equation is true from Theorem 3.3.3. It remains to show that the constant of anti-difference $c_{2,d}$ is $(-1)^{j+1} b_{2i-j-4}$.

From Eq.(3.3.20) we have

$$c_{2,d} = x_{i,j} - \Delta^{(-2)}_{\alpha_d} x_{i-1,j} \tag{3.3.21}$$

Substituting expressions for $x_{i,j}$ by using Eq.(3.3.1) and Theorem 3.3.3, Eq.(3.3.21) results as

$$c_{2,d} = (-1)^{j+1} b_{2i-j-4} \tag{3.3.22}$$

Hence the result follows.

In the next section, we shall discuss some properties of the determinants resulting from the generating matrix $[A_{d,n}]$.

3.4 PROPERTIES OF THE DETERMINANTS OF THE GENERATING MATRIX:

By Cramer's rule, we have

$$d_{k,n} = \frac{|C_{d,k}|}{|A_{d,n}|} \qquad (3.4.1)$$

where, $d_{k,n}$'s are the coefficients of the k^{th} powered term of the denominator polynomial $P_{d,n}(t)$ and $|C_{d,k}|$ is the determinant of the generating matrix $[A_{d,n}]$ with its k^{th} column replaced by the vector $[B_{dn}]$. There exist some relationships among these determinants, and hence among the coefficients $d_{k,n}$'s of the polynominal $P_{d,n}(t)$. In this section these relations are discussed.

Theorem 3.4.1

$$|C_{d,n}| = (-1)^{(n)} \; |A_{d,n+1}| \qquad (3.4.2)$$

Proof:

The first element in the matrix $[A_{d,n+1}]$ is always unity. Therefore, its determinant is the determinant of the submatrix of order n obtained by deleting the first row and the first column of the matrix $[A_{d,n+1}]$. In this submatrix if the first, the second, the third,... and the n^{th} columns are replaced respectively with the second, the third, the fourth,..., $(n-1)^{th}$ and the first columns, the matrix $[C_{d,n}]$

is obtained where the n^{th} column will have the elements of the vector $-[B_{dn}]$. When n is odd or even, there will be (n-1) number of column changes. Taking into account the negative sign of the n^{th} column, the number of column changes is n. In general, this can be put as in Eq.(3.4.2).

Hence the result follows.

<u>Theorem 3.4.2.</u>

$$d_{n,n} = (-1)^{(n)} \frac{|A_{d,n+1}|}{|A_{d,n}|} \tag{3.4.3}$$

<u>Proof:</u>

From Cramer's rule, we have

$$d_{n,n} = \frac{|C_{d,n}|}{|A_{d,n}|} \tag{3.4.4}$$

From Theorem 3.4.1, substituting for $|C_{d,n}|$, we get

$$d_{n,n} = (-1)^{(n)} \frac{|A_{d,n+1}|}{|A_{d,n}|}$$

Hence the results follows.

Using Theorems 3.3.2 and 3.3.3, the elements of the generating matrix $[A_{d,n}]$ can be shown to be sequentially related to the partial forward differences of the n^{th} element of the vector $[B_{dn}]$ with respect to α_d. This n^{th} element is designated as $G_d(\alpha_d)$ which is equal to $x_{n+1,2}$.

In order to prove further properties of the generating matrix $[A_{d,n}]$, the matrix $[C_{d,k}]$, and the relations among the coefficients $d_{k,n}$'s of the polynomial $P_{d,n}(t)$, the elements of the generating matrix $[A_{d,n}]$ are expressed as the respective partial forward differences of the function $x_{n+1,2}$ which is the n^{th} element of the column vector $[B_{dn}]$ given by $G_d(\alpha_d)$ as

$$G_d(\alpha_d) = \{f_{2n-1} + f_{2n-4}\, b_{1,d} + \sum_{k=1}^{2n-5} f_{2n-5-k}\, b_{k+2,d}\} \quad (3.4.5)$$

and the matrix $[C'_{d,n}]$ will be

$$[A_{d,n}] = \begin{bmatrix} G_d^{(2n-1)} & 0 & 0 & \cdots 0 \\ G_d^{(2n-3)} & -G_d^{(2n-2)} & G_d^{(2n-1)} & \cdots 0 \\ G_d^{(2n-5)} & -G_d^{(2n-4)} & G_d^{(2n-3)} & \cdots 0 \\ \cdot & \cdot & \cdot & \cdots 0 \\ \cdot & \cdot & \cdot & \cdots 0 \\ \cdot & \cdot & \cdot & \cdots J_{0,d} \\ \cdot & \cdot & \cdot & \cdots J_{1,d} \\ \cdot & \cdot & \cdot & \cdots \cdot \\ G_d^{(3)} & -G_d^{(4)} & G_d^{(5)} & \cdots \cdot \\ G_d^{(1)} & -G_d^{(2)} & G_d^{(3)} & \cdots J_{n,d} \end{bmatrix} \quad (3.4.6)$$

where

for n odd: $J_{0,d} = G_d^{(2n-1)},\ J_{1,d} = G_d^{(2n-3)}, \ldots,\ J_{n,d} = G_d^{(n)}$ (3.4.7)

for n even: $J_{0,d} = -G_d^{(2n-2)},\ J_{1,d} = -G_d^{(2n-4)}, \ldots,\ J_{n,d} = -G_d^{(n)}$ (3.4.8)

and $G_d^{(2n-1)}$ is $(2n-1)^{th}$ partial forward difference with respect to α_d.

The elements of the matrix $|C_{d,n}|$ are also expressed as the partial forward differences of the function $G_d(\alpha_d)$.

$$|C_{d,n}| = \begin{vmatrix} G_d^{(2n-1)} & 0 & 0 & \cdots & & G_d^{(2n-2)} \\ G_d^{(2n-3)} & -G_d^{(2n-2)} & G_d^{(2n-1)} & \cdots & & G_d^{(2n-4)} \\ \cdot & \cdot & \cdot & \cdots & & \cdot \\ \cdot & \cdot & \cdot & \cdots & & \cdot \\ \cdot & \cdot & \cdot & \cdots & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdots & L_{0,d} & \cdot \\ \cdot & \cdot & \cdot & \cdots & L_{1,d} & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ G_d^{(3)} & -G_d^{(4)} & G_d^{(5)} & \cdots & L_{k-1,d} & G_d^{(2)} \\ G_d^{(1)} & -G_d^{(2)} & G_d^{(3)} & \cdots & L_{k,d} & G_d \end{vmatrix} \quad (3.4.9)$$

where

for odd n: $L_{0,d} = -G_d^{(2n-2)}, L_{1,d} = -G_d^{(2n-4)}, .., L_{k,d} = -G_d^{(n-1)}$ (3.4.10)

and for even n: $L_{0,d} = G_d^{(2n-1)}, L_{1,d} = G_d^{(2n-3)}, ..., L_{k,d} = G_d^{(n-1)}$ (3.4.11)

Next, the relation of the determinant $|C_{d,n-1}|$ with respect to the partial forward difference of the determinant $|C_{d,n}|$ with respect to α_d is established.

Theorem 3.4.3

The determinant $|C_{d,n-1}|$ is equal to the partial forward difference of the determinant $|C_{d,n}|$ with respect to α_d.

$$|C_{d,n-1}| = \Delta^{(1)}_{\alpha_d} |C_{d,n}| \tag{3.4.12}$$

Proof:

The first order partial forward difference of the determinant $|C_{d,n}|$ with respect to α_d is the sum of n partially differed determinants. The first partially differed determinant is the determinant of the matrix $[C_{d,n}]$ with its first column elements replaced by the partial forward differences of the elements in the first column of the original matrix $[C_{d,n}]$; the second partially differed determinant is the determinant of the matrix $[C_{d,n}]$ with its second column elements replaced by the partial forward differences of the elements in the second column of the original matrix $[C_{d,n}]$;... and the n^{th} partially differed determinant is the determinant of the matrix $[C_{d,n}]$ with its n^{th} column elements replaced by the partial forward differences of the n^{th} column elements of the original matrix $[C_{d,n}]$. In the k^{th} determinant, the k^{th} column elements (partial forward differences of the original k^{th} column elememts with respect to α_d) are in equivalence to the corresponding elements of the $(k+1)^{th}$ column, except when k is (n-1). And when k is n, k+1 is 1. As the two columns have the same elememnts in order, the value of the determinant is zero. Hence, in the summation of n determinants, (n-1) determinants vanish. The non zero valued determinant is the one whose $(n-1)^{th}$ column elements are the respective partial forward differences of the $(n-1)^{th}$ column elements of the original matrix $[C_{d,n}]$. Therefore, we have

$$\Delta_{\alpha_d}^{(1)}|C_{d,n}| = \begin{vmatrix} G_d^{(2n-1)} & 0 & 0 & \cdot & G_d^{(2n-2)} \\ G_d^{(2n-3)} & -G_d^{(2n-2)} & G_d^{(2n-1)} & 0 & G_d^{(2n-4)} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & W_{0,d} & \cdot \\ \cdot & \cdot & \cdot & W_{1,d} & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ G_d^{(3)} & -G_d^{(4)} & G_d^{(5)} & W_{k-1,d} & G_d^{(2)} \\ G_d^{(1)} & -G_d^{(2)} & G_d^{(3)} & W_{k,d} & G_d \end{vmatrix} \qquad (3.4.13)$$

where

for odd n : $W_{0,d} = -G_d^{(2n-1)}$, $W_{1,d} = -G_d^{(2n-3)}, \ldots$,

$$W_{k-1,d} = -G_d^{(n+2)}, \quad W_{k,d} = -G_d^{(n)}, \qquad (3.4.14)$$

and for even n

$$W_{0,d} = 0, \ W_{1,d} = G_d^{(2n-2)}, \ldots, \ W_{k-1,d} = G_d^{(n+2)}, \ W_{k,d} = G_d^{(n)} \qquad (3.4.15)$$

Interchanging the $(n-1)^{th}$ and n^{th} column in the determinant $\Delta_{\alpha_d}^{(1)}|C_{d,n}|$ we get the determinant $|C_{d,n-1}|$.

Hence the result follows.

Next, we shall establish a relation between the n^{th} and the $n-1^{th}$ coefficients of $P_{d,n}(t)$.

Theorem 3.4.4

The n^{th} coefficient $d_{n,n}$ and the $(n-1)^{th}$ coefficient $d_{n-1,n}$ are related through

$$\frac{d_{n-1,n}}{d_{n,n}} = (-1)^{(n)} \frac{\Delta^{(1)}_{\alpha_d} |A_{d,n+1}|}{|A_{d,n+1}|} \qquad (3.4.16)$$

Proof:

From Theorem 3.4.1 we have

$$|C_{d,n}| = (-1)^{(n)} |A_{d,n+1}| \qquad (3.4.17a)$$

and from Theorem 3.4.3 we have

$$|C_{d,n-1}| = \Delta^{(1)}_{\alpha_d} |C_{d,n}| \qquad (3.4.17b)$$

From Cramer's rule, we have

$$d_{n,n} = \frac{|C_{d,n}|}{|A_{d,n}|} \qquad (3.4.18)$$

and

$$d_{n-1,n} = \frac{|C_{d,n-1}|}{|A_{d,n}|} \qquad (3.4.19)$$

Forming the ratio of Eq.(3.4.18) and Eq.(3.4.19) and substituting for $|C_{d,n}|$ and $|C_{d,n-1}|$, we obtain the required result as Eq.(3.4.16). Hence the result follows.

The determinant of the matrix $|C_{d,n-2}|$ can be expressed as function of the partial forward differences of the determinant of the matrix $|C_{d,n-1}|$ or $|C_{d,n}|$ or $|A_{d,n+1}|$.

Theorem 3.4.5

$$|C_{d,n-2}| = \frac{1}{2} \Delta^{(1)}_{\alpha_d} |C_{d,n-1}| \tag{3.4.20a}$$

$$= \frac{1}{2} \Delta^{(2)}_{\alpha_d} |C_{d,n}| \tag{3.4.20b}$$

$$= \frac{1}{2} (-1)^{(n)} \Delta^{(2)}_{\alpha_d} |A_{d,n+1}| \tag{3.4.20c}$$

Proof:

The same procedure is followed as in the previous case. That is, the determinant $|C_{d,n-1}|$ is partially differed column wise n times and their summation is considered. The partially differed k^{th} column in the determinant $|C_{d,n-1}|$ results in the equivalence of its elements with respect to the elements in $(k+1)^{th}$ column except when k is (n-1) and (n-2). Hence, in the summation of the determinants all the detrminants except two, vanish as they have same respective elements in two of their columns. Therefore we have

$$\Delta^{(1)}_{\alpha_d} |C_{d,n-1}| = |Q_{1,d}| + |Q_{2,d}| \tag{3.4.21}$$

where

$$|Q_{1,d}| = \begin{vmatrix} G_d^{(2n-1)} & 0 & 0 & \cdots & 0 & G_d^{(2n-2)} \\ G_d^{(2n-3)} & -G_d^{(2n-2)} & G_d^{(2n-1)} & \cdots & 0 & G_d^{(2n-4)} \\ G_d^{(2n-5)} & -G_d^{(2n-4)} & G_d^{(2n-3)} & \cdots & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdots & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdots & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdots 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdots U_{0,d} & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdots U_{1,d} & W_{0,d} & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ G_d^{(3)} & -G_d^{(4)} & G_d^{(5)} & \cdots U_{k-1,d} & W_{k-1,d} & G_d^{(2)} \\ G_d^{(1)} & -G_d^{(2)} & G_d^{(3)} & \cdots U_{k,d} & W_{k,d} & G_d \end{vmatrix} \tag{3.4.22}$$

and

$$
|Q_{2,d}| = \begin{vmatrix}
G_d^{(2n-1)} & 0 & 0\cdot\cdot & & & G_d^{(2n-2)} \\
G_d^{(2n-3)} & -G_d^{(2n-2)} & \cdots & & & G_d^{(2n-4)} \\
\cdot & \cdot & \cdots & & & \cdot \\
\cdot & \cdot & \cdots & & & \cdot \\
\cdot & \cdot & \cdots & & 0 & \cdot \\
\cdot & \cdot & \cdots & \Delta^{(-1)}_{\alpha_d} U_{0,d} & 0 & \cdot \\
\cdot & \cdot & \cdots & \Delta^{(-1)}_{\alpha_d} U_{1,d} & \Delta^{(1)}_{\alpha_d} W_{0,d} & \cdot \\
\cdot & \cdot & \cdots & \cdot & \Delta^{(1)}_{\alpha_d} W_{1,d} & \cdot \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\
G_d^{(3)} & -G_d^{(4)} & \cdots & \Delta^{(-1)}_{\alpha_d} U_{k-1,d} & \Delta^{(1)}_{\alpha_d} W_{k-1,d} & G_d^{(2)} \\
G_d^{(1)} & -G_d^{(2)} & \cdots & \Delta^{(-1)}_{\alpha_d} U_{k,d} & \Delta^{(1)}_{\alpha_d} W_{k,d} & G_d
\end{vmatrix}
\qquad (3.4.23)
$$

for odd n

$$U_{0,d} = 0, \quad W_{0,d} = -G_d^{(2n-1)}, \quad \Delta^{(-1)}_{\alpha_d} U_{0,d} = G_d^{(2n-1)}, \quad \Delta^{(1)}_{\alpha_d} W_{0,d} = 0$$

$$U_{1,d} = G_d^{(2n-2)}, \quad W_{1,d} = -G_d^{(2n-3)}, \quad \Delta^{(-1)}_{\alpha_d} U_{1,d} = G_d^{(2n-3)},$$

$$\Delta^{(1)}_{\alpha_d} W_{1,d} = -G_d^{(2n-2)}.$$

$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

$$U_{k-1,d} = G_d^{(n-1)}, \quad W_{k-1,d} = -G_d^{(n+2)}, \quad \Delta^{(-1)}_{\alpha_d} U_{k-1,d} = G_d^{(n)},$$

$$\Delta^{(1)}_{\alpha_d} W_{k-1,d} = -G_d^{(n+3)},$$

$$U_{k,d} = G_d^{(n-3)}, \quad W_{k,d} = -G_d^{(n)}, \quad \overset{(-1)}{\Delta}_{\alpha_d} U_{k,d} = G_{,d}^{(n-3)},$$
$$\overset{(1)}{\Delta}_{\alpha_d} W_{k,d} = -G_d^{(n+1)} \tag{3.4.24}$$

and for n even

$$U_{0,d} = -G_d^{(2n-1)}, \quad W_{0,d} = 0, \quad \overset{(-1)}{\Delta}_{\alpha_d} U_{0,d} = -G_d^{(2n-2)}, \quad \overset{(1)}{\Delta}_{\alpha_d} W_{0,d} = 0$$

$$U_{1,d} = -G_d^{(2n-3)}, \quad W_{1,d} = G_d^{(2n-2)}, \quad \overset{(-1)}{\Delta}_{\alpha_d} U_{1,d} = -G_d^{(2n-4)},$$
$$\overset{(1)}{\Delta}_{\alpha_d} W_{1,d} = G_d^{(2n-1)}$$

$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot$$
$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

$$U_{k-1,d} = -G_d^{(n-1)}, \quad W_{k-1,d} = G_d^{(n)}, \quad \overset{(-1)}{\Delta}_{\alpha_d} U_{k-1,d} = -G_d^{(n)},$$
$$\overset{(1)}{\Delta}_{\alpha_d} W_{k-1,d} = G_d^{(n+1)}$$

$$U_{k,d} = -G_{,d}^{(n-3)}, W_{k,d} = G_d^{(n-2)}, \quad \overset{(-1)}{\Delta}_{\alpha_d} U_{k,d} = -G_d^{(n-2)},$$
$$\overset{(-1)}{\Delta}_{\alpha_d} W_{k,d} = G_d^{(n-1)} \tag{3.4.25}$$

The first determinant $|Q_{1,d}|$ in Eq.(3.4.21) is the determinant $|C_{d,n-2}|$. It remains to show that the second determinant $|Q_{2,d}|$ is also same as the $|C_{d,n-2}|$.

By taking the partial difference of $(n-2)^{th}$ column and partial anti-difference of $(n-1)^{th}$ column of the second determinant $|Q_{2,d}|$ and further applying the Theorems 3.3.2 and 3.3.4, it is clearly seen the existence of the equivalence of the first determinant $|Q_{1,d}|$ to the determinant $|Q_{2,d}|$.

Hence the result follows.

We shall now establish the relation of $d_{n-2,n}$ with respect to the coefficients $d_{n-1,n}$ and $d_{n,n}$.

Theorem 3.4.6

$$\frac{d_{n-2,n}}{d_{n,n}} = \frac{|C_{d,n-2}|}{|C_{d,n}|} \tag{3.4.26a}$$

$$= \frac{1}{2}\,\frac{\Delta^{(1)}_{\alpha_d}\,|C_{d,n-1}|}{(-1)^{(n)}\,|A_{d,n+1}|} \tag{3.4.26b}$$

$$= \frac{1}{2}\,\frac{\Delta^{(2)}_{\alpha_d}\,|A_{d,n+1}|}{(-1)^{(n)}\,|A_{d,n+1}|} \tag{3.4.26c}$$

Proof:

By Cramer's rule we have

$$d_{n-2,n} = \frac{|C_{d,n-2}|}{|A_{d,n}|} \quad \text{and} \tag{3.4.27}$$

$$d_{n,n} = \frac{|C_{d,n}|}{|A_{a,n}|} \tag{3.4.28}$$

Taking the ratio of Eq.(3.4.27) and Eq.(3.4.28) and using Theorems 3.4.3 and 3.4.5, we obtain the required result as Eqs.(3.4.26a) to (3.4.26c).

Hence the results follows.

Thus Theorems 3.4.4 and 3.4.6 establish the relation among the three coefficients.

Theorem 3.4.7

The coefficient $d_{1,n}$ for any order n is always α_d. That is for all n

$$d_{1,n} = \alpha_d \quad (3.4.29)$$

Proof:

The solution vector $[X_{dn}]$ is

$$[X_{dn}] = [A_{d,n}]^{-1} [B_{dn}] \quad (3.4.30)$$

where

$$[A_{d,n}]^{-1} = \text{Adjoint of } [A_{d,n}] / |A_{d,n}| \quad (3.4.31)$$

In the generating matrix $[A_{d,n}]$, the first element is the element corresponding to the first row and first column which is always unity and the rest of the elements in the first row are all zeros. Hence

$$|A_{d,n}| = Z_{11,d} \text{ of } |A_{d,n}| \quad (3.4.32)$$

where $Z_{11,d}$ is the cofactor of $[A_{d,n}]$. Adjoint of $|A_{d,n}|$ is $|C_{d,1}|$ which is $x_{2,2}\, Z_{11,d}$. Therefore

$$d_{1,n} = x_{2,2}$$
$$= \alpha_d \quad (3.4.33)$$

Hence the result follows.

We shall next discuss the Hurwitz nature of the polynomial $P_{d,n}(t)$.

3.5 HURWITZ PROPERTIES OF THE POLYNOMIAL $P_{d,n}(t)$:

As stated earlier, two of the important requirements of a transfer function are the stability and and the realizability. For the all-pole t-domain transfer function, if the denominator polynomial $P_{d,n}(t)$ is strictly Hurwitz, the corresponding digital transfer function is always realizable by some structure.

Whenever it is not easy to generate the polynomial $P_{d,n}(t)$ in the closed or analytical form, generation of the same by numerical or other techniques is unavoidable. In such a technique, it is advantageous to incorporate stability constraints which are to be obtained as a function of a set of parameters. The advantages are the reduction in the computational effort and time.

In this section, a method to generate the stability constraints as a function of the new variables (α_d, $b_{1,d}$, $b_{3,d}$, $b_{5,d}$,...) is explained. It will be shown that these stability constraints are equivalent to the Hurwitz stability criteria.

The denominator polynomial $P_{d,n}(t)$ is expressed as sum of its odd and even parts, that is,

$$P_{d,n}(t) = O_{d,n}(t) + E_{d,n}(t) \qquad (3.5.1)$$

where $O_{d,n}(t)$ and $E_{d,n}(t)$ are odd and even parts of $P_{d,n}(t)$ respectively.

The stability constraints depend on the requirements that the principal minors of the generating matrix should satisfy

$$(-1)^{(i(i-1)/2)} \cdot |A_{d,i}| > 0 \quad \text{for}(1 \leq i \leq n+1) \qquad (3.5.2)$$

or the coefficients in the continued fraction expansion of the even part $E_{d,n}(t)$ to the odd part $O_{d,n}(t)$ shall be positive. First, the continued fraction expansion of $E_{d,n}(t)$ by $O_{d,n}(t)$ about the origin is expressed as a function of Hurwitz determinants. Then, it will be shown that these are equivalent to the determinants of the generating matrix $[A_{d,i}](1 \leq i \leq n+1)$ with respect to their absolute values.

The continued fraction expansion of $E_{d,n}(t)$ and $O_{d,n}(t)$ is

$$\frac{E_{d,n}(t)}{O_{d,n}(t)} = \frac{\gamma_1}{t} + \cfrac{1}{\cfrac{\gamma_2}{t} + \cfrac{1}{+ \cdots + \cfrac{1}{\cfrac{\gamma_{n-1}}{t} + \cfrac{1}{\cfrac{\gamma_n}{t}}}}} \qquad (3.5.3)$$

Where γ_i's are functions of the determinants known as Hurwitz determinants [44]. The coefficient γ_i is

$$\gamma_i = \frac{|H_{i,d}|^2}{|H_{i+1,d}|\ |H_{i-1,d}|} \quad \text{for } (1 \leq i \leq n) \qquad (3.5.4)$$

In Eq.(3.5.4), $|H_{i,d}|$ is the principal minor of order i in the Hurwitz matrix $[H_{n+1,d}]$ of order (n+1). This matrix $[H_{n+1,d}]$ is obtained by arranging the coefficients of the denominator polynomial

$P_{d,n}(t)$ as follows.

Eq.(3.2.2) for $P_{d,n}(t)$ is rewritten as

$$P_{d,n}(t) = 1 + \sum_{k=1}^{n} d_{k,n}\, t^k \tag{3.5.5}$$

The Hurwitz matrix is formed as follows.

For odd n,

$$[H_{n+1,d}] = \begin{bmatrix} 1 & 0 & 0 & \cdot & & \cdot & \cdot & 0 \\ 0 & d_{1,n} & d_{3,n} & \cdot & d_{n,n} & 0 & \cdot & 0 \\ 0 & 1 & d_{2,n} & & d_{n-1,n} & 0 & \cdot & 0 \\ 0 & 0 & d_{1,n} & \cdot & d_{n-2,n} & d_{n,n} & & 0 \\ 0 & 0 & 1 & \cdot & d_{4,n} & \cdot & d_{n-1,n} & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \cdot & & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0\ 0 & 0\ 0 & d_{1,n} & d_{3,n} & \cdot\ \cdot & & & d_{n,n} \\ 0\ 0 & 0\ 0 & 1 & d_{2,n} & \cdot\ \cdot & & & d_{n-1,n} \end{bmatrix} \tag{3.5.6}$$

Similarly, it is simple to construct the Hurwitz matrix $[H_{n+1,d}]$ where n is even. This is shown below.

$$[H_{n+1,d}] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & & 0 \\ 0 & d_{1,n} & d_{3,n} & \cdots d_{7,n} & d_{n-1,n} & & 0 \\ 0 & 1 & d_{2,n} & \cdots d_{6,n} & d_{n-2,n} & d_{n,n} & 0 \\ 0 & 0 & d_{1,n} & \cdots d_{5,n} & d_{n-3,n} & d_{n-1,n} & 0 \\ 0 & \cdot & \cdot & \cdot \;\; \cdot & & \cdot & 0 \\ 0 & 0 & 0 & 0 & \cdot & \cdot \;\; \cdot & 0 \\ 0 & \cdot & & 0 \;\; d_{1,n} & d_{3,n} & \cdot \; d_{n-1,n} & 0 \\ 0 & \cdot & & 1 \;\; d_{2,n} & \cdot & \cdot \; d_{n-2,n} & d_{n,n} \end{bmatrix}$$

(3.5.7)

It is well known that when all the principal minors of the matrix $[H_{n+1,d}]$ are greater than zero, then the polynomial $P_{d,n}(t)$ is strictly Hurwitz [44]. These determinants are

$$|H_{0,d}| = 1 \text{ (an assumption)}$$

$$|H_{1,d}| = 1$$

$$|H_{2,d}| = d_{1,n}$$

$$|H_{3,d}| = d_{1,n}\ d_{2,n} - d_{3,n}$$

$$|H_{4,d}| = d_{3,n}\ (d_{1,n}\ d_{2,n} - d_{3,n}) - d_{1,n}\ (d_{1,n}\ d_{2,n} - d_{3,n})$$

$$\cdot$$

$$\cdot$$

etc, (3.5.8)

The coefficients in the continued fraction expansion of $E_{d,n}(t)/O_{d,n}(t)$ are the γ_i's $(1 \leq i \leq n)$. For the polynomial $P_{d,n}(t)$, the

Routh-Hurwitz array(R-H array) can be constructed. The elements in the first column of Routh-Hurwitz array can be expressed as a ratio of the principal minors of the Hurwitz matrix $[H_{n+1,d}]$. That is, the first column elements of R-H array are

$$(R\text{-}H_d)_{i,1} = \frac{|H_{i+1,d}|}{|H_{i,d}|} \quad \text{for } (1 \le i \le n) \tag{3.5.9}$$

It can also be shown that the coefficients γ_i's can be expressed in terms of the these determinants which are the principal minors of the Hurwitz matrix $[H_{n+1,d}]$. That is,

$$\gamma_i = \frac{|H_{i,d}|^2}{|H_{i-1,d}|\ |H_{i+1,d}|} \quad \text{for } (1 \le i \le n) \tag{3.5.10}$$

and $|H_{0,d}| = |H_{1,d}| = 1$.

Our aim is to obtain γ_i's as a function of the principal minors of the generating matrix $[A_{d,n}]$. We shall now show that there exists a relation between principal minors of the Hurwitz matrix and the generating matrix $[A_{d,n}]$.

Theorem 3.5.1

The determinant of the generating matrix $[A_{d,i}]$ of order i is related to the determinant of the Hurwitz matrix $[H_{i,d}]$ of the same order i as

$$(-1)^{(i(i-1)/2)}\ |A_{d,i}| = |H_{i,d}| \tag{3.5.11}$$

(Proof is given in the Appendix C.)

Theorem 3.5.2

The coefficient γ_i in terms of the determinants of the generating matrix $[A_{d,i}]$ is given by

$$\gamma_i = \frac{|A_{d,i}|^2}{-|A_{d,i+1}|\ |A_{d,i-1}|} \text{ for } (1 \leq i \leq n) \qquad (3.5.12)$$

This result can be proved as a consequence of Theorem 3.5.1 in view of the foregoing discussion.

Theorem 3.5.3

The necessary and sufficient condition for the polynomial $P_{d,n}(t)$ to be strictly Hurwitz is

$$(-1)^{(i(i-1)/2)}\ |A_{d,i}| > 0 \text{ for } (1 \leq i \leq n+1) \qquad (3.5.13)$$

Where $|A_{d,i}|$ $(1 \leq i \leq n+1)$ is the principal minor(of order i) of the generating matrix $[A_{d,n+1}]$.

Proof:

From Theorem 3.5.1, for γ_i to be greater than zero, it is required that

$$-|A_{d,i+1}|\ |A_{d,i-1}| > 0 \text{ for } (1\leq i\leq n) \qquad (3.5.14)$$

The sign of $|A_{d,i}|$ is determined by the order i as given in Eq.(3.5.13). The $|A_{d,i}|$ will have the same negative (or positive) sign for any two consecutive orders(i,i+1), where i is even and when (i(i-1)/2) is odd (or even). In the above expression the difference between the orders of the two matrices is two. Therefore, one of the determinants either $|A_{d,i+1}|$ or $|A_{d,i-1}|$ will have a negative sign. The conclusion is, if $|A_{d,i+1}|$ is less than zero (or greater than zero), then $|A_{d,i-1}|$ is greater than zero (or less than zero). Therefore, the condition Eq.(3.5.13) is necessary and sufficient.

Hence the result follows.

3.6 GENERATION OF $P_{d,n+1}(t)$ BY RECURRENCE RELATION:

The intention is to show that in general, a polynomial of degree (n+1) can be generated from the recurrence relation

$$P_{d,n+1}(t) = P_{d,n}(t) + R_{d,n}(t)\, P_{d,n-1}(t) \qquad (3.6.1)$$

with $P_{d,0}(t) = 1$, $P_{d,1}(t) = 1 + \alpha_d\, t$ and

$$R_{d,n}(t) = t^2 \frac{|A_{d,n-1}|\ |A_{d,n+2}|}{|A_{d,n}|\ |A_{d,n+1}|} \qquad (3.6.2)$$

$$= \frac{P_{d,n+1}(t) - P_{d,n}(t)}{P_{d,n-1}(t)} \qquad (3.6.3)$$

Several authors have developed methods to generate transfer functions by recurrence relations [33-35]. A variety of phase(group delay) responses such as arbitrary phase, equidistant linear phase, maximally flat linear phase,...etc, can be satisfied with these transfer functions. From the recurrence relation, stability and realizability criteria can be established [35]. In our case, these two criteria depend on $R_{d,n}(t)$ in Eq.(3.6.2). In particular, these criteria depend on the

$$\frac{|A_{d,n-1}|\ |A_{d,n+2}|}{|A_{d,n}|\ |A_{d,n+1}|} \qquad (3.6.4)$$

Eq.(3.6.4) is a function of the coefficients of the polynomials $P_{d,n+1}(t)$, $P_{d,n}(t)$, and $P_{d,n-1}(t)$. This function is

$$\frac{d_{n+1-i,n+1} - d_{n+1-i,n}}{d_{n-i-1,n-1}} \qquad (3.6.5)$$

for $i = 0, 1, 2, \ldots, n-1$ and $d_{n+1-i,n} = 0$ for $i = 0$.

<u>Theorem 3.6.1</u>

The polynomial $P_{d,n}(t)$ can be generated from the recurrence relation

$$P_{d,n+1}(t) = P_{d,n}(t) + R_{d,n}(t)\ P_{d,n-1}(t)$$

where

$$R_{d,n}(t) = t^2 \frac{|A_{d,n+2}|\ |A_{d,n-1}|}{|A_{d,n}|\ |A_{d,n+1}|}$$

(Proof is given in Appendix D.)

This leads to a relationship among the coefficients of the polynomial $P_{d,n-1}(t)$, $P_{d,n}(t)$ and $P_{d,n+1}(t)$ which is given by the following theorem.

Theorem 3.6.2

$$\frac{d_{n-\ell,n-1}}{d_{n-k,n-1}} = \frac{d_{n+1-j,n+1} - d_{n+1-j,n}}{d_{n+1-i,n+1} - d_{n+1-i,n}} \qquad (3.6.6)$$

with $\ell \neq k$, $n-\ell \leq n-1$ and $n-k \leq n-1$

$j \neq i$, $n+1-j \leq n+1$, if $n+1-j > n$, then $d_{n+1-j,n+1} = 0$.

$n+1-i \leq n+1$, if $n+1-i > n$, then $d_{n+1-i,n+1} = 0$.

Proof:

In Eq.(3.6.3), if Eq.(3.6.5) and t^2 are taken out as common factors, the numerator and the denominator polynomial are identical. By equating the respective coefficients the relation as given by Eq.(3.6.6) is established. As $R_{d,n}(t)$ exists and is unique, so is the Eq.(3.6.6).

Hence the results follows.

The structural properties of the generating matrix $[A_{d,n}]$ are the same as those of $[A_{a,n}]$ and hence they will not be discussed here.

3.7 GENERATION OF A PHASE FUNCTION FROM THE COEFFICIENTS OF THE TRANSFER FUNCTION:

In this section we present a method for obtaining the coefficients of an odd infinite series representing the phase function of a network, given its parameters (the coefficients of the transfer function representing the network).

As a transfer function is represented by a ratio of two polynomials, it suffices to obtain the coefficients of the odd phase function, given the coefficients of a polynomial $P_{d,n}(t)$.

Our objective is to obtain α_d and $b_{i,d}$'s given $d_{k,n}$'s of the polynomial $P_{d,n}(t)$. This will enable us to obtain a new set of variables α_d and $b_{i,d}$'s whose values are used as initial values for the minimization algorithm of the optmization procedure.

The matrix equation Eq.(3.2.23) can be rearranged such that the unknown column vector $[X_{dn}]$ has the new variables $b_{2i-1,d}$'s $(1 \leq i \leq n)$ as its elements. From Theorem 3.4.7, the solution for the new variable α_d is the coefficient $d_{1,n}$. The elements of the matrix $[A_{d,n}]$ and $[B_{dn}]$ are functions of the coefficients $d_{k,n}$'s of the polynomial $P_{d,n}(t)$. The matrix equation Eq.(3.2.23) is rewritten incorporating the change of variables as

$$[\overline{A}_{d,n-1}]\,[\overline{X}_{d,n-1}] = [\overline{B}_{d,n-1}] \qquad (3.7.1)$$

where $$[\overline{X}_{d,n-1}] = (b_{1,d}\; b_{3,d}\; \ldots\; b_{2i-1,d}) \qquad (3.7.2)$$

$$[\overline{B}_{d,n-1}] = (\overline{I}_{3,d}\; \overline{I}_{5,d}\; \ldots\; \overline{I}_{2i+1,d}) \qquad (3.7.3)$$

with $\bar{I}_{2i+1,d}$'s given as

$$\bar{I}_{2i+1,d} = I_{2i+1,d} + \sum_{k=1}^{i-3} I_{2k+1,d} \; b_{2i+2-2k,d} \qquad (3.7.4)$$

and the elements $I_{i,d}$'s are given as

$$I_{i,d} = \sum_{k=0}^{i} d_{i-k,d} \, f_k \, (-1)^k \qquad (3.7.5)$$

The matrix $[A_{d,n-1}]$ is given as

$$[A_{d,n-1}] = \begin{bmatrix} 1 & 0 & 0 & 0.. & 0 \\ I_{2,d} & 1 & 0 & 0.. & 0 \\ I_{4,d} & I_{2,d} & 1 & 0 & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ I_{2n-2,d} & I_{2n-4,d} & .. & I_{2,d} & 1 \end{bmatrix} \qquad (3.7.6)$$

The lower suffix in Eq.(3.7.1) does not indicate the order of the column or the matrix except for the vector $[B_{d,n-1}]$.

The solution can be obtained in a recursive manner as shown below:

$$b_{1,d} = I_{3,d}$$
$$b_{3,d} = I_{5,d} - I_{2,d}\, b_{1,d}$$
$$b_{5,d} = I_{7,d} - I_{4,d}\, b_{1,d} - I_{2,d}\, b_{3,d}$$
$$b_{7,d} = I_{9,d} - I_{6,d}\, b_{1,d} - I_{4,d}\, b_{3,d} + I_{3,d}\, b_{4,d} - I_{2,d}\, b_{5,d}$$

.

$$b_{2i+1,d} = I_{2i+3,d} - I_{2i,d}\, b_{1,d} - \sum_{k=1}^{2i-3} I_{2i-1-k,d}\, b_{k+2,d} \qquad (3.7.7)$$

The variables with even numbers as suffixes are related to the variables with odd numbers as suffixes as can be seen below:

$$b_{2,d} = 0$$
$$b_{4,d} = (b_{1,d})^2$$
$$b_{6,d} = 2\, b_{1,d}\, b_{3,d}$$
$$b_{8,d} = 2\, b_{1,d}\, b_{5,d} + (b_{3,d})^2$$
$$b_{10,d} = 2b_{1,d}\, b_{7,d} + 2\, b_{3,d}\, b_{5,d} + (b_{1,d})^4/3 \qquad 3.7.8)$$

So when an odd numbered variable is determined from the recurrence relation, the next even numbered variable is determined. The next odd numbered variable will have terms consisting of the previous lower even numbered variables. Likewise any number of variables with odd numbers as suffixes can be generated for a given n^{th} degree polynomial $P_{d,n}(t)$.

Alternatively, instead of treating $b_{i,d}$'s as the new variables, the coefficients $\zeta_{2i+1,1}$'s of the defined phase function as an infinite series, can be considered as new parameters. We will now show how any desired number of coefficients $\zeta_{2i+1,1}$'s could be generated and hence the phase function as an a truncated series could be obtained. The relation between the odd numbered variable $b_{i,d}$'s and the coefficients $\eta_{2i+1,1}$'s of the error phase polynomial $\delta_d(t)$ is given by equation Eqs.(3.2.17) and (3.2.18).

In Eqs.(3.2.17) and (3.2.18) $\nu_{m,k}$ (m is 2i-1 or 2i) represents the coefficient of the k^{th} degree term of the variable t in the expansion of the error phase polynomial $\delta_d(t)$ raised to the power m $\{(\delta_d(t))^m\}$. The term whose suffix m is greater than one are all functions of $\nu_{2i+1,k}$'s where the odd powered suffix 2i+1 is less than or equal to k. Thus any number of coefficients $\nu_{2i+1,k}$'s can be determined(hence, $\zeta_{2i+1,1}$'s by using Eqs.(3.2.19) to (3.3.22)) in a recursive manner.

With respect to the phase function which is an infinite series several observations can be made. Firstly, for a given n^{th} order polynomial, there exists only n independent coefficients in the phase function. These are α_d and the first (n-1) coefficients $\zeta_{2i+1,1}$'s of the error phase polynomial $\delta_d(pT/2)$. The rest of the coefficients are dependent on the first n coefficients. Secondly, these can be generated recursively. Depending on the accuracy requirement this infinite series can be truncated to the required number of terms.

In the next section we shall show that an approximation technique can be developed by considering the new variables $b_{i,d}$'s as new parameters.

3.8 AN APPROXIMATION PROCEDURE:

From the foregoing discussion, the existence of the direct relation between the coefficients of the phase function(represented as an infinite odd series) and the coefficients of the polynomial $P_{d,n}(t)$ has been established. Using the above results and properties, it now remains to develop an approximation procedure for obtaining a digital low-pass transfer function approximating a specified constant group delay over a specified band of frequencies(or bandwidth Bw). Also, the stability constraints are to be incorporated in the procedure. The approximation shall be carried out using the least mean square error criterion.

The objective is to obtain a stable digital low-pass, all-pole, n^{th} order filter such that the group delay of the filter approximates a desired constant group delay in a specified bandwidth Bw. The parameters of the filter shall be the new variables namely, α_d, $b_{1,d}$, $b_{3,d}$, . . . , and $b_{2i-3,d}$ which will also define the stability constraints. It is noted that the coefficients $d_{k,n}$'s are generated indirectly. This in turn leads to obtain the coefficients of the digital transfer function by use of the bilinear transformation. Thus the coefficients of the required digital transfer function are generated in terms of the new variables.

The objective function in this approximation is defined as

$$Ob(\alpha_d,\overline{X}_{d,n-1}) = \sum_{i=1}^{m} (\tau_{dn}(\Omega_i,\alpha_d,\overline{X}_{d,n-1}) - \tau_{sd})^2 \qquad (3.8.1)$$

where

α_d and the elements of $\overline{X}_{d,n-1}$ are the paramaters

$\tau_{dn}(\Omega_i, \alpha_d, \overline{X}_{d,n-1})$ is the group delay function,

n is the order of the filter,

m is the number of points considerd in the specified bandwidth Bw,

Ω_i is the frequency interval equal to Bw/(m-1) and

τ_{sd} is the specified group delay.

The objective function is minimized subject to the stability constraints as

$$(-1)^{(i(i-1)/2)}\,|A_{d,i}| > 0 \quad \text{for}(1 \leq i \leq n+1)$$

(The actual evaluation of the determinants is discussed in Section 2.7.)

Example 3.8.1:

It is required to design a fourth order low-pass, all-pole digital filter such that its group delay response approximates a specified group delay τ_{sd} in a bandwidth of 0.35 radians per seconds normalized with respect to twice the sampling frequency.

The transfer function is

$$T_{z,4}(z) = c_{z4} \frac{(z+1)^4}{P_{z,4}(z)} \tag{3.8.2}$$

where

$$P_{z,4}(z) = u_{0,4} + u_{1,4}z + u_{2,4}z^2 + u_{3,4}z^3 + u_{4,4}z^4 \tag{3.8.3}$$

and c_{z4} is a constant which normalizes the magnitude at the origin to unity.

With $z = e^{j\Omega T}$, we have the group delay function as

$$\tau_{z,4}(\Omega,V) = \frac{M(\Omega,V)\,N'(\Omega,V) - M'(\Omega,V)\,N(\Omega,V)}{M(\Omega,V)^2 + N(\Omega,V)^2} - 4c_{z4}T/2$$

where is the digital domain frequency variable in radians per seconds, V is a vector whose elements are the coefficients $u_{k,z}$'s, T is the sampling period in seconds. And

$$M(\Omega,V) = u_{0,4} + \sum_{i=1}^{4} u_{i,4} \, Cos(i\Omega T)$$

$$N(\Omega,V) = \sum_{i=1}^{4} u_{i,4} \, Sin(i\Omega T)$$

$$M'(\Omega,V) = - \sum_{i=1}^{4} iT \, u_{i,4} \, Sin(i\Omega T)$$

$$N'(\Omega,V) = \sum_{i=1}^{4} iT \, u_{i,4} \, Cos(i\Omega T)$$

The group delay function is a function of Ω and the coefficients $u_{k,4}$'s. The hext step required is to transform the coefficients parameters to the new variables α_d, $b_{1,d}$, $b_{3,d}$, $b_{5,d}$ and $b_{7,d}$. The coefficients $u_{k,4}$'s as a function of new variables are generated as follows:

We consider the transfer function in the t-domain

$$T_{d,4}(t) = \frac{c_{d4}}{P_{d,4}(t)}$$

where c_{d4} is a positive constant and

$$P_{d,4}(t) = \sum_{i=0}^{4} d_{i,4}\, t^{i}$$

The matrix equation Eq.(3.2.18) for order four is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ x_{2,1} & -x_{2,2} & 1 & 0 \\ x_{3,1} & -x_{3,2} & x_{2,1} & -x_{2,2} \\ x_{4,1} & -x_{4,2} & x_{3,1} & -x_{3,2} \end{bmatrix} \begin{bmatrix} d_{1,4} \\ d_{2,4} \\ d_{3,4} \\ d_{4,4} \end{bmatrix} = \begin{bmatrix} x_{2,2} \\ x_{3,2} \\ x_{4,2} \\ x_{5,2} \end{bmatrix}$$

where

$$x_{2,1} = f_2$$
$$x_{3,1} = f_4 + f_1\, b_{1,d}$$
$$x_{4,1} = f_6 + f_3\, b_{1,d} + f_1\, b_{3,d} + b_{4,d}$$
$$x_{2,2} = \alpha_d$$
$$x_{3,2} = f_3 + b_{1,d}$$
$$x_{4,2} = f_5 + f_2\, b_{1,d} + b_{3,d}$$
$$x_{5,2} = f_7 + f_4\, b_{1,d} + f_2\, b_{3,d} + f_1\, b_{4,d} + b_{5,d}$$

The above matrix equation is solved analytically and the solution is

$$d_{0,4} = \frac{\alpha_d^{6}}{45} - \frac{\alpha_d^{4}}{36} + \frac{\alpha_d^{2}}{180} - \left(\frac{\alpha_d^{3}}{3} - \frac{2}{3}\alpha_d\right) b_{1,d} + \alpha_d\, b_{3,d} - (b_{1,d})^2$$

$$= |A_{d,4}|$$

$$d_{1,4} = \alpha_d$$

$$d_{2,4} = \frac{\alpha_d^{\,8}}{105} - \frac{\alpha_d^{\,6}}{15} + \frac{2}{15}\alpha_d^{\,4} - \frac{8}{105}\alpha_d^{\,2} - \left(\frac{\alpha_d^{\,5}}{5} - \frac{\alpha_d}{5}\right) b_{1,d}$$

$$\left(\frac{2}{3}\alpha_d^{\,2} + \alpha_d\right) b_{3,d} + b_{1,d}\, b_{3,d} - \alpha_d\, b_{5,d}$$

$$d_{3,4} = \frac{2}{945}\alpha_d^{\,10} - \frac{2}{9}\alpha_d^{\,7} + \frac{\alpha_d^{\,5}}{15} - \frac{44}{945}\alpha_d^{\,3} - \left(\frac{\alpha_d^{\,6}}{15} - \frac{\alpha_d^{\,2}}{15}\right) b_{1,d}$$

$$- (b_{1,d})^3 + \left(\frac{\alpha_d^{\,4}}{3} + \frac{2}{3}\alpha_d^{\,2}\right) b_{3,d} - \alpha_d^{\,2}\, b_{5,d}$$

$$+ \left(\frac{\alpha_d^{\,6}}{6} - \frac{5}{6}\alpha_d^{\,2}\right)(b_{1,d})^2$$

$$d_{4,4} = |A_{d,5}|$$

where

$$|A_{d,5}| = \frac{\alpha_d^{\,10}}{4725} - \frac{\alpha_d^{\,8}}{315} + \frac{\alpha_d^{\,6}}{75} - \frac{17}{945}\alpha_d^{\,4} + \frac{4}{525}\alpha_d^{\,2} + b_{1,d}\, b_{3,d}$$

$$- \left(\frac{\alpha_d^{\,7}}{105} + \frac{2}{15}\alpha_d^{\,2} - \frac{\alpha_d}{7}\right) b_{1,d} - \alpha_d (b_{1,d})^3$$

$$+ \left(\frac{\alpha_d^{\,5}}{15} + \frac{\alpha_d^{\,3}}{3} - \frac{2}{5}\alpha_d\right) b_{3,d}$$

$$- \left(\frac{\alpha_d^{\,3}}{3} - \frac{\alpha_d}{3}\right) b_{5,d} + \alpha_d^{\,2}\, b_{1,d}\, b_{3,d} - (b_{3,d})^2$$

$$- \frac{1}{24}(\alpha_d^{\,4} + 6\ \alpha_d^{\,3} + 13\,\alpha_d^{\,2} + 6\ \alpha_d)\ (b_{1,d})^2$$

The coefficients $d_{k,4}$'s are thus obtained as a function of a set of variables α_d, $b_{1,d}$, $b_{3,d}$, $b_{5,d}$ and $b_{7,d}$. By applying the bilinear transformation to the polynomial $P_{d,4}(t)$, the polynomial $P_{u,4}(z)$ is obtained. Therefore, now we have the group delay function τ_{z4} as a function of new variables. The objective function is

$$Ob(\alpha_d,\overline{X}_{d4}) = \sum_{i=1}^{m} (\tau_{z4}(\Omega_i,\alpha_d,\overline{X}_{d4}) - \tau_{sd}\}^2$$

and the stability constraints are

1) $|A_{d,1}| > 0$

2) $-|A_{d,2}| > 0$

3) $-|A_{d,3}| > 0$

4) $|A_{d,4}| > 0$

5) $|A_{d,5}| > 0$

The stability constraints can be further simplified as follows.

The first constraint is always unity as the determinant of the generating matrix $[A_{d,1}]$ is unity. Hence, there are actually four constraints. These are

1) $\alpha_d > 0$

2) $\frac{\alpha_d^3}{3} - \frac{\alpha_d}{3} - b_{1,d} > 0$

3) $|A_{d,4}| > 0$

4) $|A_{d,5}| > 0$

The above problem is a nonlinear least square data fitting problem. The algorithm described by Fletcher is used to minimize the objective function [48]. This minimization algorithm requires an initial approximation to the variables α_d, $b_{1,d}$, $b_{3,d}$, $b_{5,d}$ and $b_{7,d}$. These are determined as follows:

The relation between the variables and the coefficient $d_{k,4}$'s are established in Section 3.7. From equation Eq.(3.7.7) the variables α_d, $b_{1,d}$, $b_{3,d}$, $b_{5,d}$ and $b_{7,d}$ are obtained as a function of the coefficients $d_{k,4}$'s of the polynomial $P_{d,4}(t)$. They are

$$\alpha_d = \frac{d_{1,4}}{d_{0,4}}$$

$$b_{1,d} = d_{3,4} - d_{2,4} f_1 + d_{1,4} f_2 - d_{0,4} f_3$$

$$b_{3,d} = I_{5,d} - I_{2,d} b_{1,d}$$

$$b_{4,d} = (b_{1,d})^2$$

$$b_{5,d} = I_{7,d} - I_{4,d}\, b_{1,d} - I_{2,d}\, b_{3,d}$$

$$b_{6,d} = 2\, b1\, b_{3,d}$$

$$b_{7,d} = I_{9,d} - I_{6,d}\, b_{1,d} - I_{4,d}\, b_{3,d} + I_{3,d}\, b_{4,d} - I_{2,d}\, b_{5,d}$$

$$b_{9,d} = I_{10,d} - I_{8,d}\, b_{1,d} - I_{6,d}\, b_{3,d} + I_5\, b_{4,d} - I_{4,d}\, b_{5,d} + I_{3,d}\, b_{6,d} - I_{2,d}\, b_{7,d}$$

where $I_{i,d}$'s are given by equation Eq.(3.7.5).

The flowchart for the optimization is given in Figure.3.8.1. It is seen that this flowchart is for any order n. A known t-domain all-pole low-pass filter such as maximally flat delay, Butterworth, Chebyshev,... etc are chosen to determine the corresponding new variables α_d, $b_{1,d}$, $b_{3,d}$, $b_{5,d}$, and $b_{7,d}$. These variable values are used as initial guess values as required by the optmization algorithm. Figure.3.8.3 shows the various group delay responses. In the Table 3.8.2, the coefficients $u_{k,4}$'s of the transfer function, and the percentage root means square error for various bandwidths are tabulated. By adopting similar procedures, the results for the cases n = 3 and n = 5 are obtained and these are shown in Tables 3.8.1 and 3.8.3 respectively. Figures.3.8.2 and 3.8.4 show the responses corresponding to these cases. Also, the poles for different banwidths and respective orders of the transfer functions are tabulated in Tables 3.8.4 to 3.8.6. Figures.3.8.5, 3.8.6 and 3.8.7 show the magnitude responses corresponding to these cases.

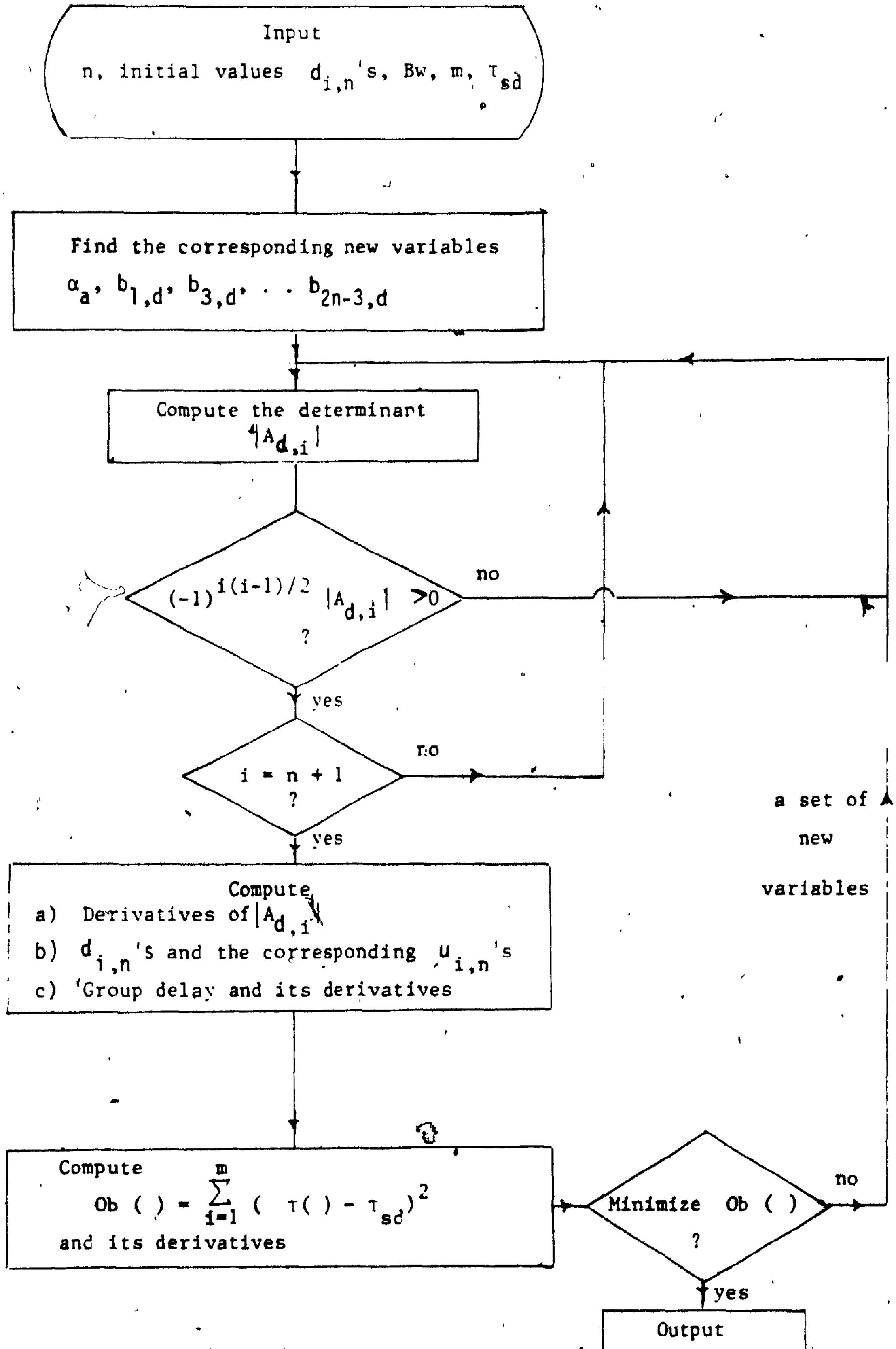
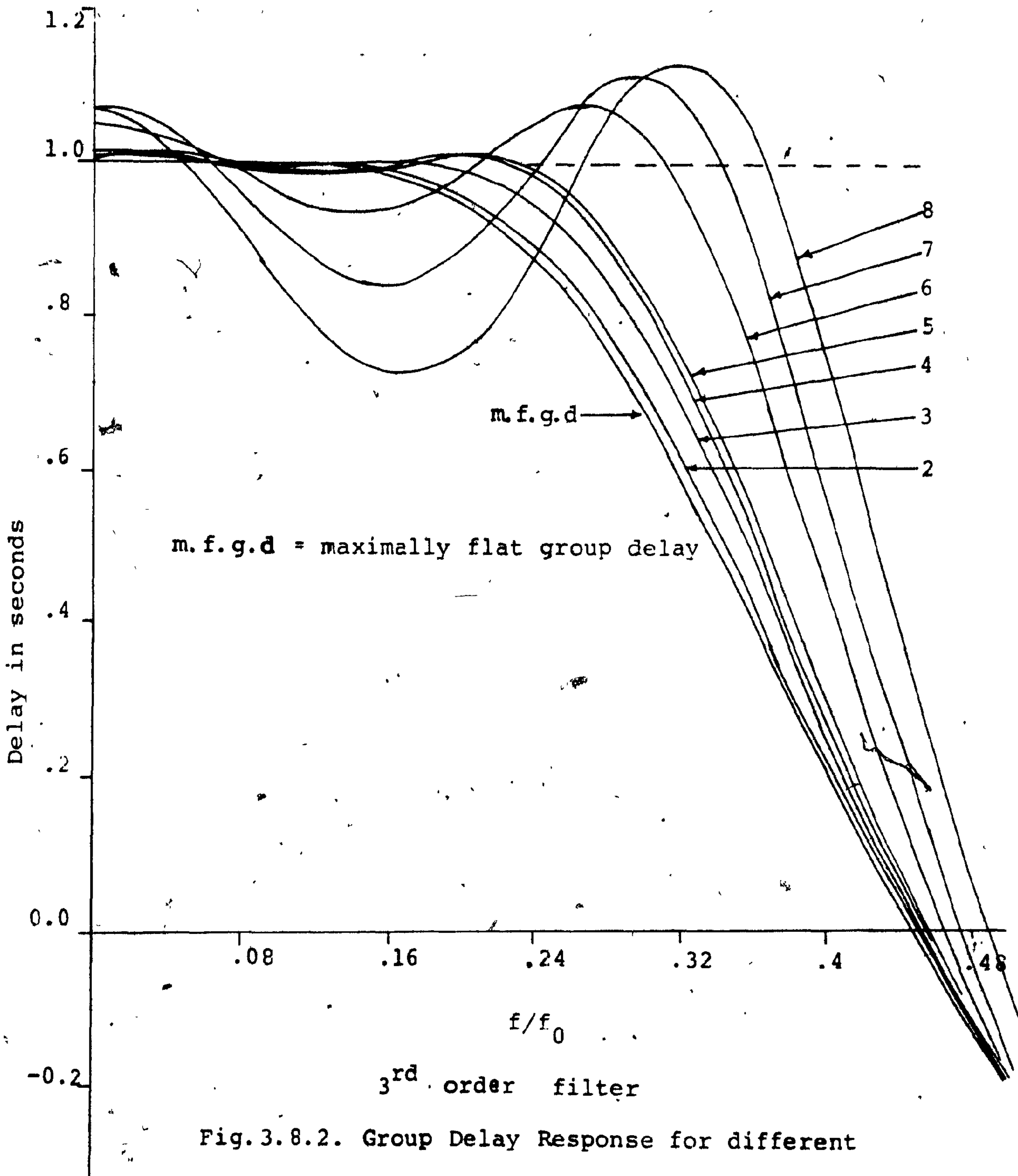
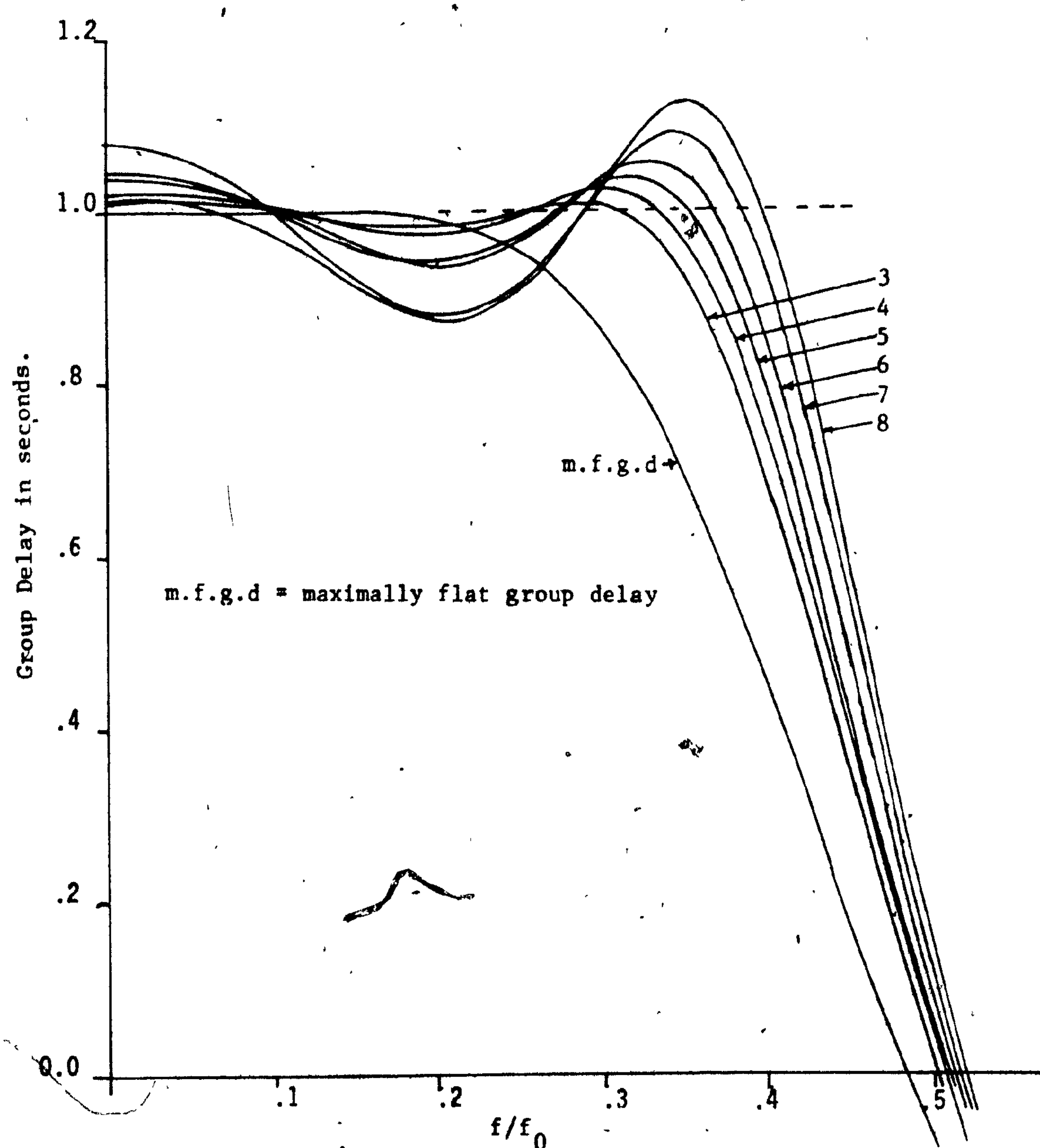


Fig. 3.8.1. Flowchart for the Approximation Procedure: Discrete Domain



3^{rd} order filter

Fig. 3.8.2. Group Delay Response for different Bandwidths



4^{th} order filter

Fig.3.8.3 Group Delay Response for different Bandwidths.

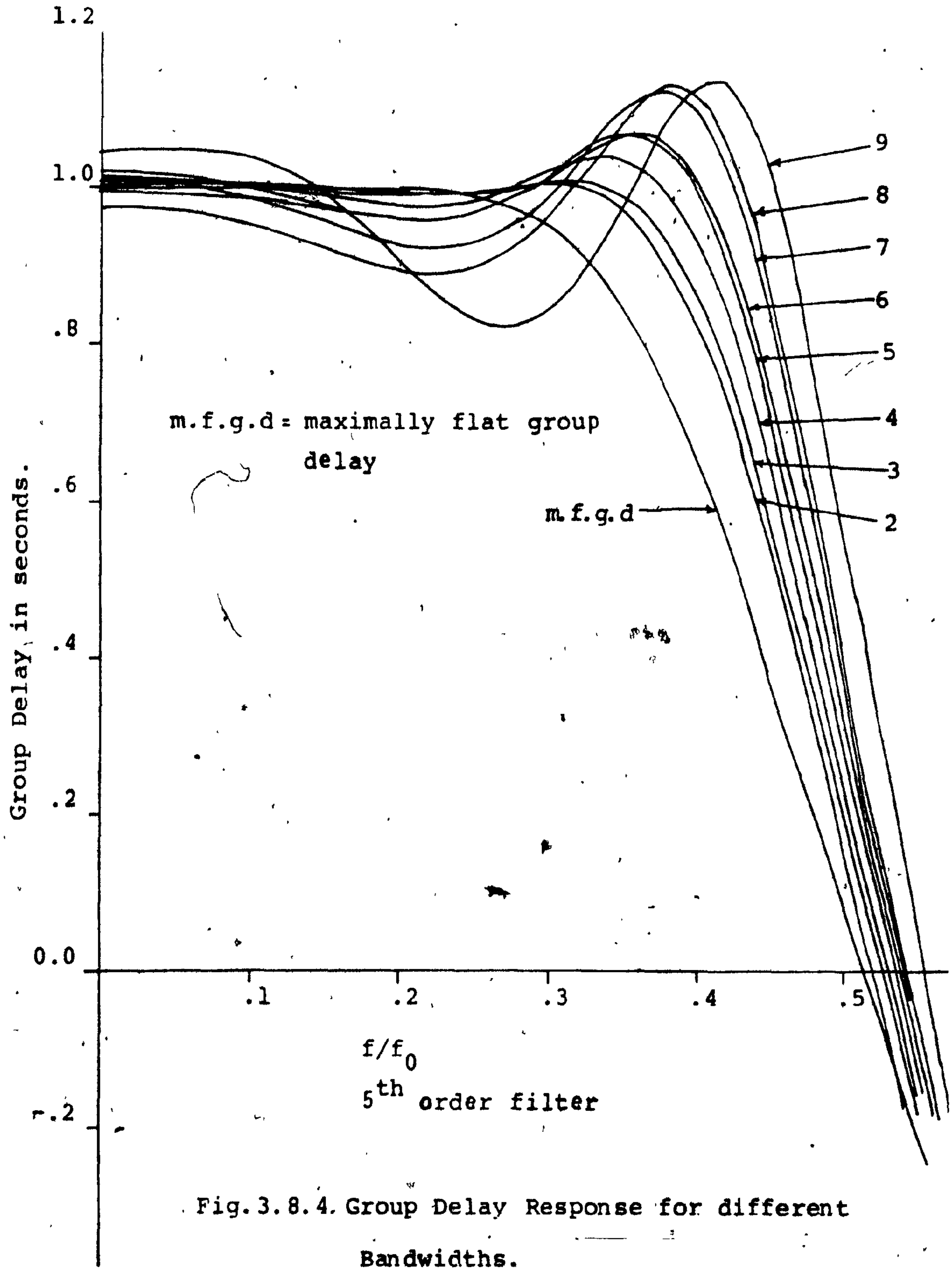


Fig. 3.8.4. Group Delay Response for different Bandwidths.

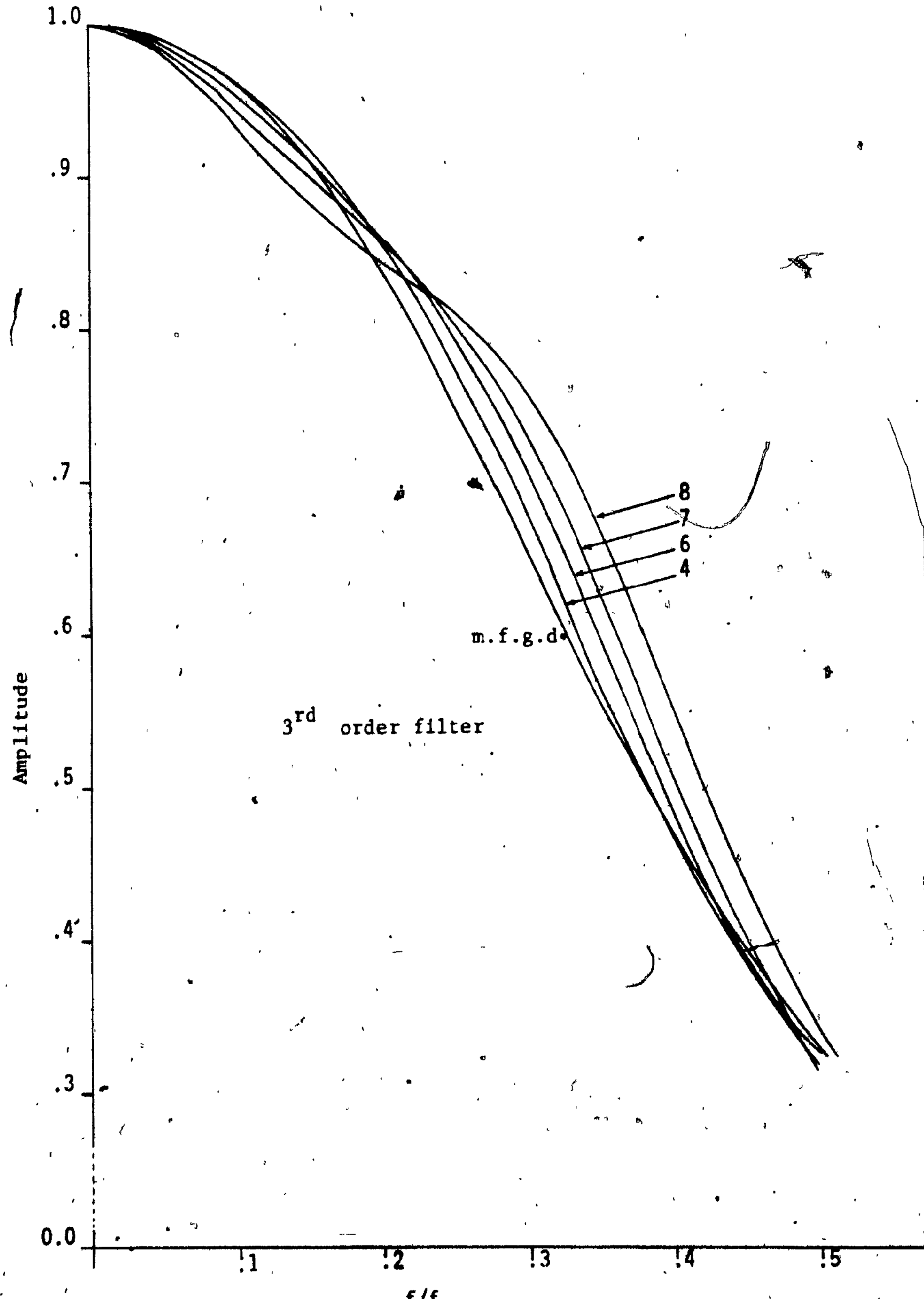


Fig.3.8.5 Amplitude Response for different Bandwidths

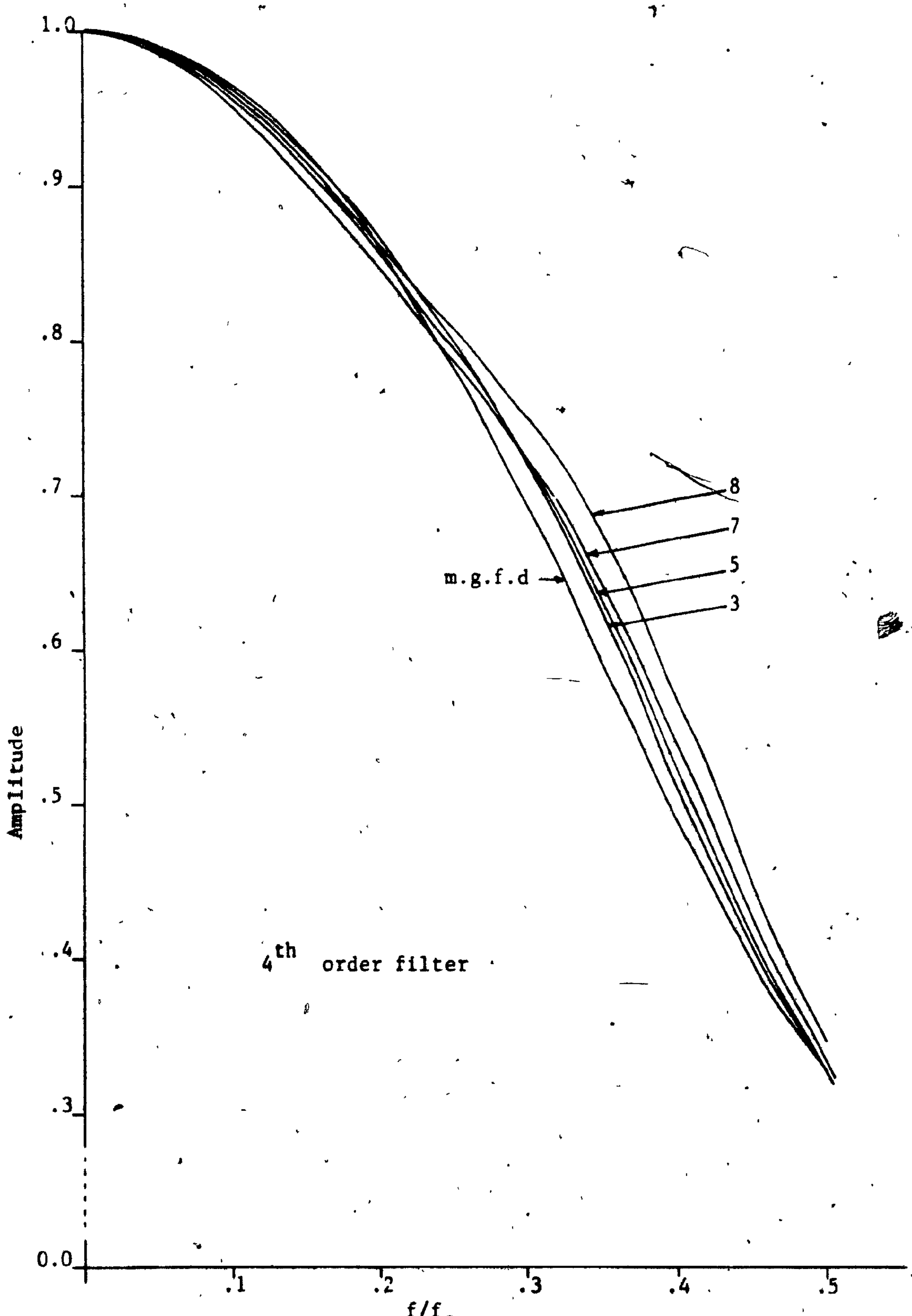


Fig.3.8.6. Amplitude Response for different Bandwidths.

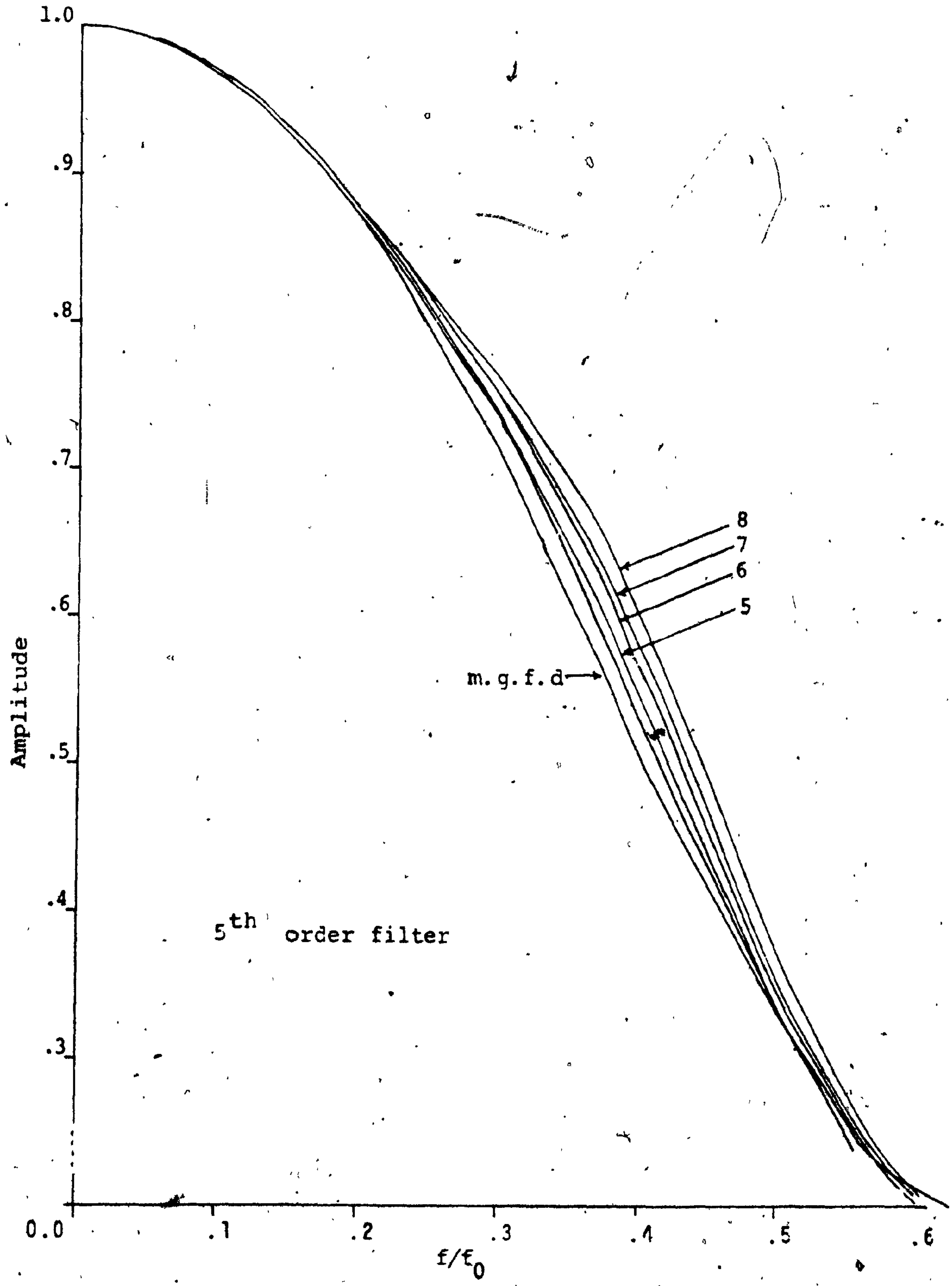


Fig.3.8.7. Amplitude Response for different Bandwidths.

Table.3.8.1

No.	Bandwidth	Coefficients				% r.m.s. error
		$u_{0,3}$	$u_{1,3}$	$u_{2,3}$	$u_{3,3}$	
1	.100000E+00	-.862031E+01	.502010E+02	-.114363E+03	.112847E+03	.266106E-02
2	.150000E+00	-.944612E+01	.528086E+02	-.117229E+03	.113804E+03	.116543E-01
3	.200000E+00	-.106288E+02	.562595E+02	-.120267E+03	.114195E+03	.180036E+00
4	.250000E+00	-.124252E+02	.613462E+02	-.125107E+03	.115590E+03	.713076E+00
5	.300000E+00	-.133694E+02	.607368E+02	-.117231E+03	.104995E+03	.221468E+01
6	.350000E+00	-.164965E+02	.682861E+02	-.124249E+03	.107985E+03	.574388E+01
7	.400000E+00	-.204583E+02	.771442E+02	-.133207E+03	.114139E+03	.125407E+02
8	.450000E+00	-.210852E+02	.731733E+02	-.122252E+03	.106936E+03	.225942E+02
9	.500000E+00	-.161784E+02	.521837E+02	-.860180E+02	.796695E+02	.342000E+02

Table. 3.8.2

No.	Bandwidth	Coefficients					% r.m.s. error
		$u_{0,4}$	$u_{1,4}$	$u_{2,4}$	$u_{3,4}$	$u_{4,4}$	
1	.300000E+00	.302623E+02	-.195478E+03	.554929E+03	-.872155E+03	.709111E+03	.834030E+00
2	.325000E+00	.368376E+02	-.224638E+03	.608105E+03	-.925063E+03	.737726E+03	.958402E+00
3	.350000E+00	.401772E+02	-.237608E+03	.624673E+03	-.931396E+03	.733921E+03	.167942E+01
4	.375000E+00	.364009E+02	-.216495E+03	.557603E+03	-.819080E+03	.639566E+03	.364418E+01
5	.400000E+00	.481059E+02	-.269144E+03	.668023E+03	-.957917E+03	.737797E+03	.438335E+01
6	.425000E+00	.410626E+02	-.230081E+03	.555584E+03	-.783887E+03	.601210E+03	.802576E+01
7	.450000E+00	.110406E+03	-.585387E+03	.137902E+04	-.190953E+04	.146655E+04	.102036E+02

Table 3.8.3

No.	Bandwidth	Coefficients						% r.m.s.error
		$U_{0,5}$	$U_{1,5}$	$U_{2,5}$	$U_{3,5}$	$U_{4,5}$	$U_{5,5}$	
1	.325000E+00	-.522222E+02	.459196E+03	-.176327E+04	.349751E+04	-.532788E+04	.402506E+04	.593546E+00
2	.350000E+00	-.867173E+02	.721060E+03	-.264860E+04	.568668E+04	-.762437E+04	.569655E+04	.389990E+00
3	.400000E+00	-.685336E+02	.576361E+03	-.210333E+04	.442011E+04	-.581528E+04	.429284E+04	.195325E+01
4	.425000E+00	-.783304E+02	.652423E+03	-.233174E+04	.477975E+04	-.616549E+04	.449931E+04	.355900E+01
5	.475000E+00	-.111416E+03	.889111E+03	-.303408E+04	.595481E+04	-.745347E+04	.539244E+04	.887542E+01
6	.500000E+00	-.100476E+03	.872454E+03	-.294252E+04	.571459E+04	-.712848E+04	.522670E+04	.135870E+02
7	.525000E+00	-.100494E+03	.720314E+03	-.218562E+04	.393943E+04	-.474070E+04	.336077E+04	.165864E+02

Table.3.8.4

No.	Bandwidth	Pole locations(real and imaginary parts)				% r.m.s.error
		Re.(z_1)	Im. (z_1)	Re.(z_2)=Re.(z_3)	Im.(z_2)=-Im.(z_3)	
1	.100000E+00	.368846E+00	.000000E+00	.322297E+00	-.321292E+00	.266106E-02
2	.150000E+00	.384750E+00	.000000E+00	.322675E+00	-.334088E+00	.116543E-01
3	.200000E+00	.403969E+00	.000000E+00	.324600E+00	-.353608E+00	.180036E+00
4	.250000E+00	.429350E+00	.000000E+00	.326491E+00	-.379168E+00	.713076E+00
5	.300000E+00	.460859E+00	.000000E+00	.327839E+00	-.410875E+00	.221468E+01
6	.350000E+00	.496687E+00	.000000E+00	.326962E+00	-.447957E+00	.574388E+01
7	.400000E+00	.529906E+00	.000000E+00	.318575E+00	-.486579E+00	.125407E+02
8	.450000E+00	.550832E+00	.000000E+00	.296200E+00	-.519833E+00	.225942E+02
9	.500000E+00	.558032E+00	.000000E+00	.260827E+00	-.543941E+00	.342000E+02

Table.3.8.5

No.	Bandwidth	Pole locations (real and imaginary parts)**				% r.m.s.error
		Re.(z_1)	Im.(z_1)	Re.(z_3)	Im.(z_3)	
1	.30000E+00	.37927E+00	.13728E+00	.23569E+00	.45471E+00	.83403E+00
2	.32500E+00	.39619E+00	.14477E+00	.23077E+00	.47685E+00	.95840E+00
3	.35000E+00	.40616E+00	.14621E+00	.22837E+00	.49155E+00	.16794E+01
4	.37500E+00	.40923E+00	.12735E+00	.23111E+00	.50640E+00	.36442E+01
5	.40000E+00	.42537E+00	.14597E+00	.22380E+00	.52182E+00	.43833E+01
6	.42500E+00	.42928E+00	.12364E+00	.22264E+00	.54099E+00	.80258E+01
7	.45000E+00	.43628E+00	.15015E+00	.21474E+00	.55367E+00	.10204E+02

** $z_2 = z_1^*$

$z_4 = z_3^*$

Table.3.8.6

No.	Bandwidth	Pole locations (real and imaginary parts)**						% r.m.s. error.
		Re.(z_1)	Im.(z_1)	Re.(z_2)	Im.(z_2)	Re.(z_4)	Im.(z_4)	
1	.325E+00	.332E+00	.000E+00	.324E+00	-.196E+00	.172E+00	-.493E+00	.594E+00
2	.350E+00	.343E+00	.000E+00	.335E+00	-.214E+00	.163E+00	-.504E+00	.390E+00
3	.400E+00	.344E+00	.000E+00	.338E+00	-.199E+00	.168E+00	-.523E+00	.195E+01
4	.425E+00	.343E+00	.000E+00	.347E+00	-.196E+00	.167E+00	-.540E+00	.356E+01
5	.475E+00	.351E+00	.000E+00	.358E+00	-.203E+00	.158E+00	-.568E+00	.888E+01
6	.500E+00	.351E+00	.000E+00	.357E+00	-.206E+00	.149E+00	-.577E+00	.136E+02
7	.525E+00	.359E+00	.000E+00	.407E+00	-.224E+00	.119E+00	-.609E+00	.166E+02

** $z_3 = z_2^*$
$z_5 = z_4^*$

3.9 SUMMARY AND CONCLUSIONS:

In this chapter, a method is developed which will generate the denominator polynomial $P_{d,n}(t)$ of an all-pole t-domain transfer function in order to approximate constant group delay. Starting from an all-pass function (obtained from the transfer function considered), a set of new variables (α_d and $b_{i,d}$'s) is obtained which determines the error between the actual and the constant group delays. This results in the generating matrix $[A_{d,n}]$ whose elements are functions of α_d and $b_{i,d}$'s. Some important properties of the elements of $[A_{d,n}]$ and the principal minors of $[A_{d,n}]$ are discussed. In particular, it is shown that the principal minors of the generating matrix are equivalent to the corresponding Hurwitz determinants formulated by the coefficients $d_{k,n}$'s of $P_{d,n}(t)$. This will enable us to obtain the various stability constraints in terms α_d and $b_{i,d}$'s. In addition, a recurrence relationship is obtained which will permit us to obtain higher order polynomials starting from lower order ones and incorporating the stability constraints.

Using the above properties, an optimization procedure is formulated in order to approximate a constant group delay in the digital domain and the corresponding digital transfer function is obtained. It is observed that the structure of $[A_{d,n}]$ is same as that of $[A_{a,n}]$. Hence the savings in number of multiplications is 68.75% as compared with that of Gaussian method of evaluating determinants.

In this procedure, it is shown that the objective function and the stability constraints are obtained as functions of the parameters that are related to the phase function expressed as an infinite series. It is shown that the maximally flat group delay function is one particular case (all $b_{i,d}$'s are equal to zero) in this procedure and a large number of responses can be obtained depending on the extent of minimization of the objective function. Also, given the derivatives of the phase function, the corresponding transfer function can be obtained.

CHAPTER FOUR

SUMMARY AND CONCLUSIONS

4.1 SUMMARY AND CONCLUSIONS:

In this thesis, a new procedure for constant group delay approximation of 1-D, low-pass analog and digital filters has been developed. Such filters so obtained possess group delay characteristics approximating the specified constant group delay in the least mean square sense and in a specified bandwidth.

Starting with a phase function as an odd infinite series, a new set of variables is defined. In terms of these variables and the unknown coefficients of the transfer function, a linear matrix equation is formed. Some properties of the elements of the matrix and the determinants are studied. These properties lead to the generation of the coefficients of the transfer function incorporating the conditions of stability. It is shown that a polynomial of a given order can be generated by polynomials of lower orders by means of recurrence relations. The structural properties of the generating matrix reduces the number of multiplications (and divisions) that are required to evaluate the values of the determinants of the generating matrix. Hence, the computational ability of the approximation procedure has been enhanced considerably by exploiting these properties of the generating matrix.

The analog domain is first considered. Several properties of the generating matrix are obtained and studied. The differential and integral properties of the elements of the matrix and of the determinants of the matrix are established. Further, some properties of the determinant of the generating matrix which enables us to obtain solutions to some of the coefficients, are obtained. In addition, solutions to the coefficients of the transfer function are expressed in terms of the determinants of the generating matrix. It is also shown how the polynomial can be generated by a recurrence relation. Thus, in order to solve the matrix equation, the requirement of inverting the matrix is avoided. That is, the ill-conditioning problem that exists in inversion techniques is eliminated [45]. The stability of the transfer function depends on the Hurwitz property of the denominator polynomial. This property of the polynomial is established by showing that the principal minors of the generating matrix are equal to the Hurwitz determinants with respect to their absolute values. Finally, the structural properties of the generating matrix lead to reduction in the number of multiplications that are required to evaluate the determinants. The computational savings is 68.75% more when compared with the complexity involved in the Gaussian method of evaluating the determinant of the same order.

Solutions to the coefficients of the transfer function, stability constraints, and the elemental values have been obtained in terms of the determinants of the generating matrix whose elements are functions of the variables. As these variables are functions of the coefficients of the error phase function and the phase slope, it can be observed that the problem has been defined in the phase domain only [46].

An approximation algorithm is developed where, in order to reduce the number of computations, the various properties developed are used.

The minimization algorithm employed is a non-linear programming which requires a set of initial guess values for the parameters. As these values have to satisfy the stability constraints, the parameters corresponding to a known, stable transfer function, are to be determined by a method. Hence, for a given set of coefficients of a stable transfer function, a method to obtain the variables is formulated. The elemental values of a ladder network terminated in resistances are obtained simultaneously. These are illustrated with examples. The elements of a doubly terminated LC-ladder network are functions of the determinants of the generating matrix and, in fact, are only the principal minors. These are evaluated in each iteration of the approximation algorithm simultaneously along with the stability constraints. On the other hand, the transfer function alone could have been generated first and by means of the continued fraction expansion, the LC-ladder network could have been obtained. This is avoided here as the elemental values so obtained cause the group delay response to vary considerably due to the high rate of convergence of the continued fraction expansion [47]. Thus, the sensitivity of frequency response characteristics due to the perturbation of the values of parameters are less when compared with the case where the parameters are the coefficients or the poles and zeros of the transfer function.

The Bessel filter is shown as a special case. Also, if the derivatives of the phase are given, the corresponding transfer function can be obtained using the above method.

In the discrete domain, the Richard's variable t has been considered as an independent variable and the corresponding z-domain polynomial is obtained from the strictly Hurwitz polynomial in t.

Various new properties similar to those in the analog domain are obtained. The difference and the anti-difference properties of the elements of the generating matrix and of the determinants of the matrix are established. Some properties of the determinant of the generating matrix, which enables us to obtain solutions to some of the coefficients, are obtained. In addition, solutions to some of the coefficients of the transfer function are expressed in terms of the determinants of the generating matrix. Finally, the structural properties of the generating matrix leads to a reduction in the number of multiplications that are required to evaluate the determinants.

As in the analog case, similar conclusions can be drawn except for the realization. It can be observed that the properties with respect to differentiability and integrability in the analog domain and the difference and anti-difference in the discrete domain are similar.

An approximation procedure is developed where, in order to enhance the computational ability, the various properties established are made use of. The minimization algorithm incorporated requires a set of initial values for the variables. A method to obtain these initial values is described. Some examples have been worked out.

As in the case of analog domain, the maximally flat group delay filter can be obtained as a special case. Also, given the derivatives of the phase function, the corresponding digital transfer function can be obtained.

In order to reduce the number of computations of the procedure further, the approximation can be carried out in the t-domain. From the resulting t-domain transfer function, the corresponding digital transfer function can be obtained.

4.2 SCOPE FOR FURTHER RESEARCH:

In the analog domain as well as discrete domain, the existence of solutions to coefficients $a_{i,n}$'s($d_{i,n}$'s) for (2<i<n-3) can be investigated. Further properties of the determinants of the generating matrix can be developed. These properties are likely to lead to further simplification of the stability constraints and enhancement of the computational ability of the approximation algorithm.

Phase function as an odd infinite series and its properties with respect to stability and realizability can be investigated. These properties are likely to lead to new algorithms for phase unwrapping techniques. Also, possibility of expressing the phase function as a rational function can be investigated. This may give clues to devlop explicit solutions to constant group delay response in the Chebyshev sense.

By considering an arbitrary set of n equations that can be formed from Eqs.(2.2.11) and (3.2.13), similar properties can be investigated. These properties are likely to lead to other approximations of constant group delay.

Further work can be carried out in order to extend the above methods to two dimensional filter design.

APPENDIX A

PROOF FOR THEOREM 2.5.1.

The generating matrix $[A_{a,n}]$ given by Eq.(2.2.20) is rewritten (without the elements in the first row and the first column, as the solution to $a_{1,n}$ is $f_{2,2}$) as

$$[A_{a,n}] = \begin{bmatrix} -f_{2,2} & 1 & 0 & 0 \cdot & 0 \\ -f_{3,2} & f_{2,1} & -f_{2,2} & 1 \cdot & 0 \\ -f_{4,2} & f_{3,1} & -f_{3,2} & \cdot\cdot & 0 \\ \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & F_0 \\ \cdot & \cdot & \cdot & \cdot & F_1 \\ \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ -f_{n,2} & f_{n-1,1} & -f_{n-1,2} & \cdot\cdot & F_n \end{bmatrix} \qquad (A.1)$$

where

for odd $n: F_0 = 1,\ F_1 = f_{2,1}, \ldots,\ F_n = f_{(n+1)/2,1}$

and for even $n: F_0 = 0,\ F_1 = -f_{2,2}, \ldots,\ F_n = -f_{(n/2),1}$

The elements in the first column are the negative of the elements of the column vector $[B_{an}]$. These elements can be expressed in terms of the coefficients and the elements themselves. That is,

$$f_{2,2} = a_{1,n}$$

$$f_{3,2} = f_{2,1}\, a_{1,n} - f_{2,2}\, a_{2,n} + a_{3,n}$$

$$f_{4,2} = f_{3,1}\, a_{1,n} - f_{3,2}\, a_{2,n} + f_{2,1}\, a_{3,n} - f_{2,2}\, a_{4,n} + a_{5,n}$$

$$\vdots$$

$$f_{j,2} = \sum_{k=1}^{k_1} f_{j-k,1}\, a_{2k-1,n} - \sum_{k=1}^{k_2} f_{j-k,2}\, a_{2k,n} \quad \text{for } (2 \leq j \leq n)$$

where for odd j: $k_1 = (j-1)/2$, $k_2 = k_1 - 1$

and for even j: $k_1 = j/2 + 1$, $k_2 = k_1 - 1$ (A.2)

Substituting Eq.(A.2) for the elements in the first column of the matrix $[A_{a,n}]$ and applying the elementary transformations on the matrix, we get the desired result. These elementary transformations are described as follows:

Step 1:

1) Row 2 - $f_{2,1}$ row 1

2) Row 3 - $f_{3,1}$ row 2

.

.

n-1) Row n - $f_{n,1}$ row n-1

Step 2:

1) Row 3 - $f_{2,1}$ row 2

2) Row 4 - $f_{3,1}$ row 3

.

.

n-2) Row n - $f_{n-1,1}$. row n-1

Step 3:

1) Row 4 - $f_{2,1}$ row 3

.

.

n-3) Row n - $f_{n-1,1}$ row n-1

Step 4:

.

.

$n-1^{th}$ Step:

1) Row n - $f_{2,1}$ row n-1

The last step is described as follows:

n^{th} Step:

1) Row 2 + row 1 $a_{2,n}$

2) Row 3 + row 2 $a_{2,n}$ + row 1 $a_{4,n}$

.

.

.

n-1) Row n + $\sum_{i=1}^{k_1}$ row n-i $a_{2i,n}$ (A.3)

for odd n, $k_1 = (n-1)/2$

and for even n, $k_1 = n/2$ (A.4)

After applying the above transformation, the matrix $[A_{a,n}]$ reduces to

$$[H_n] = \begin{bmatrix} -a_{1,n} & 1 & 0 & 0 & \cdot & & 0 \\ -a_{3,n} & a_{2,n} & -a_{1,n} & 1 & \cdot & & 0 \\ -a_{5,n} & a_{4,n} & -a_{3,n} & \cdot & & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & 0 \\ -a_{k_1,n} & a_{k_2,n} & \cdot & \cdot & & & 0 \\ 0 & a_{k_4,n} & -a_{k_3,n} & & \cdot & & 0 \\ 0 & 0 & -a_{k_1,n} & & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \cdot & \cdot & & 0 \\ 0 & 0 & 0 & 0 & & a_{2,n} & -a_{1,n} \\ 0 & 0 & 0 & & & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & 0 & -a_{k_1,n} \end{bmatrix} \quad \text{(A.5)}$$

where for odd n: $k_1 = n$, $k_2 = n-1$, $k_3 = n-2$, $k_4 = 0$

and for even n: $k_1 = n-1$, $k_2 = n-2$, $k_3 = n-3$, $k_4 = n$

The matrix in Eq.(A.5) can be transposed to have the necessary form as required. Also a row and a column is added with all the elements zero except the first one which is unity. The matrix as given by Eqs.(2.5.6 or 2.5.7) results.

Hence the result follows.

The above result is now illustrated with the following example.

Example 2.5.1:

The order of the transfer function is five, the number of coefficients is six. The order of the generating matrix is six which can be reduced to a fifth order matrix and the number of coefficients is five as the coefficients can be normalized with respect to the first coefficient, viz, $a_{0,5}$. The matrix $[A_{a,6}]$ is reduced to order five. The reduced matrix is designated as $[D_{a,5}]$ which is

$$[D_{a,5}] = \begin{bmatrix} -f_{2,2} & 1 & 0 & 0 & 0 \\ -f_{3,2} & f_{2,1} & -f_{2,2} & 1 & 0 \\ -f_{4,2} & f_{3,1} & -f_{3,2} & f_{2,1} & -f_{2,2} \\ -f_{5,2} & f_{4,1} & -f_{4,2} & f_{3,1} & -f_{3,2} \\ -f_{6,2} & f_{5,1} & -f_{5,2} & f_{4,1} & -f_{4,2} \end{bmatrix}$$

with

$$f_{2,2} = a_{1,5}$$
$$f_{3,2} = f_{2,1}a_{1,5} - f_{2,2}a_{2,5} + a_{3,5}$$
$$f_{4,2} = f_{3,1}a_{1,5} - f_{3,2}a_{2,5} + f_{2,1}a_{3,5} - f_{2,2}a_{4,5} + a_{5,5}$$
$$f_{5,2} = f_{4,1}a_{1,5} - f_{4,2}a_{2,5} + f_{3,1}a_{3,5} - f_{3,2}a_{4,5} + f_{2,1}a_{5,5}$$
$$f_{6,2} = f_{5,1}a_{1,5} - f_{5,2}a_{2,5} + f_{4,1}a_{3,5} - f_{4,2}a_{4,5} + f_{3,1}a_{5,5}$$

Let $\Gamma_{\text{Step } k} : [D_{a,5}]_{\text{Step } k-1} \rightarrow [D_{a,5}]_{\text{Step } k}$

where $\Gamma_{\text{Step } k}$ represents a set of transformations to be applied at each step k and for k = 1, $[D_{a,5}]_{\text{Step } 0} = [D_{a,5}]$. The matrix $[D_{a,5}]_{\text{Step } k}$ (k is i, ii, iii, iv) after applying each set of transformations and the corresponding elements of the matrix are as follows:

Step i:

1) Row 2 - row 1 $f_{2,1}$

2) Row 3 - row 2 $f_{3,1}$

3) Row 4 - row 3 $f_{4,1}$

4) Row 5 - row 4 $f_{5,1}$

$$[D_{a,5}]_{\text{Step i}} = \begin{bmatrix} -a_{1,5} & 1 & 0 & 0 & 0 \\ q_{2,1}^{(i)} & 0 & -a_{1,5} & 1 & 0 \\ q_{3,1}^{(i)} & 0 & q_{3,3}^{(i)} & 0 & -a_{1,5} \\ q_{4,1}^{(i)} & 0 & q_{4,3}^{(i)} & 0 & q_{4,5}^{(i)} \\ q_{5,1}^{(i)} & 0 & q_{5,3}^{(i)} & 0 & q_{5,5}^{(i)} \end{bmatrix}$$

where, the right hand upper suffix (k)(k=i, ii, iii, iv) of the element of the matrix $[D_{a,5}]_{\text{Step k}}$ indicates the particular element obtained after applying the set of transformations under the step k and these elements are

$$q_{2,1}^{(i)} = a_{1,5}\, a_{2,5} - a_{3,5}$$

$$q_{3,1}^{(i)} = f_{3,2}\, a_{2,5} - f_{2,1}\, a_{3,5} + f_{2,2}\, a_{4,5} - a_{5,5}$$

$$q_{4,1}^{(i)} = f_{4,2}\, a_{2,5} - f_{3,1}\, a_{3,5} + f_{3,2}\, a_{4,5} - f_{2,1}\, a_{5,5}$$

$$q_{5,1}^{(i)} = f_{5,2}\, a_{2,5} - f_{4,1}\, a_{3,5} + f_{4,2}\, a_{4,5} - f_{3,1}\, a_{5,1}$$

$$q_{3,3}^{(i)} = f_{2,1}\, a_{1,5} + f_{2,2}\, a_{2,5} - a_{3,5}$$

$$q_{4,3}^{(i)} = -f_{3,1}\, a_{1,5} + f_{3,2}\, a_{2,5} - f_{2,1}\, a_{3,5} + f_{2,2}\, a_{4,5} - a_{5,5}$$

$$q_{5,3}^{(i)} = f_{4,2}\, a_{2,5} - f_{4,1}\, a_{1,5} + f_{3,2}\, a_{4,5} - f_{3,1}\, a_{4,5} - f_{2,1}\, a_{5,5}$$

$$q_{3,4}^{(i)} = f_{2,1}$$

$$q_{4,4}^{(i)} = f_{3,1}$$

$$q_{5,4}^{(i)} = f_{4,1}$$

$$q_{4,5}^{(i)} = -f_{2,1}\, a_{1,5} + f_{2,2}\, a_{2,5} - a_{3,5}$$

$$q_{5,5}^{(i)} = f_{3,2}\, a_{2,5} - f_{3,1}\, a_{1,5} - f_{2,1}\, a_{3,5} - f_{2,2}\, a_{4,5} - a_{5,5}$$

Step ii:

1) Row 3 - row 2 $f_{2,1}$

2) Row 4 - row 2 $f_{3,1}$

3) Row 5 - row 2 $f_{4,1}$

$$[D_{a,5}]_{\text{Step ii}} = \begin{bmatrix} -a_{1,5} & 1 & 0 & 0 & 0 \\ q^{(ii)}_{2,1} & 0 & -a_{1,5} & 1 & 0 \\ q^{(ii)}_{3,1} & 0 & q^{(ii)}_{3,3} & 0 & -a_{1,5} \\ q^{(ii)}_{4,1} & 0 & q^{(ii)}_{4,3} & 0 & q^{(ii)}_{4,5} \\ q^{(ii)}_{5,1} & 0 & q^{(ii)}_{5,3} & 0 & q^{(ii)}_{5,5} \end{bmatrix}$$

where

$$q^{(ii)}_{2,1} = q^{(ii)}_{3,3}$$

$$q^{(ii)}_{3,1} = a_{1,5}\, a_{2,5}^{\;2} + a_{3,5}\, a_{2,5} + a_{4,5}\, a_{1,5} - a_{5,5}$$

$$q^{(ii)}_{4,1} = -f_{3,2}\, a_{2,5}^{\;2} + f_{2,1}\, a_{1,5}\, a_{3,5} - 2a_{1,5}\, a_{2,5}\, a_{4,5} + a_{2,5}\, a_{5,5} + f_{2,1}\, a_{1,5}\, a_{4,5} + a_{3,5}\, a_{4,5} - f_{2,1}\, a_{5,5}$$

$$q^{(ii)}_{5,1} = -f_{4,2}\, a_{2,5}^{\;2} + f_{3,1}\, a_{2,5}\, a_{3,5} - a_{1,5}\, a_{4,5}^{\;2} + f_{2,1}\, a_{2,5}\, a_{5,5} + f_{3,1}\, a_{1,5}\, a_{4,5} + f_{2,1}\, a_{3,5}\, a_{4,5} + a_{4,5}\, a_{5,5} - f_{3,1}\, a_{5,5}$$

$$q^{(ii)}_{4,3} = f_{2,1}\, a_{1,5}\, a_{2,5} - a_{1,5}\, a_{2,2}^{\;2}$$

$$q^{(ii)}_{5,3} = f_{3,1}\, a_{1,5}\, a_{2,5} - f_{3,2}\, a_{2,5}^{\;2} + f_{2,1}\, a_{2,5}\, a_{3,5} - a_{1,5}\, a_{2,5}\, a_{3,4} + a_{2,5}\, a_{5,5} - f_{3,1}\, a_{3,5} + f_{3,2}\, a_{4,5} - f_{2,1}\, a_{5,5}$$

$$q^{(ii)}_{4,5} = -f_{2,1}\, a_{1,5} + a_{1,5}\, a_{2,5} - f_{2,1}\, a_{3,5} + a_{1,5}\, a_{4,5} - a_{5,5}$$

Step iii:

i) Row 4 - row 3 $f_{2,1}$

ii) Row 5 - row 3 $f_{3,1}$

$$[D_{a,5}]_{\text{Step iii}} = \begin{bmatrix} -a_{1,5} & 1 & 0 & 0 & 0 \\ q^{(iii)}_{2,1} & 0 & -a_{1,5} & 1 & 0 \\ q^{(iii)}_{3,1} & 0 & q^{(iii)}_{3,3} & 0 & -a_{1,5} \\ q^{(iii)}_{4,1} & 0 & q^{(iii)}_{4,3} & 0 & q^{(iii)}_{4,5} \\ q^{(iii)}_{5,1} & 0 & q^{(iii)}_{5,3} & 0 & q^{(iii)}_{5,5} \end{bmatrix}$$

$$q^{(iii)}_{2,1} = q^{(iii)}_{3,3} = q^{(iii)}_{4,5} = q^{(ii)}_{2,1} = q^{(i)}_{2,1}$$

$$q^{(iii)}_{3,1} = q^{(iii)}_{4,3}$$
$$= -a_{1,5}\, a_{2,5}^{\,2} + a_{3,5}\, a_{2,5} + a_{1,5}\, a_{4,5} - a_{5,5}$$

$$q^{(iii)}_{4,1} = a_{1,5}\, a_{2,5}^{\,2} - a_{3,5}\, a_{2,5}^{\,3} - 2\, a_{1,5}\, a_{2,5}\, a_{4,5}$$
$$+ a_{2,5}\, a_{5,5} + a_{3,5}\, a_{5,4}$$

$$q^{(iii)}_{5,1} = f_{3,2}\, a_{2,5}^{\,3} - f_{2,1}\, a_{3,5}\, a_{2,5}^{\,2} + 3\, a_{1,5}\, a_{2,5}^{\,2}\, a_{4,5}$$
$$- a_{2,5}^{\,2}\, a_{5,5} - 2\, f_{2,1}\, a_{1,5}\, a_{2,5}\, a_{4,5} - 2\, a_{2,5}\, a_{3,5}\, a_{4,5}$$
$$+ f_{2,1}\, a_{2,5}\, a_{5,5} + f_{2,1}\, a_{3,5}\, a_{4,5} - a_{1,5}\, a_{4,5}^{\,2} + a_{4,5}\, a_{5,5}$$

$$q^{(iii)}_{5,3} = -f_{3,2}\, a_{2,5}^{\,2} + f_{2,1}\, a_{2,5}\, a_{3,5} - a_{1,5}\, a_{2,5}\, a_{4,5}$$
$$- a_{2,5}\, a_{5,5} + f_{3,2}\, a_{4,5} - f_{2,1}\, a_{5,5}$$

$$q^{(iii)}_{5,5} = f_{3,2}\, a_{2,5} - f_{2,1}\, a_{3,5} + a_{1,5}\, a_{4,5} - a_{5,5}$$

Step iv:

1) Row 5 - row $f_{2,1}$

$$[D_{a,5}]_{\text{Step iv}} = \begin{bmatrix} -a_{1,5} & 1 & 0 & 0 & 0 \\ q_{2,1}^{(iv)} & 0 & -a_{1,5} & 1 & 0 \\ q_{3,1}^{(iv)} & 0 & q_{3,3}^{(iv)} & 0 & -a_{1,5} \\ q_{4,1}^{(iv)} & 0 & q_{4,3}^{(iv)} & 0 & q_{4,5}^{(iv)} \\ q_{5,1}^{(iv)} & 0 & q_{5,3}^{(iv)} & 0 & q_{5,5}^{(iv)} \end{bmatrix}$$

where

$$q_{2,1}^{(iv)} = q_{2,1}^{(iii)}$$

$$q_{3,1}^{(iv)} = q_{4,3}^{(iv)} = q_{5,5}^{(iv)} = q_{3,1}^{(iii)}$$

$$q_{4,1}^{(iv)} = q_{5,3}^{(iv)} = q_{4,1}^{(iii)}$$

$$q_{5,1}^{(iv)} = -a_{1,5}\, a_{2,5}^{4} + a_{3,5}\, a_{2,5}^{3} - a_{5,5}\, a_{2,5}^{4} + a_{4,5}\, a_{5,5} + 3\, a_{1,5}\, a_{2,5}^{2}\, a_{4,5} - 2\, a_{2,5}\, a_{3,5}\, a_{4,5} - a_{1,5}\, a_{4,5}^{2}$$

Second sequence

1) Row 2 + row 1 $a_{2,5}$

2) Row 3 + row 2 $a_{2,5}$ + row 1 $a_{4,5}$

3) Row 4 + row 3 $a_{2,5}$ + row 2 $a_{4,5}$

4) Row 5 + row 4 $a_{2,5}$ + row 3 $a_{4,5}$

$$[D_{a,5}] = \begin{bmatrix} -a_{1,5} & 1 & 0 & 0 & 0 \\ -a_{3,5} & a_{2,5} & -a_{1,5} & 1 & 0 \\ -a_{5,5} & a_{4,5} & -a_{3,5} & a_{2,5} & -a_{1,5} \\ 0 & 0 & -a_{5,5} & a_{4,5} & -a_{3,5} \\ 0 & 0 & 0 & 0 & -a_{5,5} \end{bmatrix}$$

transposing the matrix $[D_{a,5}]$ after adding a row and a column with zero elements except the first element $f_{1,1}$ being unity, we get the desired result as follows:

$$[H_{5+1}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{1,5} & a_{3,5} & a_{5,5} & 0 & 0 \\ 0 & 1 & a_{2,5} & a_{4,5} & 0 & 0 \\ 0 & 0 & a_{1,5} & a_{3,5} & a_{5,5} & 0 \\ 0 & 0 & 1 & a_{2,5} & a_{4,5} & 0 \\ 0 & 0 & 0 & a_{1,5} & a_{3,5} & a_{5,5} \end{bmatrix} (-1)^{((5-1)/2)}$$

As the original set of equations is satisfied at every elementary transformation, it can easily be seen that $|H_{5+1}|$ is the same as $|A_{a,6}|$.

Hence the theorem.

APPENDIX B

PROOF FOR THE THEOREM 2.6.1

The intention is to show that the polynomial $P_{a,n}(p)$ can be generated by recurrence relation. We have

$$P_{a,n+1}(p) = P_{a,n}(p) + R_{a,n}(p)\, P_{a,n-1}(p) \tag{B.1}$$

with $P_{a,0}(p) = 1$, $P_{a,1}(p) = 1 + \alpha_a\, p$ and

$$R_{a,n}(p) = p^2 \frac{|A_{a,n-1}|\ |A_{a,n+2}|}{|A_{a,n}|\ |A_{a,n+1}|} \tag{B.2}$$

The existance of $R_{a,n}(p)$ depends on the factor

$$\frac{|A_{a,n-1}|\ |A_{a,n+2}|}{|A_{a,n}|\ |A_{a,n+1}|} \tag{B.3}$$

Let us assume that the relation (B.1) exists. Then we have

$$R_{a,n}(p) = \frac{P_{a,n+1}(p) - P_{a,n}(p)}{P_{a,n-1}(p)} \tag{B.4}$$

Using Theorem 2.4.7 and substituting for the polynomials we get

$$R_{a,n}(p) = p^2 \frac{\{a_{n+1,n+1}p^{(n-1)} + \sum_{i=1}^{n-1}(a_{n+1-i,n+1} - a_{n+1-i,n})p^{(n-1-i)}\}}{(\sum_{i=1}^{n-1} a_{n-i,n-1}p^{n-i} + 1)} \quad (B.5)$$

In Eq.(B.1) the polynomial $P_{a,n+1}(p)$ has (n+2) terms. Similarly the polynomials $P_{a,n}(p)$ and $P_{a,n-1}(p)$ have (n+1) and (n) terms respectively. It is obvious that $R_{a,n}(p)$ has to be a polynomial of degree two. Hence, the denominator polynomial has to be a factor of the numerator in Eq.(B.5). The term p^2 is already a factor. The terms within the parenthesis in the numerator and the denominator represent polynomials of the same degree (n-1). In order that $R_{a,n}(p)$ is to be a polynomial, it is necessary that the numerator and the denominator differ by a multiplication factor. This factor is

$$\frac{a_{n+1-i,n+1} - a_{n+1-i,n}}{a_{n-i-1,n-1}} \quad (B.6)$$

for $i = 0, 1, 2, \ldots, n-1$ and $a_{n+1-i,n} = 0$ for $i \leq 0$

The polynomial $R_{a,n}(p)$ now becomes

$$R_{a,n}(p) = p^2 \frac{a_{n+1-i,n+1} - a_{n+1-i,n}}{a_{n-i-1,n-1}} \quad (B.7)$$

We shall now consider the validity of the Eq.(B.7) for various values of i.

(a) Let i=0, we have

$$R_{a,n}(p) = p^2 \frac{a_{n+1,n+1}}{a_{n-1,n-1}} \qquad (B.8)$$

From Theorem 2.4.2 substituting for $a_{n+1,n+1}$ and $a_{n-1,n-1}$ the result follows which is rewritten for convenience as

$$R_{a,n}(p) = p^2 \frac{|A_{a,n+2}| \cdot |A_{a,n-1}|}{|A_{a,n}| \cdot |A_{a,n+1}|}, \qquad (B.9)$$

(b) Let i=1, we have

$$R_{a,n}(p) = p^2 \frac{a_{n,n+1} - a_{n,n}}{a_{n-2,n-1}} \qquad (B.10)$$

Two cases arise.

Case 1. n is odd:

$a_{n,n+1}$ corresponds to the coefficient of p^n in the polynomial $P_{a,n+1}(p)$. This polynomial can also be obtained in terms of the coefficients of the continued fraction expansion Eq.(2.5.3)

The coefficients β_i's of the continued fraction expansion are related to the coefficents $a_{k,n}$'s of the denominator polynomial $P_{a,n}(p)$ as follows:

$$a_{n,n+1} = \frac{\beta_2 + \beta_4 + \dots + \beta_n}{\prod_{i=1}^{n+1} \beta_i} \tag{B.11}$$

$$a_{n,n} = \frac{1}{\prod_{i=1}^{n} \beta_i} \tag{B.12}$$

$$a_{n-2,n-1} = \frac{\beta_2 + \beta_4 + \dots + \beta_{n-1}}{\prod_{i=1}^{n-1} \beta_i} \tag{B.13}$$

Substituting these in Eq.(B.10), we get

$$R_{a,n}(p) = \frac{p^2}{\beta_n \beta_{n+1}} \tag{B.14}$$

Substituting again for $\beta_n \beta_{n+1}$ from Eq.(2.5.12) we get

$$R_{a,n}(p) = p^2 \frac{|A_{a,n+2}|\;|A_{a,n-1}|}{|A_{a,n}|\;|A_{a,n+1}|} \tag{B.15}$$

Case 2. n is even:

We have

$$a_{n,n+1} = \frac{\beta_1 + \beta_3 + \dots + \beta_{n+1}}{\prod_{i=1}^{n+1} \beta_i} \tag{B.16}$$

$$a_{n,n} = \frac{1}{\prod_{i=1}^{n} \beta_i} \tag{B.17}$$

$$a_{n-2,n-1} = \frac{\beta_1 + \beta_3 + \dots + \beta_{n-1}}{\prod_{i=1}^{n-1} \beta_i} \tag{B.18}$$

Substituting these in Eq.(B.10) we get the same result as Eqs.(B.14) and (B.15).

c) For $i = 2$, we have

$$R_{a,n}(p) = p^2 \frac{a_{n-1,n+1} - a_{n-1,n}}{a_{n-3,n-1}} \tag{B.19}$$

Again two cases arise.

Case 1. n is odd:

In terms of the coefficients β_i's, these coefficients can be expressed as

$$a_{n-1,n+1} = \frac{\sum_{i=1}^{\frac{n+1}{2}} \beta_{2i-1} \sum_{i=1}^{\frac{n+1}{2}} \beta_{2i_1}}{\prod_{i=1}^{n+1} \beta_i} \tag{B.20}$$

$$a_{n-1,n} = \frac{\beta_1 + \beta_3 + \cdots + \beta_n}{\prod_{i=1}^{n} \beta_i} \tag{B.21}$$

$$a_{n-3,n-1} = \frac{\sum_{i=1}^{\frac{n-1}{2}} \beta_{2i-1} \sum_{i_1=i}^{\frac{n-1}{2}} \beta_{2i_1}}{\prod_{i=1}^{n-1} \beta_i} \tag{B.22}$$

Substituting the above expression in Eq.(B.19) and simplifying, we get

$$R_{a,n}(p) = p^2 \frac{1}{\beta_n \beta_{n+1}} \tag{B.23}$$

$$= p^2 \frac{|A_{a,n+2}|\;|A_{a,n-1}|}{|A_{a,n}|\;|A_{a,n+1}|} \tag{B.24}$$

The simplification is carried out as shown below.

$$a_{n-1,n+1} - a_{n-1,n} = \frac{\sum_{i=1}^{\frac{n+1}{2}} \beta_{2i-1} \sum_{i_1=i}^{\frac{n+1}{2}} \beta_{2i_1} - \beta_{n+1} \sum_{i_2=i_1}^{\frac{n+1}{2}} \beta_{2i_2-1}}{\prod_{i=1}^{n+1} \beta_i} \tag{B.25}$$

When $i_1 = (n+1)/2$, the last term becomes zero. Therefore $(n+1)/2$ reduces to $(n-1)/2$, and i_1 and i can have a maximum value of $(n-1)/2$.

Hence Eq.(B.25) can be written as

$$a_{n-1,n+1} - a_{n-1,n} = \frac{\sum_{i=1}^{\frac{n-1}{2}} \beta_{2i-1} \sum_{i_1=i}^{\frac{n-1}{2}} \beta_{2i_1}}{\prod_{i=1}^{n+1} \beta_i} \tag{B.26}$$

The numerators of Eqs.(B.26) and (B.22) are the same. Substituting these in Eq.(B.19) we have the result as Eqs.(B.14) and (B.15). Similarly for n even, same results can be obtained.

In general for any integer $i(0 \leq i \leq n+1)$, it can be shown that the expression Eq.(B.7) is true.

Hence the theorem.

APPENDIX C

PROOF FOR THEOREM 3.5.1.

The generating matrix $[A_{d,n}]$ given by Eq.(3.2.23) is rewritten (without the elements in the first row and the first column, as the solution to $d_{1,n}$ is $x_{2,2}$) as

$$[A_{d,n}] = \begin{bmatrix} -x_{2,2} & 1 & 0 & 0 & 0 \\ -x_{3,2} & x_{2,1} & -x_{2,2} & 1 & 0 \\ -x_{4,2} & x_{3,1} & -x_{3,2} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & F_{0,d} \\ \cdot & \cdot & \cdot & \cdot & F_{1,d} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -x_{n,2} & x_{n-1,1} & -x_{n-1,2} & \cdot & F_{n,d} \end{bmatrix} \qquad (C.1)$$

where

for odd n: $F_{0,d} = 1,\ F_{1,d} = x_{2,1}, \ldots, F_{n,d} = x_{(n+1)/2,1}$

and for even n: $F_{0,d} = 0,\ F_{1,d} = -x_{2,2}, \ldots, F_{n,d} = -x_{n/2,1}$

The elements in the first column are the negative of the elements of the column vector $[B_{dn}]$. These elements can be expressed in terms of the coefficients and the elements themselves. That is,

$$x_{2,2} = d_{1,n}$$

$$x_{3,2} = x_{2,1}\, d_{1,n} - x_{2,2}\, d_{2,n} + d_{3,n}$$

$$x_{4,2} = x_{3,1}\, d_{1,n} - x_{3,2}\, d_{2,n} + x_{2,1}\, d_{3,n} - x_{2,2}\, d_{4,n} + d_{5,n}$$

$$\vdots$$

$$x_{j,2} = \sum_{k=1}^{k_1} x_{n-k,1}\, d_{2k-1,n} - \sum_{k=1}^{k_2} x_{n-k,2}\, d_{2k,n}$$

for odd n: $k_1 = (n-1)/2$, $k_2 = k_1 - 1$

and for even n: $k_1 = k_2 = j/2 - 1$ (C.2)

Substituting Eq.(C.2) for the elements in the first column of the matrix $[A_{d,n}]$ and applying the elementary transformations on the matrix, we get the desired result. These elementary transformations are described as follows:

Step 1:

1) Row 2 - $x_{2,1}$. row 1

2) Row 3 - $x_{3,1}$. row 2

.

.

n-1) Row n - $x_{n,1}$. row n-1

Step 2:

1) Row 3 - $x_{2,1}$. row 2

2) Row 4 - $x_{3,1}$. row 3

.

.

n-2) Row n - $x_{n-1,1}$. row n-1

Step 3:

1) Row 4 - $x_{2,1}$. row 3

.

.

n-3) Row n - $x_{n-1,1}$. row n-1

Step 4:

.

.

$n-1^{th}$ Step:

1) Row n - $x_{2,1}$. row n-1

The last step is described as follows.

n^{th} Step :

1) Row 2 + row 1 . $d_{2,n}$

2) Row 3 + row 2 . $d_{2,n}$ + row 1 . $d_{4,n}$

.

.

.

$$\text{n-1) Row n} + \sum_{i=1}^{k_1} \text{row n-i} \cdot d_{2i,n} \qquad (C.3)$$

for odd n, $k_1 = (n-1)/2$, and even n, $k_1 = n/2$ (C.4)

After applying the above transformations, the matrix $[A_{d,n}]$ reduces to

$$[H_{n+1,d}] = \begin{bmatrix} -d_{1,n} & 1 & 0 & 0 \cdot & \cdot & 0 \\ -d_{3,n} & d_{2,n} & -d_{1,n} & 0 \cdot & \cdot & 0 \\ -d_{5,n} & d_{4,n} & -d_{3,n} & 0 \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot\cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot\cdot & \cdot & 0 \\ -d_{k_1,n} & d_{k_2,n} & \cdot & \cdot\cdot & \cdot & 0 \\ 0 & d_{k_4,n} & -d_{k_3,n} & \cdot & \cdot & 0 \\ 0 & 0 & -d_{k_1,n} & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 0 \cdot & \cdot & 0 \\ 0 & 0 & 0 & 0 & d_{2,n} & -d_{1,n} \\ 0 & 0 & 0 & & \cdot & 0 \\ 0 & 0 & 0 & \cdot\cdot & \cdot & 0 \\ 0 & 0 & 0 & \cdot\cdot & \cdot & d_{k_1,n} \end{bmatrix} \quad \text{(C.5)}$$

where for odd n: $k_1 = n$, $k_2 = n-1$, $k_3 = n-2$, $k_4 = 0$

and for even n: $k_1 = n-1$, $k_2 = n-2$, $k_3 = n-3$, $k_4 = n$

The matrix in Eq.(C.5) can be transposed to have the necessary form as required. Also a row and a column is added with all the elements zero except the first one which is unity. The matrix as given by Eq.(3.5.6) or (3.5.7) results.

Hence the result follows.

As the structures of the matrices $[A_{d,n}]$ and $[A_{a,n}]$ are same, the above property can be illustrated with the Example 2.5.1 with the following modifications: The element $f_{i,j}$ is changed to the element $x_{i,j}$ and the coefficients $a_{k,n}$'s are replaced by $d_{k,n}$'s.

APPENDIX D

PROOF FOR THEOREM 3.6.1.

The intention is to show that in general, a polynomial of degree (n+1) can be generated from the recurrence relation

$$P_{d,n+1}(t) = P_{d,n}(t) + R_{d,n}(t)\, P_{d,n-1}(t) \tag{D.1}$$

with $P_{d,0}(t) = 1$, $P_{d,1}(t) = 1 + \alpha_d t$ and

$$R_{d,n}(t) = t^2 \frac{|A_{d,n-1}|\ |A_{d,n+2}|}{|A_{d,n}|\ |A_{d,n+1}|} \tag{D.2}$$

The existance of $R_{d,n}(t)$ depends on the factor

$$\frac{|A_{d,n-1}|\ |A_{d,n+2}|}{|A_{d,n}|\ |A_{d,n+1}|} \tag{D.3}$$

Let us assume that the relation (D.1) exists. Then we have

$$R_{d,n}(t) = \frac{P_{d,n+1}(t) - P_{d,n}(t)}{P_{d,n-1}(t)} \tag{D.4}$$

Using Theorem 3.4.7 and substituting for the polynomials we get

$$R_{d,n}(t) = t^2 \frac{\{d_{n+1,n+1}\, t^{(n-1)} + \sum_{i=1}^{n-1}(d_{n+1-i,n+1} - d_{n+1-i,n})\; t^{(n-1-i)}\}}{(\sum_{i=1}^{n-1} d_{n-i,n-1}\; t^{n-i} + 1)} \tag{D.5}$$

In Eq.(D.1) the polynomial $P_{d,n+1}(t)$ has (n+2) terms. Similarly the polynomials $P_{d,n}(t)$ and $P_{d,n-1}(t)$ have (n+1) and (n) terms respectively. It is obvious that $R_{d,n}(t)$ has to be a polynomial of degree two. Hence, the denominator polynomial has to be a factor of the numerator in Eq.(D.5). The term t^2 is already a factor. The terms within the parenthesis in the numerator and the denominator represent polynomials of the same degree (n-1). In order that $R_{d,n}(t)$ is to be a polynomial, it is necessary that the numerator and the denominator differ by a multiplication factor. This factor is

$$\frac{d_{n+1-i,n+1} - d_{n+1-i,n}}{d_{n-i-1,n-1}}, \tag{D.6}$$

for i= 0, 1, 2,......,n-1 and $d_{n+1-i,n} = 0$ for $i = 0$

The polynomial $R_{d,n}(t)$ now becomes

$$R_{d,n}(t) = t^2 \frac{d_{n+1-i,n+1} - d_{n+1-i,n}}{d_{n-i-1,n-1}} \tag{D.7}$$

We shall now consider the various values of i

(a) Let i=0, we have

$$R_{d,n}(t) = t^2 \frac{d_{n+1,n+1}}{d_{n-1,n-1}} \tag{D.8}$$

From Theorem 3.4.2 substituting for $d_{n+1,n+1}$ and $d_{n-1,n-1}$ the result follows which is rewritten for convenience as

$$R_{d,n}(t) = t^2 \frac{|A_{d,n+2}|\ |A_{d,n-1}|}{|A_{d,n}|\ |A_{d,n+1}|} \tag{D.9}$$

(b) Let i=1, we have

$$R_{d,n}(t) = t^2 \frac{d_{n,n+1} - d_{n,n}}{d_{n-2,n-1}} \tag{D.10}$$

Two cases arise:

Case 1. n is odd:

$d_{n,n+1}$ corresponds to the coefficient of t^n in the polynomial $P_{d,n+1}(t)$. This polynomial can also be obtained in terms of the coefficients of the continued fraction expansion Eq.(3.5.3)

The coefficients of the continued fraction expansion γ_i's are related to the coefficents $d_{k,n}$'s of the denominator polynomial $P_{d,n}(t)$ as follows:

$$d_{n,n+1} = \frac{\gamma_2 + \gamma_4 + \cdot\ \cdot\ \cdot\ \cdot\ + \gamma_n}{\prod_{i=1}^{n+1} \gamma_i} \tag{D.11}$$

$$d_{n,n} = \frac{1}{\prod_{i=1}^{n} \gamma_i} \tag{D.12}$$

$$d_{n-2,n-1} = \frac{\gamma_2 + \gamma_4 + \cdot \cdot \cdot \cdot \cdot + \gamma_{n-1}}{\prod_{i=1}^{n-1} \gamma_i} \quad (D.13)$$

Substituting these in Eq.(D.10), we get

$$R_{d,n}(t) = \frac{t^2}{\gamma_n \gamma_{n+1}} \quad (D.14)$$

Substituting again for $\gamma_n \gamma_{n+1}$ from Eq.(3.5.12) we get

$$R_{d,n}(t) = t^2 \frac{|A_{d,n+2}| \; |A_{d,n-1}|}{|A_{d,n}| \; |A_{d,n+1}|} \quad (D.15)$$

<u>Case 2. n is even:</u>

We have

$$d_{n,n+1} = \frac{\gamma_1 + \gamma_3 + \cdot \cdot \cdot \cdot + \gamma_{n+1}}{\prod_{i=1}^{n+1} \gamma_i} \quad (D.16)$$

$$d_{n,n} = \frac{1}{\prod_{i=1}^{n} \gamma_i} \quad (D.17)$$

$$d_{n-2,n-1} = \frac{\gamma_1 + \gamma_3 + \cdot \cdot \cdot \gamma_{n-1}}{\prod_{i=1}^{n-1} \gamma_i} \quad (D.18)$$

Substituting these in Eq.(D.10) we get the same result as Eqs.(D.14) and (D.15).

c) For i = 2, we have

$$R_{d,n}(t) = t^2 \frac{d_{n-1,n+1} - d_{n-1,n}}{d_{n-3,n-1}} \tag{D.19}$$

Again two cases arise.

Case 1. n is odd:

In terms of the coefficients γ_i's, these coefficients can be expressed as

$$d_{n-1,n+1} = \frac{\sum_{i=1}^{\frac{n+1}{2}} \gamma_{2i-1} \sum_{i_1=i}^{\frac{n+1}{2}} \gamma_{2i_1}}{\prod_{i=1}^{n+1} \gamma_i} \tag{D.20}$$

$$d_{n-1,n} = \frac{\gamma_1 + \gamma_3 + \cdot \cdot + \gamma_n}{\prod_{i=1}^{n} \gamma_i} \tag{D.21}$$

$$d_{n-3,n-1} = \frac{\sum_{i=1}^{\frac{n-1}{2}} \gamma_{2i-1} \sum_{i_1=i}^{\frac{n-1}{2}} \gamma_{2i_1}}{\prod_{i=1}^{n-1} \gamma_i} \tag{D.22}$$

Substituting the above expression in Eq.(D.19) and simplifying, we get

$$R_{d,n}(t) = t^2 \frac{1}{\gamma_n \gamma_{n+1}} \tag{D.23}$$

$$= t^2 \frac{|A_{d,n+2}| \; |A_{d,n-1}|}{|A_{d,n}| \; |A_{d,n+1}|} \tag{D.24}$$

The simplification is carried out as shown below.

The term $d_{n-1,n+1} - d_{n-1,n}$ of the numerator in Eq.(D.19) reduces to

$$= \frac{\sum_{i=1}^{\frac{n+1}{2}} \gamma_{2i-1} \sum_{i_1=i}^{\frac{n+1}{2}} \gamma_{2i_1-} \gamma_{n+1} \sum_{i_2=i_1}^{\frac{n+1}{2}} \gamma_{2i_2-1}}{\prod_{i=1}^{n+1} \gamma_i} \tag{D.25}$$

When $i_1 = (n+1)/2$, the last term becomes zero. Therefore $(n+1)/2$ reduces to $(n-1)/2$ and also i_1 and i can have a maximum value of $(n-1)/2$. Hence Eq.(D.24) can be written as

$$d_{n-1,n+1} - d_{n-1,n} = \frac{\sum_{i=1}^{\frac{n-1}{2}} \gamma_{2i-1} \sum_{i_1=i}^{\frac{n-1}{2}} \gamma_{2i_1}}{\prod_{i=1}^{n+1} \gamma_i} \tag{D.27}$$

The numerators of Eqs.(D.27) and (D.22) are the same. Substituting these in Eq.(D.19) we have the result as Eq.(D.14) and Eq.(D.15).

Similarly for n even, same results can be obtained.

In general for any integer $i(0 \leq n \leq n+1)$, it can be shown that the expression Eq.(D.7) is true.

Hence the theorem.

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