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A Comparative Study of Quantization Procedures

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Abstract

A Comparative Study of Quantization Procedures

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The problem of quantization is as old as quantum mechanics itself. For more than half a century many researchers worked on quantization. Nowadays, we know different types of quantizations; for example, geometric quantization, prime quantization. The relationships between these quantizations have not been studied fully in the literature.

In this thesis we study the general problem of quantization, analysing it via geometric quantization and prime quantization. We also give a comparative analysis of the two methods with examples.

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Symbols and Notations

a	annihilation operator
a^+	creation operator
A^*	adjoint of A
$\mathcal{A}_c$	classical algebra of observables
$\mathcal{B}(\Gamma)$	Borel sets of Γ
C	the set of complex numbers
$C^\infty(M)$	the set of complex valued funcutions on M, vanishing at infinity
$C^\infty_{\mathbb{R}}(M)$	the subset of real functions on M
d	exterior derivative
$\exp(n)$	exponential of n
$\tilde{G}$	extended Galilei group
$GL(n,C)$	general linear group
h	Plank's constant
$\mathcal{H}$	abstract Hilbert space
$\mathcal{H}_K$	reproducing kernel Hilbert space
$\tilde{\mathcal{H}}$	direct integral space
$\mathcal{H}_{P_V}$	Hilbert space of vertical polarization
$\mathcal{H}_{P_h}$	Hilbert space of horizontal polarization
K	reproducing kernel
$L^2(\Gamma)$	the square integrable functions on Γ
$\mathcal{L}(\mathcal{H})$	the set of bounded linear operators on $\mathcal{H}$
$\mathcal{L}_\xi$	Lie derivative w.r.t.ξ

$\mathscr{L}(\mathscr{H})^+$	positive cone of $\mathscr{L}(\mathscr{H})$
L	line bundle
L_m	fibre over m
(M,ω)	symplectic manifold
ML(n,C)	the metalinear group
$\mathcal{P}$	projection operator
$\mathcal{P}_k$	projection corresponding to a reproducing kernel K
$\tilde{P}$	projection valued measure
p	momentum coordinate
$\hat{p}$	momentum operator
Pm	holomorphic polarization
$\overline{Pm}$	antiholomorphic polarization
q	position coordinate
$\hat{q}$	position operator
R	the set of real numbers
s_0	unit section
s	section
SO(3)	special orthogoral group
tr	trace
$\mathscr{T}(\mathscr{H})$	Banach space of trace class operators on $\mathscr{H}$
T^*Q	cotangent bundle of Q
T_m^cM	complexified tangent bundle of M at m.
U(M)	the space of all vector fields on M
$U_p(M)$	the space of polarized vector fields on M
X	locally compact, separable topological space

X_f	Hamiltonian vector field generated by f.
Z	the set of integers
$\Gamma(L)$	the set of smooth sections on L
ρ	density matrix
ω	symplectic form
∇	connection
Σ	summation
$\oplus$	direct sum
$\otimes$	tensor product
$A \subset B$	A is a subset of B
$\lVert A \rVert$	the norm of A
$\rfloor$	interior product (contraction)
$\langle . \vert . \rangle$	scalar product
χ_Δ	characteristic function
Λ	wedge product
$\Leftrightarrow$	if and only if
$[A, B]$	the commutator bracket
$\Pi_{x \epsilon X} K_x$	cartesian product of K_x

$$\vert \Phi \rangle \langle \Psi \vert \gamma = \langle \Psi \vert \gamma \rangle \Phi$$

Introduction

Quantization refers to the process of forming a quantum mechanical system from a given classical system. In other words, we can say that it is a mapping between classical and quantum observables satisfying the correspondence principle (i.e. when the Plank's constant $\hbar \to o$, the quantum system changes to the corresponding classical system). A classical phase space is usually represented by a symplectic manifold (M, ω) (a rigorous definition will be given later) and the observables are the smooth functions on M. On the other hand, a quantum phase space is represented by the rays in a Hilbert space $\mathcal{H}$ and the observables are self-adjoint operators in $\mathcal{H}$

The notion of quantization emerged in the early stages of the development of quantum mechanics. For example, canonical quantization (a quantization in which the classical observable, represented by a function $f(p_n, q^m)$ of the canonical coordinates (p_n, q^m) is mapped to the corresponding quantum mechanical observable through the operator $f(-i\hbar \frac{\partial}{\partial q^n}, q^m)$ was introduced by the originators of quantum mechanics. Since then the problem of quantization has been studied from different points of views. Nowadays, quantization is a complete subject in its own right and there exist different quantization schemes, e.g. geometric quantization, prime quantization etc. In this thesis we wish to discuss the problem of quantization by analysing it via two existing methods: 1) geometric quantization and 2) prime quantization.

The geometric approach to the problem of quantization was introduced by Kostant [1] and Souriau [2] independently. Kostant discovered these techniques through his attempts to generalize Kirillov's [3] result on nilpotent groups. Souriau's work was directly related to the problem of quantization.

The next fundamental development of geometric quantization was due to Blattner, Kostant and Sternberg [4].

Actually geometric quantization begins with the notion of a symplectic manifold. In this method, one takes a symplectic manfiold as a classical phase space (and the functions on this manifold serve as classical observables) and constructs a Hilbert space $\mathscr{H}$ (quantum phase space) and obervables (self-adjoint operators in $\mathscr{H}$) for the underlying quantum system in a geometric, coordinate-free way. The general procedure for constructing a Hilbert space representation of an algebra of classical observables is known as *prequantization*. The Hilbert space constructed by prequantization may be too big to represent a quantum system. So one has to cut down the prequantization Hilbert space so that the representation becomes irreducible (that is, no proper subspace of the Hilbert space is invariant under its action). The way one reduces the size of the Hilbert space is called a *polarization*, in the language of geometric quantization.

The name "prime quantization" is due to S.T. Ali and H. D. Doebner [5] following up an earlier work of E. Prugovecki [6]. However, a very similar type of quantization was developed independently by F.A. Berezin [6] before them. This latter type of quantization is known as Berezin quantization in the literature. In prime quantization, like in any other approach to the quantization problem, a classical observable is first mapped to a self-adjoint operator in a Hilbert space $\mathcal{H}$. Then to have an irreducible representation of the classical algebra of observables, one uses the techniques of the reproducing-kernel Hilbert space [see the appendix B] i.e., by projecting down to one of the various possible reproducing kernel Hilbert subspaces of $\mathcal{H}$ one gets the quantized Hilbert space - a procedure which is reminiscent of polarization in geometric quantization. One of the remarkable features of prime quantization is that it simulteneously provides the solution to the problem of ordering of operators [8, 9] in quantum mechanics.

In chapter I, we define quantization in a manner general enough to cover both geometric quantization and prime quantization. A detailed discussion of geometric quantization and prime quantization with examples, is given in Chapters 2 and 3 respectively. In the final chapter we give a comparative analysis of the two methods and try to find a possible relationship between them.

1. QUANTIZATION

In the introduction we have already given a brief non-mathematical description of quantization. Now we want to define quantization mathematically. We said that quantization is a mapping between a classical and a quantum phase space. But an arbitrary relationship between classical and quantum observables will not define a quantization. The mapping between these phase spaces must satisfy certain conditions such as Dirac's quantum conditions [10]. According to this condition, to each $f \in \mathscr{C} \subset C^{\infty}(M)$ there corresponds an operator $\hat{f}$ in $\mathscr{H}$ such that

1) The map $f \rightarrow \hat{f}$ is linear over $\mathbf{R}$

2) $\{f, g\}^{\wedge} = (\frac{1}{i}) [\hat{f}, \hat{g}]$ for each $f, g \in C^{\infty}(M)$ (1.1.1)

where $\{,\}$ is the Poisson bracket and $[,]$ is the commutator bracket. But the these conditions alone will not uniquely determine a quantum system from a classical one.

For this (a) $\mathscr{C}$ must contain the constant functions in $C^{\infty}(M)$ and these functions must be represented in $\mathscr{H}$ by the corresponding multiples of the identity operator, b) we have to restrict the size of $\mathscr{H}$ i.e., the operators in $\mathscr{H}$ must form an irreducible representation of some Lie algebra of observables.

Let us therefore define quantization as:

Definition 1.1.1: Let Q be any manifold. Then a quantization of Q is a map taking classical observables f (i.e., continuous function of $(q,p) \in T^*Q$) to self-adjoint operators $\hat{f}$ on a Hilbert space $\mathscr{H}$ such that,

(i) $(f + g)^\wedge = \hat{f} + \hat{g}$

(ii) $(\lambda f)^\wedge = \lambda \hat{f}\,,\ \lambda \in \mathbf{R}$

(iii) $\{f,g\}^\wedge = (1/i)\ [\hat{f}\,,\hat{g}]$ (1.1.2)

(iv) $\hat{1} = I$ (1 = constant function,
I = identity operator)

(v) $\hat{q}^i$ and $\hat{p}_j$ act irreducibly on $\mathscr{H}$.

Here $\hat{q}^i$ and $\hat{p}_j$ are ith and jth component of position and momentum operator respectively. By the word "irreducibility" we mean that no proper subspace of $\mathscr{H}$ is invariant under its action.

Unfortunately there does not exist a quantization satisfying (i) - (v) [16]. There may be a quantization satisfying only (i) - (iv) and this quantization is known as *prequantization*. We have to relax the condition (iii) i.e., (iii) would not be true for all f's and g's but only for certain types of f's and g's.

2. GEOMETRIC QUANTIZATION

2.1 Prequantization

The classical phase space M carries a natural volume element in terms of the symplectic form:

$$\omega^n = \omega \wedge \omega \wedge \cdots \wedge \omega \quad \text{(n-times)}$$
$$= dp_1 \wedge dp_2 \wedge \cdots \wedge dp_n \wedge dq^1 \wedge dq^2 \wedge \cdots \wedge dq^n \quad (2.1.1)$$
(in local coordinates)

so there is a Hilbert space $\mathscr{H} = L^2(M)$ associated to M with the inner product defined by

$$\langle\Psi|\phi\rangle = \int_M \overline{\psi}\, \phi\, \omega^n, \; \psi, \phi \in \mathscr{H} \quad (2.1.2)$$

In local coordinates, we write

$$\omega^n = d^n p\, d^n q.$$

For each $f \in C^\infty(M)$ there is an operator $\hat{f}$ such that

$$\hat{f}(\phi) = -i\hbar X_f \phi, \; \phi \in \mathscr{H} \quad (2.1.3)$$

where X_f is the Hamiltonian vector field associated to f.

Although (2.1.3) satisfies the quantum condition, yet whenever $df=0$ (i.e., if f is a constant function), $X_f = 0$. So constant function will correspond to the zero operator; which is contrary to definition (1.1.1). We could correct this by writing

$$\hat{f}(\phi) = -i\hbar X_f(\phi) + f\phi \quad (2.1.4)$$

where $f\phi$ acts by multiplication.

But this, in general, does not satisfy the bracket relation, since

$$[-i\hbar X_f + f, -i\hbar X_g + g]$$
$$= -i\hbar(-i\hbar)[X_f, X_g] - i\hbar[X_f, g] - i\hbar[f, X_g] + [f, g]$$
$$= -i\hbar(-i\hbar X_{\{f, g\}} + 2\{f, g\}), \; f, g \in C^\infty(M) \quad (2.1.5)$$

We could correct this by adding one more term to $\hat{f}$ i.e.,

$$\hat{f}(\phi) = -i\hbar[X_f(\phi) - \frac{i}{\hbar}(X_f \rfloor \theta)\phi] + f\phi \tag{2.1.6}$$

where θ is a 1-form, defined in App. A).

(2.1.6) satisfies the Lie bracket as we can see from the following calculation.

$$[\hat{f}, \hat{g}]\phi = \hbar^2[X_f, X_g]\phi - i\hbar\phi\{X_f(g - X_g \rfloor \theta) - X_g(f - X_f \rfloor \theta)\}$$

$$= -i\hbar\{f, g\}^\wedge \phi \tag{2.1.7}$$

But $\hat{f}$, as defined in (2.1.6) depends on θ and cannot be defined unless ω is exact i.e., when $\omega = d\theta$. The dependence of $\hat{f}$ on θ can be avoided if one applies a gauge transformation (i.e., $\theta \rightarrow \theta + du$ and $\phi \rightarrow \exp(iu/h)\phi$) which amounts to choosing a line bundle (see App. A) over M. Hence to give the construction of $\hat{f}$ a strong foundation, we are led to using a Hermitian line bundle over M, instead of starting simply with functions on M.

2.2 Quantizable Phase Space

Let (M,ω) be a quantizable symplectic manifold i.e. ω satisfies the integrality conditon (A47). Let (L,π,M) be a Hermitian line bundle over M with connection ∇. Then the prequantum Hilbert space $\mathscr{H}$ is the space of all smooth sections $s \in \Gamma(L)$ of L for which the integral of (s,s) over M exists and is finite. The scalar product on $\mathscr{H}$ is defined by

$$<s_1 | s_2> = \int_M (s_1, s_2)\omega^n, \quad s_1, s_2 \in \mathscr{H} \tag{2.2.1}$$

Then the operator $\hat{f}$ corresponding to a classical observable $f \in C^\infty(M)$ is

$$\hat{f}s = -i\hbar \nabla_{X_f} s + fs \tag{2.2.2}$$

where s is in some suitable subset of $\mathscr{H}$ For example, if we take Q as a configuration space of a classical system and $M = T^*Q$, then the line bundle is simply the trivial bundle (i.e., $L = M \times C$) and the connection

∇_{X_f} is defined by

$$\nabla_{X_f}s = X_f(s) - \frac{i}{\hbar}(X_f \lrcorner \theta)s \tag{2.2.3}$$

Now if $\theta = \Sigma\, p_j\, dq^j$ we have

$$\hat{f}s = \frac{\hbar}{i}(X_f s) - (X_f \lrcorner \theta)s + fs$$

where $X_f = \sum \frac{\partial f}{\partial p_j}\frac{\partial}{\partial q^j} - \frac{\partial f}{\partial q^j}\frac{\partial}{\partial p_j}$

$$X_f \lrcorner \theta = \Sigma\left(\frac{\partial f}{\partial p_j}\frac{\partial}{\partial q^j} - \frac{\partial f}{\partial q^j}\frac{\partial}{\partial p_j}\right)(\Sigma p_j dq^j) = \Sigma p_j \frac{\partial f}{\partial p_j}$$

Then (2.2.3) $\Rightarrow \hat{f}s = \frac{\hbar}{i}\left[\Sigma\left(\frac{\partial f}{\partial p_j}\frac{\partial}{\partial q^j} - \frac{\partial f}{\partial q^j}\frac{\partial}{\partial p_j}\right)\right]s$

But when $\hat{f}_j = \hat{p}_j \Rightarrow \hat{f}(p_j, q^j) = f(p_j, q^j) = p_j$

$$\Rightarrow \hat{p}_j s = \frac{\hbar}{i}\left[\left(\frac{\partial p_j}{\partial p_j}\frac{\partial}{\partial q^j} - \frac{\partial p_j}{\partial q^j}\frac{\partial}{\partial p_j}\right]s - p_j\frac{\partial p_j}{\partial p_j}s + p_j s$$

$$\Rightarrow \hat{p}_j s = -i\hbar\frac{\partial}{\partial q^j}s \Rightarrow \hat{p}_j = -i\hbar\frac{\partial}{\partial q^i} \tag{2.2.4a}$$

Again, $f(p_j, q^j) = q^j$ when $\hat{f}_j = \hat{q}^j$ and

(2.2.3) $\Rightarrow \hat{q}^j s = \frac{\hbar}{i}\left(\frac{\partial q^j}{a p_j}\frac{\partial}{\partial q^j} - \frac{\partial q^j}{\partial q^j}\frac{\partial}{\partial p_j}\right)s + q^j s - p_j\frac{\partial q^j}{\partial p_j}s$

$$= \frac{\hbar}{i}\left(-\frac{\partial}{\partial p_j}\right)s + q^j s = \left(i\hbar\frac{\partial}{\partial p_j} + q^j\right)s$$

$$\Rightarrow \hat{q}^j = i\hbar\frac{\partial}{\partial p_j} + q^j \tag{2.2.4b}$$

This can be viewed in another way:

Let $f \in C^\infty(M)$ and $\eta_f \in U(L^*)$ (vector field on $L^* = L - \{0\}$) satisfying

1) $\pi_*(\eta_f) = X_f$ and
where $\pi_*: U(L^*) \to U(M)$ (2.2.5)

2) $\eta_f \lrcorner \alpha = f \circ \pi$

where $\alpha = \theta/\hbar + i\, dz/z$ is the connection form on L and $\pi: L \rightarrow M$ is the projection.

Then if $L = \mathbf{M} \times C$, η_f is given by

$$\eta_f = X_f - \frac{i}{\hbar} z.(f - A) \frac{\partial}{\partial z} + \frac{i}{\hbar} \bar{z} (f - A) \frac{\partial}{\partial \bar{z}} \qquad (2.2.6)$$

where $A = X_f \rfloor \theta \varepsilon C^{\infty}(M)$ is the "action" function of f and θ is the symplectic potential.

Thus we have already constructed a representation of the Lie algebra $C^{\infty}(M)$ by Hermitian operators in $\mathcal{H}$

For each $f \varepsilon C^{\infty}(M)$ we would have an operator $\hat{f}$ in $\mathcal{H}$ and the domain of $\hat{f}$ is a dense subset of $\mathcal{H}$ If all these domains do not have a common intersection, we cannot construct $\hat{f}$. So the domains of definitions of these operators are some common dense subspaces of $\mathcal{H}$

The representation we constructed is not irreducible. So our next job is to reduce the space on which sections will be defined. This can be done by choosing a suitable "polarization" which we now introduce.

2.3 Polarization

The problem of reducing the space of sections involves the idea of Lagrangian submanifolds of symplectic manifolds. Let M be a 2n-dimensional symplectic manifold. Then a submanifold $P \subset M$ of M is said to be a Lagrangian submanifold if P has the following properties:

1) $\dim P = n$

2) $\omega(X, Y) = 0;\ X, Y \ \varepsilon\ U(P)$ (2.3.1)

where U(P) is the space of vector fields on P.

A real polarization (complex polarizations will be introduced later) of a symplectic manifold (M,ω) is a foliation of M by Lagrangian submanifolds. That is, it is a distribution having the following properties: (Note: For definition of foliation and distribution, see App. A).

1) P is integrable : if X, Y ε $U_p(M)$, then
$[X, Y] \subset U_p(M)$. where
$U_p(M) = \{X \varepsilon U(M) \mid X_m \varepsilon P_m \ \forall m \varepsilon M\}$ and
for each $m \varepsilon M$, $P_m \subset T_m M$.

2) P is Lagrangian: for each $m \varepsilon M$, P_m is a Lagrangian subspace of $T_m M$.

A vector field $\xi \varepsilon U_p(M)$ is characterized by the condition

$$\xi(f o Pr) = 0 \ \forall \ f \varepsilon \ C^{\infty}(Q) \tag{2.3.2}$$

where Pr: $M = T^*Q \rightarrow Q$ is the natural projection, θ has the further property

$$\xi \lrcorner \theta = 0 \tag{2.3.3}$$

where $\theta \ \varepsilon \ \Omega^1(M)$ is the canonical 1-form (symplectic potential).

Then for ξ, $\eta \ \varepsilon \ U_p(M)$,

$$[\xi,\eta] \ (f o Pr) = \xi(\eta(f o Pr)) - \eta \ (\xi(f o Pr)) = 0 \ \forall \ f \ \varepsilon \ C^{\infty}(Q) \ \text{(by 2.3.2)} \tag{2.3.4}$$

so that $[\xi,\eta] \subset U_p(M)$. That implies P is integrable.

Also $\omega(\xi,\eta) = \xi \ (\eta \lrcorner \theta) - \eta(\xi \lrcorner \theta) - [\xi,\eta]\theta$

$= 0$ (by (2.3.3) and (2.3.4))

Now if $M = T^*Q$ and $L = M \times C$

$\hat{f}s = -i\hbar \ \nabla_{X_f} \ s + fs$ (Eq. (2.2.2)) can be written as

$$\hat{f}s = i\hbar[X_f s - \frac{i}{\hbar}(X_f \lrcorner \theta)s] + fs.$$

But upon polarization

$X_f \lrcorner \theta = 0$ (by (2.3.3))

So then $\hat{f}s = -i\hbar X_f s + fs$ (2.3.5)

For $M = T^*Q$, the vector fields $\frac{\partial}{\partial p_j}$ define the *vertical polarization*. and the space of leaves can be identified with Q.

If $M = \mathbf{R}^{2n}$, the vector fields $\frac{\partial}{\partial q^j}$ define the *horizontal polorization*.

In the case of a general cotangent bundle, these fields cannot be naturally extended from Q to the bundle.

Let us consider the *vertical polarization* on $M = \mathbf{R}^{2n}$. If s_0 is a unit section and θ is the symplectic potential, then

$$\nabla_X \Psi(p,q)s_o = X(\Psi(p,q))s_o - \frac{i}{\hbar}(X \lrcorner \theta)\Psi(p,q)$$

$$= X(\Psi(p,q))\, s_0 \quad \text{since } X \lrcorner \theta = 0$$

where $X = \{\frac{\partial}{\partial p_j}, \frac{\partial}{\partial q^j}\}$.

$$= (\frac{\partial \Psi(p,q)}{\partial p_j}, \frac{\partial \Psi(p,q)}{\partial q^j})(s_o)$$

$$\Rightarrow \nabla(\partial/\partial pj)\Psi(p,q)s_o = (\frac{\partial \Psi(p,q)}{\partial p_j})s_o \qquad (2.3.6)$$

This will vanish for each j if Ψ depends only on q. In this case

$$\Psi(p,q)s_o = \Psi(q)s_o \qquad (2.3.7)$$

Hence the quantum operators are:

$$\hat{q}^j(\Psi(q)s_o) = q^j\, \Psi(q)\, s_o$$

$$\hat{p}_j(\Psi(q))s_o) = -i\hbar\, (\partial\Psi/\partial q^j)s_o \qquad (2.3.8)$$

which is the usual Schrodinger representation.

Now we want to introduce *complex polarization*. But this is usually associated with an important class of symplectic manifolds, known as **Kähler manifolds**. Before defining complex polarization we should have some idea of Kähler manifolds. A Kähler manifold is a triple (M, ω, J), where

1) (M, ω) is a symplectic manifold;

2) J is a tensor field on M i.e., for each $m \in M$,

$$J: T_m M \to T_m M$$

is a linear map having the following properties:

a) $J^2 = -1$ (2.3.9)

b) The bilinear form g defined by

$$g(\xi,\eta) = \omega(\xi, J\eta) \; ; \quad \xi,\eta \in T_m M \tag{2.3.10}$$

is a positive definite Riemannian metric.

c) For each $m \in M$,

$$\omega(J\xi, J\eta) = \omega(\xi,\eta) \; , \; \forall \, \xi, \eta \in T_m M \tag{2.3.11}$$

In local complex analytic coordinates $\{z^j\}$ one can introduce

$$J\left(\frac{\partial}{\partial z^j}\right) = i \frac{\partial}{\partial \bar{z}^j}, \; J\left(\frac{\partial}{\partial \bar{z}^j}\right) = -i \frac{\partial}{\partial z^j} \tag{2.3.12}$$

It is possible to find [11] a smooth real function f such that

$$\omega = i \frac{\partial^2 f}{\partial z^j d\bar{z}^k} dz^j \wedge d\bar{z}^k \tag{2.3.13}$$

A **complex polarization** on a symplectic manifold (M, ω) is a complex distribution P having the properties:

1) P_m, for each $m \in M$, is a complex Lagrangian subspace of $T_m^c M$.

2) P is involutive, that is, for $X, Y \in U_P(M)$, $[X, Y] \in U_P(M)$.

3) $D_m = P_m \cap \overline{P_m} \cap T_m M$ must have a constant dimension for each $m \in M$. Here $\overline{P_m}$ is the complex conjugate of P_m.

A Kähler manifold has two natural polarizations:

1) holomorphic polarization P_m

2) antiholomorphic polarization $\overline{P_m}$

In the local coordinates $\{z_j\}_{j=1,...,n}$ P_m is the linear span of the set $\{\partial/\partial\overline{z_j}\}$ of antiholomorphic coordinate vectors at m and $\overline{P_m}$ is spanned by the holomorphic coordinate vectors $\{\partial/\partial z_j\}$. In this case

$$D_m = P_m \cap \overline{P_m} = \{0\}, \forall\ m \in M.$$

2.4 Examples

Example 1

Let us consider a free particle moving in the space $\mathbf{Q} = \mathbf{R}$. The corresponding phase space is $M = \mathbf{R}^2$ with the symplectic form

$$\omega = dp \wedge dq \qquad (2.4.1)$$

where q is the coordinate on $\mathbf{R}$ and p is the corresponding momentum.

ω can be written as

$$\omega = d\theta, \text{ where } \theta = pdq \qquad (2.4.2)$$

so that (M,ω) is quantizable. In particular: the line bundle (L, π, M) is simply the trivial bundle (i.e. $L= M\times C$) , the space of sections $\Gamma(L)$ is identified with $C^\infty(M)$ and the connection ∇ is defined as

$$\nabla_X\Psi = X(\Psi) - i\ (X \rfloor \theta)\Psi,\ X\varepsilon U(M),\ \Psi\varepsilon C^\infty(M) \qquad (2.4.3)$$

where θ is the canonical 1-form defined as in (2.4.2). The corresponding connection form is

$$\alpha = pdq + i\ dz/z \quad \text{(see Appendix A)} \qquad (2.4.4)$$

Any vector field on TM (tangent bundle on M) will be of the form:

$$\xi = \xi^1 \frac{\partial}{\partial p} + \xi^2 \frac{\partial}{\partial q} \qquad \text{and let}$$

$$TM \ni \eta = \eta^1 \frac{\partial}{\partial p} + \eta^2 \frac{\partial}{\partial q} \qquad (2.4.5)$$

Then for $\xi, \eta \in P_m \subset T_mM,\ m\varepsilon M,$

$$\omega(\xi, \eta) = \xi^1\eta^2 - \xi^2\eta^1 = 0 \qquad (2.4.6)$$

This is only possible when:

A) each $\xi \varepsilon T_mM$ is of the form

$$\xi = \xi^1 \frac{\partial}{\partial p} \qquad (2.4.7)$$

B) each $\xi \varepsilon T_mM$ is of the form

$$\xi = \xi^2 \frac{\partial}{\partial q} \qquad (2.4.8)$$

and

C) $$\xi^1\eta^2 = \xi^2\eta^1 \Rightarrow \frac{\xi^1}{\xi^2} = \frac{\eta^1}{\eta^2}, \qquad (2.4.9)$$

that means all the vectors in T_mM will be scalar multiples of one fixed vector.

Therefore, $M = \mathbf{R}^2$ admits two naturally defined polarizations. The polarization spanned by $\frac{\partial}{\partial p}$ will give us the *vertical polarization* and the polarized Hilbert space $\mathcal{H}_{p_v} = L^2(\mathbf{R}, dq)$. The polarization spanned by $\frac{\partial}{\partial q}$ is known as *horizontal polarization* and the corresponding Hilbert space

$$\mathcal{H}_{p_h} = L^2(\mathbf{R}, dp).$$

If we introduce analytic complex coordinates $z = x + iy$ on $M = \mathbf{R}^2$, then

$$2\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \qquad (2.4.10)$$

and $$2\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \qquad (2.4.11)$$

also $\omega\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) = \omega\left(\frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial \bar{z}}\right) = 0$ (2.4.12)

Hence $M = \mathbf{R}^2$ has two natural Kähler polarizations:

A) One (P_m) spanned by the antiholomorphic coordinate vector $(\partial/\partial\bar{z})_m$ is known as *holomorphic polarization.*

B) Another $(\bar{P}_m)$ spanned by the holomorphic coordinate vector $(\partial/\partial z)_m$ is known as *antiholomorphic polarization.*

The Hilbert space corresponding to Kähler polarization is $\mathcal{H}_{p_k} \subset L^2(\mathbf{R}^2, dp\, dq)$.

Example 2: Quantization on a sphere.

Let $M = \{ (x_1, x_2, x_3) \in R^3 | \; x_1^2 + x_2^2 + x_3^2 = r^2 \}$ (2.4.13)

be a two dimensional sphere with radius r with centre at the origin. In polar coordinates, we have

$x_1 = r\sin\theta\cos\phi,\ x_2 = r\sin\theta\sin\phi,\ x_3 = r\cos\theta$ (2.4.14)

where $0 \le \theta \le \pi,\ 0 \le \phi < 2\pi$, (r is fixed).

Then surface element is $dA = r^2\sin\theta d\theta d\phi$ (2.4.15)

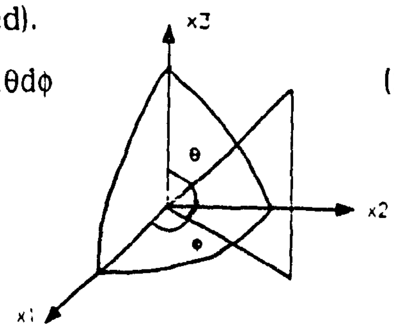

Figure 1

Here $\theta = 0$ and $\theta = \pi$ are the bad points. We can construct a mapping $(x_1, x_2, x_3) \rightarrow (\theta, \phi)$

everywhere on the sphere except at $\theta = 0, \pi$. Interchanging the coordinate axes we can also construct another mapping

$(x_1, x_2, x_3) \to (\theta^1, \phi^1)$ on everywhere except $\theta^1 = 0, \pi$. The two mappings would cover the whole sphere

We can take $dA = \omega = r^2 \sin\theta d\theta \wedge d\phi$ (2.4.16)

where ω is a 2-form. But in this basis the corresponding matrix would not be $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ so we need a transformation.

Let $u = r\cos\theta, \quad v = r\phi,$ then $\omega = du \wedge dv$ (2.4.17)

Let β be the one-form such that

$$d\beta = -\omega \Rightarrow \beta = r^2 \cos\theta d\phi = u dv \tag{2.4.18}$$

The integrality condition:

$$\int \omega = 2n\pi, \; n \in Z \quad \text{(over any closed oriented 2-surface)}$$

$$\Rightarrow \int r^2 \sin\theta d\theta d\phi = r^2 . 2\pi \int_0^\pi \sin\theta d\theta = 2\pi \; r^2 [\cos\theta]_0^\pi$$

$$= 4\pi r^2.$$

$$4\pi r^2 = 2n\pi \Rightarrow r^2 = \frac{n}{2}$$

So we can quantize only those spheres with radius r such that

$$r^2 = \frac{n}{2} \tag{2.4.19}$$

Let f be a classical observable and X_f and ∇_x be the corresponding Hamiltonian vector field and connection respectively.

Let $$X_f = X_{fu} \frac{\partial}{\partial v} + X_{fv} \frac{\partial}{\partial v} \tag{2.4.20}$$

Then $$X_f \rfloor \omega = -df = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv \tag{2.4.21}$$

But $$X_f \rfloor \omega = X_{fu} dv - X_{fv} du \tag{2.4.22}$$

$$\therefore \quad X_{fu} dv - X_{fv} du = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv$$

$$\Rightarrow X_{fu} = \frac{\partial f}{\partial u} \quad , \quad X_{fv} = - \frac{\partial f}{\partial u} \tag{2.4.23}$$

$$\text{and } X_f = \frac{\partial f}{\partial u} dv - \frac{\partial f}{\partial v} \ du$$

Now $$f \rightarrow -i \nabla_{X_f} + f \tag{2.4.24}$$

$i\nabla_x s_o = (X\lrcorner\beta)s_o$ (where s_o is a unit section)

$$= (X_u \frac{\partial}{\partial u} + X_v \frac{\partial}{\partial v})\lrcorner udv$$

$$= uX_v s_o$$

$$\Rightarrow \qquad \nabla_x s_o = -iuX_v s_o \tag{2.4.25}$$

Let $S = \chi so$ be any other section, then

$$(\nabla_{X_f}\chi s_o) = X_f(\chi)s_o - iu\chi\ X_v so$$

$$= (\frac{\partial f}{\partial u}\frac{\partial}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial}{\partial u})s + iu\frac{df}{du}s \tag{2.4.26}$$

Now (2.4.24) implies that

$$\hat{f}s = - i\nabla_{X_f}s + fs$$

$$= -i\ (\frac{\partial f}{\partial u}\frac{\partial}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial}{\partial u})s - u\frac{\partial f}{\partial u} + f\ s$$

$$\therefore\ \hat{u} = -i\frac{\partial}{\partial v} \text{ and } \hat{v} = i\frac{\partial}{\partial u} + v \tag{2.4.27}$$

Here one can define two polarizations:

1) When $\nabla_{\frac{\partial}{\partial u}} s=0$, that is, it is spanned by $\partial/\partial v$ and the corresponding Hilbert space is $(L^2([0,2\pi r], dv)$ (this polarization is independent of θ).

2) When $\nabla_{\frac{\partial}{\partial v}} s=0$, that is, it is spanned by $\partial/\partial u$ and the corresponding Hilbert space is L^2 ([-r, r], du). This polarization is independent of ϕ).

2.5 1/2-Densities, 1/2-P-density, 1/2-P-form

What we have discussed in the previous sections is not all about the geometric quantization scheme. We still have something more to discuss, to give this scheme a strong foundation by overcoming some serious drawbacks and ambiguities.

Let us go back, for example, to the case where the phase space (M,ω) is the cotangent bundle (i.e., $M = T^*Q$) of some configuration space Q and we use the vertical polarization to pick out the sections of the prequantum bundle that depend only on the position coordinates (i.e., they are constant on the fibres on T^*Q). These can be thought of as wave functions in the configuration space. The scalar product of these sections with respect to the volume element ω^n will diverge. To overcome this situation we should construct the quantum states with the help of 1/2-densities. A very brief description is given below.

Definition 2.5.1 Let $r\epsilon\mathbf{R}$. An r-density of a vector space V is a function ν that assigns to each basis $\{X_j\}$ in V a complex number $\nu\{X_j\}$ such that

$$\nu(\{X_j A^j_k\}) = |\Delta_A|^r \nu(\{X_j\}) \qquad (2.5.1)$$

Where Δ_A is the determinant of A. The set of r-densities form a one-dimensional vector space and this is equally true for a manifold. There is a natural scalar product $<. | .>$ on the space of 1/2-densities.

If Φ and Ψ are 1/2-densities, the product $\Phi.\Psi$ is a density (i.e., $r = 1$) and the scalar product is defined by integrating the product $\Phi.\Psi$ over Q. The square integrable 1/2-densities will form the prequantum

Hilbert space corresponding to the configuration space Q. In the general case, that is, when $M \neq T^*Q$ for a general configuration space Q or P is not a vertical polarization, we have to find 1/2-densities on Q with some suitable objects on T^*Q, the 1/2-P-densities.

Let B^P_m be the set of all bases for $P_m \subset (T_m M)^C$ at $m = (q, p) \varepsilon M = T^*Q$ and

$$B^P(T^*Q) = \bigcup_{m \varepsilon T^*Q} \{m\} \times B^P_m \tag{2.5.2}$$

is called the *frame bundle* of P and it is a principal GL(n, C) bundle.

Definition 2.5.2 A 1/2-P-density is a function

$$v: B^P(T^*Q) \to C, \text{ such that}$$

$$v o g = |\Delta g|^{-\frac{1}{2}} v \tag{2.5.3}$$

where g: $B^P(T^*Q) \to B^P(T^*Q)$; $g \,\varepsilon\, GL\,(n, C)$.

Each basis $\eta_1, \eta_2, \ldots, \eta_n \,\varepsilon\, (B^P_q Q)$ at $q \varepsilon Q$ defines a basis $\xi_1, \xi_2, \ldots, \xi_n$ at $m = (q,p)$ such that

$$\xi_j \rfloor \omega + Pr^* (\alpha_j) = 0 \tag{2.5.4}$$

Where Pr: $M = T^*Q \to Q$ is the natural projection and Pr^* is its pullback (see App. A).

Therefore each 1/2-density μ on Q defines a 1/2-P-density v_μ on T^*Q given by

$$v_\mu(m, \xi_1, \text{--}, \xi_n) = \mu\,(q, \eta_1, \text{---}, \eta_n) \tag{2.5.5}$$

But not every 1/2-P-density is a 1/2-density on Q. The 1/2-P-densities v which are of the form $v = v_\mu$ for some 1/2-density μ on Q must satisfy $\mathcal{L}_\zeta v = 0$ (2.5.6)

(where $\mathcal{L}_\zeta$ is the Lie derivative (see App. A) with respect to ζ) for every locally Hamiltonian vector field $\zeta \,\varepsilon\, Up\,(T^*Q)$.

So the wave functions of the Schrodinger prescription are the 1/2-P-densities which are constant on the integral surfaces of the vertical polarization.

By analogy, in the general case, the Hilbert space of a quantizable symplectic manifold should be constructed from the products of sections of L with the 1/2-P- densities of some polarization P, a subset of these which is constant in the direction P will have a natural pre-Hilbert space structure. This construction can be done in a better way if we use 1/2-P-forms instead of "1/2-P-densities".

Definition 2.5.3 A 1/2-P-form on a symplectic manifold (M, ω) with a polarization P is a function

$$v: \quad B^P(M) \to C \text{ such that}$$

$$v \circ g = (\Delta g)^{-1/2} v; \qquad (2.5.7)$$

where $g: \quad B^P(M) \to B^P(M), \ g \in GL(n, C)$.

In (2.5.7) we have an ambiguity in the square root and this can be overcome by replacing the general linear group by its *double cover*, the *metalinear group* and in this case one takes the square root rather than the determinant.

The metalinear group ML(n,C) is the subgroup of GL(n+1,C) of matrices of the form

$$\begin{vmatrix} g & o \\ o & z \end{vmatrix}, \ g \in GL(n, \mathbf{C}), \ z^2 = \Delta g \qquad (2.5.8)$$

The covering

$$\sigma: \ ML(n, C) \to GL(n, C) \text{ is given by } \sigma\left(\begin{bmatrix} g & o \\ o & z \end{bmatrix}\right) = g \qquad (2.5.9)$$

and the double cover

$$\sigma^{-1}(g) = \begin{bmatrix} g & o \\ o & \pm z \end{bmatrix}, \ g \in GL(n, C), \ z = (\Delta g)^{\frac{1}{2}} \qquad (2.5.10)$$

There exists also a natural homomorphism χ: ML(n, C) $\rightarrow$ C*
(C* = C - {0}) defined by $\chi \begin{pmatrix} g & o \\ o & z \end{pmatrix} = z$
for which the following diagram will be commutative and hence X will be a well defined "square root of the determinant".

$$\begin{array}{ccc} ML(n,C) & \xrightarrow{\chi^2} & \\ \downarrow & & C^* \\ GL(n,C) & \xrightarrow[\Delta_g]{} & \end{array}$$

Note that the 1/2-P- forms are functions defined not on the frame bundle $B^P(M)$, but on a double covering $\tilde{B}^P(M)$, known as a metalinear frame bundle. For P a metalinear frame bundle

Pr: $\tilde{B}^P(M) \rightarrow M$ is a principal ML(n, C) bundle together with a covering map ρ: $\tilde{B}^P(M) \rightarrow B^P(M)$ which makes the diagram

$$\begin{array}{ccc} \tilde{B}^P_{(M)} \times ML(n,\ C) & \rightarrow & \tilde{B}^P(M) \\ \downarrow & & \downarrow \\ B^P(M) \times GL(N,\ C) & \rightarrow & B^P(M) \end{array}$$

commutative.

Alternatively, 1/2-P-forms can be regarded as sections of the line bundle

$$\pi: L^P \rightarrow M \text{ where } L^P = \bigcup_{m \varepsilon M} \{m\} \times L^P m$$

and L^P_m is the set of functions

v: $\tilde{B}^P_m \rightarrow C$ ($\tilde{B}^P_m$ is the set of meta-frames at $m \varepsilon M$)

such that

$$v(b\tilde{g}) = \chi(\tilde{g})^{-1}.\ v(b);\ b\ \varepsilon\ \tilde{B}^P_m,\ \tilde{g}\ \varepsilon\ ML(n,\ C)$$

Now to construct the Hilbert space for the quantum system one has to use sections on $L \otimes L^P$ (where $L \otimes L^P$ is the tensor product of L and L^P) that is from the space $\Gamma(L \otimes L^P)$ of

$$s.v;\ s \in \Gamma(L),\ v \in \Gamma(L^P) \text{ such that } (\phi s).v = s.(\phi v);\ \ \phi \in C^\infty(M).$$

To have the analogy with the Schrodinger prescription one has to define, at this stage, the Lie derivative (App. A) of a 1/2-P-form along a vector field η which preserves P. Since the double covering $\tilde{B}^P_{(m)} \to B^P_{(m)}$ is a local diffeomorphism, one can define Lie derivative as

$$\mathcal{L}_\eta v = \eta' v,\ \eta \in B^P(M),\ \eta' \in \tilde{B}^P_{(M)} \tag{2.5.11}$$

where $v: B^P(M) \to C$ is a 1/2-P-form. But for a locally Hamiltonian vector field ξ to be in Up(M) one must have

$$\mathcal{L}_\xi v = o \text{ and } \nabla_\xi s = o \text{ for } \Psi = s.\ v \in \Gamma(L \otimes L^P). \tag{2.5.12}$$

The set of all the sections $\Psi \varepsilon \Gamma(LxL^P)$ satisfying (2.5.12) form a complex vector space. We shall denote this space by W^P.

Then we define the scalar product $<\Psi_1 | \Psi_2>$ on W^P by integrating (Ψ_1, Ψ_2) over Q, that is,

$$<\Psi_1 | \Psi_2> = \int_Q (\Psi_1, \Psi_2),\ \Psi_1, \Psi_2 \in W^P \tag{2.5.13}$$

The subspace of W^P of wave functions Ψ for which $<\Psi|\Psi>$ is finite forms a pre-Hilbert space H^P_o and the completion of this pre-Hilbert space will be the quantum Hilbert space H^P.

Now question may arise: to what extent is the quantization procedure independent of the choice of polarization? Is it possible to relate two or more sections which are associated with different polarizations?

In the general case, at the present state of knowledge of the subject, the answer is no. If we ignore some problems such as problem of convergence, the answer is yes for real polarizations. Under certain conditions it is possible to write down an expression for the unitary isomorphism U: $H^{P_1} \rightarrow H^{P_2}$ which will generalize the familar Fourier transform between the p and q representations of elementary quantum mechanics.

If P_1 and P_2 are real and transverse i.e., they span the whole tangent space $T_m M$ at each point m∈M, it is possible to construct a "pairing"

$$W^{P_1} \times W^{P_2} \rightarrow C^{\infty}(M) : (\Psi_1, \Psi_2) \rightarrow \Psi_1 * \Psi_2$$

This mapping is linear in Ψ_1, antilinear in Ψ_2 and is related by

$\Psi_1 * \Psi_2 = \overline{\Psi_2 * \Psi_1}$, where "$\overline{\quad}$" indicates complex conjugate (2.5.14)

If W^{P_1} and W^{P_2} are finite dimensional, then it is possible to define the following unique linear transformation

$$U_{P_1 P_2} : W^{P_1} \rightarrow W^{P_2}$$

which satisfies:

$$<\Psi_1|\Psi_2> = \int_M \Psi_1^*(U\Psi_1).\omega^n \ \forall \ \Psi_1 \in W^{P_1} \qquad (2.5.15)$$

where <.|.> is the scalar product in W^{P_1}.

The above transformation is known as BKS transform. For the infinite dimensional case it is very difficult to say whether or not $U_{P_1 P_2}$ exists and it can only be dealt with case by case [4].

Example 3 We have again consider the case where Q = **R** as in example 1, chapter 2. We have already seen in the aforecited example that M(= T*Q = $\mathbf{R}^2$) admits two naturally defined polarizations

A) The vertical polarization P_1, spanned by $\partial/\partial p$

B) The horizontal polarization P_2, spanned by $\frac{\partial}{\partial q}$.

Here P_1 and P_2 have only the trivial joint metalinear structure: the vector field $\partial/\partial p$ defines a global trivialization of $B^{P_1}(M)$ and so any point of $B^{P_1}(M)$ can be represented by a pair

$$(m, \begin{bmatrix} g & o \\ o & z \end{bmatrix}) ; \begin{bmatrix} g & o \\ o & z \end{bmatrix} \varepsilon \, ML(n, C) \tag{2.5.16}$$

The $\frac{1}{2}$ - P_1-form ν_1 is defined by

$$\nu_1(m, \begin{bmatrix} 1 & o \\ o & 1 \end{bmatrix}) = 1 \ \forall \ m \varepsilon M \tag{2.5.17}$$

does not vanish anywhere and it is constant along P_1. Therefore any wave function $\Psi_1 \ \varepsilon \ W^{P_1}$ can be written as

$$\Psi_1 = \phi_1 . \nu_1 \ ; \quad \phi_1 \ \varepsilon \ C^\infty(M) \tag{2.5.18}$$

where $\nabla_\xi \phi_1 = \xi \phi_1 = 0 \ \forall \ \xi \in U_{P_1}(M)$ (2.5.19)

Note that ϕ_1 is independent of p.

Similarly, any $\Psi_2 \ \varepsilon \ W^{P_2}$ can be uniquely written in the form

$$\Psi_2 = \phi_2 \cdot \nu_2 \tag{2.5.20}$$

where ν_2 is the 1/2-P_2-form on $\tilde{B}^{P_1}(M)$ defined by

$$\nu_2(m, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = 1 \quad \forall \quad m \varepsilon M \tag{2.5.21}$$

and $\nabla_\xi \phi_2 = \xi \phi_2 - i \, (\xi \lrcorner \theta) . \phi_2 \ = 0 \ \forall \ \xi \varepsilon U P_2(M)$

Here θ = pdq, so $\phi_2 = \exp \,(+i \ p \ q) . \ \chi_2(p)$ (2.5.22)

where $\chi_2 \varepsilon C^\infty(M)$ is independent of q.

Since $\nu_1 . \ \nu_2(m) = 1 \ \forall \ m \varepsilon M$ so the pairing $W^{P_1} \times W^{P_2} \rightarrow C^\infty(M)$ will be given by

$$\Psi_1 {*} \Psi_2(q, \ p) = \exp \, (+ \frac{i}{8}) . \exp \ (ipq) . \Psi_1(q) . \chi_2(p) \tag{2.5.23}$$

From(2.5.23) it follows that the BKS transform is given by

$\phi_1 \rightarrow \chi_2 = F(\phi_1)$ where F is the Fourier transform.

3 PRIME QUANTIZATION

In the prime quantization program, like any other quantization scheme, one constructs a quantum system from a given classical system. The additional feature of this program is that it provides a solution to the problem of ordering of operators [8, 9] in quantum mechanics. In this program the quantization is effected by two steps:

1) Realizing a classical observable as an operator of multiplication on the Hilbert spaces of (phase space functions) $L^2(\Gamma)$.

2) Projecting down to one of the various possible reproducing-kernel Hilbert subspaces (see App. B) of $L^2(\Gamma)$ which carry an irreducible representation of the appropriate kinematic group.

The step 1 can be considered as some sort of a prequantization and the step 2 as a polarization in the language of geometric quantization (Chapter 2).

To study prime quantization one needs the idea of ordering of operators in quantum mechanics. One can state this as follows:

Definition: 3.1 An ordering of operators on $L^2(\Gamma)$, corresponding to the classical algebra of observables $L^\infty(\Gamma)$, is a positive linear map

$$\pi_k^*: L^\infty(\Gamma) \to \mathscr{L}(\mathscr{H}_k) \qquad (3.1)$$

such that

$$\pi_k^*(f) = \mathcal{P}_k F^c \mathcal{P}_k, \; f \in L^\infty(\Gamma) \qquad (3.2)$$

where $(F^c \Psi)(p,q) = f(q,p)\Psi(q,p), \; \Psi \in L^2(\Gamma)$ (3.3)

and $\mathscr{H}_k = \mathcal{P}_k L^2(\Gamma)$ is a reproducing-kernel Hilbert space, with kernel K and associated projector $\mathcal{P}_k$.

If we consider, for example, a classical observable (on the phase space $\Gamma = T^*\mathbf{R} = \mathbf{R}^2$)

$$f(q, p) = \sum_{m,n=0}^{\infty} C_{mn}\, q^m p^n \tag{3.4}$$

which is a finite-degree polynomial with coefficients $C_{m,n}$, then its quantized version F is, in general, again a polynomial in the operators Q and P,

$$F = \sum_{m,n=0}^{\infty} C_{m,n}\, Q^m P^n \tag{3.5}$$

and Q and P would satisfy the canonical commutation relations

$[Q, P] = i, \qquad (\hbar = 1)$

We have to decide a certain ordering (e.g. normal, antinormal etc.) of the non-commuting operators Q and P when their product appears in (3.5).

Let us suppose that a classical system is moving on the manifold M. Its phase space Γ is the cotangent bundle T*M and its classical algebra of observables $\mathscr{A}_c$ is the set of all complex continuous functions on T*M, vanishing at infinity. Then $\mathscr{A}_c$ is a commutative C* algebra [18].

In this program the quantization map π^* is defined, exactly in the same way as we did in definition (1.1.1), from $\mathscr{A}_c$ to the set of all bounded operators on a Hilbert space.

That is $\qquad \pi^* : \mathscr{A}_c \rightarrow \mathscr{L}(\mathscr{H}) \qquad$ (3.6)

Like definition (1.1.1), π^* would satisfy $\pi^*(1) = I$, where 1 is a constant function and I is the identity operator in $\mathscr{L}(\mathscr{H})$. Unlike definition (1.1.1), (3.6) is silent about bracket relation and it has one additional property: the C* algebra generated by the set $\pi^*(\mathscr{A}_c)$ should be dense in $\mathscr{L}(\mathscr{H})$ (nondegenaracy condition).

The map (3.6) gives some ordering of the operators Q and P. This can be understood from the following example:

If we consider the antinormal ordering i.e., $z^m z^{*n} \leftrightarrow a^m a^{+n}$ where $z = (\frac{1}{\sqrt{2}})$ (q + ip), (q,p) ε $T^*\mathbf{R}$ and a is the annihilation and a^+ the creation operator. Then this ordering of operators can be characterized [9] by a one-dimensional projection operator in the Hilbert space of the system

$$2\pi\, T_A(q,p) = |q,p\rangle\langle q,p| \tag{3.7}$$

where $|q,p\rangle$ is a Glauber coherent state [12] which satisfies

$$a|q,p\rangle = z|q,p\rangle\ ,\ z = (1/\sqrt{2})(q+ip) \tag{3.8}$$

When written in terms of the operator T_A, the antinormal ordering assumes the following fcrm

$$f \rightarrow F_A = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T_A(q,p) f(q,p)\,dq\,dp \tag{3.9}$$

for each $f \in \mathscr{A}_c$.

F_A in (3.9) can be written as $F_A = F_A(Q,P)$ i.e., Q and P are antinormally ordered. (3.9) will define a quantization map if we set

$$\pi_A^*(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T_A(q,p) f(q,p)\,dq\,dp \tag{3.10}$$

Since π_A^* is linear (because of its integral representation), from the properties of coherent states it follows that $\pi_A^*(1) = I$ and as proved by S.T. Ali and E. Prugovecki [13] the classical position and momentum observables q and p on $T^*(\mathbf{R})$ are mapped by π_A^* to two operators which satisfy the canonical commutation relations.

$$[\pi_A^*(q),\ \pi_A^*(p)] = iI \tag{3.11}$$

Finally the C^* algebra generated by $\pi_A^*(\mathscr{A}_c)$ is dense in $\mathscr{L}(\mathscr{H})$ (follows from the theorem 3.1 given below).

$f \rightarrow F(Q,P)$ also sets up a mapping from the states ρ of the quantum system i.e., normalized density matrices to normalized measure μ on the phase space according to the relation.

$$\mathrm{tr}[F(Q,P)\rho] = \mathrm{tr}[\pi^*(f)\rho] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(q,p)\,d\mu(q,p) \tag{3.12}$$

For the antinormal ordering π_A^*, the measure μ_A corresponding to any density matrix ρ will be positive and in this case

$$\begin{aligned} d\mu_A(q,p) &= \mathrm{tr}[T_A(q,p)\rho]\,dq\,dp \\ &= \langle q,p|\rho|q,p\rangle \frac{dq\,dp}{2\pi} \end{aligned} \tag{3.13}$$

The positivity condition imposed on μ can be interpreted in terms of localization on phase space [14]. In fact, if the system is localized in Γ, then for any Borel subset Δ of Γ representing a localization volume, there exists an observable $a(\Delta) \in \mathscr{L}(\mathscr{H})$ such that $\mathrm{tr}[a(\Delta)\rho]$ will give the probability of finding the quantum system (in the state ρ) localized in the volume Δ of phase space.

If we insist on the probability interpretation of $\mathrm{tr}[a(\Delta)\rho]$ we must have (after comparing with (3.13))

$$\mathrm{tr}[a(\Delta)\rho] = \int_\Gamma \chi_\Delta(\zeta)\,d\mu(\zeta) = \mu\ (\Delta),\ \zeta \in \Gamma \tag{3.14}$$

$$\mu(\Gamma) = 1 \tag{3.15}$$

where χ_Δ is the characteristic function of the set Δ. Therefore μ is a probability measure.

The phase space Γ can be equipped with a natural positive measure ν (when Γ considered as the spectrum of $\mathscr{A}_c$), as the volume form on $T^*(M)$ (given earlier). Moreover, if the number of quantum particles in a unit phase space cell is finite, then it is possible to prove the existence [13, 14] of a positive operator-valued function.

$$T: \Gamma \to \mathcal{L}(\mathcal{H}) \tag{3.16}$$

such that

$$a(\Delta) = \int_\Delta T(\zeta)\, dv(\zeta) \tag{3.17}$$

Therefore

$$d\mu(\zeta) = \mathrm{tr}[T(\zeta)\rho]dv(\zeta) \tag{3.18}$$

(by comparing (3.14) and (3.17))

A comparison of (3.13) with (3.17) gives us

$$a_A(\Delta) = \frac{1}{2\pi}\int_\Delta |q,p\rangle\langle q,p|\, dqdp \tag{3.19}$$

which are clearly positive and which defines a positive operator valued (POV) measure.

Written in terms of the POV measure a, the quantization map (3.6) becomes

$$\pi^*(f) = \int_\Gamma f(\zeta)da(\zeta) = \int_\Gamma T(\zeta)f(\zeta)dv(\zeta) \tag{3.20}$$

which is clearly the generalization of the antinormal rule of ordering (3.9).

The mapping π^* may be given an alternative mathematical description which is more useful in some sense. Let $\mathcal{T}(\mathcal{H})$ be the Banach space of the trace-class operators on $\mathcal{H}$. Then $\mathcal{L}(\mathcal{H})$, equipped with the strong operator topology, is the Banach space dual of $\mathcal{T}(\mathcal{H})$. Also if $L^\infty(\Gamma,v)$ is the dual of the Banach space $L^1(\Gamma,v)$, then the measure μ has a density (follows from (3.18)) which is an element of $L^1(\Gamma,v)$. Hence for a fixed POV measure a, the relations (3.17) and (3.18) imply a mapping π from $\mathcal{T}(\mathcal{H})^+$ to $L^1(\Gamma,v)$, the dual of which is π^* (and hence the notation),

$$\pi: \mathcal{T}(\mathcal{H}) \to L^1(\Gamma,v) \tag{3.21}$$

The dual,

$$\pi^*: L^\infty(\Gamma,\nu) \to \mathscr{L}(\mathscr{H}) \tag{3.22}$$

is the bounded linear map which in (4.20) defines an ordering of operators. Given a POV measure a and its extension $\tilde{P}$, we can write the quantization (3.20) in terms of $\tilde{P}$:

$$\tilde{P}(f) = \int_\Gamma f(\zeta)d\tilde{P}(\zeta), \; f\varepsilon\mathscr{A}_c \tag{3.23}$$

the integral on the right is weakly defined. Comparing (3.20) and (B38) the quantization map π^* can be written as

$$f \to \pi^*(f) = \mathscr{P}\tilde{P}(f)\mathscr{P} = W\pi^*(f)W^{-1} \tag{3.24}$$

Quantization Map

We first embed $\mathscr{A}_c$ into an algebra of operators on $L^2(\Gamma,\nu)$ using the PV measure $\tilde{P}$ such that

$$[\tilde{P}(\Delta)\Psi](\zeta) = \chi_\Delta(\zeta)\Psi(\zeta), \; \Psi\varepsilon\tilde{\mathscr{H}} \tag{3.25}$$

The domain of $\tilde{P}$ may be extended to the set $L^\infty[\Gamma,\nu)$ of all bounded ν-measurable functions of Γ and its range to a commutative von Neumann algebra. For any ν-measurable function $f\varepsilon L^\infty(\Gamma,\nu)$ we have

$$\tilde{P}(f) = \int_\Gamma f(\zeta)d\tilde{P}(\zeta) \tag{3.26}$$

In this case K_ζ=C, for all $\zeta\varepsilon\Gamma$, the direct integral (B10) so that $\tilde{\mathscr{H}} = L^2(\Gamma,\nu)$.

Definition 3.2: A prime quantization (hence, an ordering of operators) of the classical algebra $\mathscr{A}_c$ is a positive linear map

$$\pi^*: L^\infty(\Gamma,\nu) \to \mathscr{L}(\mathscr{H}) \tag{3.27}$$

such that

1) the C* algebra generated by the set,

$$\pi^*(\mathscr{A}_c) = \{\pi^*(f) \,|\, f\varepsilon\mathscr{A}_c\} \tag{3.28}$$

is weakly dense in $\mathscr{L}(\mathscr{H})$;

2) $\mathcal{H} = \mathcal{H}_K$ is a reproducing kernel Hilbert subspace of $L^2(\Gamma,\nu)$, that is, the projection operator $\mathcal{P}$ should have kernel K: $\Gamma \times \Gamma \to \mathbf{C}$, which is separately continous in each variable, and

$$\pi^*(f) = \mathcal{P}\,\tilde{P}(f)\mathcal{P}, \quad f \varepsilon L^\infty(\Gamma,\nu) \tag{3.29}$$

where π^* depends on the kernel K.

According to our discussion at the beginning of this section we can say that any prime quantization gives rise to a POV measure a_k (B32) on the Borel sets of the Γ, for which

$$\pi^*(f) = \int_\Gamma f(\zeta) da_K(\zeta) = \int_\Gamma f(\zeta)(E_\zeta^K)^* E_\zeta^K d\nu(\zeta) \tag{3.30}$$

Moreover, $\mathcal{H}$ and a_K determine $L^2(\Gamma,\nu)$ uniquely in the sense of Theorem (B5).

The constant function $f(\zeta) = 1$, $\zeta \varepsilon \Gamma$ is mapped to the identity operator on $\mathcal{H}$ since $\pi^*(1) = a_K(\Gamma) = \mathcal{P}$. The condition which will ensure the nondegeneracy of the map π^* is stated in the following theorem.

Theorem 3.1. If the phase space Γ, considered as a topological space, has no discrete part, then the prime quantization map π^* is nondegenerate.

For proof see Ref.[5].

Example:

Let a free particle be moving in the configuration space $\mathbf{R}^3$. Then its phase space is $\Gamma \equiv T^*(\mathbf{R}^3) = \mathbf{R}^6$ and kinematical group is the extended Galilei group $\tilde{G}$. The reproducing kernel Hilbert space $\mathcal{H}_K$ should carry an irreducible representation of $\tilde{G}$ and the kernel would be $\tilde{G}$ covariant.

Let us suppose that $C^\infty(\Gamma)$ is the set of all continuous functions on Γ vanishing at infinity. Note that then $C^\infty(\Gamma)$ is a classical C^* algebra. If dqdp is the Lebesgue measure for a measure ν on Γ and θ is the phase subgroup and T the subgroup of time translations of $\tilde{G}$, then

$$\Gamma \equiv \tilde{G}/\theta \otimes T \otimes SO(3) \tag{3.31}$$

is a homogeneous space and dqdp is the corresponding invariant measure [15] on Γ.

Let $\tilde{\mathcal{H}} = L^2(\mathbf{R}^6, d\mathbf{q}d\mathbf{p})$ (3.32)

be a Hilbert space. We need to construct a subspace $\mathcal{H}_{e,l}$ of $\tilde{\mathcal{H}}$ which admits a reproducing kernel $K_{e,l}$. If e is a square integrable, rotationally invariant function $\mathbf{R}^3$ such that

$$\int_{\mathbf{R}^3} |e(\mathbf{k})|^2 dk = 1, \tag{3.33}$$

$$e(R\mathbf{k}) = e(\mathbf{k}),\ R \in SO(3) \tag{3.34}$$

Then we define

$$K_{e,l}(\mathbf{q},\mathbf{p};\mathbf{q}^1,\mathbf{p}^1) = \frac{2l+1}{(2\pi)^3}\int_{\mathbf{R}^3}\exp[i\mathbf{k}.(\mathbf{q}-\mathbf{q}^1)] \times \mathcal{P}_l\left[\frac{(\mathbf{k}-\mathbf{p}).(\mathbf{k}-\mathbf{p}^1)}{||\mathbf{k}-\mathbf{p}||\ ||\mathbf{k}-\mathbf{p}^1||}\right] \times \overline{e(\mathbf{k}-\mathbf{p})}\, e(\mathbf{k}-\mathbf{p}^1)d\mathbf{k} \tag{3.35}$$

$l = 0, 1, 2, 3, \ldots$

where $\mathcal{P}_l$ is a Legendre polynomial of order l.

Then $K_{e,l}$ is $\tilde{G}$ covariant [15] and satisfies all the properties (App. B) of a reproducing kernel. If $\mathcal{P}_{e,l}$ is the projector operator

$$(\mathcal{P}_{e,l}\tilde{\Psi})(q,p) = \int_{\mathbf{R}^6} K_{e,l}(q,p;\, q^1,p^1)\tilde{\Psi}(q^1,p^1)dq^1dp^1,\ \tilde{\Psi} \in \tilde{\mathbf{H}} \tag{3.36}$$

and $\mathcal{H}_{K_{e,l}} \equiv \mathcal{H}_{e,l} = \mathcal{P}_{e,l} L^2(\mathbf{R}^6, d\mathbf{q}d\mathbf{p})$ (3.37)

then by using (3.35) we have

$$\Psi_{e,l} = \mathcal{P}_{e,l}\tilde{\Psi},\ \tilde{\Psi} \in \mathcal{H} \tag{3.38}$$

where $\Psi_{e,l}$ is the projected continuous functions in $\mathcal{H}_{e,l}$. Also $\mathcal{H}_{e,l}$ carries a unitary irreducible representation [15] of the extended Galilei group $\tilde{G}$ corresponding to a particle of mass m and spin l. In particular when $l = 0$ and

$$e(\mathbf{k}) = \pi^{-3/4} \exp(-\mathbf{k}^2/2) \tag{3.39}$$

we would have antinormal ordering (discussed earlier). Here we want to explain the different ordering possibilities when e may assume some other form rather than (3.39).

If we define $f^j(q,p) = q^j$ and $g^j(q,p)=p^j$, $j = 1, 2, 3$ (3.40)

Then the corresponding quantum operators

$$\begin{aligned} Q^j &= \mathcal{P}_{e,l}\, \tilde{P}(f^j)\, \mathcal{P}_{e,l} = q^j + i\frac{\partial}{\partial q^j} \quad \text{and} \\ P^j &= \mathcal{P}_{e,l}\, \tilde{P}(g^j)\mathcal{P}_{e,l} = -i\partial/\partial q^j \end{aligned} \tag{3.41}$$

would satisfy the canonical commutation relation

$$[Q^j, P^k] = i\delta_{jk}I \tag{3.42}$$

on a stable dense domain $\mathcal{D} \subset \mathcal{H}_{e,l}$. Further, using (3.25), for $\tilde{\Psi} \in \tilde{\mathcal{H}}$, we can show that

$$[a_{e,l}(\Delta)\, \tilde{\Psi}\,]\, (\mathbf{q},\mathbf{p}) = \int_{\mathbf{R}^6} K^{\Delta}_{e,l}\, (\mathbf{q},\, \mathbf{p};\, \mathbf{q}^1,\mathbf{p}^1)\tilde{\Psi}(\mathbf{q}^1,\mathbf{p}^1)d\mathbf{q}d\mathbf{p} \tag{3.43}$$

where $a_{e,l} = \mathcal{P}_{e,l}\, \tilde{P}(\Delta)\, \mathcal{P}_{e,l}$ (3.44) is the POV measure canonically associated to $K_{e,\iota}$ (see App. B) and

$$K^{\Delta}_{e,\iota}(\mathbf{q},\mathbf{p};\ \mathbf{q}^1,\mathbf{p}^1) = \int_{\Delta} K_{e,\iota}(\mathbf{q},\mathbf{p};\ \mathbf{q}'',\mathbf{p}'')K_{e,\iota}(\mathbf{q}'',\mathbf{p}'';\ \mathbf{q}^1,\mathbf{p}^1)\ d\mathbf{q}''d\mathbf{p}'' \qquad (3.45)$$

If we use the notation $F_{e,\iota}(Q,P)$ for the quantized form of the classical observable $f(q,p)$ then in view of (3.41) we can write

$$F_{e,\iota}(Q,\ P) = \mathcal{P}_{e,\iota}\ \tilde{P}(f)\ \mathcal{P}_{e,\iota} \qquad (3.46)$$

We can also see from (3.40) and (3.41) that $F_{e,\iota}$ gives an ordering of the operators Q and P in the quantization of the classical observable f. Now we want to explain what a Galilean covariant ordering is. We have already seen that $\mathscr{H}_{e,\iota}$ carries an irreducible representation of $\tilde{G}$, corresponding to a particle of mass m and spin ι. Moreover, it can be shown [15] that the representation in question is given by the unitary operators U(g), g = (θ,b,**a**,**v**,R) ε $\tilde{G}$, on $\tilde{\mathscr{H}}$,

$$[U(g)\tilde{\Psi}](\mathbf{q},\mathbf{p}) = \exp\{i[\theta+(p2/2m)b+m\mathbf{v}.(q-\mathbf{a}]\}$$

$$\times\ \tilde{\Psi}(R^{-1}(\mathbf{q}-\mathbf{a}),\ R^{-1}(\mathbf{p}-m\mathbf{v})), \qquad (3.47)$$

where θ is the phase translation, b the time translation, **a** the space translation, **v** the velocity boost, R the spatial rotation, and

$$P^2 = -\nabla^2_q \qquad (3.48)$$

This representation is highly reducible and each $\mathscr{H}_{e,\iota} \subset \tilde{\mathscr{H}}$ carries an irreducible subrepresentation $U_{e,\iota}$ of U and $\tilde{\mathscr{H}}$ is the direct sum of the subspaces $\mathscr{H}_{e,\iota}$ that is, $\tilde{\mathscr{H}} = \oplus \sum_e \sum_\iota \mathscr{H}_{e,\iota}$.

If we consider only the isochronous subgroup (i.e a subgroup of $\tilde{G}$ when t is fixed = 0) $\tilde{G}'$ of $\tilde{G}$, then it can be shown [15] that

$$U_{e,\iota}(g)\ a_{e,\iota}\ (\Delta)\ U_{e,\iota}(g)^* = a_{e,\iota}(g[\Delta]) \qquad (3.49)$$

where g[Δ] is the translation of the set $\Delta \varepsilon \mathscr{B}(\Gamma)$ by g and the action of $g\varepsilon\tilde{G}$ on (q,p) $\varepsilon\Gamma(=\mathbf{R}^6)$ is given by

$$g(q,p) = (\ Rq+\mathbf{a},\ Rp+m\mathbf{v}) \qquad (3.50)$$

For a classical observable f if we define

$$g[f]\ (\mathbf{q},\mathbf{p}) = f(g^{-1}(\mathbf{q},\mathbf{p})) \tag{3.51}$$

then (3.46) would imply that

$$U_{e,l}(g)F_{e,l}\,[U_{e,l}(g)]^* = \mathcal{P}_{e,l}\ \widetilde{P}(g[f])\mathcal{P}_{e,l},\ \forall g\varepsilon\widetilde{G} \tag{3.52}$$

So the Galilean transformed classical observable (g[f]) corresponds to the Galilean- tranformed quantum observable ($U_{e,l}(g)F_{e,l}[U_{e,l}(g)]^*$ and hence establishes the Galilean covariance of the ordering procedure (3.46).

4 CONCLUDING CHAPTER

We observed in Chapter 2 that geometric quantization proceeds in two stages:

1. Prequantization:

In this stage a map is found from a classical algebra of observables to the self-adjoint operators in a Hilbert space. In other words, at this stage one constructs the Hilbert space representation of an algebra of classical observables.

2. Polarization:

At this state, the size of the prequantum Hilbert space is reduced to have an irreducible representation of the set of quantum observables.

In Chapter 3, we have seen that a prime quantization is also effected in two stages:

1. Realizing a classical observable as an operator of multiplication on a Hilbert space.

2. Projecting down to one of the various possible reproducing - kernel Hilbert subspaces, consisting of phase space functions, and carrying an irreducible representation of the quantized observables. So the choice of a reproducing-kernel Hilbert space is analogous to the choice of a polarization.

Therefore to find an explicit relationship between geometric quantization and prime quantization one has to find the relationship between the polarization and the corresponding reproducing-kernel Hilbert space. It is our intention to proceed in this direction.

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Appendix - A

This appendix contains an amalgamation of various objects from Differential Geometry and Hamiltonian Mechanics, required to describe geometric quantization.

1. Symplectic Manifolds: Some Properties

Symplectic manifolds play an important role in studying the problem of geometric quantization. It would not be an exaggeration to say that the scheme of geometric quantization begins with the concept of a symplectic manifold. Let M be a smooth manifold of dimension 2n. A symplectic manifold is a pair (M,ω) where ω is a bilinear 2-form on M, that is, for each $m \in M$, ω defines a mapping

ω_m: $T_mM \times T_mM \to \mathbf{R}$ which is antisymmetric and non-degenerate. If x, y, z are vector fields on M, then ω satisfies:

1) $d\omega = 0$ (closedness)

2) $\omega(X, Y) = -\omega(Y, X)$ (antisymmetry)

3) $\omega(X, Y) = 0$, for fixed vector field X and all $Y \Rightarrow X = 0$ (non-degeneracy).

One of the simplest examples of a symplectic manifold is $M = \mathbf{R}^{2n}$ with the symplectic form $\omega = \sum_{i=1}^{n} dp_k \wedge dq^k$, where $(q^1, q^2, \cdots q^n; p_1, p_2, \cdots, p_n)$ are the canonical coordinates.

If ω is a symplectic form on M there is a basis [17] $\{U_i \mid 1 \le i \le 2n\}$ for M such that

$$\omega = \sum_{i=1}^{n} U_i^* \wedge U_{n+i} \quad \text{(A2)}$$

where $\{U^* \mid 1 \le i \le 2n\}$ is the dual basis for M^*. We call such basis *symplectic*. The matrix of ω with respect to a symplectic basis is

$$\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \quad \text{(A3)}$$

where I_n is the n x n identity matrix.

If Q is any manifold and $M = T^*Q$ its cotangent bundle, then one has the natural projection map $T(M) \rightarrow M = T^*Q$ from tangent bundle T(M) of M to the cotangent bundle T^*Q of Q and $\pi_Q^*: T^*Q \rightarrow Q$ from the cotangent bundle T^*Q of Q to Q. The derivative of π_Q^* is the tangent linear map: $T\pi_Q^*: T(M) \rightarrow T(Q)$.

Now, points in M conists of pairs (q,p), where $q \varepsilon Q$ and p is a covector at q, i.e., $p \varepsilon T^*(Q)$. Let X be a vector field on M. Then we define the 1-form θ on M (by giving its action on vector fields X on M) as:

$$\theta(q,p)\ (X) = p \rfloor T\pi_Q^*(X) \quad \text{(A4)}$$

considering $\frac{\partial}{\partial q^j}$ and $\frac{\partial}{\partial p_j}$ as tangent vectors in $T_{(q,p)}(M)$ we have

$$\theta(q^j,p_j)(\partial/\partial q^j) = (\sum p_j \rfloor dq^j)(\partial/\partial q^j) = p_j \quad \text{(A5)}$$

and $\theta(q^j,p_j)(\partial/\partial p_j) = (\sum p_j \rfloor dq^j)(\partial/\partial p_j) = 0$ (A6)

(A5) and (A6) together imply that

$$\theta = \sum_{j=1}^{n} p_j dq^j \quad \text{(A7)}$$

This 1-form θ is called the symplectic potential. Then ω is given by

$$\omega = d\theta = d\sum_{j=1}^{n} p_j dq^j$$

$$\Rightarrow \omega = \sum_{j=1}^{n} dp_j \wedge dq^j \quad \text{(A8)}$$

which is exact, hence closed.

2 Poisson Brackets

On any symplectic manifold (M,ω) the symplectic form ω defines an isomorphism

$$T_mM \to T_m^*M : X \to X\rfloor\omega \qquad \text{(A9)}$$

between the tangent and the cotangent spaces at each point $m\varepsilon M$. Physically, a symplectic manifold (M,ω) represents the phase space of a classical system and a smooth real-valued function on M represents a classical observable. In classical mechanics, an observable plays two roles:

1) It is a measurable quantity, represented by a smooth function on the phase space.

2) It generates, at least locally, a one-parameter family of canonical transformations.

Let (M,ω) and (N,ρ) be symplectic manifolds. Then by a canonical transformation we mean a C^∞-mapping $F: M \to N$ such that $F^*\rho = \omega$, where

$F^*: T^*N \to T^*M$ is given by

$(F^*\phi)x = (\phi o F)x$, $\phi\varepsilon TM$ and $x\varepsilon M$.

The roles played by a classical observable are connected geometrically by the following relation: for $f\varepsilon C^\infty_{\mathbf{R}}(M)$ the Hamiltonian vector field $X_f\varepsilon U(M)$ determined by $X_f\rfloor\omega + df = 0$ (A10)

preserves ω in the sense

$$\mathcal{L}_{X_f}\omega = X_f\rfloor d\omega + d(X_f\rfloor\omega) = -d(df) = 0 \qquad \text{(A11)}$$

where $\mathcal{L}_{X_f}\omega$ is the Lie derivative of ω with respect to the vector field X_f.

[Before defining Lie derivative one needs to define a local one-parameter group action or flow on a manifold.

Definition: A1: A local **one-parameter group** action or flow on a manifold M is a C^∞ map F: N→M, where $N \subset \mathbf{R}\times M$ is an open set, which satisfies the following two conditions:

1) $\Phi_o(p) = p \;\; \forall \; p\varepsilon M$

2) For all s, $t\varepsilon\mathbf{R}$, $p\varepsilon M$, $\Phi_s o \Phi_t(p) = \Phi_{s+t}(p) = \Phi_t o \Phi_s(p)$.

Each 1-parameter group of transformation $\Phi = (\Phi_t)$ induces a vector field X as follows. Let $x\varepsilon M$. Then X(x) is the tangent vector to the curve $t \to \Phi_t(X)$ (called the *orbit* of x) at $x = \Phi_o(x)$. Hence the orbit $\Phi_t(x)$ is an integral curve of X starting at x. X is called the *infinitesimal generator of* Φ_t. Let X be a vector field on M and Φ_t a local 1-parameter group of local transformations generated by X. Then we define the Lie derivative $\mathscr{L}_x\Omega$ of a k-form Ω with respect to X as follows:

$$(\mathscr{L}_x\Omega)(q,p) = \lim_{t\to o} (1/t)[\Omega(q,p) - (\Phi^*_{-t}\,\Omega)(q,p)] \qquad \text{(A12)}$$

$(q,p)\varepsilon M$, where for each $t\varepsilon\mathbf{R}$, $\Phi^*_{-t}:\Lambda M \to \Lambda M$ is an automorphism of the exterior algebra ΛM.

We know that an anti-symmetric covariant tensor field of degree p on a manifold M is a differential form of degree p [17]. The set $\Lambda^p M$ of all such forms is module over $C^\infty(M)$. If $\omega\varepsilon\Lambda^p M$ and $\tau\Lambda^q M$, we define the exterior product $\omega\Lambda\tau\,\varepsilon\,\Lambda^{p+q}M$ by

$$(\omega\Lambda\tau)(x) = \omega(x)\,\Lambda\,\tau(x),\; x\varepsilon M \qquad \text{(A13)}$$

If we set $\Lambda M = \oplus_{p=o}^{M}\Lambda^P M$, where dim M=m, then ΛM is an associative algebra such that

$$\omega\Lambda\tau = (-1)^{pq}\tau\Lambda\omega,\; \omega\varepsilon\Lambda^p M,\; \tau\varepsilon\Lambda^q M;$$

ΛM is called exterior algebra on M. Here by automorphism on ΛM we mean a mapping $\Lambda M \rightarrow \Lambda M$ preserving the algebraic structure of ΛM.]

Then the flow

$$\Phi_f: U(M) \subset M x \mathbf{R} \rightarrow M \qquad (A14)$$

generated by X_f will define a local 1-parameter family of canonical transformations of M. Φ_f is called the canonical flow and X_f the Hamiltonian vector field generated by f.

In local coordinates $\{p_j, q^j\}$ we have

$\omega = \sum_{j=1}^{n} dp_j \Lambda dq^j$ (see (A3)) and

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial q^j} dq^j + \frac{\partial f}{\partial p_j} dp_j \qquad (A15)$$

Now substituting (A15) in (A11) we obtain

$$X_f \rfloor \sum_{j=1}^{n} dp_j \wedge dq^j + \sum_{j=1}^{n} \frac{\partial f}{\partial q^j} dq^j + \frac{\partial f}{\partial p_j} dp_j = 0$$

$$\Rightarrow X_f = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial p_j} \frac{\partial}{\partial q^j} - \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_j} \right) \qquad (A16)$$

(using $\frac{\partial}{\partial p} \rfloor dp \Lambda dq = dq$ etc.)

Definition A2: Let $f, g \varepsilon C^{\infty}_{\mathbf{R}}(M)$. Then the Poisson bracket of f and g, denoted by $[f, g]$, is a function defined by

$$[f, g] = X_f(g) \qquad (A17)$$

<u>Some properties of Poisson brackets:</u>

P1) In the local coordinates $\{q^j, p_j\}$ we can write using (A16) and (A17)

$$[f, g] = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q^j} - \frac{\partial g}{\partial p_j} \frac{\partial f}{\partial q^j} \right) \qquad (A18)$$

which is the familiar form of Poisson bracket.

P2) $[f,g] = X_f(g) = 2\omega(X_f, X_g)$ (A19)

Proof: $[f, g] = X_f(g) = -X_f \rfloor (X_g \rfloor \Psi)$

$= 2\omega(X_f, X_g)$

$\Rightarrow X_f(g) = 2\omega(X_f, X_g)$

(A19) implies that the Poisson bracket is antisymmetric (since ω is antisymmetric).

P3) $[X_f, X_g] = X_{[f,\, g]}$; $f, g \in C^\infty_{\mathbf{R}}$

To prove (P3) we need to use the following two relations and two lemmata.

$$\mathcal{L}_\zeta = \zeta \rfloor d + d\zeta \rfloor \quad \text{(A20)}$$

$$[\zeta,\eta] = \mathcal{L}_\zeta \eta \rfloor - \eta \rfloor \mathcal{L}_\zeta \quad \text{(A21)}$$

where $\eta, \zeta \in U(M)$ (U(M) denotes the space of vector fields on M).

Elements of U(M) corresponding to closed 1-forms are called **locally Hamiltonian vector fields** and those corresponding to exact 1-forms are called **globally Hamiltonian vector field.**

Lemma A1: If $\eta \varepsilon U(M)$ then η belongs to locally Hamiltonian vector fields ($L(M)$ on M if and only if $\mathcal{L}_\eta \omega = 0$.

Proof: $\eta \varepsilon$ (M) $\Leftrightarrow d(\eta \rfloor \omega) = 0$

$\Leftrightarrow \mathcal{L}_\eta \omega - \eta \rfloor d\omega = 0$ (using (A15))

$\Leftrightarrow \mathcal{L}_\eta \omega = 0$ (since ω is closed)

Lemma A2: If $\zeta, \eta \in L(M)$, then

$[\zeta, \eta] = 2\xi_{\omega(\zeta, n)} \in g(M)$ (globally Hamiltonian vector fields on M)

Proof: $[\zeta, \eta] \rfloor \omega = \mathcal{L}_\zeta(\eta \rfloor \omega) - \eta \rfloor \mathcal{L}_\zeta \omega$ (using (A21))

$= d(\zeta \rfloor (\eta \rfloor \omega)) + \zeta \rfloor (d(\eta \rfloor \omega))$

(using (A20) and Lemma A1)

$= 2d(\omega(\eta, \zeta)) = 2(\zeta_{\omega(\zeta,\eta)}) \rfloor \omega$

$$\Rightarrow [\zeta, \eta] = 2\xi_{\omega(\zeta,\, \eta)} \quad \text{(A22)}$$

Proof of P3 :

$[X_f, X_g] = 2X_{\omega(X_f, X_g)}$ (using A22)

$\Rightarrow [X_f, X_g] = X_{[f, g]}$ (using P2)

From (P3) we can conclude that the map

$f \rightarrow X_f$ preserves brackets.

P4) The Poisson bracket satisfies the Jacobi identity. That is,

$$\Sigma[f, [g, h]] = 0 \qquad \text{(A23)}$$

where Σ denotes cyclic summation over f, g, h ε $C^\infty(M)$.

Proof: Since ω is closed, we can write

$d\omega(X_f, X_g, X_h) = 0$; f, g, h ε $C^\infty(M)$

$\Rightarrow \Sigma(X_f(\omega(X_g, X_h)) - \omega([X_g, X_h], X_f)) = 0$

but $\omega(X_g, X_h) = \frac{1}{2}[g, h]$

and $X_f(\frac{1}{2}[g, h]) = \frac{1}{2}[f, [g, h]]$ (by P2)

$\Rightarrow \frac{1}{2}\Sigma([f, [g, h]] - [[g, h], f]) = 0$

$\Rightarrow \Sigma [f, [g, h]] = 0.$

3 Line Bundles and Sections

A **line bundle** over a smooth manifold M is a triple (L, π, M) where:

1) L is a smooth manifold and π is a smooth map of L onto M.

2) For each mεM, $L_m = \pi^{-1}(m)$ has the structure of a one dimensional complex vector space. L_m is called the *fibre over* m.

3) There is an open cover $\{U_j | j\varepsilon\Lambda\}$ of M(here Λ is some index set) and a collection of C^∞ maps

$S_j : U_j \rightarrow L$ such that

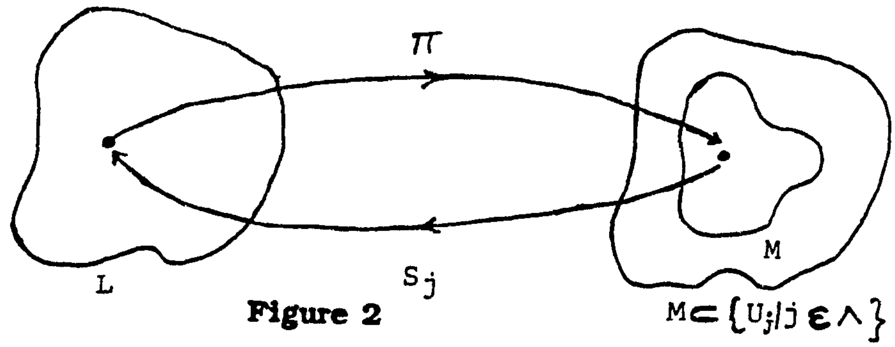

Figure 2

a) For each $j \epsilon \Lambda$, $\pi o S_j = I_{U_j}$ (the identity map on U_j)

b) For each $j \epsilon \Lambda$, the map

$$\varphi_j: U_j x\ C \to \pi^{-1}(U_j): (m, z) \to z.S_j(m)$$

is a diffeomorphism.

The collection {Uj, Sj} is called a **local system** for L. A smooth map S: U⊂M → L from some subset of M into L which satisfies $\pi o S = I_U$ is called a **local section**. When U = M it is called simply a section.

The simplest example of a line bundle is the bundle with L = M x C (Cartesian product) and the map π: L = MxC → M the projection onto the first factor. This bundle is some times called the "**trivial bundle**" or "**product bundle**".

As a local system for L, we can take the set $\{(M, S_0)\}$ where S_0: M→L s.t. $S_0(m) = (m, 1)$ is the unit section of L and any section can be identified with a function φ: M → **C**, since any other section S is uniquely of the form

$$S = \phi \circ S_0 = (m, \phi(m)) \text{ for some } \phi \epsilon C^{\infty}(M) \qquad \text{(A24)}$$

i.e., it is obtained by changing smoothly in the fibre. In order for a line bundle to be used in geometric quantization it needs to have two additional structures:

1) A Hermitian metric

2) A connection

1) **A Hermitian metric:** On each fibre there is a Hilbert space metric $<\cdot|\cdot>$ (C is considered as a 1-dimensional Hilbert space) such that for any two sections S_1, S_2 the function

$$<S_1|S_2> \text{ defined by } <S_1|S_2>: M \to C$$
$$: m \to <S_1(m)|S_2(m)> \qquad \text{(A25)}$$

is smooth.

Here we denote by $\Gamma(L)$ the set of smooth sections on L.

2) **A connection:** There is a map ∇ which assigns to each vector field $\zeta \varepsilon U(M)$ an endomorphism (i.e. a structure preserving map) $\nabla_\zeta: \Gamma(L) \to \Gamma(L)$ satisfying

i) $\nabla_{\zeta+\eta}s = \nabla_\zeta s + \nabla_\eta s$ (A26)

ii) $\nabla_{\phi.\zeta}s = \phi\nabla_\zeta s$ (A27)

iii) $\nabla_\zeta(\phi.s) = (\zeta\phi).s + \phi.\nabla_\zeta s$ (A28)

for each $s\varepsilon\Gamma(L)$, ζ, $\eta\varepsilon U(M)$ and $\phi\varepsilon C^\infty(M)$.

In addition we need compatibility of these two structures. They are said to be compatible if for each real $\zeta\varepsilon U(M)$ we have

$$\zeta<S_1|S_2> = <S_1|\nabla_\zeta S_2> + <\nabla_\zeta S_1|S_2> \qquad \text{(A29)}$$

for all S_1, $S_2 \ \varepsilon\ \Gamma(L)$.

For the trivial bundle we have the natural metric:

$$(S_1, S_2) \to <S_1|S_2>_m = <S_1(m)|S_2(m)>$$
$$= \overline{S_1(m)}\, S_2(m) \qquad \text{(A30)}$$

Let α be a smooth one-form and S_0 the unit section coming from the local trivialization, then

$$\nabla_\zeta S = \nabla_\zeta(\phi S_0) = (\zeta\phi)S_0 - i\phi(\zeta \rfloor \alpha)S_0 \qquad \text{(A31)}$$

where $S\varepsilon\Gamma(L)$, $\phi\varepsilon C^\infty(M)$ and $S = \phi\circ S_0$, constitute a connection, since it satisfies all the conditions required to be a connection. Indeed,

a) $\nabla_{\zeta+\eta}(\phi)S_0 = (\zeta+\eta)\ (\phi)S_0 - i\phi[(\zeta+\eta)\rfloor\alpha]S_0$

$$= (\zeta\phi)S_0 + (\eta\phi)S_0 - i\phi(\zeta\rfloor\alpha)S_0 - i\phi(\eta\rfloor\alpha)S_0$$

$$\Rightarrow \nabla_{\zeta+\eta}(\phi S_0) = (\nabla_\zeta\phi)S_0 + (\nabla_\eta\phi)S_0$$

b) $\nabla_{f.\zeta}S = \nabla_{fo\zeta}(\phi S_0) = (fo\zeta\phi)S_0 - i\phi(fo\zeta\rfloor\alpha)S_0$

$$= f(\zeta\phi)S_0 - if\phi(\zeta\rfloor\alpha)S_0$$

$$= f[(\zeta\phi)S_0 - i\phi(\zeta\rfloor\alpha)S_0]$$

$\Rightarrow \nabla_{f.\zeta}S = f\nabla_\zeta s$ for each $S\varepsilon\Gamma(L)$, $\zeta\varepsilon U(M)$, $f\varepsilon C^\infty(M)$.

c) $\nabla_\zeta(f.S) = (\zeta f\phi)S_0 - if\phi(\zeta\rfloor\alpha)S_0$

$$= f(\zeta\phi)S_0 + \phi(\zeta f)S_0 - if\phi(\zeta\rfloor\alpha)S_0$$

$$= (\zeta f)\phi S_0 + f[(\zeta\phi)S_0 - i\phi(\zeta\rfloor\alpha)S_0]$$

$$= (\zeta f)S + f\,\nabla_\zeta S$$

The α in (A31) is called the **connection form**. Note that this will be compatible with the natural Hermitian metric if and only if α is real. In general

$$\zeta(f^*g) = \langle\nabla_\zeta(fS_0)|gS_0\rangle + \langle fS_0|\nabla_\zeta(gS_0)\rangle + if^*(\zeta\rfloor\alpha^*)g - if^*(\zeta\rfloor\alpha)g \tag{A32}$$

Note that locally any connection on a line bundle (not necessarily trivial bundle) has the form (A31). To see this, let $\{(U_j, S_j)\}$ be a local system for L and for each j let us define the map

$$U(M) \to C^\infty(U_j) : \zeta \to -i\,\frac{\nabla_\zeta S_j}{S_j} \tag{A33}$$

Clearly the map defined in (A33) is linear in ζ(since ∇_Z is linear in ζ) and so we can define a 1-form α_j on U_j.

Now using (A28) we can write

$$(\nabla_\zeta S)|_{U_j} = [(\zeta\phi_j - i(\zeta\rfloor\alpha_j)\phi_j]S_j \tag{A34}$$

where $S = \phi_j.S_j\ \varepsilon\ \Gamma(L)$ and $\phi_j\ \varepsilon\ C^\infty(U_j)$.

To find the connection form on a nonempty intersection $U_j \cap U_k$ one needs the idea of *transition functions*. Let (U_j, S_j) and (U_k, S_k) be two local systems of L. Then the function $C_{jk} \varepsilon\, C^{\infty}(U_j \cap U_k)$defined by

$$S_k(m, z) = C_{jk}(m)\, S_j(m, z) \qquad \text{(A35)}$$

is called the transition function between (U_j, S_j) and (U_k, S_k). On $U_j \cap U_k$ we can write

$$S_j = C_{jk}\, S_k \qquad \text{(A36)}$$

where $C_{jk} \varepsilon C^{\infty}(U_j \cap U_k)$ is the transition function. Then it can be shown [1] that α_j and α_k are related by

$$\alpha_j = \alpha_k + i\,\frac{dC_{jk}}{C_{jk}} \text{ on } U_j \cap U_k \qquad \text{(A37)}$$

and any α_j in (A37) will define a connection on L.

Choosing new coordinates z a given connection form can be extended [1] to $U_j x C^*$ ($C^* = C - \{o\}$) as follows:

$$\alpha_j \to \alpha_j + idz/z \qquad \text{(A38)}$$

Then it is possible to define a global connection form.

4. Curvature

Let (L, π, M) be a line bundle with a connection ∇. If $\zeta, \eta \varepsilon\, U(M)$, then the corresponding operators ∇_ζ and ∇_η do not, in general, commute and the connection will have a curvature.

Curvature, denoted by $Curv(L, \nabla)$, is defined by

$$Curv(L,\nabla)\,(\zeta, \eta)(S) = i\,([\nabla_\zeta, \nabla_\eta] - \nabla[\zeta, \eta])S \qquad \text{(A39)}$$

where $\zeta, \eta\ \varepsilon\ U(M)$ and $S\ \varepsilon\ \Gamma(L)$.

The right hand side of (A39) is skew symmetric and linear in ζ, η, so defines a 2-form. Hence we can write

$$Curv(L, \nabla)(\zeta, \eta)(S) = \omega(\zeta, \eta)S \qquad \text{(A40)}$$

where $\omega \varepsilon \Omega(M)$ is a 2-form on M

Also $\omega|_{U_j} = d\alpha_j$ where α_j is connection form on U_j [1]. (A41)

Let (L, π, M) be a line bundle with connection ∇ and Hermitian metric $<\cdot|\cdot>$ and let $\gamma:[a,b] \rightarrow M$ be a smooth curve with tangent ζ.

A section S over γ is said to be parallel if $\nabla_\zeta S = 0$. In the local system (U_j, S) with unit section S_o and connection form α we have

$$\nabla_\zeta S = (\zeta\phi)S_o - i\phi(\zeta \lrcorner \alpha)S_o \quad \text{(A42)}$$

where $S = \phi S_o$.

then $\nabla_\zeta S = o \Rightarrow (\zeta\phi)S_o = i\phi(\zeta \lrcorner \alpha)$

$$\Rightarrow \dot{\phi} = i\phi(\zeta \lrcorner \alpha) \text{ (since } \zeta\phi = \dot{\phi}, \zeta \text{ is a tangent)}$$

$$\Rightarrow \frac{\dot{\phi}}{\phi} = i\,(\zeta \lrcorner \alpha) \quad \text{(A43)}$$

Now integrating (A43), assuming that γ is a closed curve, we have

$$\frac{\phi(b)}{\phi(a)} = \exp[i\oint_\gamma(\zeta(t) \lrcorner \alpha)dt] = \exp(i\oint_\gamma \alpha) \quad \text{(A44)}$$

If the curve lies in one patch U_j we can write using Stoke's theorem [17]

$$\phi(b)/\phi(a) = \exp(i\int_{\Sigma_1} d\alpha) \quad \text{(A45)}$$

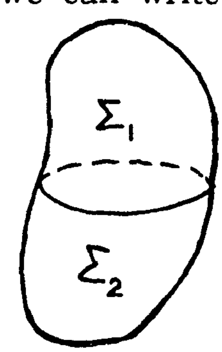

Figure 3.

Note that this is well defined also if we require more than one patch, since at transitions,

$$\alpha_j = \alpha_k + i\frac{dC_{jk}}{C_{jk}} = \alpha_k + id(\log C_{jk}) \quad \text{(A46)}$$

and $d\alpha_j = d\alpha_k$

So, if the surface is closed this should be true for either part Σ_1, or Σ_2 (see Fig. 3). Then using $d\alpha = \omega$ (equation (A41)) (A45) can be written as

$$\phi(b)/\phi(a) = \exp(i\int_{\Sigma_1}\omega) = \exp(-i\int_{\Sigma_2}\omega)$$

$$= \exp(i\int_{\Sigma_1 U \Sigma_2}\omega) = 1$$

$$\Rightarrow \int_{\Sigma_1 U \Sigma_2}\omega = 2n\pi,\ n \varepsilon Z \qquad \text{(A47)}$$

(A47) is known as **integrality condition**.

Actually geometric quantization begins with the converse of this result and it is known as Weil's Theorem.

Theorem of Weil [11]

If ω is a closed real integral 2-form on a manifold M then there exists a line bundle (L, π, M) with a Hermitian metric <.|.> and a compatible connection ∇ such that Curv(L, ∇) = ω.

Foliation: A foliation of dim-k on an m-dimensional manifold M is a decomposition of M into disjoint connected subsets $F = \{L_\alpha | \alpha \varepsilon \Lambda\}$ (Λ is some index set) called leaves of the foliation, such that each point of M has a coordinate chart (U, X^i) [17] such that for each leaf L_α, the components of $L_\alpha \cap U$ are locally given by the equations

$$X^{K+1} = \text{constant}, \ldots., X^m = \text{constant}.$$

Distribution: Let M be a manifold of dimension m. A k-dim. *distribution* D on M is a choice of a k-dim, subspace D(x) of T_xM for each x in M.

Appendix B

In this appendix we have described the mathematical preliminaries required for the formulation of prime quantization.

Borel set: An algebra $\mathscr{A}$ of sets is called a σ-algebra if it is closed under countable union of sets; that is, $\bigcup_{i=1}^{\infty} A_i$ is in $\mathscr{A}$ whenever the countable collection $\{A_i\}$ of sets; is in $\mathscr{A}$. The σ-algebra generated by the family of all open sets in **R**, denoted by $\mathscr{B}$, is called the class of Borel sets in **R**. The sets in $\mathscr{B}$ are called *Borel sets* in **R**.

Let $\mathscr{H}$ be a separable Hilbert space, $\mathscr{L}(\mathscr{H})$ the set of all bounded linear operators on $\mathscr{H}$, $\mathscr{L}(\mathscr{H})^+$ the positive cone (i.e., the set of all positive operators) of $\mathscr{L}(\mathscr{H})$, X a locally compact, separable topological space, $\mathscr{B}(X)$ the Borel sets of X, μ a regular positive Borel measure on X such that its support is the whole of X. Whenever μ is the measure of the topologial space T one defines **the support of** μ as the closed set

$$\text{Support}(\mu) = X \setminus \cup\{V : V \varepsilon T \text{ and } \mu(V) = 0\}.$$

Definition B1: A normalized positive operator-valued (POV) measure on X, with values in $\mathscr{L}(\mathscr{H})$, is a map

$$\alpha : \mathscr{B}(x) \to \mathscr{L}(\mathscr{H})^+ \tag{B1}$$

such that

(1) $\alpha(\Phi) = 0$, where Φ denotes the null set (B2)

(2) $\alpha(x) = I$, I is the identity operator on $\mathscr{H}$ (B3)

(3) $\alpha(\bigcup_{k\varepsilon J} \Delta_k) = \sum_{k\varepsilon J} \alpha(\Delta_k)$ (B4)

where J is a countable index set, and for k, l ε J, k$\neq$l, $\Delta_k \cap \Delta_l = \phi$. The sum is (3) is assumed to converge weakly. A sequence $\{X_n\}$ of elements of E is said to be *weakly convergent* to the element XεE

when $\lim_{n \to \infty} f(Xn) = f(x)$ for every bounded linear functional f defined on the given space E. $A_n \to A$ if and only if, for $\Phi, \Psi \; \varepsilon \; \mathscr{H}$,

$$\langle \Phi | A_n \Psi \rangle \to \langle \Phi | A \Psi \rangle.$$

A POV- measure which satisfies the additional property

$$a(\Delta) = a(\Delta)^* = [a(\Delta)]^2 \tag{B5}$$

is called a **projection valued** (PV) measure.

Example of a POV measure on $\mathscr{H} = L^2(\mathbf{R}, dx)$

Let χ_Δ be the characteristic function for each

$$\Delta \varepsilon \mathscr{B}(\mathbf{R}) \text{ i.e., } \chi_\Delta(x) = \begin{cases} 1 \text{ if } x \; \varepsilon \; \Delta \\ 0, \text{ otherwise} \end{cases} \tag{B6}$$

Then the map $\Delta \to P(\Delta)$ defined by

$$(P(\Delta)\Psi)(x) = \chi_\Delta(x)\Psi(x), \; \forall \Psi \varepsilon L^2(\mathbf{R}, \; dx) \tag{B7}$$

for almost all xε**R** is a normalized PV measure. This is verified below:

(1) $(P(\Phi)\Psi)(x) = \chi_\Phi(x)\Psi(x) = 0, \Phi$-null set, $\Psi \varepsilon L^2(\mathbf{R}, dx)$

(2) $(P(\mathbf{R})\Psi)(x) = \chi_{\mathbf{R}}(x)\Psi(x) = \Psi(x) \Rightarrow P(\mathbf{R}) = I, \; \Psi \varepsilon L^2(\mathbf{R}, \; dx)$

(3) Let $\Delta_1, \Delta_2, \; \varepsilon \; \mathscr{B}(\mathbf{R})$ and $\Delta_1 \cap \Delta_2 = \Phi$, then

$$(P(\Delta_1 U \Delta_2)\psi)(x) = \chi_{\Delta_1 U \Delta_2}(x) \; \Psi(x) = \chi_{\Delta_1}(x)\Psi(x) + \chi_{\Delta_2}(x)\Psi(x)$$
$$= (P(\Delta_1)\Psi)(x) + (P(\Delta_2)\Psi)(x).$$

(4) $\langle P(\Delta)\Psi | \xi \rangle = \int_\Delta \overline{P(\Delta)\Psi(x)} \xi(x) dx = \int_\Delta \overline{\chi_\Delta(x)\Psi(x)} \; \xi(x) dx$

$$= \int_{\Delta} \overline{\Psi}(x)\chi_{\Delta}\xi(x)dx = \langle\Psi | P(\Delta)\xi\rangle$$

$\Rightarrow$ $P(\Delta) = P(\Delta)^*$

and $(P^2(\Delta)\Psi)(x) = P(P(\Delta)\Psi)(x) = P(\chi_{\Delta}(x)\Psi(x))$

$$= \chi_{\Delta}(x)\ (\chi_{\Delta}(x)\Psi(x)) = \chi_{\Delta}(x)\Psi(x)$$

$$= (P(\Delta)\Psi)(x)$$

$\Rightarrow$ $[P(\Delta)]^2 = P(\Delta)$.

The POV-measure α is said to admit a bounded positive μ-density if there exists a μ-measurable function

$F: X \to \mathscr{L}(\mathscr{H})^+$ such that

$$\alpha(\Delta) = \int_{\Delta} F(x)d\mu(x)\ ,\ \forall\ \Delta\varepsilon\mathscr{B}(x) \qquad \text{(B8)}$$

the integral being assumed to converge weakly.

The Direct Integral Space

Let X be a Borel space equipped with a σ-finite Borel measure[1] μ. A measurable field of Hilbert space on $\{X, \mu\}$ is a family $\{K_x: x\varepsilon X\}$ of Hilbert spaces index by Γ together with a subspace M of the product vector space $\Pi_{x\varepsilon X} K_x$ with the following properties:

(1) For any $\zeta\varepsilon M$, the function $x\varepsilon X \to ||\zeta(x)||$ is μ-measurable.

(2) For any $\eta\varepsilon\Pi_{x\varepsilon X} K_x$, if the function

$x\varepsilon X \to \langle\zeta(x)|\eta(x)\rangle\ \varepsilon\ C$ is μ-measurable for every $\zeta\varepsilon M$, then η belongs to M.

1. If E is a Borel set, then E is said to be *of σ-finite Borel* measure if E is the union of a countable collection of measurable sets of finite measure.

(3) There exists a countable subset $\{\zeta_1, ---, \zeta_n, ---\}$ of M such that for every $x \varepsilon X$, $\{\zeta_n(x): n= 1, 2, ---\}$ is total in K_x.

Let $\tilde{\mathscr{H}}$ be the collection of measurable vector fields ζ such that

$$||\zeta|| = \int_X ||\zeta(x)||^2 d\mu(x)\}^{\frac{1}{2}} < +\infty \qquad \text{(B9)}$$

where $<\zeta| \eta> = \int_X <\zeta(x)|\eta(x)> d\mu(x)$, $\zeta, \eta \varepsilon \tilde{\mathscr{H}}$, is the scalar product in K_x.

We would identify two vector fields $\zeta, \eta \varepsilon \tilde{\mathscr{H}}$ if $\zeta(x) = \eta(x)$ μ-almost everywhere. We call this Hilbert space $\tilde{\mathscr{H}}$ the direct integral of measurable fields of Hilbert spaces and denote it by

$$\tilde{\mathscr{H}} = \int_X^{\oplus} K_x d\mu(x) \qquad \text{(B10)}$$

The elements $\Phi \varepsilon \tilde{\mathscr{H}}$ are equivalence classes of mappings $x \varepsilon X \rightarrow \tilde{\Phi}(x) \varepsilon K_x$, which are μ-measurable and satisfy

$$\int_X ||\tilde{\Phi}(x)||_x^2 d\mu(x) < \infty \qquad \text{(B11)}$$

where $||...||_x$ is the norm in K_x.

Let $\Pi_{x \varepsilon X} K_x$ be the cartesian product space of K_x for each $x \varepsilon X$ and $\mathscr{H}$ a fixed subspace of $\tilde{\mathscr{H}}$.

Definition B2:

A μ-selection σ for $H \subset \tilde{\mathscr{H}}$ is a linear map

$$\sigma : H \rightarrow \Pi_{x \varepsilon X} K_x \qquad \text{(B12)}$$

which associates to each μ-equivalence class $[f]$ of functions in H, a function $x \rightarrow \sigma([f])(x)$ in $\Pi_{x \varepsilon X} K_x$ such that $[f] = [\sigma([f])]$ (B13)

Let $L(K_x, K_y)$ be the set of all bounded linear maps from K_x to K_y. For any $A \varepsilon L(K_x, K_y)$, its adjoint A^* is again a bounded linear map, in $L(K_y, K_x)$. That is, $<V|Au>_y = <A^*|U>_x \ \forall\ U \varepsilon K_x$ (B14)

and $V \varepsilon K_y$, where $<\cdot|\cdot>_x$ is the scalar product in K_x, $x\varepsilon X$.

Definition B3:

A reproducing kernel K on $\tilde{\mathscr{H}}$ is a mapping,

$(x, y) \varepsilon X \times X \rightarrow (K(x, y) \varepsilon L(K_y, K_x)$ such that

I) $K(x, y) = K(y, x)^*, (x, y) \varepsilon X \times X$ (B15)

II) $<U|K(x, x)U> > 0, \forall\ U\varepsilon K_x, U \neq 0, \forall x \varepsilon X$ (B16)

III) the integral operator $\mathcal{P}_k$ on $\tilde{\mathscr{H}}$ such that

$$(\mathcal{P}_k\tilde{\Psi})(x) = \int_X K(x,y)\ \tilde{\Psi}\ (y)d\mu(y) \quad \forall\ \tilde{\Psi}\ \varepsilon\ \tilde{\mathscr{H}} \tag{B17}$$

exists and is bounded.

IV) for all $U\varepsilon K_x$ and $V\varepsilon K_z$

$$\int_X \langle U|K(x,\ y)K(y,z)v\rangle_x d\mu(y) = \langle U|K(x,\ z)v\rangle_x \ \forall\ (x, z)\ \varepsilon\ X \times X \tag{B18}$$

Lemma B1:

The operator $\mathcal{P}_k$ in (B17) is a projection in $\mathscr{L}(\tilde{\mathscr{H}})$, that is,

$$\mathcal{P}_k = \mathcal{P}_k^* = \mathcal{P}_k^2 \tag{B19}$$

Proof:

$$(\mathcal{P}_k\tilde{\Psi})(x) = \int_X K(x,\ y)\tilde{\Psi}(y)d\mu(y),\ \ \tilde{\Psi}\ \varepsilon\ \tilde{\mathscr{H}}$$

Then

$$\left\langle \mathcal{P}_k\tilde{\Psi}|\tilde{\Phi}\right\rangle = \int_X \overline{\mathcal{P}_k\tilde{\Psi}(z)}\tilde{\Phi}(z)dz$$

$$= \int_X \int_X \overline{K(x, y)\tilde{\tilde{\Psi}}d\mu(y)}\tilde{\Phi}(z)dz$$

$$= \int_X \int_X K(y, x)\, \tilde{\tilde{\Psi}}(y)\tilde{\Phi}(z)d\mu(y)dz$$

$$= \int_X \int_X \tilde{\tilde{\Psi}}(z)k(x, y)\tilde{\Phi}(z)d\mu(y)dz$$

$$= \int_X \tilde{\tilde{\Psi}}(z)\int_X K(z, x)\tilde{\Phi}(y)d\mu dz$$

$$= \int_X \tilde{\tilde{\Psi}}(z)\ (\mathcal{P}_K\tilde{\Phi})(z)dz = \langle\tilde{\Psi}|\mathcal{P}_K\tilde{\Phi}\rangle$$

$$\Rightarrow \mathcal{P}_K = \mathcal{P}^*_K \tag{B20}$$

Again,

$$(\mathcal{P}_K(\mathcal{P}_K\tilde{\Psi}))(x) = \int_X K(x, z)(\mathcal{P}_K\tilde{\Psi})(z)d\mu(z)$$

$$= \int_X K(x, z)\int_X K(z, y)\tilde{\Psi}(y)d\mu(y)d\mu(z)$$

$$= \int_X \int_X K(x, z)K(z, y)d\mu(z)\tilde{\Psi}(y)d\mu(y)$$

$$= \int_X K(x, y)\tilde{\Psi}(x)d\mu(y) \quad \text{(by B18)}$$

$$= (\mathcal{P}_K\tilde{\Psi})(x)$$

$$\Rightarrow \mathcal{P}_K{}^2 = \mathcal{P}_K \tag{B21}$$

From (B20) and (B21) it follows that $\mathcal{P}_K = \mathcal{P}_K{}^* = \mathcal{P}_K{}^2$, which completes the proof of the lemma.

Let $\mathcal{H}_K \subset \tilde{\mathcal{H}}$ be the subspace onto which $\mathcal{P}_K$ projects

$$\mathcal{H}_K = \mathcal{P}_K \tilde{\mathcal{H}} \tag{B22}$$

Definition B4:

A reproducing kernel Hilbert space is a subspace $\mathcal{H}_K$ of $\tilde{\mathcal{H}}$, for which the projection $\mathcal{P}_K$ is defined via a reproducing kernel K.

If d(x) is the dimension of the Hilbert space K_x, V_x^i, i = 1, 2, ..., d(x) an orthonormal basis in K_x. Then for a fixed vector $V \varepsilon K_y$, the vector K(., y) V is an element of $\mathcal{H}_K$. Hence let us define $\zeta_y^V(x) \varepsilon \mathcal{H}_K$ for given $y \varepsilon X$ and $V \varepsilon K_y$, $||V|| = 1$ as

$$\zeta_y^V(x) = K(x, y)V \tag{B23}$$

Let $$G_K = \{ \zeta_y^V \varepsilon \mathcal{H}_K | y\varepsilon X, V\varepsilon K_y, ||V|| = 1\} \tag{B24}$$

$$G_K' = \{\zeta_y^i \varepsilon G_K | \zeta_y^i(x) V_y^i, i = 1, 2, ..., d(y)\} \tag{B25}$$

Lemma B2:

The set of vectors G_K is over complete $\mathcal{H}_K$.

Then writing $$K_{ij}(x, y) = \langle \zeta_x^i | \zeta_y^j \rangle_{\mathcal{H}_K} \tag{B26}$$

We may express K(x, y) in terms of the generating set G_K' of G_K as

$$K(x, y) = \sum_{i=1}^{d(x)} \sum_{j=1}^{d(y)} |V_x^i\rangle Kij(x, y) \langle V_y^j| \tag{B27}$$

Theorem B1:

There exists a μ-selection σ on $\mathcal{H}_K$, such that, for each $x \varepsilon X$, the linear mapping $E_x^k: \mathcal{H}_K \to K_x$, defined by

$$E_x^k(\Psi_K) = \sigma([\Psi_k])(x), \ \Psi_k \ \varepsilon \ \mathcal{H}_K \tag{B28}$$

is continuous and has dense range in K_x.

Also

$$\sigma([\Psi_k])(x) = \int_X K(x, y)\Psi_k(y) \ d\mu(y) \tag{2.29}$$

and $K(x, y) = E_x^k E_y^{k*}$ (B30)

where $E_x^{k*} : K_y \rightarrow \mathscr{H}_K$ is the adjoint of E_y^k .

The map $E_x^k: \mathscr{H}_K \rightarrow K_x$ defined by (B28) is called an **evaluation map**.

Lemma B3:

The expression,

$F_k(x) = E_x^{k*} \; E_x^k, x \varepsilon X$ (B31)

defines a bounded positive operator on $\mathscr{H}_K$, such that $\Delta \rightarrow a_K(\Delta)$, $\Delta \varepsilon \mathscr{B}(X)$, where

$$a_K(\Delta) = \int_\Delta F_K(x) d\mu(x) \quad \text{(B32)}$$

defines a normalized POV-measure on $\mathscr{H}_K$, having μ-density $x \rightarrow F_K(x)$.

The nomalized POV-measure a_K in (B32) is said to be the **canonically associated** to the reproducing kernel Hilbert space $\mathscr{H}_K$.

Remarks:

I) We may also write E_x^k and E_x^{k*} interms of the set G_K' in (B25) and the basis sets V_x^i , i = 1, 2 .., d(x) as

$$E_x^k = \sum_{i=1}^{d(x)} |V_x^i><\zeta_x^i| \quad \text{(B33)}$$

and $$E_x^{k*} = \sum_{i=1}^{d(x)} |\zeta_x^i><V_x^i| \quad \text{(B34)}$$

II) In the special case where K_x is isomorphic to

$K, \forall x \varepsilon X \; E_x^k(\Psi_k) = \Psi_k(x)$ (B35)

III) Using (B33) and (B34) we can write

$$F_K(x) = \sum_{i=1}^{d(x)} |\zeta_x^i><\zeta_x^i|, \text{ since } ||V||_x^2 = 1 \quad \text{(B36)}$$

Theorem B2:

(Extension Theorem of Naimark)

Any normalized POV measure a on an abstract Hilbert space $\mathcal{H}$ can be extended to a projection valued (PV) measure $\tilde{P}$ on a larger Hilbert space $\tilde{\mathcal{H}}$, in the following sense. If $\mathcal{B}(\Gamma)$ is the set of all Borel sets of Γ, then there exists

I) A Hilbert space $\tilde{\mathcal{H}}$ on which there is defined a PV measure $\tilde{P}(\Delta)$, $\Delta\varepsilon\, \mathcal{B}(\Gamma)$.

II) A subspace $\hat{\mathcal{H}}$ of $\tilde{\mathcal{H}}$, with projection operator $\mathcal{P}$ such that

$$\mathcal{P}\tilde{\mathcal{H}} = \hat{\mathcal{H}} \tag{B37}$$

III) A unitary map

W: $\mathcal{H} \to \hat{\mathcal{H}}$ such that

$$\mathcal{P}\tilde{P}(\Delta)\mathcal{P} = Wa(\Delta)W^{-1},\ \Delta\varepsilon\, \mathcal{B}\varepsilon(\Gamma) \tag{B38}$$

Moreover, $\tilde{\mathcal{H}}$ may be chosen to be minimal in the sense that every other extended space $\tilde{\mathcal{H}}$ having properties (B37) and (B38) contains a subspace which is unitarily equivalent to $\tilde{\mathcal{H}}$.

As before, let $\mathcal{H}$ be a separable Hilbert space and a as in (B8). For each $x\varepsilon X$, the operator $F(x)$ is bounded and positive, and hence its square root $F(x)^{\frac{1}{2}}$ exists.

Let N_X denote the null space of $F(x)^{\frac{1}{2}}$,

$$N_X = \{\Phi\varepsilon\mathcal{H} \mid F(x)^{\frac{1}{2}}\Phi = 0\} \tag{B39}$$

Then K_x is obtained by closing the quotient space $\mathcal{H}/Nx$ with respect to the scalar product

$$<[\Psi]_X \mid [\Phi]_X>_X = <\Psi \mid F(x)\Phi>_{\mathcal{H}} \tag{B40}$$

where $[\Psi]_X$ and $[\Phi]_X$ are the equivalence classes in $\mathcal{H}/N_X$ of Ψ and Φ respectively.

For each $x \in X$, let $U(x)$ be a unitary operator on K_X and let the operator valued function $x \to U(x) \in L(k)$ defined by $x \to \langle u | U(x) v \rangle_X$ for each $u, v \in K$ be μ-measurable.

Theorem B3:

The mapping $W_K: \mathscr{H} \to \tilde{\mathscr{H}}$, defined by

$$(W_K\Phi)(x) = \Phi_K(x) = U(x)[\Phi]_X, \; x \in X \tag{B41}$$

for all $\Phi \in \mathscr{H}$, is a linear isometry between $\mathscr{H}$ and a proper subspace $\mathscr{H}_K$ of $\tilde{\mathscr{H}}$.

Lemma B4: the following norm estimates hold:

$$||K(x, y)|| = ||F(x)^{\frac{1}{2}} F(y)^{\frac{1}{2}}|| \tag{B42}$$

$$||E_x^k|| = ||(E_x^k)^*|| = ||K(x, x)||^{\frac{1}{2}} = ||F(x)||^{\frac{1}{2}} \tag{B43}$$

Theorem B5:

A subspace $\mathscr{H}_K$ of $\tilde{\mathscr{H}}$ is a reproducing-kernel Hilbert space if and only if, for every $x \in X$, there exists a continuous linear evaluation map $E_x^k: \mathscr{H}_K \to K_X$ with dense range in K_X. In this case, $\tilde{\mathscr{H}}$ is the unique minimal extension of $\mathscr{H}_K$, in the sense of Naimark, for the canonically associated POV measure α_K.

Theorem B6:

Let $\mathscr{H}$ be an abstract, separable Hilbert space, on which there exists a normalized POV measure α, defined on the Borel sets of a locally compact space X, and admitting a μ-density F. Then there exists a reproducing kernel Hilbert space $\mathscr{H}_K$, with canonically associated POV measure α_K, and a unitary map W_K such that

$$W_K\mathscr{H} = \mathscr{H}_K \tag{B44}$$

$$W_K\alpha(\Delta)\, W_K^{-1} = \alpha_K(\Delta), \; \Delta \in \mathscr{B}(x) \tag{B45}$$

$$W_K F(x)\, W_K^{-1} = (E_x^k)^* E_x^k, \; x \in X. \tag{B46}$$

Strong and Weak operator topology:

Let $\mathscr{H}$ be a Hilbert space and $\mathscr{L}(\mathscr{H})$ the set of all bounded linear operators on $\mathscr{H}$. Then the strong operator topology is the locally convex topology determined by the seminorms:

$$x \in \mathscr{L}(\mathscr{H}) \to ||x\zeta||, \ \zeta \in \mathscr{H}.$$

The locally convex topology determined by the seminorms:

$x \in \mathscr{L}(\mathscr{H}) \to |\langle x\zeta|\eta\rangle|$, $\zeta, \eta \in \mathscr{H}$, is called the weak operator topology.

Von Neumann Algebra :

Let M be any subset of $\mathscr{L}(\mathscr{H})$. We shall name the *commutant* of M, to be denoted by M', the set of those elements of $\mathscr{L}(\mathscr{H})$ that commute with all the elements of M. We denote (M')' = M" (the double commutant). If a subalgebra of $\mathscr{L}(\mathscr{H})$ is invariant under the *-operations then it is called a *-subalgebra of $\mathscr{L}(\mathscr{H})$.

Let $\xi, \eta \in \mathscr{L}(\mathscr{H})$ and $\alpha \in C$, the the *- operations have the following properties:

(i) $(\xi^*)^* = \xi$;

(ii) $(\xi+\eta)^* = \xi^* + \eta^*$;

(iii) $(\alpha\xi)^* = \bar{\alpha}\xi^*$; ($^-$ denotes complex conjugate)

(iv) $(\xi\eta)^* = \eta^* \xi^*$;

(v) $||\xi^*|| = ||\xi||$

Definition B5: A von Neumann algebra on $\mathscr{H}$ is a *-subalgebra M of $\mathscr{L}(\mathscr{H})$ such that

$$M = M''.$$

Trace class operator

Let $||A||_1 = tr([A^*A]^{\frac{1}{2}}) = \Sigma <\Phi_i|(A^*A)^{\frac{1}{2}}\Phi_i>$

where $\{\Phi_i\}$ is an orthonormal basis. If $||A||_1 < \infty$ then it is independent of choice of orthonormal basis. Then $||A||_1$ is called the **trace norm** of A and A itself is called a **trace-class operator**.