SOME CHARACTERIZATION THEOREMS AND NON-PARAMETRIC TESTS OF THE EXPONENTIAL DISTRIBUTION

STELLA ANN CHI-HSING CHANG

A THESIS

in

THE DEPARTMENT

of

MATHEMATICS

Presented in Partial Fulfillment of the Requirement
for the degree of Master of Science at
Concordia University
Montréal, Québec, Canada

September, 1976

TO

MY PARENTS

大學之道，在明明德，在親民，在止於至善。知止而后有定，定而后能靜，靜而后能安，安而后能慮，慮而后能得。物有本末，事有終始；知所先後，則近道矣。

ABSTRACT

STELLA ANN CHI-HSING CHANG

SOME CHARACTERIZATION THEOREMS AND NON-PARAMETRIC TESTS

OF

THE EXPONENTIAL DISTRIBUTION†

Supposed that it is desired to test whether or not a random sample of size $n \geq 3$ are from the exponential distribution with location parameter $\mu = 0$ and unknown scale parameter θ. Inferences based on any test statistic which is not independent of θ would depend on θ, which is unknown, in some form. Hence it is desirable to construct test statistics which are independent of θ. Some of these statistics and their non-parametric tests are discussed. A new characterization theorem of the exponential distribution together with the resultant non-parametric tests are proposed.

† A portion of this thesis has been presented in the conference on The Theory and Application of Reliability with Emphasis on Baysian and Non-parametric Methods at The University of South Florida, Tampa, Florida in December, 1975 and shall be appearing in its proceedings.

ACKNOWLEDGEMENTS

"No man is an island, entire of itself; every man
is a piece of the continent, a part of the main ..."

—— John Donne

I am gratefully indebted to Dr. Y.H. Wang for his constant counsel and encouragement throughout the fours years of my undergraduate and graduate studies; also for his time, understanding, willingness to listen and help, and his tolerance of my "ignorence".

My special thanks to Dr. M. Cohen for his time, encouragement, help and especially for the boost of self-confidence when it was desperately needed.

To my parents, to whom this thesis is dedicated to with love, I am thankful for their love, understanding and guidance; for their sacrifices and the hardships that they have had to go through for the benifits of my brother and myself; and most of all, for being my parents.

Thanks also to my brother and my friends for being friends in need, and to Mr. H. Markovits for his typing.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

Often, in real life situations, it is desired to ascertain whether a set of data or a series of observations collected from a particular population possesses certain characteristics. In statistics, this is parallel to testing whether the population from which the random sample is taken has a particular distribution. Specifically, suppose that $X_1, X_2, \ldots, X_n$ is a random sample of size n from a population with distribution function (c.d.f.) F. By utilizing the sample, the following null hypothesis is to be tested:

$$F(x) = 1 - e^{-(x-\mu)/\theta} , \quad x \geq \mu , \quad \theta > 0 , \tag{1.1}$$

That is, based on the sample, it is desirable to test whether or not F is the exponential distribution function with location parameter μ and unknown scale parameter θ, denoted by $\mathrm{Exp}(\mu,\theta)$.

Among others, there are two possible approaches to a given problem in statistical inferences - the parametric and the non-parametric methods. While the emphasis of this thesis is on the non-parametric methods in testing exponentiality, a brief comparison of these two approaches will be given in Section 1.1. Section 1.2 serves as an introduction to the exponential distribution and its importances.

Chapter II deals with reviews of past literature and incorporates those test statistics which are pertinent to the theme of this thesis. The main characterization theorem of the exponential distribution is discussed and proven in Chapter III. Along with the resultant non-parametric tests, results from simulations are discussed and concluded in Chapter IV.

1.1 THE COMPARISONS OF THE PARAMETRIC AND THE NON-PARAMETRIC METHODS IN STATISTICAL INFERENCES.

As Silvey [33] has pointed out, of the two above mentioned approaches, the parametric methods entail much stronger assumptions in regarding the family of possible distributions on the sample space than the non-parametric methods. The term non-parametric is used in the sense that one is not concerned with the parameters of a given population, consequently, no assumptions pertaining to the population from which the samples are taken are made, except possible mere postulates which may be self-evident in a given situation. To this extent, the non-parametric methods are more realistic and hence have a greater intuitive appeal. The following example [5, pp. 139-140] best illustrates these qualities:

Suppose two random samples $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ of sizes m and n are taken from two populations with distribution functions F and G respectively. One may be prepared to assume that

$$G(z) = F(z-\xi)$$

where ξ is an unknown constant to be estimated from the data. One approach to this problem of estimation is to assume normality of the underlying distributions. The problem then becomes parametric in character and a "best" estimate of ξ may be found by classical estimation theories. A confidence interval for ξ may also be derived; but one's confidence in this interval will strongly depend on the confidence one has in the normality assumption. If this assumption is based on the grounds of expediency, then a robustness study would be in order.

An alternative approach is to make far weaker assumptions s about the nature of the underlying distributions, for instance, that they are continuous. The problem now becomes non-parametric with the labelling parameter θ taking the form $\theta = (F,\xi)$ and F ranges over the space of continuous distribution functions, while ξ ranges over the real numbers.

Although the advantage of virtually no assumptions is sometimes offset by weaker efficiencies, the non-parametric methods are generally quite easy to perform and require fewer computations than their counterparts, if they exist, in the parametric approach.

1.2 PRELIMINARIES.

The exponential distribution is a prominent distribution in its own right. Its "memoryless property" plays a key role in probability theory as its counterpart - constant failure rate in reliability theory. The former property refers to the feature of a phenomenon in which the probability of an event occuring in a given time interval $(t,t+\Delta t)$ does not depend on the history proceeding the time t and depends solely on the length of the interval Δt. In reliability, an object possessing the latter property develops no major propensity towards failure as time elapses, and hence has a constant failure rate*. Physically, this represents a situation in which the object malfunctions only if a sufficiently large environmental stress occurs. These two important properties shall be formally stated later in this section.

Another feature of the exponential distribution may be found in statistical modeling. Often, the failure rate of an object may not be constant; however, a slight modification of the exponential distribution may give rise to an adequate representation of the true underlying distribution, which would result in proper description of the failure rate or the distribution of the object under study.

Before proceeding any further, let us examine how one may eliminate the location parameter μ of the exponential distribution.

LEMMA 1.1: Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n \geq 2$ from a population with c.d.f. (1.1) and let $X_{1,n} < X_{2,n} < \cdots < X_{n,n}$ denote the corresponding *order statistics*.

(a) If the location parameter μ is a known constant, then $X_1-\mu$, $X_2-\mu, \ldots, X_n-\mu$ is a random sample of size n from a population with c.d.f. (1.1) and $\mu = 0$.

(b) If μ is an unknown, then $X_1-X_{1,n}, X_2-X_{1,n}, \ldots, X_n-X_{1,n}$, after eliminating the zero value, is a random sample of size $n-1$ from a population with c.d.f. (1.1) and $\mu = 0$.

<u>Proof</u>. (a) The result may be shown by letting $Y_i = X_i-\mu$, $i = 1,2,\ldots,n$

---

*The failure rate is also known as the hazard rate, the intensity rate and the force of mortality.

followed by a straight-forward calculation.

(b) Since $X_{1,n}$ is a maximum likelihood estimator of μ; hence, by replacing μ by $X_{1,n}$ in (1.1), letting $Y_i = X_i - X_{1,n}$, $i=1,2,\ldots,n$, after eliminating the zero value, and applying a calculation which is similar to that of (a), the results are readily obtained.

Lemma 1.1 shows that without loss of generality, one may assume $\mu = 0$ in (1.1), resulting in

$$(1.2) \qquad F(x) = 1 - e^{-x/\theta} , \quad x \geq 0 , \ \theta > 0 .$$

Wherefore, (1.2) shall be taken to represent the exponential distribution and the null hypothesis H_0 henceforth.

We now proceed to state and prove the two properties of the exponential distribution.

THEOREM 1.2: If a random variable $X \sim Exp(0,\theta)$, then

$$(1.3) \qquad P\{X > x + \Delta x \mid X > x\} = P\{X > \Delta x\} , \ \forall x \geq 0 .$$

<u>Proof</u>. By the definition of conditional probability,

$$\begin{aligned} P\{X > x + \Delta x \mid X > x\} &= P\{X > x + \Delta x , \ X > x\} \ / \ P\{X > x\} \\ &= P\{X > x + \Delta x\} \ / \ P\{X > x\} \end{aligned}$$

The required result is then obtained by utilizing (1.2).

The converse of the memoryless property is also true as seen from the following theorem:

THEOREM 1.3: Let F be an c.d.f. of a non-degenerate, non-negative and continuous random variable which satisfies

$$(1.4) \qquad \frac{1 - F(x+y)}{1 - F(y)} = 1 - F(x) \ , \quad \forall x,y > 0 \ .$$

Then for some $\theta > 0$, F is the exponential distribution (1.2).

Proof. Let $h(\cdot) = 1 - F(\cdot)$, then (1.4) may be written as

$$h(x+y) = h(x)h(y)$$

which is the Cauchy functional equation and has solution

$$h(x) = e^{cx} \ , \quad c \in \mathbb{R}$$

for F is continuous. But since F is an c.d.f. of a non-negative random variable, i.e. $F(\infty) = 1$ or $h(\infty) = 0$, therefore, for some $\theta > 0$, F is the exponential distribution (1.2).

The constant failure rate phenomenon uniquely characterizes the exponential distribution in the following sense.

THEOREM 1.4: Let F be an c.d.f., then F is the exponential distribution (1.2) if and only if the failure rate is constant.

Proof. The failure rate is defined to be $r(t) = f(t)/[1-F(t)]$. If F satisfies (1.2), then clearly $r(t) = \theta^{-1}$ which is constant in time.

Suppose $h(t) = \theta^{-1}$, then $\ln[1 - F(t)] = -\theta^{-1}t$. Consequently, $F(t) = 1 - \exp(-t/\theta)$.

The essence of Theorem 1.4 is that if an object has an exponential life distribution, then the age of the object is irrelevant to its failure, it is as good as new.

Several distributions will be used in the forthcoming pages, they shall be introduced and discussed below.

By the gamma distribution, it is meant that a random variable X having density function (p.d.f.)

$$(1.5) \qquad f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}, & 0 < x < \infty, \quad \alpha,\beta > 0 \\ 0, & \text{otherwise,} \end{cases}$$

and characteristic function

$$(1.6) \qquad \phi_X(t) = (1 - i\beta t)^{-\alpha}$$

where α is the shape parameter and β is the scale parameter. Denoted symbolically by $X \sim \Gamma(\alpha,\beta)$. Several distributions arise from the gamma distribution, such as the Erlang distribution - with α being a positive integer; the exponential distribution - with $\alpha = 1$; and the chi-square distribution - with $\alpha = n/2$, $n \geq 1$ and $\beta = 2$. In reliability, the gamma failure rate is monotone over time, it is a decreasing failure rate (DFR) for $0 < \alpha \leq 1$ and is an increasing failure rate (IFR) for $\alpha \geq 1$.

A well-known property of the gamma distribution may be seen in the following corollary.

LEMMA 1.5: Let $X_1, X_2, \ldots, X_n$ be n independent random variables with $X_i \sim \Gamma(\alpha_i, \beta)$, $i=1,2,\ldots,n$. Then

$$Y_j = \sum_{i=1}^{j} X_i \sim \Gamma\left(\sum_{i=1}^{j} \alpha_i, \beta \right), \quad j=1,2,\ldots,n .$$

The next corollary ensures how one may obtain a random variable having the F-distribution from two independent chi-square random variables.

LEMMA 1.6: Let X_1 and X_2 be two independent chi-square random variables with ν_1 and ν_2 degrees of freedom, denoted by $\chi^2(\nu_1)$ and $\chi^2(\nu_2)$ respectively. Then

$$Y = \frac{X_1 / \nu_1}{X_2 / \nu_2}$$

has the F-distribution with ν_1 and ν_2 degrees of freedom.

The Weibull distribution, one of the most widely employed distributions, is used to describe experimentally observed variations in the fatigue resistance and elastic limits of steel, in lengths of service time of electronic components, etc. Its p.d.f. is

$$(1.7) \qquad f(x) = \begin{cases} \frac{\gamma}{\theta}\left(\frac{x-\mu}{\theta}\right)^{\gamma-1} \exp\left\{-\left(\frac{x-\mu}{\theta}\right)^{\gamma}\right\}, & x \geq \mu,\ \theta,\gamma > 0,\ \mu \in R \\ 0 & , \text{otherwise} \end{cases}$$

where γ, θ and μ are the shape, scale and location parameters respectively.[a] Special cases are the Raleigh distribution - with $\gamma = 2$ and the exponential distribution - with $\gamma = 1$. The Weibull failure rate is also monotone over time, IFR for $\gamma \geq 1$ and DFR for $0 < \gamma \leq 1$.

CHAPTER II

REVIEW OF LITERATURE ON THE TEST OF EXPONENTIALITY

Among the numerous non-parametric tests of exponentiality, one of the most celebrated is the chi-square test for goodness of fit, originated by Karl Pearson in 1900. The test is easy to employ, it may be utilized for discrete or continuous data and it is flexible - in the sense that it may be modified to allow estimation of parameters from the data. However, in reducing the problem to a parametric form, grouping and discretizing the data is required, resulting in the loss of information. Hence, it is more suitable for large samples and is less powerful for certain families of distributions.

Another well-known non-parametric method is the Kolmogorov-Smirnov test. Although it requires the assumption that the underlying distribution function is continuous and is less flexible, it does have the appeal of giving a refined analysis of the data and is applicable to small samples.

Three main categories may be assigned to the methods of obtaining test statistics with distributions independent of the unknown scale parameter θ and consequently, significant points independent of θ. Namely, "the ratio type methods", "the Kolmogorov-Smirnov type methods" and "the rank type methods".

2.1 "THE RATIO TYPE METHODS".

As the name suggests, test statistics in this category are formed by taking the ratio of two other statistics. By grouping the samples appropriately, the distribution of these test statistics may be made to be independent of the scale parameter θ .

Csörgö, Seshadri and Yalosky [6], Epstein [10], Gnedenko, Belyayev and Solovyeu [13], Hartley [15] and Shapiro and Wilk [32] are among those who have taken such an approach. Except for [32], the others based their statistics on the normalized spacings (2.2) and whose distributions depend on the well-known facts which shall be stated below as a lemma.

LEMMA 2.1: Let $X_1, X_2, \ldots, X_n$ be $n \geq 2$ independent and identically distributed random variables (i.i.d. r.v.'s) with c.d.f. (1.2). Define $X_{0,n} \equiv 0$ and let $X_{1,n} < X_{2,n} < \cdots < X_{n,n}$ denote the corresponding order statistics. Then each and every one of the following is true.

(a) The spacings

$$\bar{D}_i = X_{i,n} - X_{i-1,n} , \quad i=1,2,\ldots,n \tag{2.1}$$

are independent exponential random variables with parameters $(n-i+1)/\theta$, $i=1,2,\ldots,n$.

(b) The normalized spacings

$$D_i = (n-i+1)(X_{i,n} - X_{i-1,n}) , \quad i=1,2,\ldots,n \tag{2.2}$$

are i.i.d. r.v.'s with c.d.f. (1.2).

(c) $2D_i/\theta$, $i=1,2,\ldots,n$ are i.i.d. $\chi^2(2)$ r.v.'s.

<u>Proof.</u> The joint p.d.f. of the order statistics $X_{1,n}, X_{2,n}, \ldots, X_{n,n}$

is given by

$$g(x_{1,n},x_{2,n},\dots,x_{n,n}) = \begin{cases} n! \prod_{i=1}^{n} f(x_{i,n}) \ , & 0 < x_{1,n} < \cdots < x_{n,n} < \infty \\ 0 & , \text{ otherwise} \end{cases}$$

where f is the p.d.f. of the r.v.'s $X_1, X_2, \dots, X_n$. By (2.1), $x_{k,n} = \sum_{i=1}^{k} \bar{D}_i$, $k=1,2,\dots,n$, with Jacobian $|J| = 1$. Hence, the joint p.d.f. of the spacings is

$$\bar{h}(\bar{d}_1,\bar{d}_2,\dots,\bar{d}_n) = \frac{n!}{\theta^n} \exp\left(-\frac{1}{\theta} \sum_{i=1}^{n} (n-i+1)\bar{d}_i\right), \quad 0 < \bar{d}_i < \infty .$$

Clearly, the marginal p.d.f. of each of the spacings is

$$\bar{h}_j(\bar{d}_j) = \int_0^\infty \cdots \int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{n!}{\theta^n} \exp\left(-\frac{1}{\theta} \sum_{i=1}^{n} (n-i+1)\bar{d}_i\right) d\bar{d}_1 \dots d\bar{d}_{j-1} d\bar{d}_{j+1} \dots d\bar{d}_n \tag{2.3}$$

$$= \frac{(n-j+1)}{\theta} \exp\{-\frac{1}{\theta}(n-j+1)\bar{d}_j\} \ , \quad \bar{d}_j > 0 \ , \quad \theta > 0$$

and the spacings $\bar{D}_1, \bar{D}_2, \dots, \bar{D}_n$ are independent since

$$\bar{h}(\bar{d}_1,\bar{d}_2,\dots,\bar{d}_n) = \prod_{i=1}^{n} \bar{h}_i(\bar{d}_i) .$$

(b) The right hand side of (2.2) may be written as $(n-i+1)\bar{D}_i$, so $\bar{d}_i = d_i/(n-i+1)$ and $|J| = 1/(n-i+1)$, $i=1,2,\dots,n$. Applying the transformation into (2.3) yields

$$h_i(d_i) = \theta^{-1}\exp(-d_i/\theta) \ , \quad d_i > 0 \ , \quad \theta > 0 \ , \quad i=1,2,\dots,n$$

which implies that the normalized spacings $D_1, D_2, \dots, D_n$ are identically distributed random variables and their independence follows from the independence of the spacings.

(c) For each $i=1,2,\ldots,n$, the characteristic function of D_i is $\phi_{D_i}(t) = (1-i\theta t)^{-1}$, therefore the characteristic function of $2D_i/\theta$ is $(1-2it)^{-1}$ which is the characteristic function of a $\chi^2(2)$ random variable.

Csörgö et al [6] proposed several tests based on the statistic

$$Z_{r:n-1} = \frac{Y_r}{Y_n} \quad \text{where} \quad Y_r = \sum_{i=1}^{r} D_i \, , \quad r = 1,2,\ldots,n-1 \, .$$

It is evident that under H_0, $Z_{r:n-1}$, $r=1,2,\ldots,n-1$ are the $n-1$ order statistics of $n-1$ i.i.d. $U(0,1)$ r.v.'s, since the joint density of $(D_1,D_2,\ldots,D_n)$ is $\exp(-\sum_{i=1}^{n} D_i/\theta)/\theta^n$, if one is to let $Z_{n:n-1} = Y_n$, then the Jacobian $|J| = z_{n:n}^{n-1}$ and the joint density of $(Z_{1:n-1},Z_{2:n-1},\ldots,Z_{n:n-1})$ becomes

$$\frac{z_{n:n}^{n-1} e^{-z_{n:n}/\theta}}{\theta^n} \, , \quad 0 < z_{1:n-1} < \cdots < z_{n-1:n-1} < 1,$$
$$0 < z_{n:n-1} < \infty \, ,$$

integrating over $z_{n:n-1}$ gives us the joint marginal of $(Z_{1:n-1},z_{2:n-1},\ldots,z_{n-1:n-1})$ as

$$\Gamma(n) \, , \quad 0 < z_{1:n-1} < z_{2:n-1} < \cdots < z_{n-1:n-1} < 1$$

which is the joint p.d.f. of the order statistics of $n-1$ i.i.d. $U(0,1)$ r.v.'s.

Based on the statistic defined as follows

$$(2.4)\qquad Q(r,n-r) = \frac{\sum_{i=1}^{r} D_i/r}{\sum_{j=r+1}^{n} D_j/(n-r)}\ ,\quad 1 \le r < n$$

Gnedenko et al [13] proposed the "G-B-S test". Under H_0, by Lemma 2.1c and Lemma 1.5, $2\sum_{i=1}^{r} D_i/\theta \sim \chi^2(2r)$, $2\sum_{j=r+1}^{n} D_j/\theta \sim \chi^2(2(n-r))$, and the two variables are independent. Consequently, by virtue of Lemma 1.6, $Q(r,n-r)$ has the F-distribution with $2r$ and $2(n-r)$ degrees of freedom.

Let k and r be positive integers so that $n=kr$. Define

$$G_m = \sum_{i=(m-1)r+1}^{mr} D_i\ ,\quad m=1,2,\ldots,k$$

and

$$F_{max} = \frac{\max_{1 \leqslant j \leqslant k} G_j}{\min_{1 \leqslant m \leqslant k} G_m}\ .$$

Then, under H_0, by Lemma 2.1c and Lemma 1.5, $2G_m/\theta$, $m=1,2,\ldots,k$ are i.i.d. r.v.'s having the chi-square distribution with $2r$ degrees of freedom and hence F_{max} has the maximum F-ratio distribution with $2r$ and k degrees of freedom. Hartley [15] developed his test based on this statistic. At 95% and 99% levels of significance, and for some values of k and r, the critical values of the maximum F-ratio distribution may be found in Pearson and Hartley [26], page 202.

Tests which were based on the statistics

$$E_r = \frac{2rk\left(\ln \frac{\sum_{m=1}^{k} G_m}{k} - \frac{1}{k}\sum_{m=1}^{k} \ln G_m\right)}{1 + \frac{k+1}{6rk}} \quad , \quad r \geq 1$$

were proposed by Epstein [10]. To which he showed that under H_0, E_r are distributed approximately as chi-square variables with $k-1$ degrees of freedom.

The statistic, W-exponential, proposed by Shapiro and Wilk [32] is defined as

$$W = \frac{n(\bar{X} - X_{1,n})^2}{(n-1)S^2}$$

where $\bar{X}$ and S^2 are the sample mean and the sample variance respectively.

2.2 "THE KOLMOGOROV-SMIRNOV TYPE METHODS".

Let F be the c.d.f. of the random sample $X_1, X_2, \ldots, X_n$ which is completely specified and assumed to be continuous, and let F_n be their sample distribution function, that is,

$$F_n(x) = (\text{the number of } X_1, X_2, \ldots, X_n \leq x)/n$$

$$= \begin{cases} 0 & \text{if } x < X_{1,n} \\ i/n & \text{if } X_{i,n} \leq x < X_{i+1,n}, \quad i=1,2,\ldots,n-1 \\ 1 & \text{if } x \geq X_{n,n}. \end{cases}$$

Define the random variables

$$D_n = \sup\{|F_n(x) - F(x)| ; x \in R\}$$

(2.5) $$D_n^+ = \sup\{ F_n(x) - F(x) ; x \in R\}$$

$$D_n^- = \sup\{ F(x) - F_n(x) ; x \in R\} .$$

Then, by the Glivenko-Cantelli Theorem, as $n \to \infty$, D_n, D_n^+ and D_n^- tend to zero almost surely under H_0. Hence H_0 is rejected if $D_n > c$, $D_n^+ > c^+$ and $D_n^- > c^-$ respectively, where the constants c, c^+ and c^- are determined by $P\{D_n > c \mid H_0\} = \alpha'$, $P\{D_n^+ > c^+ \mid H_0\} = \alpha'$ and $P\{D_n^- > c^- \mid H_0\} = \alpha'$, where α' is the level of significance. Such is the well-known Kolmogorov-Smirnov one sample test.

In testing exponentiality, if θ is known, then F is replaced by (1.2). Suppose θ is not known, then the critical values for the above mentioned conventional Kolmogorov-Smirnov test no longer applies, instead, new critical values must be used. Investigations of methods of calculating such critical values have been done by, among others, Durbin [8], Finklestein and Schafer [12], Lilliefors [19], Srinivasan [34] and Stephens [27].

Since θ uniquely determines the mean of the exponential distribution and it is well-known that $\bar{X}$, the sample mean, is the minimum variance unbiased estimate of the population mean; hence, in the case when θ is not known, $\bar{X}$ may be used to replace θ in (1.2). Consequently, (2.5) becomes

$$D_n^* = \sup\{|F_n(x) - (1 - e^{-x/\bar{X}})| ; x > 0\}$$

$$D_n^{*+} = \sup\{ F_n(x) - (1 - e^{-x/\bar{X}}) ; x > 0\}$$

$$D_n^{**} = \sup\{(1-e^{-x/\bar{X}}) - F_n(x) \; ; \; x > 0\} .$$

As a result of using Monte Carlo simulations for modest sample sizes, Lilliefors [19] tabulated some critical values of D_n^*. Stephens [27], supplemented by smoothing and other devices, carried out a similar but much more extensive experiment.

Define

$$\tilde{D}_n = \max_{1 \le i \le n} |F_n(x_i) - \tilde{F}(x_i;\theta)|$$

where $\tilde{F}(x_i;\theta) = 1 - (1 - x_i/n\bar{X})^{n-1}$ is the conditional expectation of the indicator function of the event $\{X_i \le x_i\}$ given that $\bar{X} = \bar{x}$, Srinivasan [34] used a Monte Carlo simulation to calculate and tabulate the critical values of $\tilde{D}_n$. Note that the distribution of $\tilde{D}_n$ under H_0 is independent of θ since for each i, by the transformation $Y_i = X_i/\theta$, Y_i has the standard exponential distribution.

Finklestein and Schafer [12] used the statistic

$$\tilde{S}_n = \sum_{i=1}^{n} |\delta_i|$$

where $\delta_i = \max_{1 \le i \le n} \{i/n - [1 - \exp(-X_i/\bar{X})] , \; 1 - \exp(-X_i/\bar{X}) - (i-1)/n\}$. Percentage points were also calculated by means of Monte Carlo methods and tabulated.

Durbin [8] developed a method of calculating the distribution function of D_n^*, D_n^{+*} and D_n^{-*} under H_0 by means of the Fourier transform. Percentage points for sample sizes $n = 2(1)10(2)30(5)50(10)100$ were tabulated.

2.3 "THE RANK TYPE METHODS".

This approach was taken by, among others, Proschan and Pyke [28], Bickel and Doksum [3] and Bickel [2].

In the case of constant failure rate versus monotone increasing failure rate, Proschan and Pyke [26] proposed the test statistic

$$V_n = \sum_{\substack{i,j=1 \\ i<j}}^{n} V_{i,j}$$

where $V_{i,j}$ is the indicator function of the event $\{D_i \geq D_j \ ; \ i,j=1,2,\ldots,n\}$. The distribution of V_n is independent of θ and is known. Tables for $P\{V_n \leq k\}$, $k > 0$ for $n \leq 10$ are given in Kendall [17] and Mann [21].

Let $R_1,R_2,\ldots,R_n$ be the ranks of the normalized spacings $D_1,D_2,\ldots,D_n$. Bickel and Doksum [3] proposed test statistics which are linear functions of $-\ln[1 - (n+1)^{-1}R_i]$, $i=1,2,\ldots,n$. As an example of an application, Bickel and Doksum considered four specific alternatives - Makeham, linear failure rate, Weibull and gamma, and eight specific statistics, among which are

$$W_0 = \sum_{i=1}^{n} \frac{i}{n+1} \frac{R_i}{n+1}$$

$$W_1 = \sum_{i=1}^{n} - \frac{i}{n+1} [-\ln(1 - \frac{R_i}{n+1})]$$

$$W_3 = \sum_{i=1}^{n} [-\ln(1 - \frac{i}{n+1})][-\ln(1 - \frac{R_i}{n+1})] .$$

Their efficiencies and Monte Carlo powers are tabulated in [3] .

CHAPTER III

A CHARACTERIZATION THEOREM OF THE EXPONENTIAL DISTRIBUTION

The development of the main theorem, Theorem 3.5, was motivated by Lukacs' celebrated characterization theorem of the gamma distribution [20], Theorem 3.1. Several well-known facts are required in the proofs of Theorem 3.1 and Theorem 3.5, they shall be stated below without proof.

F1. The characteristic function of a random variable always exists.

F2. There is a one-to-one correspondance between the characteristic function and the distribution function of a random variable.

F3. The Laplace transform of $q(\cdot)$ is defined as

$$Q(t) = \int_0^\infty e^{-tp} q(p)dp$$

where $0 \leq p < \infty$ and $t = a+ib$. Then, for some non-negative number c, Q and $Q^{(k)}$, $k=1,2,...$ are analytic in the half-plane $Re(t) > c$.

F4. All moments of a bounded random variable exist.

F5. If $X_1, X_2, \ldots, X_n$ are n independent random variables and ϕ_{X_i} denotes the characteristic function of X_i, $i=1,2,\ldots,n$. Then

$$\phi_{\sum_{i=1}^n X_i}(t) = \prod_{i=1}^n \phi_{X_i}(t).$$

F6. The characteristic functions f and g, defined in (3.3), do not vanish in the half-plane $Im(t) \geq 1$.

F7. If g_1 and g_2 are measurable functions of two independent r.v.'s X and Y, respectively, then $g_1(X)$ is independent of $g_2(Y)$.

THEOREM 3.1: (LUKACS). Let X and Y be two non-degenerate and positive random variables, and suppose that they are independently distributed. The random variables $U = X+Y$ and $V = X/Y$ are independently distributed if and only if both X and Y have gamma distributions with the same scale parameter.

Proof. To prove that the independence of U and V implies that X and Y have gamma distributions with a common scale parameter, define a new random variable

(3.1) $$W = \frac{1}{1+V} = \frac{Y}{X+Y}$$

then $0 \leq W \leq 1$, consequently, by F4, all moments of W exist. Denote by

(3.2) $$\theta_1 = E(W) \quad \text{and} \quad \theta_2 = E(W^2)$$

where E is the expectation operator.

Let F, G, H_1 and H_2 denote the distribution functions of the random variables X, Y, U and W respectively, and let H denote the joint c.d.f. of the random variables U and W. The non-negativity of X and Y implies that their characteristic functions are

(3.3) $$f(t) = \int_0^\infty e^{itx} dF(x) , \quad g(t) = \int_0^\infty e^{ity} dG(y)$$

which exist not only for real t but also for $t = a + ib$, $b \geq 0$;
By F3, f and g are analytic for $b = \mathrm{Im}(t) > 0$ and so are

$$
(3.4)\qquad
\begin{aligned}
f'(t) &= i\int_0^\infty xe^{itx}dF(x) , \quad f''(t) = -\int_0^\infty x^2e^{itx}dF(x) \\
g'(t) &= i\int_0^\infty ye^{ity}dG(y) , \quad g''(t) = -\int_0^\infty y^2e^{ity}dG(y) .
\end{aligned}
$$

Since $U = X+Y$ and $V = X/Y$ are assumed to be independent, by virtue of (3.1), so are U and W ; therefore by F5,

$$E\{\exp(itU + isW)\} = E\{\exp(itU)\}E\{\exp(isW)\}$$

or

$$
(3.5)\qquad
\begin{aligned}
&\int_0^\infty\int_0^\infty \exp\left(it(x+y) + \frac{isy}{x+y}\right) dF(x)dG(y) \\
&= \int_0^\infty\int_0^\infty \exp\left(it(x+y)\right)dF(x)dG(y) \int_0^\infty\int_0^\infty \exp\left(\frac{isy}{x+y}\right)dF(x)dG(y)
\end{aligned}
$$

which are analytic in t and in s if $\mathrm{Im}(t) > 0$. We shall assume that $\mathrm{Im}(t) > 0$ and restrict ourselves to the half-plane $\mathrm{Im}(t) \geq 1$.

To establish the first of the two relations

$$(3.6)\qquad [g(t)]^{1-\theta_1} = [f(t)]^{\theta_1} ,$$

differentiate (3.5) twice, first with respect to t and then with respect to s to obtain

$$(3.7) \qquad \int_0^\infty\int_0^\infty y \exp\left(it(x+y) + \frac{isy}{x+y}\right)dF(x)dG(y)$$

$$= \int_0^\infty\int_0^\infty (x+y)\exp\left(it(x+y)\right)dF(x)dG(y) \cdot \int_0^\infty\int_0^\infty \frac{y}{x+y} \exp\left(\frac{isy}{x+y}\right)dF(x)dG(y) .$$

Next, set $s=0$ and use the notation in (3.2) to arrive at

$$\int_0^\infty\int_0^\infty y \exp[it(x+y)]dF(x)dG(y)$$

$$= \theta_1 \int_0^\infty\int_0^\infty (x+y)\exp[it(x+y)]dF(x)dG(y) .$$

Making use of (3.3) and (3.4), the following is obtained

$$ig'(t)f(t) = \theta_1[f'(t)g(t) + g'(t)f(t)]$$

or

$$(1-\theta_1)g'(t)f(t) = \theta_1 f'(t)g(t) , \quad \mathrm{Im}(t) > 0 .$$

By F6, one may divide the above equation to get

$$(3.8) \qquad (1-\theta_1)\frac{g'(t)}{g(t)} = \theta_1 \frac{f'(t)}{f(t)}$$

Solving the above differential equation with the initial conditions $f(0) = g(0) = 1$, (3.6) is obtained.

The second relation

$$\text{(3.9)} \qquad \frac{g''(t)}{g(t)} = \theta_2\left(\frac{f''(t)}{f(t)} + 2\,\frac{f'(t)}{f(t)}\,\frac{g'(t)}{g(t)} + \frac{g''(t)}{g(t)}\right)$$

is established in a similar manner - differentiating (3.7) with respect to t then with respect to s to get

$$\int_0^\infty\int_0^\infty y^2 \exp\left(it(x+y) + \frac{isy}{x+y}\right) dF(x)dG(y)$$

$$= \int_0^\infty\int_0^\infty (x+y)^2 \exp[it(x+y)]dF(x)dG(y) \cdot \int_0^\infty\int_0^\infty\left(\frac{y}{x+y}\right)^2 \exp\left(\frac{isy}{x+y}\right)dF(x)dG(y) .$$

Then, set $s=0$ and the notation in (3.2) is incorporated to obtain

$$\int_0^\infty\int_0^\infty y^2 \exp[it(x+y)]dF(x)dG(y)$$

$$= \theta_2 \int_0^\infty\int_0^\infty (x+y)^2 \exp[it(x+y)]dF(x)dG(y) .$$

Finally, by substituting in (3.3) and (3.4) then dividing by $f(t)g(t)$, (3.8) is attained.

Next, the following notations are introduced

$$\text{(3.10)} \qquad \begin{array}{ll} \phi(t) = \ln f(t) & \Phi(t) = \ln g(t) \\ \dfrac{f'(t)}{f(t)} = \phi'(t) & \dfrac{g'(t)}{g(t)} = \Phi'(t) \\ \dfrac{f''(t)}{f(t)} = \phi''(t) + [\phi'(t)]^2 & \dfrac{g''(t)}{g(t)} = \Phi''(t) + [\Phi'(t)]^2 \end{array}$$

By making use of (3.10), (3.8) may be developed into

(3.11) $$\phi'(t) = \frac{\theta_1}{1-\theta_1}\,\phi'(t) \quad \text{and} \quad \phi''(t) = \frac{\theta_1}{1-\theta_1}\,\phi''(t)$$

and (3.9) may be rewritten as

(3.12) $$(1-\theta_2)\{\phi''(t) + [\phi'(t)]^2\} = \theta_2\{\phi''(t) + [\phi'(t)]^2 + 2\phi'(t)\phi'(t)\}$$

By substituting (3.11) into (3.12), the following differential equation is obtained

(3.13) $$(1-\theta_1)(\theta_1 - \theta_2)\phi''(t) = (\theta_2 - \theta_1)^2[\phi'(t)]^2 .$$

Clearly, if either $\theta_1 = \theta_2$ or $\theta_2 = \theta_1^2$ then $\phi'(t) = 0$ or $\phi''(t) = 0$ which in turn implies that X and Y are degenerate random variables. Consider the case where $\theta_2 \neq \theta_1$ and $\theta_2 \neq \theta_1^2$, more precisely the case where $0 < \theta_1^2 < \theta_2 < \theta_1 < 1$. Rewrite (3.13) as

(3.14) $$\frac{\phi''(t)}{[\phi'(t)]^2} = \frac{1}{\alpha}$$

where $\alpha = (1-\theta_1)(\theta_1 - \theta_2)/(\theta_2 - \theta_1^2) > 0$ and denote

$$k_1 = E(e^{-X}) , \quad k_2 = E(Xe^{-X}) \quad \text{and} \quad \beta = k_2/(k_1\alpha - k_2) .$$

By (3.3) and (3.4), it is evident that $f(i) = k_1$ and $f'(i) = ik_2$ hence $\phi'(i) = ik_2/k_1$, using this as the initial condition, integrate (3.14) to obtain

$$\phi'(t) = i\beta\alpha(1 - i\beta t)^{-1} .$$

Integrating the above equation and keeping in mind the first set of notations in (3.10) together with the initial condition $f(0) = 1$ would give rise to

$$f(t) = (1 - i\beta t)^{-\alpha}$$

and by the first relation (3.6),

$$g(t) = (1 - i\beta t)^{-\alpha\theta_1/(1-\theta_1)} .$$

Removing the restriction and by F2, the sufficiency is proven.

To show that the converse is also true, one begins by finding the joint density of U and V. Suppose that $X \sim \Gamma(\alpha_1,\beta)$ and $Y \sim \Gamma(\alpha_2,\beta)$, then the joint density of X and Y is

$$h(x,y) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}} x^{\alpha_1-1} y^{\alpha_2-1} e^{-(x+y)/\beta} , \quad 0 < x,y < \infty .$$

Applying the transformation $U = X+Y$ and $V = X/Y$ with Jacobian $|J| = u(1+v)^{-2}$, the joint density of U and V is

$$h^*(u,v) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}} u^{\alpha_1+\alpha_2-1} e^{-u/\beta} \frac{v^{\alpha_1-1}}{(1+v)^{\alpha_1+\alpha_2}} ,$$

$$0 < u < \infty , \quad 0 < v < 1 .$$

By the assumption and Lemma 1.5, $U \sim \Gamma(\alpha_1+\alpha_2,\beta)$, that is, the p.d.f. of U is

$$\eta(u) = \frac{1}{\Gamma(\alpha_1+\alpha_2)\beta^{\alpha_1+\alpha_2}} u^{\alpha_1+\alpha_2-1} e^{-u/\beta} , \quad 0 < u < \infty .$$

The p.d.f of V may be found by first applying the transformation $V = X/Y$ and Y with Jacobian $|J| = y$ and joint density

$$\zeta^*(v,y) = \frac{1}{\Gamma(\alpha_1+\alpha_2)\beta^{\alpha_1+\alpha_2}} v^{\alpha_1-1} y^{\alpha_1+\alpha_2-1} e^{-(1+v)y/\beta} ,$$

$$0 < y < \infty, \; 0 < v < 1 .$$

Then the p.d.f. of V is found to be

$$\zeta(v) = \frac{v^{\alpha_1-1}}{\Gamma(\alpha_1+\alpha_2)} \int_0^\infty \frac{1}{\beta^{\alpha_1+\alpha_2}} y^{\alpha_1+\alpha_2-1} e^{-(1+v)y/\beta} dy , \quad 0 < v < 1$$

$$= \frac{v^{\alpha_1-1}\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)(1+v)^{\alpha_1+\alpha_2}} \int_0^\infty \frac{1}{\Gamma(\alpha_1+\alpha_2)\beta^{\alpha_1+\alpha_2}} z^{\alpha_1+\alpha_2-1} e^{-z/\beta} dz$$

$$= \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \frac{v^{\alpha_1-1}}{(1+v)^{\alpha_1+\alpha_2}} , \quad 0 < v < 1 .$$

Since

$$\eta(u)\zeta(v) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}} u^{\alpha_1+\alpha_2-1} e^{-u/\beta} \frac{v^{\alpha_1-1}}{(1+v)^{\alpha_1+\alpha_2}} ,$$

$$0 < u < \infty, \; 0 < v < 1$$

$$= h^*(u,v)$$

therefore, the random variables U and V are independent. This establishes the necessary condition and the proof is complete.

COROLLARY 3.2: The condition " U and V are independent" in Theorem 3.1 is equivalent to " U and $Z = X/(X+Y)$ are independent" .

Proof. Let

$$Z = \frac{V}{1+V} = \frac{X}{X+Y}$$

Since Z is a function of V alone, therefore the independence of U and V is valid if and only if U and Z are independent.

In 1974, Marsaglia [23] proposed a slightly more general version of Theorem 3.1, for comparison purposes, it shall be cited below as Theorem 3.3.

THEOREM 3.3: If X and Y are independent and non-degenerate random variables, then $X+Y$ is independent of $X/(X+Y)$ if and only if X and Y or $-X$ and $-Y$ have gamma distributions with the same scale parameter.

The last required result is a theorem relating the gamma distribution to the beta distribution.

THEOREM 3.4: Let X and Y be two independent random variables having gamma distributions with β as their common scale parameter, and α_1 and α_2 as their location parameters respectively. Then $Z = X/(X+Y)$

has the beta distribution with parameters α_1 and α_2, denoted by $B(\alpha_1,\alpha_2)$. Its p.d.f. is given by

$$f(z) = \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} z^{\alpha_1-1}(1-z)^{\alpha_2-1}, \quad 0 < z < 1 .$$

3.1 THE MAIN THEOREM.

Having established the necessary foundations, we proceed to state and prove the main theorem.

THEOREM 3.5: Let $X_1,X_2,\ldots,X_n$ be $n \geq 3$ i.i.d. r.v.'s. Define $Z_k = (S_k/S_{k+1})^k$, $k=1,2,\ldots,n-1$, where $S_m = \sum_{i=1}^{m} X_i$, $m=1,2,\ldots,n$. Then, Z_i and Z_j are i.i.d. $U(0,1)$ r.v.'s, $1 \leq i < j \leq n-1$, if and only if $X_1,X_2,\ldots,X_n$ or $-X_1,-X_2,\ldots,-X_n$ are from the exponential distribution (1.2).

Proof. The necessary condition is proven as follows. With $Z_k = (S_k/S_{k+1})^k$, $k=1,2,\ldots,n-1$, set $Z_n = S_n$. Then, clearly,

$$s_k = z_n \prod_{i=k}^{n-1} z_i^{1/i}, \quad k=1,2,\ldots,n-1 \text{ and } s_n=z_n . \tag{3.15}$$

Define

$$S_0 \equiv 0 \text{ and } Z_0 \equiv 0 . \tag{3.16}$$

Since S_m, $m=1,2,\ldots,n$ are the partial sums of the random variables $X_1,X_2,\ldots,X_n$ hence X_i, $i=1,2,\ldots,n$ may be written as

(3.17) $$X_i = S_i - S_{i-1}, \quad i=1,2,\ldots,n .$$

Incorporating (3.15), (3.16) and (3.17) yields

$$x_i = z_n \prod_{m=i}^{n-1} z_m^{1/m} - z_n \prod_{j=i-1}^{n-1} z_j^{1/j}, \quad i=1,2,\ldots,n-1$$

and

$$x_n = z_n - z_n z_{n-1}^{1/(n-1)} .$$

To calculate the Jacobian of the transformation $T(X_1,X_2,\ldots,X_n) = (Z_1,Z_2,\ldots,Z_n)$ is to find the absolute value of the determinant of the following $n \times n$ matrix.

$$\begin{bmatrix} \frac{\partial x_1}{\partial z_1} & \cdots & \frac{\partial x_1}{\partial z_i} & \cdots & \frac{\partial x_1}{\partial z_n} \\ \vdots & & \vdots & & \vdots \\ \frac{\partial x_i}{\partial z_1} & \cdots & \frac{\partial x_i}{\partial z_i} & \cdots & \frac{\partial x_i}{\partial z_n} \\ \vdots & & \vdots & & \vdots \\ \frac{\partial x_n}{\partial z_1} & \cdots & \frac{\partial x_n}{\partial z_i} & \cdots & \frac{\partial x_n}{\partial z_n} \end{bmatrix}$$

$$
= \begin{bmatrix}
z_n \prod_{m=2}^{n-1} z_m^{1/m} & \cdots & \frac{1}{i} z_n z_i^{-1+1/i} \prod_{\substack{m=1 \\ m \neq i}}^{n-1} z_m^{1/m} & \cdots & \prod_{m=1}^{n-1} z_m^{1/m} \\
\vdots & & \vdots & & \vdots \\
\delta_{i1} & \cdots & \frac{1}{i} z_n z_i^{-1+1/i} \prod_{m=i+1}^{n-1} z_m^{1/m} - \frac{1}{i} z_n z_i^{-1+1/i} \prod_{\substack{j=i-1 \\ j \neq i}}^{n-1} z_j^{1/j} & \cdots & \prod_{m=i}^{n-1} z_m^{1/m} - \prod_{j=i-1}^{n-1} z_j^{1/j} \\
\vdots & & \vdots & & \vdots \\
0 & \cdots & \delta_{ni} & \cdots & 1 - z_{n-1}^{1/(n-1)}
\end{bmatrix}
$$

where

$$\delta_{i1} = \begin{cases} -z_n \prod_{m=2}^{n-1} z_m^{1/m} & \text{if } i=2 \\ 0 & \text{if } i=3,4,\ldots,n \end{cases}$$

and

$$\delta_{ni} = \begin{cases} -\frac{1}{n-1} z_n z_{n-1}^{-1+1/(n-1)} & \text{if } i=n-1 \\ 0 & \text{if } i=1,2,\ldots,n-2 \ . \end{cases}$$

After (n-1) elementary row operations, the above matrix becomes an upper triangular matrix, namely

$$\begin{bmatrix} z_n \prod_{m=2}^{n-1} z_m^{1/m} & \cdots & \frac{1}{i} z_n z_i^{-1+1/i} \prod_{\substack{m=1 \\ m\neq i}}^{n-1} z_m^{1/m} & \cdots & \prod_{m=1}^{n-1} z_m^{1/m} \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & \frac{1}{i} z_n z_i^{-1+1/i} \prod_{m=i+1}^{n-1} z_m^{1/m} & \cdots & \prod_{m=i}^{n-1} z_m^{1/m} \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

where $\prod_{m=i+1}^{n-1} z_m^{1/m}$ is defined to be 1 if $i=n-1$. Since the determinant of an upper triangular matrix is the product of the elements in the main diagonal, therefore, the Jacobian $|J|$ is

$$|J| = \left| \prod_{i=1}^{n-1} \frac{1}{i} z_n z^{-1+1/i} \prod_{m=i+1}^{n-1} z_m^{1/m} \right|$$

$$= \frac{1}{\Gamma(n)} z_n^{n-1} .$$

By the assumptions, the joint density of $X_1, X_2, \ldots, X_n$ is given by

$$f(x_1, x_2, \ldots, x_n) = \frac{1}{\theta^n} \exp\left(- \sum_{i=1}^{n} x_i/\theta \right) , \quad \sum_{i=1}^{n} x_i > 0 , \quad \theta > 0 .$$

Hence, the joint density of $Z_1, Z_2, \ldots, Z_n$ is

$$(3.18) \qquad g(z_1, z_2, \ldots, z_n) = \frac{1}{\Gamma(n)\theta^n} z_n^{n-1} \exp(-z_n/\theta)$$

for $0 \le z_k \le 1$, $k=1,2,\ldots,n-1$, $0 \le z_n < \infty$ and $\theta > 0$. By integrating (3.18), it is evident that $Z_1, Z_2, \ldots, Z_{n-1}$ are identically distributed $U(0,1)$ random variables.

$$g_0(z_1, z_2, \ldots, z_{n-1}) = \prod_{k=1}^{n-1} g_k(z_k) = 1$$

where g_0 denotes the joint marginal density of $Z_1, Z_2, \ldots, Z_{n-1}$ and g_k denotes the p.d.f. of Z_k , $k=1,\ldots,n-1$ implies that the random variables $Z_1, Z_2, \ldots, Z_{n-1}$ are also independent.

We have established the fact that if $X_1, X_2, \ldots, X_n$ are i.i.d. r.v.'s with c.d.f. (1.2) then $Z_1, Z_2, \ldots, Z_{n-1}$ are i.i.d. $U(0,1)$ r.v.'s.

To prove the sufficient condition, that if $Z_1, Z_2, \ldots, Z_{n-1}$ are i.i.d. $U(0,1)$ r.v.'s then X_i, $i=1,2,\ldots,n$ are i.i.d. $\exp(0,\theta)$ r.v.'s, one may assume without loss of generality that X_i, $i=1,2,\ldots,n$ are non-degenerate and decompose S_{k+1} into

$$(3.19) \qquad S_{k+1} = -(X_{k+2} + \cdots + X_m) + \frac{X_{m+1} Z_m^{1/m}}{1 - Z_m^{1/m}}, \quad 1 \le k < m \le n-1$$

with $(X_{k+2} + \cdots + X_m) \equiv 0$ if $m = k+1$.

Define

$$U = \frac{S_k}{S_{k+1}}$$

and consider the random variables U and S_{k+1}, the former one may be decomposed into

$$U = \frac{S_k}{S_{k+1}} = \frac{S_k}{S_k + X_{k+1}}$$

$$(3.20) \qquad = \frac{X_1 + X_2 + \ldots + X_k}{X_1 + X_2 + \ldots + X_k + X_{k+1}} = Z_k^{1/k}$$

and express the latter by (3.19). Since the terms involving the random variables X_i, $i=1,2,\ldots,n$ in (3.19) and (3.20) are disjoint, furthermore, Z_k and Z_m, $1 \le k < m \le n-1$, are assumed to be independent, therefore by F7, U and S_{k+1} are independent.

By virtue of Theorem 3.3, S_k and X_{k+1} have the gamma distributions with the same scale parameter, say θ, i.e. $S_k \sim \Gamma(\alpha,\theta)$

and $X_{k+1} \sim \Gamma(\alpha_1,\theta)$. Since $S_{k+1} = S_k + X_{k+1}$ and X_{k+1} is independent of S_k, hence S_{k+1} must be an $\Gamma(\alpha_1+\alpha,\theta)$ random variable; and by Theorem 3.4, $U \sim B(\alpha,\alpha_1)$, i.e. the p.d.f. of U is

$$(3.21) \qquad h(u) = \frac{\Gamma(\alpha_1+\alpha)}{\Gamma(\alpha)\Gamma(\alpha_1)} u^{\alpha-1} (1-u)^{\alpha_1-1}, \quad 0 < u < 1 .$$

Using the fact that $Z_k \sim U(0,1)$, $k=1,2,\ldots,n-1$ and by applying the transformation $U = Z_k^{1/k}$ with Jacobian; $|J| = ku^{k-1}$, the p.d.f. of U is found to be

$$(3.22) \qquad h(u) = ku^{k-1}, \quad 0 < u < 1 ,$$

Equating the right-hand side of (3.21) to that of (3.22) yields

$$(3.23) \qquad \frac{\Gamma(\alpha_1+\alpha)}{\Gamma(\alpha)\Gamma(\alpha_1)} u^{\alpha-1} (1-u)^{\alpha_1-1} = ku^{k-1}, \quad 0 < u < 1 .$$

It is evident that (3.23) holds if and only if $\alpha_1 = 1$ and $\alpha = k$, i.e. $X_{k+1} \sim \Gamma(1,\theta)$ and $S_k \sim \Gamma(k,\theta)$, $k=1,2,\ldots,n-1$ but since $S_1 = X_1$, therefore $X_i \sim \Gamma(1,\theta)$ or equivalently, $X_i \sim Exp(0,\theta)$, $i=1,2,\ldots,n$. X_i, $i=1,2,\ldots,n$ are independent since the joint density of $X_1,X_2,\ldots,X_n$ is also the p.d.f. of S_n.

The sufficiency condition is established and the theorem is proven.

The following corollary is an immediate consequence of Theorem 3.5, of which several non-parametric tests originate.

COROLLARY 3.6: Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n \geq 3$ from a population having distribution function F such that $F(x) = 0$ for $x \leq 0$. Define $Z_k = (S_k/S_{k+1})^k$, $k=1,2,\ldots,n-1$ with $S_m = \sum_{i=1}^{m} X_i$, $m=1,2,\ldots,n$. Then F is the exponential distribution (1.2) if and only if $Z_1, Z_2, \ldots, Z_{n-1}$ are $n-1$ mutually independent and identically distributed $U(0,1)$ random variables.

It should be noted here that the condition, "$n \geq 3$", in both Theorem 3.5 and Corollary 3.6 is essential, for the results need not be true for $n < 3$. It is easily seen that both F, defined by (1.2), and $K(x) = (1/\theta x^2)\exp(-1/\theta x)$, $x > 0$, $\theta > 0$ would give rise to $Z_1 = X_1/(X_1 + X_2)$ being an $U(0,1)$ random variable.

CHAPTER IV

TEST PROCEDURES AND SIMULATIONS

4.1 TEST PROCEDURES.

The essence of Corollary 3.6 is that to test the null hypothesis (1.2) based on the random sample $X_1, X_2, \ldots, X_n$ is equivalent to using the i.i.d. r.v.'s $Z_1, Z_2, \ldots, Z_{n-1}$ to test the null hypothesis H_0' :

$$g(z) = \begin{cases} 1 & \text{if } 0 \le z \le 1 \\ 0 & \text{otherwise} . \end{cases} \tag{4.1}$$

In doing the actual simulations of the tests, it is found that when the alternative distribution is IFR then the values after transformation $z_1, z_2, \ldots, z_{n-1}$ tend to cluster around the ends of the interval [0,1] ; whereas when the alternative distribution is DFR then they tend to centre around the mid-point region of [0,1] . In either case, it would result in large values of the test statistics and consequently, less sensitive tests. In order to sensitize the tests, the following transformation is applied to the statistics $Z_1, Z_2, \ldots, Z_{n-1}$

$$h(Z) = \begin{cases} 2Z & \text{if } 0 \le Z \le \frac{1}{2} \\ 2(1-Z) & \text{if } \frac{1}{2} < Z \le 1 . \end{cases} \tag{4.2}$$

The tests are invariant under transformation (4.2) as it is shown in Lemma 4.1 infra.

LEMMA 4.1: A r.v. X is distributed uniformly over the interval [0,1] if and if the random variable Y is, where Y is defined to be

$$Y = \begin{cases} 2X & \text{if } 0 \le X \le \frac{1}{2} , \\ 2(1-X) & \text{if } \frac{1}{2} < X \le 1 . \end{cases}$$

Proof. ($\Rightarrow$) With $X \sim U(0,1)$ if and only if

$$P\{X \le x\} = x , \quad 0 \le x \le 1$$

also that the intervals $[0,\frac{1}{2}]$ and $(\frac{1}{2},1]$ are disjoint, we have

$$\begin{aligned} P\{Y \le y\} &= P\{2X \le y\} + P\{2(1-X) \le y\} \\ &= P\{X \le \tfrac{1}{2}y\} + P\{X \ge 1-\tfrac{1}{2}y\} \\ &= \tfrac{1}{2}y + \tfrac{1}{2}y = y , \qquad 0 \le y \le 1 . \end{aligned}$$

The converse ($\Leftarrow$) may be shown by means of a straightforward transformation.

LEMMA 4.2: If $X \sim U(0,1)$ then $-2 \ln X \sim \chi^2(2)$.

Proof. Let $Y = -2 \ln X$, then

$$x = \exp(-\tfrac{1}{2}y) \quad \text{and} \quad |J| = \tfrac{1}{2}\exp(-\tfrac{1}{2}y) .$$

Since

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

therefore the p.d.f. of Y' is

$$g(y) = \begin{cases} \frac{1}{2} e^{-y/2} , & y > 0 \\ 0 , & \text{otherwise} \end{cases}$$

which is the chi-square density function with two degrees of freedom.

Let H denote the c.d.f. of the uniform distribution over $[0,1]$ and let H_n denote the sample distribution function of $h(Z_1)$, $h(Z_2),\ldots,h(Z_{n-1})$. Define

(4.3) $$\chi_n = -2 \sum_{i=1}^{n-1} \ln(h(Z_i)) .$$

Then under the null hypothesis H_0' , χ_n has the chi-square distribution with $2(n-1)$ degrees of freedom by Lemma 4.2 and Lemma 1.5 .

Based on H , H_n and χ_n , the following two groups of test procedures are developed:

I. C_n : Two-sided test based on χ_n .

C_n^+ : Upper one-sided test based on χ_n .

C_n^- : Lower one-sided test based on χ_n .

II. D_n : Two-sided Kolmogorov-Smirnov test based on $\sup_x |H_n(x)-H(x)|$.

D_n^+ : One-sided Kolmogorov-Smirnov test based on $\sup_x (H_n(x)-H(x))$.

D_n^- : One-sided Kolmogorov-Smirnov test based on $\sup_x (H(x)-H_n(x))$.

4.2 SIMULATIONS AND CONCLUSIONS.

The gamma distribution and the Weibull distribution, both with shape parameters $\alpha = .5, .8, 1.5, 2.0, 3.0$, were chosen to be the alternative distributions in the computer simulations. Subroutines from the IMSL packages were used to generate random samples of sizes $n = 4$, 6, 10, 16, 20(10)40, 60 from each of the alternative distributions. After having repeated the experiment one thousand times for each of the eighty combinations of α and n at 95% level of significance, the powers were calculated. These results, along with those corresponding to the tests proposed by Gnedenko et al (with $r = n/2$) and Durbin, may be found in Tables I and II.

From the results of the simulations, it is found that with the two above mentioned alternatives, C_n is the overall best test for the alternative hypothesis $F(x) \neq 1 - \exp(-x/\theta)$, except when $\alpha = 1.5$, 2.0 and $n = 4, 6, 10$ for which it is slightly weaker than Durbin's D_n^*. C_n^+ and C_n^- are, respectively, the overall best tests for alternative hypotheses

$$F \text{ is IFR} \quad \text{and} \quad F \text{ is DFR} .$$

Also, both C_n and D_n , like "the G-B-S test", provide information on whether F is IFR or DFR in the following way: Suppose the test procedure C_n is applied, then $C_n > \chi^2_{1-\alpha'/2}$ would imply that F is DFR whereas $C_n < \chi^2_{\alpha'/2}$ would imply that F is IFR.

For $n = 30$ and all values of α , Figures I and II (III and IV) are the power curves of C_n , C_n^+ and C_n^- (D_n , D_n^+ and D_n^-) for the Weibull and the gamma alternatives respectively.

A copy of the simulation programme may be found in the Appendix.

TABLE I

POWER COMPARISONS OF TESTING EXPONENTIALITY AT 5% LEVEL
WHEN THE ALTERNATIVE IS WEIBULL WITH SHAPE PARAMETER α

	NEW PROCEDURES						EXISTING PROCEDURES			
n	C_n	C_n^+	C_n^-	D_n	D_n^+	D_n^-	G-B-S	D_n^*	D_n^{*+}	D_n^{*-}
				α =	.5					
4	.398	.498	.005	.212	.342	.006	.341	.206	.400	.006
6	.566	.636	.001	.310	.436	.003	.453	.356	.497	.002
10	.722	.788	.002	.440	.558	.004	.570	.528	.682	.002
15	.893	.941	.000	.667	.773	.000	.779	.773	.874	.000
20	.954	.974	.000	.761	.840	.000	.852	.854	.926	.000
30	.997	.993	.000	.913	.950	.000	.950	.972	.959	.000
40	1.000	1.000	.000	.976	.987	.000	.989	.996	.998	.000
60	1.000	1.000	.000	.994	.999	.000	1.000	1.000	1.000	.000
				α =	.8					
4	.094	.136	.016	.068	.102	.019	.081	.057	.131	.023
6	.111	.162	.031	.073	.097	.026	.072	.059	.116	.022
10	.177	.250	.016	.102	.162	.019	.146	.110	.194	.024
16	.200	.286	.005	.128	.200	.008	.167	.135	.238	.012
20	.229	.328	.004	.128	.200	.008	.206	.159	.260	.005
30	.351	.468	.005	.170	.250	.007	.263	.255	.366	.010
40	.397	.506	.001	.226	.324	.004	.313	.297	.425	.005
60	.539	.654	.000	.281	.397	.003	.428	.414	.565	.001

TABLE I

(CONTINUED)

n	NEW PROCEDURES						EXISTING PROCEDURES			
n	C_n	C_n^+	C_n^-	D_n	D_n^+	D_n^-	G-B-S	D_n^*	D_n^{*+}	D_n^{*-}
				α = 1.5						
4	.084	.003	.144	.089	.005	.148	.072	.115	.004	.141
6	.127	.000	.227	.122	.006	.202	.081	.140	.001	.193
10	.201	.001	.313	.154	.001	.256	.128	.205	.001	.281
16	.343	.000	.485	.223	.004	.345	.260	.339	.000	.441
20	.411	.001	.560	.269	.002	.394	.296	.384	.001	.496
30	.590	.000	.725	.395	.001	.545	.473	.562	.001	.660
40	.731	.000	.827	.489	.000	.637	.612	.712	.005	.800
60	.890	.000	.947	.677	.001	.803	.789	.877	.011	.931
				α = 2						
4	.147	.000	.267	.146	.001	.246	.106	.213	.001	.255
6	.261	.000	.403	.224	.001	.354	.143	.300	.000	.385
10	.495	.000	.635	.395	.000	.529	.346	.506	.001	.596
16	.729	.000	.844	.560	.000	.705	.608	.734	.012	.818
20	.866	.000	.933	.678	.000	.801	.756	.867	.023	.930
30	.961	.000	.991	.851	.000	.928	.947	.977	.055	.993
40	.993	.000	.996	.936	.000	.970	.981	.993	.120	.999
60	1.000	.000	1.000	.992	.000	.997	.998	1.000	.341	1.000

TABLE I

(CONTINUED)

	NEW PROCEDURES						EXISTING PROCEDURES			
n	C_n	C_n^+	C_n^-	D_n	D_n^+	D_n^-	G-B-S	D_n^*	D_n^{*+}	D_n^{*-}
				$\alpha = 3$						
4	.313	.000	.481	.314	.001	.474	.216	.445	.000	.501
6	.581	.000	.736	.509	.000	.674	.425	.632	.002	.713
10	.865	.000	.948	.753	.000	.877	.761	.889	.024	.937
16	.982	.000	.993	.932	.000	.978	.965	.987	.162	.993
20	.993	.000	.999	.978	.000	.992	.994	.999	.297	1.000
30	1.000	.000	1.000	.998	.000	.999	1.000	1.000	.694	1.000
40	1.000	.000	1.000	1.000	.000	1.000	1.000	1.000	.914	1.000
60	1.000	.000	1.000	1.000	.000	1.000	1.000	1.000	.999	1.000

TABLE II

POWER COMPARISONS OF TESTING EXPONENTIALITY AT 5% LEVEL
WHEN THE ALTERNATIVE IS GAMMA WITH SHAPE PARAMETER α

	NEW PROCEDURES						EXISTING PROCEDURES			
n	C_n	C_n^-	C_n^-	D_n	D_n^+	D_n^-	G-B-S	D_n^*	D_n^{*+}	D_n^{*-}
					$\alpha =$	.5				
4	.281	.349	.013	.123	.201	.009	.200	.117	.257	.012
6	.336	.425	.009	.171	.250	.011	.222	.163	.274	.012
10	.485	.586	.003	.282	.367	.010	.337	.272	.418	.007
16	.628	.730	.000	.363	.488	.004	.404	.398	.545	.007
20	.726	.799	.000	.435	.549	.003	.481	.477	.637	.005
30	.855	.897	.000	.591	.699	.001	.602	.649	.759	.004
40	.931	.959	.000	.694	.791	.001	.696	.792	.865	.001
60	.989	.995	.000	.856	.925	.000	.851	.921	.960	.002
					$\alpha =$	.8				
4	.091	.112	.027	.060	.074	.040	.085	.043	.097	.030
6	.095	.118	.035	.070	.093	.038	.083	.055	.092	.037
10	.108	.157	.015	.080	.123	.019	.087	.069	.117	.027
16	.128	.180	.014	.074	.108	.021	.085	.067	.124	.021
20	.140	.216	.010	.075	.124	.018	.118	.087	.154	.020
30	.177	.253	.008	.119	.168	.022	.136	.131	.195	.018
40	.196	.321	.002	.120	.189	.011	.146	.137	.215	.006
60	.268	.378	.004	.153	.253	.010	.181	.167	.276	.012

TABLE II

(CONTINUED)

n	C_n	C_n^-	$C_n^{\pm}$	D_n	D_n^+	D_n^-	G-B-S	D_n^*	D_n^{*+}	D_n^{*-}
	NEW PROCEDURES						EXISTING PROCEDURES			
				$\alpha = 1.5$						
4	.053	.004	.100	.054	.013	.098	.038	.096	.011	.108
6	.063	.004	.119	.066	.010	.120	.051	.090	.009	.120
10	.096	.003	.195	.084	.012	.150	.069	.111	.004	.153
16	.165	.002	.258	.122	.005	.200	.120	.166	.002	.232
20	.181	.001	.296	.126	.006	.218	.119	.158	.004	.234
30	.302	.000	.435	[illegible]	.005	.282	.152	.242	.002	.327
40	.364	.000	.498	.229	.001	.351	.196	.291	.001	.411
60	.557	.000	.698	.237	.000	.476	.309	.467	.004	.587
				$\alpha = 2$						
4	.063	.003	.124	.071	.010	.130	.052	.106	.006	.130
6	.124	.000	.209	.108	.006	.173	.090	.135	.005	.191
10	.200	.000	.319	.164	.003	.253	.123	.207	.003	.272
16	.359	.000	.510	.267	.003	.400	.199	.329	.002	.430
20	.465	.000	.624	.342	.002	.485	.283	.423	.001	.536
30	.654	.000	.774	.465	.001	.616	.397	.582	.001	.695
40	.802	.000	.895	.569	.000	.719	.615	.723	.004	.808
60	.942	.000	.975	.774	.000	.874	.755	.891	.005	.959

TABLE II

(CONTINUED)

n	NEW PROCEDURES C_n	C_n^+	C_n^-	D_n	D_n^+	D_n^-	EXISTING PROCEDURES G-B-S	D_n^*	D_n^{*+}	D_n^{*-}
				$\alpha = 3$						
4	.126	.000	.237	.128	.000	.223	.080	.178	.000	.220
6	.238	.000	.393	.191	.002	.333	.155	.300	.000	.371
10	.483	.000	.649	.354	.000	.533	.267	.475	.000	.602
16	.732	.000	.851	.557	.000	.714	.461	.702	.006	.797
20	.860	.000	.936	.653	.001	.788	.566	.805	.011	.884
30	.960	.000	.987	.869	.000	.929	.808	.951	.020	.973
40	.994	.000	.999	.944	.000	.978	.933	.991	.046	.996
60	1.000	.000	1.000	.992	.000	.999	.992	1.000	.127	1.000

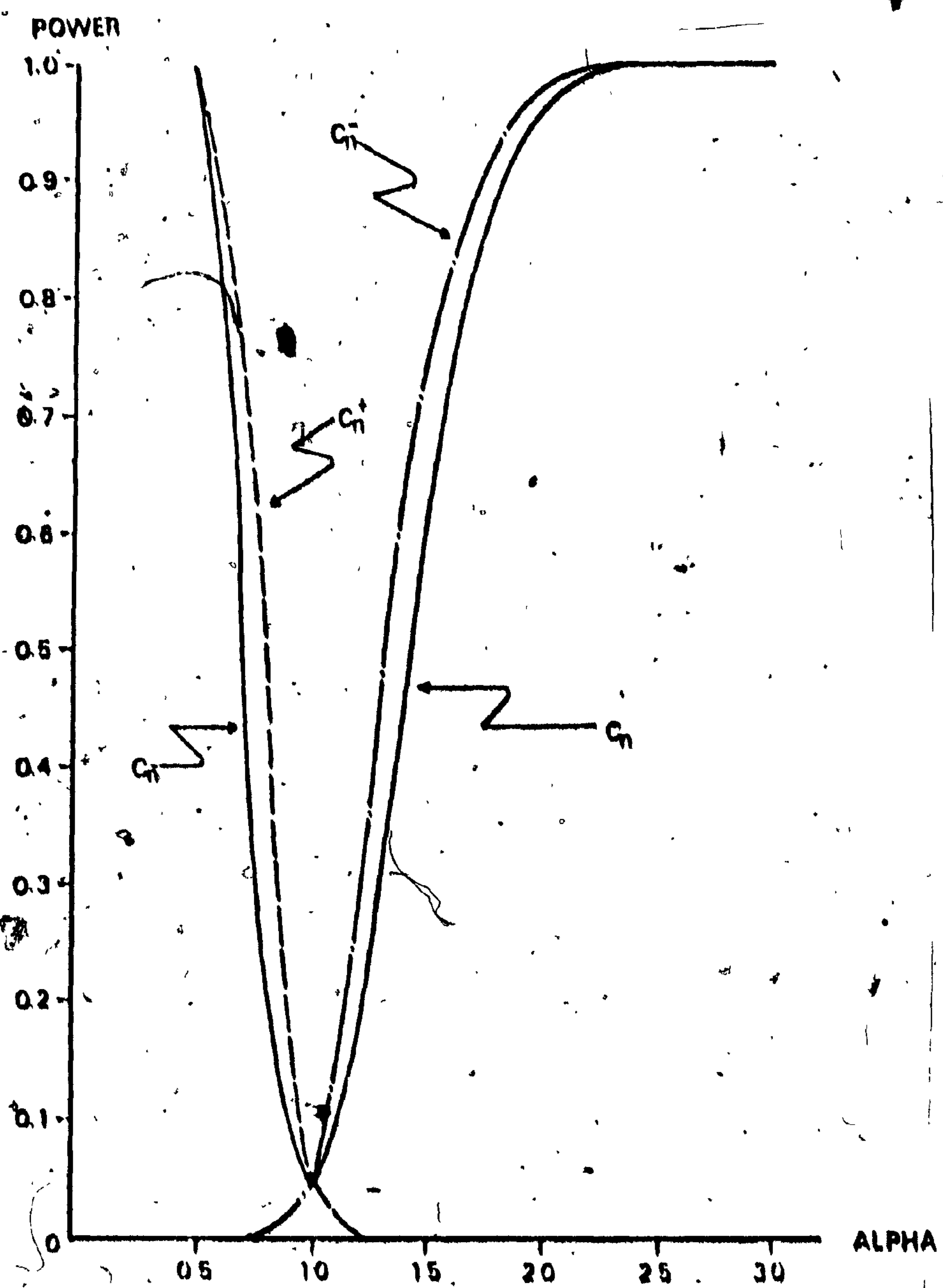

FIGURE I. POWER CURVES FOR C_n, C_n^+, C_n^- AT 5% LEVEL WITH WEIBULL ALTERNATIVE AND n = 30

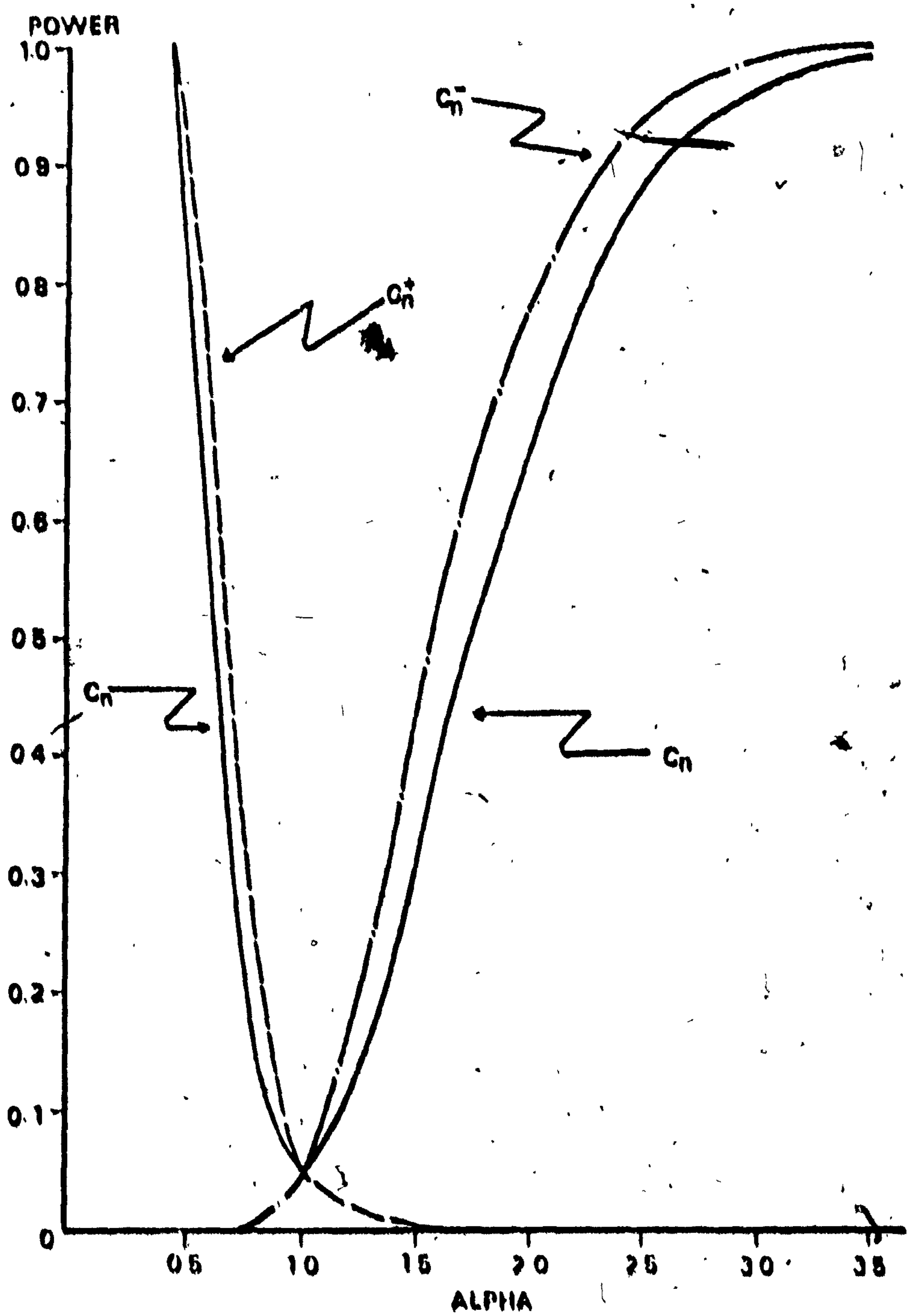

FIGURE II. POWER CURVES FOR C_n, C_n^+, C_n^- AT 5% LEVEL WITH GAMMA ALTERNATIVE AND n = 30

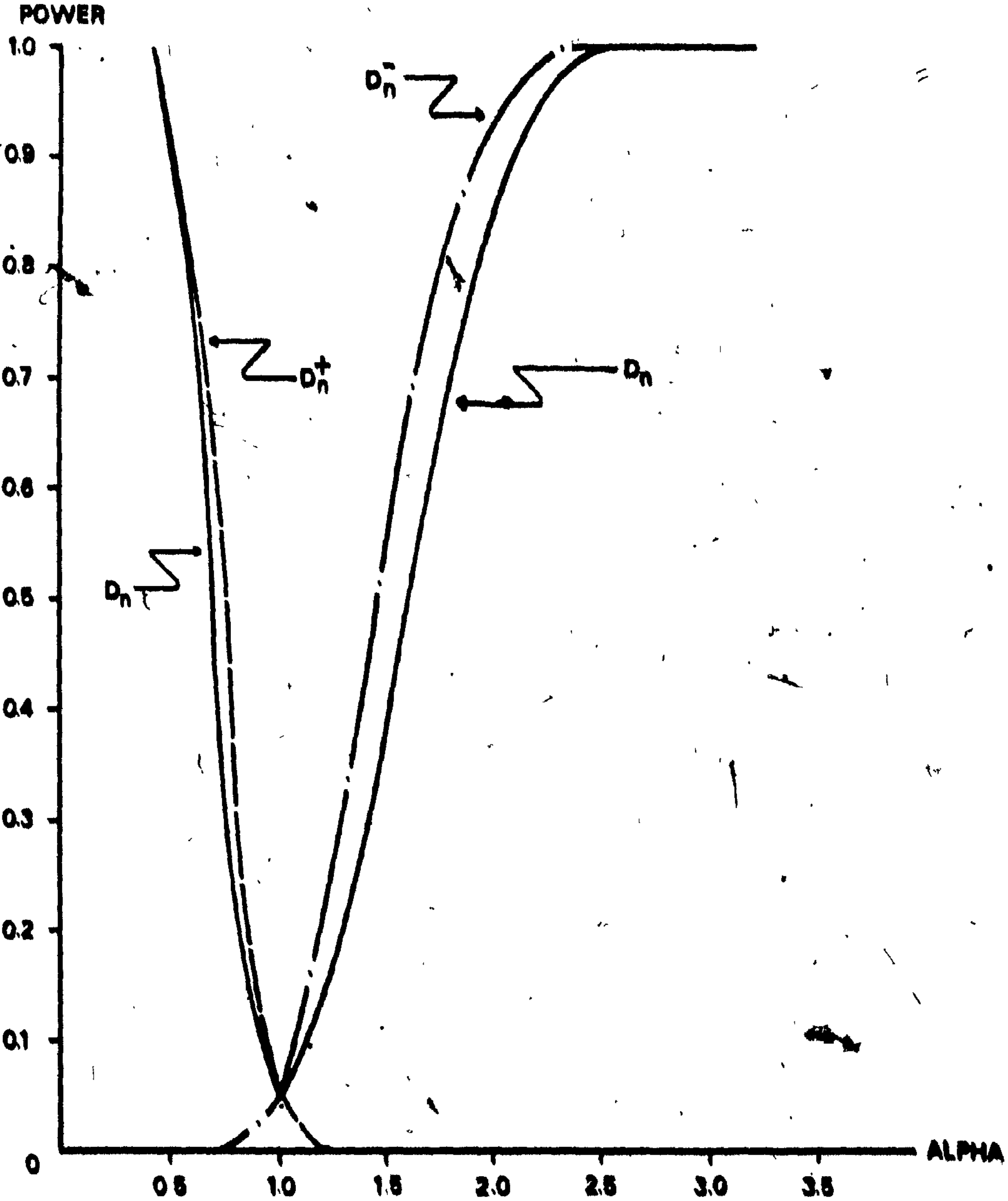

FIGURE III. POWER CURVES FOR D_n, D_n^+, D_n^- AT 5% LEVEL WITH WEIBULL ALTERNATIVE AND $n = 30$.

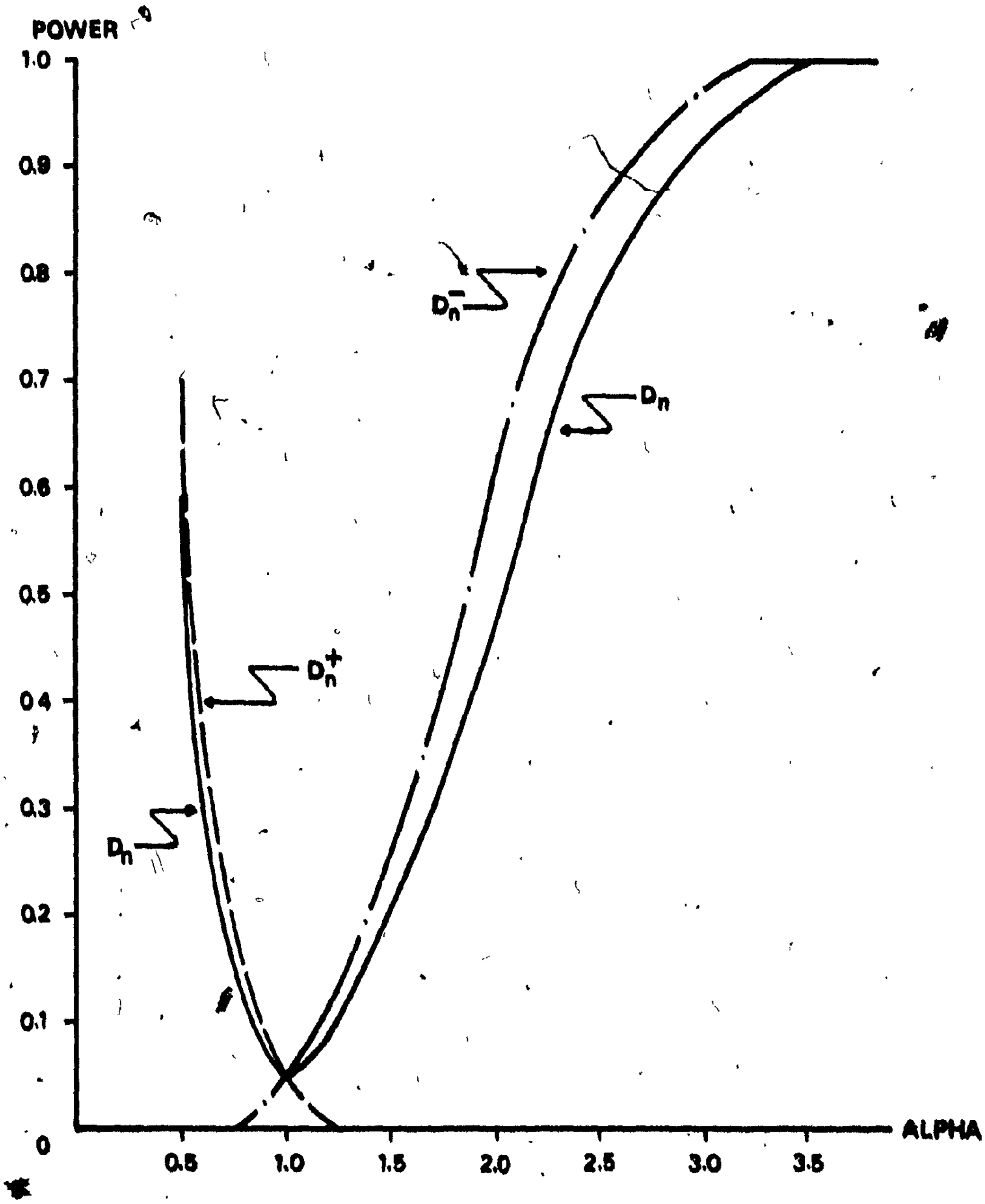

FIGURE IV. POWER CURVES FOR D_n, D_n^+, D_n^- AT 5% LEVEL WITH GAMMA ALTERNATIVE AND n = 30.

REFERENCES

[1] BARLOW, R.E., MARSHALL, A.W. and PROSCHAN, F., "Properties of probability distributions with monotone hazard rate", Ann. Math. Statist., 34, (1963), 375-389.

[2] BICKEL, P.J., "Tests for monotone failure rate II", Ann. Math. Statist., 40, (1969), 1250-1260.

[3] BICKEL, P.J. and DOKSUM, K.A., "Tests for monotone failure rate based on normalized spacings", Ann. Math. Statist., 40, (1969), 1216-1235.

[4] BLUMENTHAL, S., "Logarithms of sample spaces", SIAM, J. Appl. Math., 16, (1968), 1184-1191.

[5] BURY, K.V., Statistical Methods in Applied Science, John Wiley & Sons, Inc., (1975).

[6] CSÖRGÖ, M., SESHADRI, V. and YALOVSKY, M., "Applications of characterizations in the area of goodness of fit", A Modern Course on Statistical Distribution, D. Reidel Publishing Co., (1974), 79-90.

[7] DARLING, D.A., "The Kolmogorov-Smirnov, Cramér-Von Mises tests", Ann. Math. Statist., 28, (1957), 823-838.

[8] DURBIN, J., "Some methods of constructing exact tests", Biometrika, 48, (1961), 41-55.

[9] ________, "Kolmogorov-Wmirnov tests when parameters are estimated with applications to tests of exponentiality and tests on spacings", Biometrika, 62, (1975), 5-22.

[10] EPSTEIN, B., "Testing for the validity of the assumption that the underlying distribution of life is exponential", Technometrics, 2, (1960), 83-101 (Part I), 167-183 (Part II).

[11] FERCHO, W.W. and RINGER, L.J., "Small sample power of some tests of the constant failure rate", Technometrics, 14, (1972), 713-724.

[12] FINKLESTEIN, J.M. and SCHAFER, R.E., "Improved goodness of fit tests", Biometrika, 58, (1971), 641-645.

[13] GNEDENKO, B.V., BELYAYEV, Yu.K. and SOLOVYEV, A.D., Mathematical Methods of Reliability Theory, Academic Press, (1969).

[14] HAJEK, J., A Course in Nonparametric Statistics, Holden-Day, Inc., (1969).

[15] HARTLEY, H.O., "The maximum F ratio as a short-cut test for homogeneity of variance", Biometrika, 37, (1950), 308-312.

[16] HOLLANDER, M. and WOLFE, D.A., Nonparametric Statistical Methods, John Wiley & Sons, Inc., (1973).

[17] KENDALL, M.G., "A new measure of rank correlation", Biometrika, 30, (1938), 81-93.

[18] KOTLARSKI, I., "On characterizing the gamma and the normal distributions", Pacific J. Math., 20, (1967), 69-76.

[19] LILLIEFORS, H.W., "On the Kolmogorov-Smirnov test for the exponential distribution with mean unknown", JASA, 64, (1969), 387-389.

[20] LUKACS, E., "A characterization of the gamma distribution", Ann. Math. Statist., 26, (1955), 319-324.

[21] MANN, H.B., "Nonparametric tests against trend", Econometrica, 13, (1945), 245-249.

[22] MANN, N.R., SCHAFER, R.E. and SINGPURWALLA, N.D., Methods for Statistical Analysis of Reliability and Life Data, John Wiley & Sons, Inc., (1974).

[23] MARSAGLIA, G., "Extension and applications of Lukacs' characterization of the gamma distribution", Proc. Symp. Statist. and Related Topics, (Oct. 24-26, 1974), Carleton University, Ottawa, (1974), 9.01-13.

[24] MORAN, P.A.P., "The random division of an interval - Part II", J. Royal Statist. Soc., Ser. B, 13, (1951), 147-150.

[25] PATIL, G.P. and SESHADRI, V., "Characterization theorems for some univariate probability distributions", J. Royal Statist., Ser. B, 27, (1965), 286-292.

[26] PEARSON, E.S. and HARTLEY, H.O., Biometrika Tables for Statisticians, Vol. 1 (3rd edition), Cambridge University Press, (1966).

[27] PEARSON, E.S. and HARTLEY, H.O., Biometrika Tables for Statisticians, Vol. 2, Cambridge University Press, (1972).

[28] PROSCHAN, F. and PYKE, R., "Test for monotone failure rate", Fifth Berkeley Symp. on Math. Statist., and Prob., Vol. 3, (1967), 293-312.

[29] ROHATGI, V.K., An Introduction to Probability and Mathematical Statistics, John Wiley & Sons, Inc., (1976).

[30] ROUSSAS, G.G., A First Course in Mathematical Statistics, Addison-Wesley Publishing Co., (1973).

[31] SESHADRI, V., CSORGO, M. and STEPHENS, M.A., "Tests for the exponential distribution using Kolmogorov-type statistics", J. Royal Statist. Soc., Ser. B, 31, (1969), 449-509.

[32] SHAPIRO, S.S. and WILK, M.B., "An analysis of variance test for the exponential distribution (complete samples), Technometrics, 14, (1972), 355-370.

[33] SILVEY, S.D., Statistical Inference, Penguin Books Ltd., (1970).

[34] SRINIVASAN, R., "An approach to testing the goodness of fit of incomplete specified distributions", Biometrika, 57, (1970), 605-611.

[35] WANG, Y.H. and CHANG, S.A., "A new approach to the nonparametric tests of the exponential distribution with unknown parameters", Proc. of the conference on The Theory and Applications of Reliability with Emphasis on Baysian and Nonparametric Methods, (Dec. 2 - 5, 1975), University of South Florida, Tampa, Florida, (to appear).

APPENDIX

```
C
C        GAMMA ALTERNATIVE
C
      PROGRAM CHANG (INPUT, OUTPUT, TAPE1, TAPE2=OUTPUT)
      COMMON LAMDA
      DOUBLE PRECISION SEED
      DIMENSION E(60), S(60), U(60), H(60), PDIFE(6), PDIFU(6), DBN1(8,3
     $), DBN2(8,3), DBN3(8,3), UT1(8,3), UT2(8,3), CL(8,3), CU(8,3), FL(
     $8,3), FU(8,3), AL(16), M(8), GS(60), W(60)
      INTEGER EN1(3), EN2(3), EN3(3), CN(3), UN1(3), UN2(3), UN3(3), GN(
     $3)
      REAL LAMDA
      EXTERNAL PDFE, PDFU
      DATA SEED /7.422190-1/
      B = 2.0
      AL(1) = 0.5
      AL(2) = 0.8
      AL(3) = 1.5
      AL(4) = 2.0
      AL(5) = 3.0
      M(1) = 4
      M(2) = 6
      M(3) = 10
      M(4) = 16
      M(5) = 20
      M(6) = 30
      M(7) = 40
      M(8) = 60
      NUM = 1000
      FL(1,1) = 1.0/11.07
      FL(1,2) = 1.0/5.82
      FL(1,3) = 1.0/4.28
      FU(1,1) = 11.07
      FU(1,2) = 5.82
      FU(1,3) = 4.28
```

```
FL(2 ,1) = 1.0/5.85
FL(2,2) = 1.0/3.72
FL(2,3) = 1.0/2.98
FU(2 ,1) = 5.85
FU(2,2) = 3.72
FU(2,3) = 2.98
FL(3,1) = 1.0/3.879
FL(3,2) = 1.0/2.764
FL(3,3) = 1.0/2.335
FU(3,1) = 3.879
FU(3,2) = 2.764
FU(3,3) = 2.335
FL(4,1) = 1.0/3.32
FL(4,2) = 1.0/2.46
FL(4,3) = 1.0/2.12
FU(4,1) = 3.32
FU(4,2) = 2.46
FU(4,3) = 2.12
FL(5,1) = 1.0/2.63
FL(5,2) = 1.0/2.07
FL(5,3) = 1.0/1.84
FU(5 ,1) = 2.63
FU(5 ,2) = 2.07
FU(5 ,3) = 1.84
FL(6,1) = 1.0/23.15
FL(6,2) = 1.0/9.60
FL(6,3) = 1.0/6.39
FU(6,1) = 23.15
FU(6,2) = 9.60
FU(6,3) = 6.39
FL(7,1) = 1.0/2.30
FL(7,2) = 1.0/1.88
FL(7,3) = 1.0/1.69
FU(7,1) = 2.30
FU(7,2) = 1.88
FU(7,3) = 1.69
```

```
FL(8,1) = 1.0/1.84
FL(8,2) = 1.0/1.67
FL(8,3) = 1.0/1.53
FU(8,1) = 1.84
FU(8,2) = 1.67
FU(8,3) = 1.53
CL(1,1) = 2.15586
CL(1,2) = 3.24697
CL(1,3) = 3.94030
CU(1,1) = 25.1882
CU(1,2) = 20.4832
CU(1,3) = 18.3070
CL(2 ,1) = 6.26480
CL(2 ,2) = 8.23075
CL(2 ,3) = 9.39046
CU(2 ,1) = 37.1565
CU(2 ,2) = 31.5264
CU(2 ,3) = 28.8693
CL(3 ,1) = 13.7867
CL(3 ,2) = 16.7908
CL(3 ,3) = 18.4927
CU(3 ,1) = 53.6720
CU(3 ,2) = 46.9792
CU(3 ,3) = 43.7730
CL(4 ,1) = 19.2888
CL(4 ,2) = 22.8798
CL(4 ,3) = 24.8824
CU(4 ,1) = 64.182
CU(4 ,2) = 56.886
CU(4 ,3) = 53.39
CL(5 ,1) = 34.0257
CL(5 ,2) = 38.8568
CL(5 ,3) = 41.50326
CU(5 ,1) = 89.4594
CU(5 ,2) = 80.9222
CU(5 ,3) = 76.7665
CL(6,1) = 0.676
CL(6,2) = 1.237
```

```
CL(6,3) = 1.635
CU(6,1) = 18.548
CU(6,2) = 14.449
CU(6,3) = 12.592
CL(7,1) = 49.582
CL(7,2) = 55.466
CL(7,3) = 58.654
CU(7,1) = 113.911
CU(7,2) = 104.316
CU(7,3) = 99.617
CL(8,1) = 82.185
CL(8,2) = 89.827
CL(8,3) = 93.918
CU(8,1) = 161.314
CU(8,2) = 149.957
CU(8,3) = 144.354
DBN1(1,1) = 1.1631
DBN1(1,2) = 1.0007
DBN1(1,3) = 0.9141
DBN2(1,1) = 1.0573
DBN2(1,2) = 0.8386
DBN2(1,3) = 0.7286
DBN3(1,1) = 1.1216
DBN3(1,2) = 0.9554
DBN3(1,3) = 0.8652
DBN1(2 ,1) = 1.2057
DBN1(2 ,2) = 1.0258
DBN1(2 ,3) = 0.9343
DBN2(2 ,1) = 1.1032
DBN2(2 ,2) = 0.8826
DBN2(2 ,3) = 0.7727
DBN3(2 ,1) = 1.1523
DBN3(2 ,2) = 0.9686
DBN3(2 ,3) = 0.8719
DBN1(3 ,1) = 1.2304
DBN1(3 ,2) = 1.0424
DBN1(3 ,3) = 0.9482
DBN2(3 ,1) = 1.1315
DBN2(3 ,2) = 0.9103
```

```
DBN2(3 ,3) = 0.7995
DBN3(3 ,1) = 1.1702
DBN3(3 ,2) = 0.9765
DBN3(3 ,3) = 0.8763
DBN1(4 ,1) = 1.2392
DBN1(4 ,2) = 1.0486
DBN1(4 ,3) = 0.9536
DBN2(4 ,1) = 1.1418
DBN2(4 ,2) = 0.9205
DBN2(4 ,3) = 0.8095
DBN3(4 ,1) = 1.1764
DBN3(4 ,2) = 0.9793
DBN3(4 ,3) = 0.8778
DBN1(5 ,1) = 1.2519
DBN1(5 ,2) = 1.0580
DBN1(5 ,3) = 0.9[illegible]7
DBN2(5 ,1) = 1.1569
DBN2(5 ,2) = 0.9357
DBN2(5 ,3) = 0.8249
DBN3(5 ,1) = 1.1852
DBN3(5 ,2) = 0.9833
DBN3(5 ,3) = 0.8800
DBN1(6,1) = 1.1148
DBN1(6,2) = 0.9687
DBN1(6,3) = 0.8884
DBN2(6,1) = 0.9851
DBN2(6,2) = 0.7969
DBN2(6,3) = 0.6852
DBN3(6,1) = 1.0874
DBN3(6,2) = 0.9377
DBN3(6,3) = 0.8554
DBN1(7,1) = 1.2588
DBN1(7,2) = 1.0633
DBN1(7,3) = 0.9665
DBN2(7,1) = 1.1654
DBN2(7,2) = 0.9444
```

```
DBN2(7,3)  = 0.8338
DBN3(7,1)  = 1.1900
DBN3(7,2)  = 0.9855
DBN3(7,3)  = 0.8812
DBN1(8,1)  = 1.2665
DBN1(8,2)  = 1.0694
DBN1(8,3)  = 0.9720
DBN2(8,1)  = 1.1750
DBN2(8,2)  = 0.9544
DBN2(8,3)  = 0.8440
DBN3(8,1)  = 1.1951
DBN3(8,2)  = 0.9878
DBN3(8,3)  = 0.8826
UT1(1,1)  = 0.66853
UT1(1,2)  = 0.56328
UT1(1,3)  = 0.50945
UT2(1,1)  = 0.62718
UT2(1,2)  = 0.50945
UT2(1,3)  = 0.44698
UT1(2  ,1)  = 0.51332
UT1(2  ,2)  = 0.43001
UT1(2  ,3)  = 0.38746
UT2(2  ,1)  = 0.47960
UT2(2  ,2)  = 0.38746
UT2(2  ,3)  = 0.33910
UT1(3  ,1)  = 0.40420
UT1(3  ,2)  = 0.33760
UT1(3  ,3)  = 0.30397
UT2(3  ,1)  = 0.37713
UT2(3  ,2)  = 0.30397
UT2(3  ,3)  = 0.26588
UT1(4  ,1)  = 0.36117
UT1(4  ,2)  = 0.30143
UT1(4  ,3)  = 0.27136
UT2(4  ,1)  = 0.33685
UT2(4  ,2)  = 0.27136
UT2(4  ,3)  = 0.23735
UT1(5  ,1)  = 0.29466
```

```
      UT1(5 ,2) = 0.24571
      UT1(5 ,3) = 0.22117
      UT2(5 ,1) = 0.27471
      UT2(5 ,2) = 0.22117
      UT2(5 ,3) = 0.19348
      UT1(6,1) = 0.82900
      UT1(6,2) = 0.70760
      UT1(6,3) = 0.63604
      UT2(6,1) = 0.75456
      UT2(6,2) = 0.63604
      UT2(6,3) = 0.56481
      UT1(7,1) = 0.25518
      UT1(7,2) = 0.21273
      UT1(7,3) = 0.19148
      UT2(7,1) = 0.23786
      UT2(7,2) = 0.19148
      UT2(7,3) = 0.16753
      UT1(8,1) = 0.20844
      UT1(8,2) = 0.17373
      UT1(8,3) = 0.15639
      UT2(8,1) = 0.19627
      UT2(8,2) = 0.15639
      UT2(8,3) = 0.13686
      DO 103 I = 1, 5
      ALPHA = AL(I)
      DO 102 J = 1, 8
      REWIND 2
      N = M(J)
      WRITE (1,60) N, ALPHA
   60 FORMAT (//, 10X, "N= ", I5, 10X, "GAMMA(", F3.1, ")", /)
      KNTC = 0
      KNTG = 0
      DO 10 K1 = 1,3
      GN(K1) = 0
      CN(K1) = 0
      UN1(K1) = 0
      UN2(K1) = 0
```

```
      UN3(K1) = 0
      EN1(K1) = 0
      EN2(K1) = 0
      EN3(K1) = 0
   10 CONTINUE
      DO 101 L = 1, NNM
      CALL GGTMAJ (SEED, ALPHA, B, N, W, E)
      S(1) = E(1)
      DO 20 II = 2,N
      S(II) = S(II-1) + E(II)
   20 CONTINUE
      CALL VSORTA (E(1), N)
      LAMDA = S(N)/N
      CALL MKS1(PDFE, E(1), N, PDIFE, IER)
      SAMEN = FLOAT(N)
      SON = SQRT (SAMEN)
      BN1 = SON * PDIFE(1)
      BN2 = SON * PDIFE(2)
      BN3 = -SON * PDIFE(3)
      H(1) = N * E(1)
      GS(1) = H(1)
      DO 30 JJ = 2,N
      H(JJ) = (N-JJ+1) * (E(JJ)-E(JJ-1))
      GS(JJ) = GS(JJ-1) + H(JJ)
   30 CONTINUE
      G = GS(N/2)/(GS(N)-GS(N/2))
      C = 0.
      N1 = N-1
      DO 40 KK = 1,N1
      U(KK) = (S(KK)/S(KK+1)) ** KK
      IF (U(KK) .LE. 0.5)   GO TO 35
      U(KK) = 2.0 * (1-U(KK))
      GO TO 37
   35 U(KK) = 2.0 * U(KK)
   37 C = C - 2.0 * ALOG(U(KK))
   40 CONTINUE
```

```
      CALL VSORTA (U(1), N1)
      CALL NKS1 (PDFU, U(1), N1, PDIFU, IER)
      DN1 = PDIFU(1)
      DN2 = PDIFU(2)
      DN3 = -PDIFU(3)
      DO 50 K = 1,3
      IF(C .LT. CL(J,K) .OR. C .GT. CU(J,K))    CN(K) = CN(K) + 1
      IF(G .LT. FL(J,K) .OR. G .GT. FU(J,K))    GN(K) = GN(K) + 1
      IF (BN1 .GT. DBN1(J,K))    EN1(K) = EN1(K) + 1
      IF (BN2 .GT. DBN2(J,K))    EN2(K) = EN2(K) + 1
      IF (BN3 .GT. DBN3(J,K))    EN3(K) = EN3(K) + 1
      IF(DN1 .GT. UT1(J,K))   UN1(K) = UN1(K) + 1
      IF(DN2 .GT. UT2(J,K))   UN2(K) = UN2(K) + 1
      IF(DN3 .GT. UT2(J,K))   UN3(K) = UN3(K) + 1
   50 CONTINUE
      IF (C .LT. CL(J,3))     KNTC = KNTC + 1
      IF (G .LT. FL(J,3))     KNTG = KNTE + 1
      WRITE (1,70)  L, C, (CN(L1), L1=1,3), DN1, (UN1(L2), L2=1,3), DN2,
     $(UN2(L3), L3=1,3), DN3, (UN3(L4), L4=1,3)
   70 FORMAT (2X, "I=", I4, 1X, "CHI-SQ C=", F6.2, 1X, "SP=", 3I4, 3X, "
     $K-S-UNIF DN=", F5.3, 1X, "SP=", 3I4, 1X, "DN+=", F6.3, 1X, "SP=",
     $3I4, 1X, "DN-=", F6.3, 1X, "SP=", 3I4)
      WRITE (1,80)  G, (GN(L1), L1=1,3), BN1, (EN1(L2), L2=1,3), BN2, (E
     $N2(L3), L3=1,3), BN3, (EN3(L4), L4=1,3)
   80 FORMAT (9X, "GNDKD  G=", F6.2, 1X, "SP=", 3I4, 3X, "K-S-EXP  DN=",
     $F5.3, 1X, "SP=", 3I4, 1X, "DN+=", F6.3, 1X, "SP=", 3I4, 1X, "DN-="
     $, F6.3, 1X, "SP=", 3I4)
  101 CONTINUE
      WRITE (1,90) KNTC, KNTG
   90 FORMAT (/, 3X, "CN-= ", I5, 2X, "GBS-= ", I5, /)
  102 CONTINUE
  103 CONTINUE
      STOP
      END
      SUBROUTINE PDFE (Y,F)
      COMMON LAMDA
      REAL LAMDA
```

```
      IF (Y .GT. 0.)    GO TO 5
      F = 0.0
      RETURN
    5 F = 1.0 - EXP(-Y/LAMDA)
      RETURN
      END
      SUBROUTINE PDFU (Z,F)
      IF (Z .GT. 0.)    GO TO 5
      F = 0.
      RETURN
    5 IF (Z .LT. 1.)    GO TO 10
      F = 1.0
      RETURN
   10 F = Z
      RETURN
      END
```

```
C
C        WEIBULL ALTERNATIVE
C
      PROGRAM CHANG (INPUT, OUTPUT, TAPE1, TAPE2=OUTPUT)
      COMMON LAMDA
      DOUBLE PRECISION SEED
      DIMENSION E(60), S(60), U(60), H(60), PDIFE(6), PDIFU(6), DBN1(8,3
     $), DBN2(8,3), DBN3(8,3), UT1(8,3), UT2(8,3), CL(8,3), CU(8,3), FL(
     $8,3), FU(8,3), AL(16), M(8), GS(60) , W(9)
      INTEGER EN1(3), EN2(3), EN3(3), CN(3), UN1(3), UN2(3), UN3(3), GN(
     $3)
      REAL LAMDA
      EXTERNAL PDFE, PDFU
      DATA SEED /7.422190D-1/
      AL(1) = 0.5
      AL(2) = 0.8
      AL(3) = 1.5
      AL(4) = 2.0
      AL(5) = 3.0
      M(1) = 4
      M(2) = 6
      M(3) = 10
      M(4) = 16
      M(5) = 20
      M(6) = 30
      M(7) = 40
      M(8) = 50
      NUM = 1000
      XM = 5.0
      FL(1,1) = 1.0/11.07
      FL(1,2) = 1.0/5.82
      FL(1,3) = 1.0/4.28
      FU(1,1) = 11.07
      FU(1,2) = 5.82
      FU(1,3) = 4.28
      FL(2,1) = 1.0/5.85
      FL(2,2) = 1.0/3.72
```

```
FL(2,3) = 1.0/2.98
FU(2,1) = 5.85
FU(2,2) = 3.72
FU(2,3) = 2.98
FL(3,1) = 1.0/3.879
FL(3,2) = 1.0/2.764
FL(3,3) = 1.0/2.335
FU(3,1) = 3.879
FU(3,2) = 2.764
FU(3,3) = 2.335
FL(4,1) = 1.0/3.32
FL(4,2) = 1.0/2.46
FL(4,3) = 1.0/2.12
FU(4,1) = 3.32
FU(4,2) = 2.46
FU(4,3) = 2.12
FL(5,1) = 1.0/2.63
FL(5,2) = 1.0/2.07
FL(5,3) = 1.0/1.84
FU(5,1) = 2.63
FU(5,2) = 2.07
FU(5,3) = 1.84
FL(6,1) = 1.0/23.15
FL(6,2) = 1.0/9.60
FL(6,3) = 1.0/6.39
FU(6,1) = 23.15
FU(6,2) = 9.60
FU(6,3) = 6.39
FL(7,1) = 1.0/2.30
FL(7,2) = 1.0/1.88
FL(7,3) = 1.0/1.69
FU(7,1) = 2.30
FU(7,2) = 1.88
FU(7,3) = 1.69
FL(8,1) = 1.0/1.84
FL(8,2) = 1.0/1.67
```

```
FL(8,3) = 1.0/1.53
FU(8,1) = 1.84
FU(8,2) = 1.67
FU(8,3) = 1.53
CL(1,1) = 2.15586
CL(1,2) = 3.24697
CL(1,3) = 3.94030
CU(1,1) = 25.1882
CU(1,2) = 20.4832
CU(1,3) = 18.3070
CL(2,1) = 6.26480
CL(2,2) = 8.23075
CL(2,3) = 9.39046
CU(2,1) = 37.1565
CU(2,2) = 31.5264
CU(2,3) = 28.8693
CL(3,1) = 13.7867
CL(3,2) = 16.7908
CL(3,3) = 18.4927
CU(3,1) = 53.6720
CU(3,2) = 46.9792
CU(3,3) = 43.7730
CL(4,1) = 19.2898
CL(4,2) = 22.8798
CL(4,3) = 24.8824
CU(4,1) = 64.182
CU(4,2) = 56.886
CU(4,3) = 53.39
CL(5,1) = 34.0257
CL(5,2) = 38.8568
CL(5,3) = 41.50326
CU(5,1) = 89.4594
CU(5,2) = 80.9222
CU(5,3) = 76.7665
CL(6,1) = 0.676
CL(6,2) = 1.237
```

```
CL(6,3) = 1.635
CU(6,1) = 18.548
CU(6,2) = 14.449
CU(6,3) = 12.592
CL(7,1) = 49.582
CL(7,2) = 55.466
CL(7,3) = 58.654
CU(7,1) = 113.911
CU(7,2) = 104.316
CU(7,3) = 99.617
CL(8,1) = 82.185
CL(8,2) = 89.827
CL(8,3) = 93.918
CU(8,1) = 161.314
CU(8,2) = 149.957
CU(8,3) = 144.354
DBN1(1,1) = 1.1631
DBN1(1,2) = 1.0007
DBN1(1,3) = 0.9141
DBN2(1,1) = 1.0573
DBN2(1,2) = 0.8386
DBN2(1,3) = 0.7286
DBN3(1,1) = 1.1216
DBN3(1,2) = 0.9554
DBN3(1,3) = 0.8652
DBN1(2,1) = 1.2057
DBN1(2,2) = 1.0258
DBN1(2,3) = 0.9343
DBN2(2,1) = 1.1032
DBN2(2,2) = 0.8826
DBN2(2,3) = 0.7727
DBN3(2,1) = 1.1523
DBN3(2,2) = 0.9686
DBN3(2,3) = 0.8719
DBN1(3,1) = 1.2304
DBN1(3,2) = 1.0424
```

```
DBN1(3,3) = 0.9482
DBN2(3,1) = 1.1315
DBN2(3,2) = 0.9103
DBN2(3,3) = 0.7995
DBN3(3,1) = 1.1702
DBN3(3,2) = 0.9765
DBN3(3,3) = 0.8763
DBN1(4,1) = 1.2392
DBN1(4,2) = 1.0486
DBN1(4,3) = 0.9536
DBN2(4,1) = 1.1418
DBN2(4,2) = 0.9205
DBN2(4,3) = 0.8095
DBN3(4,1) = 1.1764
DBN3(4,2) = 0.9793
DBN3(4,3) = 0.8778
DBN1(5,1) = 1.2519
DBN1(5,2) = 1.0580
DBN1(5,3) = 0.9617
DBN2(5,1) = 1.1569
DBN2(5,2) = 0.9357
DBN2(5,3) = 0.8249
DBN3(5,1) = 1.1852
DBN3(5,2) = 0.9833
DBN3(5,3) = 0.8800
DBN1(6,1) = 1.1148
DBN1(6,2) = 0.9687
DBN1(6,3) = 0.8884
DBN2(6,1) = 0.9851
DBN2(6,2) = 0.7969
DBN2(6,3) = 0.6852
DBN3(6,1) = 1.0874
DBN3(6,2) = 0.9377
DBN3(6,3) = 0.8554
DBN1(7,1) = 1.2588
```

```
DBN1(7,2) = 1.0633
DBN1(7,3) = 0.9665
DBN2(7,1) = 1.1654
DBN2(7,2) = 0.9444
DBN2(7,3) = 0.8338
DBN3(7,1) = 1.1900
DBN3(7,2) = 0.9855
DBN3(7,3) = 0.8812
DBN1(8,1) = 1.2665
DBN1(8,2) = 1.0694
DBN1(8,3) = 0.9720
DBN2(8,1) = 1.1750
DBN2(8,2) = 0.9544
DBN2(8,3) = 0.8440
DBN3(8,1) = 1.1951
DBN3(8,2) = 0.9878
DBN3(8,3) = 0.8826
UT1(1,1) = 0.66953
UT1(1,2) = 0.56328
UT1(1,3) = 0.50945
UT2(1,1) = 0.62718
UT2(1,2) = 0.50945
UT2(1,3) = 0.44698
UT1(2,1) = 0.51332
UT1(2,2) = 0.43001
UT1(2,3) = 0.38746
UT2(2,1) = 0.47960
UT2(2,2) = 0.38746
UT2(2,3) = 0.33910
UT1(3,1) = 0.40420
UT1(3,2) = 0.33760
UT1(3,3) = 0.30397
UT2(3,1) = 0.37713
UT2(3,2) = 0.30397
UT2(3,3) = 0.26588
UT1(4,1) = 0.36117
```

```
UT1(4,2) = 0.30143
UT1(4,3) = 0.27136
UT2(4,1) = 0.33685
UT2(4,2) = 0.27136
UT2(4,3) = 0.23735
UT1(5,1) = 0.29466
UT1(5,2) = 0.24571
UT1(5,3) = 0.22117
UT2(5,1) = 0.27471
UT2(5,2) = 0.22117
UT2(5,3) = 0.19348
UT1(6,1) = 0.82900
UT1(6,2) = 0.70760
UT1(6,3) = 0.63604
UT2(6,1) = 0.78456
UT2(6,2) = 0.63604
UT2(6,3) = 0.56481
UT1(7,1) = 0.25518
UT1(7,2) = 0.21273
UT1(7,3) = 0.19148
UT2(7,1) = 0.23786
UT2(7,2) = 0.19148
UT2(7,3) = 0.16753
UT1(8,1) = 0.20844
UT1(8,2) = 0.17373
UT1(8,3) = 0.15639
UT2(8,1) = 0.19427
UT2(8,2) = 0.15639
UT2(8,3) = 0.13686
DO 103 I = 1, 5
ALPHA = AL(I)
BETA = 1.0/ALPHA
DO 102 J = 1, 8
REWIND 2
N = M(J)
WRITE (1,60)  N, ALPHA
```

```
60 FORMAT (//, 10X, "N= ", I5, 10X, "WEIBULL(", F3.1, ")", /)
   KNTC = 0
   KNTG = 0
   DO 10 K1 = 1,3
   GN(K1) = 0
   CN(K1) = 0
   UN1(K1) = 0
   UN2(K1) = 0
   UN3(K1) = 0
   EN1(K1) = 0
   EN2(K1) = 0
   EN3(K1) = 0
10 CONTINUE
   DO 101 L = 1,NUM
   CALL GGEXP (SEED, XM, N, E)
   E(1) = E(1) ** BETA
   S(1) = E(1)
   DO 20 II = 2,N
   E(II) = E(II) ** BETA
   S(II) = S(II-1) + E(II)
20 CONTINUE
   CALL VSORTA (E(1), N)
   LAMDA = S(N)/N
   CALL NKS1(PDFE, E(1), N, PDIFE, IER)
   SAMEN = FLOAT(N)
   SQN = SQRT (SAMEN)
   BN1 = SQN * PDIFE(1)
   BN2 = SQN * PDIFE(2)
   BN3 = -SQN * PDIFE(3)
   H(1) = N * E(1)
   GS(1) = H(1)
   DO 30 JJ = 2,N
   H(JJ) = (N-JJ+1) * (E(JJ)-E(JJ-1))
   GS(JJ) = GS(JJ-1) + H(JJ)
30 CONTINUE
   G = GS(N/2)/(GS(N)-GS(N/2))
```

```
      C = 0.
      N1 = N-1
      DO 40 KK = 1,N1
      U(KK) = (S(KK)/S(KK+1)) ** KK
      IF (U(KK) .LE. 0.5)   GO TO 35
      U(KK) = 2.0 * (I-U(KK))
      GO TO 37
   35 U(KK) = 2.0 * U(KK)
   37 C = C - 2.0 * ALOG(U(KK))
   40 CONTINUE
      CALL VSORTA (U(1), N1)
      CALL NKS1 (PDFU, U(1), N1, PDIFU, IER)
      DN1 = PDIFU(1)
      DN2 = PDIFU(2)
      DN3 = -PDIFU(3)
      DO 50 K = 1,3
      IF(C .LT. CL(J,K) .OR. C .GT. CU(J,K))   CN(K) = CN(K) + 1
      IF(G .LT. FL(J,K) .OR. G .GT. FU(J,K))   GN(K) = GN(K) + 1
      IF (BN1 .GT. DBN1(J,K))   EN1(K) = EN1(K) + 1
      IF (BN2 .GT. DBN2(J,K))   EN2(K) = EN2(K) + 1
      IF (BN3 .GT. DBN3(J,K))   EN3(K) = EN3(K) + 1
      IF(DN1 .GT. UT1(J,K))   UN1(K) = UN1(K) + 1
      IF(DN2 .GT. UT2(J,K))   UN2(K) = UN2(K) + 1
      IF(DN3 .GT. UT2(J,K))   UN3(K) = UN3(K) + 1
   50 CONTINUE
      IF (C .LT. CL(J,3))    KNTC = KNTC + 1
      IF (G .LT. FL(J,3))    KNTG = KNTG + 1
      WRITE (1,70)  L, C, (CN(L1), L1=1,3), DN1, (UN1(L2), L2=1,3), DN2,
     $(UN2(L3), L3=1,3), DN3, (UN3(L4), L4=1,3)
   70 FORMAT (2X, "I=", I4, 1X, "CHI-SQ C=", F6.2, 1X, "SP=", 3I4, 3X, ",
     $K-S-UNIF DN=", F5.3, 1X, "SP=", 3I4, 1X, "DN+=", F[illegible]3, 1X, "SP=",
     $3I4, 1X, "DN-=", F6.3, 1X, "SP=", 3I4)
      WRITE (1,80)  G, (GN(L1), L1=1,3), BN1, (EN1(L2), L2=1,3), BN2, (E
     $N2(L3), L3=1,3), BN3, (EN3(L4), L4=1,3)
```

```
 80 FORMAT (9X, "GNPKO  G=", F6.2, 1X, "SP=", 3I4, 3X, "K-S-EXP  DN=",
   $F5.3, 1X, "SP=", 3I4, 1X, "DW+=", F6.3, 1X, "SP=", 3I4, 1X, "DN-="
   $, F6.3, 1X, "SP=", 3I4)
101 CONTINUE
    WRITE (1,90)  KNTC, KNTG
 90 FORMAT (/, 3X, "CN-= ", I5, 2X, "GBS-= ", I5, /)
102 CONTINUE
103 CONTINUF
    STOP
    END
    SUBROUTINE PDFE (Y,F)
    COMMON LAMDA
    REAL LAMDA
    IF (Y .GT. 0.0)   GO TO 5
    F = 0.0
    RETURN
  5 F = 1.0 - EXP(-Y/LAMDA)
    RETURN
    END
    SUBROUTINE PDFU (Z,F)
    IF (Z .GT. 0.)   GO TO 5
    F = 0.
    RETURN
  5 IF (Z .LT. 1.)   GO TO 10
    F = 1.0
    RETURN
 10 F = Z
    RETURN
    END
```