ABSTRACT

OPTIMAL CONSTANT OUTPUT FEEDBACK CONTROL
FOR LINEAR SYSTEMS

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This report discusses the optimal control of a linear time-invariant system with respect to a quadratic cost functional (performance criterion). The problem is posed with the constraint that the control vector is a linear time-invariant function of the output vector

$$[u(t) = G\, y(t)]$$

The cost functional is then minimized to find an optimal gain $G^*$. Two algorithms for computing $G^*$ are presented, of which the first is simpler to implement, while the second ensures convergence for a broader class of problems. A numerical example is solved.

# ACKNOWLEDGEMENTS

## ACKNOWLEDGEMENTS

The author is deeply indebted to Dr. M. Vidyasagar for suggesting the project, his criticism, advice and invaluable suggestions. He has been a constant source of inspiration during the course of the preparation of this thesis for which I express my gratitude.

TABLE OF CONTENTS

## TABLE OF CONTENTS

LIST OF FIGURES

## LIST OF FIGURES

# I. INTRODUCTION

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In a control system, if the input or the commanding signal is predetermined and will not change no matter what the outcome of the control is, the system is said to be an open-loop control system. It is clear that an open-loop control system is not a good system. A desirable control signal should react to the behavior of the system. If the system behaves well, no change in the control signal is necessary; otherwise, a proper change in the control signal is required to bring the response of the system to a desired one.

Such a system whose input signal depends on the outcome of the control is called a feedback control system.

If a dynamical-equation description of a system is available, it is reasonable to base the choice of the input on the value of the state $x(t)$, the reference input $v(t)$ and possibly on $t$, because the state and the input determine completely the future behavior of the system. Hence, a good control signal should be determined by an equation of the form

$$u(t) = f[v(t), x(t), t] \tag{1.1}$$

This relation is called a control law. Present-day optimal control theory is mainly concerned with how to find the best control law.

In this paper, we study only linear time-invariant dynamical systems. Therefore, it is reasonable to assume that the control law depends linearly on v and x, and is of the form

$$u(t) = v(t) + Kx(t) \tag{1.2}$$

where v stands for a desired reference input and K is some real constant matrix called a feedback gain matrix.

Practically, however, the state variables are not always available for feedback purposes. In this case, the output variables may be used to generate these controls instead of reconstructing the state.

The distinction between state feedback and output feedback should be made. In output feedback, the output y(t) is fed back into the input; in state feedback, the state x(t) is fed back into the input, as shown on the following page in Figures 1.1 and 1.2.

There is more room for manipulation in state feedback than in output feedback since:

(1) The number of state variables is generally larger than the number of output variables;

(2) A solution G exists in K = GC for any K if and only if C is square and non-singular, and hence for any constant matrix G there exists a constant

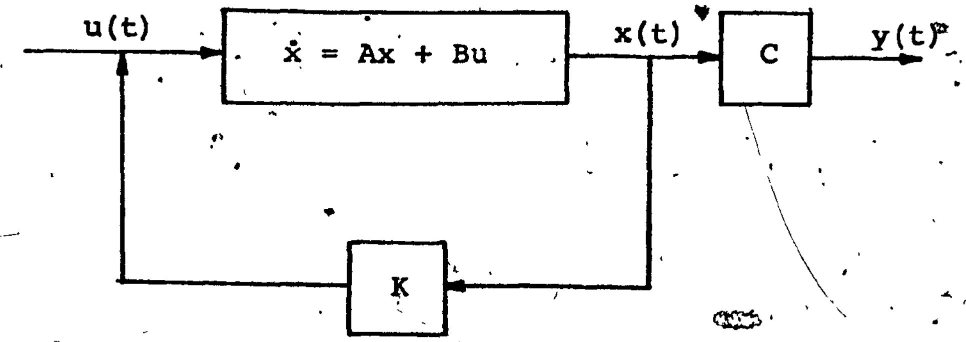


FIG. 1.1 STATE FEEDBACK

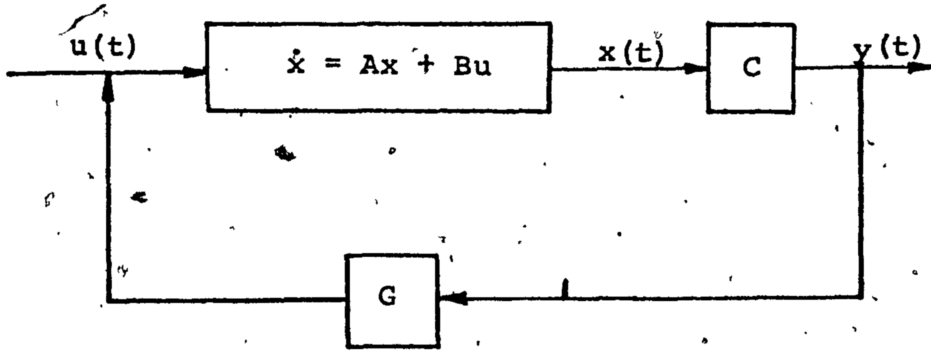


FIG. 1.2 OUTPUT FEEDBACK

matrix K, but the converse is not necessarily true. Therefore, whatever can be achieved by output feedback can always be achieved by state feedback, but the converse is not always true.

In this report, the Linear Regulator Problem, a typical state feedback case, is discussed briefly. An output feedback problem is presented together with a solution given by Levine, Johnson and Athans [4]. Then this last problem is treated differently, by choosing a different cost functional (performance criterion.) The condition for optimality is derived, a simple numerical problem is solved using this condition, and computer algorithms are presented to solve the general case.

# II. LINEAR REGULATOR PROBLEM (STATE FEEDBACK CONTROL)

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Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t), \; x(t_0) \triangleq x_0 \tag{2.1}$$

The design problem is to find the optimal control $u^*(t)$ such that the performance criterion

$$J = \frac{1}{2} x'(T)Px(T) + \frac{1}{2} \int_0^T [x'(t)Q(t)x(t) + u'(t)R(t)u(t)] \, dt \tag{2.2}$$

is minimized, where the matrices $P$ and $Q$ are symmetric and positive semidefinite, and $R$ is symmetric and positive definite.

The solution of this problem is given by

$$u^*(t) = M^*(t)\, x(t) = - R^{-1}(t)\, B'(t)\, K^*(t)\, x(t) \tag{2.3}$$

where the gain $K$ is found from the Ricatti equation.

$$\dot{K} = - Q - A'K - KA + KBR^{-1} B'K \tag{2.4}$$

$$K(T) = P \tag{2.5}$$

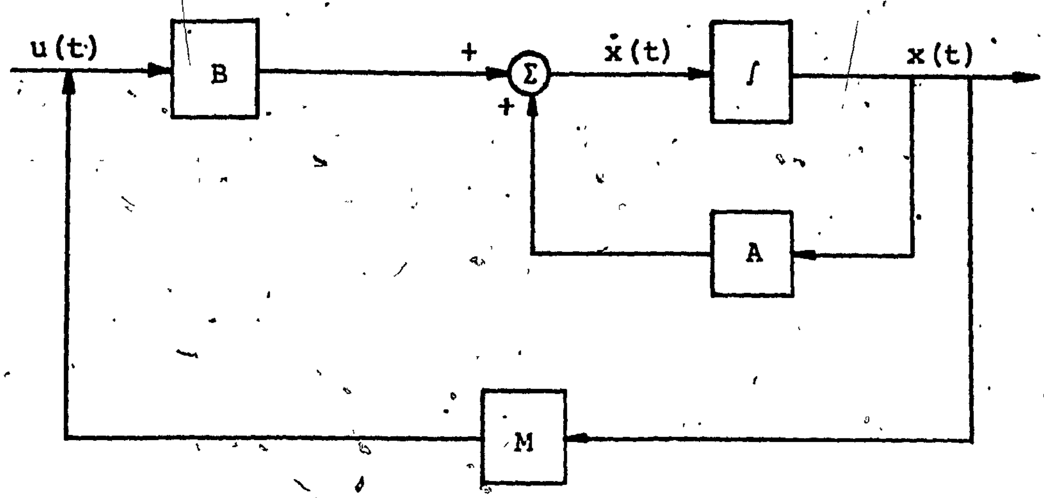


FIG. 2.1 BLOCK DIAGRAM OF STATE FEEDBACK CONTROL SYSTEM

In the time-invariant case ($T = \infty$), Q, R, A, B and K are constant. Therefore,

$$\dot{K} = 0 \tag{2.6}$$

Hence, the Ricatti equation becomes

$$0 = -Q - A'K - KA + KBR^{-1} B'K \tag{2.7}$$

This gives a constant feedback gain

$$M = -R^{-1} B'K \tag{2.8}$$

The optimal control becomes

$$u^*(t) = -R^{-1} B'K^* x(t) \tag{2.9}$$

The state equation of the optimal closed-loop system is given by

$$\dot{x}(t) = (A - BR^{-1} B'K)x(t) \tag{2.10}$$

which is always stable regardless of the stability of A, as long as the system is controllable, i.e.,

$$\text{rank}\,[B|AB|\ldots\ldots..|A^{n-1} B] = n \tag{2.11}$$

## III. METHOD OF LEVINE, JOHNSON AND ATHANS

## III. METHOD OF LEVINE, JOHNSON AND ATHANS[1]

The case considered is the control of a time-invariant linear system

$$\dot{x}(t) = A\,x(t) + B\,u(t), \quad x(t_0) \triangleq x_0 \tag{3.1}$$

with outputs

$$y(t) = C\,x(t) \tag{3.2}$$

and a control law

$$u(t) = G\,y(t) \tag{3.3}$$

where the elements of the (mxr) constant gain matrix G are to be chosen so as to minimize the infinite-time quadratic criterion

$$J(G) = \tfrac{1}{2} \int_0^\infty [x'(t)\,Q\,x(t) + u'(t)\,R\,u(t)]\,dt \tag{3.4}$$

From equations (3.1) to (3.3), the closed-loop system is governed by

$$\dot{x}(t) = (A + BGC)x(t), \quad x(0) = x_0 \tag{3.5}$$

which has a solution

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[1] IEEE Transactions on Automatic Control, December, 1971.

$$x(t) = \phi(t,0)x_0$$

$$= e^{(A + BGC)t} . x_0 \qquad (3.6)$$

Theorem:

Any matrix $G^*$ which satisfies

$$J(G^*) \leq J(G) \qquad (3.7)$$

for the above cost functional, also satisfies the matrix equations:

$$G^* = - R^{-1} B'K^*L^*C'(CL^*C')^{-1} \qquad (3.8)$$

where

$$K^* = \int_0^\infty e^{A^{*'}\tau}(Q+C'G^{*'}RG^*C)e^{A^*\tau}\, d\tau \qquad (3.9)$$

$$L^* = \int_0^\infty e^{A^*\sigma} X_0 e^{A^{*'}\sigma}\, d\sigma \qquad (3.10)$$

$$A^* = A + BG^*C \qquad (3.11)$$

and the condition that $A^*$ be stable (i.e., have all eigenvalues with negative reals parts.) Furthermore, if $X_0$ is positive definite and $G^*, K^*$ and $L^*$ are solutions of (3.8) to (3.11), then $K^*$ is a positive semidefinite (definite if $[A,Q^{\frac{1}{2}}]$ is observable) solution of

$$0 = K^*A^* + A^{*'}K^* + Q + C'G^{*'} RG^*C \qquad (3.12)$$

and $L^*$ is a positive definite solution of

$$0 = L^*A^{*\prime} + A^*L^* + X_0 \tag{3.13}$$

The following computational algorithm is suggested:

(1) Choose a matrix $K_0$,

e.g., the solution of the Ricatti equation

$$0 = K^*A + A'K^* + Q - K^*BR^{-1} B'K^* \tag{3.14}$$

(2) Solve the nonlinear algebraic matrix equation

$$0 = L_{n-1}(A+BG_{n-1}C)' + (A+BG_{n-1}C)L_{n-1} + X_0 \tag{3.15}$$

where

$$G_{n-1} = - R^{-1} B' K_{n-1}L_{n-1} C'(CL_{n-1}C')^{-1} \tag{3.16}$$

for $n = 1$ (giving $L_0$ and $G_0$).

(3) Solve the linear algebraic equation

$$0 = K_n(A+BG_{n-1}C) + (A+BG_{n-1}C)'K_n +$$

$$+ C'G'_{n-1}RG_{n-1}C + Q \tag{3.17}$$

for $n = 1$ (giving $K_1$).

(4) Repeat steps (2) and (3) for $n = 2,3,\ldots,$ giving sequences $\{K_n\},\{L_n\},\{G_n\}$.

For this computational algorithm, the following result is shown in [4]:

Corollary:

Consider the algorithm (3.14) to (3.17) at step n. Assuming Q is positive definite and $(A + BG_{n-1}C)$ is stable, there exists a unique positive definite $K_n$ satisfying (3.17). Furthermore, if a positive definite solution, $L_{n-1}$, of (3.15) exists, then

$$J(G_n) \leq J(G_{n-1}) \tag{3.18}$$

Comments:

Several comments may be made regarding this theorem.

(1) The conditions are only necessary, i.e., there may exist solutions to (3.8) to (3.11), or (3.12) and (3.13) which are not (globally) optimal.

(2) If no matrix G will stabilize (A + BGC), the problem is meaningless, since J(G) is infinite.

(3) If we take complete state feedback, with $C = I$ (invertible), the matrix $L^*$ (and hence any initial-state dependence) drops out of (3.8), and we obtain the standard gains for the linear-quadratic regulator problem:

$$G^* = -R^{-1}B'K^* \tag{3.19}$$

where

$$0 = K^*A + A'K^* + Q - K^*BR^{-1}B'K^* \tag{3.20}$$

(4) Convergence of the solution process is not implied by either the theorem or the computational algorithm. In particular, if at any stage $A + BG_nC$ is not stable, the algorithm comes to a halt.

# IV. NEW METHOD

# IV. NEW METHOD

## 4.1 CASE

The problem discussed here is for a time-invariant system, and an appropriate matrix of feedback gains is to be chosen.

The time-invariant system has an $n^{th}$ order state vector x(t), an $m^{th}$ order control vector u(t), and an $r^{th}$ order output vector y(t).

These are related as follows:

$$\dot{x}(t) = A\, x(t) + B\, u(t) \tag{4.1}$$

$$y(t) = C\, x(t) \tag{4.2}$$

$$u(t) = G\, y(t) \tag{4.3}$$

This problem was treated by Levine and Athans, choosing a standard infinite-final-time quadratic cost functional

$$J = \tfrac{1}{2} \int_0^{\infty} [x'(t) Q\, x(t) + u'(t) R\, u(t)]\, dt \tag{4.4}$$

This cost functional has two disadvantages:

(i) In a physical feedback control problem, the quantity to be minimized is the norm of the feedback gain, not the norm of the control function.

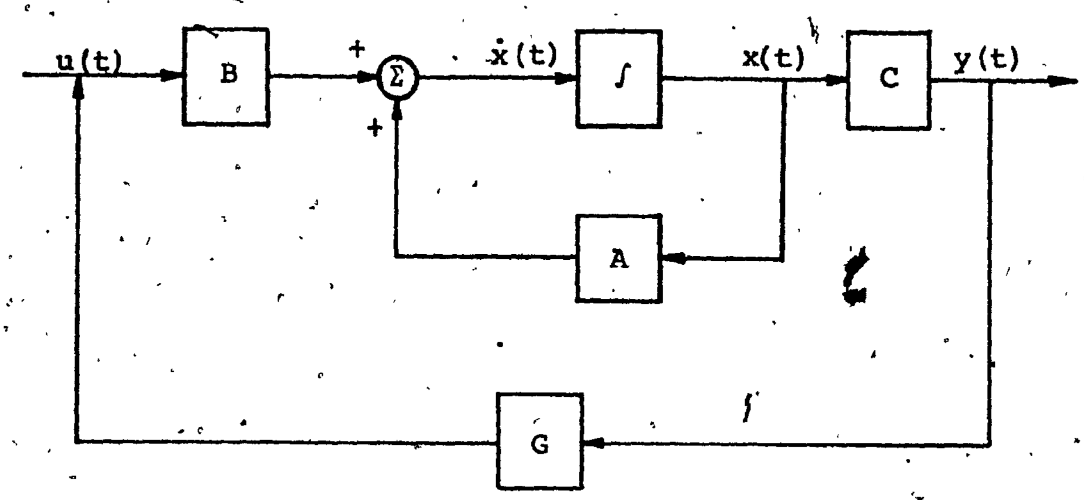


FIG.4.1 BLOCK DIAGRAM OF OUTPUT FEEDBACK CONTROL SYSTEM

(ii) For the objective function (4.4), it is not clear whether an optimal $G$ exists.

This is why in this paper, another cost functional is considered, namely

$$J = \frac{1}{2} \int_0^\infty x'(t) Q\, x(t)\, dt + \frac{1}{2} \operatorname{tr} (G'G) \tag{4.5}$$

where "tr" denotes the trace of a matrix.

The system (4.1), (4.2) and (4.3) can be rewritten as

$$\dot{x}(t) = (A+BGC)x(t) \tag{4.6}$$

Therefore,

$$x(t) = \phi(t,0)x(0) \tag{4.7}$$

where $\phi(t,0)$ is the fundamental transition matrix of the system, given by

$$\phi(t,0) = \exp\,[(A+BGC)t] \tag{4.8}$$

This enables us to rewrite the cost functional as

$$J = \frac{1}{2}\, x'(0)\left[\int_0^\infty \phi'(t,0) Q\, \phi(t,0)dt\right] x(0) + \frac{1}{2} \operatorname{tr}(G'G) \tag{4.9}$$

Averaging out the effects of the initial condition $x(0)$, and making use of the trace identity

$$z'Mz = tr(Mzz') \tag{4.10}$$

we get

$$J = \frac{1}{2}\, tr \int_0^\infty \phi'(t,0)Q\, \phi(t,0)dt + \frac{1}{2}\, tr(G'G)$$

$$= \frac{1}{2}\, tr \int_0^\infty e^{(A+BGC)'t} Q\, e^{(A+BGC)t} dt + \frac{1}{2}\, tr(G'G) \tag{4.11}$$

## 4.2 FORMULATION OF THE PROBLEM

Given the time-invariant linear system

$$\dot{x}(t) = (A+BGC)\, x(t) \tag{4.6}$$

where G is the control parameter.

Given also the cost functional

$$J = \frac{1}{2}\, tr \int_0^\infty e^{(A+BGC)'t}\, Q\, e^{(A+BGC)t} dt + \frac{1}{2}\, tr(G'G) \tag{4.11}$$

Find G* which minimizes the cost functional (4.11), subject to the constraint imposed by the system (4.6).

## 4.3 DERIVATION OF NECESSARY CONDITION FOR OPTIMALITY

The averaged cost functional in (4.11) can be written as

$$J = \frac{1}{2}\, tr\, P + \frac{1}{2}\, tr(G'G) \tag{4.12}$$

where

$$P = \int_0^\infty e^{(A+BGC)'t}\, Q\, e^{(A+BGC)t}\, dt \tag{4.13}$$

Therefore P is the solution of the Lyapunov matrix equation

$$(A+BGC)'P + P(A+BGC) = -Q \qquad (4.14)$$

The gradient of J with respect to G, is found by changing G to $G + \Delta G$; consequently P changes to $P + \Delta P$ and J to $J + \Delta J$. Therefore

$$[A + B(G+\Delta G)C]'(P+\Delta P) + (P+\Delta P)[A + B(G+\Delta G)C] = -Q$$

$$(A + BGC + B\Delta GC)'(P+\Delta P) + (P+\Delta P)(A + BGC + B\Delta GC) = -Q \qquad (4.15)$$

Neglecting second order terms in $\Delta P$ and $\Delta G$, we get

$$(A+BGC)'P + (A+BGC)'\Delta P + (B\Delta GC)'P + P(A+BGC) + \\ + \Delta P(A+BGC) + P(B\Delta GC) = -Q \qquad (4.16)$$

Subtracting (4.14) from (4.16), we get

$$(A+BGC)'\Delta P + \Delta P(A+BGC) + (B\Delta GC)'P + P(B\Delta GC) = 0$$

i.e., $\Delta P$ is the solution of the equation

$$(A+BGC)'\Delta P + \Delta P(A+BGC) = -[(B\Delta GC)'P + P(B\Delta GC)] \qquad (4.17)$$

which is a Lyapunov matrix equation. Therefore,

$$\Delta P = {}_0\int^{\infty} e^{(A+BGC)'t}\,[(B\Delta GC)'P + P(B\Delta GC)]\,e^{(A+BGC)t}\,dt$$

$$= {}_0\int^{\infty} e^{(A+BGC)'t}\,(C'\Delta G'B'P + PB\Delta GC)\,e^{(A+BGC)t}\,dt$$

$$= {}_0\int^{\infty} e^{(A+BGC)'t}\,C'\Delta G'B'P\;e^{(A+BGC)t}\,dt +$$

$$+ {}_0\int^{\infty} e^{(A+BGC)'t}\,PB\Delta GC\;e^{(A+BGC)t}\,dt \qquad (4.18)$$

$$\mathrm{tr}\,\Delta P = \mathrm{tr}\,{}_0\int^{\infty} (e^{(A+BGC)'t}\,C'\Delta G')(B'P\,e^{(A+BGC)t})\,dt$$

$$+ \mathrm{tr}\,{}_0\int^{\infty} (e^{(A+BGC)'t}PB\Delta G)(C\,e^{(A+BGC)t})\,dt \qquad (4.19)$$

But,

$$\mathrm{tr}(MN') = \mathrm{tr}(M'N) = \mathrm{tr}(M'N)' = \mathrm{tr}(N'M) \qquad (4.20)$$

Making use of (4.20), we get

$$\mathrm{tr}\,\Delta P = \mathrm{tr}\,{}_0\int^{\infty} B'P\,e^{(A+BGC)t}\,.\,e^{(A+BGC)'t}\,C'\Delta G'\,dt +$$

$$+ \mathrm{tr}\,{}_0\int^{\infty} \Delta G'B'P'\,e^{(A+BGC)t}\,.\,e^{(A+BGC)'t}\,C'\,dt$$

(4.21)

Now, let us define the matrix L to be the solution of the Lyapunov equation

$$L(A+BGC)' + (A+BGC)L = -I \qquad (4.22)$$

Therefore,

$$L = {}_0\int^{\infty} e^{(A+BGC)t}\,e^{(A+BGC)'t}\,dt \qquad (4.23)$$

Then, equation (4.21) becomes:

$$\begin{aligned} tr\Delta P &= tr(B'PLC'\Delta G') + tr(\Delta G'B'PLC') \\ &= tr[(CLPB)'\Delta G'] + tr[\Delta G'(CLPB)'] \\ &= 2\ tr[\Delta G'(CLPB)'] \end{aligned} \qquad (4.24)$$

Denoting the inner product of two matrices $M$ and $N$ as $\langle M,N\rangle$, and defining it as $\langle M,N\rangle = tr(MN') = tr(M'N)$, (4.24) becomes

$$tr\Delta P = 2\ \langle \Delta G, (CLPB)'\rangle \qquad (4.25)$$

In (4.12), changing $J$ to $J+\Delta J$, $P$ to $P+\Delta P$ and $G$ to $G+\Delta G$, we get

$$J + \Delta J = \tfrac{1}{2}\ tr(P+\Delta P) + \tfrac{1}{2}\ tr\ [(G+\Delta G)'(G+\Delta G)] \qquad (4.26)$$

Neglecting the second order term in $\Delta G$, and subtracting (4.12) from (4.26), we get

$$\begin{aligned} \Delta J &\doteq \tfrac{1}{2}\ tr\ \Delta P + \tfrac{1}{2}\ tr(G'\Delta G + \Delta G'G) \\ &\doteq \tfrac{1}{2}\ tr\ \Delta P + tr(\Delta G'G) \\ &= \langle \Delta G, (CLPB)'\rangle + \langle \Delta G, G\rangle \end{aligned} \qquad (4.27)$$

Therefore, the gradient of $J$ with respect to $G$ is

$$\frac{\partial J}{\partial G} = (CLPB)' + G \tag{4.28}$$

where P and L are the solutions, respectively, of the equations

$$(A+BGC)'P + P(A+BGC) = -Q \tag{4.29}$$

and

$$(A+BGC)L + L(A+BGC)' = -I \tag{4.30}$$

Since J is a continuous function of G and since $J \to \infty$ as $||G|| \to \infty$, it is clear that J has a global minimum.

For optimal G*

$$\left.\frac{\partial J}{\partial G}\right|_{G^*} = 0 \tag{4.31}$$

Therefore, from equation (4.28), we get

$$G^* = -(CLPB)' \tag{4.32}$$

## 4.4 MAIN RESULT

Theorem:

For the problem described in Section 4.2, assuming (A+BGC) is stable, in order for G* to be optimal, it is necessary that

$$G^* = -(CLPB)' \tag{4.32}$$

where P and L are solutions, respectively, of

$$(A+BG^*C)'P + P(A+BG^*C) + Q = 0 \tag{4.33}$$

and

$$(A+BG^*C)L + L(A+BG^*C)' + I = 0 \tag{4.34}$$

# V. NUMERICAL EXAMPLE

## V. NUMERICAL EXAMPLE

$$A = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c = [0 \quad 1]$$

$$J = \tfrac{1}{2}\,\mathrm{tr}\left[\int_0^{\infty} e^{(A+bgc)'t}\, Q\, e^{(A+bgc)t}\, dt + g^2\right]$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

On the $n^{\underline{th}}$ iteration:

$$g_n = -[0 \quad 1]\, PL \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

where P and L are solutions of

$$\left\{\begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} g_{n-1} [0 \quad 1]\right\} P + P\left\{\begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} g_{n-1} [0 \quad 1]\right\} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$\left\{\begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} g_{n-1} [0 \quad 1]\right\} L + L\left\{\begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} g_{n-1} [0 \quad 1]\right\} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

For the first iteration, an initial gain $g_0 = 0$ is chosen.

Initial value chosen as $g_0 = 0$.

<u>**First iteration:**</u>

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{12} \end{bmatrix}$$

$$L = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{2} \end{bmatrix}$$

$$g_1 = -[0 \quad 1] \begin{bmatrix} \frac{1}{4} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{12} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-1}{12} & \frac{-1}{12} \end{bmatrix} \begin{bmatrix} \frac{1}{6} \\ \frac{1}{2} \end{bmatrix}$$

$$= -\frac{1}{72} - \frac{1}{24}$$

$$= \frac{-1}{18}$$

$$g_1 = -0.05555$$

$$g_1 = -0.05555$$

Second Iteration:

$$P = \begin{bmatrix} \frac{1}{4} & \frac{9}{110} \\ \frac{9}{110} & \frac{81}{1045} \end{bmatrix}$$

$$L = \begin{bmatrix} \frac{37}{76} & \frac{162}{1045} \\ \frac{162}{1045} & \frac{9}{19} \end{bmatrix}$$

$$g_2 = -[0 \quad 1] \; P \; L \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= -\begin{bmatrix} \frac{9}{110} & \frac{81}{1045} \end{bmatrix} \begin{bmatrix} \frac{162}{1045} \\ \frac{9}{19} \end{bmatrix}$$

$$= -\left(\frac{9 x 162}{110 x 1045} + \frac{81 x 9}{1045 x 19}\right)$$

$$= -(0.01268 + 0.03671)$$

$$= -0.04939$$

$$g_2 = -\ 0.04939$$

<u>Third Iteration:</u>

$$P = \begin{bmatrix} 0.25 & 0.08198 \\ 0.08198 & 0.07812 \end{bmatrix}$$

$$L = \begin{bmatrix} 0.32812 & 0.15624 \\ 0.15624 & 0.47646 \end{bmatrix}$$

$$g_1 = -\begin{bmatrix} 0 & 1 \end{bmatrix} \; P \; L \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= -\begin{bmatrix} 0.08198 & 0.07812 \end{bmatrix} \cdot \begin{bmatrix} 0.15624 \\ 0.47646 \end{bmatrix}$$

$$= -(0.01280 + 0.03722)$$

$$= -\ 0.05002$$

$$g_3 = -\ 0.05002$$

**Fourth Iteration:**

$$P = \begin{bmatrix} 0.25 & 0.08196 \\ 0.08196 & 0.07805 \end{bmatrix}$$

$$L = \begin{bmatrix} 0.32806 & 0.15612 \\ 0.15612 & 0.47618 \end{bmatrix}$$

$$g_4 = -[0 \quad 1]\ P\ L \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= -[0.08196 \quad 0.07805] \begin{bmatrix} 0.15612 \\ 0.47618 \end{bmatrix}$$

$$= -(0.01279 + 0.03716)$$

$$= -\ 0.04995$$

$$g_4 = -\ 0.04995$$

Fifth Iteration

$$P = \begin{bmatrix} 0.25 & 0.08196 \\ 0.08196 & 0.07806 \end{bmatrix}$$

$$L = \begin{bmatrix} 0.32806 & 0.15613 \\ 0.15613 & 0.47621 \end{bmatrix}$$

$$g_5 = -[0 \quad 1]\ P\ L \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= -[0.08196 \quad 0.07806] \begin{bmatrix} 0.15613 \\ 0.47621 \end{bmatrix}$$

$$= -(0.01279 + 0.03717)$$

$$= -\ 0.04996$$

$$g_5 = -0.04996$$

<u>Sixth Iteration</u>

$$P = \begin{bmatrix} 0.25 & 0.08196 \\ 0.08196 & 0.07806 \end{bmatrix}$$

$$L = \begin{bmatrix} 0.32806 & 0.15613 \\ 0.15613 & 0.47620 \end{bmatrix}$$

$$g_6 = -[0 \quad 1]\ P\ L \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= -[0.08196 \quad 0.07806] \begin{bmatrix} 0.15613 \\ 0.47620 \end{bmatrix}$$

$$= -(0.01279 + 0.03717)$$

$$g_6 = -0.04996 = g_5$$

Therefore

$$g^* = -0.04996$$

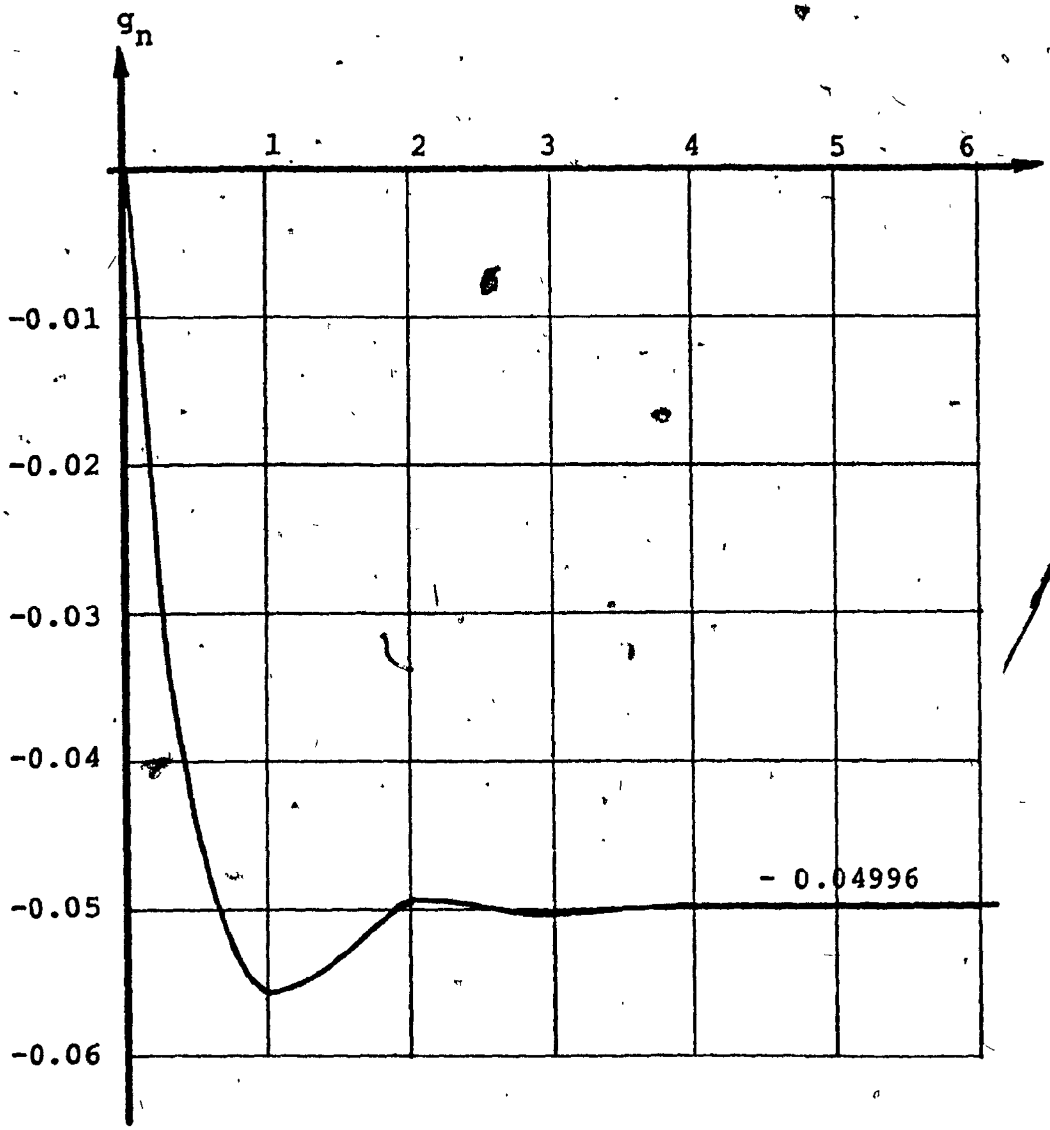


FIG. 5.1 PROGRESS OF ITERATIONS
(NUMERICAL EXAMPLE)

# VI. COMPUTER ALGORITHMS

## VI. COMPUTER ALGORITHMS

(A) A simple stable system is solved using the following algorithm:

(1) Set $G_0 = 0$

$i = 0$

(2) Solve

$$(A+BG_iC)'P_i + P_i(A+BG_iC) + Q = 0 \quad \text{and}$$

$$(A+BG_iC)L_i + L_i(A+BG_iC)' + I = 0$$

for $P_i$ and $L_i$.

(3) $G_{i+1} = -(CL_iP_iB)^{-1}$

(4) If $G_{i+1}$ is very close to $G_i$, stop.

If not, return to step 2, with $i \rightarrow i+1$.

During this procedure, the system is checked for stability at every iteration by checking P for positive definiteness.

(B) The problem discussed in (A) is a very simple one, dealing with a system that remains stable during iterations.

For a more general problem, an iteration process has to be used to ensure stability while iterating.

The following algorithm is proposed for this case.

(1) Choose $G_0$ so that $(A+BG_0C)$ is stable.

Set $i = 0$.

(2) Find $J_i$ using

$$J_i = \tfrac{1}{2}\, tr\, P_i + \tfrac{1}{2}\, tr(G_i'G_i)$$

where $P_i$ is the solution of

$$(A+BG_iC)'P_i + P_i(A+BG_iC) + Q = 0$$

(3) Find $\delta J_i$ using

$$\delta J_i = (CL_i\ P_iB)' + G_i$$

where $L_i$ is the solution of

$$(A+BG_iC)L_i + L_i(A+BG_iC)' + I = 0$$

(4) If $\delta J_i$ is very close to zero, stop.

$$G_i = G^*$$

If not continue.

(5) A step $S_i$ is chosen as follows:

$$S_i = -\delta J_i\ , \quad \text{for } i = 0 \qquad \text{(steepest descent)}$$

$$S_i = -\delta J_i + H_iS_{i-1}\ , \text{ for } i \neq 0$$

where,

$$H_i = \frac{\langle \delta J_i, \delta J_i - \delta J_{i-1} \rangle}{\langle \delta J_{i-1}, \delta J_{i-1} \rangle} \quad \text{(Polak-Ribière)}$$

(6) $$G_{i+1} = G_i + \alpha_i S_i$$

where $\alpha_i$ is chosen by minimizing the function

$$J(G_i + \alpha_i S_i)$$

in one dimension.

(7) Go to (2) with $i \rightarrow i+1$.

# VII. CONCLUSIONS

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In this report, a method has been proposed for determining the optimal output feedback gain matrix for a linear time-invariant system. The method has been illustrated with a simple example, and two computational algorithms have been proposed for the general case. The first one is simpler to implement, while the second ensures convergence for a broader class of problems.

Further work is needed to test and perfect the proposed algorithms.

# VIII. REFERENCES