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On the Domination Number of Grid Graphs

Thi Nhu Mai Vo

A thesis

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Computer Science

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ABSTRACT

On the Domination Number of Grid Graphs

Thi Nhu Mai Vo

A *dominating set* D of a graph $G = (V, E)$ is a subset of V such that every vertex of G is either in D or is adjacent to a vertex in D. The domination number $\gamma(G)$ is the minimum size of a dominating set. Recent work by Cockayne et al. introduce an upper bound for $\gamma(G)$ using star-center patterns. This work presents a new construction for dominating sets on $k \times n$ grid graphs, which relaxes in certain ways the condition that no neighbourhoods overlap in the interior of the graph. For widths up to 12, these sets are smaller in the limit than those obtained using star-center patterns. The constructions cannot give improved bounds if $(k - 13)(n - 13) > 45$. We give a conjecture on the structure of dominating sets which would prove optimal the bounds obtained. We also present two algorithms to generate optimal dominating sets for $k \times n$ grid graphs under our conjecture. Some computational results are given.

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Chapter 1

INTRODUCTION

The $k \times n$ *complete grid graph* $G(k,n)$ has vertex set $V_k \times V_n$ where $V_k = \{1,2,\ldots,k\}$ and $V_n = \{1,2,\ldots,n\}$. Two vertices (i,j) and (i',j') are adjacent when they are consecutive on a row or column (see Fig. 1.1) i.e. when

$$(i = i' \text{ and } j = j' \pm 1) \text{ or } (j = j' \text{ and } i = i' \pm 1)$$

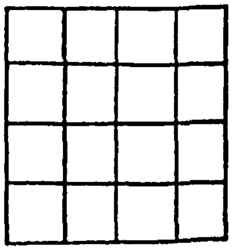


Fig. 1.1: A $G(5,5)$ grid graph

A *dominating set* D of a graph $G = (V,E)$ is a subset of $V(G)$ such that every vertex of G is either in D or is adjacent to a vertex in D. D is a *minimal dominating set* if no proper subgraph of D is a dominating set. The *domination number* $\gamma(G)$ is the minimum size of a dominating set.

The computation of the domination number for graphs is an NP-complete problem [13]. Cockayne et al. [4] have established upper and lower bounds for the domination number of complete grid graphs. Hare et al. [11] present an algorithm to compute the domination number of $k \times n$ complete grid graphs for fixed k. However, the domination number of $G(k,n)$ remains unknown for grid graphs of widths > 15.

The *neighbourhood* $N(v_i)$ of a vertex v_i in G is the set consisting all vertices adjacent to v_i. The *closed neighbourhood* $N[v_i]$ is $N[v_i] = N(v_i) \cup \{v_i\}$. A vertex in common between neighbourhoods is an *overlap*.

We define the *closed neighbourhood* $N[S]$ of S as follows:

$$N[S] = \{v_j \in V(G) : v_j \in N[v_i] \text{ for some } v_i \in S\}$$

We say that S ***dominates*** $N[S]$. Informally, we say that a ***dominator*** v is any member of a dominating set, and that it dominates all vertices in $N[v]$.

In a grid graph, each vertex dominates at most a subgraph of the form indicated in Fig. 1.2.

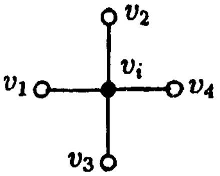


Fig. 1.2: v_i dominates the subgraph formed by $\{v_1, v_2, v_3, v_4, v_i\}$

A ***star center*** set S of a grid graph $G(k,n)$ is a dominating set of G with no overlaps in the interior (Fig. 1.3).

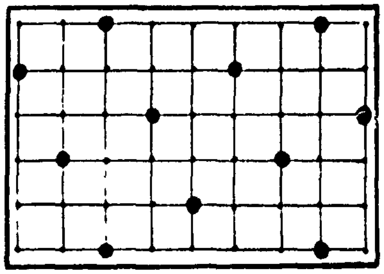


Fig. 1.3: A star-center set of $G(6,9)$

In this paper, we present a new construction for dominating sets on complete $k \times n$ grid graphs which relaxes in certain ways the condition that no neighbourhoods overlap in the interior of the graph. For widths $k \leq 12$, these sets are smaller in the limit than those obtained using the star center pattern. The constructions cannot give improved bounds if

$$(k-13)(n-13) > 45.$$

We conjecture that the smalllest dominating set of $G(k,n)$ realized under the above relaxed overlap condition is optimal. We call this the *relaxed overlap conjecture.*

We describe an algorithm to construct the optimal dominating sets for $k \times n$ grid graphs by exhaustive search under the relaxed overlap conjecture. Our results agree with the optimal results in all cases in which these have been obtained, thus our conjecture has yet to be disproved.

Chapter 2

BACKGROUND

2.1 History of grid graph problems

The study of grid graphs started in the 1890's with the Five Queen Problem on the chessboard [17]. We want to place five queens on the board in such position that they dominate the whole board. A solution is indicated in Fig. 2.1: no smaller number of queens will suffice, so that $\gamma(G) = 5$.

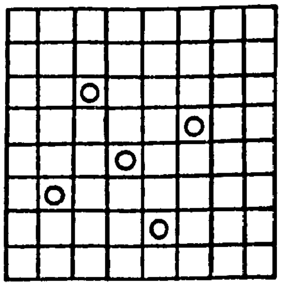


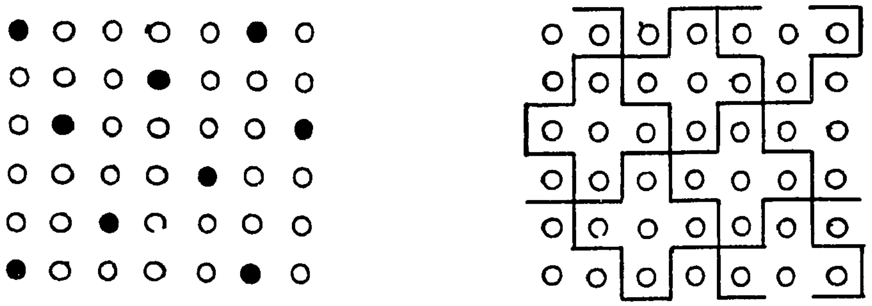
Fig. 2.1: A solution to the Five Queen problem

In more recent times, grid graphs have been used to model a variety of routing problems in street networks. Berge [3] mentions the problem of keeping all points in a network under surveillance by a set of radar stations. In a similar vein, Liu [16] discusses the application of dominance to communications in a network, where a dominating set represents a set of cities which, acting as transmitting stations, can transmit messages to every city in the network.

Consequently, the importance of studying the graph-theoretic properties of grids has attracted more interest. Farley and Hedetneimi [9], Peck [18], Van Scoy [20], Liestman [15], and Ko [14] have studied the problem of fast transmission of information in grid graphs.

Cockayne et al. [4] have established upper and lower bounds for the domination number of complete grid graphs, determined exact values of the domination number for $G(2,n)$, $G(3,n)$, and some $G(4,n)$, and computed $\gamma(G(k,n))$ for several values of n when $k \leq 7$.

Bange et al. [2] introduce the concept of efficient domination in grid graphs. A dominating set $D \in V(G)$ is *efficient* if the distance between every pair of vertices of D is at least 3, i.e., there are no overlaps between neighbourhoods. For an unbounded planar grid graph, it is proved that up to symmetry, there is only one way to choose an efficient dominating set, namely the *tiling pattern* (Fig. 2.2).



(a) The Tiling pattern

(b) Tiling the plane

Fig. 2.2

We observe that the tiling pattern is in fact Cockayne's star-center pattern.

D is an ***efficient near-domination*** of G if D is an efficient dominating set such that the number of uncovered vertices in $V(G) - D$ is minimum. Bange et al. [2] have proved that the tiling pattern produces an optimal efficient near-domination of all $k \times n$ grid graphs where k, $n \geq 7$.

Hare et al. [11] have developed a linear algorithm to compute the domination number of $k \times n$ complete grid graphs for fixed k. This algorithm is based on the recursive definition of the family of $k \times n$ grid graphs and the theory of linear computation. According to this theory, we can solve certain NP-complete problems in linear time when these problems are restricted to some particular family of recursively defined graphs [21]. Any $k \times n$ grid graph $G(k, n)$ can be recursively defined as a composition of $G(k, n-1)$ and the basis graph $G(k, 1)$. This algorithm produces dominating sets for complete grid graphs of heights $k \leq 12$ which are known to be of minimum size.

2.2 Dominating sets and related concepts in graph theory

2.2.1 Dominating sets and independent sets

An *independent set* of G is a subset of V in which no vertices in the set are adjacent.

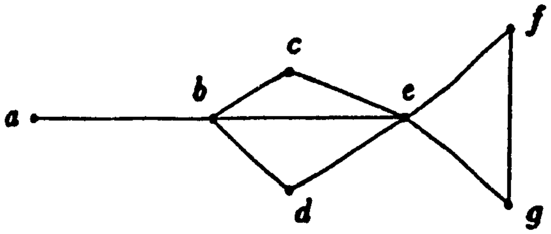


Fig. 2.3

In Fig. 2.3, $\{a, c, d\}$ is an independent set. A *maximal independent set* is an independent set to which no other vertex can be added without destroying its independence property. The set $\{a, c, d, f\}$ in Fig. 2.3 is a maximal independent set, so are sets $\{b, g\}$ and $\{b, f\}$. In general, a graph has many maximal independent sets of different sizes. The *independence number* $\beta_0(G)$ is the size of the largest maximal independent set of G. In Fig. 2.3, $\beta_0(G) = 4$.

There is a close relationship between the dominating sets and the independent sets of a graph G. We observe that:

1. A minimum dominating set may or may not be independent. In Fig. 2.3, the dominating set $\{b, e\}$ is minimal but not independent (b and e are adjacent).
2. Every maximal independent set is a dominating set.

Proof (by contradiction):

Let S be a maximal independent set of $G = (V, E)$. Assume S does not dominate the graph. Then

$$\exists v_i \in V(G) : v_i \notin S \text{ and } v_i \text{ is not adjacent to any } v_j \in S$$

Therefore, we can add v_i to S without destroying S's independence. But then S could not have been maximal $\Rightarrow$ *contradiction.* □

Theorem 2.1 *An independent set is maximal if and only if it is a dominating set.*

Proof:

- Since any vertex not in a maximal independent set is adjacent to one or more vertices in the set, a maximal independent set is also a dominating set.
- An independent set that is also a dominating set must be maximal since any vertex not in a dominating set is adjacent to one or more vertices in the set.

□

Corollary 2.1 *For any graph G,*

$$\gamma(G) \leq \beta_0(G).$$

2.2.2 Dominating sets and edge coverings

In a graph G, a set g of edges is said to *cover G* if every vertex in G is incident on at least one edge in g. A set of edges that covers the graph G is called an *edge covering* of G. In a *minimal covering*, no edge can be removed without destroying its ability to cover the graph. The *covering number* of G is the size of a minimum edge covering.

An edge covering is somewhat similar to a dominating set of edges in the sense that every edge in the graph is either in a covering or is adjacent to some edge in the covering.

2.2.3 Independent dominating number and chromatic number

An *independent dominating set* $I(G)$ of a graph $G = (V, E)$ is a dominating set which is also an independent set of vertices. The *independent dominating number* $i(G)$ is the size of a minimum independent dominating set of G.

A *clique* $I(G)$ of a graph G is a maximal complete subgraph of G.

In the coloring problem, a *proper coloring* of a graph $G = (V, E)$ is an assignment of colors to the vertices of G such that no two adjacent vertices have the same color. A graph G that requires κ different colors for its proper coloring, and no less, is call a *κ-chromatic graph.* We call κ the *chromatic number* of G.

Cockayne et al. [5] show that for any graph G:

$$\kappa(G) = i(I(G)).$$

2.2.4 Dominating sets and matching

A *matching* in a graph $G = (V, E)$ is a subset of edges in which no two edges are adjacent. A *maximal matching* is a matching to which no edge in the graph can be added. The number of edges in a largest matching is called the *matching number* $\beta_1(G)$ of G.

The problem of finding a maximum matching, the so-called matching problem, is closely related to that of finding a minimum dominating set except that in the matching problem, we want to dominate vertices in the graph with edges instead of vertices.

2.2.5 The domination number and the domatic number

A *D-partition* of a graph $G = (V, E)$ is a partition of $V(G)$ into dominating sets. The *domatic number* $d(G)$ of G is the maximum order of a D-partition of G.

Cockayne et al. [6] have observed that if a graph G has domatic number $d(G)$ then every vertex must be adjacent to at least $d(G)-1$ vertices, 1 in each dominating subset of a D-partition of order $d(G)$. Allan and Laskar [1] have established a theorem on the relationship between the domatic number of the complement of a graph G, denoted as $\bar{G}$, $d(\bar{G})$ and the domination number $\gamma(G)$ as follows:

Theorem 2.2 *For any graph G:*

$$\gamma(G) \leq d(\bar{G}).$$

2.3 Existing methods to determine the domination number

2.3.1 Boolean method (1973)

Deo [8] describes a method for obtaining all minimal dominating sets in a graph using Boolean arithmetic on the vertices.

Let n be the number of vertices in the graph $G = (V, E)$.

Let each vertex $v_i \in V(G)$ be treated as a Boolean variable.

$a + b$ denotes the operation of including vertex a or b or both.

ab denotes the operation of including both vertices a and b.

To dominate a vertex $v_i \in V(G)$, we must either include v_i or any of the vertices adjacent to v_i, i.e., we have

$$S(v_i) = v_i + v_{i1} + v_{i2} + \cdots + v_{ik} = 1$$

where v_{i1}, v_{i2}, $\cdots$, v_{ik} are the neighbors of v_i for every v_i in G.

To dominate all vertices $v_i \in V(G)$, we must include all $S(v_i)$ in the dominating set.

Therefore, we form a Boolean product of sums:

$$\theta = \prod_{i=1}^{n} S(v_i) \tag{1}$$

When θ is expressed as a sum of products, each term in it will represent a minimal dominating set. A term with the smallest number of variables represents a minimum dominating set.

Consider the graph in Fig. 2.4.

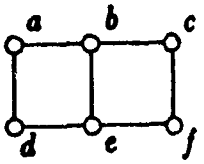


Fig. 2.4

We derive the following expression θ from (1):

$$\begin{aligned}\theta &= (a+b+d)(b+a+c+e)(c+b+f) \\ &\quad (d+a+e)(e+b+d+f)(f+c+e)\end{aligned}$$

Using the absorption law, we arrive finally at

$$\theta = af + be + cd + abc + ace + abf + bce + bfd + dfe$$

Each of the above terms represents a minimal dominating set. Clearly, $\gamma(G) = 2$ for this example.

To apply the Boolean method to find all the minimal dominating sets for a $k \times n$ grid graph, we must work out a $k \times n$-term Boolean product in $k \times n$ variables. Therefore, this method, requiring enumeration of all minimal dominating sets to determine the domination number of a graph, is inefficient and needs prohibitively large amounts of computer memory.

2.3.2 Linear programming method

Linear programming deals with problems in which a linear objective function of several variables is to be maximized or minimized subject to linear equality and inequality constraints on the variables.

We can express such a problem in the following form:

$$
\begin{array}{lll}
\text{minimize} & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n & \text{Objective function} \\
\text{subject to} & \left.\begin{array}{l} a_{11}x_1 + \cdots + a_{1n}x_n \geq b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n \geq b_2 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \geq b_m \end{array}\right\} & \text{Explicit constraints} \\
 & \quad x_1, x_2, \ldots, x_n \geq 0 & \text{(implicit) non-negativity} \\
 & & \text{constraints}
\end{array}
$$

Depending on the domain $D(x_i)$ of x_i, we have a *linear program* if $D(x_i) = \mathbf{R}$ or an *integer program* if $D(x_i) = \mathbf{N}$.

Consider the graph $G = (V, E)$ where $|V| = n$.

Let D be a dominating set of G.

Define integer variables:

$$
x_i = \begin{cases} 1 & \text{if } v_i \in D \\ 0 & \text{otherwise} \end{cases}
$$

$$
a_{ij} = \begin{cases} 1 & \text{if } v_j \in N[v_i] \\ 0 & \text{otherwise} \end{cases}
$$

To cover a vertex $v_i \in V(G)$, either v_i or any of its neighbours must be included in D i.e.

$$\sum_{j=1}^{n} a_{ij} x_j \geq 1 \tag{2}$$

To determine the domination number of graph G, we want to minimize the sum $z = \sum_{i=1}^{n}$ subject to constraints (2) for all $v_i \in V(G)$.

Therefore, the problem of finding the domination number of a graph G is equivalent to the following linear programming problem:

$$\begin{array}{lrl} \text{minimize} & z = & x_1 + x_2 + \cdots + x_n \\ \text{subject to} & & a_{11}x_1 + \cdots + a_{1n}x_n \geq 1 \\ & & a_{21}x_1 + \cdots + a_{2n}x_n \geq 1 \\ & & \vdots \\ & & a_{m1}x_1 + \cdots + a_{mn}x_n \geq 1 \\ & & x_1, x_2, \ldots, x_n \geq 0 \\ & & \text{with all } a_{ij} = 0 \text{ or } 1 \end{array}$$

Solving the above problem with x_i real for all $v_i \in V(G)$, we obtain the lower bound on domination number for any given graph G. The optimum value of the objective function in the 0-1 problem (integer variables) of the above defined linear program is the dominating number of G.

Example: Fig. 2.6 shows the values of a_{ij} for the graph G in Fig. 2.5.

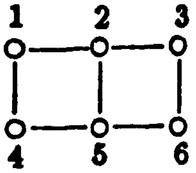


Fig. 2.5

a_{ij}		i					
		1	2	3	4	5	6
j	1	1	1	0	1	0	0
	2	1	1	1	0	1	0
	3	0	1	1	0	0	0
	4	1	0	0	1	1	0
	5	0	1	0	1	1	1
	6	0	0	1	0	1	1

Fig. 2.6: Values of a_{ij} for $G(2,6)$

Interpreted in linear programming terms, our objective is to:

$$\begin{array}{lll}
\text{minimize} & z = & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \\
\text{subject to} & & x_1 + x_2 + x_4 \geq 1 \\
& & x_1 + x_2 + x_3 \geq 1 \\
& & x_2 + x_3 + x_6 \geq 1 \\
& & x_1 + x_4 + x_5 \geq 1 \\
& & x_2 + x_4 + x_5 + x_6 \geq 1 \\
& & x_3 + x_5 + x_6 \geq 1 \\
& & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{array}$$

We obtain the optimal solution

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, 0)$$

with an objective value of $z = 2$. This is a lower bound of the domination number of the graph shown in Fig. 2.5.

The optimum value of the objective function in the 0-1 problem is $z_I = 2$ where

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 0, 0, 0, 0, 1)$$

We have $\gamma(G) = 2$.

In general, for a graph $G(k, n)$ of kn vertices, we must solve kn equations. For problems with large number of variables and/or constraints, integer programming becomes less attractive. Cockayne et al. [7] have developed a branch and bound algorithm to compute the dominating partitions of a graph G using the integer programming approach. The algorithm appears to be exponential in worst case.

2.3.3 Dynamic programming method

The basic idea underlying dynamic programming is *decomposition*. To solve a problem with many variables, the dynamic programming approach determines the variables one at a time (sequentially), decomposes the problem into a series of stages, each corresponding to a subproblem in only one variable, and solves these single-variable subproblems separately.

The families of graphs to which this approach applies must have standard recursive definitions, in terms of a finite set of *basis* graphs and a finite set of *rules of* *composition.* Each rule of composition, however, must be defined in terms of a finite set of k vertices, for some fixed integer k. These vertices are called *terminals.* In this context, we think of graphs as consisting of triples:

$$G = (V, E, T) \quad \text{where} \quad \begin{array}{l} V : \text{set of vertices} \\ E : \text{set of edges} \\ T : \text{set of terminals, } T \subseteq V, |T| = k \end{array}$$

Hare et al. [11] have applied the dynamic programming methodology, developed from the theory of linear computation [21], to generate an algorithm to compute the domination number of $k \times n$ complete grid graphs for fixed k.

This table-driven, dynamic programming algorithm is based on the following recursive definition of the family of $k \times n$ grid graphs:

1. *Basis graph*: the path P_k on k vertices is a grid graph with terminals v_1, $v_2, \cdots$, v_k. In fact, $G(k, 1) = P_k$. Fig. 2.7 represents the basis graph for the family of $2 \times n$ grid graphs.

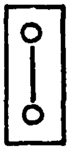


Fig. 2.7: Basis graph for the family of 2 × n grid graphs

2. *Rules of composition*: if $G(k, n-1)$ is a grid graph with terminals u_1, u_2, $\cdots$, u_k, then the graph $G(k, n) = G(k, n-1) \circ P_k$ can be defined as follows:

   $G(k, n) = (V, E, T)$

$$\begin{aligned}
\text{where } V(G(k,n)) &= V(G(k,n-1)) \cup V(P_k) \\
E(G(k,n)) &= E(G(k,n-1)) \cup E(P_k) \cup \{u_iv_i\} \text{ with } i = 1 \cdots k \\
T(G(k,n)) &= \{v_1, v_2, \cdots, v_k\}
\end{aligned}$$

The terminals of $G(k,n)$ are the vertices v_1, v_2, $\cdots$, v_k of the composed $G(k,n)$.

Fig. 2.8 illustrates the composition of $G(k,n-1)$ and P_k to yield $G(k,n)$.

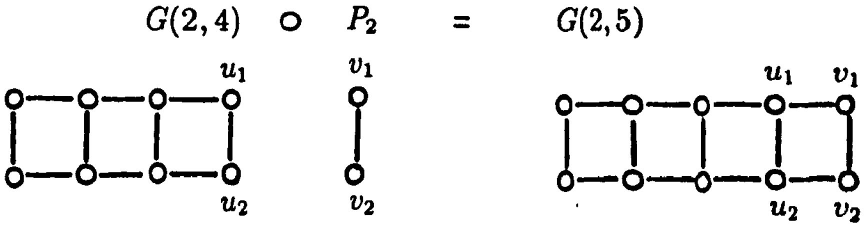


Fig. 2.8: The composition of $G(2,4)$ and P_2 to form $G(2,5)$

The problem of determining the minimum dominating set of $G(k,n)$ can be decomposed into the following 2 problems:

1. determining the minimum subsets $S \subseteq V(G(k,n-1))$ such that $N[S] \supseteq V - T_G$.
2. determining the minimum dominating set of the $T_G \circ P_k$ where T_G denotes the terminals of $G(k,n-1)$.

Since $N[S] \supseteq V - T_G$, for any vertex $v \in T_G$ either

i)	$v \in S$	v is in S
ii)	$v \in N[S] - S$	v is not in S but is dominated by S
iii)	$v \notin N[S]$	v is not dominated by S

To form the dominating set S_n of $G(k,n)$, we combine dominating sets S_{n-1} of $G(k,n-1)$ with vertices of P_k. If a vertex in $V(G(k,n-1)) - T_G$ is not dominated by any vertex in S_{n-1}, then it will not be dominated by any vertex in P_k. Thus we need not consider sets S of $G(k,n-1)$ for which $N[S]$ does not contain $V - T_G$. Consider the grid graph $G(2,2)$ in Fig. 2.9 and all its subsets S. We have:

$$\begin{aligned} V &= \{1,2,3,4\} \\ T_G &= \{2,4\} \\ V - T_G &= \{1,3\} \end{aligned}$$

The subsets $S = \{2\}$ and $S = \{4\}$ can be eliminated in this case.

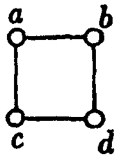
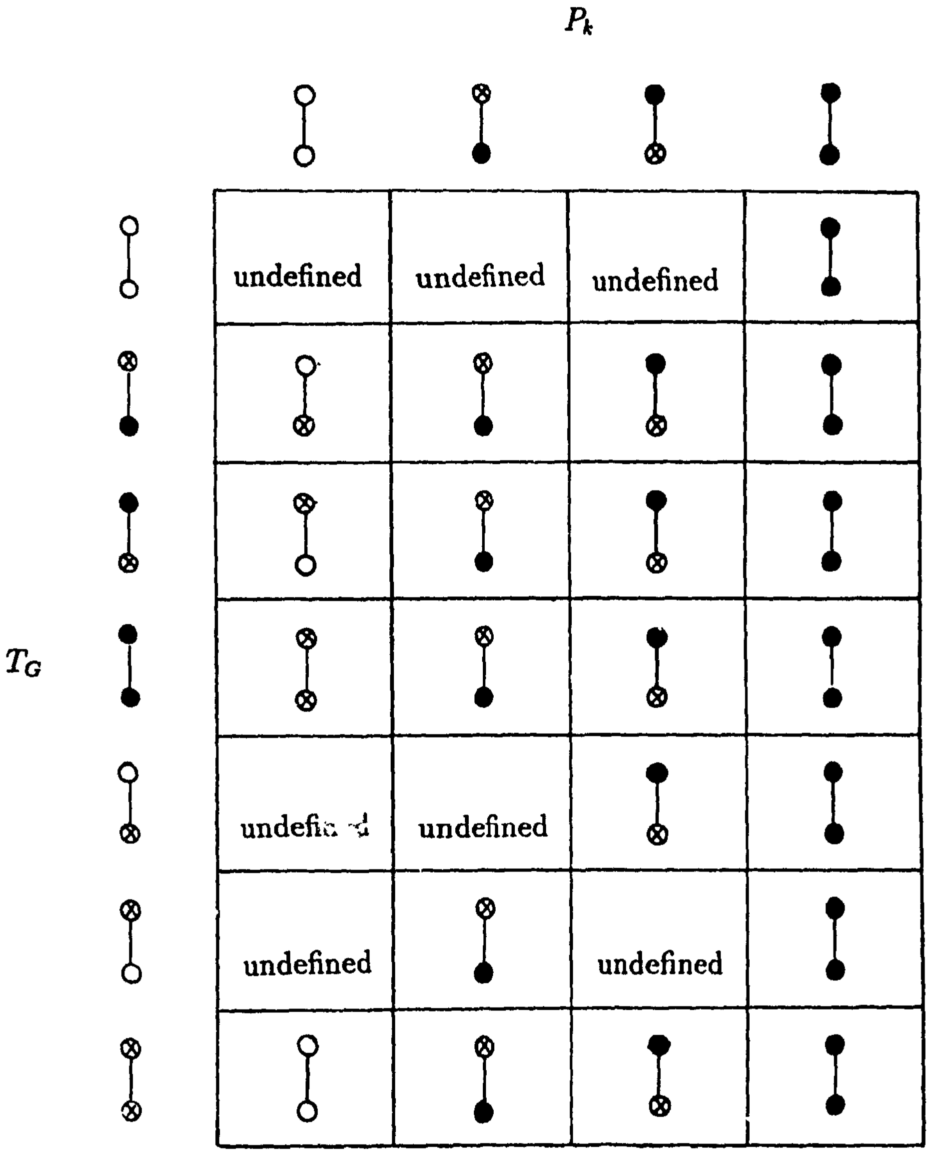


Fig. 2.9: The $G(2,2)$ grid graph

For fixed k, we can construct a state table which represents all possible placements of dominators in $T_G \circ P_k$.

Example: State table for $k = 2$.

In this table, the absence of an entry in a slot represents an *undefined* composition, one in which $V - T_G$ is not contained in $N[S]$.



● v is in S

⊗ v is not in S but is dominated by S

○ v is not dominated by S

Fig. 2.10: Composition of T_G with P_k for $k = 2$

To find the minimum dominating set of $G(2,3)$, we must solve 2 subproblems:

1. determining the minimum subset $S \subset V(G(2,2))$ such that $N[S] \supseteq V - T_G$. We obtain 3 subsets (Fig. 2.11):

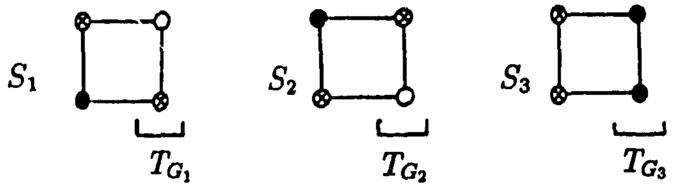


Fig. 2.11: 3 subsets S_1, S_2, and S_3 of $V(G(2,2))$

2. determining the minimum dominating set for $T_G \circ P_k$.

The state table in Fig. 2.10 gives the placements of dominators in P_k which produces the minimum dominating set (Fig. 2.12):

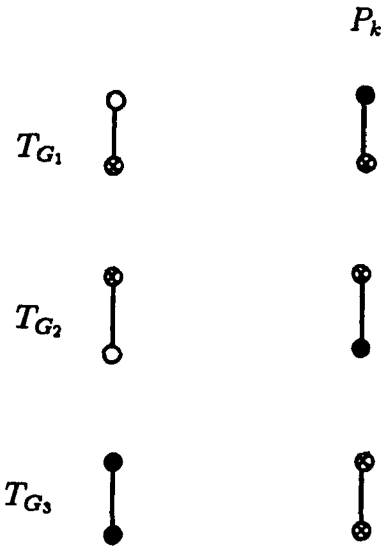


Fig. 2.12: Optimal placements of dominators in P_k

Combining solutions to the above 2 subproblems, we have 3 minimum dominating sets for $G(2,3)$ (see Fig. 2.13):

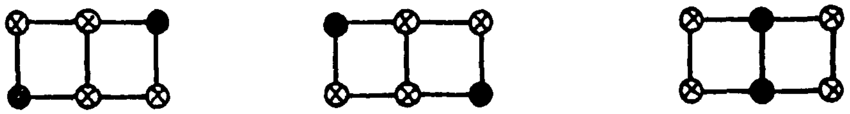


Fig. 2.13: 3 minimum dominating sets for $G(2,3)$

Therefore $\gamma(G(2,3)) = 2$.

This linear algorithm is far superior than any of the existing methods to determine the domination number of complete $k \times n$ grid graphs of fixed k. It has produced domination numbers of $G(k,n)$ for $k \leq 12$ which appear to be minimum.

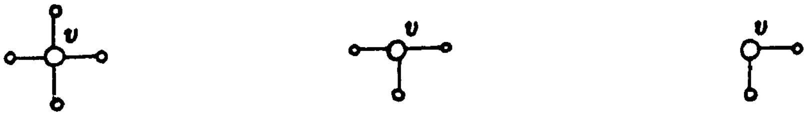
However, the exhaustive state table construction is very time and space consuming. The size of the table grows exponentially with k. For $k = 7$, the table size is 577×128.

Chapter 3

THEORY OF DOMINATION IN GRID GRAPHS

3.1 Star-center pattern

In a grid graph, each vertex dominates a subgraph of one of the following forms:



(a) Interior vertex $|N[v]| = 5$

(b) Edge vertex $|N[v]| = 4$

(c) Corner vertex $|N[v]| = 3$

Fig. 3.1

Let $D = D_I \cup D_E \cup D_C$ where

$$\begin{aligned} D_I &= \{v_i \in Dv_i \text{ in interior }\} \\ D_F &= \{v_i \in Dv_i \text{ on edge }\} \\ D_C &= \{v_i \in Dv_i \text{ in corner }\} \end{aligned}$$

The total number of vertices covered by D is

$$T_c(D_G) = 5|D_I| + 4|D_E| + 3|D_C| = |V| + O_I + O_E + O_C$$

where O_I, O_E, O_C are the number of overlaps generated by D_I, D_E, D_C respectively.

For D to be minimum, we want to minimize the number of overlaps. We are tempted to assume the following:

Assumption 1 *The dominating set D of $G(k,n)$ has no overlaps in the interior:*

$$\forall v, w \in D : v, w \text{ interior vertices } \Rightarrow N[v] \cap N[w] = \emptyset$$

The above assumption forces the star-center pattern in the interior. By extending $G(k,n)$ by a 1-wide strip on all 4 edges, all vertices of $G(k,n)$ are in the interior of $G(k+2,n+2)$. They can be completely covered by the star center set for $G(k+2,n+2)$ (Fig. 3.2). To determine the dominating set of $G(k,n)$, we apply a *pulling algorithm* to pull all edge dominators of $G(k+2,n+2)$ in onto the edge vertices of $G(k,n)$ which they dominate (Fig. 3.3).

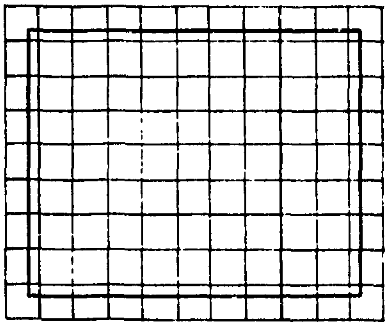


Fig. 3.2: $G(8,10)$ embedded in $G(10,12)$

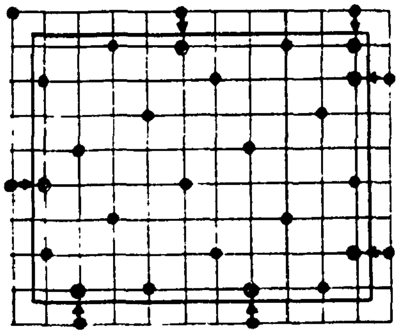


Fig. 3.3: Covering $G(8,10)$ with the star center

We derive an upper bound for $\gamma(G(k,n))$:

Lemma 1 *The minimum number $m(G(k,n))$ of star points of the star-center pattern contained in $G(k,n)$ is $\lfloor \frac{kn}{5} \rfloor$.*

Note: The points concerned do not generally constitute a dominating set. We are simply constructing an orientation of the star-center pattern which minimizes the star points within $G(k,n)$.

Proof:

In the star center pattern, of every 5 horizontally or vertically adjacent vertices, one is a dominator.

Therefore, in 5 consecutive columns of $G(k,n)$, for $n \geq 5$, there will be k star centers, one in each row. So

$$m(G(k,n)) = m(G(k,n-5)) + k$$

for $n \geq 5$.

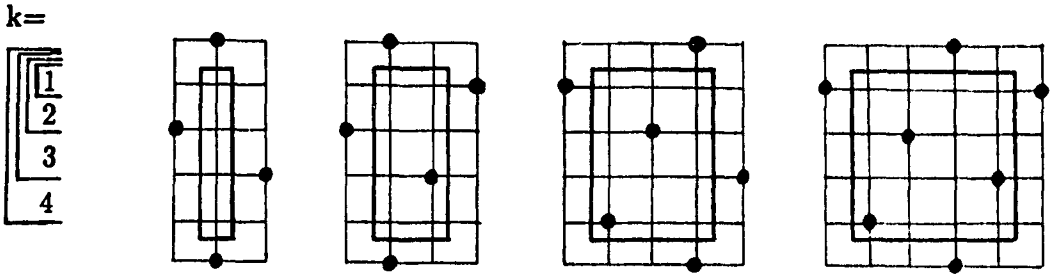
Similarly,

$$m(G(k,n)) = m(G(k-5,n)) + n$$

for $k \geq 5$.

Since $\lfloor \frac{kn}{5} \rfloor = \lfloor \frac{k(n-5)}{5} \rfloor + k$ for $n \geq 5$ and similarly exchanging the roles of k and n, the conclusion follows inductively if we can prove it for $k, n < 5$.

The result for $k, n < 5$ follows by inspection of the graphs below (Fig. 3.4), which give for each $n \in \{1, 2, 3, 4\}$ a placement of the star-center pattern such that the top k rows give a grid graph $G(k,n)$ with $\lfloor \frac{kn}{5} \rfloor$ dominators.



$n = 1$ 2 3 4

Fig. 3.4

Theorem 3.1 *For $G(k,n)$ with $k,n \geq 8$:*

$$\gamma(G(k,n)) \leq \lfloor \frac{(k+2)(n+2)}{5} \rfloor - 4. \tag{3}$$

Proof:

1. $\lambda(G(k,n)) \leq \lfloor \frac{(k+2)(n+2)}{5} \rfloor$

   As discussed earlier, we can cover all vertices in $G(k,n)$ with star-center pattern be extending the graph with a 1-wide strip on all 4 edges giving a rectangle of size $(k+2) \times (n+2)$. Therefore, by Lemma 1, we need at most $\lfloor \frac{(k+2)(n+2)}{5} \rfloor$ dominators to cover $G(k,n)$.

2. $\gamma(G(k,n)) \leq \lfloor \frac{(k+2)(n+2)}{5} \rfloor - 4$

   In the star-center pattern, every 5th vertex on a given row or column is a dominator. Choosing any point and an orientation forces the whole pattern. Therefore, there are 5 possible placements of dominators at each corner of the star-center pattern as shown in Fig. 3.4. Each of these can be re-dominated to save 1 dominator (Fig. 3.5). Hence, to cover $G(k,n)$, we can eliminate 1 dominator each at the 4 corners of the extended graph $G(k+2,n+2)$, i.e.

$$\gamma(G(k,n)) \leq \lfloor\frac{(k+2)(n+2)}{5}\rfloor - 4.$$

□

The upper bound of $\gamma(G(k,n))$ in (3) is consistent with Cockayne et al. upper bound for square grid graphs [4] where

$$\gamma(G(k,n)) \leq \begin{cases} \frac{1}{5}(k^2+4k-16) & k=5a-2 \\ \frac{1}{5}(k^2+4k-17) & k=5a-1 \\ \frac{1}{5}(k^2+4k-20) & k=5a \\ \frac{1}{5}(k^2+4k-20) & k=5a+1 \\ \frac{1}{5}(k^2+4k-17) & k=5a+2 \end{cases}$$

for $k \geq 8$.

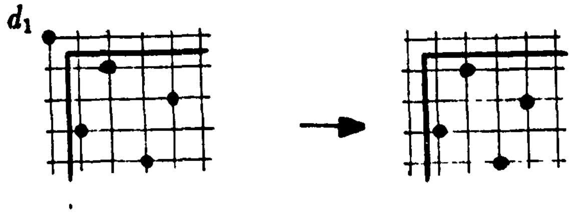
Justification:

For $G(k,n)$ where $k=n$, we have:

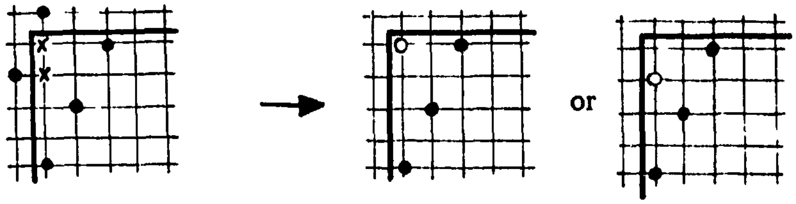
$$\lfloor\frac{(k+2)(n+2)}{5}\rfloor - 4 = \lfloor\frac{(k+2)^2}{5}\rfloor - 4$$

- For $k=5a-2$:

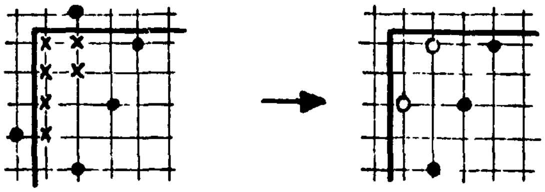
$$\begin{aligned} \lfloor\frac{(k+2)^2}{5}\rfloor - 4 &= \lfloor\frac{(5a-2+2)^2}{5}\rfloor - 4 \\ &= 5a^2 - 4 \\ &= 5(\frac{k+2}{5})^2 - 4 \\ &= \frac{1}{5}(k^2+4k-16) \end{aligned}$$



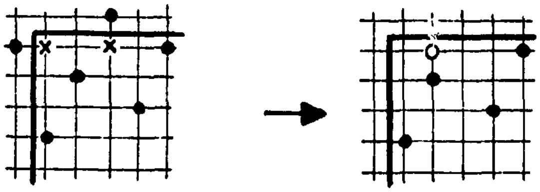
Case 1: d_1 does not cover any vertices of $G(k,n)$ ⇒ d_1 eliminated.



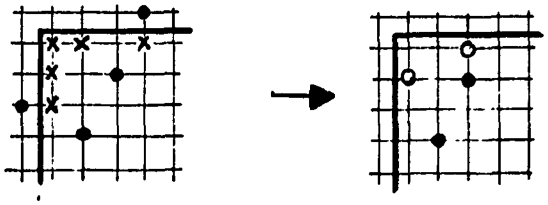
Case 2: We need only 1 dominator to cover the vertices marked 'x' ⇒ one dominator can be eliminated.



Case 3: We need 2 dominators to cover the vertices marked 'x' ⇒ one dominator can be eliminated.



Case 4: We need 1 dominator to cover the vertices marked 'x' ⇒ one dominator can be eliminated.



Case 5: We need 2 dominators to cover the vertices marked 'x' ⇒ one dominator can be eliminated.

Fig. 3.5: Re-arrangements of corner dominators of the star-center pattern

- For $k = 5a - 1$:

$$\begin{aligned}
\lfloor\frac{(k+2)^2}{5}\rfloor - 4 &= \lfloor\frac{(5a-1+2)^2}{5}\rfloor - 4 \\
&= \lfloor\frac{(5a+1)^2}{5}\rfloor - 4 \\
&= 5a^2 + 2a - 4 \\
&= a(5a+2) - 4 \\
&= \frac{k+1}{5}(k+1+2) - 4 \\
&= \frac{1}{5}(k^2 + 4k - 17)
\end{aligned}$$

- For $k = 5a$:

$$\begin{aligned}
\lfloor\frac{(k+2)^2}{5}\rfloor - 4 &= \lfloor\frac{(5a+2)^2}{5}\rfloor - 4 \\
&= 5a^2 + 4a - 4 \\
&= a(5a+4) - 4 \\
&= \frac{k}{5}(k+4) - 4 \\
&= \frac{1}{5}(k^2 + 4k - 20)
\end{aligned}$$

- For $k = 5a + 1$:

$$\begin{aligned}
\lfloor\frac{(k+2)^2}{5}\rfloor - 4 &= \lfloor\frac{(5a+1+2)^2}{5}\rfloor - 4 \\
&= \lfloor\frac{(5a+3)^2}{5}\rfloor - 4 \\
&= 5a^2 + 2a + 1 - 4 \\
&= a(5a+6) - 3 \\
&= \frac{k-1}{5}(k-1+6) - 3 \\
&= \frac{1}{5}(k^2 + 4k - 20)
\end{aligned}$$

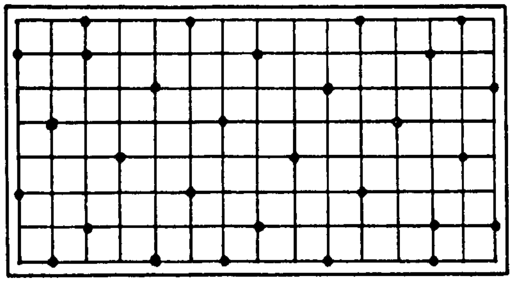
- For $k = 5a + 2$:

$$\begin{aligned}
\lfloor\frac{(k+2)^2}{5}\rfloor - 4 &= \lfloor\frac{(5a+2+2)^2}{5}\rfloor - 4 \\
&= \lfloor\frac{(5a+4)^2}{5}\rfloor - 4 \\
&= 5a^2 + 8a + 3 - 4 \\
&= a(5a+8) - 1 \\
&= \frac{k-2}{5}(k-2+8) - 1 \\
&= \frac{1}{5}(k^2 + 4k - 17)
\end{aligned}$$

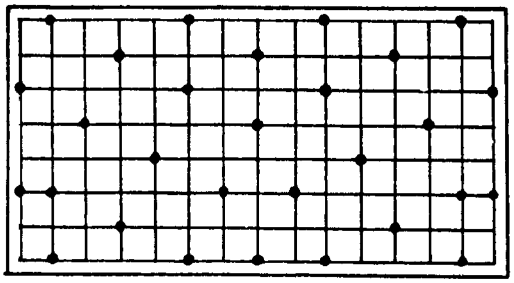
We denote by $\gamma_1(G(k,n))$ the size of the smallest dominating set realized under assumption 1.

3.2 Knight's move pattern

It is tempting to conjecture that $\gamma_1(G(k,n)) = \gamma(G(k,n))$. This conjecture is false however, as the following example (Fig. 3.6) shows:



(a) $\lambda_1(G(8,15)) = 30$ (star-center pattern)



(b) $\lambda(G(8,15)) = 29$

Fig. 3.6: Star-center pattern does not generate minimum dominating set for $G(8,15)$.

It follows that a smaller dominating set may be obtained in spite of overlaps in the interior of G. We are led to relax the overlap condition of assumption 1 as follows:

Assumption 2 *(relaxed overlap condition): There can be at most 1 overlap between* *any 2 interior dominators of a dominating set D.*

$$\forall v, w \in D : v, w \textit{ interior vertices } \Rightarrow |N[v] \cap N[w]| \leq 1$$

Note: This assumption cannot be extended beyond the interior. For example, as indicated in Fig. 3.7, the unique minimal dominating set for $G(3,6)$, apart from symmetry transformations, is:

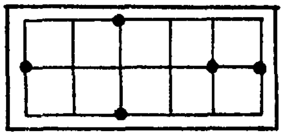


Fig. 3.7: The unique minimal dominating set for $G(3,6)$

We now introduce a convention to represent grid graphs in the Cartesian coordinate system. Any vertex $v_i \in G(k,n)$ can be represented by its coordinates (x,y) if we place $G(k,n)$ in a x-y plane with respect to some origin O. In Fig. 3.8, the Cartesian coordinates $(5,3)$ represent the marked vertex h.

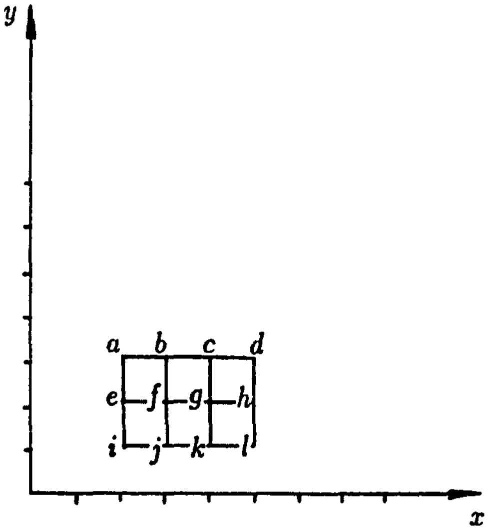


Fig. 3.8: $G(k,n)$ in the Cartesian x-y plane

The entire graph is determined by the coordinates of all its vertices. For convenience, we select a x-y plane such that

y-axis || columns of $G(k,n)$

and x-axis || rows of $G(k,n)$

We define the *Cartesian distance* $d_C = (d_x, d_y)$ between 2 vertices $v_i = (x_i, y_i)$ and $v_j = (x_j, y_j)$ of $G(k,n)$ as:

$$(d_x, d_y) = (x_i - x_j, y_i - y_j).$$

The Cartesian distance of marked vertices e and g in Fig. 3.8 is $(-2,0)$.

Let $\gamma_2(G)$ be the size of the smallest dominating set of G realized under assumption 2. As the star-center does not always produce the optimal dominating set for $G(k,n)$ in general, we are led to the following conjecture:

Conjecture 3.1

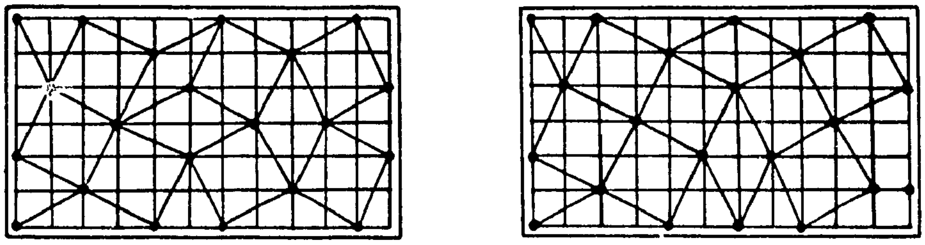
$$\gamma_2(G) = \gamma(G).$$

We have so far found no counterexample to this conjecture. Unfortunately, it seems difficult to prove. One approach we have tried is to consider all possible patterns of dominators which violate the conjecture and show how to transform them to patterns which satisfy the conjecture and cover the same area with either

- the same number of dominators, or
- a smaller dominating set.

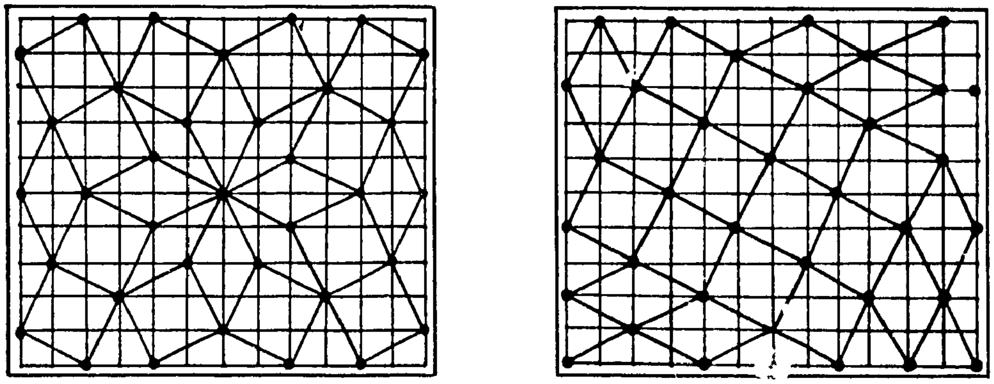
The principle of transformation is to move regions with overlaps not satisfying the conjecture out towards the edge until a Knight's move pattern is obtained. Fig.

3.9 shows that a fairly extensive system of transformation would be necessary to account for all possibilities, and it is not clear how such a system would be built.



Before transformation After transformation

(a) $G(7,12)$



Before transformation After transformation

(b) $G(11,13)$

Fig. 3.9: Transformations of minimal dominating sets of $G(7,12)$ and $G(11,13)$ to satisfy the conjecture.

3.2.1 Implications of assumption 2 (relaxed neighbourhood condition)

The following two theorems establish conditions on interior dominators in a dominating set D realized under assumption 2.

Theorem 3.2 *Interior dominators under assumption 2 cannot be at Cartesian distance* $(\pm 2, \pm 2), (0, \pm 3)$, *or* $(\pm 3, 0)$.

Proof:

Let D be a dominating set of $G(k, n)$.

Let $d_1, d_2 \in D$ be 2 interior dominators.

1. Suppose d_1 and d_2 are at Cartesian distance $(\pm 2, \pm 2)$ as shown in Fig. 3.10. Then vertex A is not covered by any dominators in D. Under assumption 2, neither A nor any of its neighbours can be a dominator. Thus D does not dominate all vertices of the interior $\Rightarrow$ *contradiction.*

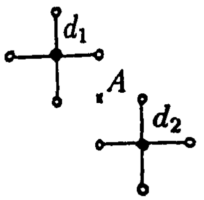


Fig. 3.10: Dominators d_1 and d_2 at distance $d_C = (\pm 2, \pm 2)$

2. Suppose d_1 and d_2 are at distance $(\pm 3, 0)$ as shown in Fig. 3.11. Neither A nor B can be a dominator under assumption 2, hence A

can be dominated only by S, and B only by T. For S and T both to be dominators contradicts assumption 2 as well.

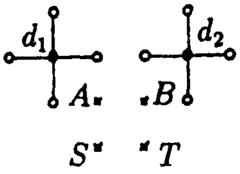


Fig. 3.11: Dominators d_1 and d_2 at distance $d_C = (\pm 3, 0)$

The same argument applies for d_1 and d_2 at distance $(0, \pm 3)$. □

Theorem 3.3 *Given any dominators $d \in G$, under Assumption 2, there must be a dominator d' at distance $(i, 2j)$ or $(2i, j)$ from d, for each choice of (i, j), where $i = \pm 1, j = \pm 1$.*

Proof:

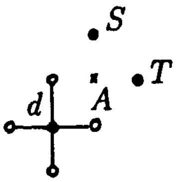


Fig. 3.12: Vertex A in the NE quadrant

We prove the case $i = j = 1$ (NE quadrant in Fig. 3.12).

Vertex A must be covered. Hence, either S (at distance $(1, 2)$ from d) or T (at distance $(2, 1)$ from d) must be a dominator.

The same argument applies for vertex A in the NW, SE, and SW quadrants. □

We say that dominator d' at distance $(\pm1, \pm2)$ from a vertex d is a *Knight's move dominator* of d.

We observe that from any given point, there exist 8 possible Knight's moves, 2 in each of the 4 directions NE, NW, SW, SE as shown in Fig. 3.13.

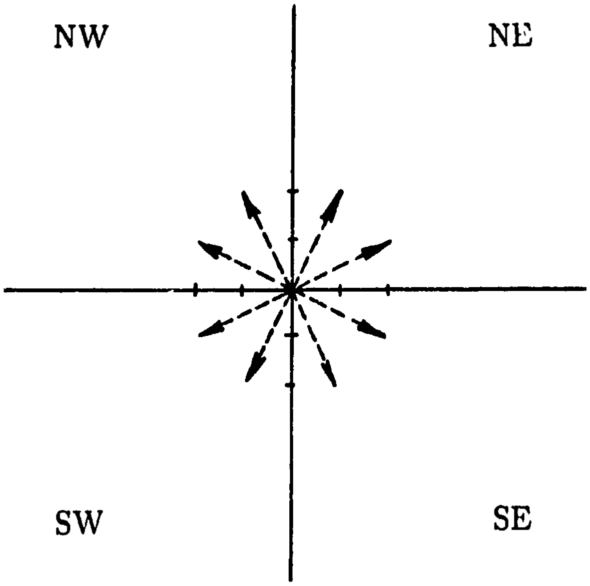


Fig. 3.13: Knight's moves in 4 directions NE, NW, SW, and SE.

Iterating theorem 3, we find that from any given interior vertex in a given direction (NE, NW, SW, SE), there must be a sequence of dominators, each at Knight's move distance from the next (Fig. 3.14).

Any Knight's move dominator can be represented by the Cartesian distance $v = (d_x, d_y)$ from its immediate predecessor. Therefore, a Knight's move dominator sequence $M = (P, V)$ is a sequence of Cartesian distances $V = (v_1, v_2, \cdots)$ from origin $P = (x_0, y_0)$.

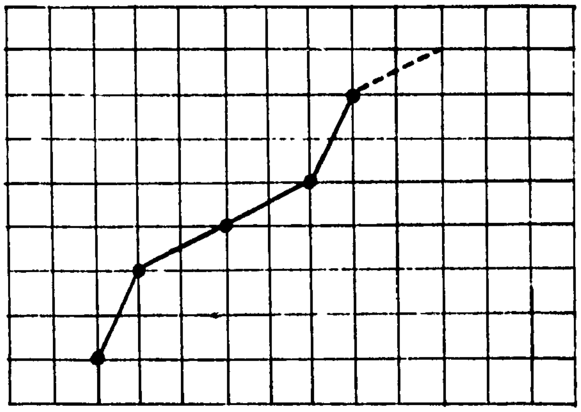


Fig. 3.14: A Knight's move dominator sequence in NE direction.

We define a direction as $S = (s, s') = (\pm 1, \pm 1)$. $S_1 = (s_1, s_1')$ designates a quadrant adjacent to $S_2 = (s_2, s_2')$ if and only if

$$s_1 \; s_2 \; s_1' \; s_2' = -1.$$

Then, the Knight's move sequence M can be abbreviated by:

$$\begin{aligned} M = (P, A, S) \quad \text{where} \quad P &= (x_0, y_0), \\ S &= (s, s') = (\pm 1, \pm 1), \ \text{a direction} \\ A &= (a_1, a_2, a_3, \cdots, a_r), \ a_i \in [1, 2] \\ v_i &= (i(3 - a_i), j a_i) \end{aligned}$$

For example, the Knight's move sequence M in Fig. 3.15 can be represented as follows:

$$M = (P, A, S) \quad \text{where} \quad \begin{aligned} P &= (x_0, y_0), \\ S &= (s, s') = (1, 1) \\ A &= (2, 2, 1, 2, 1) \end{aligned}$$

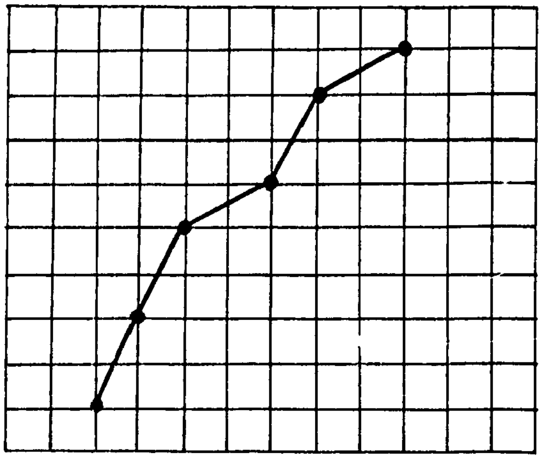


Fig. 3.15: Knight's move sequence $M = (P, A, S)$

We call A the *Knight's descriptor sequence* of M and a_1, a_2, $\cdots$, a_r *descriptor elements* of A. We observe that for any pair of consecutive dominators (Q_{i-1}, Q_i) on M, $a_i = |y_i - y_{i-1}|$. So, a_i represents the y-distance between Q_{i-1} and Q_i.

Two Knight's move dominators in adjacent quadrants will force the third dominator as stated in the following theorem:

Theorem 3.4 *Under Assumption 2, a dominator P together with 2 Knight's move dominators Q, R in adjacent quadrants determines a dominator $S = Q + R - P$ (vector arithmetic).*

Proof:

Consider the NE and SE quadrants of P without loss of generality. There are 3 possible configurations of P, Q, and S, as indicated in Fig. 3.16.

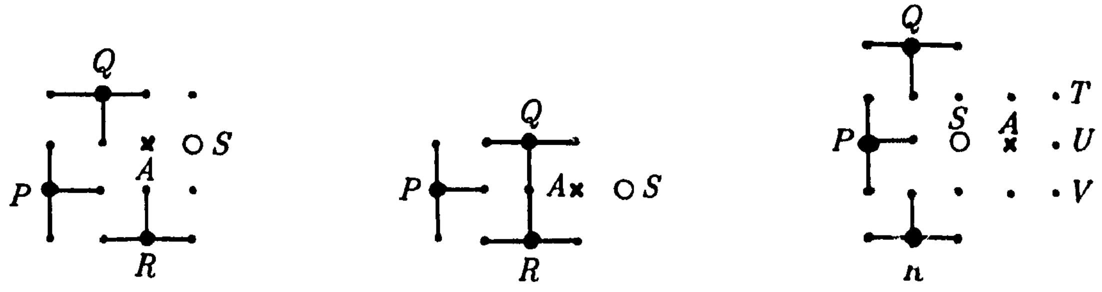


Fig. 3.16: Three configurations of dominators (P, Q, R)

Case (a): $Q = P + (1,2), R = P + (2,-1)$

To cover A, $S = Q+(R-P) = P+(3,1)$ must be a dominator. S results from the Knight's move $(2,-1)$ from Q or $(1,2)$ from R.

We observe that P, Q, R, S forms an ***oblique square.*** Another oblique square results from $Q = P+(2,1), R = P+(1,-2), S = P+(3,-1)$.

Case (b): $Q = P + (2,1), R = P + (2,-1)$

To cover A, $S = Q + R - P = P + (4,0)$ must be a dominator.

We observe that P, Q, R, S forms a ***horizontal diamond.***

Case (c): $Q = P + (1,2), R = P + (1,-2)$

If A is not a dominator, then $T, U,$ and V must be dominators to assure the dominance property of D. This fact violates assumption 2. Therefore, $A = Q + R - P = P + (2,0)$ must be a dominator. Here P, Q, R, S forms a *vertical diamond.*

In each case, the implied dominator is at Knight's distance from Q and R. □

We now have sufficient conditions to state a theorem which results in a new construction of dominating sets for complete $k \times n$ grid graphs.

Let $M = (P, A, S_M)$ and $N = (P, B, S_N)$ be 2 sequences of adjacent Knight's move dominators where

$$\begin{aligned} P &= (x_0, y_0) \\ A &= \{a_1, a_2, \cdots, a_m\} \\ S_M &= \{s_1, s_2\} \\ B &= \{b_1, b_2, \cdots, b_n\} \\ S_N &= \{s'_1, s'_2\} \\ \text{and } s_1\ s_2\ s'_1\ s'_2 &= -1 \end{aligned}$$

Let $Q_1, Q_2, \cdots, Q_m$ be consecutive dominators on M.

Let $R_1, R_2, \cdots, R_n$ be consecutive dominators on N.

We have:

Theorem 3.5 *A dominator P together with 2 sequences of adjacent Knight's move dominators $M = (P, A, S_M)$ and $N = (P, B, S_N)$ completely determine all dominators within the area bounded by M, N, M', N' where $M' = (R_n, A, S_M)$ and $N' = (Q_m, B, S_N)$.*

Proof:

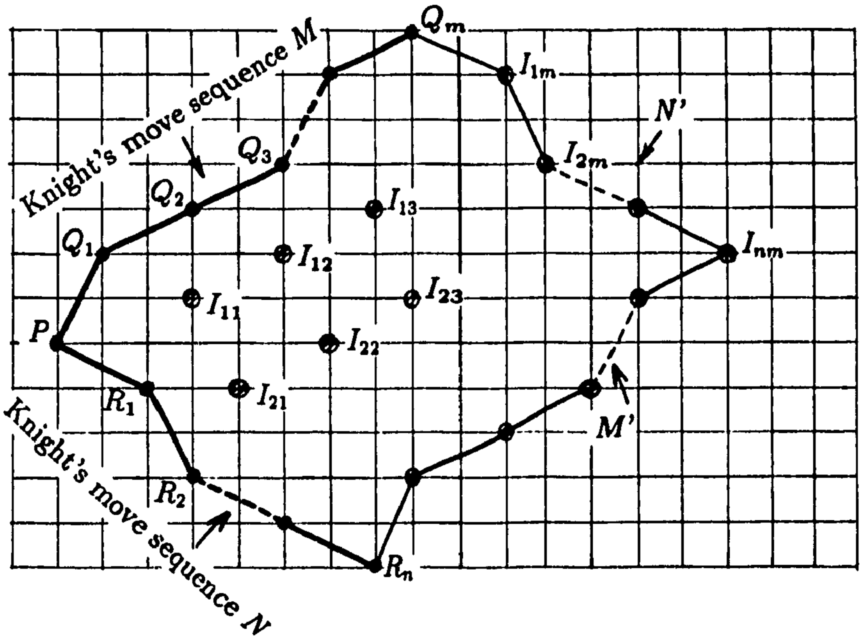


Fig. 3.17

Consider the three dominators P, Q_1, and R_1 in Fig. 3.17. By theorem 3.4, we have:

$$\begin{aligned} I_{11} &= Q_1 + R_1 - P \\ I_{12} &= Q_2 + I_{11} - Q_1 \\ I_{13} &= Q_3 + I_{12} - Q_2 \end{aligned}$$

$$\vdots$$

$$I_{1m} = Q_m + I_{1(m-1)} - Q_{m-1}$$

which determines $M_1 = (R_1, A, S_M)$.

By repeating the same operation on every remaining dominator $R_i, i = 2, 3, \cdots, n$ of N, all Knight's move sequences $M_i, i = 2, 3, \cdots, n$ are determined.

It is easy to see that the dominators Q_i, I_{1i}, I_{2i}, $\cdots$, I_{ni} determine the Knight's move sequence $N_i = (Q_i, B, S_N)$. Therefore, by repeating application of theorem 3.4 on Knight's move dominators on M and N, we force all dominators in the area bounded by M, N, M', N' to cover the entire area. □

3.2.2 Covering characteristics of Knight's move pattern

The area forced by 2 sequences of adjacent Knight's move dominators forms a distorted rectangle where dominators are laid out in a distorted rectilinear pattern. By theorem 3.5, this area is completely covered by the Knight's move pattern forced by these Knight's move sequences. To cover the infinite plane grid $G(k, n)$ where $k = \infty$ and $n = \infty$ (see Fig. 3.18), we can apply uniform translation of this area, considered as a *primitive cell.*

We define the *covering factor* σ of a dominating set D over a graph G the average number of vertices of G each dominator in D can dominate. We have:

$$\sigma = \frac{|V(G)|}{\text{number of dominators in } D}$$

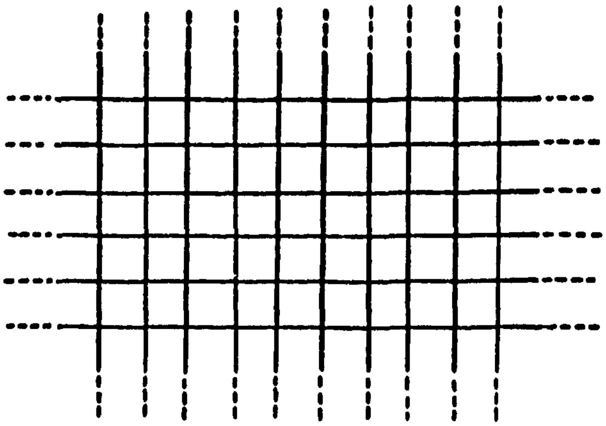


Fig. 3.18: The infinite plane grid

The area of the primitive cell is constant, and so is the number of dominators. Therefore, we can determine the covering factor of this pattern on the infinite plane by determining the covering factor of the pattern on the primitive cell.

In this section, we will compute the covering factor of Knight's move pattern on the primitive cell.

Consider the primitive cell formed by 2 adjacent Knight's move sequences $M = (P, A, S_M)$ and $N = (P, B, S_M)$ as in Fig. 3.17. We have:

1. *The total number of dominators in the primitive cell:*

   M has m dominators:

   $$Q_1, Q_2, \cdots, Q_m$$

   which is also the number of dominators on $M_i = (R_i, A, S_M), i = 1, 2, \cdots, n$.

Let T_d be the total number of dominators in the primitive cell. We have:

$$\begin{aligned} T_d &= \sum_{i=1}^{n} \text{number of dominators on } (R_i, A, S_M) \\ &= \sum_{i=1}^{n} m \\ &= nm \end{aligned}$$

2. *The area of the primitive cell:*

Fig. 3.19 represents the oblique squares and the diamonds formed by 2 adjacent Knight's moves originating from some origin P. Each pair (a_i, b_j) determines

- an oblique square if $(a_i, b_j) = (1,2)$ or $(2,1)$
- a diamond if $(a_i, b_j) = (1,1)$ or $(2,2)$

Each square has 4 interior points covered while each diamond has only 3 interior points covered resulting in 1 overlap.

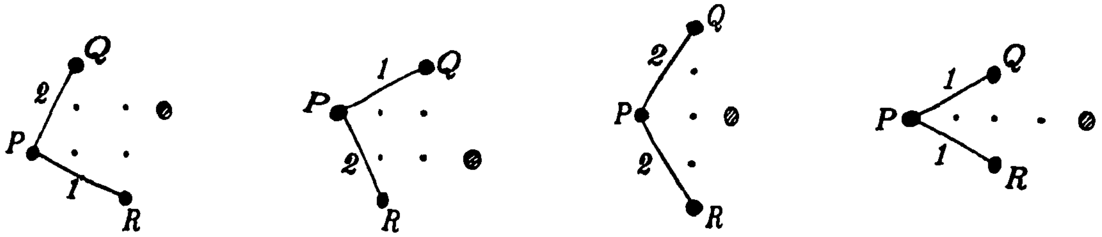


Fig. 3.19: Knight's move dominators in adjacent quadrants

Let r_M = number of $a_i = 1, i = 1, 2, \cdots, m$ in M

s_M = number of $a_i = 2, i = 1, 2, \cdots, m$ in M

r_N = number of $b_i = 1, i = 1, 2, \cdots, n$ in N

s_N = number of $b_i = 2, i = 1, 2, \cdots, n$ in N.

We have: $m = r_M + s_M$ and $n = r_N + s_N$.

Any $a_i = 1$ determines r_N diamonds and s_N oblique squares in the pattern.

Any $a_i = 2$ determines s_N diamonds and r_N oblique squares in the pattern.

Therefore, there are $(r_M r_N + s_M s_N)$ diamonds and $(r_M s_N + r_N s_M)$ oblique squares which cover an area of:

$$\begin{aligned} A &= 4(r_M s_N + r_N s_M) + 3(r_M r_N + s_M s_N) + T_d \\ &= 4(r_M s_N + r_N s_M) + 4(r_M r_N + s_M s_N) - (r_M r_N + s_M s_N) + mn \\ &= 4(r_M + s_M)(r_N + s_N) - (r_M r_N + s_M s_N) + (r_M + s_M)(r_N + s_N) \\ &= 5(r_M + s_M)(r_N + s_N) - (r_M r_N + s_M s_N) \end{aligned}$$

3. *The covering factor on the primitive cell:*

$$\begin{aligned} \text{Covering factor } \sigma &= \frac{\text{Area}}{\text{number of dominators}} \\ &= A/T_d \\ &= \frac{5(r_M + s_M)(r_N + s_N) - (r_M r_N + s_M s_N)}{(r_M + s_M)(r_N + s_N)} \\ &= 5 - \frac{(r_M r_N + s_M s_N)}{(r_M + s_M)(r_N + s_N)} \qquad (4) \end{aligned}$$

3.2.3 Knight's move pattern vs Star-center pattern

As described in section 3.1, the pulling algorithm must be applied to cover the 4 edges of a grid graph $G(k, n)$ with the star-center pattern. This algorithm results in additional dominators on the edges to completely cover G.

In the covering of G with Knight's move patterns, we need not apply this edge covering algorithm. In fact, we have the flexibility to arrange Knight's move dominators to efficiently cover G, without the necessity of adding extra edge dominators, resulting in a *perfect* edge covering. However, while the star-center pattern gives optimum covering of the interior of $G(k, n)$ with no overlaps, the Knight's move pattern may have overlaps in the interior due to the constraints necessary to obtain a perfect edge covering. Therefore, such a construction does not necessarily lead to an optimal dominating set. In the following sections, we will study conditions under which Knight's move patterns may give smaller dominating sets than star-center patterns.

3.3 Covering infinite grids with Knight's move pattern

In an infinite grid G, we have $|N[v_i]| = 5$ for all $v_i \in V(G)$. In the equation (4) of section 3.2, $(r_M r_N + s_M s_N)$ represents the number of overlaps in the primitive cell. Covering is perfect if there is no overlap i.e.

$$\begin{aligned} r_M r_N + s_M s_N &= 0 \\ \Rightarrow \sigma &= 5 \end{aligned}$$

which means that there must be no diamond in the pattern.

We have the following theorem [2]:

Theorem 3.6 *On an infinite grid, there is a perfect covering if and only if it is covered with star-center patterns.*

3.4 Covering strip graphs with Knight's move pattern

A *strip graph* $G(k, \infty)$ is a strip of height k and infinite width (Fig. 3.20).

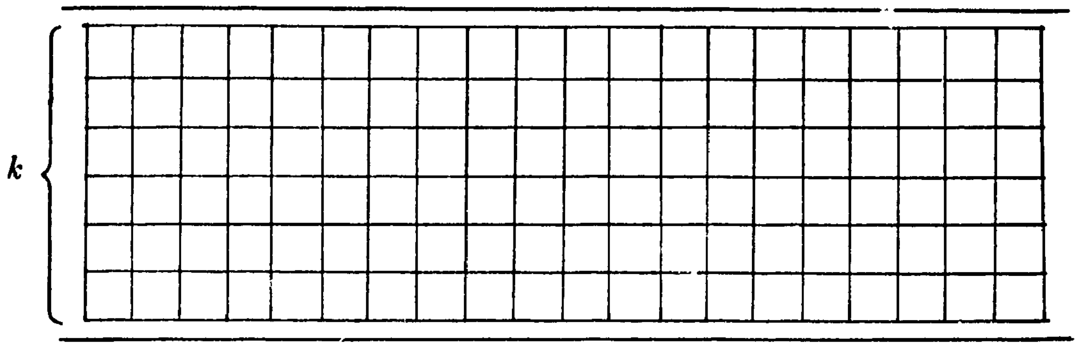


Fig. 3.20: The infinite strip graph $G(7, \infty)$

As mentioned in section 3.2, Knight's move patterns may provide a better covering of $G(k, n)$ than star-center patterns for certain widths k. A better covering of $G(k, n)$ implies an upper bound for $\gamma(G)$ better than the upper bound introduced by Cockayne et al. [4] using the star-center pattern. In this section, we will prove that Knight's move covering is asymptotically superior to star-center covering for certain widths. Therefore, the study of best covering in strip graphs helps in establishing improved upper bounds for $\gamma(G)$.

3.4.1 Definitions

A *column* of $G(k, \infty)$ is the set of vertices $v_i \in G(k, \infty)$ having the same y-coordinate. We assume the columns of $G(k, \infty)$ are numbered from some starting column.

A *rectangle* $G(k,\infty)[i,j]$ is the finite strip obtained by cutting $G(k,\infty)$ from column i to j, $i \leq j$ (Fig. 3.21).

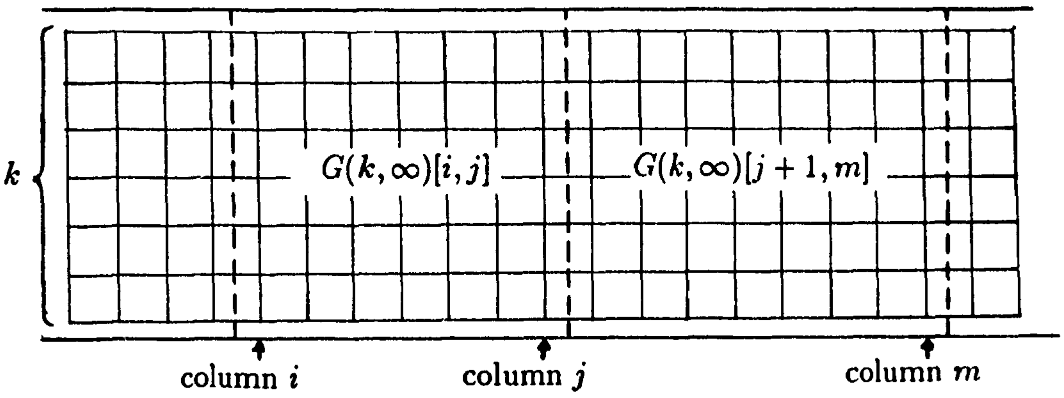


Fig. 3.21: Columns and rectangles in $G(k,\infty)$

3.4.2 Periodicity of patterns in strip graphs

We have the following theorem on optimal covering of strip graphs.

Theorem 3.7 *In strip graphs $G(k,\infty)$ of fixed k, optimum covering can be achieved using a periodic pattern.*

Proof:

> There are 2^k possible configurations of dominators on every column of $G(k,\infty)$, since any vertex may or may not be a dominator. Suppose there exists an optimum covering of $G(k,\infty)$ with a non-periodic pattern. However, by the pigeonhole principle, given any column C_i, there exists a smallest $p > 0$ so that $C_{i+p} = C_i$, and $C_{i+p+1} = C_{i+1}$ (there are only 2^{2k} possibilities for C_i and C_{i+1}).

We must have:

$$\sigma(G(k,\infty)) = \sigma(G(k,\infty)[i,i+p])$$

because otherwise we can remove the rectangle $G(k,\infty)[i,i+p]$ and get a better covering factor. Therefore, repeating the rectangle over $G(k,\infty)$ must give the optimum covering factor. □

We call p the *period* of $G(k,\infty)$. Fig. 3.22 illustrates a periodic pattern on $G(7,\infty)$ where $p = 6$.

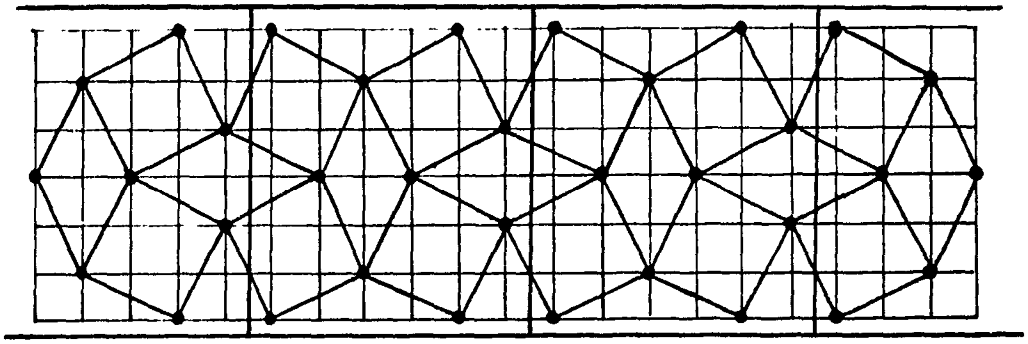


Fig. 3.22: A periodic pattern on $G(7,\infty)$

3.4.3 Covering factor in strip graphs

To efficiently covering all edge vertices without adding extra dominators to the pattern, we assume the following:

Assumption 3 *No extra dominators need be inserted to cover the edges.*

This assumption is plausible though we shall see that it is not optimal for widths $k > 8$.

Implications:

1. The pattern must meet all edges on half diamond boundaries. Consider Fig. 3.23.

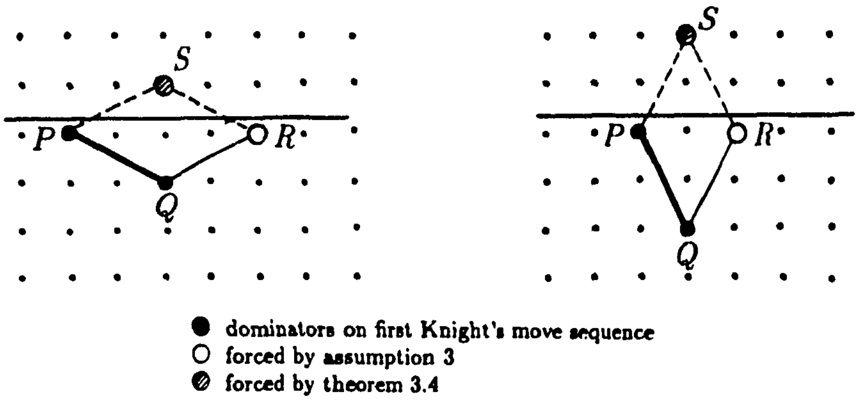


Fig. 3.23: Edge covering under assumption 3

- *Case (a):* $Q = P + (2, -1)$ or $a = 1$ in the SE quadrant.

  To cover R, assumption 3 forces a Knight's move from Q. We have $R = Q + (2, 1)$ or $b = 1$ in the NE quadrant. By theorem 4, $S = P + R - Q$ must be a dominator. We observe that P, Q, R forms a horizontal *half-diamond.*

- *Case (b):* $Q = P + (1, -2)$ or $a = 2$ in the SE quadrant.

  To cover R, assumption 3 forces a Knight's move from Q. We have $R = Q + (1, 2)$ or $b = 2$ in the NE quadrant. By theorem 4, $S = P + R - Q$ must be a dominator. We observe that P, Q, R forms a vertical *half-diamond.*

The same principle must be applied along the edge to form a half-diamond boundary without adding extra dominators to the edge.

2. Consider 2 adjacent Knight's move sequences $M = (P, A, S_M)$ and $N = (P, B, S_N)$ as defined in section 3.2.2. $P = (x_0, y_0)$ is a vertex on the upper edge of $G(k, \infty)$. In one period of the descriptor sequence, we must have $r_M = r_N, s_M = s_N$.

   Proof:

   Let $S_M = (1, -1)$ and $S_N = (1, 1)$ without loss of generality.

   Let $A = (a_1, a_2, \cdots, a_m)$.

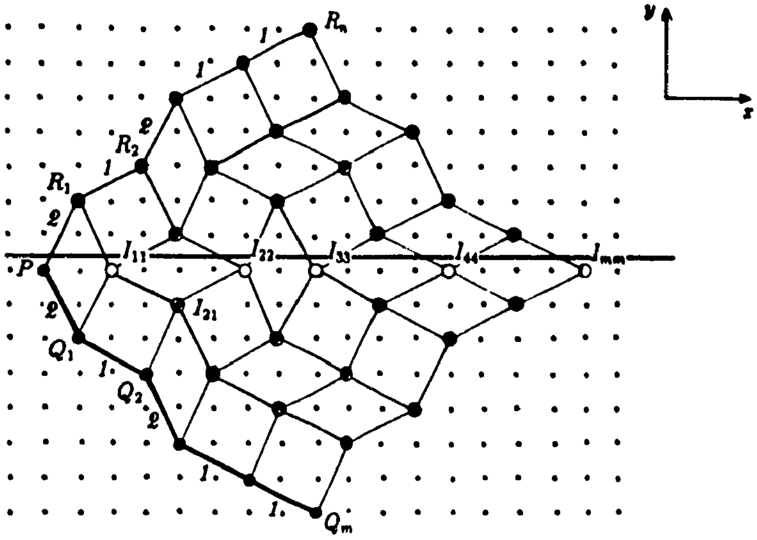
   Let $B = (b_1, b_2, \cdots, b_n)$.

   Consider the primitive cell determined by M and N (Fig. 3.24). We will prove that M completely determines N under assumption 3.

   (a) Assumption 3 forces the edge dominator I_{11}. By theorem 3.4, $R_1 = P + I_{11} - Q_1$ is determined to form a diamond where $b_1 = a_1$.

   (b) By theorem 3.4, $I_{21} = I_{11} + Q_2 - Q_1$ is determined. Assumption 3 forces $I_{22} \Rightarrow b_2 = a_2$.

   By repeating the same operation on all Q_i on $M, i = 1, 2, \cdots, m$, all dominators $R_1, R_2, \cdots, R_m$ on N are determined, forced by assumption 3 and theorem 3.4. We have $(a_i, b_i) \in \{(1,1), (2,2)\}$ for all $i = 1, 2, \cdots, m$. Therefore, the 2 descriptor sequences A of M and B of N are identical, i.e., $r_M = r_N, s_M = s_N$. The number of dominators in the primitive cell is $m(m+1)$. □



● dominators on first Knight's move sequence
○ forced by assumption 3
⊘ forced by theorem 3.4

Fig. 3.24

As discussed in section 3.2, we can cover the entire infinite grid using the primitive cell determined by M and N. This covering, restricted to the strip $G(k,\infty)$, determines a periodic pattern on the strip as stated in the following theorem:

Theorem 3.8 *A sequence of Knight's move dominators* $M = (P, A, S_M)$ *along the strip graph* $G(k,\infty)$, *starting from some vertex* P *on the upper edge and ending on the lower edge of* $G(k,\infty)$ *with* $A = (a_1, a_2, \cdots, a_m)$, *determines a periodic Knight's move pattern, having a period of* $p = 2(3m - k + 1)$.

Proof:

We have:

$$A = (a_1, a_2, \cdots, a_m)$$

$$\text{where } a_i = 1 \text{ or } 2, i = 1, 2, \cdots, m, \text{ and}$$

$$\sum_{i=1}^{m} a_i = k - 1$$

- For $a_i = 1$, the next edge dominator is at distance 4 (horizontal diamond).
- For $a_i = 2$, the next edge dominator is at distance 2 (vertical diamond).

Let r and s be the number of 1's and 2's in A, respectively. We have:

$$\begin{aligned} \sum_{i=1}^{m} &= r + 2s = m + s = k + 1 \\ \Rightarrow s &= k - 1 - m \\ r &= m - s = m - (k - 1 - m) = 2m - k + 1 \end{aligned}$$

On the boundary, there are r horizontal half diamonds with length 4 and s vertical half diamonds with length 2 (see Fig. 3.25), i.e.,

$$\text{period } p = 4s + 2s = 4(2m - k + 1) + 2(k - 1 - m) = 2(3m - k + 1) \quad (5)$$

Since the period of the pattern along the length of the strip is p, we can consider as primitive cell any cell having this period, say the $k \times p$ rectangle whose upper left corner is at P. □

Now, we can compute the covering factor in a primitive cell of $G(k, \infty)$:

$$\begin{aligned} \sigma(G(k,p)) &= \frac{pk}{m(m+1)} \\ &= \frac{2k(3m - k + 1)}{m(m+1)} \end{aligned} \qquad (6)$$

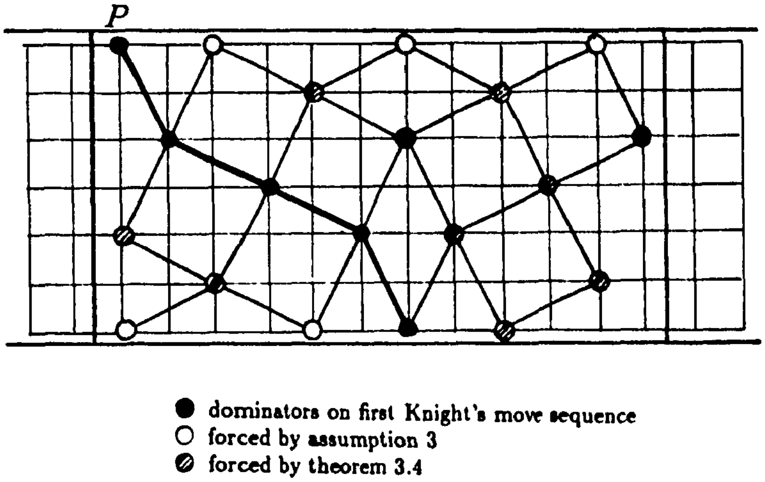


Fig. 3.25: A periodic pattern on $G(k,\infty)$.

This is also the covering factor on the strip graph $G(k,\infty)$.

We observe that $\sigma(G(k,\infty))$ is a function of variable m:

$$\sigma(G(k,\infty)) = f(m) = \frac{2k(3m-k+1)}{m(m+1)}$$

To maximize the covering factor on the strip graph, we want to find the value of m for which $\sigma(G(k,\infty))$ is maximum. Let $f'(m)$ be the derivative of $f(m) = \sigma(G(k,\infty))$. For $f(m)$ to be maximum, we must have:

$$\begin{aligned} f'(m) &= 0 \\ \text{i.e.} \quad \left(\frac{2k(3m-k+1)}{m(m+1)} \right)' &= 0 \\ 2k[(m(m+1)]^{-2}[-3m^2+2m(k-1)+(k-1)] &= 0 \\ \Rightarrow m &= \frac{1}{3}(k-1 \pm \sqrt{(k-1)(k+2)}) \end{aligned}$$

So, we obtain the best Knight's move covering of strip graphs $G(k,\infty)$ under assumption 3 with Knight's move sequences $M = (P, A, S_M)$ where the descriptor sequence M has $\lfloor m \rfloor$ or $\lceil m \rceil$ descriptor elements, and

$$m = \frac{1}{3}(k - 1 + \sqrt{(k-1)(k+2)}) \tag{7}$$

Fig. 3.26 represents the actual best covering factors compare with star-center patterns for strip graphs $G(k, \infty)$ of widths k ranging from 7 to 12.

k	Covering factor σ	
	Best values found	Star-center values
7	4.200	4.083
8	4.267	4.000
9	4.304	4.050
10	4.333	4.167
11	4.342	4.172
12	4.340	4.114

Fig 3.26: Table of best values of σ compare with star-center values for $G(k, \infty)$ $k = 7$ to 12.

Though assumption 3 is plausible, it is not optimal for widths $k > 8$, as the following example shows (Fig. 3.27 and Fig. 3.28). The point of introducing it is to show that Knight's move pattern can give asymptotically better dominating sets than star-center pattern.

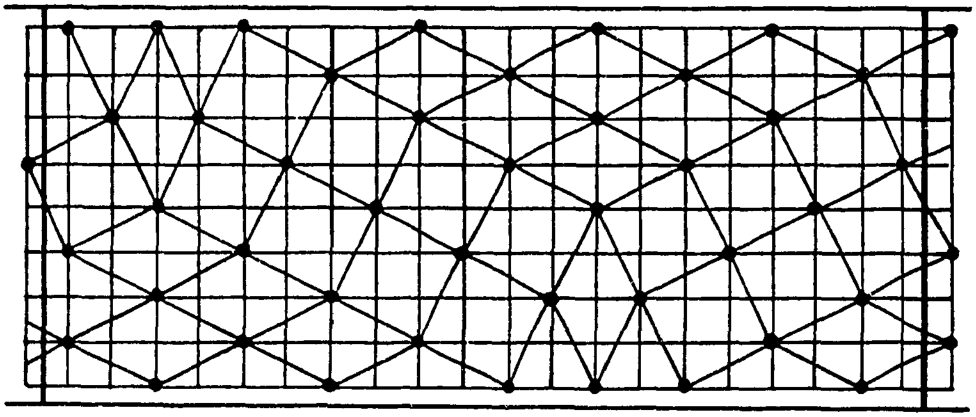


Fig. 3.27: $G(9, \infty)$ with $\sigma = 4.286$ (under assumption 3)

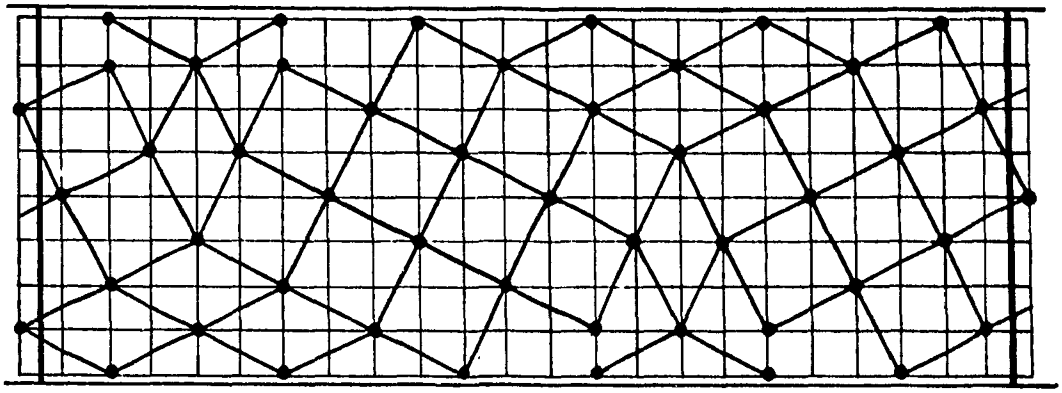


Fig. 3.28: $G(9, \infty)$ with $\sigma = 4.304$ (assumption 3 not applied)

3.4.4 Limit on possible superiority of Knight's move pattern

As discussed in the above sections, Knight's move patterns may cover certain grid graphs better than star-center patterns. However, on the infinite plane grid, star-center patterns provide best covering with a covering factor of 5 (section 3.3). We are interested in defining a limit size of $G(k, n)$ up to which Knight's move patterns may be superior to star-center patterns.

Let γ_K be the minimum size of a dominating set of $G(k, n)$ realized under assumptions 2 and 3.

Let γ_S = minimum size of a dominating set of $G(k, n)$ realized under assumption 1.

Theorem 3.9 *For* $k, n \geq 8$, *we have*

$$(k - 13)(n - 13) > 45 \Rightarrow \gamma_K \geq \gamma_S.$$

Proof:

Under assumption 3, we have

$$r_M = r_N = r, s_M = s_N = s \tag{8}$$

by implication 2 (of assumption 3) in section 3.4.3. Recall that

$$\sigma = 5 - \frac{(r_M r_N + s_M s_N)}{(r_M + s_M)(r_N + s_N)} \tag{9}$$

Substituting (8) in (9), we obtain:

$$\sigma_K \leq 5 - \frac{r^2 + s^2}{(r+s)^2}$$

This is maximized when $r = s$, so

$$\sigma_K \leq 5 - \frac{1}{2} = \frac{9}{2}$$

Consider the basic cell $G(4,4)$ where covering is optimum with $r = s = 1$ (Fig. 3.29).

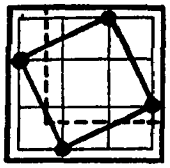
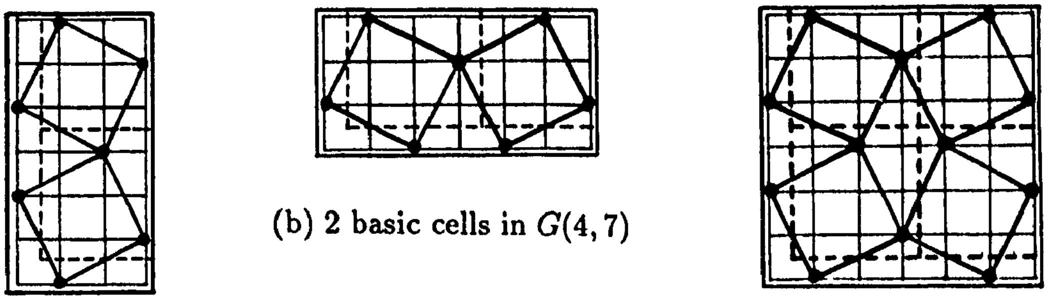


Fig. 3.29: Basic cell $G(4,4)$.

A *perfect rectangle* is a graph $G(k,n)$ of which the dominating set satisfies assumption 3. The dominating set is not necessarily optimal. From the basic cell in Fig. 3.29, we can produce a perfect rectangle for any $G(k,n)$ with $k = 3a+1$ and $n = 3b+1$ as illustrated in Fig. 3.30.

We observe that there are $(2ab + a + b)$ dominators in each dominating set formed from the basic cell for $G(k,n)$ with $k = 3a+1$ and $n = 3b+1$.



(a) 2 basic cells in $G(7,4)$

(b) 2 basic cells in $G(4,7)$

(c) 4 basic cells in $G(7,7)$

Fig. 3.30

For general $G(k,n)$, extra dominators may need be inserted to completely cover the graph. We have:

$$\gamma_K \geq 2ab + a + b = \frac{1}{2}(4ab + 2a + 2b) = \frac{1}{2}[(2a+1)(2b+1) - 1] \quad (10)$$

Substituting $a = \dfrac{k-1}{3}$ and $b = \dfrac{n-1}{3}$ in (10):

$$\begin{aligned} \gamma_K &\geq \frac{1}{2}\left[(2\frac{k-1}{3} + 1)(2\frac{n-1}{3} + 1) - 1 \right] \\ &\geq \frac{1}{2}\left[\frac{(2k+1)(2n+1)}{9} - 1 \right] \qquad (11) \end{aligned}$$

Under assumption 1,

$$\gamma_S \leq \frac{(k+2)(n+2)}{5} - 4 \qquad (12)$$

For $\gamma_K \geq \gamma_S$, we must have:

$$\begin{aligned} \frac{1}{2}\left[\frac{(2k+1)(2n+1)}{9} - 1 \right] &\geq \frac{(k+2)(n+2)}{5} - 4 \\ 5[(2k+1)(2n+1) - 9] &\geq 18[(k+2)(n+2) - 20] \\ 5(2k+1)(2n+1) - 18(k+2)(n+2) &\geq 45 - 360 = -315 \\ 2kn - 26k - 26n &\geq -315 + 69 = -248 \end{aligned}$$

$$\begin{aligned} kn - 13k - 13n &\geq -124 \\ kn - 13k - 13n + 169 &\geq -124 + 169 = 45 \\ \Rightarrow (k - 13)(n - 13) &\geq 45 \end{aligned}$$

□

Now we can prove that Knight's move patterns are asymptotically superior to star-center patterns for $G(k,n)$ for certain widths k.

Corollary 3.1 *For any strip graph* $G(k,n)$*,* $k > 13$*:*

$$\exists n_0 \; \forall n \leq n_0 : \gamma_K(G(k,n)) < \gamma_S(G(k,n)).$$

Proof:

From theorem 3.9, we conclude that for any $G(k,n)$ where

$$(k-13)(n-13) < 45 \tag{13}$$

there exists a Knight's move dominating set for which $\gamma_K < \gamma_S$.

For $k \leq 13$, (13) is always satisfied with any value of n.

For $k > 13$, let $a = k - 13 > 0$. We have

$$\begin{aligned} (12) \Rightarrow (n-13) &< \frac{45}{a} \\ n &< \frac{45}{a} + 13 = \frac{45}{k-13} + 13 = n_0 \end{aligned}$$

So, for $k \geq 13$, there always exist some n_0 for which Knight's move patterns with $n < n_0$ may cover $G(k,n)$ better than star-center patterns.

□

Fig. 3.31 represents the table of values of n_0 for $G(k,n)$, $k \geq 13$.

k	n_0
≤ 13	any
14	58
15	36
16	28
17	25
18	22
19	20

Fig. 3.31: Table of values of n_0 for $G(k,n)$, $k \geq 13$.

3.5 Covering finite $k \times n$ grid graphs with Knight's move pattern

As discussed in section 3.2, 2 adjacent Knight's move sequences $M = (P, A, S_M)$ and $N = (P, B, S_N)$ completely determine all Knight's move dominators within a distorted rectangle, two sides of which are M and N. To cover the rectangle $G(k, n)$, we consider concatenating 4 *partial* distorted rectangles, each of which is generated by 2 adjacent Knight's move sequences (see Fig. 3.32).

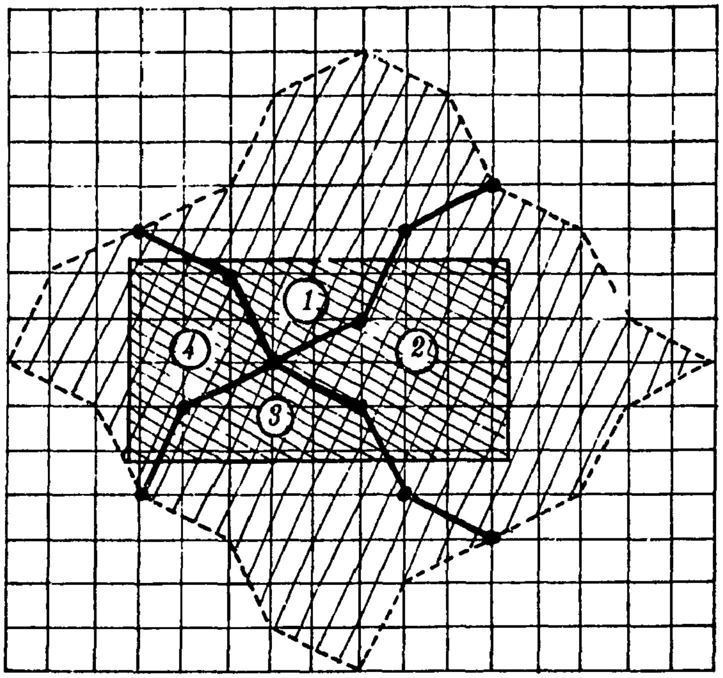


Fig. 3.32: $G(k, n)$ is formed from 4 partial distorted rectangles in 4 quadrants.

To determine the dominating set of $G(k, n)$ using the Knight's move pattern, we must generate 4 adjacent Knight's move sequences M_1, M_2, M_3, and M_4, 1 in each quadrant NE, SE, NW, and SW.

To generate all Knight's move patterns over $G(k, n)$, from some interior vertex

P_0 of $G(k,n)$, we consider all possible Knight's move sequences in the 4 quadrants:

- NW quadrant: $M_1 = (P_0, A_1, S_1), S_1 = (-1, 1)$.
- NE quadrant: $M_2 = (P_0, A_2, S_2), S_2 = (1, 1)$.
- SE quadrant: $M_3 = (P_0, A_3, S_3), S_3 = (1, -1)$.
- NW quadrant: $M_4 = (P_0, A_4, S_4), S_4 = (-1, -1)$.

Any interior vertex of $G(k,n)$ is a potential candidate for P_0. However, we need consider only the set of 5 centers:

$$P_0 \in S_0 = (x_0, y_0), (x_0 \pm 1, y_0), (x_0, y_0 \pm 1)$$

where (x_0, y_0) is the center vertex of $G(k,n)$. Clearly, any dominating set must contain a point in S_0 if (x_0, y_0) is to be covered (see Fig. 3.33).

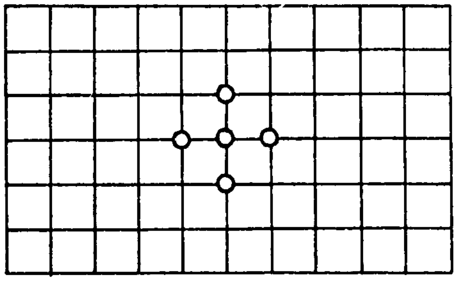


Fig. 3.33: Set S_0 of 5 centers of $G(k,n)$.

Therefore, we need consider only patterns originating from each of the 5 centers in S_0.

First, we will determine how such a pattern must be extended in order to cover the whole rectangle, by determining how far each Knight's move sequence in a given quadrant must extend to cover the corner of $G(k,n)$ which lies in that quadrant.

3.5.1 Definitions

Consider the finite grid graph $G(k,n)$ in Fig 3.34.

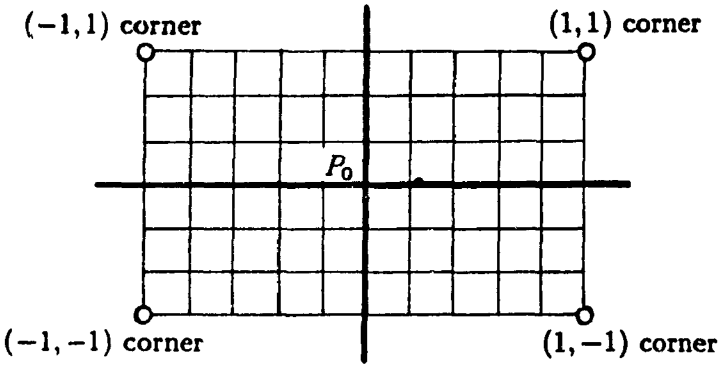


Fig. 3.34

Let $v_c = (x_c, y_c)$ be a corner of $G(k,n)$ such that the points of $G(k,n)$ adjacent to P_0 are $(x_c - s_x, y_c)$ and $(x_c, y_c - s_y)$, where $s_x, s_y = \pm 1$. Recall that $S = (s_x, s_y)$ is the direction vector of a Knight's move sequence $M = (P, A, S)$. We call v_c the (s_x, s_y) *corner*. For example, the upper right corner is the $(1,1)$ corner.

Given the corner vertex v_c, we define the *Knight's boundary rays* C and D as follows:

Let ray C originate from v_c and pass through the point $v_c(C) = (x_c - s_x, y_c + 2s_y)$.

Let ray D from P_0 pass through $v_c(D) = (x_c + 2s_x, y_c - s_y)$.

Note that each of $v_c(C)$ and $v_c(D)$ is a Knight's move distant from v_c.

There are 8 Knight's boundary rays for $G(k,n)$, 2 rays in each corner as indicated in Fig. 3.35.

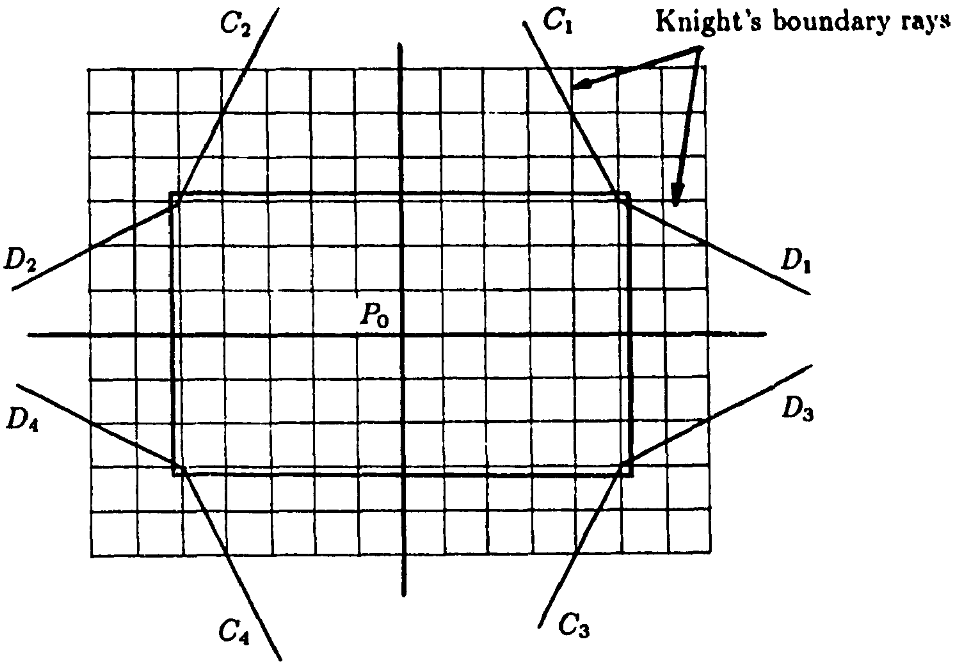


Fig. 3.35: 8 Knight's boundary rays of $G(k,n)$, 2 in each corner.

3.5.2 Theorem on Knight's move sequences

We can now establish conditions under which a Knight's move sequence $M = (P_0,A,S)$ must terminate to ensure covering of the corner vertex.

Theorem 3.10 *To cover the* (s_x, s_y) *corner in some quadrant of* $G(k,n)$*, the Knight's move sequence in that quadrant must terminate on or beyond the corresponding Knight's boundary rays originating from that corner.*

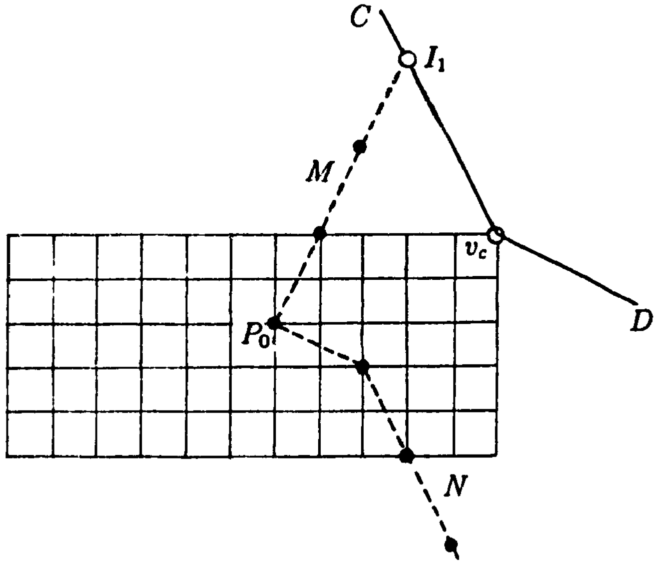
Proof:

Let (s_x, s_y) be the direction vector for the Knight's move sequence M.

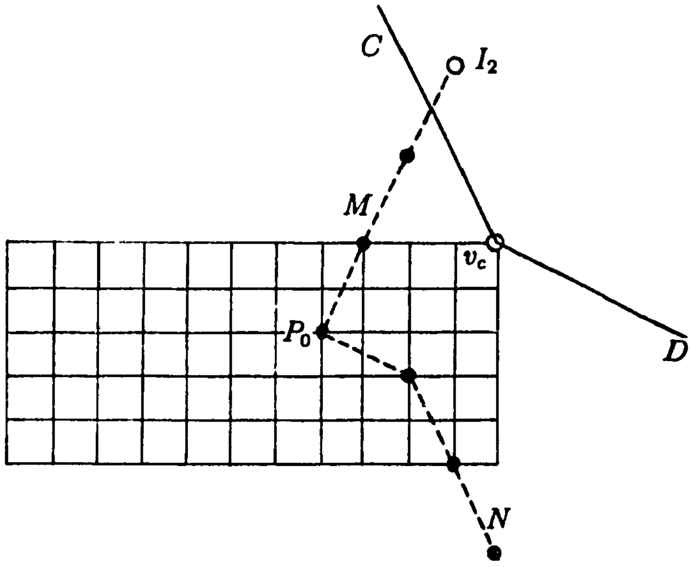
Let v_c be the (s_x, s_y) corner of $G(k,n)$.

Let C, D be the boundary rays as defined above.

Let $N = (P_0, B, S_N)$ be the Knight's move sequence adjacent to M such that v_c lies between M and N (Fig. 3.36).



(a) M terminates on the Knight's boundary ray



(b) M terminates beyond the Knight's boundary ray

Fig. 3.36

1. *M terminates on a Knight's boundary ray*, say on C, without loss of generality: there is a Knight's move dominator I_1 at the intersection of M and C. Since any dominator on M is a Knight's move distant from the previous dominator and C originates from v_c, v_c is a certain number of Knight's move distant from I_1. Therefore, it must be at worst on the boundary of the area covered by M and N (see Fig. 3.36a).
2. *M terminates beyond a Knight's boundary ray*, say on C, without loss of generality: there is a Knight's move dominator I_2 beyond the intersection of M and C. The area determined by M and N includes v_c. Therefore, v_c is covered by some Knight's move dominator in this area (see Fig. 3.36b). □

Theorem 3.10 gives immediately:

Corollary 3.2 *The Knight's move dominators within $G(k,n)$ are completely determined by any set of 4 Knight's move sequences, 1 in each direction, originating from some center P_0 interior to G, each of which terminates on or beyond the corresponding Knight's boundary rays of $G(k,n)$.*

Proof:

The four Knight's move sequences M_1, M_2, M_3, and M_4 divide $G(k,n)$ into 4 areas, numbered from 1 to 4 as in Fig. 3.37.

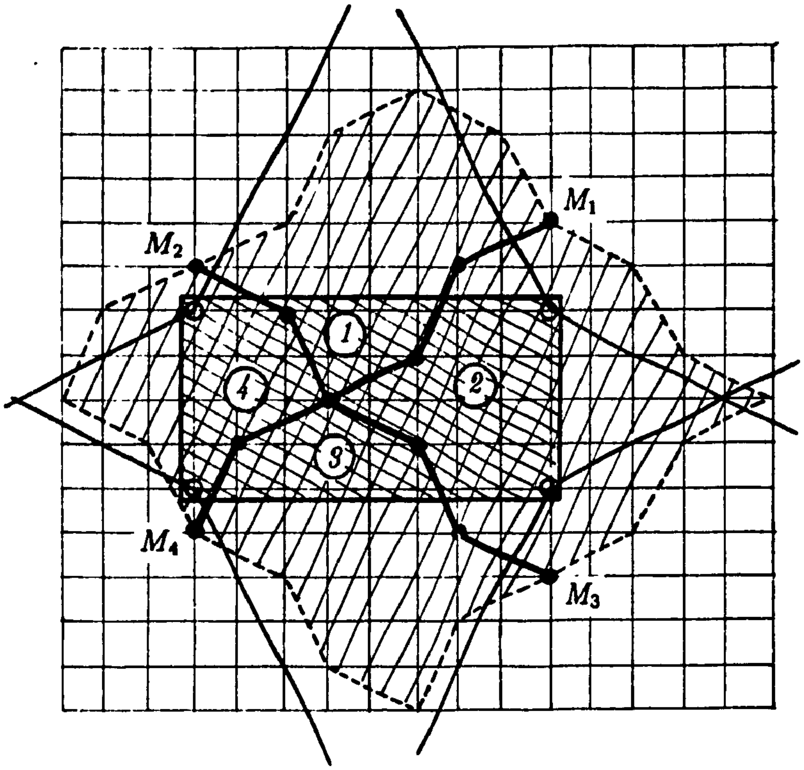


Fig. 3.37

Consider M_1 and M_2. The distorted rectangle formed by M_1 and M_2 includes all vertices of $G(k,n)$ in area 1. By theorem 3.5, these vertices must be covered by Knight's move dominators generated from M_1 and M_2.

Applying the same argument for the remaining 3 areas, we have a complete covering of $G(k,n)$ by M_1, M_2, M_3, and M_4. $\square$

3.5.3 Covering $G(k,n)$ by covering its 2 subgraphs

Consider 2 adjacent Knight's move sequences M and N in Fig. 3.38. The line (M, P_0, N) which divides $G(k,n)$ into 2 subgraphs G_1 and G_2 is called the *Knight's dividing line $\mathcal{D}$* of $G(k,n)$.

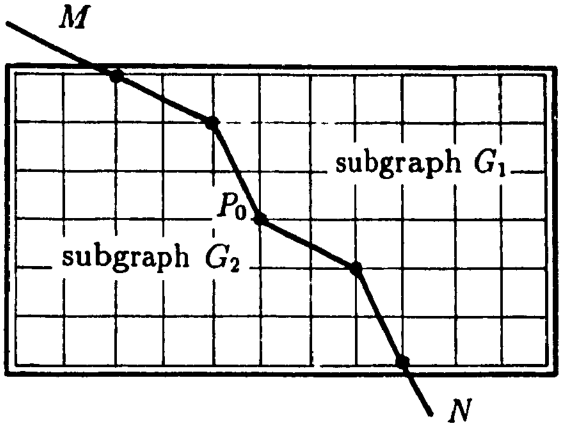


Fig. 3.38: The Knight's dividing line $\mathcal{D}(M, P_0, N)$ of $G(k,n)$

As the dividing line cuts the rectangle into 2 halves, we may save considerable computing time by solving the domination problem on each half independently and combining the solution. However, as the dividing line may or may not terminate on the edges, we must examine conditions under which there are dominators which dominate vertices on both sides of the dividing line.

1. *The dividing line terminates on the edge:* we can generate the dominating set for each subgraph of $G(k,n)$ separately (see Fig. 3.39). The total number of dominators of $G(k,n)$ will be the sum of the number of dominators on $\mathcal{D}$, G_1, and G_2.

2. *The dividing line terminates beyond the edge:* Fig. 3.40 represents three possible configurations of dominators on the dividing line and dominators on 2 subgraphs G_1 and G_2 of $G(k,n)$. S denotes the edge vertex that can be covered by dominators on both subgraphs.

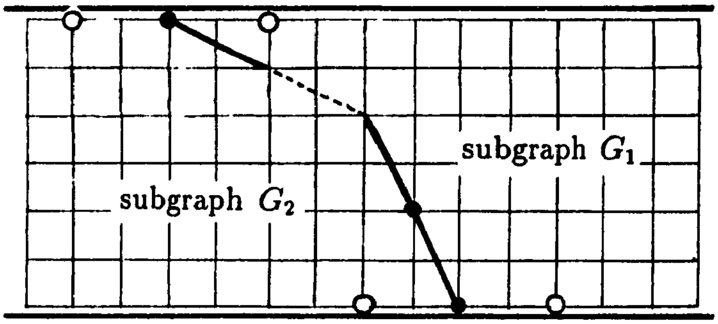
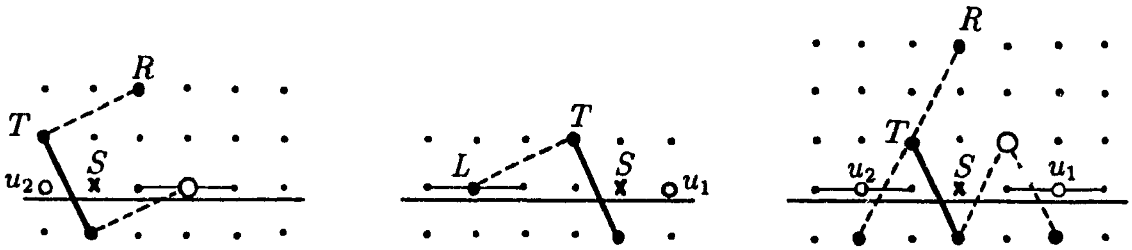


Fig. 3.39: $\mathcal{D}$ terminates on the edge: G_1 and G_2 are independent



(a) $R = T + (2,1)$

(b) $L = T + (-2,-1)$

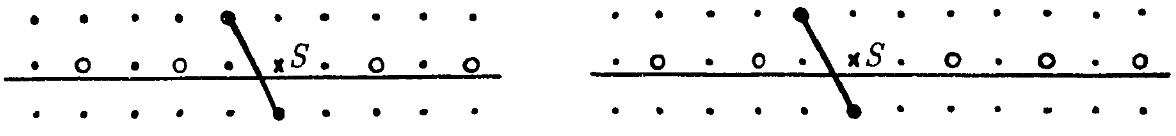
(c) $R = T + (1,2)$, $L = T + (-1,-2)$

Fig. 3.40: $\mathcal{D}$ terminates beyond the edge

After generating the dominating sets for G_1 and G_2, we may need to insert an extra dominator to cover S. We call this number the *connection constant* of $\mathcal{D}$ and denote it e. In Fig. 3.39, $e = 0$. In Fig. 3.40, we have:

- *Case a:* $R = T + (2,1)$: S cannot be covered by any dominator in G_1. To cover S, either S or U_2 in G_2 must be a dominator.

- *Case b:* $L = T + (-2, -1)$: S cannot be covered by any dominator in G_2. To cover S, either S or U_1 in G_1 must be a dominator.
- *Case c:* $L = T + (-1, -2)$ and $R = T + (1, 2)$: We call U_1 and U_2 the *open points* on the edge. In this case, the placement of edge dominators on the two subgraphs is important. If n is the number of consecutive open points on the edge, including S, at distance 2 from each other, then we need $\lceil \frac{n}{2} \rceil$ edge dominators (see Fig. 3.41).



(a) n odd (b) n even

Fig. 3.41

To overcome the difficulty of separating G_1 and G_2 in case (c), we generate the dominating sets for each subgraph under the assumption that S is covered.

Let n_1 and n_2 be the number of consecutive open points on the edge of G_1 and G_2, respectively.

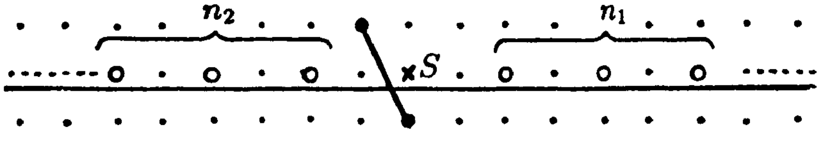


Fig. 3.42: Final placement of edge dominators.

If the final placement of edge dominators is as shown in Fig. 3.42, and:

1. If every best dominating set for G_2 has n_2 odd and every best dominating set for G_1 has n_1 even, then 1 extra dominator must be adjoined to the best dominating sets for G_1 and G_2 in order to cover S. We have $e = 1$.
2. In other cases, $e = 0$.

This reasoning must be applied on both ends of the dividing line (upper and lower edges), giving connection constants e_U and e_L. We are led to the following theorem:

Theorem 3.11 *Let γ_0 be the number of dominators on a given Knight's dividing line of $G(k, n)$. Let γ_1 and γ_2 be the size of the smallest dominating sets of the Knight's subgraphs G_1 and G_2 of $G(k, n)$, respectively. Then*

$$\gamma = \gamma_0 + \gamma_1 + \gamma_2 + e_U + e_L \tag{14}$$

is the size of the smallest dominating set of $G(k, n)$ for the given Knight's dividing line.

Chapter 4

ALGORITHMS TO CONSTRUCT A MINIMUM DOMINATING SET IN A K × N GRID GRAPH

In the previous chapter, we have introduced a new construction of the dominating set of $k \times n$ grid graphs using Knight's move patterns. We have also discussed the periodicity of the Knight's move pattern on strip graphs $G(k, \infty)$. In this chapter, we present two algorithms to generate dominating sets for strip graphs using Knight's move patterns. Algorithm 1 is a linear algorithm generating the optimal periodic Knight's move pattern which will produce the maximum covering factor on strip graphs $G(k, \infty)$ of given width k under assumptions 2 and 3. Algorithm 2 generates all Knight's move patterns which might lead to an optimal dominating set of $G(k, n)$ under assumption 2.

4.1 Algorithm 1

This algorithm is based on assumptions 2 and 3 on periodic Knight's move patterns on $G(k, \infty)$ strip graphs. For a given height k, we construct the set of all SE Knight's move sequences M starting from some origin P_0 on the upper border of the strip and terminating at the lower border:

$M = (P_0, A, S)$ where $P_0 = (0,0)$

$$A = (a_1, a_2, \cdots, a_m), a_i \in [1,2] \text{ for } i = 1, 2, \cdots, m$$

$$S = (1, -1)$$

By theorem 3.8, for each sequence M, we have:

$$\begin{aligned}\sum_{i=1}^{m} a_i &= k-1 \\ \text{period } p &= 2(3m-k+1) \quad \text{recall from (5)}\end{aligned}$$

There are $\mathcal{K} = m(m+1)$ Knight's move dominators in a period determined by M.

We select coordinates with origin $O = P_0 = (0,0)$ as shown in Fig. 4.1.

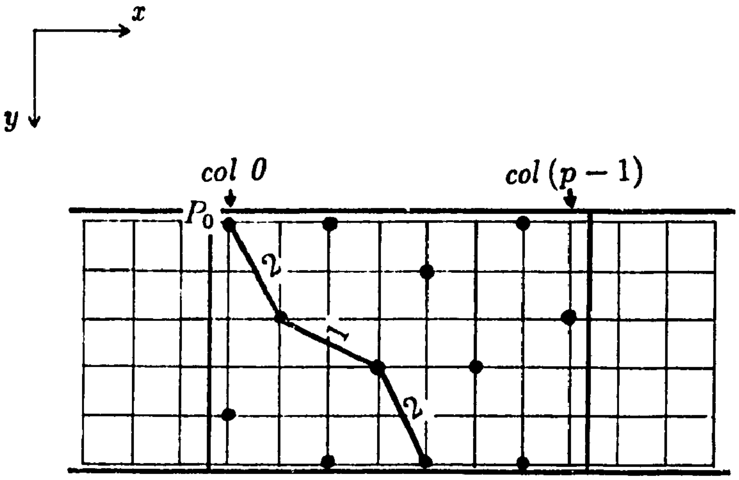


Fig. 4.1: A period of $G(6, \infty)$ determined by $A = (2, 1, 2)$.

To generate all sequences M for a given k, we must generate all descriptor sequences $A = (a_1, a_2, \cdots, a_m)$ for which $\sum_{i=1}^{m} a_i = k - 1$. Therefore, we first find all ordered partitions of $(k-1)$ employing only $a_i \in \{1, 2\}$. We denote by $\mathcal{P}_{k-1}$ the number of descriptor sequences A for $G(k, \infty)$.

For every Knight's move sequence, we construct the periodic pattern $\mathcal{P}$ for $G(k, \infty)$

as follows:

1. From $P_0 = (0,0)$, we can recursively determine all Knight's move dominators on the Knight's move sequence:

$$P_j = P_{j-1} + (3 - a_j, a_j), \quad \forall j \in [1..m].$$

2. By assumption 3, we generate the next Knight's move dominators from current edge dominators to form half-diamond boundaries.

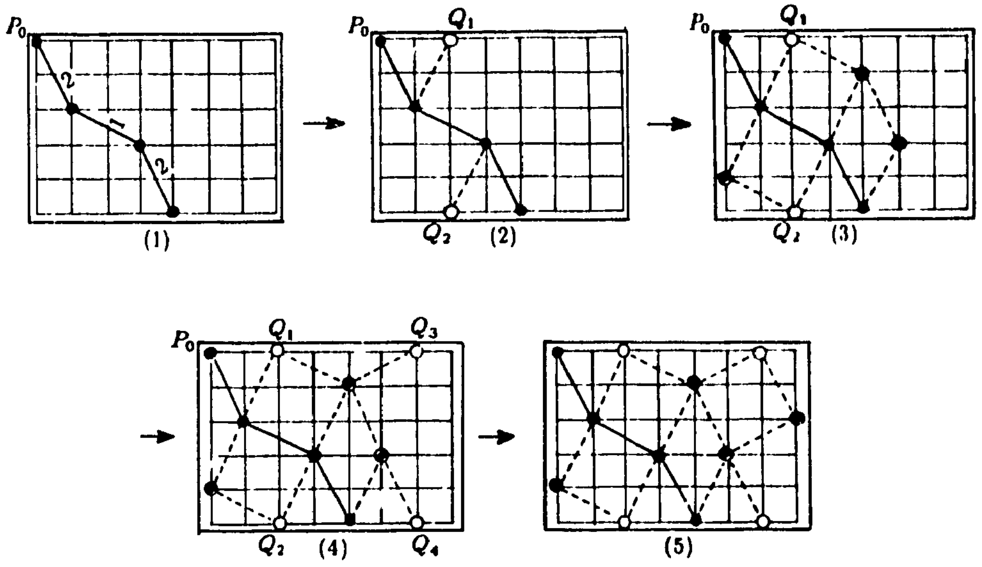


Fig. 4.2: 5 steps to form the periodic Knight's move pattern for $G(6,\infty)$ with $A = (2,1,2)$:

(1) the first Knight's move sequence
(2) 2 half diamonds (by assumption 3) $\Rightarrow Q_1, Q_2$
(3) new Knight's move dominators generated from Q_1, Q_2 (theorem 3.4)
(4) 2 half diamonds (by assumption 3) $\Rightarrow Q_3, Q_4$
(5) new Knight's move dominators generated from Q_3, Q_4 (theorem 3.4)

3. From the new edge dominators, we apply theorem 3.4 to obtain new Knight's move dominators, forming new Knight's move sequences translated from the original sequences. Again, by assumption 3, half diamonds will terminate these sequences on the edges.

   Fig. 4.2 illustrates the process of generating the Knight's move pattern in a period of $G(6, \infty)$ with $A = (2, 1, 2)$.

By theorem 3.8, a period of the pattern in Fig. 4.2 has length

$$p = 2(3m - k + 1) = 2(3.3 - 6 + 1) = 8.$$

Once the Knight's move pattern for a period has been constructed, we can cover any $G(k, \infty)$ by repeating the same pattern on every block of p columns of G (see fig 4.3). We denote SG_A the strip graph covered by the periodic pattern determined by the descriptor sequence A.

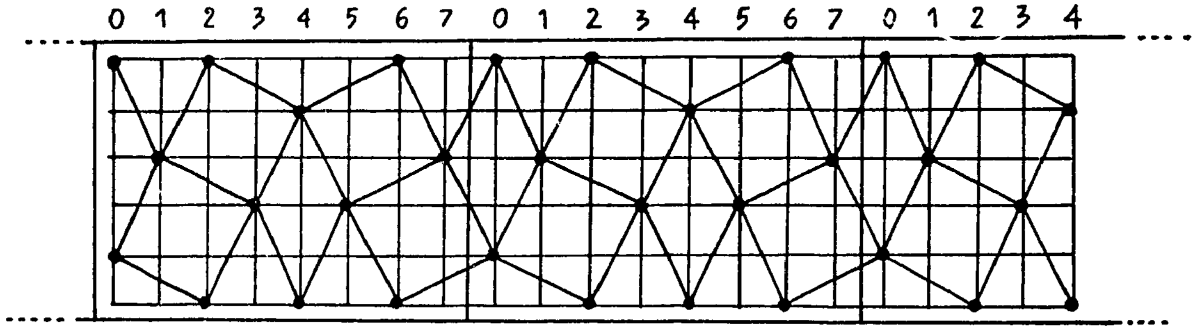


Fig. 4.3: Covering $C(6, \infty)$ using the periodic Knight's move pattern generated by $A = (2, 1, 2)$. We have $p = 8$.

We now introduce a cutting algorithm to obtain the smallest dominating set for $G(k, n)$, given a periodic pattern $\mathcal{P}$ on $G(k, \infty)$.

Let the graph $G(k,n) = G[I,J]$, where $n = J - I + 1$, be the graph composed of columns I to J inclusive of the strip graph SG_A determined by the descriptor sequence A, with period p.

Let $i = I \bmod p, j = J \bmod p, b = \lfloor \frac{n}{p} \rfloor$. Graph $G(k,n)$ is the concatenation of $G[i,p-1]$, b blocks of pattern $\mathcal{P}$, and $G[0,j]$ (see Fig. 4.4).

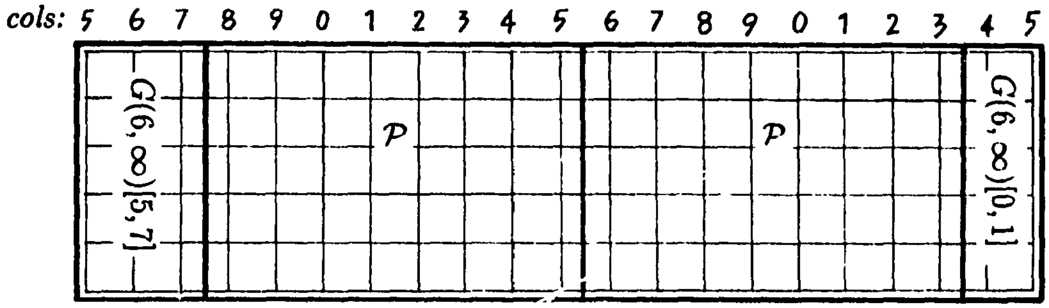


Fig. 4.4: $G(6,21) = G(6,\infty)[5,25] = G(6,\infty)[5,7] \mid\mid 2 \times \mathcal{P} \mid\mid G(6,\infty)[0,1]$.

We say that the pattern $\mathcal{P}$ is *left-cut* on the i^{th} column and *right-cut* on the j^{th} column to form $G(k,n)$. Some vertices on column i are covered only by dominators on the previous column of $\mathcal{P}$; therefore, we must pull dominators on column $(i-1) \bmod p$ in onto it to have the left edge of $G(k,n)$ covered. Similarly, we must pull dominators of $\mathcal{P}$ on column $(j+1) \bmod p$ in onto the right edge of $G(k,n)$. To obtain the smallest dominating set for $G(k,n)$ of a given pattern $\mathcal{P}$, we want to minimize the number of dominators in $G[i,p-1]$ and $G[0,j]$, i.e., to minimize the total number of interior dominators and the number of extra dominators added to the vertical edges, i.e., to minimize the number of dominators in the rectangle

$$G(k,t) = G[i,j] = G[i,p-1] \mid\mid G[0,j]$$

with $t = i + j - 1$.

Let d_c be the number of dominators on column c of $\mathcal{P}$.

Let l_c be the number of dominators pulled in onto column i from column $(i - 1) \bmod p$ of $\mathcal{P}$ (left-cut).

Let r_c be the number of dominators pulled in onto column j from column $(j + 1) \bmod p$ of $\mathcal{P}$ (right-cut).

We have:

$$\gamma(G(k,t)) = \sum_{c=i}^{p} d_c + \sum_{c=0}^{j} d_c + l_i + r_j. \tag{15}$$

For a given length n, I determines J. Since $i = I \bmod p$, $j = J \bmod p$, $I = i.p+j$. Therefore, we conclude that i determines I, J, and j. We may select each column $c \in [0..(p-1)]$ of $\mathcal{P}$ in turn as the starting column of the rectangle $G[i,j]$ and compute $\gamma(G(k,t))$ for this cutting using (15). Any choice of column $i \in [0..(p-1)]$ which produces the smallest $\gamma(G(k,t))$ is considered as the best cutting for given $G(k,t)$ and $\mathcal{P}$.

Then

$$\begin{aligned} \gamma(G(k,n)) &= \gamma(G(k,t)) + b.\mathcal{K} \\ \text{where } t &= n \bmod p \\ b &= \lfloor \frac{n}{p} \rfloor \\ \mathcal{K} &= \text{number of dominators in a period} \end{aligned}$$

$$= m(m+1),\ m = \text{length of the descriptor sequence } A$$

The algorithm to compute d_i, l_i, r_i, $i = 0, 1, \cdots, p-1$ and the cutting algorithm to obtain the best cutting for $G(k,t)$, $t = 1, 2, \cdots, p-1$ are given in Appendix A. Both algorithms are of order $\mathcal{O}(n\mathcal{P}_{k-1})$. Therefore, once all the partitions are determined, the computation proceeds very quickly. Refer to Appendix C for best domination numbers of $G(k,n)$, k from 7 to 16, obtained from this algorithm. We observe that these results are equal to the optimal $\gamma(G(k,n))$ for width $k \leq 10$. This leads us to conclude that a minimum dominating set does not necessarily have half-diamond boundaries as stated in assumption 3 (see Fig. 4.5).

Theorem 4.1 *Assumption 3 does not necessarily lead to the best dominating number.*

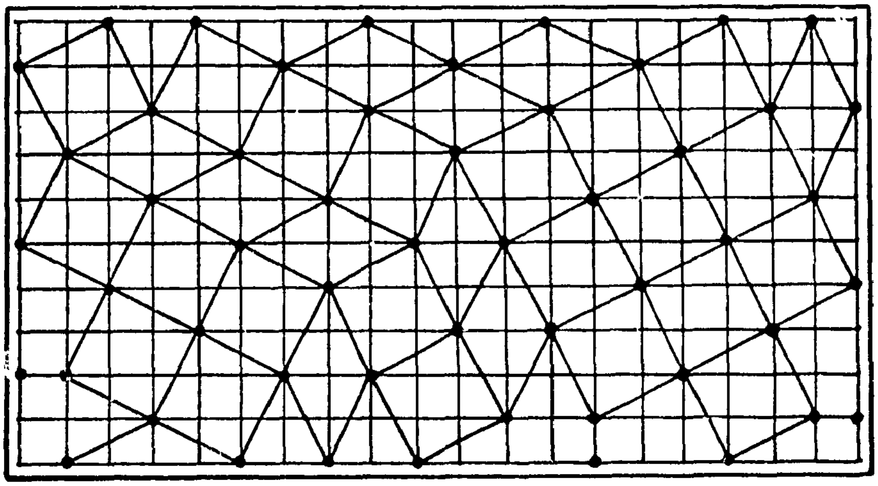


Fig. 4.5: Minimum dominating set for $G(11,20)$ (assumption 3 not applied).

4.2 Algorithm 2

In this section, we present a general algorithm to compute $\gamma(G(k,n))$ for finite strip graphs $G(k,n)$. This algorithm recursively generates all Knight's move patterns which might lead to minimum covering of $G(k,n)$ by generating all possible Knight's move sequences, one in each of the 4 quadrants NE, SE, NW, and SW of the rectangle $G(k,n)$. We use the domination number obtained from star-center patterns as the initial upper bound of $\gamma(G(k,n))$. The recursive solution procedure is as follows:

1. Select an interior vertex P_0 as starting point: To assure that we consider all possible Knight's move patterns for $G(k,n)$, every vertex interior to $G(k,n)$ is a potential candidate for P_0. However, as discussed in section 3.5, we need to consider only the set of 5 starting points:
$$P_0 \in S_0 = \{(x_0, y_0), (x_0 \pm 1, y_0), (x_0, y_0 \pm 1)\}$$
where $x_0 = \lfloor \frac{n}{2} \rfloor$ and $y_0 = \lfloor \frac{k}{2} \rfloor$.
2. For each starting point $P_0 \in S_0$, recursively generate all possible Knight's dividing lines of $G(k,n)$ in NW and SE quadrants. Each dividing line divides $G(k,n)$ into 2 Knight's subgraphs G_1 and G_2.
3. Let γ_u be the best domination number obtained for $G(k,n)$ so far (initially, $\gamma_u = \gamma_s$). For each Knight's dividing line, we proceed as follows:
    - *Step 1:* Count the number of dominators γ_0 on the dividing line.
    - *Step 2:* To cover subgraph G_1, recursively generate all possible Knight's move sequences M_1 originating from P_0 in the NE direction. From a

Knight's move dominator on M_1, we can extend M_1 by generating the next Knight's move in this direction (see Fig. 4.6). Each new Knight's move on M_1 results in new dominators for $G(k, n)$.

Let γ_1 be the number of dominators generated so far to cover G_1.

A branch-and-bound technique is used to backtrack from Knight's moves on M_1 which lead to $\gamma(G) > \gamma_u$. Therefore, we stop trying to extend M_1 if $\gamma_0 + \gamma_1 > \gamma_u$. Otherwise, M_1 will terminate on or beyond the Knight's boundary rays in the NE quadrant.

We select the sequence M_1 for which γ_1 is minimum and $\gamma_0 + \gamma_1 < \gamma_u$.

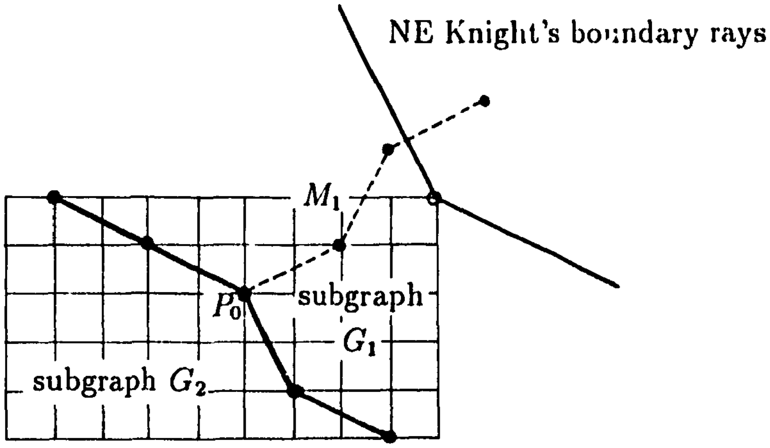


Fig. 4.6: Covering subgraph G_1 of $G(k, n)$.

- *Step 3:* To cover subgraph G_2, apply step 2 to obtain all possible Knight's move sequences M_2 originating from P_0 in the SW direction (see Fig. 4.7). Let γ_2 be the number of dominators generated so far to cover G_2. Branch-and-bound technique is used to backtrack from Knight's moves on M_2 which lead to $\gamma(G) > \gamma_u$. At this stage, γ_1 is known. Therefore, we can use γ_1 to enforce the upper bound i.e to stop trying to extend M_2 if $\gamma_0 + \gamma_2 > \gamma_u - \gamma_1$. Otherwise, M_2 will terminate on or beyond the Knight's boundary rays in the SW quadrant.

We select the sequence M_2 for which γ_2 is minimum and $\gamma_0 + \gamma_2 < \gamma_u - \gamma_1$.

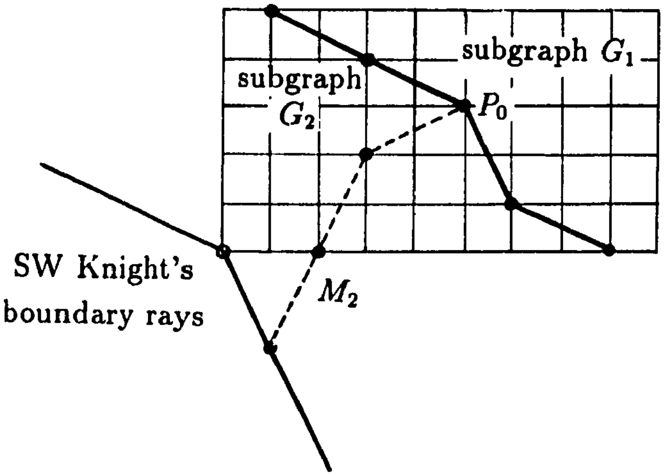


Fig. 4.7: Covering subgraph G_2 of $G(k,n)$.

4. $\gamma(G(k,n))$ is the minimum $\gamma_K(G(k,n))$ for all Knight's dividing lines of $G(k,n)$.

The algorithm to generate the smallest dominating set for $G(k,n)$ given a starting point P_0 is described in Appendix B. Appendix C gives results obtained from this general program for finite grid graphs of widths up to 20. These results agree with $\gamma(G(k,n))$ as produced by Hare's dynamic programming method [11]. This fact strongly justifies the conjecture on optimum covering factor with Knight's move patterns (assumption 2) introduced in chapter 3.

The construction of all possible Knight's move patterns for a given $G(k,n)$ by backtracking is exhaustive. The time required to compute $\gamma(G_(k,n))$ grows exponentially as we increase k and n. By applying the branch-and-bound technique to stop the recursive procedure to extend the Knight's move sequences as the next move from the current move does not lead to any solution better than the optimal solution found so far, we eliminate a considerable number of Knight's move sequence in a given quadrant. In addition, computing time is significantly reduced by theorem 3.11 which allows the decomposition of this problem into 2 independent subproblems to cover 2 independent Knight's subgraphs of $G(k,n)$.

Chapter 5

CONCLUDING REMARKS

The problem of determining the domination number of $k \times n$ grid graphs is a complex problem. In this paper, we have presented a new algorithm to construct a minimum dominating set for $G(k,n)$ using Knight's move patterns.

So far, there are 3 known algorithms aiming to build the smallest dominating set for $G(k,n)$. They are:

1. Algorithm to cover $G(k,n)$ with star-center patterns introduced by Cockayne et al. [4] based on the assumption that ther is no overlap in the interior of $G(k,n)$.
2. Algorithm to cover $G(k,n)$ with Knight's move patterns introduced in this paper based on the relaxed overlap condition.
3. A general algorithm to cover $G(k,n)$ introduced by Hare et al. [11] using the dynamic programming method.

Results obtained show that the star-center patterns do not produce the optimal $\gamma(G(k,n))$ for $G(k,n)$ with finite k and n, while the Knight's move patterns cover very well $G(k,n)$ satisfying the condition $(k-13)(n-13) < 45$. We are able to generate dominating sets for such $G(k,n)$ with a number of dominators equal to the domination number for $G(k,n)$ considered to be optimal so far produced by Hare's general algorithm.

However, we have yet to prove our results are optimal. We may only hope that the conjecture is valid or attempt to prove it. Therefore, as the problem of determining $\gamma(G)$ for general grid graphs G is NP-complete, it remains an open problem, as well as the complexity of the domination problem for complete $k \times n$ grid graphs. A large number of problems is open for study in this area, among which we find the followings to be particularly worthwhile:

1. Optimal domination in 3-dimensional grids.
2. Optimal domination in cylinder graphs: a cylinder graph is a complete grid graph $G(k, n)$ where vertices on 2 vertical edges are neighbours to each other (product of a cycle and a path).

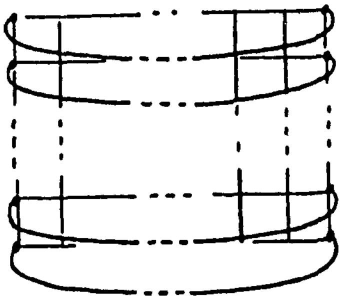


Fig. 5.1: A cylinder graph

3. Optimal domination in torus graphs: a torus graph is a complete grid graph $G(k, n)$ where vertices on parallel edges are neighbours to each other (product of 2 cycles) (see Fig. 5.2).

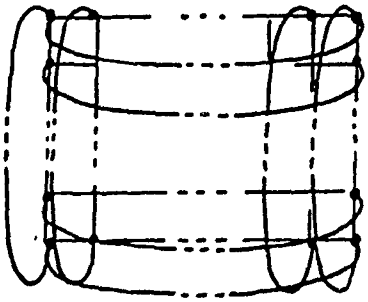


Fig. 5.2: A torus graph

The last 2 graphs are important in VLSI.

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APPENDIX A

Algorithm 1

Generating periodic Knight's move patterns over $G(k, \infty)$

```
algorithm generate (A);
{ Algorithm to compute $d_i$, $l_i$, $r_i$, $i = 0, 1, \cdots, p-1$ for the periodic pattern $\mathcal{P}$ with a period of p of $G(k, \infty)$ generated from the descriptor sequence A.
Variables:
. $A = \{a_1, a_2, \cdots, a_m\}$ : descriptor sequence.
. d = array [0..(p−1)] of integer: d[i] indicates the number of dominators on column i of $\mathcal{P}$.
. l = array [0..(p − 1)] of integer: l[i] indicates the number of dominators pulled in onto column i from column (i − 1) mod p of $\mathcal{P}$ (left-cut).
. r = array [0..(p − 1)] of integer: r[i] indicates the number of dominators pulled in onto column i from column (i + 1)modp of $\mathcal{P}$ (right-cut).
. dom = array [0..(k − 1), 0..(p − 1)] of Boolean: dom[i, j] is true if vertex (i, j) is a dominator, false otherwise. }

begin
    { Initialize arrays d, l, r }
    for col ← 0 to p − 1 do
```

```
        $d[col] \leftarrow 0$;
        $l[col] \leftarrow 0$;
        $r[col] \leftarrow 0$;
        for $row \leftarrow 0$ to $k-1$ do
                $dom[row, col] \leftarrow false$
endfor
{ Compute period p=2(m-k+1) }
$p \leftarrow 2(m-k+1)$
"Generate the periodic pattern $\mathcal{P}$ over the period $p$"
{ Compute $d[i]$, $i = 0, 1, \cdots, p-1$ }
for $col \leftarrow 0$ to $p-1$ do
        for $row \leftarrow 0$ to $k-1$ do
                if dom[row,col] then $d[col] \leftarrow d[col] + 1$
        endfor
endfor
{ Compute l, r }
for $col \leftarrow 0$ to $p-1$ do
        $pre\_col \leftarrow col - 1$;
        if $pre\_col < 0$ then $pre\_col \leftarrow p - 1$;
        $nxt\_col \leftarrow col + 1$;
        if $nxt\_col \geq p$ then $nxt\_col \leftarrow 0$;
```

```
            for row ← 0 to $k-1$ do
                if dom[row,pre_col]
                then if "vertex[row,col] uncovered" then $l[col] \leftarrow l[col]+1$;
                if dom[row,nxt_col]
                then if "vertex[row,col] uncovered" then $r[col] \leftarrow r[col]+1$
            endfor
        endfor
    end
```

GENERATING THE BEST CUTTING OVER A PERIODIC PATTERN OF $G(k,\infty)$

```
algorithm cutting (A);
{ Algorithm to obtain the best cutting for $G(k,t), t = 1,2,\cdots,p-1$ given the
periodic pattern $\mathcal{P}$ of $G(k,\infty)$ generated from the descriptor sequence $A$.
Variables:
. $A = a_1, a_2, \cdots, a_m$ : descriptor sequence.
. $d$ = array $[0..(p-1)]$ of integer: $d[i]$ indicates the number of dominators on column
$i$ of $\mathcal{P}$.
. $l$ = array $[0..(p-1)]$ of integer: $l[i]$ indicates the number of dominators pulled in
onto column $i$ from column $(i-1)$ mod $p$ of $\mathcal{P}$ (left-cut).
. $r$ = array $[0..(p-1)]$ of integer: $r[i]$ indicates the number of dominators pulled in
onto column $i$ from column $(i+1)$ mod $p$ of $\mathcal{P}$ (right-cut).
. Best_dom = array $[1..(p-1)]$ of integer: Best_dom$[i]$ indicates the smallest number
of dominators for the rectangle $G(k,i)$.
. Start = array $[0..(p-1)]$ of integer: Start$[i]$ indicates the column of $\mathcal{P}$ at which a
left-cut will generate the smallest number of dominators for $G(k,i)$, i.e., Best_dom$[i]$.
. no_dom = accumulated number of dominators for the current $G(k,t)$. }
begin
    { Initialize arrays Best_dom and Start }
    for $t \leftarrow 1$ to $(p-1)$ do
        Best_dom$[t] \leftarrow k \times t$;
```

```
                Start[t] ← 0;
                { Find Best_dom[t] and Start[t] for $G(k,t)$, $t = 1, 2, \cdots, p-1$ }
                for start_col ← 0 to $p-1$ do
                    end_col ← start_col + $t - 1$;
                    right_col ← end_col mod $p$;
                    { $G(k,t)$ is left-cut on column start_col of $\mathcal{P}$
                    and right-cut on column right_col of $\mathcal{P}$ }
                    no_dom ← l[start_col] + r[end_col];
                    for col ← start_col to $p-1$ do
                        no_dom ←no_dom + d[col];
                    for col ← 0 to end_col do
                        no_dom ←no_dom + d[col];
                    if no_dom < Best_dom[t]
                    then begin
                        Best_dom[t] ← no_dom;
                        Start[t] ← start_col
                    endif
                endfor
            endfor
        end
```

APPENDIX B

Algorithm 2

COVERING A KNIGHT'S SUBGRAPH OF $G(k,n)$

```
algorithm cover (  snum: subgraph of G (1: $G_1$, 2: $G_2$);
                   P: starting point;
                   A: descriptor sequence for the Knight's move sequence
                       in NW quadrant;
                   B: descriptor sequence for the Knight's move sequence
                       in SE quadrant;
                   var $\gamma[snum]$: total number of dominators
                               in subgraph $G_{snum}$);
```

{ *Algorithm to generate the smallest dominating set for the Knight's subgraph* G_{snum} *of the strip graph* $G(k,n)$*, given the starting point* P *and a dividing line defined by* $M_1 = (P_0, A, S_1)$, P_0, *and* $M_3 = (P_0, B, S_3)$ *where* $P_0 = (x_0, y_0)$, $S_1 = (-1, 1)$, *and* $S_3 = (1, -1)$.

The minimum Knight's move domination number for this subgraph is returned in $\gamma[snum]$.

Variables:

. $A = a_1, a_2, \cdots, a_m$: descriptor sequence.

. $B = b_1, b_2, \cdots, b_n$: descriptor sequence.

```
. $\gamma_0$: total number of dominators on the dividing line defined by $M_1$, $P_0$, $M_3$.
. $Move$ = array [1..2, 1..2] of vector:
    - Subgraph 1: $Move[1,1] = (1,2)$; $Move[1,2] = (2,1)$
    - Subgraph 2: $Move[2,1] = (-1,-2)$; $Move[2,2] = (-2,-1)$
. $Best\gamma[snum]$: the minimum domination number found so far for subgraph $G_{snum}$.
. $Opt\gamma$: the optimal domination number for $G(k,n)$ found so far. Initially, $Opt\gamma = \gamma_s$.
. $DomP$ = total Knight's move dominators generated from a dominator $P$ on the Knight's move sequence generated in the subgraph. }

begin
    if "$P$ is on or beyond the Knight's boundary rays of subgraph $G_{snum}$"
    then begin
            GenDomP($P, A, B, DomP$);
            $\gamma[snum] \leftarrow \gamma[snum] + DomP$;
            "Apply edge covering algorithm to reduce $\gamma[snum]$";
            if $\gamma[snum] < Best\gamma[snum]$
            then { we have a better solution than $Best\gamma[snum]$}
                $Best\gamma[snum] \leftarrow \gamma[snum]$
                return
            endif ;
    else { generate the next Knight's move from $P$ }
```

```
            for $i \leftarrow 1$ to 2 do
                $P \leftarrow P + Move[snum, i]$;
                GenDomP($P, A, B, DomP$); {generate $DomP$ }
                { Branch-and-Bound on $Opt\gamma$ }
                if $(\gamma[snum] + DomP + \gamma_0) < Opt\gamma$
                then Cover($snum, P, A, B, \gamma[snum] + DomP$) {continue}
            endfor
        endif
    end
```

GENERATING THE NUMBER OF DOMINATORS IN A KNIGHT'S SUBGRAPH

```
algorithm GenDomP ( $P$ : starting point;
                    $A$ : descriptor sequence for the Knight's move sequence
                          in NW quadrant;
                    $B$ : descriptor sequence for the Knight's move sequence
                          in SE quadrant;
                    var $DomP$: number of dominators generated from $P$);

{ Algorithm to return $DomP$, the number of Knight's move dominators generated
from a dominator $P$ on the Knight's move sequence generated in the subgraph.
Variables:
. $A = a_1, a_2, \cdots, a_m$ : descriptor sequence.
. $B = b_1, b_2, \cdots, b_n$ : descriptor sequence. }

begin
    $DomP \leftarrow 0$;
    if "$P$ is in the subgraph" then $DomP \leftarrow DomP + 1$;
    $P_1 \leftarrow$ "the number of Knight's move dominators on the Knight's move
            sequence $M = (P, A, S_1)$ interior to the subgraph"
    $P_2 \leftarrow$ "the number of Knight's move dominators on the Knight's move
            sequence $M = (P, B, S_3)$ interior to the subgraph"
    $DomP \leftarrow DomP + P_1 + P_3$
end
```

MAIN ROUTINE TO GENERATE THE SMALLEST KNIGHT'S MOVE DOMINATING SET FOR $G(k,n)$

```
algorithm GenDom ($P_0$: starting point);
{ Algorithm to generate the smallest dominating set for $G(k,n)$ given the starting
point $P_0$.
All the 8 Knight's boundary rays are defined.
Variables:
. $C_1$ = set of all Knight's move sequence $M_1$ in the NW quadrant terminating on or
beyond the upper edge of $G(k,n)$; $M_1 = (P_0, A, S_1)$.
. $C_3$ = set of all Knight's move sequence $M_3$ in the SE quadrant terminating on or
beyond the lower edge of $G(k,n)$; $M_3 = (P_0, B, S_3)$.
. $\gamma_0$: total number of dominators on the dividing line defined by $M_1, P_0, M_3$.
. Best$\gamma$[snum]: the minimum domination number found so far for subgraph $G_{snum}$.
. Opt$\gamma$: the optimal domination number for $G(k,n)$ found so far. Initially, Opt$\gamma$ =
$\gamma_s$. }
begin
    for each $M_1$ in $C_1$ do
        for each $M_3$ in $C_3$ do
            { form the Knight's dividing line $\mathcal{D} = (M_1, P_0, M_3)$ }
            $\gamma_0$ ← "number of Knight's move dominators
                on $\mathcal{D}$ interior to $G(k,n)$"
```

```
            { initialize the optimal domination number in subgraphs $G_1$ and $G_2$ }
            $Best\gamma[1] \leftarrow 999$;
            $Best\gamma[2] \leftarrow 999$;
            { find the smallest dominating sets in 2 subgraphs }
            Cover(1,$P_0$, $A$, $B$,0);
            $\gamma_0 \leftarrow \gamma_0 + Best\gamma[1]$;
            Cover(2,$P_0$, $A$, $B$,0);
            { compute the total number of dominators }
            $Tot\_dom \leftarrow \gamma_0 + Best\gamma[1] + Best\gamma[2]$;
            "Apply the edge covering algorithm to reduce $Tot\_dom$";
            if $Tot\_dom < Opt\gamma$
            then $Opt\gamma \leftarrow Tot\_dom$
        endfor
    endfor
end
```

APPENDIX C

SUMMARY OF RESULTS

Values of $\gamma(G(k,n))$ obtained for widths k from 7 to 20

LEGEND

Area $= k \times n$

$\gamma_1(G)$ = domination number obtained from star-center pattern

$\gamma_{a1}(G)$ = domination number obtained from Algorithm 1

$\gamma_{a2}(G)$ = domination number obtained from Algorithm 2

$\gamma_H(G)$ = optimal domination number obtained from Hare's general algorithm [11]

Empty slot = data not available or not calculated

n	Area	Results				Best overall		Run time (secs) for algorithm 2
		$\gamma_1(G)$	$\gamma_{a1}(G)$	$\gamma_{a2}(G)$	$\gamma_H(G)$	$\gamma(G)$	$\sigma(G)$	
Width $k = 7$								
7	49	12	12	12		12	4.0833	2
8	56	14	14	14		14	4.0000	3
9	63	15	16	16		15	4.2000	4
10	70	17	17	17		17	4.1176	6
11	77	19	19	19		19	4.0526	7
12	84	21	21	21		21	4.0000	10
13	91	23	22	22		22	4.1364	12
14	98	24	24	24		24	4.0833	17
15	105	26	26	26		26	4.0385	22
16	112	28	27	27		27	4.1481	31
17	119	30	29	29		29	4.1034	40
18	126	32	31	31		31	4.0645	58
19	133	33	32	32		32	4.1563	77
20	140	35	35	34		34	4.1176	111
21	147	37	36	36		36	4.0833	145
22	154	39	37	37		37	4.1622	218
23	161	41	39	39		39	4.1282	293
24	168	42	41			41	4.0976	
25	175	44	42			42	4.1667	
26	182	46	45			45	4.0444	
27	189	48	46			46	4.1087	
28	196	50	47			47	4.1702	
29	203	51	49			49	4.1429	
30	210	53	51			51	4.1176	

n	Area	Results				Best overall		Run time (sec for algorithm
		$\gamma_1(G)$	$\gamma_{a1}(G)$	$\gamma_{a2}(G)$	$\gamma_H(G)$	$\gamma(G)$	$\sigma(G)$	
Width $k = 8$								
8	64	16	16	16		16	4.0000	5
9	72	18	18	18		18	4.0000	7
10	80	20	20	20		20	4.0000	9
11	88	22	22	22		22	4.0000	12
12	96	24	24	24		24	4.0000	16
13	104	26	26	26		26	4.0000	20
14	112	28	28	28		28	4.0000	25
15	120	30	29	29		29	4.1379	34
16	128	32	31	31		31	4.1290	45
17	136	34	33	33		33	4.1212	61
18	144	36	35	35		35	4.1143	77
19	152	38	37	37		37	4.1081	106
20	160	40	39	39		39	4.1026	139
21	168	42	41	41		41	4.0976	203
22	176	44	43	43		43	4.0930	279
23	184	46	44	44		44	4.1818	401
24	192	48	46	46		46	4.1739	328
25	200	50	48	48		48	4.1667	412
26	208	52	50	50		50	4.1600	1060
27	216	54	52	52		52	4.1538	1513
28	224	56	54	54		54	4.1481	2019
29	232	58	56	56		56	4.1429	2840
30	240	60	58	58		58	4.1379	2330
31	248		59	59		59	4.2034	2891
32	256		61	61		61	4.1967	4266
33	264		63	63		63	4.1905	5408

n	Area	Results				Best overall		Run time (secs) for algorithm 2
		$\gamma_l(G)$	$\gamma_{a1}(G)$	$\gamma_{a2}(G)$	$\gamma_H(G)$	$\gamma(G)$	$\sigma(G)$	
Width $k = 9$								
9	81	20	20	20	20	20	4.0500	4
10	90	22	22	22	22	22	4.0909	8
11	99	24	24	24	24	24	4.1250	11
12	108	26	26	26	26	26	4.1538	15
13	117	29	29	29	29	29	4.0345	19
14	126	31	31	31	31	31	4.0645	27
15	135	33	33	33	33	33	4.0909	32
16	144	35	35	35	35	35	4.1143	45
17	153	37	37	37	37	37	4.1351	53
18	162	40	39	39	39	39	4.1538	77
19	171	42	41	41	41	41	4.1707	91
20	180	44	43	43	43	43	4.1860	144
21	189	46	45	45	45	45	4.2000	165
22	198	48	47	47	47	47	4.2128	248
23	207	51	49	49	49	49	4.2245	292
24	216	53	52	52	52	52	4.1538	500
25	225	55	54	54	54	54	4.1667	571
26	234	57	56	56	56	56	4.1786	974
27	243	59	58	58	58	58	4.1897	1146
28	252	62	60	60	60	60	4.2000	1848
29	261	64	62	62	62	62	4.2097	2199
30	270	66	64	64	64	64	4.2188	3724
31	279	68	66	66	66	66	4.2273	4356

n	Area	Results				Best overall		Run time (secs) for algorithm 2
		$\gamma_1(G)$	$\gamma_{a1}(G)$	$\gamma_{a2}(G)$	$\gamma_H(G)$	$\gamma(G)$	$\sigma(G)$	
Width $k = 10$								
10	100	24	24	24	24	24	4.1667	13
11	110	27	27	27	27	27	4.0741	17
12	120	29	29	29	29	29	4.1379	24
13	130	32	31	31	31	31	4.1935	31
14	140	34	34	34	34	34	4.1176	41
15	150	36	36	36	36	36	4.1667	49
16	160	39	38	38	38	38	4.2105	65
17	170	41	41	41	41	41	4.1463	81
18	180	44	43	43	43	43	4.1860	109
19	190	46	45	45	45	45	4.2222	136
20	200	48	48	48	48	48	4.1667	188
21	210	51	50	50	50	50	4.2000	234
22	220	53	52	52	52	52	4.2308	318
23	230	56	54	54	54	54	4.2593	387
24	240	58	57	57	57	57	4.2105	582
25	250	60	59	59	59	59	4.2373	724
26	260	63	62	62	62	62	4.1935	1086
27	270	65	64	64	64	64	4.2188	1412
28	280	68	66	66	66	66	4.2424	2143
29	290	70	69	69	69	69	4.2029	2666
30	300	72	71	71	71	71	4.2254	4277
31	310	75	73	73	73	73	4.2466	5524
32	320	77	75	75	75	75	4.2667	8292
33	330		78	78	78	78	4.2308	10342
34	340		80	80	80	80	4.2500	16500
35	350		82	82	82	82	4.2683	20912
36	360		84	84	84	84	4.2857	30668
37	370		87	87	87	87	4.2529	38766
38	380		89	89	89	89	4.2697	57878
39	390		92	92	92	92	4.2391	72282
40	400		94		94	94	4.2553	

n	Area	Results				Best overall		Run time (secs) for algorithm 2
		$\gamma_1(G)$	$\gamma_{a1}(G)$	$\gamma_{a2}(G)$	$\gamma_H(G)$	$\gamma(G)$	$\sigma(G)$	
Width $k = 11$								
11	121	29	29	29	29	29	4.1724	17
12	132	32	32	32	32	32	4.1250	36
13	143	35	35	35	35	35	4.0857	48
14	154	37	37	37	37	37	4.1622	64
15	165	40	40	40	40	40	4.1250	82
16	176	42	42	42	42	42	4.1905	111
17	187	45	45	45	45	45	4.1556	137
18	198	48	47	47	47	47	4.2128	193
19	209	50	50	50	50	50	4.1800	222
20	220	53	53	52	52	52	4.2308	310
21	231	55	55	55	55	55	4.2000	379
22	242	58	58	57	57	57	4.2456	550
23	253	61	60	60	60	60	4.2167	645
24	264	63	63	63	63	63	4.1905	986
25	275	66	65	65	65	65	4.2308	1147
26	286	68	68	68	68	68	4.2059	1756
27	297	71	70	70	70	70	4.2429	2003
28	308	74	73	73	73	73	4.2192	3465
29	319	76	75	75	75	75	4.2533	3949
30	330	79	78	78	78	78	4.2308	6483
31	341	81	80	80	80	80	4.2625	7635
32	352	84	83	83	83	83	4.2410	13165
33	363	87	86	85	85	85	4.2706	15234
34	374		88	88	88	88	4.2500	24806
35	385		91	91	91	91	4.2308	29069
36	396		93		93	93	4.2581	
37	407		96		96	96	4.2396	
38	418		98		98	98	4.2653	
39	429		101		101	101	4.2475	
40	440		103		103	103	4.2718	

n	Area	Results				Best overall		Run time (secs) for algorithm 2
		$\gamma_1(G)$	$\gamma_{a1}(G)$	$\gamma_{a2}(G)$	$\gamma_H(G)$	$\gamma(G)$	$\sigma(G)$	
					Width $k = 12$			
12	144	35	35	35	35	35	4.1143	56
13	156	38	38	38	38	38	4.1053	72
14	168	40	40	40	40	40	4.2000	96
15	180	43	43	43	43	43	4.1860	127
16	192	46	46	46	46	46	4.1739	171
17	204	49	49	49	49	49	4.1633	219
18	216	52	52	51	51	51	4.2353	289
19	228	54	54	54	54	54	4.2222	355
20	240	57	57	57	57	57	4.2105	472
21	252	60	60	60	60	60	4.2000	573
22	264	63	63	62	62	62	4.2581	784
23	276	66	65	65	65	65	4.2462	960
24	288	68	68	68	68	68	4.2353	1294
25	300	71	71	71		71	4.2254	1591
26	312	74	74	74		74	4.2162	2242
27	324	77	76	76		76	4.2632	2716
28	336	80	79	79		79	4.2532	4064
29	348	82	82	82		82	4.2439	5038
30	360	85	85	85		85	4.2353	7322
31	372	88	87	87		87	4.2759	9201
32	384	91	90	90		90	4.2667	14526
33	396	94	93	93		93	4.2581	18270
34	408	96	96	96		96	4.2500	28975
35	420		99	98		98	4.2857	37030
36	432		102	101		101	4.2772	57241

n	Area	Results				Best overall		Run time (secs) for algorithm 2
		$\gamma_1(G)$	$\gamma_{a1}(G)$	$\gamma_{a2}(G)$	$\gamma_H(G)$	$\gamma(G)$	$\sigma(G)$	
Width $k = 13$								
13	169	41	40	40		40	4.2250	79
14	182	44	44	44		44	4.1364	152
15	195	47	47	47		47	4.1489	202
16	208	50	49	49		49	4.2449	273
17	221	53	53	53		53	4.1698	343
18	234	56	56	55		55	4.2545	464
19	247	59	58	58		58	4.2586	589
20	260	62	62	62		62	4.1935	792
21	273	65	65	64		64	4.2656	981
22	286	68	67	67		67	4.2687	1366
23	299	71	71	70		70	4.2714	1605
24	312	74	74	73		73	4.2740	2256
25	325	77	76	76		76	4.2763	2645
26	338	80	80	79		79	4.2785	3870
27	351	83	83	82		82	4.2805	4558
28	364	86	85	85		85	4.2824	6513
29	377	89	88	88		88	4.2841	7283
30	390	92	92	91		91	4.2857	12133
31	403	95	94	94		94	4.2872	13984
32	416	98	98	97		97	4.2887	23543
33	429	101	101	100		100	4.2900	25964

n	Area	Results				Best overall		Run time (secs) for algorithm 2
		$\gamma_1(G)$	$\gamma_{a1}(G)$	$\gamma_{a2}(G)$	$\gamma_H(G)$	$\gamma(G)$	$\sigma(G)$	
Width $k = 14$								
14	196	47	47	47		47	4.1702	246
15	210	50	50	50		50	4.2000	327
16	224	53	53	53		53	4.2264	422
17	238	56	56	56		56	4.2500	525
18	252	60	60	60		60	4.2000	696
19	266	63	63	63		63	4.2222	899
20	280	66	66	66		66	4.2424	1204
21	294	69	69	69		69	4.2609	1508
22	308	72	73	72		72	4.2778	2010
23	322	76	76	76		76	4.2368	2471
24	336	79	79	79		79	4.2532	3266
25	350	82	82	82		82	4.2683	3955
26	364	85	85	85		85	4.2824	5468
27	378	88	88	88		88	4.2955	6680
28	392	92	92			92	4.2609	
29	406	95	95	95		95	4.2737	10977
30	420	98	98	98		98	4.2857	15228
31	434	101	101	101		101	4.2970	18477
32	448	104	105	104		104	4.3077	27266
33	462	108	108			108	4.2778	
34	476	111	111			111	4.2883	
35	490	114	114			114	4.2982	
36	504	117	118			117	4.3077	
37	518	120	121			120	4.3167	
38	532	124	124			124	4.2903	
39	546	127	127			127	4.2992	
40	560	130	130			130	4.3077	

n	Area	Results				Best overall		Run time (secs) for algorithm 2
		$\gamma_1(G)$	$\gamma_{a1}(G)$	$\gamma_{a2}(G)$	$\gamma_H(G)$	$\gamma(G)$	$\sigma(G)$	
Width $k = 15$								
15	225	53	53	53		53	4.2453	347
16	240	57	57	57		57	4.2105	658
17	255	60	60	60		60	4.2500	856
18	270	64	63	64		63	4.2857	1135
19	285	67	67	67		67	4.2537	1470
20	300	70	71	70		70	4.2857	1963
21	315	74	74	74		74	4.2568	2418
22	330	77	78	77		77	4.2857	3324
23	345	81	81	81		81	4.2593	3930
24	360	84	84	84		84	4.2857	5672
25	375	87	88			87	4.3103	
26	390	91	91			91	4.2857	
27	405	94	94			94	4.3085	
28	420	98	98			98	4.2857	
29	435	101	101			101	4.3069	
30	450	104	105			104	4.3269	

n	Area	Results				Best overall		Run time (secs) for algorithm 2
		$\gamma_1(G)$	$\gamma_{a1}(G)$	$\gamma_{a2}(G)$	$\gamma_H(G)$	$\gamma(G)$	$\sigma(G)$	
Width $k = 16$								
16	256	60	60	60		60	4.2667	971
17	272	64	64	64		64	4.2500	1312
18	288	68	68	68		68	4.2353	1796
19	304	71	71	71		71	4.2817	2318
20	320	75	75	75		75	4.2667	3015
21	336	78	79	78		78	4.3077	3768
22	352	82	82	82		82	4.2927	4998
23	368	86	86	86		86	4.2791	6216
24	384	89	90			89	4.3146	
25	400	93	93	93		93	4.3011	10615
26	416	96	97			96	4.3333	
27	432	100	101			100	4.3200	
28	448	104	104			104	4.3077	
29	464	107	108			107	4.3364	
30	480	111	112			111	4.3243	
Width $k = 17$								
19	323	75		75		75	4.3067	3523
20	340	79		79		79	4.3038	4754
21	357	83		83		83	4.3012	6136
22	374	87		87		87	4.2989	8173
Width $k = 20$								
20	400	92		92		92	4.3478	18023