HIGHER DIMENSIONAL ANALOGUES OF THE TENT MAPS

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A Thesis

in

The Department

of

Mathematics

Presented in Partial Fulfillment of the requirements
for the degree of Master of Science at
Concordia University
Montreal, Quebec, Canada

August, 1987

ISBN 0-315-37070-X

ABSTRACT

Higher dimensional analogues of the tent maps

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A family of point transformations:

$$T(x,y) : [0,1] \times [0,1] \longrightarrow [0,1] \times [0,1]$$

is defined, and the main characteristics of this family are investigated. It shall be shown that this family satisfies a sufficient condition for the existence of continuous ergodic invariant measures and, in particular , that a sub-class of this family has Lebesgue measure as its invariant measure.

ACKNOWLEDGEMENT

The author would like to express his gratitude to his thesis supervisor, Dr. Abraham Boyarsky of Concordia University, for his many helpful comments and suggestions. Also, I would like to thank Dr. H. Proppe and Dr. W. Byers of Concordia University for their valuable assistance.

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CHAPTER I

Introduction

Recently there has been considerable interest in simple low dimensional dynamical systems, both continuous and discrete. This interest stems from the fact that these systems show features which mimic physical phenomena such as turbulence. In fluids, turbulence can be thought of as the chaotic mixing of viscous flows induced by large-scale eddies which produce ever smaller-scale eddies. All laminar flows become unstable and change into turbulence when a flow parameter, called Reynolds number, becomes large enough. The term turbulence is also applicable in a wide range of other fields, such as mathematical biology where the complexity and extreme non-linearity of biological systems makes chaos likely to occur. Statistical mechanics and Hamiltonian dynamics also provide a rich source of examples of systems that can exhibit complicated behavior. Therefore physicist, biologists, engineers and mathematicians all have reason to understand this type of phenomena.

A recent concept that has led to a greater understanding of such chaotic systems, is the so called strange attractor. Mathematically this is a complicated set with a Cantor-like cross-section. A simple example of a strange attractor was given in 1976 by M.Henon [14], who investigated the properties of the two-dimensional family of quadratic mappings

$$T(x,y) = (1 + y - ax^2 , bx)$$

For $a = 1.4$ and $b = 0.3$, he found that points under this transformation tended towards a limit set with a Cantor-like cross-section, as shown in Fig. 1.1 .

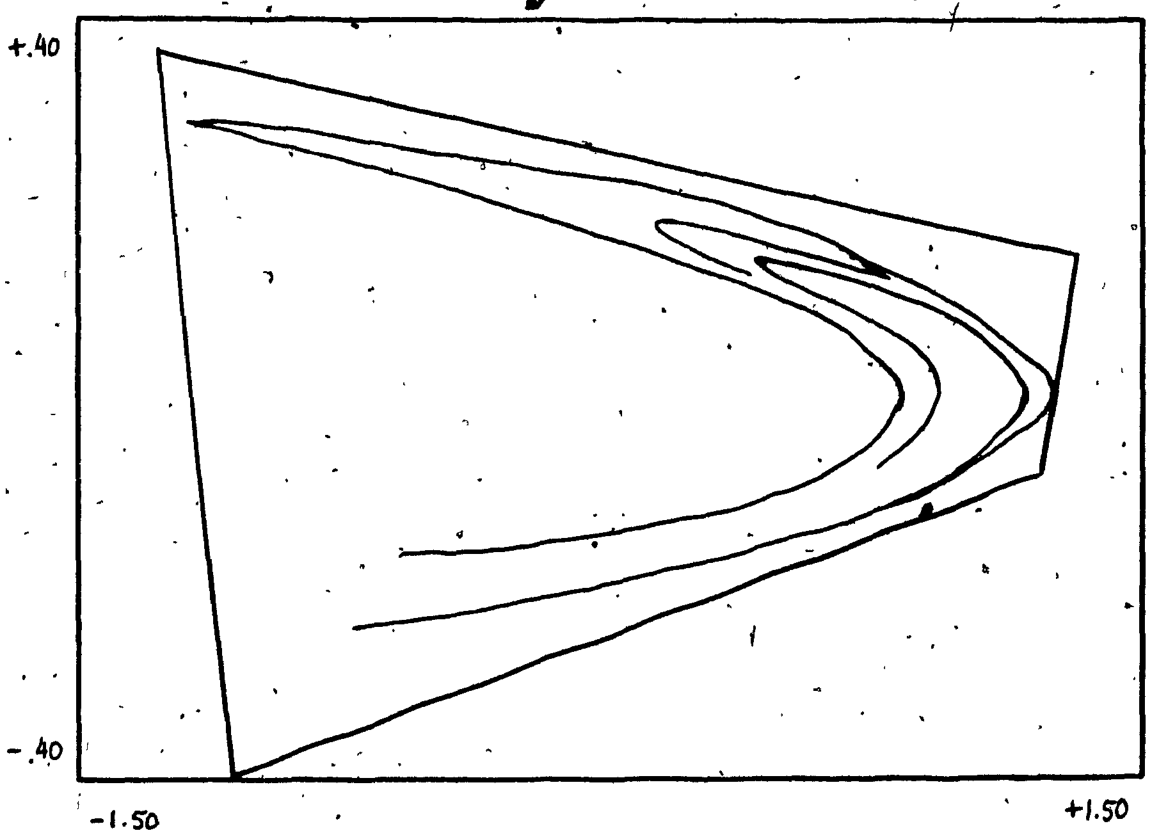

Fig. 1.1

The importance of strange attractors can be seen in fluid mechanics, where the discovery of coherent structures has engendered new excitement in turbulence research. The discovery of these strange attractor-like structures is quite exciting because the time-evolution of a turbulent flow might now be mathematically tractable [16].

Currently, most of our knowledge of the complicated behavior of such systems is obtained numerically by computer experiments. Above all, what seems to be lacking is a simple generic two-dimensional map upon which mathematical questions may be focused and insight gained towards the study of the above mentioned systems. In this thesis a class of two-dimensional point mappings is presented as such a generic model. A detailed analytic description of their dynamics is given along with a method of generalizing these maps to higher dimensions. First, to motivate the construction of this class of transformations, a one-dimensional example is presented.

1.2 The one-dimensional family of tent-maps

One of the most studied one-dimensional systems [1,2] is defined by

$$T_a(x) = \begin{cases} x/a & , \ 0 < x < a \\ (x-1)/(a-1) & , \ a < x < 1 \end{cases}$$

where a denotes the turning point, as shown in Fig. 1.2 . This family of point transformations is simple and has extraordinary rich dynamical behavior [1]. It has therefore become the prototype for one-dimensional discrete dynamical systems.

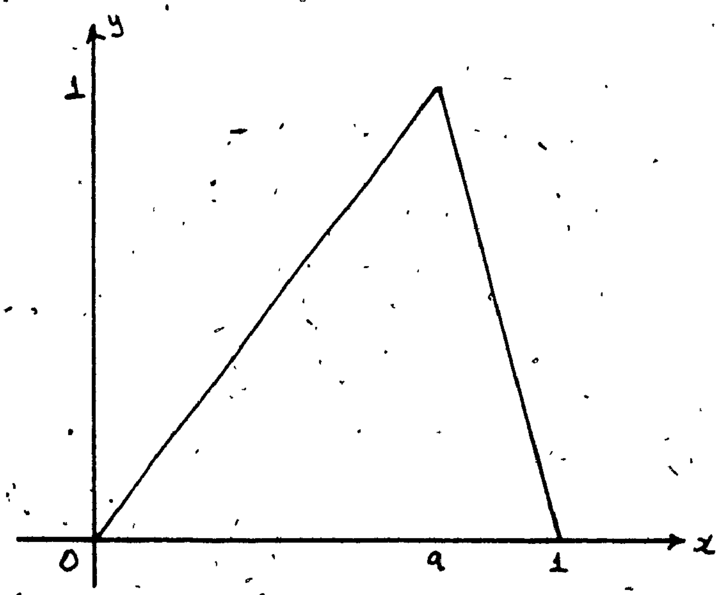

Fig.1.2

Some characteristics of this family are

1. Restricted to either [o,a] or [a,1], T_a is a homeomorphism onto [0,1].

2. Inf $|T_a'| > 1$, that is T_a is a piecewise expanding transformation.

3. Given $0 < a < b < 1$, there exists a homeomorphism $h_{a,b}$ such that

$$T_b(h_{a,b}) = h_{a,b}(T_a)$$

that is the maps are topologically conjugate, so rich dynamical behavior is shared by each map in the family [3].

4. The map $T_a(x)$ preserves Lebesgue measure. that is

$$m(T_a^{-1}(A)) = m(A)$$

for all Lebesgue measurable subsets A in [0,1], (where m denotes Lebesgue measure).

1.3 Scope of the Thesis

In Chapter II a two-dimensional analogue of the family of tent-maps will be introduced [18]. They will be referred to as 'spine-maps'.

In Chapter III a sufficient condition for the existence of measures invariant with respect to continuous mappings on topological spaces is presented and proved. The existence of an ergodic, continuous invariant measure for the spine-maps is then shown. The main result of [9] is then used to establish the existence of an absolutely continuous invariant measure for a certain sub-class of the family of spine-maps. We end this chapter by showing that another sub-class of the spine-maps preserves Lebesgue measure.

In Chapter IV the family of spine-maps is generalized to higher dimensions.

In Chapter V some numerical examples are given.

CHAPTER II

Spine-maps

2.1 Definition

Let

$$T(x,y) : [0,1]x[0,1] \longrightarrow [0,1]x[0,1]$$

be defined by $T(x,y) = (t_1(x,y), t_2(x,y))$, where

$$t_1(x,y) : [0,1]x[0,1] \longrightarrow [0,1]$$
$$t_2(x,y) : [0,1]x[0,1] \longrightarrow [0,1]$$

t_1 and t_2 are shown in Fig.2.1 . Refering to Fig.2.1 (a), we see that the continuous function $x = g(y)$ acts as a 'spine' at level $z = 1$, and that all of the cross-sections of the surfaces $t_1(x,y)$ are tent-maps having their peaks on the spine. Clearly as y varies over [0,1], the tent-maps will continuously change their shape. Thus the sides of the surface $t_1(x,y)$ will not be planar, rather, a curved surface.

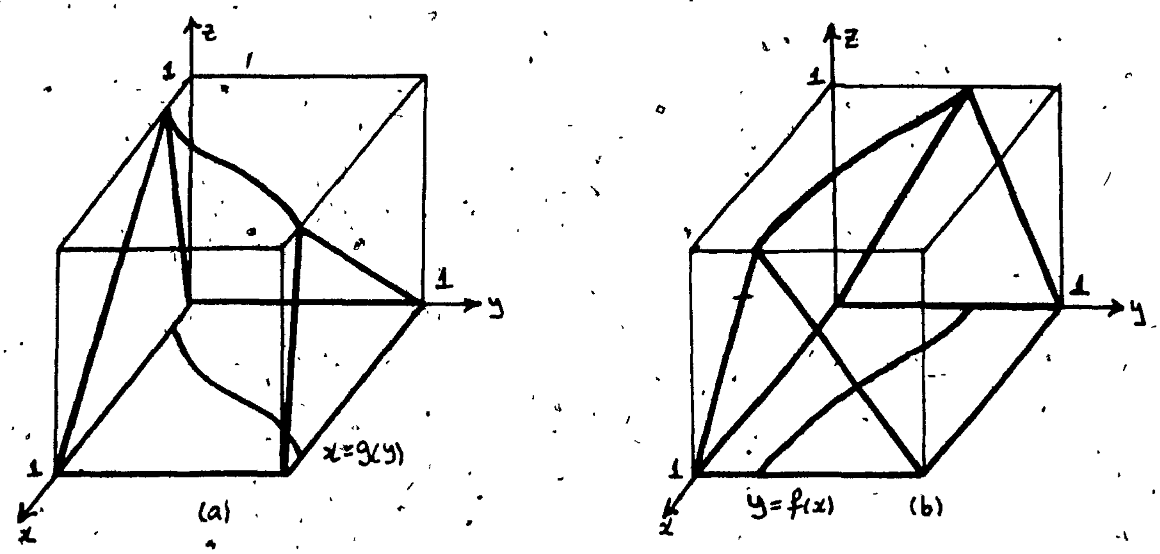

Fig.2.1

We shall assume that $x = g(y)$ is a monotonic function on [0,1]. Similarly, we let $y = f(x)$, also monotonic on [0,1], serve as a 'spine' for $t_2(x,y)$. Projecting these two spines down onto the xy-plane will divide the unit square into four regions. Referring to Fig.2.2, we may then write the following definition:

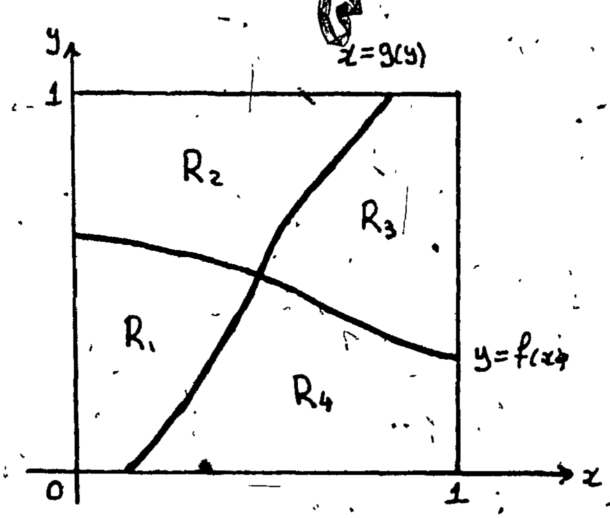

Fig 2.2

$$T(x,y) = \begin{cases} (x/g(y) , y/f(x)) & , (x,y) \in R_1 \\ (x/g(y) , (y-1)/(f(x)-1)) & , (x,y) \in R_2 \\ ((x-1)/(g(y)-1) , (y-1)/(f(x)-1)) & , (x,y) \in R_3 \\ ((x-1)/(g(y)-1) , y/f(x)) & , (x,y) \in R_4 \end{cases}$$

Note that each spine-map is uniquely determined by the two spine-functions $f(x)$ and $g(y)$.

The family of spine-maps share some important properties with the one-dimensional tent-maps. These properties are:

1. Restricted to one of the four regions $R_i, i=1,2,3,4$ (see Fig.2.2), the spine-maps are homeomorphisms onto the unit square.

2. Certain spine-maps are expanding.

3. Spine-maps are topologically conjugate to each other.

We shall demonstrate each of these properties. The unit square $[0,1]x[0,1]$ will be denoted by X .

2.2 Spine-maps are piecewise homeomorphic to X.

Let

$$T : X \longrightarrow X$$

Theorem 2.2.1 [18]. Let $g(y)$ be monotonically decreasing and $f(x)$ be monotonically increasing (or,vise-versa). Then

$$T : Ri \longrightarrow X$$

is a homeomorphism for i=1,2,3,4.

Proof: (This will be proved for i=1)
To show that T is one-one on R1, we must show that there exists a unique solution to the pair of equations

$$x/g(y) = a \quad \text{and} \quad y/f(x) = b$$

with a and b elements of [0,1]. That is, the curves $x = ag(y)$ and $y = bf(x)$ must intersect at only one point. But this follows from the monotonicity of $f(x)$ and $g(y)$. Since this is true for every $a,b \in [0,1]$, T is also onto.

Q.E.D.

2.3 Piecewise expanding spine-maps.

We have seen that the spine-functions, $f(x)$ and $g(y)$, divide the unit square into four regions: R_i, $i=1,2,3,4$, and that, restricted to each of the R_i, the corresponding spine-map is a homeomorphism to $[0,1]\times[0,1]$. We now wish to show that some of these spine-maps are also expanding maps on each of the R_i. We first define the operator norm:

$$|A| = \mathrm{Sup}(|Av| : |v| = 1)$$

Where $v = (v_1, \ldots, v_n)$ and $|v| = \sqrt{v_1^2 + \ldots + v_n^2}$.
Also, we will need the following result:

Lemma 2.3.1

$$\max_{|v| = 1} |Av| = \sqrt{\pi_1}$$

Where π_1 is the largest eigenvalue of the matrix A^TA.

Proof: The matrix A^TA is symmetric. Let $\pi_1 \geq \ldots \geq \pi_n$ be its eigenvalues and $v_1, \ldots, v_n$ be the orthonormal system of eigenvectors belonging to these eigenvalues. Now, any vector u, with norm equal to one, may be written:

$$u = c_1v_1 + \ldots + c_nv_n .$$

Then,

$$c_1^2 + \ldots + c_n^2 = 1$$

Moreover,

$$\begin{aligned}\langle Av, Av\rangle &= \langle v, A^TAv\rangle \\ &= \pi_1 c_1^2 + \ldots + \pi_n c_n^2 \\ &\leq \pi_1 (c_1^2 + \ldots + c_n^2) \\ &= \pi_1\end{aligned}$$

For the eigenvector v_1, we have:

$$\langle Av_1, Av_1\rangle = \langle v_1, A^TAv_1\rangle = \langle v_1, \lambda v_1\rangle = \pi_1$$

Thus:

$$\sup_{|u| = 1} |Au| = \sqrt{\pi_1}$$

Q.E.D.

The following result gives a sufficient condition for a given spine-map to be expanding on each of the R_i.

<u>Proposition 1</u>

Let T_i be the restriction of a given spine-map to the region R_i, $i=1,2,3,4$. Let $S_i = T_i^{-1}$ and JS_i be the Jacobian matrix associated with S_i. If

$$\sup_{(x,y)\in R_i} \left| JS_i(x,y) \right| \leq \alpha^{-1} < 1,$$

then

$$d(T_i(x_0,y_0), T_i(x_1,y_1)) > \alpha\, d((x_0,y_0),(x_1,y_1))$$

where $(x_0,y_0),(x_1,y_1) \in R_i$.

Proof: Consider the line segment joining the two points $u = T_i(x_0, y_0)$ and $v = T_i(x_1, y_1)$ in $[0,1] \times [0,1]$. Since T_i is a homeomorphism and $[0,1] \times [0,1]$ is convex then this line segment may be represented by a differentiable function $h: [0,1] \longrightarrow [0,1] \times [0,1]$, in fact, $h(t) = (1-t)u + tv$. Now, the length of this line segment is

$$d(u,v) = \int_0^1 \langle h'(t), h'(t) \rangle^{1/2} \, dt$$

Now, $S_i \cdot h$ is a curve in R_i whose length is

$$L = \int_0^1 \langle (S_i \cdot h)'(t), (S_i \cdot h)'(t) \rangle^{1/2} \, dt$$

$$= \int_0^1 \langle J_{Si}(h(t)) \, h'(t), J_{Si}(h(t)) \, h'(t) \rangle^{1/2} \, dt$$

Let $A_t = J_{Si}(h(t))$, $X_t = h'(t)$ then,

$$L = \int_0^1 (X_t^T A_t^T A_t X_t)^{1/2} \quad dt$$

Since $A_t^T A_t$ is a symmetric matrix there exists an orthogonal matrix O_t such that $O_t^T A_t^T A_t O_t = D_t$, where D_t is a diagonal matrix with real entries $\pi_1 \leq \ldots \leq \pi_n$. Now,

$$X_t^T A_t^T A_t X_t = X_t^T O_t O_t^T A_t^T A_t O_t O_t^T X_t$$
$$= (X_t^T O_t)\, D_t\, (O_t^T X_t)$$
$$= Y_t^T D_t Y_t$$

where $Y_t = O_t^T X_t$. Therefore,

$$(Y_t^T D_t Y_t)^{1/2} = (\Sigma\, y_i^2\, \pi_i)^{1/2}$$
$$< \left|Y_t\right| (\max \pi_i)^{1/2}$$
$$= \sqrt{\pi_1}\, \left|Y_t\right|$$
$$= \sqrt{\pi_1}\, \left|X_t\right|$$

By Lemma 2.3.1

$$\left|A_t\right| = \sqrt{\pi_1}$$

Therefore,

$$\sqrt{\pi_1} \leq \alpha^{-1} < 1$$

Thus,

$$(Y_t^T D_t Y_t)^{1/2} < \alpha^{-1} \left|X_t\right|$$

Thus,

$$L \leq \alpha^{-1} \int_0^1 X_t \, dt$$

$$= \alpha^{-1} \; d(u,v)$$

That is

$$d((x_0,y_0),(x_1,y_1)) \leq \alpha^{-1} \; d(T_1(x_0,y_0),T_1(x_1,y_1))$$

and the proposition is proved.

To show that a given spine-map satisfies the conditions of Proposition 1, we have to show that the largest eigenvalue of $J_{Si}{}^T J_{Si}$ is less than unity. (Or the converse, that the smallest eigenvalue of $J_{Ti}{}^T J_{Ti}$ is greater than 1.) To illustrate this, we shall consider the spine-map resulting from the two constant spine-functions

$$g(y) = m \quad \text{and} \quad f(x) = k \,, \quad 0 < m,k < 1$$

Therefore, in R_1, we have:

$$T_1(x,y) = (x/m, y/k)$$

which gives us:

$$J_{T1}(x,y) = \begin{pmatrix} 1/m & 0 \\ 0 & 1/k \end{pmatrix}$$

And, thus:

$$J_{T1}{}^T J_{T1} = \begin{pmatrix} 1/m^2 & 0 \\ 0 & 1/k^2 \end{pmatrix}$$

Therefore the eigenvalues of this matrix are:

$$\pi_1 = 1/m^2 \quad \text{and} \quad \pi_2 = 1/k^2$$

So, $0 < m,k < 1$ implies that $1/m^2, 1/k^2 > 1$. Thus, all spine-maps defined by constant spine-functions are piecewise expanding on $[0,1]x[0,1]$.

If we now consider the most general spine-map, that is where $f(x)$ and $g(y)$ are arbitrary, monotonic, spine-functions, we obtain in R_1:

$$T_1(x,y) = (x/g(y), y/f(x))$$

Therefore: $J_{Ti}^{T} J_{Ti}$

$$= \begin{pmatrix} \frac{1}{(g(y))^2} + \frac{y^2(f'(x))^2}{(f(x))^4} & -\frac{xg'(y)}{(g(y))^3} - \frac{yf'(x)}{(f(x))^3} \\ -\frac{xg'(y)}{(g(y))^3} - \frac{yf'(x)}{(f(x))^3} & \frac{1}{(f(x))^2} + \frac{x^2(g'(x))^2}{(g(y))^4} \end{pmatrix}$$

Therefore, setting the smallest eigenvalue of this matrix to be greater than 1, we obtain:

$$= \frac{1}{2}\left(\frac{1}{(g)^2} + \frac{y^2(f')^2}{(f)^4} + \frac{1}{(f)^2} + \frac{x^2(g')^2}{(g)^4} \right)$$

$$- \left(\frac{1}{4}\left(\frac{1}{(g)^2} + \frac{y^2(f')^2}{(f)^4} - \frac{1}{(f)^2} - \frac{x^2(g')^2}{(g)^4} \right)^2 + \left(\frac{xg'}{(g)^3} + \frac{yf'}{(f)^3} \right)^2 \right)^{1/2}$$

$$> 1$$

Thus, given the two spine-functions, we may determine if the resulting spine-map is piece-wise expanding.

2.4 Topological conjugacy of expanding spine-maps.

In this section we show that any two piecewise expanding spine-maps are topologically conjugate [10,11,18]. Ergodicity and weak mixing are examples of properties which are conjugacy invariant. That is, two conjugate transformations either both have the weak mixing property or both do not have this property. We will see later that the spine-maps admit continuous, ergodic invariant measures, thus every map in the family must share this property.

We shall call the union of the boundaries of all the regions Ri (i=1,2,3,4) the 1-skeleton. That is,

$$\{0,1\} \times [0,1] \cup [0,1] \times \{0,1\} \cup \{(x,f(x))\} \cup \{(g(y),y)\}$$

will be refered to as the 1-skeleton. On the 1-skeleton the family of spine-maps, as defined in section 2.1, satisfy the following conditions

1. $T(\{0,1\} \times [0,1]) \subset \{0\} \times [0,1]$
2. $T([0,1] \times \{0,1\}) \subset [0,1] \times \{0\}$
3. $T(x,f(x)) \in [0,1] \times \{1\}$
4. $T(g(y),y) \in \{1\} \times [0,1]$

Furthermore, as shown in sections 2.2 and 2.3 , the spine-maps are piecewise homeomorphic to the whole unit square X and piecewise expanding.

<u>Theorem 2.4.1</u> [18]. Let T1 and T2 be any two piecewise expanding spine-maps, with spine functions f1(x),g1(y) and f2(x),g2(x) . Then T1 and T2 are topologically conjugate.

<u>Proof:</u> We must show that there exists a unique homeomorphism h of the unit square, such that

$$h \cdot T1 = T2 \cdot h$$

Note first of all that T1 (or T2) when restricted to the boundary of the square, or to one of the spines, is a 2-1 expanding tent-map. And, any two such mappings are well known to be topologically conjugate [10]. Then Ti (i=1,2) restricted to {0} x [0,1] and [0,1] x {0} are topologically conjugate tent-maps. Let h0 be the homeomorphism such that

$$T2 \cdot h0 = h0 \cdot T1 \quad \text{on} \quad \{0\} \times [0,1] \cup [0,1] \times \{0\}$$

We may extend h0 to the remaining boundary of the unit square by setting

$$h0 = T2-1\left(h0 \cdot T1\right) \quad \text{on} \quad \{1\} \times [0,1]$$

where $h0 \cdot T1$ is already defined and we take the branch of $T2^{-1}$ corresponding to the position of y. Similarly we define h0 for $x \in [0,1]$. It is straight forward to verify that h0 is a continuous conjugacy on the boundary of the unit square. Now, to define h0 on the rest of the 1-skeleton , that is, on the spines, we set

$$h0(g1(y),y) = T2^{-1}\left(h0 \cdot T1(g1(y),y)\right) \quad \text{on} \quad graph(g2).$$

Once again, being careful to choose the appropriate branch of $T2^{-1}$. In this way we extend the conjugacy h0 on the 1-skeleton of T1 onto the 1-skeleton of T1 . Now let

$$C = \{h:X \to X : h \text{ continuous and } h=h0 \text{ on the 1-skeleton of } T1\}$$

Then C is a complete metric space with respect to the uniform topology. And C is not empty by the Tietze Extension Theorem. We now define a contraction mapping

$$F : C \longrightarrow C,$$

from which it follows that F must have a unique fixed point in C ; which will be our required conjugacy. We define the contraction mapping F, by setting

$$F(h(x,y)) = T2^{-1}(h(T1(x,y))). \quad \text{for} \quad (x,y) \in X,$$

being careful to choose the appropriate branch of T2-1. Since T2 is piecewise homeomorphic onto the whole unit square, the above inverse is well defined for each branch. For points on the spines portion of the 1-skeleton, we have $h = h0$. Thus,

$$h0(T1(x,y)) = T2(h0(x,y))$$

Hence,

$$T2\text{-}1(h(T1(x,y))) = h0(x,y)$$

Therefore, $F(h(x,y))$ is well defined and equal to h0 when restricted to the 1-skeleton of T1.

To show that F is indeed a contraction mapping, choose $(x,y) \in X$, h1 and h2 $\in C$, and the sup norm on C.

$$d(F(h1),F(h2)) = d(T2\text{-}1(h1(T1)),T2\text{-}1(h2(T1)))$$

$$\leq 1/k \; d(h1(T1),h2(T1))$$

where $k > 1$.

Since T2 is piecewise expanding, thus,

$$\text{Sup}\,\{d(F(h1),F(h2))\} \leq 1/k \;\text{Sup}\,\{d(h1(T1),h2(T1))\}$$

$$= 1/k \;\text{Sup}\,\{d(h1,h2)\}$$

$$= 1/k \; d(h1,h2)$$

Thus, taking the Supremum over each region, R_i, $i=1,2,3,4$ we obtain

$$d(F(h_1),F(h_2)) \leq 1/k \; d(h_1,h_2) \qquad 0 < k < 1.$$

Thus F is a contraction on the complete metric space C which establishes the existence of a unique fixed point h^*, such that

$$h^* \cdot T_1(x,y) = T_2 \cdot h^*(x,y)$$

Note that all we have shown so far, is the existence of a continuous map h^*, such that

$$h^* \cdot T_1 = T_2 \cdot h^*$$

We have not yet found a homeomorphism. Hence we have not yet obtained our conjugacy, only a semi-conjugacy. We must now obtain another semi-conjugacy k^* (in the other direction) such that

$$k^* \cdot T_2 = T_1 \cdot k^*$$

Then, using arguments analogous to those in [11], we can show that $h^{*-1} = k^*$. Therefore h^* is indeed the required homeomorphism.

CHAPTER III

Continuous ergodic invariant measures

In this Chapter, we consider the more general case where X is a topological Hausdorff space and T a continuous mapping from X into itself [12]. Later we apply our results to the spine-maps of section 2.1 [18].

3.1 Definitions.

The space X is the collection of all possible states of the system (T,X). The evolution of this system is governed by the transformation

$$T : X \longrightarrow X,$$

where Tx is taken as the state of the system at time 1, which at time 0 is at $x \in X$. By a measure we mean any regular probabilistic measure defined on the sigma algebra B of Borel subsets of X. A measure m is called invariant under T if

$$m(T^{-1}(E)) = m(E) \quad \text{for every } E \in B.$$

We say that the measure m is continuous if

$$m(\{x\}) = 0 \quad \text{for each singleton } \{x\}.$$

A measure m is supported on E if

$$m(E) = 1.$$

Now, let (X,B,m) be a complete probability space where m and T satisfy all of the above conditions.

The orbit of a point $x \in X$, under T, can be thought of as a history of the above system, and shall be denoted by

$$\{T^k x\ ,\ k \in Z\} = \{x, Tx, T^2x, T^3x, \ldots\}$$

where $T^n x = T \cdot T \cdot T \ldots\ldots Tx$ (n times).

3.2 Continuous measures invariant under spine-maps.

Given any $E \in B$, the measure associated with T represents the probability that the orbit of a point in X will eventually enter E, where this probability is m(E). We will now present, and prove, a sufficient condition for the existence of such invariant measures corresponding to continuous mappings on topological spaces.

Theorem 3.2.1 [12]. Let T be a continuous mapping from a topological Hausdorff space X into itself. Suppose that there exists two non-empty, compact, disjoint sets A0 and A1 such that

$$T(A0) \cap T(A1) \supseteq A0 \cup A1$$

Then, there exists a continuous measure which is invariant with respect to T.

Proof: The method of proof will be to construct a set D00 such that $T^{-1}D00 = D00$, which implies (via Kryloff and Bogoliuboff) that the set of measures, invariant under T, supported on D00 is non-empty. We then show that there exists a subset of the set of invariant measures, containing exclusively continuous measures. To start, let $Y = T(A0) \cap T(A1)$ and define S0 and S1 by the following:

(1) $T\ S_0 = T\ S_1 =$ Identity.

(i.e. right inverse mappings of T)

(2) $S_0\ (Y) \subseteq A_0$ and $S_1\ (Y) \subseteq A_1$

Now, we write

$$D_{k1\ldots kn} = \overline{S_{k1\ldots kn}\ (Y)}$$

where $k_i = 0$ or 1 for $i=1,\ldots,n$. Using the above notation, we obtain then, a decreasing family of compact non-empty subsets of X, for each combination of the k_i's. Referring to the above definition, we notice that

$$T(D_{k1\ldots kn}) = D_{k2\ldots kn}$$

and, also that

$$D_{k1\ldots kn} \subset D_{k1\ldots kn-1} \bigcap T^{-1}(D_{k2\ldots kn}) \qquad -(1)-$$

For each n, there are 2^n possible combinations of 0's and 1's. Thus $D_{k1\ldots kn}$ represents 2^n disjoint, compact non-empty sets. For each n, the intersection of all these sets will also be non-empty. With this in mind, we will define the set D_{00} as

$$D_{00} = \bigcap_{n\geq 1} \bigcup_{k_1\ldots k_n} D_{k1\ldots kn}$$

where the union is taken over every possible sequence $\{k_1 \ldots k_n\}$. By -(1)- D_{00} enjoys the following property

$$T^{-1}(D_{00}) = D_{00}$$

Thus, by Kryloff-Bogoliuboff's theorem, the set of measures invariant with respect to T is non-empty. Let M be the set of these invariant measures supported on D_{00}. Denote by M_0, the subset of M which consists of all measures satisfying

$$m(D_{k1 \ldots kn}) = 1/2^n$$

Note: if $m \in M_0$ then

$$m(\sum_{k_1 \ldots k_n} D_{k1 \ldots kn}) = \bigcup m(D_{k1 \ldots kn})$$

$$= 2^n \cdot 1/2^n$$

$$= 1$$

The set M may be considered as a subset of the space of linear functional defined on the space of continuous functions on D_{00}. That is

$$M \subset C^*(D_{00})$$

Now, M is contained in the closed unit-ball of $C^*(D_{00})$, since the measures in M are supported on D_{00}. By the theorem of Alaoglu, the closed unit-ball of $C^*(D_{00})$ is compact in the weak topology. Thus, M weakly closed implies that M is also compact in the weak topology.

The set M is also a convex set, for, let $m^1, m_2 \in M$ and $0 < k < 1$, then

$$\begin{aligned}[k\, m_1 + (1-k)\, m_2](D_{00}) &= k\, m_1(D_{00}) + (1-k)\, m_2(D_{00}) \\ &= k + (1-k) \\ &= 1\end{aligned}$$

and

$$T^{-1}(D_{00}) = D_{00}$$

The set M_0 is a non-empty convex, weakly* compact subset of M. This follows from the convexity of M and from the fact that M_0 is a weakly closed subset of the weakly* compact set M.

We now show that every measure $m \in M_0$ is continuous. To do this we must show that

$$m(\{x\}) = 0$$

for every $m \in M_0$ and for every singleton $\{x\}$ in X. Let $x \in X$ and assume that $x \in D_{00}$. If not $m(\{x\})=0$ automatically.

Now, $x \in D00$ means that, for each n

$x \in Dk1...kn$ for some sequence $\{k1,...,kn\}$.

Thus,

$$m(\{x\}) < m(Dk1...kn) = 1/2n \longrightarrow 0$$

Thus, we have shown that there exists a continuous measure, which is invariant with respect to T. This completes the proof of Theorem 3.2.1.

Now, to add to the result obtained in Theorem 3.2.1, we will use the fact that M0 is an extremal subset of M in order to show that there exists a continuous measure, invariant with respect to T, which is ergodic.

We have seen that M0 is a non-empty, convex, weakly* compact set. Therefore, by the Krein-Milman Theorem there then exists an extremal point $me \in M0$. Thus, since the set M0 is an extremal subset of M, me is then an extremal point of M.

We are now in a position to prove the following Theorem.

<u>Theorem 3.2.2</u> Let (X,B,m) be a complete probability space and T : X —> X. Then, the probability measures invariant with respect to T form a convex set and the ergodic measures are exactly the extreme points of this convex set.

(That is, invariant measures are uniquely representable as combinations of ergodic measures.)

<u>Proof:</u> We will first prove the following claim: A measure m_1 is ergodic if and only if there is no other measure m_2 invariant under T, that is distinct from m_1 and where $m_2 << m_1$. We will use the symbolism $m_2 << m_1$ for m_2 absolutely continuous with respect to m_1. To show this, suppose that there is another ergodic measure m_2 and that m_1 and m_2 do not coincide. Then, there is a set $E \varepsilon B$ such that

$$m_1(E) \neq m_2(E)$$

Now, if A_1 is the limit set where

$$\lim \; 1/n \sum_{k=0}^{n-1} \chi_E (T^k(a_1)) = m_1(E)$$

where $a_1 \varepsilon A_1$. χ_E denotes the characteristic function of the set E. Let A_2 be the limit set where

$$\lim \; 1/n \sum_{k=0}^{n-1} \chi_E (T^k(a_2)) = m_2(E)$$

where $a_2 \varepsilon A_2$. Then,

$$A_1 \cap A_2 = \emptyset \quad \text{and} \quad m_1(A_1) = 1 = m_2(A_2)$$

which implies that m_1' and m_2' are mutually singular.

Thus, if m_1 and m_2 are both ergodic then they either coincide or are mutually singular. So, if we have an ergodic measure m such that $m_1 \ll m$ (hence m_1 is ergodic) then since m and m_1 cannot be mutually singular they must coincide: $m = m_1$. On the other hand, assume that m is not ergodic, then there must be an invariant set A (i.e. $T^{-1}A = A$) such that $0 < m(A) < 1$. Now, define a measure m_1 by

$$m_1(D) = m(D/A)$$

where $D \in B$.

Then

$$m_1(T^{-1}D) = m(T^{-1}(D/A)) = m(D/A) = m_1(D)$$

Hence, m_1 is invariant with respect to T and m_1 is distinct from m. Moreover

$$m(E) = 0 \text{ implies that } m_1(E) = 0$$

Thus, $m_1 \ll m$. This concludes the proof of the claim that m is ergodic if and only if there is no other measure m_1 invariant under T, that is distinct from m and $m_1 \ll m$. Now, suppose that m is a convex combination of invariant measures m_1 and m_2, that is $m_1 \neq m_2$ and

$$m = k\, m_1 + (1-k)\, m_2 \quad , \quad 0 < k < 1.$$

Then,

$$m(E) = k\, m_1(E) + (1-k)\, m_2(E) \quad \text{for every} \quad E \in B.$$

Now, $m_1 \ll m$ (or $m_2 \ll m$) since $m(E) = 0$ implies that

$$m_1(E) = m_2(E) = 0$$

and m_1 is distinct from m. Therefore, m is not ergodic. There is a non-trivial set A such that

$$m(A) = k \quad \text{and} \quad m(A^c) = 1-k$$

Then, with

$$m_1(D) = m(D/A)$$

and,

$$m_2(D) = m(D/A^c)$$

we have

$$\begin{aligned} m(D) &= m(A)\, m(D/A) + m(A^c)\, m(D/A^c) \\ &= k\, m_1(D) + (1-k)\, m_2(D) \end{aligned}$$

a convex combination. Therefore, m is an ergodic measure if and only if it cannot be expressed as a convex combination of invariant measures. So, the ergodic measures are exactly the extremal points of M.

We may now state a stronger version of Theorem 3.2.1.

<u>Theorem 3.2.3</u> Assume that

$$T : X \longrightarrow X$$

is a mapping satisfying the conditions of Theorem 3.2.1. Then, there exists a continuous measure which is ergodic and invariant with respect to T.

<u>Proof:</u> By Theorem 3.2.1 there exists a continuous measure invariant under T and, by Theorem 3.2.2 ergodic measures are the extreme points of the set of continuous measures invariant under T.

Q.E.D.

In the following section, we apply these results to spine-maps.

3.3 Spine-maps have continuous, ergodic invariant measures.

Choosing any two regions R_i and R_j, $i \neq j$ (as defined in 2.1) we obtain

$$T(R_i) = T(R_j) = X \supset R_i \cup R_j$$

Thus, by Theorem 3.2.3, there exists a continuous ergodic measure which is invariant with respect to the spine-maps. If we now restrict spine-functions to be analytic, we can show that the spine-maps admit an absolutely continuous invariant measure.

3.4 Absolutely Continuous Invariant Measures.

If we assume that the spine-functions f(x) and g(y) are analytic, then the resulting spine-map will be piece-wise analytic on the unit square. Hence, we may use the main result of [9] to establish the existence of a unique, finite invariant measure, absolutely continuous with respect to Lebesgue measure. This result is stated, without proof, in the following Theorem.

<u>Theorem 3.4.1</u>[9] Let T be a piecewise expanding and piecewise analytic on X. Let the spine-functions be analytic on [0,1]. Then there is a finite measure, absolutely continuous with respect to Lebesgue measure on X that is T invariant, which is unique.

Later, we shall prove that if one of the spine-functions is parallel to an axis, then Lebesgue measure is invariant for the resulting spine-map. The only restriction on the other spine-function will be continuity.

First, we will formally introduce the Frobenius-Perron operator, which is of considerable use in studying the behavior of dynamical systems [13].

3.5 The Frobenius-Perron operator [13].

A transformation

$$T : X \longrightarrow X$$

is called non-singular if

$$m(T^{-1}A) = 0 \quad \text{whenever} \quad m(A) = 0$$

for every Borel subset A. Let (X,B,m) be a measure space. If

$$T : X \longrightarrow X$$

is a non-singular transformation, the unique operator

$$P_T : L1 \longrightarrow L1$$

defined by,

$$\int_A P_T f(x) \, dm = \int_{T^{-1}A} f(x) \, dm \qquad -(2)-$$

where $A \in B$ and $f \in L1[X]$, is called the Frobenius-Perron operator corresponding to T. It is straight forward to show, from this definition, that P_T has the following properties:

(1) $P_T(a_1 f_1 + a_2 f_2) = a_1 P_T f_1 + a_2 P_T f_2$

(2) $P_T f \geq 0$ if $f \geq 0$

(3) $\int_X P_T f(x)\, dm = \int_X f(x)\, dm$

In some special cases we may obtain an explicit form of the Frobenius-Perron operator. If $X = [a,b]$ and $A = [a,x]$ then we may obtain the following from -(2)-

$$\int_a^x P_T f(t)\, dt = \int_{T^{-1}[a,x]} f(t)\, dt$$

Differentiation yields

$$P_T f(x) = \frac{d}{dx} \int_{T^{-1}[a,x]} f(t)\, dt$$

Note: if T is monotonic on [a,x] then

$$T^{-1}[a,x] = [T^{-1}a, T^{-1}x] \quad \text{(for increasing T)}$$

Hence,

$$P_T f(x) = \frac{d}{dx} \int_{T^{-1}(a)}^{T^{-1}(x)} f(t) \; dt = f(T^{-1}x) \frac{d}{dx}[T^{-1}x]$$

In general,

$$P_T f(x) = f(T^{-1}x) \left| \frac{d}{dx}[T^{-1}x] \right|$$

In our case, however, we wish to use the Frobenius-Perron operator on the unit square in R^2. We take $X = [0,1]^2$ and $A = [0,x] \times [0,y]$. Hence we have

$$\int^x ds \int^y P_T f(s,t) \; dt = \iint_{T^{-1}(A)} f(s,t) \; dsdt$$

Differentiating, first with respect to x, then with respect to y, yields:

$$P_T f(x,y) = \frac{d^2}{dxdy} \iint_{T^{-1}(A)} f(s,t) \; dsdt$$

Analogous formulae may be derived for $X \in R^n$. Similarly to the one-dimensional case, we may obtain yet another form for P_T when T is piecewise diffeomorphic.

That is

$$P_T f(x,y) = \Sigma \; f(T^{-1}(x,y)) \cdot \left| J^{-1}(x,y) \right|$$

where the summation is taken over each piece of T and care is taken in choosing the appropriate branch of T^{-1}, and J^{-1} is the Jacobian of T^{-1}.

3.6 Spine-maps have Lebesgue measure as Invariant Measure

In the case where one spine-function is parallel to an axis, the definitions of section 2.1 are then modified to:

$$T(x,y) = \begin{cases} (x/k\ ,\ y/f(x)) & ,\ (x,y)\ \varepsilon\ R_1 \\ (x/k\ ,\ (y-1)/(f(x)-1)) & ,\ (x,y)\ \varepsilon\ R_2 \\ ((x-1)/(k-1)\ ,\ (y-1)/(f(x)-1)) & ,\ (x,y)\ \varepsilon\ R_3 \\ ((x-1)/(k-1)\ ,\ y/f(x)) & ,\ (x,y)\ \varepsilon\ R_4 \end{cases}$$

as shown in Fig. 3.1.

Note that T may not be expanding (if one spine is longer than one) yet we have existence of an absolutely comtinuous invariant measure.

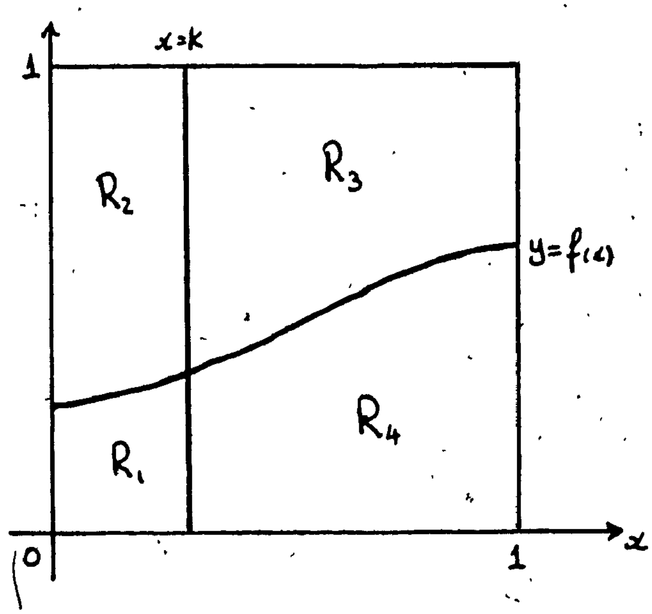

Fig 3.1

In this special case, we are able to calculate the Frobenius-Perron operator corresponding to T.

<u>Theorem 3.6.1</u> [18]. Let

$$T : X \longrightarrow X$$

have one spine-function parallel to an axis and let the other spine-function be continuous. Then Lebesgue measure is invariant under T.

<u>Proof:</u> We will need the following fact: If (X,B,m) is a measure space with Lebesgue measure m, and

$$T : X \longrightarrow X$$

is a non-singular transformation, and PT is the Frobenius-Perron operator corresponding to T, then the measure defined by:

$$mf(A) = \int_A f \ dm$$

is invariant if and only if f is a fixed point of PT [13].

$$\text{i.e.} \quad PTf = f.$$

Note that if $f = 1$ then,

$$m_1(A) = \int_A dm = m(A)$$

Thus, Lebesgue measure is invariant under T if and only if $f = 1$ is a fixed point of P_T. We will now calculate the Frobenius-Perron operator for the above case and show that $P_T 1 = 1$. The Frobenius-Perron operator is:

$$P_T h(x,y) = \frac{d^2}{dydx} \iint_{T^{-1}([0,x]\times[0,y])} h(s,t) \; dsdt$$

where $h \varepsilon L^1$. In this case, the inverse transformation is easily found for each region R_i $(i=1,2,3,4)$.

R_1: $T^{-1}(x,y) = (kx, yf(kx))$

R_2: $T^{-1}(x,y) = (kx, yf(kx-1)+1)$

R_3: $T^{-1}(x,y) = ((k-1)x+1, y(f((k-1)x+1)-1)+1)$

R_4: $T^{-1}(x,y) = ((k-1)x+1, yf((k-1)x+1))$

Thus, the inverse image of the rectangle $[0,x]\times[0,y]$ will be the four shaded regions of Fig.3.2.

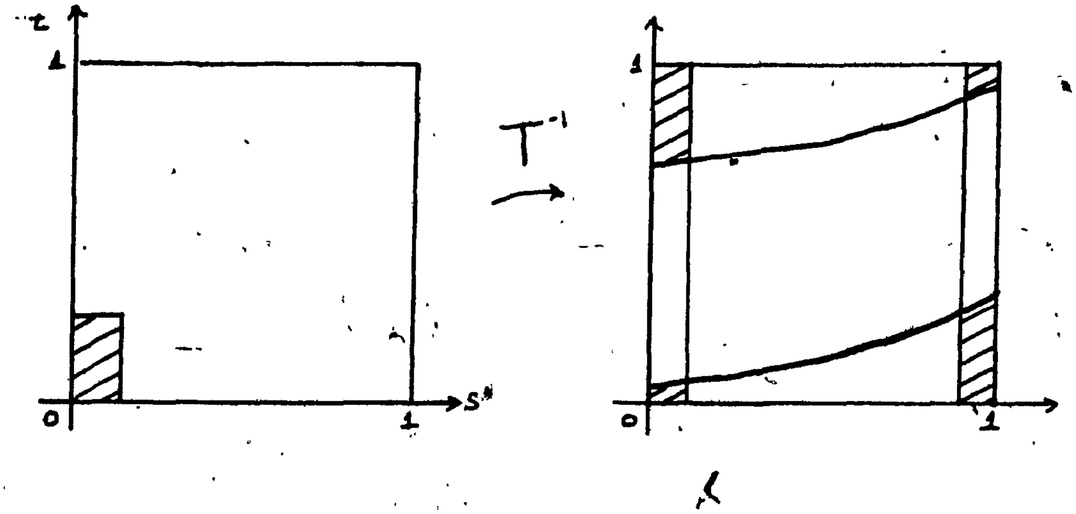

Fig 3.2

We have,

$$P_T h(x,y) = \frac{d^2}{dy\,dx} \int_0^{kx} \left(\int_0^{yf(x)} h(s,t)\, dt + \int_{yf(x)+1-y}^{1} h(s,t)\, dt \right) ds$$

$$+ \int_{kx+1-x}^{1} \left(\int_{yf(x)+1-y}^{1} h(s,t)\, dt + \int_0^{yf(x)} h(s,t)\, dt \right) ds$$

$$= kf(kx)\, h(kx, yf(kx)) - k(f(kx)-1)\, h(kx, yf(kx)+1-y)$$

$$+ (k-1)(f(kx+1-x)-1)\, h(kx+1-x, yf(kx+1-x)+1-y)$$

$$- (k-1)f(kx+1-x)\, h(kx+1-x, yf(kx+1-x))$$

Applying this Frobenius-Perron operator to the function $h(x,y) = 1$ we obtain,

$$P_T 1 = kf(kx)-k(f(kx)-1)+(k-1)(f(kx+1-x)-1)-(k-1)f(kx+1-x)$$
$$= 1.$$

Hence, $h(x,y) = 1$ is a fixed point of P_T, which implies that Lebesgue measure is invariant under this particular sub-class of the family of spine-maps.

To conclude this chapter, we wish to present an example of a spine-map T, where the spine-functions are linear, but where Lebesgue measure is not invariant under T.

3.7 Example

Consider the spine-map determined by the spine-functions

$$y = f(x) = cx + d \quad \text{and} \quad x = g(y) = ay + b$$

as in Fig. 3.3.

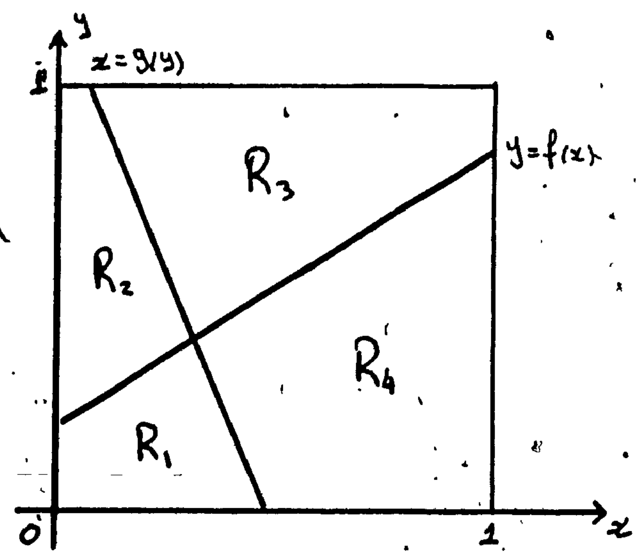

Fig. 3.3

We will use the alternate form of the corresponding Frobenius-Perron operator, as it applies in this case.

Thus,

$$P_T h(x,y) = \Sigma \quad h(T^{-1}(x,y)) \cdot \left| J^{-1}(x,y) \right|$$

Hence,

$$P_T 1 = \Sigma \left| J^{-1}(x,y) \right|$$

Calculating the determinant of the Jacobian, for each branch of T^{-1}, we obtain:

$$P_T 1 = -\frac{ac(y-1)(x-1) - (acy-1)(acx-1)}{(1 - acxy)^3} \neq 1$$

for all $(x,y) \in X$, which implies that Lebesgue measure is not invariant for this type of spine-map.

CHAPTER IV

Generalized spine-maps

4.1 Spine-maps generalized on the unit square.

So far we have presented the spine-maps in two-dimensions. Now, we shall show that we may generalize the previous ideas to higher dimensions [18]. This generalization may be accomplished in two directions. The first direction will involve staying with the unit square $[0,1]x[0,1]$ and simply increasing the number of spine-functions. Thus, these spine-maps will be constructed from the spine- functions:

$$x = gi(y) \quad , \quad i=1,\ldots,m$$

and,

$$y = fj(x) \quad , \quad j=1,..,n$$

Where the numbering of the spine-functions is done according to the following:

$$g1(y) < \ldots < gm(y)$$

and,

$$f1(x) < \ldots < fn(x)$$

The unit square is then divided into m×n regions, each of which is mapped homeomorphically onto [0,1]x[0,1] under the resulting spine-map.

To preserve the topological conjugacy between any two such piecewise expanding spine-maps, we need to modify condition (3) of section 2.4. This modified condition will be:

(3^*) $\quad T(x, f_j(x)) \in [0,1] \times \{1\}$, $j=1,3,5,\ldots,$odd

and,

$T(x, f_j(x)) \in [0,1] \times \{0\}$; $j=2,4,6,\ldots,$even

Similarly,

$T(g_i(y), y) \in \{0\} \times [0,1]$, $i=1,3,5,\ldots,$odd

and,

$T(g_i(y), y) \in \{1\} \times [0,1]$, $i=2,4,6,\ldots,$even

For example, the cross-section through the spine-functions $f_j(x)$ at $x = x_0$ would be as in Fig.4.1.

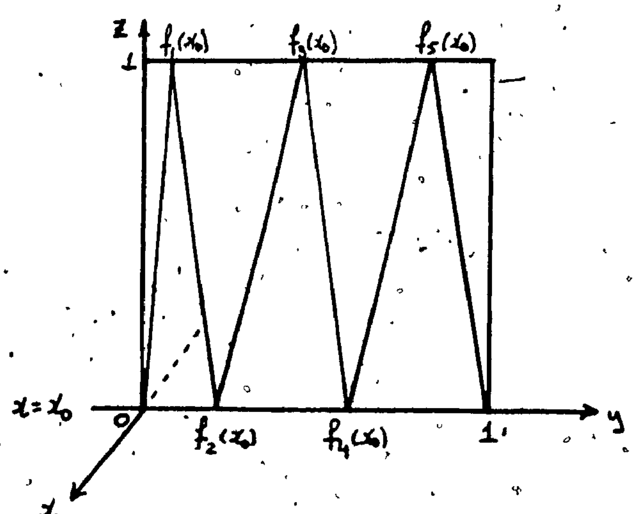

Fig.4.1

Note that the cross-sections are still 'tent-maps', so the resulting spine-maps may be determined in a similar manner as described in section 2.1.

Also, choosing any two regions $R_{m,n}$ and $R_{u,v}$ we obtain

$$[0,1]\times[0,1] = T(R_{m,n}) = T(R_{u,v}) \supset R_{m,n} \cup R_{u,v}$$

Hence these spine-maps admit continuous ergodic invariant measures, by Theorem 3.2.3 of section 3.2.

4.2 The n-dimensional unit cube $[0,1]^n$

Finally, let us briefly consider generalizing the spine-maps from the unit square $[0,1]^2$ to the n-dimensional cube $I^n = [0,1]^n$. To obtain results analogous to those of Chapters 2 and 3, we denote the faces of I^n in the following way:

$$A_i = [0,1] \times [0,1] \times \ldots \times \underset{i}{\{0\}} \times \ldots \times [0,1] \times [0,1]$$

Where the {0} occurs in the ith position, for i=1,2,...,n. Similarly, let

$$B_i = [0,1] \times [0,1] \times \ldots \times \underset{i}{\{1\}} \times \ldots \times [0,1] \times [0,1]$$

where the {1} also appears in the ith position for i=1,2,...,n. If we now consider the following continuous functions from I^{n-1} into (0,1),

$$x_i = f_i(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$$

for i=1,...,n. Then we may generalize the two-dimensional spine-maps of section 2.1 in the following manner:

$$T(x_1,\ldots,x_n) = (t_1(x_1,\ldots,x_n), \;\ldots\ldots\;, t_n(x_1,\ldots,x_n))$$

The t_i, $i=1,\ldots,n$, are obtained by the continuous functions f_i in the following way:

$$t_i(x) = \begin{cases} x_i/f_i(x) & , \quad x_i < f_i \\ (x_i-1)/(f_i(x)-1) & , \quad x_i > f_i \end{cases}$$

where $x = (x_1,\ldots,x_n)$ and $i=1,\ldots,n$. Under the appropriate conditions on

$$f_i : I^{n-1} \longrightarrow (0,1),$$

it may be shown that these mappings satisfy the following conditions, analogous to those found in section 2.4.

(1^{**}) $T(A_i) \subset A_i$, $i=1,\ldots,n$

(2^{**}) $T(B_i) \subset B_i$, $i=1,\ldots,n$

(3^{**}) $T(x_1,\ldots\ldots,f_j(x_1,\ldots,x_n),\ldots\ldots,x_n) \subset B_j$

(4^{**}) When the mapping T is restricted to any of the 2^n regions given by the inequalities:

$$x_i < f_i(x_1,\ldots,x_n)$$

and,

$$x_i > f_i(x_1,\ldots,x_n)$$

T is a homeomorphism.

(5^{**}) Under condition analogous to those of section 2.3 some of these mappings are expanding on each of the 2^n regions defined in (4^{**}).

CHAPTER V

Numerical results.

A simple Fortran code was written to calculate the orbit of a point under a given spine-map. The unit square is divided into a grid of 100 squares and an initial point, or seed, is given. The program then counts the number of times the orbit of this point 'visits' each small square.

In Chapter III, Theorem 3.6.1 tells us that Lebesgue measure is invariant under a spine-map where one of its spine-functions is parallel to an axis. So, as a first example, we will consider the following spine-map:

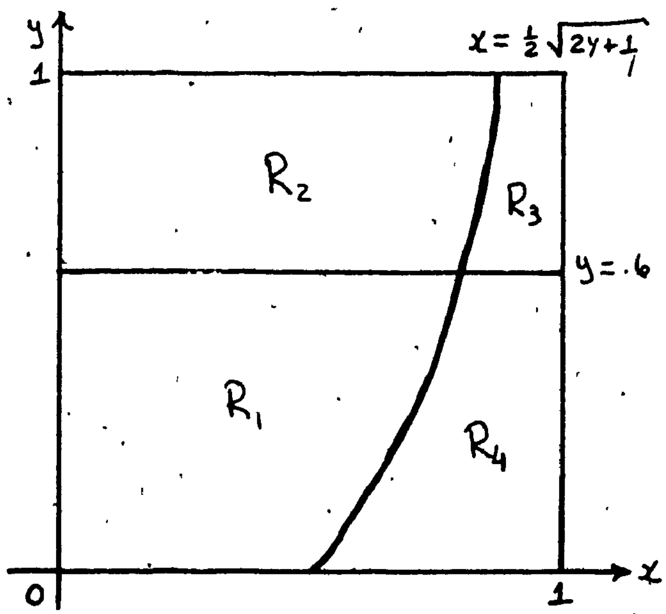

Fig. 4.1

$$
T(x,y) = \begin{cases} \left(\dfrac{2x}{\sqrt{2y+1}} , \dfrac{5y}{3} \right) , & (x,y) \in R_1 \\ \left(\dfrac{2x}{\sqrt{2y+1}} , \dfrac{5}{2}(1-y) \right) , & (x,y) \in R_2 \\ \left(\dfrac{2(x-1)}{\sqrt{2y+1} - 2} , \dfrac{5}{2}(1-y) \right) , & (x,y) \in R^3 \\ \left(\dfrac{2(x-1)}{\sqrt{2y+1} - 2} , \dfrac{5y}{3} \right) , & (x,y) \in R_4 \end{cases}
$$

Thus, starting at the point (.35 , .91), we obtain the following distribution of 100,000 orbit points:

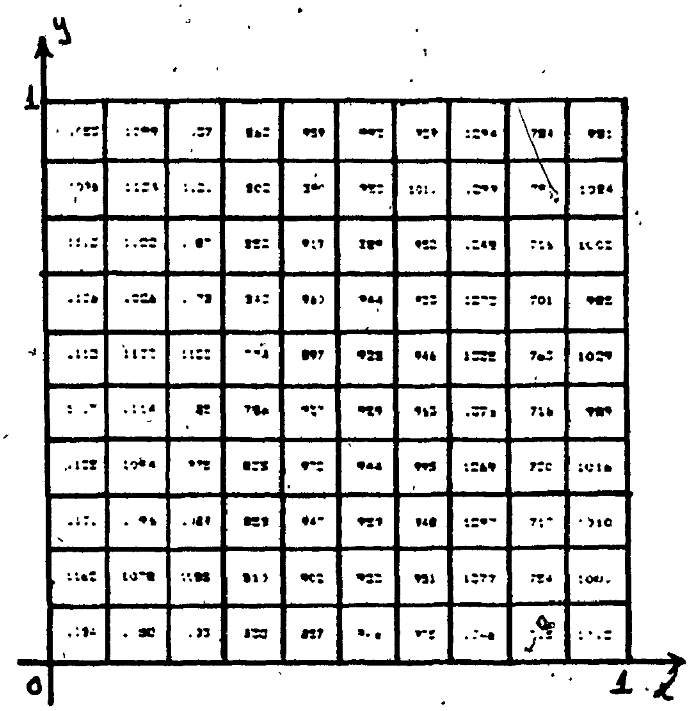

Fig. 4.2

In Section 3.7 it wás shown that Lebesgue measure is not invariant under a spine-map with linear spine-functions where neither is parallel to an axis. That is, we should not expect a uniform distribution of points under such a spine-map. To illustrate this we consider the following spine-map:

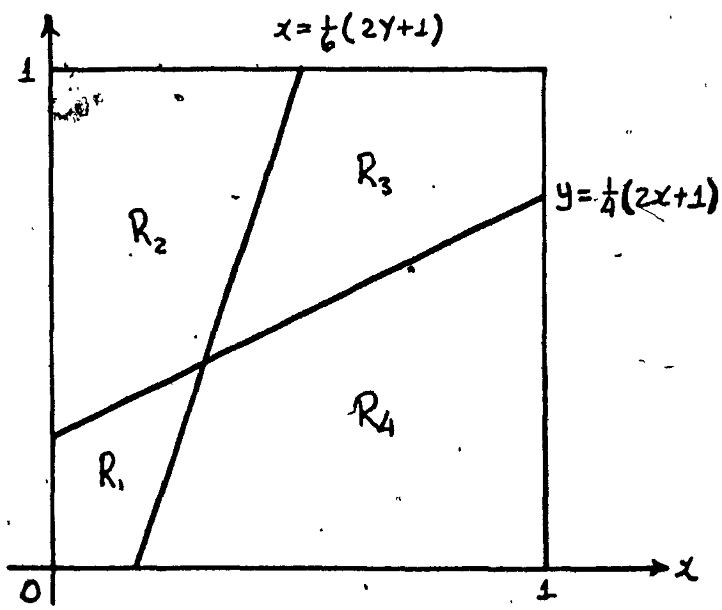

Fig. 4.3

$$T(x,y) = \begin{cases} \left(\dfrac{6x}{2y+1}, \dfrac{4y}{2x+1}\right), & (x,y) \in R_1 \\ \left(\dfrac{6x}{2y+1}, \dfrac{4(y-1)}{2x-3}\right), & (x,y) \in R_2 \\ \left(\dfrac{6(x-1)}{2y-5}, \dfrac{4(y-1)}{2x-3}\right), & (x,y) \in R_3 \\ \left(\dfrac{6(x-1)}{2y-5}, \dfrac{4y}{2x+1}\right), & (x,y) \in R_4 \end{cases}$$

Thus, with seed (.11 , .37) we obtain the following distribution of 100,000 orbit points:

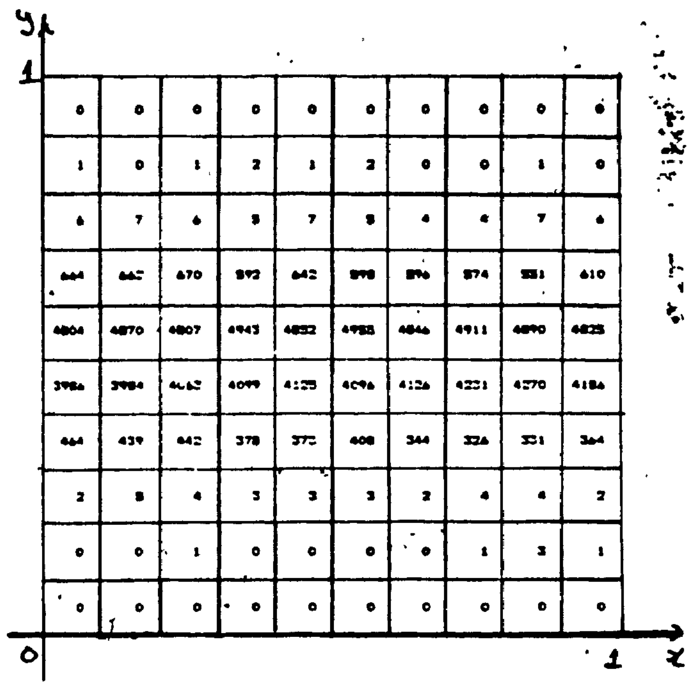

Fig. 4.4

The following example will illustrate the fact that there are spine-maps which are not expanding but preserve Lebesgue measure. Consider the spine-map corresponding to the spine-functions in Fig.4.5 . The part of f(x) that bounds the region R1 has arc-length:

$$L = \int_0^{.9} \sqrt{1 + (f(x))2} \, dx = 1.15$$

The corresponding spine-map will map this part of $f(x)$ onto the upper boundary of the unit square, which has unit length. Thus this spine-map is not expanding. However, by Theorem 3.6.1 this spine-map preserves Lebesgue measure.

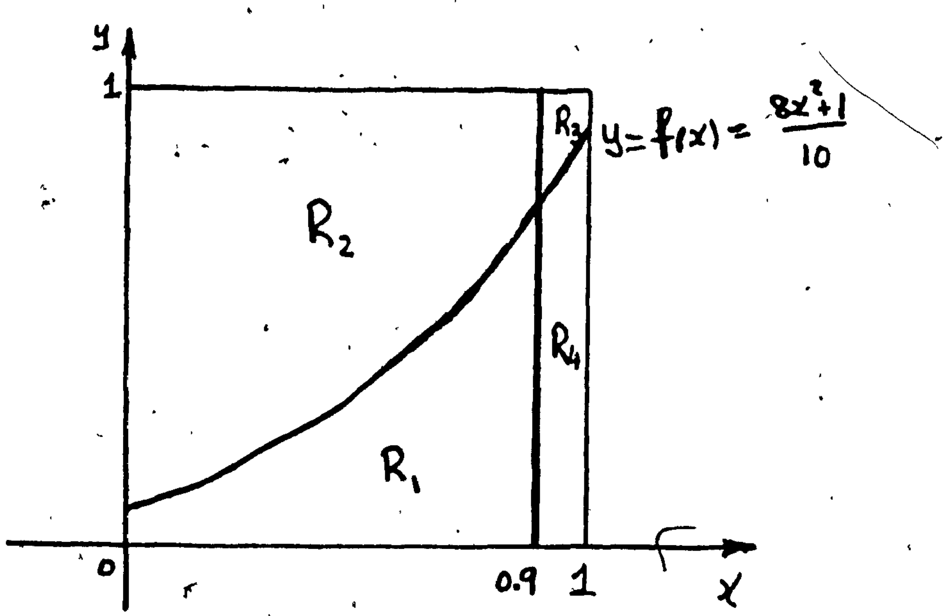

Fig. 4.5

The last example will be of a spine-map ~~which~~ is expanding. Consider the spine-map associated with the spine-functions of Fig. 4.6 . In the region R1 the spine-map will be ~~given by~~:

$$T(x,y) = (2x , 3y/(x+1))$$

Thus, using the results of section 2.3, with JT representing the Jacobian matrix of T, the smallest eigenvalue of the matrix JTTJT will be:

$$\lambda = \frac{4(x+1)^4 + 9y^2 + 9(x+1)^2}{(x+1)^4}$$

$$- \frac{1}{2} \left(\frac{4(x+1)^2 - 9y^2(x+1)^2 - 9(x+1)^4 + y^2}{(x+1)^6} \right)^{1/2}$$

now, in R1 : $0 \leq x \leq .5$ and $0 \leq y \leq .5$.

Therefore,

$$\frac{4(x+1)^4 + 9y^2 + 9(x+1)^2}{(x+1)^4} \geq \frac{4 + 9}{(1.5)^4} = 2.57$$

and,

$$-\frac{1}{2} \left(\frac{4(x+1)^2 - 9y^2(x+1)^2 - 9(x+1)^4 + y^2}{(x+1)^6} \right)^{1/2}$$

$$\geq -\frac{1}{2} \left(\frac{4(x+1)^2 - 9(x+1)^4 + 1}{(x+1)^6} \right)^{1/2}$$

$$\geq -\frac{1}{2} \left(4(1.5)2 - 9(1) + 1 \right)^{1/2} = -\frac{1}{2}$$

Thus,

$$\lambda \geq 2.57 - 0.5 = 2.07 > 1$$

which, by Proposition 1 , shows that this spine-map is expanding on R1.

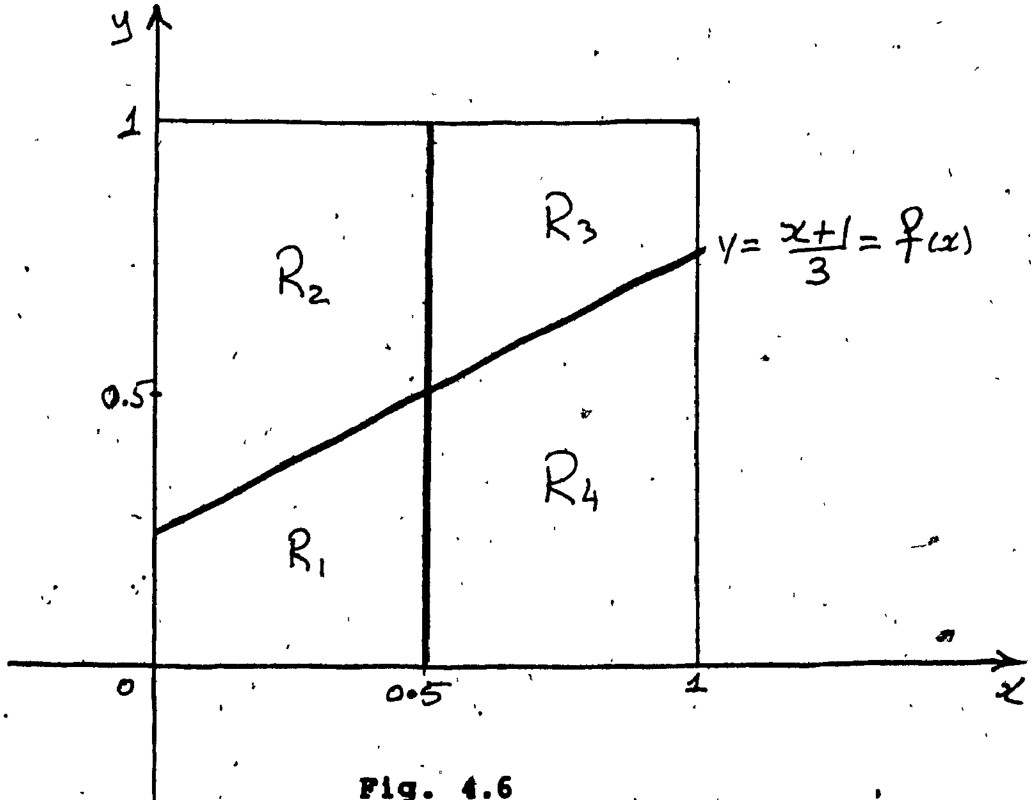

Fig. 4.6

REFERENCES

[1] Li, T.Y. and Yorke, J.A., "Period 3 implies chaos", Amer. Math. Monthly., 82, (1975), 985-992.

[2] Lasota, A and Yorke, J.A., "On the existence of invariant measures for piecewise monotonic transformations", Trans. Amer. Math. Soc., 183, (1973), 481-485.

[3] Proppe, H., Byers, W. and Boyarsky, A., "Singularity of topological conjugacies between certain unimodal maps of the interval", Israel Jour. of Math., vol. 44, no.4, (1983), 277-288.

[4] Curry, J. and Yorke, J.A., "A transition from Hopf bifurcation to chaos: computer experiments on maps in R^2. The structure of attractors in dynamical systems", Lecture notes in Math., 668, 48-68, Springer, N.Y., (1978).

[5] Levine, S.H. and Scudo, F.M., Plunkett, D.J., "Persistence and convergence of ecosystems: an analysis of some second-order difference equations", J. Math. Biol., 4, 171-182, (1977).

[6] Maynard-Smith, J., "Mathematical ideas in biology", Cambridge University Press, (1971).

[7] May, R. and Oster, G., "Bifurcations and dynamical stability in simple ecological models:, Am.Naturalist. 110, 573-599, (1976).

[8] Guckenheimer, J., Oster, G. and Ipaktchi, A., "The dynamics of density dependent population models", J. Math. Biol., 4, 101-147, (1977).

[9] Keller, G., "Ergodicite et mesure invariante pour les transformations dilatantes par morceaux d'une region bornee du plan", C.R. Acad. Sc. Paris., t289, 12 nov., 1979. Series A-625.

[10] Byers, W., "Topological semiconjugacy of piecewise monotone maps of the interval", Trans. Amer. Math. Soc., 276, 489-495, (1983).

[11] Shub, M., "Endomorphisms of compact differentiable manifolds", Amer. Jour. Math., 91, 175-199, (1969).

[12] Lasota, A. and Piagianni, G., "Invariant measures on topological spaces", Bullettino. U.M.I., 5, 14-B, 592-603, (1977).

[13] Lasota, A. and Mackey, M., "Probabilistic Properties of Deterministic Systems", Cambridge University Press., (1985).

[14] Henon, M., "A two-dimensional mapping with a strange attractor", Comm. Math. Phys., 50, 69-77, (1976).

[15] Haller, H., "Rectangle exchange transformations", Monats. Math., 91, 215-232, (1981).

[16] Hussain, A.K.M.F., "Coherent structures--reality and myth", Phys. Fluids., 10, Oct. 1983.

[17] Kosyakin, A.A. and Sandler, E.A., "Ergodic properties of a class of piecwise smooth transformations of a segment", Izvestiya. VUZ. Mathematica., 13, 32-40, (1972).

[18] Boyarsky, A. , Byers, W. and Gauthier, P. "Higher dimensional analogues of the tent maps." Non-linear analysis. Theory. Methods and Applications. (to appear).