DIFFERENCE EQUATIONS:

STABILITY THEORY AND APPLICATIONS

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ABSTRACT

DIFFERENCE EQUATIONS: STABILITY THEORY AND APPLICATIONS

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This thesis considers three methods for studying the stability of difference equations. The first is the classical Lyapunov theory approach. The second is based on the non-Lyapunov work, due to Perron and Bellman. Finally, the third method involves important extensions of the classical Lyapunov theory, due to LaSalle and Hurt.

The various stability concepts are illustrated by examples and a chapter of applications, which emphasizes the important role that stability theory plays in the understanding of physical systems. Some of the theoretical results obtained are new.

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> If man was not made for God, why is he only happy in God? If man was made for God, why is he opposed to God?
>
> Pascal

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CHAPTER I

INTRODUCTION

1.1 OUTLINE.

Many phenomena in nature have the property that their dynamics change only at discrete times. For instance, many biological species (May [30], Hassell and May [16]) have been observed to change only at specific times. Between these changes, the biological population does not change. The discrete time dynamics of such phenomena are modelled by a difference equation of the form

$$x_{k+1} = f(x_k, k) , \tag{1.1}$$

where x_k is an n-dimensional vector and k is an integer ($k \geq 0$).

As with differential equations, qualitative analysis of such difference equations is of importance. The purpose of this thesis is to study the stability properties of difference equations. Basically, stability involves analyzing the behavior of solutions about an equilibrium point x^*, where

$$x^* = f(x^*, k) \tag{1.2}$$

for all $k \geq 0$. In stability analysis, we show that solutions near x^* stay near x^*, converge to x^*, or move away from x^*.

1.2 THE NEED TO STUDY DIFFERENCE EQUATIONS APART FROM DIFFERENTIAL EQUATIONS.

Innis [19] advocates the use of difference equations instead of differential equations in the representation of the dynamics of "soft science" systems-biology, ecology, sociology, economics, political science, etc. These disciplines are distinguished by two factors. Firstly, there exists difficulty in making precise measurements of the variables of interest. Secondly, the laws which govern their dynamics are not generally known with great precision.

Differential equations is a tool of precision. This tool has proven to be useful in precise applications, such as engineering and physics. However, the less precise the application, the less useful the tool. In reality differential equations may hinder rather than help. This is revealed in the following argument.

A difference equation can be expressed mathematically as

$$x(t+h) = x(t) + hf(x,v,t,h) \tag{1.3}$$

where x is a state vector of the system, t is the time, h is the time step, f is the rate vector describing the average rate of change of x over the time interval [t,t+h] and v is a vector of other variables (exogeneous) that affect the rates of change of x.

It is simple to express the difference equation as a differential equation, provided the necessary limits exist. We see that

$$(1.4) \qquad \frac{x(t+h)-x(t)}{h} = f(x,v,t,h)$$

and letting $h \to 0$, we have

$$(1.5) \qquad \frac{dx}{dt} = f(x,v,t,0)$$

The time step h, important information for the difference equation, has been lost.

Innis illustrates his argument by considering a biological system (carcinogenic model) and modelling it as a simple Lotka-Volterra system (differential equation). Lotka-Volterra equations model many predator-prey systems. The biologist who formulated the model was seeking a description of the dynamics of those populations which displayed cyclic behavior, since this was observed naturally. A mathematician descretized the differential equation system into a difference equation system and used a time step that was too large. The results showed no cyclical behavior. Once the biologist considered more obvious reactions between the predator-prey (i.e. host-parasite) and a smaller time step, the new difference equation led to nearly cyclical behavior. Thus not only did the difference equation model give the biologist the desired results, the re-thinking on the model gave him a more appropriate mathematical representation of the physical system which the more appealing differential equation model could not give.

Greenspan [12] reinforces Innis' argument. He notes that scientific experimentation leads to discrete data which is then modelled by some continuous model. Should the equations of the continuous model be non-linear (which is often the case), these would be solved by numerical methods, which result again in discrete data. Figure I summarizes this reasoning. The middle step of this scientific activity is inconsistent with the other two and should be replaced by the appropriate model, as Figure II indicates.

1.3 QUALITATIVE ASPECTS OF DIFFERENCE EQUATIONS.

System (1.1) exhibits a variety of qualitative behavior. As with differential systems, (1.1) may be stable around the equilibrium point. System (1.1) also exhibits behavior that is peculiar to discrete-time systems. Solutions may either approach or are themselves cycles of various periodicities. Or, solutions may behave in a "chaotic" fashion. That is, solutions may have all periods, and there exists a set S such that solutions starting in S do not approach any cycle at all.

May [30] illustrates these different qualitative properties in the case of biological populations obeying difference equations. He observes that several biological populations are modelled by the non-linear difference equation

$$(1.6) \qquad x_{t+1} = x_t \exp\{r(1-x_t/K)\} .$$

where r is the growth rate, K is a carrying capacity, and x_t is the population size at time t. Depending on r, the model in Table I demonstrates all the properties discussed in the previous paragraph.

Hoppensteadt [17] reveals similar properties by considering the simplest of non-linear difference equations:

$$(1.7) \qquad x_{t+1} = m\, x_t(1-x_t) .$$

The behavior of this equation is outlined in Table II.

These two examples illustrate unexpected dynamics inherent in difference equations. Generally, one assumes that if a natural system could be modelled by a discrete-time system, many qualitative aspects of such a system could be determined. May summarizes what really occurs:

> For population biology in general, and for temperate zone insects in particular, the implication is that even if

the natural world was 100% predictable, the dynamics of populations with "density dependent" regualtion could nonetheless in some circumstances be indistinguishable from chaos, if the intrinsic growth rate r is large enough.

As interesting as chaos may be, the aim of this thesis is to present stability theory for difference equations which do not exhibit chaotic behavior. Chapter II presents the necessary theory for linear difference equations. This will serve as a basis for attacking non-linear equations in subsequent chapters.

Chapter III formulates the notion of a Lyapunov function for discrete systems. This function is then used to establish several powerful stability results.

Chapter IV investigates the stability properties of difference systems by a non-Lyapunov technique. New results are presented and extensions of older ones are also given.

Chapter V expands on the ideas presented in Chapter III. This chapter is important in developing criteria for establishing stability domains.

Finally, Chapter VI looks into three applications of the theory introduced in Chapters III-V. Another aim of this chapter is to show how stability questions arise from physical problems.

CHAPTER II

LINEAR DIFFERENCE EQUATIONS

2.1 OUTLINE.

In this chapter, an outline of the basic theory of linear difference equations is presented. In a way that is analogous to differential equations, we shall present theories for homogeneous and non-homogeneous difference equations. As with differential equations, the fundamental matrix solution of difference equations will be introduced and developed.

The chapter ends with two important results. Firstly, an existence and uniqueness result for non-homogeneous equations is shown. Secondly, Gronwall's Inequality for discrete systems is proved. Both these results are used to establish several stability theorems in Chapter IV.

2.2 HOMOGENEOUS AND NON-HOMOGENEOUS DIFFERENCE EQUATIONS.

Consider the linear difference equation

$$y_t = A(t)y_{t-1} + w_t \tag{2.1}$$

where $A(t)$ is non-singular for $t \in I_{a+1}$ and $I_a = \{a, a+1, a+2, \ldots\}$ for $a \in R$. We call (2.1) a <u>non-homogeneous difference equation</u> if w_t is not identically zero on I_{a+1}. Related to (2.1) is the associate <u>homogeneous</u> equation

$$(2.2) \qquad y_t = A(t)\, y_{t-1}$$

for $t \in I_{a+1}$. By a <u>solution</u> of (2.1) on I_a we mean a vector Ψ_t with the properties that Ψ_a is defined and that it satisfies

$$(2.3) \qquad \Psi_t = A(t)\Psi_{t-1} + w_t$$

for $t \in I_{a+1}$.

THEOREM 2.1: Let $a \in R$ and c be an n-dimensional constant vector. Then there exists one and only one vector Φ_t defined on I_a such that

$$(2.4) \qquad \Phi_a = c \quad \text{and} \quad \Phi_t = A(t)\Phi_{t-1} + w_t$$

for all $t \geq a+1$.

<u>Proof</u>. Let $\Phi_a = c$, and define $\Phi_{a+1} = A(a+1)c + w_{a+1}$. We claim that

$$(2.5) \qquad \Phi_t = A(t)\, A(t-1) \ldots A(a+1)c + \sum_{r=a+1}^{t-1} A(t)\, A(t-1) \ldots A(r+1) w_r + w_t$$

for $t = a+2, a+3, a+4, \ldots$ is the unique solution that satisfies

(2.4). This result is shown by induction. Observe that for $t = a+2$, (2.4) gives

$$(2.6) \quad \Phi_{a+2} = A(a+2)\Phi_{a+1} + w_{a+2}$$
$$= A(a+2)\, A(a+1)c + A(a+1)w_{a+1} + w_{a+2} .$$

Thus, (2.5) is true for $t = a+2$. Assume (2.5) is true for $t = a+n$, that is

$$(2.7) \quad \Phi_{a+n} = A(a+n)\; A(a+n-1)\ldots A(a+1)c + \sum_{r=a+1}^{a+n-1} A(a+n)\; A(a+n-1)\ldots A(r+1)w_r + w_{a+n} .$$

Since,

$$(2.8) \qquad \Phi_{a+n+1} = A(a+n+1)\Phi_{a+n} + w_{a+n+1}$$

and substituting (2.7) into (2.8), we have

$$(2.9) \quad \Phi_{a+n+1} = A(a+n+1)\; A(a+n)\ldots A(a+1)c + \sum_{r=a+1}^{a+n} A(a+n+1)\; A(a+n)\ldots A(r+1)w_r + w_{a+n+1} .$$

Thus, we have shown recursively that (2.5) satisfies (2.4).

Equation (2.5) shows uniqueness. If we had defined Ψ_t on I_a, which satisfies (2.4) for $t \geq a+1$, then from (2.4) we have

$$\Psi_{a+1} = A(a+1)\Psi_a + w_{a+1}$$

(2.10)
$$= A(a+1)c + w_{a+1}$$
$$= \Phi_{a+1} \, .$$

Using (2.4), one can show recursively that

(2.11) $$\Psi_{a+n} = \Phi_{a+n} \quad \text{for } n \geq a+2 \, .$$

Q.E.D.

Let $\{y_i(t),\ 1 \leq i \leq q\}$ be n-dimensional column vector solutions of (2.2) on I_a. Then the nxq matrix

(2.12) $$Y_t = [\, y_1(t), \ldots, y_q(t) \,]$$

whose colums are the $y_i(t)$ vectors is a matrix solution of (2.2) on I_a. Thus we have

(2.13) $$Y_a = [\, y_1(a), \ldots, y_q(a) \,] \text{ and}$$
$$Y_t = A(t) Y_{t-1}$$

for $t \geq a+1$.

As in the theory of linear differential equations, we introduce the notion of linear dependence for linear difference equations.

Definition 2.1: Let $f_i(t)$ $1 \leq i \leq q$, be n-dimensional vectors defined on I_a. We say $f_i(t)$ are linearly dependent on I_a

if there exist constants c_i $1 \le i \le q$, not all zero, such that

$$c_1 f_1(t) + c_2 f_2(t) + \ldots + c_q f_q(t) = 0 \tag{2.14}$$

for all $t \in I_a$. If any relation of the form (2.14) implies $c_1 = c_2 = \ldots = c_q = 0$ for all t in I_a, then $f_i(t)$ are <u>linearly independent</u> on I_a. This leads to the following definition.

<u>Definition 2.2</u>: If $y_i(t)$, $1 \le i \le n$ are n-dimensional linearly independent vectors on I_a which are solutions of (2.2) on I_a, then $Y_t = [y_1(t), y_2(t), \ldots y_n(t)]$ is called a <u>fundamental matrix set of solutions of (2.2) on I_a</u>.

To show that Y_t is linearly independent of I_a, Miller [32] establishes the following equivalent criteria.

THEOREM 2.2: Let $Y_t = [y_1(t), y_2(t), \ldots y_n(t)]$ be an nxn matrix solution of (2.2) on I_a. Then the following three statements are equivalent:

(i) Y_t is linearly independent on I_a;

(ii) Y_t is non-singular, for all $t \in I_a$;

(iii) $|Y_a| \neq 0$.

As with differential equations, one would like to find the solution of (2.1) from the solution of (2.2). The following theorem allows us to do this.

THEOREM 2.3: Let $a \in R$. Let w_t be an n-dimensional vector defined on I_{a+1}, and $A(t)$ be an nxn non-singular matrix on I_{a+1}. Let Y_t be a fundamental matrix set of solutions of (2.2) on I_a. Let c be an arbitrary constant vector. Then Φ_t, where

$$\Phi_a = c$$

$$(2.15) \qquad \Phi_t = Y_t Y_a^{-1} c + \sum_{s=a+1}^{t} Y_t Y_s^{-1} w_r$$

for $t \geq a+1$, is the unique solution of (2.1) on I_a.

Proof. We observe that

$$(2.16) \qquad Y_t = A(t) Y_{t-1} .$$

Therefore, we obtain

$$(2.17) \qquad Y_t Y_{t-s}^{-1} = A(t)\, A(t-1) \ldots A(t-s+1) .$$

Hence (2.5) becomes $\Phi_a = c$ and

$$(2.18) \qquad \begin{aligned} \Phi_t &= Y_t Y_a^{-1} c + \sum_{r=a+1}^{t-1} Y_t Y_r^{-1} w_r + w_t \\ &= Y_t Y_a^{-1} c + \sum_{r=a+1}^{t} Y_t Y_r^{-1} w_r \end{aligned}$$

for $t \geq a+1$. Thus we have shown (2.15). By virtue of Theorem 2.1, we know that this solution is unique.

Q.E.D.

We conclude this chapter with a result that will be referred to frequently in the sequel.

THEOREM 2.4 (Gronwall's Inequality): Let $u(t)$ and $v(t)$ be defined and non-negative on I_a. Let c be a positive constant. If

$$u(a) \le c$$
$$u(t) \le c + \sum_{s=a}^{t-1} v(s)u(s) \tag{2.19}$$

for $t \ge a+1$, then

$$u(t) \le c \exp\{\sum_{s=a}^{t-1} v(s)\} \tag{2.20}$$

for $t \ge a+1$.

Proof. We define $\sigma(t)$ on I_a as:

$$\sigma(a) = 0 \text{ and}$$
$$\sigma(t) = \sum_{s=a}^{t-1} v(s)u(s) \;. \tag{2.21}$$

Then for $t \in I_a$, we have

(2.22) $$u(t) \le \sigma(t) ,$$

which implies that

(2.23) $$\frac{1}{c+\sigma(t)} \le \frac{1}{u(t)} .$$

Multiplying both sides of (2.23) by u(t)v(t), we have

(2.24) $$\frac{u(t)v(t)}{c+\sigma(t)} \le v(t)$$

and

(2.25) $$1 + \frac{u(t)v(t)}{c+\sigma(t)} \le 1+v(t) \le \exp\{v(t)\} .$$

If we change the variable t to s and taking logarithms, we have

(2.26) $$\log[\, c+\sigma(s+1)\,] - \log[\, c+\sigma(s)\,] \le v(s)$$

for $s \in I_a$. Now sum the above equation from s=a to s=t-1. This results in

(2.27) $$\log[\, c+\sigma(t)\,] - \log c \le \sum_{s=a}^{t-1} v(s) \quad \text{or,}$$

$$c+\sigma(t) \le c \exp\{\sum_{s=a}^{t-1} v(s)\} .$$

By hypothesis, $u(t) \leq c+\sigma(t)$ for $t \in I_a$. Thus, the theorem is proved.

Q.E.D.

CHAPTER III

CLASSICAL LYAPUNOV THEORY

3.1 OUTLINE.

An important problem of system theory centers around the concept of stability. Briefly, stability refers to the behavior of the variables of a system as $t \to \infty$.

Freeman [11] observes that there are two major points of view with respect to the stability of a system. In one, the system is presumed to possess an equilibrium point and concern is with the ability of the system to maintain a state in the vicinity of this equilibrium in the absence of any control or input u_k as $t \to \infty$. Such systems are referred to as free systems.

In the other, we assume that the system is stable if a bounded input $u(t_k)$ yields a bounded output $x(t_{k+1})$.

In the first case we are considering a system

$$x(t_{k+1}) = F(x(t_k), t_k) \tag{3.1}$$

for $k \geq 0$. As Figure III illustrates, one starts at x_o in a δ-neighbourhood of the equilibrium x_e and wishes to show that $x(t_k)$ stays in an ε-neighbourhood of x_e as $t_k \to \infty$. In the second case we are considering the system

$$(3.2) \qquad x(t_{k+1}) = G(x(t_k),u(t_k),t_k)$$

for $k \geq 0$. As Figure IV illustrates, one starts with an initial state x_o and input u_o and wishes to show that $x(t_k)$ is bounded.

In the following theory we shall make the assumption that if a discrete-time system is stable at discrete instants t_k, for $k \geq 0$, the system is stable for all time $t \geq t_o$. This assumption ignores the possibility that the state of a system may remain bounded at the discrete instants t_k and yet increases without bound in the intervals between successive instants.

3.2 BASIC STABILITY DEFINITIONS AND THEOREMS.

In this section, the basic stability concepts for free systems are presented. Firstly, the stability definitions for free systems are stated and then some stability results using Lyapunov's direct method are developed.

We consider the system (3.1) where t_k is an independent, discrete-time variable, $t_{k+1} > t_k$ for all integers k, and $t_k \to \infty$ as $k \to \infty$. We shall denote the solution of (3.1) for any initial state x_o and an initial time to by

$$(3.3) \qquad x(t_k) = B(t_k; x_o, t_o)$$

for all $t_k \geq t_o$. We assume that the vector valued function B is continuous for all t_k and for each t_o , x_o , and that

(3.4) $$F(0,t_k) = 0$$

for all t_k . Then (3.1) possesses the trivial solution $x = 0$. This trival solution is referred to as the <u>equilibrium point</u> of the system (3.1). We are now ready to give the definitions of stability.

<u>Definition 3.1</u>: The equilibrium point 0 is said to be <u>stable</u> ("stable in the sense of Lyapunov") if, for any t_o and any $\varepsilon>0$, there exists a $\delta(\varepsilon,t_o)>0$ such that if $\|x_o\| < \delta(\varepsilon,t_o)$, then $\|B(t_k;x_o,t_o)\| < \varepsilon$ for all $t_k \geq t_o$.

<u>Definition 3.2</u>: The equilibrium point 0 is said to be <u>uniformly stable with respect to t_o</u> , if for any $t_1 \geq t_o$ and any $\varepsilon>0$, there corresponds a $\delta(\varepsilon)>0$ (N.B. ε does not depend on t_1) such that if $\|x_o\| < \delta(\varepsilon)$, then $\|B(t_k;x_o,t_1)\| < \varepsilon$ for all $t_k \geq t_1$.

<u>Definition 3.3</u>: The equilbrium point 0 is said to be <u>asymptotically stable</u> if:

(i) it is stable and

(ii) if there exists an $\eta(t_o) > 0$ such that

(3.5) $$\lim_{k\to\infty} \| B(t_k;x_o,t_o) \| = 0$$

for all $\|x_o\| < \eta(t_o)$.

<u>Definition 3.4</u>: The equilibrium point 0 is said to be <u>uniformly asymptotically stable</u> if:

(i) the origin is asymptotically stable,

(ii) $\eta(t_o)$ is independent of t_o, and

(iii) for any $\varepsilon>0$, there exists $T(\varepsilon)$ such that $\| B(t_k;x_o,t_o) \| \leq \varepsilon$ for all $t_k \geq t_o+T(\varepsilon)$ whenever $\| x_o \| \leq r$ ($r>0$, r being some fixed constant which does not depend on ε or x_o) .

From the definitions, we see that for a stable equilibrium point 0, x_k remains in the vicinity of the origin; while, for an asymptotically stable point, the system converges to the origin.

<u>Definition 3.5</u>: The equilibrium point 0 is said to be <u>asymptotically stable in the large</u> if $\| B(t_k;x_o,t_o) \| \to 0$ for all x_o .

Before we proceed to develop sufficient conditions for a given system to be stable in the various senses discussed above, we will need the following additional definitions.

Definition 3.6: A scalar function $V(x,t_k)$ is said to be positive definite in a neighbourhood N of the point $x = 0$ if $V(0,t_k) = 0$ and if there exists a continuous, non-decreasing, scalar function $w(\cdot)$ such that

$$w(0) = 0 \quad \text{and} \qquad (3.6)$$
$$V(x,t_k) \geq w(\|x\|)$$

for all x in N and for all values of t_k .

Definition 3.7: A positive function is said to be decrescent in a neighbourhood N if there exists a continuous nondecreasing scalar function $s(\cdot)$ such that

$$s(0) = 0 \quad \text{and} \qquad (3.7)$$
$$V(x,t_k) \leq s(\|x\|)$$

for all t_k and all $x \neq 0$ in N.

Definition 3.8: A positive definite function $V(x,t_k)$ is said to be radially unbounded if $|V(x,t_k)| \to \infty$ as $\|x\| \to \infty$ for all t_k .

We are now ready to present a set of important stability theorems using Lyapunov theory for discrete systems. The ideas inherent in these theorems were introduced by Li [26] and later extended by Hahn [13] , Kalman and Bertram [21] .

Consider the system (3.1) for which 0 is the equilibrium point. Let $V(x,t_k)$ be a continuous positive definite function. Let $\Delta V(x,t_k)$ denote the first forward difference in $V(x,t_k)$, that is

$$(3.8) \qquad \Delta V(x,t_k) = \frac{V(x(t_{k+1}),t_{k+1}) - V(x(t_k),t_k)}{t_{k+1} - t_k}$$

Then we have

THEOREM 3.1 (Lyapunov Stability Theorem): The equilibrium point 0 is stable if there exists a continuous positive definite function $V(x,t_k)$ possessing a non-positive forward difference $\Delta V(x,t_k)$.

<u>Proof</u>. For the sake of convenience, we let $t_{k+1}-t_k = 1$. Given a particular $\varepsilon>0$, we select $\delta(\varepsilon,t_o)\in(0,\varepsilon)$ such that for $\|x_o\| < \delta(\varepsilon,t_o)$ we obtain $V(x_o,t_o) < w(\varepsilon)$. This is possible because of the continuity of $V(x,t_k)$ in x. Since $\Delta V(x,t_k) \leq 0$,

$$(3.9) \qquad V(x_o,t_o) \geq V(B(t_k;x_o,t_o),t_k) .$$

From the positive definiteness of $V(x,t_k)$ it follows that

$$(3.10) \qquad V(x_o,t_o) \geq w(\|B(t_k;x_o,t_o)\|)$$

and therefore

(3.11) $$w(\varepsilon) > V(x_o,t_o) \geq w(\| B(t_k;x_o,t_o)\|) \quad .$$

Since $w(\cdot)$ is a nondecreasing function, it follows

$$\| B(t_k;x_o,t_o)\| < \varepsilon$$

for all $t_k \geq t_o$ and all $\| x_o \| < \delta(\varepsilon,t_o)$.

Q.E.D.

THEOREM 3.2 (Lyapunov's Asymptotic Stability Theorem): The equilibrium point 0 is asymptotically stable if there exists a decrescent positive definite function $V(x,t_k)$ possessing a negative definite forward difference $\Delta V(x,t_k)$.

Proof. From the proof of Lyapunov's Stability Theorem we know that the positive definite function $V(x,t_k)$ has a non-negative limit as $t_k \to \infty$. We denote this limit by V^* . Since $V(x,t_k)$ is decrescent by hypothesis,

(3.12) $$V(x,t_k) \leq s(\| x \|) \quad .$$

Hence $V^*>0$ implies that $\| x(t_k) \| = \| B(t_k;x_o,t_o)\|$ will always be larger than some positive number μ . Since $\Delta V(x,t_k)$ is negative definite,

(3.13) $$\Delta V(x,t_k) \leq -r(\| x \|) \; ,$$

where r is a continuous, nondecreasing scalar function. Then $V^*>0$ implies

$$(3.14) \qquad \Delta V(x,t_k) \leq -r(\mu) < 0 \ .$$

We now write $V(x,t_k)$ in terms of its forward difference $\Delta V(x,t_k)$

$$(3.15) \qquad V(x(t_k),t_k) = V(x_o,t_o) + \sum_{i=0}^{k-1} \Delta V(x(t_i),t_i)$$

It follows that

$$(3.16) \qquad V(x(t_k),t_k) \leq V(x_o,t_o) - kr(\mu) \ .$$

Since $V(x,t_k)$ is positive definite, the right-hand side of (3.16) may not become negative. The only way this can be satisfied for large k is to have $r(\mu) = 0$. Hence $\mu = 0$ and $\|B(t_k;x_o,t_o)\| \to 0$ as $k\to\infty$.

Q.E.D.

THEOREM 3.3 (Lyapunov's Asymptotic Stability in the Large Theorem): The equilibrium point 0 is asymptotically stable in the large if $V(x,t_k)$ is positive definite, radially unbounded, and $\Delta V(x,t_k)$ is negative definite.

Proof. The proof of this theorem follows directly from the proofs of the previous two theorems.

Q.E.D.

The first two theorems reveal both an advantage and a drawback of Lyapunov's direct method. The drawback is that it is often difficult to construct a Lyapunov function; an important feature of the method is that it yields a region for local stability. However, the region of local stability will vary with the Lyapunov function constructed. One wishes to construct a Lyapunov function to obtain the largest region of local stability. Unfortunately, this is only possible in special cases. One must bear in mind that the foregoing theorems give only sufficient conditions for stability. In section 3.4, we develop necessary conditions for stability.

3.3 STABILITY OF LINEAR, STATIONARY SYSTEMS.

Let us consider the linear, stationary system described by

$$x_{k+1} = Ax_k \tag{3.17}$$

with an arbitrary initial state x_o. We will establish a fundamental result in stability theory by considering the following two cases:

<u>Case 1</u>: n independent eigenvectors of A exist.

If this is so, there exists a non-singular matrix H such that

(3.18) $$\Lambda = H^{-1}AH ,$$

where

(3.19) $$\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

and λ_i are the eigenvalues of A . Now, consider the transformation

(3.20) $$x_k = Hy_k .$$

Then we have

(3.21) $$Hy_{k+1} = AHy_k .$$

Pre-multiplying both sides of (3.21) by H^{-1}, we obtain

(3.22) $$y_{k+1} = \Lambda y_k .$$

Thus

(3.23) $$y_k = \Lambda^k y_o ,$$

where

$$(3.24)\qquad \Lambda^k = \begin{bmatrix} \lambda_1^k & & & 0 \\ & \lambda_2^k & & \\ & & \ddots & \\ 0 & & & \lambda_n^k \end{bmatrix}$$

It follows that $\lim_{k\to\infty} \|y_k\| = 0$ if and only if $|\lambda_i| < 1$ for all $i = 1,2,\ldots n$, and that $\lim_{k\to\infty} \|y_k\| = \infty$ if one or more $|\lambda_i| > 1$.

<u>Case 2</u>: less than n independent eigenvectors of A exist.

In this case we can find a non-singular matrix P such that

$$(3.25)\qquad x_k = Pz_k$$

and by the same manipulation as in Case 1 we obtain

$$(3.26)\qquad z_{k+1} = P^{-1}APz_k ,$$

where

$$(3.27)\qquad P^{-1}AP = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ 0 & & & a_{nn} \end{bmatrix}$$

and $a_{ii} = \lambda_i$ are the eigenvalues of A for $i = 1,2,\ldots n$.

From (3.26) we have

$$\begin{bmatrix} z_{k+1,1} \\ z_{k+1,2} \\ \vdots \\ z_{k+1,n} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & 0 & \ddots & \vdots \\ & & & a_{nn} \end{bmatrix} \begin{bmatrix} z_{k,1} \\ z_{k,2} \\ \vdots \\ z_{k,n} \end{bmatrix} \tag{3.28}$$

or,

$$z_{k+1,i} = a_{ii} z_{k,i} + \sum_{j=i+1}^{n} a_{ij} z_{k,j} \qquad i = 1,2,\ldots n-1 \tag{3.29}$$

$$z_{k+1,n} = a_{nn} z_{k,n} . \tag{3.30}$$

Hence, for any $k \geq 0$ and any initial state $z_o = P^{-1} x_o$,

$$z_{k,n} = (a_{nn}{}^{k}) z_{o,n} . \tag{3.31}$$

Clearly $z_{k,n}$ will converge to zero as $k \to \infty$ if and only if $|a_{nn}| < 1$.

Let us examine ,

$$z_{k+1,n-1} = a_{n-1,n-1}\, z_{k,n-1} + a_{n-1,n}\, z_{k,n} . \tag{3.32}$$

It is apparent from the same reasoning applied to (3.31) that $z_{k,n-1}$ will converge to zero as $k\to\infty$ if and only if $|a_{n-1,n-1}| < 1$ and simultaneously $z_{k,n}\to 0$ as $k\to\infty$.

By induction, it then follows from (3.33) that all $z_{k,i}\to 0$ as $k\to\infty$ if and only if all $|a_{ii}| < 1$. Since the a_{ii}'s are the eigenvalues of A and since they are invariant under the transformation (3.27), we have established the following result.

THEOREM 3.4: A linear stationary system

$$x_{k+1} = Ax_k \tag{3.17}$$

is asymptotically stable if and only if all the eigenvalues of A are of magnitude less than unity.

Observe that Theorem 3.4 is true for all x_o . Thus, we have asymptotic stability in the large. For convenience, it will be understood that (3.17) is asymptotically stable in the large whenever we state that (3.17) is asymptotically stable.

3.4 A CONVERSE THEOREM.

In section 3.2 (Theorem 3.1) we saw that a Lyapunov function with a non-positive forward difference ΔV implies that the equilibrium point is stable. Often, stability analysis concerns itself with the converse problem: it is known that the system is stable and one seeks a Lyapunov function to determine the region of local stability.

The problem is to establish the existence of such functions. In general, this is difficult. For linear systems, however, Lyapunov functions can be constructed. Let us consider

$$(3.33) \qquad V(x) = x' Qx ,$$

where Q is positive definite and x' is the transpose of x . Then with respect to (3.17) and (3.8) we have,

$$(3.34) \qquad \Delta V(x) = V(Ax)-V(x) = x' (A' QA-Q)x .$$

Hence, if $A' QA-Q$ is negative definite, equation (3.17) is asymptotically stable. Conversely, suppose (3.17) is asymptotically stable and consider the equation

$$(3.35) \qquad A' QA-Q = -R .$$

If it has a solution, then

$$(3.36) \qquad -\sum_{k=0}^{n} (A')^k RA^k = (A')^{n+1} QA^{n+1} - Q .$$

Since the eigenvalues of A are in magnitude less than unity, we have that $(A')^n \to 0$ and $A^n \to 0$ as $n \to \infty$. Letting $n \to \infty$, we see that the solution Q of (3.35) is

$$Q = \sum_{k=0}^{\infty} (A')^k R A^k . \tag{3.37}$$

In [11,pp.164-165], Freeman shows that (3.35) has a unique solution Q provided the eigenvalues of A satisfy the condition

$$\lambda_i \lambda_j \neq 1 \quad \text{for all } i,j \leq n . \tag{3.38}$$

This condition is met, since Theorem 3.4 shows that the eigenvalues of A are all less than unity. Also Q is positive definite if R is positive definite since

$$x' Qx = x' Rx + \sum_{k=1}^{\infty} x' (A')^k R A^k x , \tag{3.39}$$

where $x' Rx>0$ and each term in the summation of (3.39) is greater than or equal to zero for $x \neq 0$. Thus, $x' Qx>0$ for $x \neq 0$, which implies Q is positive definite. Thus we have shown the following theorem, which can be found in LaSalle [23] :

THEOREM 3.5: If there are positive definite matrices Q and R satisfying (3.35), then (3.17) is asymptotically stable. Conversely, if (3.17) is asymptotically stable, then R, (3.35) has a unique solution Q. If R is positive definite then, Q is positive definite.

This result shows that if a linear, stationary system is asymptotically stable (i.e. eigenvalues are in magnitude less than unity) and we have a positive definite matrix R, then a positive definite matrix Q can be found by solving (3.35). Q is such that $V(x) = x'Qx$ is a Lyapunov function of the system (3.17).

This result plays an important role in the theory of discrete control systems and is useful in non-linear discrete systems. The following theorem from LaSalle [23] is an example of such an application.

THEOREM 3.6 (Stability by Linear Approximation): Consider the following system

$$x_{k+1} = Ax_k + f(x_k) \ , \tag{3.40}$$

where f(x) is o(x) (i.e. $\lim_{\|x\| \to 0} \frac{\|f(x)\|}{\|x\|} = 0$) . If (3.17) is asymptotically stable, then the origin is an asymptotically stable point of (3.40). If one of the eigenvalues has magnitude greater than unity, the origin is unstable.

Proof. We shall only prove the first part of this theorem. The second part is slightly more difficult to prove and follows from results in Chapter V. From Theorem 3.5, there is a positive definite matrix Q satisfying (3.35). For convenience, take R = I , and let $V(x) = x'Qx$. Then relative to (3.40)

$$(3.41)\qquad \begin{aligned}\Delta V(x) &= [Ax+f(x)]'Q[Ax+f(x)] - x'Qx \\ &= [x'A'+f(x)']Q[Ax+f(x)] - x'Qx \\ &= x'A'QAx + x'A'Qf(x) + f(x)'QAx \\ &\quad + f(x)'Qf(x) - x'Qx \\ &= x'(A'QA-Q)x + 2x'Qf(x) + V(f(x)) \\ &= x'(-I)x + 2x'Qf(x) + V(f(x)) \\ &= -x'x + 2x'Qf(x) + V(f(x)) \ .\end{aligned}$$

Using the fact that $f(x)$ is $o(x)$, it follows that for any $0<\alpha<1$ there exists a δ sufficiently small that

$$(3.42)\qquad \Delta V(x) \le -\alpha x'x \quad \text{for all } \|x\| < \delta \ .$$

Hence, V and $-\Delta V$ are positive definite, and the origin is asymptotically stable by Theorem 3.2.

Q.E.D.

Notice that this theorem tells us that the stability domain $\{\|x\| < \delta\}$ comes out of the Lyapunov function constructed. That is, δ is a function of the Lyapunov function.

3.5 REDUCIBLE NONSTATIONARY SYSTEMS.

The asymptotic behavior of

$$(3.43)\qquad z_{k+1} = A(k)z_k$$

is difficult to establish. Professor LaSalle, in a private communication, informed the author of an example where the eigenvalues of A(k) are constants and lie inside the unit circle and yet there are unbounded solutions.

If (3.43) can be "reduced" to a form like (3.17), then perhaps the stability properties could be related. Freeman [11] shows how this can be done through the careful use of Lyapunov's results.

Let S(k) be an nxn matrix with bounded components that is non-singular for all $k \geq 0$ and $S^{-1}(k)$ is bounded. Let

(3.44) $$y_{k+1} = B(k) y_k ,$$

where (3.44) is related to (3.43) in accordance with

(3.45) $$y_k = S(k) z_k$$

for all $k \geq 0$. Upon combining (3.45) and (3.43), we obtain

(3.46) $$y_{k+1} = S(k+1)\ A(k)\ S^{-1}(k) y_k .$$

From (3.44), we have

$$B(k) = S(k+1)\, A(k)\, S^{-1}(k) \quad (3.47)$$

Freeman indicates that if S(k) is as specified, the system corresponding to (3.44) has precisely the same stability properties as that corresponding to (3.43). This fact can be used to advantage in establishing the stability of certain non-stationary systems.

Let us suppose that for a given linear, non-stationary system (3.43), an equivalence transformation S(k) exists such that

$$S(k+1)\, A(k)\, S^{-1}(k) = A \quad (3.48)$$

for all $k \geq 0$. We can determine the stability of the given non-stationary system by studying the stability of the transformed stationary system

$$y_{k+1} = Ay_k \quad (3.49)$$

using the methods of section 3.3. Systems that can be transformed in this manner into stationary systems are known as reducible systems.

3.6 MODIFIED SCHUR-COHN CRITERION.

We have seen that for all linear, stationary systems, and reducible non-stationary systems, the necessary and sufficient condition for asymptotic stability is that the magnitudes of the eigenvalues be less than one. For an almost linear system we have sufficient conditions. To determine the eigenvalues of an n^{th} order system we must solve the n^{th} degree characteristic equation $|A-\lambda I| = 0$. Without a digital computer, difficulties may occur if $n \geq 3$. Fortunately, a number of tests exist that enable us to determine whether or not all the roots of a characteristic equation are of magnitude less than one without requiring us to solve for the roots themselves. Such tests are known as stability criteria.

One of the simplest and most effective criterion is the modified Schur-Cohn criterion. The modified criterion was introduced by Tsypkin [37] and Jury [20] and represents a considerable simplification of an older criterion due to Schur and Cohn (Marden [29]) . The modified Schur-Cohn criterion was developed for discrete-time systems. Other criterion such as the Routh-Hurwitz, Liénard-Chipart, or Nyquist are designed to test for stability of continuous-time systems, and thus require some modification before they can be used for discrete-time systems.

Let us consider the polynomial

(3.50) $$F(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$$

with $a_n > 0$, a_i real for $i = 1,2,\ldots n$, we define its inverse polynomial:

$$(3.51) \qquad F^{-1}(x) = x^n F(x^{-1}) = a_o x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n \, .$$

With the polynomials in the form (3.50) and (3.51), we divide $F^{-1}(x)$ by $F(x)$, beginning at the left (highest power) end, to obtain a quotient term and a remainder.

$$(3.52) \qquad \frac{F^{-1}(x)}{F(x)} = \alpha_o + \frac{F_1^{-1}(x)}{F(x)}$$

The division is now repeated using the remainder polynomial $F_1^{-1}(x)$ and its inverse polynomial $F_1(x)$ in accordance with the recursive relation

$$(3.53) \qquad \frac{F_i^{-1}(x)}{F_i(x)} = \alpha_i + \frac{F_{i+1}^{-1}(x)}{F_i(x)}$$

for $i = 0,1,\ldots,n-2$ and where $F_o(x) = F(x)$. The necessary and sufficient condition that the roots of the equation $F(x) = 0$ lie in the interior of the unit circle is that all of the following conditions are met:

(a) $F(1) > 0$ for n odd

(b) $F(-1) < 0$ for n odd

> 0 for n even, and

(c) $|\alpha_i| < 1 \quad i = 0,1,\ldots n-2$.

The following example [11,pp.173] uses this criterion.

EXAMPLE 3.1: We wish to determine whether a system having the following characteristic equation is asymptotically stable.

$$|A-\lambda I| = 10\lambda^3 - 41\lambda^2 + 54\lambda - 5 = 0 .$$

Letting $F(\lambda) = |A-\lambda I|$, we have

(a) $F(1) = 18 > 0$

(b) $F(-1) = -110 < 0 \quad (n = 3,\ \text{odd})$

(c) $F^{-1}(\lambda) = -5\lambda^3 \quad 54\lambda^2 - 41\lambda + 10$

$$\frac{F^{-1}(\lambda)}{F(\lambda)} = -0.5 + \frac{33.5\lambda^2-14\lambda+7.5}{10\lambda^3-41\lambda^2+54\lambda-5}$$

$$\frac{F_1^{-1}(\lambda)}{F_1(\lambda)} = \frac{33.5}{7.5} + \frac{48.5\lambda-142.3}{7.5\lambda^2-14\lambda+7.5} .$$

Since $n = 3$, we need only determine α_0 and α_1

$$|\alpha_0| = |0.5| < 1$$

$$|\alpha_1| = \left|\frac{33.5}{7.5}\right| > 1 \quad \text{(violated)}.$$

Hence the system in unstable (the actual roots are $\lambda = 2-i,\ 2+i,$ and 0.1).

CHAPTER IV

STABILITY ANALYSIS BY MEANS OF LINEARIZATION - A NON-LYAPUNOV TECHNIQUE

4.1 OUTLINE.

As was demonstrated in the last chapter, the existence of Lyapunov functions can be used to prove powerful stability results. Even if somehow, a Lyapunov function is known to exist for a system, it may be difficult to construct it. Often, one may not want to know whether a system is stable, but rather wishes only to show boundedness of solutions, or what is termed Lagrangian stability. In view of the foregoing, it is desirable to have an alternative to Lyapunov theory.

This chapter develops a non-Lyapunov technique (linearization) that can be used to establish uniform asymptotic stability and boundedness results for difference equations. This technique is due mainly to the work of Perron [34] and Bellman [3,4,5] . However, some new results will be presented using several ideas from Struble [36] . Examples will illustrate these results.

4.2 PREREQUISITES.

The notion of an "impulsively small" perturbation is used to solve particular linearization problems in the continuous-time case. It is our aim to apply this idea to the discrete-time case.

Suppose that we wish to analyze the stability properties of the system

(4.1) $$x_t = A(t)x_{t-1} .$$

One would like to show that the stability properties of the system (4.1) are similar to those of the simpler linear system

(4.2) $$x_t = Ax_{t-1} ,$$

where $A = \lim_{t\to\infty} A(t)$.

The following two definitions will be used to establish many of the results in Section 4.3.

Definition 4.1: If $A = \lim_{t\to\infty} A(t)$, then $B(t) = A(t)-A$ is called the perturbation matrix of A(t).

Definition 4.2: B(t) is impulsively small as $t\to\infty$,

if $$\sum_{s=-\infty}^{\infty} \| B(s) \| = \sum_{s=-\infty}^{-1} \| B(s) \| + \sum_{s=0}^{\infty} \| B(s) \| < \infty .$$

One result that will be used often in this section, can be stated as follows:

LEMMA 4.1: If in the system (4.2), the origin is asymptotically stable, then

(i) $\| Y_t Y_a^{-1} \| \leq \| A \|^{t-a}$ for $t \geq a+1$, and

(ii) $\| Y_t Y_a^{-1} \| \to 0$ as $t \to \infty$.

Proof. (i) This first result is easily derived from equation (2.13).

(ii) By Theorem 3.4, the hypothesis indicates that the eigenvalues of A are of magnitude less than one. This fact plus the result in [8,p.27] allows us to find a norm $\| \cdot \|$ such that

$$\| A \| < 1 . \tag{4.3}$$

From the result obtained in the first part of the lemma, it is clear that $\| Y_t Y_a^{-1} \| \to 0$ as $t \to \infty$.

Q.E.D.

4.3 LINEARIZATION THEOREMS.

The following theorem illustrates how the impulsive smallness of the perturbing term determines the boundedness of (4.1).

THEOREM 4.1: If all solutions of the system (4.2) are bounded as $t \to \infty$, the same is true of (4.1), provided the perturbation matrix B(t) is impulsively small.

Proof. We wish to find a constant M_a with the property that:

$$(4.4) \qquad \|\phi_{t,a}\| \leq M_a$$

for all $t \in I_a$. $\phi_{t,a}$ indicates that the solution begins at time a and point $\phi_{a,a}$ and continues along the orbit determined by $\phi_{t,a}$ for $t \geq a$. Now suppose that $\phi_{t,a}$ is any nonzero solution of (4.1)on I_a . We can readily verify that

$$(4.5) \quad \phi_{t,a} = Y_t Y_a^{-1} \phi_{a,a} + \sum_{s=a}^{t-1} Y_{t-s-1+a} Y_a^{-1} B(s+1)\phi_{s,a} ,$$

where $t \in I_{a+1}$ and Y_t is a fundamental matrix solution of (4.2), determines the solution of (4.1). From the properties of norms

$$(4.6) \quad \|\phi_{t,a}\| \leq \|Y_t\| \, \|Y_a^{-1}\| \, \|\phi_{a,a}\| + \sum_{s=a}^{t-1} \|Y_{t-s-1+a}\| \, \|Y_a^{-1}\| \, \|B(s+1)\| \, \|\phi_{s,a}\| .$$

Since $\|Y_t\| \leq c(a)$ (by hypothesis), $\|Y_a^{-1}\| = b(a)$, and $\|\phi_{a,a}\| = d(a)$, (4.6) becomes

$$(4.7) \quad \|\phi_{t,a}\| \leq c(a)b(a)d(a) + \sum_{s=a}^{t-1} (c(a)b(a)\|B(s+1)\|) \, \|\phi_{s,a}\| .$$

Thus, the hypotheses of Gronwall's Inequality are met with

(4.8) $$\| \phi_{t,a} \| = u(t)$$

(4.9) $$c(a)b(a) \| B(s+1) \| = v(s) \ .$$

Therefore,

(4.9) $$\| \phi_{t,a} \| \leq c(a)b(a)d(a) \cdot \exp \{c(a)b(a) \sum_{s=a}^{t-1} \| B(s+1) \| \} \ .$$

But, $\sum_{s=-\infty}^{\infty} \| B(s+1) \| = M < \infty$. Thus, $\phi_{t,a}$ is bounded by M_a, where

(4.10) $$M_a = c(a)b(a)d(a) \exp \{M \, c(a)b(a)\} \ .$$

Q.E.D.

The following theorem establishes an asymptotic stability result for (4.1).

THEOREM 4.2: If (4.2) is asymptotically stable and $\| B(k) \| \to 0$ as $k \to \infty$, then (4.1) can be shown to be asymptotically stable.

<u>Proof</u>. Define $\rho(A)$ to be

(4.11) $$\rho(A) = \max \{ |\lambda_i| : \lambda_i \text{ are eigenvalues of } A\} \ .$$

[8,p.27] shows that there exists a norm $\| \cdot \|$ such that

$\|A\| < 1$, since $\rho(A) < 1$. Choose a positive ε to be smaller than $1-\rho(A)$. There is a K_o such that

$$\|A\| + \|B(k)\| \leq 1-\varepsilon \tag{4.12}$$

for all $k \geq K_o$. The solution of (4.1) can also be expressed by

$$x_k = \left(\prod_{i=0}^{k-1} \{A+B(i)\}\right)x_o \tag{4.13}$$

$$= \left(\prod_{i=0}^{K_o-1} \{A+B(i)\}\right)\left(\prod_{i=K_o}^{k-1} \{A+B(i)\}\right)x_o$$

for $k \geq 1$. Taking norms of both sides, we have

$$\|x_k\| \leq \left(\prod_{i=0}^{K_o-1} \|A+B(i)\|\right)\left(\prod_{i=K_o}^{k-1} \|A+B(i)\|\right)\|x_o\| . \tag{4.14}$$

Let $\prod_{i=0}^{K_o-1} \|A+B(i)\| = M$ and with the inequality from (4.12), we have

$$\|x_k\| \leq M\|x_o\|(1-\varepsilon)^{k-K_o} . \tag{4.15}$$

Thus, $\|x_k\| \to 0$ as $k \to \infty$. In fact (4.1) is globally asymptotically stable, since it is asymptotically stable for each x_o .

Q.E.D.

As the following theorem indicates, we can show the stronger result that (4.1) is uniformly asymptotically stable if B(t) is impulsively small.

THEOREM 4.3: If (4.2) is asymptotically stable, then (4.1) is uniformly asymptotically stable if B(t) is impulsively small as $t \to \infty$.

Proof. Consider any $a \in R$. Let Y_t be a fundamental matrix solution of (4.2). As in Theorem 4.1, the solution of the system (4.1) can be expressed by the equation (4.5).

It follows that

$$(4.16) \qquad Y_t Y_a^{-1} = A^{t-a}$$

for $t \in I_{a+1}$. Thus, (4.5) can be expressed by

$$(4.17) \qquad \phi_{t,a} = A^{t-a} \phi_{a,a} + \sum_{s=a}^{t-1} A^{t-s-1} B(s+1) \phi_{s,a} .$$

We see that

$$(4.18) \quad \| \phi_{t,a} \| \leq \| A \|^{t-a} \| \phi_{a,a} \| + \sum_{s=a}^{t-1} \| A \|^{t-s-1} \| B(s+1) \| \, \| \phi_{s,a} \| .$$

Since (4.2) is asymptotically stable, $\| A \| < 1$. Letting $\alpha = \| A \|$ and multiplying both sides by α^{-t+a}; we have

$$(4.19) \quad \| \phi_{t,a} \| \alpha^{-t+a} \leq \| \phi_{a,a} \| + \sum_{s=a}^{t-1} (\alpha^{-1} \| B(s+1) \|) \| \phi_{s,a} \|^{-s+a}$$

We can use Gronwall's Inequality to obtain

$$(4.20) \quad \| \phi_{t,a} \| \alpha^{-t+a} \leq \| \phi_{a,a} \| \exp \{ \sum_{s=a}^{t-1} \alpha^{-1} \| B(s+1) \| \} .$$

Since $\sum_{s=-\infty}^{\infty} \| B(s) \| = M < \infty$, we have that

$$(4.21) \qquad \| \phi_{t,a} \| \leq \| \phi_{a,a} \| \left[\exp \{ \frac{M}{\alpha} \} \right] \alpha^{t-a} .$$

To show that the origin is uniformly asymptotically stable, we must verify Definition 3.4. Clearly parts (i) and (ii) are true for equation (4.21). It remains to show that for any $\varepsilon > 0$ there corresponds $T(\varepsilon)$ such that $\| B(t, \phi_{a,a}, a) \| \leq \varepsilon$ for all $t \geq a + T(\varepsilon)$ whenever $\| \phi_{a,a} \| \leq r$, $r<0$ (where r does not depend on a, ε, $\phi_{a,a}$). Choose any $\varepsilon > 0$ and $r > 0$. We wish to find $T(\varepsilon) \ni \| \phi_{t,a} \| \leq \varepsilon$ for all $t \geq a + T(\varepsilon)$ whenever $\| \phi_{a,a} \| \leq r$. Thus, we wish to show

$$(4.22) \qquad \| \phi_{t,a} \| \leq r \left[\exp \{\tfrac{M}{\alpha}\} \right] \alpha^{t-a} \leq \varepsilon \; .$$

whenever $\| \phi_{a,a} \| \leq r$. Equation (4.22) becomes

$$(4.23) \qquad \alpha^{t-a} \leq \frac{\varepsilon}{r \exp \{\frac{M}{\alpha}\}} \; .$$

Since $\alpha<1$, we can find $T(\varepsilon)$ such that

$$(4.24) \qquad \alpha^{T(\varepsilon)} \leq \frac{\varepsilon}{r \exp \{\frac{M}{\alpha}\}} \; .$$

Thus, the origin in uniformly asymptotically stable.

Q.E.D.

Having established several stability results for the linear, non-stationary system (4.1), we can obtain boundedness results for the following system:

$$(4.25) \qquad x_t = A(t)\, x_{t-1} + f(t) \; .$$

THEOREM 4.4: If all solutions of (4.2) are bounded as $t \to \infty$, then the same is true of the inhomogeneous equation (4.25) provided $B(t)$ and $f(t)$ are both impulsively small.

Proof. Consider any $a \in R$. The solution of the above equation can be expressed as

$$(4.26)\quad \phi_{t,a} = Y_t Y_a^{-1}\phi_{a,a} + \sum_{s=a}^{t-1} Y_{t-s-1+a} Y_a^{-1}(B(s+1)\phi_{s,a}+f(s))$$

for $t \geq a+1$ and where Y_t is the fundamental matrix solution of (4.2). Thus, we have

$$(4.27)\quad \|\phi_{t,a}\| \leq \|Y_t\|\,\|Y_a^{-1}\|\,\|\phi_{a,a}\| + \sum_{s=a}^{t-1}\|Y_{t-s-1+a}\|\,\|Y_a^{-1}\|\,(\|B(s+1)\|\,\|\phi_{s,a}\|+\|f(s)\|)\,.$$

This results in the following inequality

$$(4.28)\quad \|\phi_{t,a}\| \leq c(a)b(a)\|\phi_{a,a}\| + \sum_{s=a}^{t-1} c(a)b(a)\|B(s+1)\|\,\|\phi_{s,a}\| + \sum_{s=a}^{t-1} c(a)b(a)\|f(s)\|\,,$$

where $\|Y_t\| \leq c(a)$ and $b(a) = \|Y_a^{-1}\|$. Since

$$\sum_{s=-\infty}^{\infty}\|f(s)\| = M<\infty\,,$$

$$(4.29)\quad \|\phi_{t,a}\| \leq (c(a)b(a)\|\phi_{a,a}\| + M) + c(a)b(a)\sum_{s=a}^{t-1}\|B(s+1)\|\,\|\phi_{s,a}\|\,.$$

We have by Gronwall's Inequality

(4.30) $$\|\phi_{t,a}\| \le (c(a)b(a)\|\phi_{a,a}\|+M)\exp\{c(a)b(a)\sum_{s=a}^{t-1}\|B(s+1)\|\}.$$

Since $B(t)$ is impulsively small as $t\to\infty$, we have that $\phi_{t,a}$ is bounded as $t\to\infty$.

Q.E.D.

If the system (4.2) is asymptotically stable, one can derive the same results as in Theorem 4.4 by weakening the condition on $f(t)$.

THEOREM 4.5: If all solutions of (4.2) approach zero as $t\to\infty$, then all solutions of (4.29) are bounded as $t\to\infty$, provided $f(t)$ is bounded as $t\to\infty$ and $B(t)$ is impulsively small as $t\to\infty$.

Proof. To prove this theorem, we begin by considering the system

(4.31) $$y_t = Ay_{t-1} + f(t-1).$$

Any solution of this system can be expressed as

(4.32) $$\phi_{t,a} = Y_tY_a^{-1}\phi_{a,a} + \sum_{s=a}^{t-1} Y_{t-s-1+a}Y_a^{-1}f(s)$$

for $t\geq a+1$. In view of Lemma 4.1, the hypothesis implies that

(4.33) $$\| Y_t Y_a^{-1} \| \leq \| A \|^{t-a} \quad \text{and}$$

(4.34) $$\| f(t) \| \leq k(a)$$

for $t \geq a+1$. Thus, we have

(4.35) $$\| \phi_{t,a} \| \leq \| A \|^{t-a} \| \phi_{a,a} \| + \sum_{s=a}^{t-1} \| A \|^{t-s-1} k(a) \; .$$

Letting $t \to \infty$ and summing the geometric series we have

(4.36) $$\| \phi_{t,a} \| \leq \| \phi_{a,a} \| + k(a) \left(\frac{1}{1-\alpha} \right) ,$$

where $\alpha = \| A \|$. Therefore the solution y_t of (4.31) is bounded. If we subtract (4.31) from (4.25), we obtain

(4.37) $$x_t - y_t = A(t) x_{t-1} - A y_{t-1} \; .$$

Letting $z_t = x_t - y_t$, we have

(4.38) $$\begin{aligned} z_t = x_t - y_t &= A(t)(x_{t-1} - y_{t-1}) + (A(t) - A) y_{t-1} \\ &= A(t) z_{t-1} + B(t) y_{t-1} \; . \end{aligned}$$

Since $B(t)$ is impulsively small as $t \to \infty$ and y_t is bounded, the product $B(t)y_{t-1}$ is impulsively small as $t\to\infty$. Therefore the hypotheses of Theorem 4.4 are satisfied and we conclude that every solution z_t is bounded as $t\to\infty$. Clearly then, every solution $x_t = z_t + y_t$ of the system (4.25) is bounded as $t\to\infty$.

Q.E.D.

Let us consider the more general difference system

$$(4.39) \qquad x_t = f(x_{t-1}, t-1) \ .$$

For simplicity, assume (4.39) has an equilibrium point at the origin. Then, we may also express the right-hand side of (4.41) in the following manner

$$(4.40) \qquad f(x_{t-1}, t-1) = Cx_{t-1} + r(x_{t-1}, t-1) \ ,$$

where C is (any) constant matrix. Here $r(x_{t-1}, t-1)$ is the difference $f(x_{t-1}, t-1) - Cx_{t-1}$. Suppose C is a nonzero matrix and that $r(x,t)$ is $o(x)$ uniformly with respect to $t \geq a$. That is, given $\varepsilon > 0$ there is a $\delta > 0$ such that $\| r(x,t) \| \leq \varepsilon \| x \|$ for all $t \geq a$ and all $\| x \| < \delta$. Then, the equation

$$(4.41) \qquad x_t = Cx_{t-1}$$

is called the <u>linear approximation</u> of the system (4.39). The fact that $r(x,t)$ is $o(x)$ uniformly with respect to $t \geq a$ indicates that the linear term in (4.40) is the dominant part of x_{t-1} near the origin.

To determine the stability of (4.39), the following theorem due to Perron [34] and Bellman [3] demonstrates that in some cases, one can draw certain stability conclusions by studying the linearized part.

THEOREM 4.6: If the linear approximation (4.41) is asymptotically stable at the origin, then (4.39) is uniformly asymptotically stable at the origin.

<u>Proof</u>. For any $a \in R$, the solution of (4.40) can be expressed as:

$$(4.42) \quad \phi_{t,a} = Y_t Y_a^{-1} \phi_{a,a} + \sum_{s=a}^{t-1} Y_{t-s-1+a} Y_a^{-1} (r(\phi_{s,a}, s)) ,$$

where Y_t is the fundamental matrix solution of (4.41) for $t \geq a+1$. Since (4.41) is asymptotically stable,

$$(4.43) \qquad \| Y_t Y_a^{-1} \| \leq \| A \|^{t-a} ,$$

where $\| A \| < 1$ as shown in proof of Lemma 4.1 letting $\| A \| = \alpha$, we have

$$(4.44) \qquad \|\phi_{t,a}\| \leq \alpha^{t-a}\|\phi_{a,a}\| + \sum_{s=a}^{t-1}\alpha^{t-s-1}\|r(\phi_{s,a},s)\| .$$

Since $r(\phi,t)$ is $o(\phi)$ uniformly with respect to $t \geq a$, there exists M and a $\delta > 0$ such that

$\alpha \exp(1/M) < 1$ and $\|r(\phi_{s,a},s)\| < \frac{\alpha}{M}\|\phi_{s,a}\|$, where $\|\phi_{s,a}\| < \delta$. Substituting into (4.44), we obtain

$$(4.45) \qquad \|\phi_{t,a}\|\alpha^{-t+a} \leq \|\phi_{a,a}\| \exp(\frac{t-a}{M})$$

$$(4.47) \qquad \|\phi_{t,a}\| \leq \|\phi_{a,a}\| (\alpha \exp(1/M))^{t-a} .$$

As in Theorem 4.3, we observe that equation (4.47) has the same behavior as equation (4.21). Thus, $\phi_{t,a}$ is uniformly asymptotically stable about the origin.

Q.E.D.

4.4 EXAMPLES.

In this section we present some examples of the linearization results established in Section 4.3. In these examples the norm of a matrix D is denoted by

$$(4.48) \qquad \|D\| = \sum_{i=1}^{n} \sum_{j=1}^{n} |d_{ij}| \, .$$

EXAMPLE 4.1: Consider the difference equation

$$z_{n+2} + \left(1+\frac{1}{1+n^2}\right) z_n = 0 . \tag{4.49}$$

We wish to show that all solutions are bounded. One can represent this equation by the following system of difference equations. Let

$$x_{n-1} = \begin{bmatrix} z_n \\ z_{n+1} \end{bmatrix} = \begin{bmatrix} x_{n-1,1} \\ x_{n-1,2} \end{bmatrix}$$

then
$$x_n = \begin{bmatrix} z_{n+1} \\ z_{n+2} \end{bmatrix} = \begin{bmatrix} x_{n-1,2} \\ -\left(1+\frac{1}{1+n^2}\right) x_{n-1,1} \end{bmatrix}$$

$$x_n = \begin{bmatrix} 0 & 1 \\ -\left(1+\frac{1}{1+n^2}\right) & 0 \end{bmatrix} x_{n-1} = A_1(n) x_{n-1} .$$

Observe that $\lim_{n\to\infty} A_1(n) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = A_1$. Therefore

$B_1(n) = A_1(n) - A_1 = \begin{bmatrix} 0 & 0 \\ -\frac{1}{1+n^2} & 0 \end{bmatrix}$. We note that

$\sum_{n=-\infty}^{\infty} \| B_1(n) \| = \sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} < \infty$. Thus $B_1(n)$ is impulsively small as $t\to\infty$. If one considers the system $x_n = A_1 x_{n-1}$,

then for any initial value x_o , the solution is $x_n = A_1{}^n x_o$.
For A_1 , we have

$$A_1{}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} , \quad A_1{}^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} , \quad A_1{}^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \quad A_1{}^5 = A_1 .$$

Thus any solution is <u>periodic</u> and is clearly bounded. Since all the hypotheses of Theorem 4.1 have been met, the difference equation (4.49) has bounded solutions.

EXAMPLE 4.2: If we re-arrange the equation (4.49) to

$$z_{n+2} + \left(\frac{1}{1+n^2}\right) z_n = 0 , \tag{4.50}$$

we now have $x_n = \begin{bmatrix} 0 & 1 \\ -\left(\frac{1}{1+n^2}\right) & 0 \end{bmatrix} x_{n-1} = A_2(n)x_{n-1}$.

Observe that $\lim\limits_{n\to\infty} A_2(n) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = A_2$. $B_2(n) = B_1(n)$ and thus it is impulsively small as $n\to\infty$. Since the repeated eigenvalue of A_2 is zero, we have that (4.50) is uniformly asymptotically stable by Theorem 4.3.

EXAMPLE 4.3: We will show that the following difference equation

$$z_{n+2} + \left(1 + \frac{1}{1+n^4}\right) z_n = e^{-|n|} \tag{4.51}$$

has bounded solutions. Defining x_{n-1} as in Example 4.1, we obtain

$$x_n = \begin{bmatrix} 0 & 1 \\ -\left(1+\frac{1}{1+n^4}\right) & 0 \end{bmatrix} x_{n-1} + \begin{bmatrix} 0 \\ e^{-|n|} \end{bmatrix}$$

$$= A_3(n)x_{n-1} + f(n-1)$$

One finds that $A_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $B_3(n) = \begin{bmatrix} 0 & 0 \\ -\left(\frac{1}{1+n^4}\right) & 0 \end{bmatrix}$

Clearly $B_3(n)$ is impulsively small as $n\to\infty$. Observe that $e^{-|n|} \leq 1$. As in Example 4.1, the solutions of $x_n = A_3x_{n-1}$ are bounded, since they are periodic. Therefore, the solutions of equation (4.51) are bounded by Theorem 4.5.

CHAPTER V

EXTENSION OF CLASSICAL LYAPUNOV THEORY

5.1 OUTLINE.

In Chapter III, we were able to establish with the aid of Lyapunov functions, several stability properties of an equilibrium point, Lyapunov theory enabled us to find a domain of stability of an equilibrium point.

Now, consider the following difference equation

(5.1) $$x_{k+1} = (x_k)^{-2}$$

for $x>0$. Equation (5.1) has the solution

(5.2) $$x_k = (x_o)^{(-2)^k} .$$

Then if $x_o = 1$, $x_k = 1$ for all k.

If $x_o<1$, then $x_k \to 0$ for even k.
$x_k \to \infty$ for odd k.

If $x_o>1$, then $x_k \to 0$ for odd k.
$x_k \to \infty$ for even k.

If we allow ∞ to be in the vector space, we then have that some <u>subsequences</u> of the solutions approach a set $A^* = \{0,1,\infty\}$. Under the previous definitions of stability presented thus far we would have concluded that the equilibrium point 1 is unstable. However, if we could regard $\{0,1,\infty\}$ as the limit set of the solutions and observing that subsequences of the solutions converge to this set, then the previously empty domain of stability becomes the set $G = \{x>0\}$. The limit set A^* and the domain G play important roles in stability analysis.

The theory that follows was formalized by several authors, most notably LaSalle [23] , Hurt [18] , and Hale [15] . This chapter formalizes the theory which extends classical Lyapunov theory. We first examine this theory in regards to non-autonomous systems. Finally, we study the properties that apply to autonomous systems.

5.2 GENERAL STABILITY THEOREM.

Consider the non-autonomous system

$$x_{k+1} = f(x_k, k) \quad . \tag{5.3}$$

For any non-empty set A, denote the distance from x to A by d(x,A):

$$d(x,A) = \{\| x-y \| : y\in A\} \, . \tag{5.4}$$

We add the vector ∞ to the vector space X and define

(5.5) $$d(x,\infty) = \|x\|^{-1} .$$

Letting $A^* = A \cup \{\infty\}$, we define $d(x,A^*)$ as:

(5.6) $$d(x,A^*) = \min\{d(x,A),\ d(x,\infty)\} .$$

Let G be any set in the vector space X, where G may be unbounded. If $V(x,k)$ and $W(x)$ are continuous in x, $V(x,k)$ is bounded from below, and

(5.7) $$\Delta V(x,k) = V(f(x,k),k+1) - V(x) \leq -W(x) \leq 0$$

for all $k \geq k_0$ and all x in G; then <u>V is a Lyapunov function for (5.3) on G</u>. Let $\overline{G}$ be the closure of G, including ∞ if G is unbounded, and define the set A by

(5.8) $$A = \{x \in \overline{G} : W(x) = 0\} .$$

Hurt shows the discrete analogue of LaSalle's general stability theorem for differential systems.

THEOREM 5.1 (General Stability Theorem): If there exists a Lyapunov function for (5.3) on G, then each solution of (5.3) which remains in G for all $k \geq k_0$ approaches the set $A^* = A \cup \{\infty\}$.

<u>Proof</u>. Let x_k be a solution (5.3) that remains in G for all

$k \geq k_o$. Then, by assumption $V(x_k,k)$ is a monotone non-increasing function which is bounded from below. Hence, $V(x_k,k)$ must approach a limit as $k\to\infty$, and $W(x_k)$ must approach 0 as $k\to\infty$. If $\{x_k\}$ is unbounded, then $\| x_k \|^{-1}\to 0$ and we have $d(x_k',A^*)\to 0$. If $\{x_k\}$ is bounded, then there exists some subsequence $\{x_k'\}\to y$ as $k\to\infty$. By continuity, we have $W(x_k')\to W(y) = 0$. Thus, $y\in A$ and $d(x_k,A^*)\to 0$ as $k\to\infty$. Therefore, $x_k\to A^*$ as $k\to\infty$.

Remarks. (1) Theorem 5.1 reveals several important points. If G is unbounded and there exists a sequence $\{x_n\}$ such that $\| x_n \|\to\infty$ and $W(x_n)\to 0$ as $n\to\infty$, then it is possible to have an unbounded solution under the terms of the theorem. If G is bounded then there exists a set B in A such that B is compact. This is shown by observing that B is bounded, and since we consider all subsequences that converge to $y\in B$, then B is closed.

(2) Theorem 3.4 can be proved using Theorem 5.1. For instance, if G is the entire space X and W(x) is positive definite, then $A = \{0\}$ and all solutions approach the origin as $k\to\infty$. Thus if each eigenvalue of the linear system

(5.9) $$x_{k+1} = Ax_k$$

has modulus less than unity, we can find a positive definite function $W(x)$ (Theorem 3.5) and thus (5.9) is asymptotically stable in the large.

EXAMPLE 5.1 [18 p.584] : Let us apply the results of Theorem 5.1 to the system (5.1). Let G be the set of positive numbers. Then, if $x>0$ we obtain $x_k>0$, and all solutions which start in G stay in G. The function $V(x,k) = V(x)$, where

$$V(x) = \frac{x}{1+x^2}$$

is a Lyapunov function for (5.1) on G since $V(x)\geq 0$ and

$$\Delta V(x) = V(f(x))-V(x) = \frac{x^{-2}}{1+x^{-4}} - \frac{x}{1+x^2} = \frac{x(x^3-1)(1-x)}{(1+x^4)(1+x^2)}$$

$$= -W(x)\leq 0 .$$

We have $W(x) = 0$ where $x = 0$ and 1 and $W(x)\to 0$ as $x\to\infty$. This implies that $A^* = \{0,1,\infty\}$. Comparing this with the actual case shown in the introduction, we have that this Lyapunov function gives the smallest A^* and largest domain of stability G.

Example 5.1 illustrates the importance of finding a Lyapunov function for which A^* is as small as possible and G as large as possible. Hurt [18] shows that a set G can be constructed so that all solutions which start in some smaller set G, remain in G. From Theorem 5.1, if we have a Lyapunov function on G then all solutions which start in G, approach A^* as $k \to \infty$.

COROLLARY 5.1: Let $u(x)$ and $v(x)$ be continuous real-valued functions. Let $V(x,k)$ be such that

$$u(x) \leq V(x;k) \leq v(x) \tag{5.10}$$

for all $k \geq k_o$. For some η , define the sets $G_1 = G_1(\eta)$ and $G = G(\eta)$ as

$$G(\eta) = \{x : u(x) < \eta\} \tag{5.11}$$

$$G_1(\eta) = \{x : v(x) < \eta\} \ . \tag{5.12}$$

If V is a Lyapunov function for (5.3) on $G(\eta)$, then all solutions which start in $G_1(\eta)$ remain in $G(\eta)$ and approach A as $k \to \infty$.

<u>Proof</u>. Let $x(k)$ be a solution of (5.3) with $x(k_o) \in G_1(\eta)$, then $u(x(k)) \leq V(x_k,k) \leq V(x_o,k_o) \leq v(x_{k_o}) < \eta$ for all $k \geq k_o$ implying $x(k) \in G(\eta)$ for all $k \geq k_o$.

5.3 PRELIMINARIES FOR AUTONOMOUS SYSTEMS.

Let us consider the autonomous system

$$x_{n+1} = f(x_n) \ , \tag{5.13}$$

where x_o is the starting point. The solution of (5.13) is

$$x_n = f^n(x_o) \,, \tag{5.14}$$

where $f^n(x_o) = f^{n-1}(f(x_o))$ for $n \geq 1$. The following definitions and theorems form the backbone of LaSalle's "Invariance Principle", which is discussed in the next section.

Definition 5.1: A point y is a *limit point* of x if there is a sequence n_i such that $f^{n_i}x \to y$ and $n_i \to \infty$ as $i \to \infty$. The *limit set* $\Omega(x)$ of the motion $\{f^n x\}$ starting at x is the set of all limit points of x.

Definition 5.2: Relative to (5.13), or to f, a set H is said to be *positively (negatively) invariant* if $f(H) \subset H$ $(H \subset f(H))$ H is said to be *invariant* if $f(H) = H$.

Definition 5.3: A closed invariant set A is said to be *invariantly connected* if it is not the union of two non-empty disjoint closed invariant sets.

Definition 5.4: A motion $f^n(x)$ is said to be *periodic (or cyclic)* if for some $k>0$, $f^k(x) = x$. The least such integer k is called the *period* of the motion or the *order* of the cycle if $f^j(x) \neq x$ for $1<j<k$. If k=1, x is a fixed point of f and is called an equilibrium point of (5.13). The following results are well known.

THEOREM 5.2: Every limit set $\Omega(x)$ is closed and positively invariant.

THEOREM 5.3: If $\{f^n(x)\}$ is bounded for all $n \geq 1$, then $\Omega(x)$ is non-empty, compact, invariant, invariantly connected, and is the smallest closed set that $\{f^n(x)\}$ approaches as $n \to \infty$.

THEOREM 5.4: An invariant set with finite number of elements is invariantly connected if and only if it is a periodic motion.

5.4 LASALLE'S INVARIANCE PRINCIPLE FOR AUTONOMOUS SYSTEMS.

For V a Lyapunov function of (5.13) on G, LaSalle defines

$$(5.15) \qquad E = \{x : \Delta V(x) = 0,\ x \in \overline{G}\} .$$

We use M to denote the largest invariant set in E, and

$$(5.16) \qquad V^{-1}(c) = \{x : V(x) = c,\ x \in R^m\} .$$

LaSalle establishes the following improvement of Hurt's Theorem 5.1.

THEOREM 5.5 (Invariance Principle): If (i) V is a Lyapunov function of (5.13) on G and (ii) $\{x_n\}$ is a solution of (5.13), bounded, and in G for all $n \geq 0$, then there is a number c such that $x_n \to M \cap V^{-1}(c)$ as $n\to\infty$.

Proof. We know $x_n = f^n(x_o)$ is a solution of (5.13). Our assumptions imply that $V(x_n)$ is nonincreasing and bounded from below and hence $V(x_n)\to c$ as $n\to\infty$. Let $y\in\Omega(x_o)$, then there is a sequence n_i such that $n_i\to\infty$ and $x_{n_i}\to y$. Since V is continuous, $V(x_{n_i})\to V(y) = c$. Therefore $\Omega(x_o) \subset V^{-1}(c)$. Since $\Omega(x_o)$ is invariant, $V(f(y)) = c$ and $\Delta V(y) = 0$. Therefore $\Omega(x_o) \subset E$ and hence is in M. Since $x_n\to\Omega(x_o)$, $x_n \to M \cap V^{-1}(c)$ (larger set) as $n\to\infty$.

One of the results that comes out of this theory is that solutions approach periodic cycles, the equivalent of limit cycles in the continuous time case. The following example [23, pp.6-7] illustrates this point.

EXAMPLE 5.2: Consider the two-dimensional system

(5.17)
$$x(n+1) = \frac{ay_n}{1+x_n^2}$$
$$y(n+1) = \frac{bx_n}{1+y_n^2}$$

or, with the obvious notation,

$$x' = \frac{ay}{1+y_n^2}$$
$$y' = \frac{bx}{1+y^2}.$$

where $a^2 = b^2 = 1$. Take $V(x,y) = x^2 + y^2$.

$$\Delta V(x,y) = V(x',y') - V(x,y)$$

$$= \left(\frac{b^2}{(1+y^2)^2} - 1\right)x^2 + \left(\frac{a^2}{(1+x^2)^2} - 1\right)y^2 .$$

V is a Lyapunov function of (5.17) on the entire space R^2. Here $E = M$ is the union of the two coordinate axis and by Theorem 5.5, we know each solution approaches $\{(c,0),(0,c),(-c,0),(0,-c)\}$ for some c - the intersection of E with a circle of radius c. There are two subcases

(i) $ab = 1$. Then $f(c,0) = (0,bc)$, $f^2(c,0) = f(0,bc) = (abc,0) = (c,0)$. Since limit sets are invariantly connected, every solution approaches one of periodic motion of period 2 (Theorem 5.3, 5.4).

(ii) $ab = -1$. Hence $f(c,0) = (0,bc)$

$f^2(c,0) = f(0,bc) = (abc,0) = (-c,0)$

$f^3(c,0) = f(-c,0) = (0,0bc)$

$f^4(c,0) = f(0,-bc) = (-abc,0) = (c,0)$.

As in (i), each solution approaches the origin or if $c \neq 0$, it approaches a periodic motion of period 4.

CHAPTER VI

APPLICATIONS

6.1 OUTLINE.

The aim of this chapter is to illustrate some applications of the theory developed in Chapters III, IV, and V. In the process, we will get some insight into how physical problems motivate the mathematical theory.

The three areas that are examined in this chapter are quite different. However, each model indicates that a stability problem is present. Each stability problem will be solved using the methods of Chapters III, IV, and V. The mathematical resolution of the stability problem is then interpreted in physical terms. The whole exercise indicates the beautiful interplay between the questions that come out of physical systems and the mathematical machinery used to model and resolve them.

6.2 POPULATION DYNAMICS AND STABILITY.

Beddington [2] investigates the effect of age structure on the dynamics of populations. He illustrates that several authors used linear, stationary systems to model this particular aspect of the population. Thus, models are of the form

$$x_{t+1} = Ax_t , \tag{6.1}$$

where x_t is a column vector containing the n groupings of a population at time t and A is a square matrix of order n.

In this model the first row of A has the n age specific coefficients of fecundity and its sub-diagonal contains the n-1 survival probabilities. Further, it was assumed that the fecundity and survival coefficients were <u>constant</u>.

The model is able to predict the stability properties around the equilibrium point for some types of biological species. However, it becomes more and more ineffective, as fluctuation in the fecundity and survival coefficients, which vary with age, population and time, increases.

In an attempt to represent the biological phenomenon more accurately, Beddington suggests that one consider the following system:

$$x_{t+1} = N(x_t) x_t , \tag{6.2}$$

where $N(x_t)$ is now a linear, non-stationary matrix. The fecundity and survival coefficients are no longer constant.

As was mentioned in Chapter III, it is very difficult to obtain stability results for linear non-stationary systems. However, Beddington shows that (6.2) can be represented by the system

$$y_{t+1} = By_t + f(y_t) , \tag{6.3}$$

where f(y) is o(y). Thus Theorem 3.6 states that it is sufficient to show that

$$y_{t+1} = By_t \tag{6.4}$$

is stable (i.e. eigenvalues of B are in magnitude less than unity) to demonstrate that the more complicated system (6.2) is stable.

Beddington proceeds as follows. The system will be in equilibrium when for some vector x* and matrix N(x*)

$$x^* = N(x^*)x^* . \tag{6.5}$$

Clearly the equilibrium can be found when one solves

$$|N(x^*)-I| = 0 . \tag{6.6}$$

Now, consider a deviation from the equilibrium such that

$$x_t = x^* + y_t . \tag{6.7}$$

Then, substituting into (6.2), we have

$$(6.8) \qquad x^* + y_{t+1} = N(x^*+y_t)(x^*+y_t) \; .$$

Expanding $N(x^*+y_t)$ to the first order, we obtain

$$(6.9) \qquad N(x^*+y_t) = N(x^*) + \sum_{i=1}^{n} y_i N_i'(x^*)$$
$$+ \text{ second order terms of } y_t \; ;$$

where we define $N_i'(x^*)$ as the partial derivative of the matrix N with respect to the i^{th} element of the vector x_t evaluated at x^*, and y_i as the i^{th} element of the vector y_t. Substituting into (6.8), we have

$$(6.10) \quad x^* + y_{t+1} = [\, N(x^*) + \sum_{i=1}^{n} y_i N_i'(x^*) \,] (x^*+y_t)$$
$$+ \text{ second order terms of } y_t \; .$$

Ignoring second order terms and noting that $N(x^*)x^* = x^*$, we obtain

$$(6.11) \qquad y_{t+1} = N(x^*)y_t + \sum_{i=1}^{n} y_i N_i'(x^*)x^*$$
$$+ \text{ second order terms of } y_t \; .$$

If we let H_i be a square matrix whose i^{th} column is x^*, we have

$$(6.12) \qquad N_i'(x^*)\, H_i y_i = y_i N_i'(x^*) x^* .$$

Substituting into (6.11), we finally have

$$(6.13) \qquad y_{t+1} = [\, N(x^*) + \sum_{i=1}^{n} N_i'(x^*)\, H_i \,]\, y_t$$
$$+ \text{ second order terms of } y_t .$$

Let us define

$$(6.14) \qquad B = N(x^*) + \sum_{i=1}^{n} N_i'(x^*)\, H_i .$$

Thus, (6.2) is of the same form as (6.3), where B is defined as in (6.14) and $f(y)$ is $o(y)$.

Thus the stability problem of the system (6.2) has been reduced to the calculation of the eigenvalues of the matrix B. Theorem 3.6 enables us to determine when we have stability or instability. Beddington indicates that the type of populations where the survival rate is dependent on the present population (i.e. $N(x_t)$) is found in the varying hare, the vole Microtus agrestis, and the red grouse.

As an example of the application of this criterion, Usher [38] considers a simpler model. In this model the elements of the matrix $N(x_t)$ are functions of n defined as the sum of the elements of the vector x_t. Here B is given by

$$B = N(x^*) + N'(x^*)\,H\,, \tag{6.15}$$

where $N'(x^*)$ is the derivative of $N(x_t)$ with respect to n evaluated at x^* and H is a matrix whose columns are the vector x^*.

Usher considers a model where there are three female age groups and the matrix $N(x_t)$ is given by

$$N(x_t) = \begin{bmatrix} 0 & 9 & 12 \\ s_1(n) & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \tag{6.16}$$

where

$$s_I(n) = \frac{1}{3\{1+\exp\left((n/\alpha)-\beta\right)\}} \tag{6.17}$$

and $\alpha\beta = 500$. After completing the necessary calculations, he found that

$$B = \begin{bmatrix} 0 & 9 & 12 \\ \frac{1}{15}-\theta & -\theta & -\theta \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \tag{6.18}$$

where $\theta = 8(\log 4 + \beta)/165$. B has the characteristic equation

$$(6.19) \quad \lambda^3 + \theta\lambda^2 + \lambda[19\theta/2 - 3/5] + 6[\theta - 1/15] = 0 .$$

The relationship between the modulus of the dominant eigenvalue of B and the parameter β is given in Figure V.

6.3 THE STABILITY OF EPIDEMIC MODELS.

Epidemics have long been a natural motivator of stability analysis. Many models have been used to describe this phenomenon. One of the earliest and most flexible of models remains the Kermack-McKendrick model (McKendrick [31]). The model essentially divides a population into three subpopulations: <u>susceptibles</u>, <u>infectives</u>, and <u>removed individuals</u>. Susceptibles are exposed to the disease through contact with infectives. They then become infectives themselves, and then are eventually removed either through expiration of infectiousness, by treatment and cure, or by death. Hoppenstead [17] summarizes this process by the relation:

$$S \rightarrow I \rightarrow R .$$

He goes on to develop the discrete-time Kermack-McKendrick model:

$$(6.20) \quad \begin{aligned} S_{n+1} &= \exp\{-a_n I_n\} S_n \\ I_{n+1} &= \alpha_n I_n + (1 - \exp\{-a_n I_n\}) S_n \\ R_{n+1} &= (1 - \alpha_n) I_n + R_n , \end{aligned}$$

where S_n = susceptible population size in the n^{th} time period.

I_n = infective population size in the n^{th} time period

R_n = removed population size in the n^{th} time period

and $1-\alpha_n$ denotes the proportion of infectives that are cured in the n^{th} time period. We also have

$$a_n = -\log (1-p_n) , \tag{6.21}$$

where p_n = probability of an effective contact between any given infective and susceptible in the population in the n^{th} time period. Further, we assume that the population is bounded by P. If $a_n = a$ and $\alpha_n = \alpha$ are constants (6.20) becomes

$$\begin{aligned} S_{n+1} &= \exp\{-aI_n\} S_n \\ I_{n+1} &= \alpha I_n + (1 - \exp\{-aI_n\}) S_n \\ R_{n+1} &= (1-\alpha) I_n + R_n . \end{aligned} \tag{6.22}$$

Observe that if we let $x_n = [S_n, I_n, R_n]'$, then, we have

$$x_{n+1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & \alpha & 0 \\ 1-\alpha & 0 & 1 \end{bmatrix} x_n + f(x_n) . \tag{6.23}$$

In view of the fact that we are seeking qualitative results on epidemics, (6.23) brings out an important problem in stability - that of determining stability of a linear system when the largest eigenvalue is <u>one</u>. One would like to use Theorem 3.6 to show stability or instability. However, even if $f(x)$ is made to be $o(x)$, the third equation cannot be changed without destroying the effectiveness of the model. It is precisely this equation which gives the largest eigenvalue as <u>one</u> (i.e. if $\alpha<1$). However, as the next theorem indicates, we still may have stability.

THEOREM 6.1 (Threshold Theorem): Suppose $R_o = 0$ and $\alpha \neq 1$. Then we have these two results.

(i) The susceptible population approaches a limiting value; $S_n \to S^*$ as $n \to \infty$.

(ii) If $F = S^*/S_o$, then $F = \exp\{-\frac{aS_o}{1-\alpha}(1 + \frac{I_o}{S_o} - F)\}$.

<u>Proof</u>. (i) Since $S_{n+1} = \exp\{-aI_n\} S_n \leq S_n$ and $S_n \geq 0$, for all n, we see that $\lim_{n\to\infty} S_n$ exists. Denote this by S^*.

Since $R_{n+1} = (1-\alpha) I_n + R_n \geq R_n$ and $R_n \leq P$, we see that $\lim_{n\to\infty} R_n$ exists, call it R^*. Since $I_n = \frac{1}{1-\alpha}(R_{n+1}-R_n)$ for $\alpha \neq 1$, we have $\lim_{n\to\infty} I_n = 0$ and so $R^* = P-S^*-I^* = P-S^*$.

(ii) Since $S_{n+1} = \exp\{-aI_n\} S_n$, we have that

$$(6.24) \quad S_n = S_o \exp\{-a \sum_{k=0}^{n-1} I_k\}$$

$$= S_o \exp\{\frac{-a}{1-\alpha} \sum_{k=0}^{n-1} (R_{k+1}-R_k)\}$$

$$= S_o \exp\{\frac{-a}{1-\alpha} R_{n-1}\} .$$

Letting $n \to \infty$, we have

$$(6.25) \qquad S^* = S_o \exp\{\frac{-a}{1-\alpha} (P - S^*)\} .$$

Letting $F = S^*/S_o$ gives

$$(6.26) \qquad F = \exp\{\frac{-aS_o}{1-\alpha} (1 + \frac{I_o}{S_o} - F)\}$$

Q.E.D.

F is a measure of the epidemic's final severity. It is interesting to study F's dependence on the parameters of the system. The case of interest is where $I_o/S_o \ll 1$, since this is the usual situation. The solution for F is described in Figure VI from Hoppensteadt. This shows the critical dependence of F on the parameter $\gamma = \frac{aS_o}{1-\alpha}$. If $\gamma < 1$, then few susceptibles are exposed, but if $\gamma \gg 1$, many fewer susceptibles are exposed and implies that they are either infected or removed (i.e. a significant epidemic occurs).

6.4 ESTABLISHING A DOMAIN OF STABILITY.

The following discrete-time system studied by Vidal and Laurent [39] was used to model control systems:

(6.27) $$x_{k+1} = L(x_k,k)x_k \ ,$$

where L(x,k) is a matrix. Hurt [18] studied this system and used the results of Theorem 5.1 and Corollary 5.1 to find a domain of stability. Hurt proceeds in the following manner: for any vector x, $|x|$ defines the norm of x. The norm of the matrix L is given by:

(6.28) $$\|L(x,k)\| = \min\{b: |L(x,k)y| \le b|y| \text{ for all } y\neq 0\} \ .$$

Then clearly,

(6.29) $$|L(x,k)x| \le \|L(x,k)\|\,|x| \ .$$

For (6.27), Hurt tried the Lyapunov function

(6.30) $$V(x,k) = |x|$$

(6.31) $$\Delta V(x,k) = |L(x,k)x| - |x| \le (\|L(x,k)\| - 1)|x| \ .$$

Letting $u(x) = v(x) = V(x,k) = |x|$, then

(6.32) $$G_1(\eta) = G(\eta) = \{x: |x| < \eta\} \ .$$

Letting $\|L(x,k)\| \le a(x)$ and $W(x) = (1-a(x))|x|$. Then we have

(6.33) $$\Delta V(x,k) \leq -W(x) .$$

Hurt observed that if $a(x)<1$ for all x in $G(\eta)$, then $-W(x)\leq 0$, the set A is the origin and possibly something on the boundary of $G(\eta)$. Since $V(x,k)$ is a nonincreasing function of k and the boundary of $G(\eta)$ is a level surface $V(x,k)$, the solutions cannot approach the boundary of $G(\eta)$. Thus the solution which starts in $G(\eta)$ remains in $G(\eta)$ and approaches the origin as $k\to\infty$. The set $G(\eta)$is called a domain of stability for (6.27). The best $G(\eta)$ is chosen by picking η as large as possible without violating $a(x)<1$ for all x in $G(\eta)$.

Various choices for the vector norm and Lyapunov functions will result in various $G(x)$, and different domains of stability. Since each is sufficient, the union of all these domains of stability is also a domain of stability. Hurt finally shows through the use of a result from Ostrowski [33] that if $L(0,k)$ is a constant matrix, independent of k, and the largest eigenvalue of $L(0,k)$ is less than one, then there is a vector norm such that $a(x)$ is continuous in x and $a(0)<1$. This indicates that there is a non-empty domain of stability. Thus, if Hurt's criteria can be met, one could establish a stability domain about the origin.

TABLE I

$$x_{t+1} = x_t \exp\{r(1-x_t/K)\} .$$

Value of Growth Rate, r	Dynamical Behavior
$0 < r < 2$	globally stable equilibrium point
$2 < r < 2.526$	globally stable 2-point cycle
$2.526 < r < 2.656$	globally stable 4-point cycle
$2.656 < r < 2.692$	stable cycles, period 8, giving way in time to cycles of period 16, 32, etc., as r increases
$r > 2.692$	chaos (cycles of arbitrary period, or aperiodic behavior, depending on initial condition)

TABLE II

$$x_{t+1} = mx_t(1-x_t) .$$

Value of m	Dynamical Behavior
$1 < m < 3$	stable equilibrium point
$3 < m < 3.449$	stable 2-point cycle
$3.449 < m < 3.544$	stable 4-point cycle
$3.544 < m < 3.570$	stable cycles, period 8, then 16, 32, etc., as m increases
$m > 3.570$	chaos

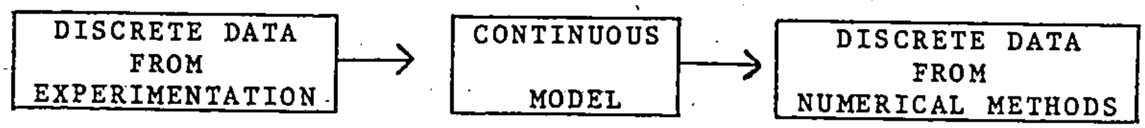

FIGURE I.

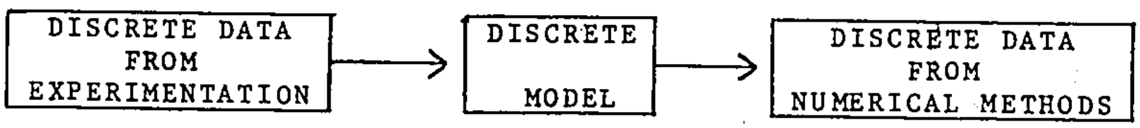

FIGURE II.

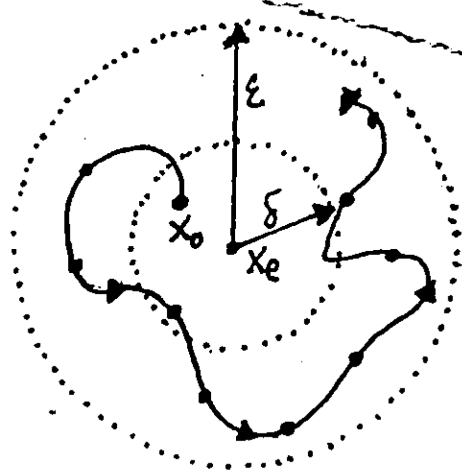

FIGURE III. STABILITY OF SYSTEM (2.1) ABOUT THE EQUILIBRIUM POINT x_e .

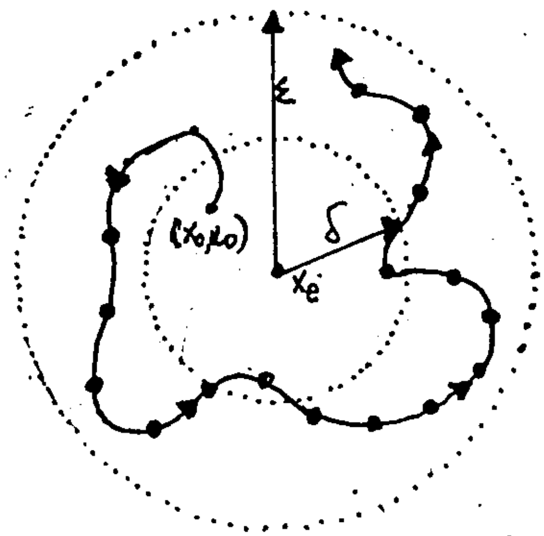

FIGURE IV. STABILITY OF SYSTEM (2.2) ABOUT THE EQUILIBRIUM POINT x_e .

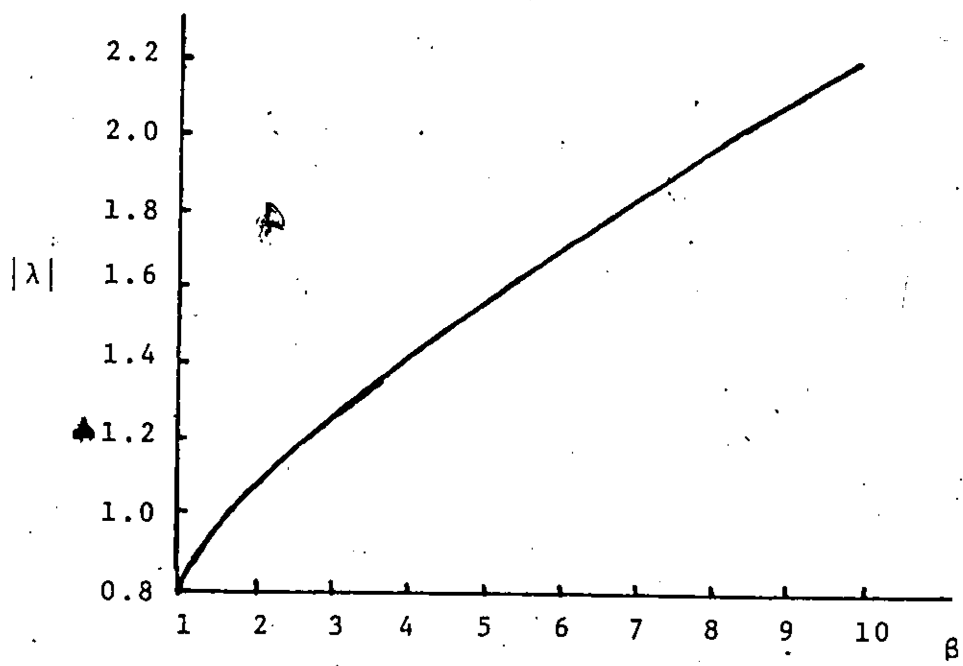

FIGURE V. RELATIONS BETWEEN THE MODULES OF THE DOMINANT EIGENVALUES λ OF MATRIX B AND THE PARAMETER β .

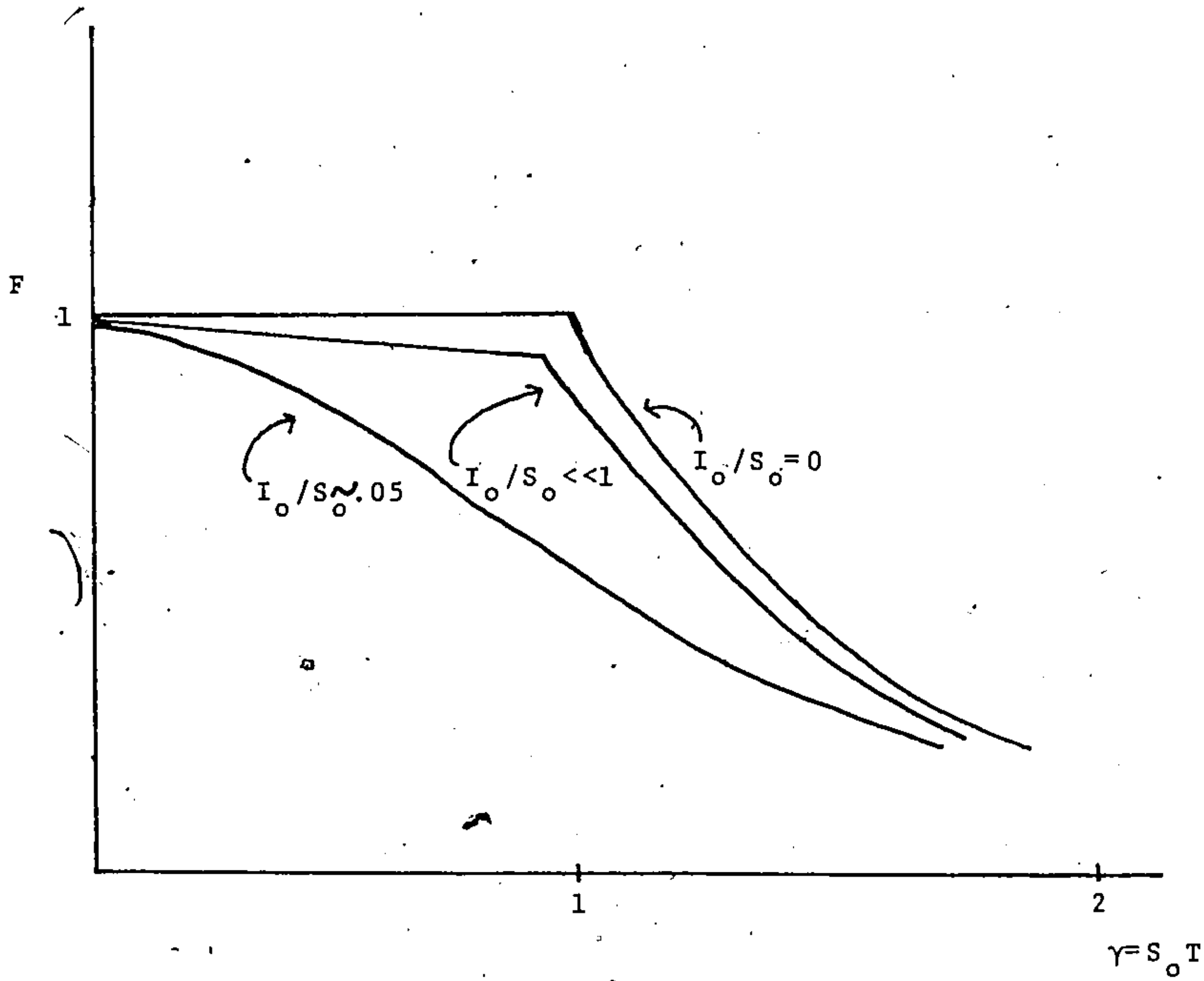

FIGURE VI. THIS FIGURE SHOWS THE CRITICAL DEPENDENCE OF F, A MEASURE OF THE EPIDEMIC'S FINAL SIZE, ON THE PARAMETER $\alpha = AS_o/(1-\alpha)$. If $\gamma < 1$, THEN FEW SUSCEPTIBLES ARE EXPOSED. BUT IF $\gamma >> 1$, A SIGNIFICANT EPIDEMIC OCCURS.

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