BIFURCATIONS IN SQUIDS

Deming Li

A thesis
in
The Department
of
Computer Science

Presented in Partial Fulfillment of the Requirements
For the Degree of Master of Computer Science
Concordia University
Montréal, Québec, Canada

July 1994

ISBN 0-315-97670-5

CONCORDIA UNIVERSITY
School of Graduate Studies

This is to certify that the thesis prepared

By: **Deming Li**

Entitled: **Bifurcations In SQUIDs**

and submitted in partial fulfillment of the requirements for the degree of

Master of Computer Science

complies with the regulations of this University and meets the accepted standards with respect to originality and quality.

Signed by the final examining commitee:

____________________ Chair

____

____

____

____

____________________ Supervisor

Approved ____________________

Chair of Department or Graduate Program Director

__________ 19 ____ ____________________

Dean of Faculty

Abstract

Bifurcations In SQUIDs

Deming Li

A theoretical study of bifurcations in a system modelling Superconducting Quantum Interference Devices (SQUIDs) is presented. This system also serves as a model of a pair of coupled pendula. Maginu [1983] numerically studied bifurcations of the damped SQUID system. His results indicate that the in-phase rotation is unstable for a certain range of the coupling strength. Out-of-phase rotations bifurcate from the in-phase rotation. In the unstable range, there is a chaotic motion. Doedel, Aronson and Othmer [1988, 1991] studied the bifurcations of the system in both the damped and the undamped case. The system undergoes period-doubling bifurcations for suitable coupling strengths. 2T-periodic rotations bifurcate from the T-periodic rotation. In the damped case, the system gets to a chaotic motion via period-doubling cascades. These computational results yield some qualitative insight on the nature of the solutions. They also motivate the theoretical analysis in this thesis.

Our study is divided into two cases: damped and undamped. In the undamped case, the system is Hamiltonian. When the coupling strength is small, there are chaotic motions on certain energy manifolds. When the coupling strength is not small, the Hamiltonian system is not integrable. With Group Theory and Singularity Theory, we have investigated the bifurcations from a family of in-phase rotations. It is found that the system undergoes period-doubling (Floquet multiplier = -1) and fixed-point (Floquet multiplier = +1) bifurcations. In the case of the period-doubling bifurcation, 2T-periodic out-of-phase rotations bifurcate from the T-periodic in-phase rotations. Depending on the solutions of the linearization, the system exhibits different kinds of bifurcations. The system has Hopf bifurcation if the linearization has odd 2T-periodic solutions. If the linearization has no odd 2T-periodic solutions

then the system has a degenerate $\mathbf{Z_2} \oplus \mathbf{Z_2}$ -bifurcation. In this case, the nonlinear $\mathbf{Z_2} \oplus \mathbf{Z_2}$ -symmetry bifurcation equation is shown to be degenerate and under certain conditions it is equivalent to a simplified normal form. Bifurcation diagrams are presented. In the case of fixed-point bifurcation, T-periodic out-of-phase rotations bifurcate from the T-periodic in-phase rotations. The normal form and bifurcation diagrams are qualitatively the same as the ones in the period-doubling bifurcation.

In the damped case, there is a period-doubling and there might be fixed-point bifurcations. Under certain conditions, a 2T-periodic out-of-phase rotation bifurcates from the T-periodic in-phase rotation in a period-doubling bifurcation, and a T-periodic out-of-phase rotation bifurcates from the T-periodic in-phase rotation in the fixed-point bifurcation. The normal form and bifurcation diagrams are presented.

In all cases, the theoretical conclusions are compatible with the numerical results.

This thesis shows a systematic way to apply Group Theory and Singularity Theory to bifurcation problems. It can be used to analyze the bifurcation in other dynamical systems such as periodically forced oscillators with one degree of freedom.

Acknowledgments

It is hard to imagine that I could have finished this thesis without my supervisor's, Dr. E.J. Doedel, excellent guidance and moral support. I really enjoyed the research and appreciate his profound knowledge and kindness

I would like to thank Mr. Xianjun Wang and Mr. Pankaj Kamthan for helpful discussions.

I would also like to thank Dr. Xinming Yu for his wonderful help in my course study and in using computer systems.

Contents

List of Figures

Chapter 1

Introduction

The Nobel prize in Physics was awarded to B. Josephson in 1973 for his discovery of the quantum-mechanical tunnelling of carriers through an insulator sandwiched between two superconducting metals. The phenomenon has been termed the Josephson effect since its discovery in 1962, and electrical devices which use this effect for conduction are called Josephson-junction devices [23]. Circuits consisting of several coupled current-biased Josephson point junctions are commonly referred as SQUIDs (Superconducting Quantum Interference Devices) [1, 2]. They provide highly sensitive field detectors and make the neuromagnetic technique for studies of the human brain practical [3]. In the simplest detectors, two-point junctions are used, and the dynamics are governed by the equations

$$\begin{aligned} \ddot{\phi}_1 + \varepsilon\dot{\phi}_1 + \sin(\phi_1) &= I_1 + \gamma(\phi_2 - \phi_1) + H \\ \ddot{\phi}_2 + \varepsilon\dot{\phi}_2 + \sin(\phi_2) &= I_2 + \gamma(\phi_1 - \phi_2) + H. \end{aligned} \tag{1}$$

Here $\phi_j (j = 1,2)$ denotes the phase difference in the electron wave function across the jth junction, I_i denotes the constant bias current, and $H = H(t)$ is proportional to the applied magnetic field. The dimensionless parameter ε measures the dissipation in the individual junctions, while γ measures the strength of the coupling.

The system (1) also serves as a model for a pair of coupled pendula subject to elastic and torsional coupling as well as some aspects of charge-density waves in anisotropic crystals [24]. Mathematically, it can be obtained as the two-point discretization of the damped sine-Gordon equation on the finite interval with Neumann boundary conditions [25, 26].

In the uncoupled case ($\gamma = 0$), the dynamics of (1) is classical and quite well understood [27]. It is well known that if the forcing depends periodically on time the problem is beyond the possibility of complete analysis. In fact, the simple example of the periodically forced pendulum exhibits virtually all the behavior that has been understood so far in low-dimensional geometrical theory of dynamical system: KAM theory [Moser, 1973], Aubry-Mather theory [Aubry & LeDaeron, 1983; Mather, 1982],hyperbolic theory (horseshoes) [Moser, 1973; Guckenheimer and Holmes, 1983, Salam and Sastry,1984; Salam, 1987], bifurcation theory[Arnold, 1983], etc, with questions still unanswered.

The case of two or more coupled pendula offers even less hope for a complete understanding. However, in certain parameter ranges, one can still obtain a good understanding of the behavior; for instance, if the coupling strength γ is sufficiently large and $I_2 = 0$, the system (1) has beating solutions; i.e it behaves as a single damped pendulum [Imry & Schulman, 1978; Zimmerman & Sullivan,1977]. In the opposite extreme, that of small γ, the dynamics is more complicated but the system is understood virtually completely in this range as well. In the range of moderate coupling γ, the system has a family of homoclinic solutions in part of the parameter space and exhibits rich dynamics [Henderson, Levi & Odeh, 1991].

A slightly different model is that where the two junction points have identical bias currents, i.e $I_1 = I_2$ in the system (1). This system has been extensively studied when $H(t) \equiv 0$ [4, 5, 6]. It is so complicated that this case is also the subject of this thesis.

Thus the system we deal with is

$$\begin{aligned}\ddot{\phi}_1 + \varepsilon\dot{\phi}_1 + \sin(\phi_1) &= I + \gamma(\phi_2 - \phi_1)\\ \ddot{\phi}_2 + \varepsilon\dot{\phi}_2 + \sin(\phi_2) &= I + \gamma(\phi_1 - \phi_2).\end{aligned} \tag{2}$$

The equations (2) possess an in-phase (or synchronous) *rotation* [5]. A rotation $\phi_i(t)$ here means that it is a solution of (2) and satisfies $\phi_i(t+T) = \phi_i(t) + 2\pi$ for all $t \in R^1$ and some $T > 0$. Maginu [5] observed that this in-phase rotation is asymptotically stable when γ is small and is large, but unstable for an intermediate range of γ-values for suitable values of ε and I. His numerical results indicate the existence of chaotic motion in the unstable range, but he did not study the transitions through chaotic behavior as γ passes through this range. Doedel, Aronson and Othmer [4] numerically obtained that the in-phase rotation is unstable in $(-\infty, 0) \cup (\gamma_1(\varepsilon), \gamma_2(\varepsilon))$ for some nonnegative $\gamma_1(\varepsilon)$ and $\gamma_2(\varepsilon)$, and stable otherwise. They also numerically investigated the transitions to the chaotic motion. They found that there are period-doubling cascades and infinitely many multiple-pulse homoclinic solutions that exist in the unstable range. Aronson, Doedel and Othmer [6] also numerically studied the rotations of damped and undamped pendula of the system (2). In the undamped case, the system (2) is Hamiltonian . It is found that it undergoes period-doubling bifurcations and 2T-periodic rotations bifurcating from a family of T-periodic rotations. In the damped case, the beating solutions are presented. These numerical results yield some qualitative insight on the nature of the solutions. This is used in this thesis to do a theoretical analysis of the problem.

There are different ways to study SQUIDs, mainly numerical computation and theoretical analysis. For the numerical computation, there are basically two approaches. The first is to integrate the equations numerically. With appropriate initial conditions, this allows stable periodic solutions to be found. The second approach is to formulate a nonsingular, boundary-value problem for the periodic motions [Doedel et al., 1984; Holodniok & Kubicek, 1984; Keller & Jepson, 1984; and others] and to solve it using an iterative solver. This method has the advantage that both the stable and unstable motions are obtained, and that the period is explicitly computed.

A variety of analytic methods can be applied to SQUIDs, such as equilibrium analysis [31], perturbation method [29, 30], Poincaré map [19, 20], Melnikov's method [19, 20], implicit function theorem and group theory and singularity theory [32, 33]. In general, one can only obtain local information of the system based on these analytic approaches. Numerical computations are indispensable to get global views of the systems. Guckenheimer and Holmes [19], Wiggins [20] describe a systematic procedure to investigate chaotic motions of two degrees of freedom of Hamiltonian systems. We will follow this way to study the chaotic motion of the SQUIDs in Chapter 2. Aronson, Doedel and Othmer [6] apply Implicit Function Theorem to discuss the continuation of rotational solutions to nonzero damping and bias current in the system (2). Aronson, Golubitsky and Krupa [22] discuss bifurcations in coupled arrays of Josephson junctions. They show that in-phase rotations lose their stability via fixed-point bifurcations and period-doubling bifurcations.

In this thesis, we analytically study chaotic motion and bifurcations from in-phase rotations of the system (2). Throughout Chapters 2,3 and 4, we consider the system (2) without damping and bias current. In this case it is a Hamiltonian system. The bifurcations that we consider are the ones from in-phase rotations. This is due to the fact that applications such as microwave generators and parametric amplifiers [22] are desired to operate SQUID circuits in stable synchronous (i.e. in-phase) oscillation. Thus it is of interest to determine where in parameter space the synchronous oscillations are stable and how the stability is lost.

In Chapter 2 it is found that there exist chaotic motions (Smale horseshoe) on certain energy manifolds when the coupling strength is small.

In Chapter 3, symmetries and linearization of the system (2) are investigated. There exist two distinct nondecreasing infinite series of bifurcation points at which the Floquet multipliers are +1 or −1. Bifurcations from the in-phase rotations are studied in Chapter 4. There are period-doubling (Floquet multipliers = -1) and fixed-point bifurcations (Floquet multipliers = +1). In the case of period-doubling bifurcation, it is found that 2T-periodic out-of-phase rotations bifurcate from the T-periodic in-phase rotations. Depending on the solutions of the linearization, the system exhibits different kinds of bifurcations. The system has Hopf bifurcation if the linearization has odd 2T-periodic solutions. If the linearization has no odd 2T-periodic solutions then the system has a degenerate $\mathbf{Z_2} \oplus \mathbf{Z_2}$ -bifurcation. In this case, the nonlinear $\mathbf{Z_2} \oplus \mathbf{Z_2}$ -symmetry bifurcation equation is shown to be degenerate and under certain conditions it is equivalent to a simplified normal form. Bifurcation diagrams are presented. In the case of fixed-point bifurcation, it is shown that T-periodic out-of-phase rotations bifurcate from the T-periodic in-phase rotations. The normal form and bifurcation diagrams are qualitatively the same as the ones in the period-doubling bifurcation.

In Chapter 5, we theoretically investigate bifurcations from the T-periodic in-phase rotation when bias current and damping do exist. In section 5.2, we discuss the period-doubling bifurcations. It is shown that the bifurcation equation has $\mathbf{Z_2}$ symmetry on R^2. Under certain conditions it is equivalent to a simplified normal form. The bifurcation diagrams of the normal form are given. There are 2T-periodic out-of-phase rotations bifurcating from the in-phase rotation. In section 5.3, we study the fixed-point bifurcations. There are T-periodic out-of-phase rotations bifurcating from the in-phase rotation under certain conditions. The normal form and bifurcation diagram are qualitatively the same as in section 5.2.

In this thesis, we theoretically obtained a set of results and indeed have new contributions. A concise summary of the contributions and new results is given below.

- Results

  1. Undamped System

     i. $\gamma << 1$, there exist chaotic motions on each energy manifold of energy larger than 2.

     ii. $\gamma = O(1)$, There exist two distinct nondecreasing infinite series of bifurcation points at which the Floquet multipliers are +1 (fixed-point bifurcation) or −1 (period-doubling);

Period-doubling: 2T-periodic out-of-phase rotations bifurcate from the T-periodic in-phase rotations.The system has Hopf bifurcation if the linearization has odd 2T-periodic solutions. If the linearization has no odd 2T-periodic solutions then the system has a degenerate $\mathbf{Z_2} \oplus \mathbf{Z_2}$ -bifurcation. In this case, the nonlinear $\mathbf{Z_2} \oplus \mathbf{Z_2}$ -symmetry bifurcation equation is shown to be degenerate and under certain conditions it is equivalent to a simplified normal form. Bifurcation diagrams are presented.

Fixed-point Bifurcation: T-periodic out-of-phase rotations bifurcate from the T-periodic in-phase rotations. Bifurcation diagrams are presented.

2. Damped System

   Bias current and damping are present. The uncoupled system has an in-phase rotation. There are also period-doubling and fixed-point bifurcations.

   i. The bifurcation equation has $\mathbf{Z_2}$ symmetry on R^2. Under certain conditions it is equivalent to a simplified normal form. For the period-doubling, there are 2T-periodic out-of-phase rotations bifurcating from the T-periodic in-phase rotations. For the fixed-point bifurcation, there are T-periodic out-of-phase rotations bifurcating from the T-periodic in-phase rotations. Bifurcation diagrams are present.

- Contributions

  1. Confirming numerical results such as period-doubling bifurcations, chaotic motions etc.
  2. Rigorously revealing some results which are not given by numerical computations such as chaotic motions on each energy manifolds of energy larger than 2, infinite series of bifurcation points and $\mathbf{Z_2} \oplus \mathbf{Z_2}$ -symmetric bifurcations.
  3. Proving a Theorem (Main Theorem 4.2.3) which presents a normal form for a type of degenerate $\mathbf{Z_2} \oplus \mathbf{Z_2}$-symmetric bifurcation equations. To our

knowledge, it is new and has not given anywhere.

4. Presenting a systematic way to apply Group and Singularity theories to some physical problems. For instance, the method can be used to study bifurcations in oscillators with one degree of freedom.

Chapter 2

Chaotic Motion When Weakly Coupled

Chaotic motions of Hamiltonian systems have been extensively studied. An analytical method to deal with chaos in two degrees of freedom systems is described by Guckenheimer and Holmes [19]. Holmes and Marsden[8, 9] generalize this method to n-degree of freedom systems. Numerical computation of chaos in Hamiltonian system is given by Lichtenberg and Lieberman[40]. Under certain circumstances, SQUIDs are modeled as a Hamiltonian system. In this case, numerical results in[6] show that a variety of homoclinic orbits and heteroclinic cycles exist when the coupling strength is small. In this Chapter, chaotic motion of the SQUIDs is investigated when bias current and dissipation do not exist and the coupling strength is small. It is found that the system has a Smale horseshoe on each energy manifold of energy larger than 2.

2.1 The Reduced Hamiltonian System

Consider the system

$$\begin{aligned} \ddot{\phi}_1 + \sin(\phi_1) &= \gamma(\phi_2 - \phi_1) \\ \ddot{\phi}_2 + \sin(\phi_2) &= \gamma(\phi_1 - \phi_2) \end{aligned} \tag{3}$$

where the coupling coefficient γ is small.

Let

$$q_1 = \phi_1 \tag{4}$$

$$q_2 = \phi_2 \tag{5}$$

$$p_1 = \dot{\phi}_1 \tag{6}$$

$$p_2 = \dot{\phi}_2. \tag{7}$$

Then (3) becomes

$$\begin{aligned} \dot{q}_1 &= p_1 \\ \dot{p}_1 &= -\sin q_1 + \gamma(q_2 - q_1) \\ \dot{q}_2 &= p_2 \\ \dot{p}_2 &= -\sin q_2 + \gamma(q_1 - q_2) \end{aligned} \tag{8}$$

The system (8) is Hamiltonian. Its Hamiltonian is given by

$$H^\gamma(q_1, p_1, q_2, p_2) = \frac{1}{2}p_1^2 + 1 - \cos q_1 + \frac{p_2^2}{2} + 1 - \cos q_2 + \frac{\gamma}{2}(q_1 - q_2)^2. \tag{9}$$

Then

$$\begin{aligned} \dot{q}_1 &= \frac{\partial H^\gamma}{\partial p_1} \\ \dot{p}_1 &= -\frac{\partial H^\gamma}{\partial q_1} \\ \dot{q}_2 &= \frac{\partial H^\gamma}{\partial p_2} \\ \dot{p}_2 &= -\frac{\partial H^\gamma}{\partial q_2}. \end{aligned} \tag{10}$$

Define new variables I and θ by

$$\begin{aligned} q_2 &= 2\arcsin\left(\sqrt{\frac{G(I)}{2}}sn(\frac{2}{\pi}K(\sqrt{\frac{G(I)}{2}})\theta,\sqrt{\frac{G(I)}{2}})\right) \\ p_2 &= \sqrt{2G(I)}\,cn\left(\frac{2}{\pi}K(\sqrt{\frac{G(I)}{2}})\theta,\sqrt{\frac{G(I)}{2}}\right) \end{aligned} \tag{11}$$

where $G(I)$ satisfies

$$\frac{dG(I)}{dI} = \frac{\pi}{2K(\sqrt{\frac{G(I)}{2}})} \tag{12}$$

$$G(0) = 0 \tag{13}$$

and sn(t,h) , dn(t,h) are elliptic functions. $K(m)$ is the complete elliptic integral of the first kind. As a reference for the rest of this chapter, some formulas of elliptic functions are given below [39]:

$$sn^2(u,m) + cn^2(u,m) = 1$$

$$dn^2(u,m) + m^2 sn^2(u,m) = 1$$

$$sn'(u,m) = cn(u,m)dn(u,m)$$

$$cn'(u,m) = -sn(u,m)dn(u,m)$$

$$sn(-u,m) = -sn(u,m) \qquad cn(-u,m) = cn(u,m)$$

$$sn(u+4K(m),m) = sn(u,m) \qquad cn(u+4K(m),m) = cn(u,m)$$

$$cn(u,m) = \frac{2\pi}{mK(m)}\sum_{s=0}^{\infty}\nu_s \cos\frac{(2s+1)\pi}{2K(m)}u \tag{14}$$

$$\nu_s = \frac{\xi^{s+1/2}}{1+\xi^{2s+1}} \tag{15}$$

$$\xi = \exp(-\pi\frac{K(1-m)}{K(m)}). \tag{16}$$

A transformation $q = q(\theta, I)$ and $p = p(\theta, I)$ is called an action_angle transformation if it satisfies

(i) There exists a function $H(q,p)$ such that $H(q(\theta, I), p(\theta, I)) = G(I)$.

(ii) $q(\theta, I)$ and $p(\theta, I)$ are 2π–periodic.

(iii) The Jacobian matrix satisfies $\mid \frac{\partial(q,p)}{\partial(\theta,I)} \mid = 1$.

Then we have:

Lemma 2.1 *The equation (11) is an action_angle transformation.*

Proof:

1. Let

$$H(q_2, p_2) = \frac{1}{2}p_2^2 + 1 - \cos q_2$$

Then

$$\begin{aligned}
H(q_2,p_2) &= \frac{1}{2}\left(\sqrt{2G(I)}\, cn(\frac{2}{\pi}K(\sqrt{\frac{G(I)}{2}})\theta, \sqrt{\frac{G(I)}{2}})\right)^2 \\
&\quad + 1 - \cos\left(2\arcsin\,(\sqrt{\frac{G(I)}{2}}sn(\frac{2}{\pi}K(\sqrt{\frac{G(I)}{2}})\theta, \sqrt{\frac{G(I)}{2}}))\right) \\
&= G(I)\, cn^2\left(\frac{2}{\pi}K(\sqrt{\frac{G(I)}{2}})\theta, \sqrt{\frac{G(I)}{2}}\right) + \\
&\quad 2\sin^2\left(\arcsin(\sqrt{\frac{G(I)}{2}}sn(\frac{2}{\pi}K(\sqrt{\frac{G(I)}{2}})\theta, \sqrt{\frac{G(I)}{2}}))\right) \\
&= G(I)\, cn^2\left(\frac{2}{\pi}K(\sqrt{\frac{G(I)}{2}})\theta, \sqrt{\frac{G(I)}{2}}\right) + G(I)\, sn^2\left(\frac{2}{\pi}K(\sqrt{\frac{G(I)}{2}})\theta, \sqrt{\frac{G(I)}{2}}\right) \\
&= G(I)\left(cn^2(\frac{2}{\pi}K(\sqrt{\frac{G(I)}{2}})\theta, \sqrt{\frac{G(I)}{2}}) + sn^2(\frac{2}{\pi}K(\sqrt{\frac{G(I)}{2}})\theta, \sqrt{\frac{G(I)}{2}})\right) \\
&= G(I).
\end{aligned}$$

2.

$$q_2(I, \theta + 2\pi) = 2\arcsin\left(\sqrt{\frac{G(I)}{2}}sn(\frac{2}{\pi}K(\sqrt{\frac{G(I)}{2}})(\theta + 2\pi), \sqrt{\frac{G(I)}{2}})\right)$$

$$
\begin{aligned}
&= 2\arcsin\left(\sqrt{\frac{G(I)}{2}} sn(2\frac{K}{\pi}\theta + 4K, \sqrt{\frac{G(I)}{2}})\right) \\
&= q_2(I,\theta) \\
p_2(I,\theta+2\pi) &= \sqrt{2G(I)}\, cn\left(\frac{2}{\pi}K(\sqrt{\frac{G(I)}{2}})(\theta+2\pi), \sqrt{\frac{G(I)}{2}}\right) \\
&= p_2(I,\theta).
\end{aligned}
$$

3.

$$
\begin{aligned}
\frac{\partial q_2}{\partial \theta} &= 2\frac{1}{\sqrt{1-\frac{G}{2}sn^2(2\frac{K}{\pi}\theta, \sqrt{\frac{G}{2}})}} \sqrt{\frac{G}{2}} cn(2\frac{K}{\pi}\theta, \sqrt{\frac{G}{2}}) dn(2\frac{K}{\pi}\theta, \sqrt{\frac{G}{2}}) 2\frac{K}{\pi} \\
&= 2\frac{1}{\sqrt{dn^2(2\frac{K}{\pi}\theta, \sqrt{\frac{G}{2}})}} \sqrt{\frac{G}{2}} cn(2\frac{K}{\pi}\theta, \sqrt{\frac{G}{2}}) dn(2\frac{K}{\pi}\theta, \sqrt{\frac{G}{2}}) 2\frac{K}{\pi} \\
&= 2\sqrt{\frac{G}{2}} cn(2\frac{K}{\pi}\theta, \sqrt{\frac{G}{2}}) 2\frac{K}{\pi}.
\end{aligned}
$$

By (13)

$$
\begin{aligned}
\frac{\partial q_2}{\partial \theta} &= p_2(I,\theta)\frac{1}{G'(I)} \\
&= \frac{1}{G'(I)}\frac{\partial H}{\partial p_2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial p_2}{\partial \theta} &= \sqrt{2G}\,2\frac{K}{\pi}\left(-sn(2\frac{K}{\pi}\theta, \sqrt{\frac{G}{2}}) dn(2\frac{K}{\pi}\theta, \sqrt{\frac{G}{2}})\right) \\
&= -\frac{1}{G'(I)}\sqrt{2G}\, sn(2\frac{K}{\pi}\theta, \sqrt{\frac{G}{2}}) dn(2\frac{K}{\pi}\theta, \sqrt{\frac{G}{2}})
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial H}{\partial q_2} &= \sin q_2 = 2\sin\frac{q_2}{2}\cos\frac{q_2}{2} \\
&= 2\sin\left(\arcsin(\sqrt{\frac{G}{2}} sn(2\frac{K}{\pi}\theta, \sqrt{\frac{G}{2}}))\right)\cos\left(\sqrt{\frac{G}{2}}\arcsin(sn(\frac{K}{\pi}\theta, \sqrt{\frac{G}{2}}))\right) \\
&= 2\sqrt{\frac{G}{2}} sn(2\frac{K}{\pi}\theta, \sqrt{\frac{G}{2}})\sqrt{1-\sin^2\left(\arcsin(\sqrt{\frac{G}{2}} sn(2\frac{K}{\pi}\theta, \sqrt{\frac{G}{2}}))\right)}
\end{aligned}
$$

$$
\begin{aligned}
&= 2\sqrt{\frac{G}{2}}sn(2\frac{K}{\pi}\theta,\sqrt{\frac{G}{2}})\sqrt{1-\frac{G}{2}sn^2(2\frac{K}{\pi}\theta,\sqrt{\frac{G}{2}})} \\
&= 2\sqrt{\frac{G}{2}}sn(2\frac{K}{\pi}\theta,\sqrt{\frac{G}{2}})dn(2\frac{K}{\pi}\theta,\sqrt{\frac{G}{2}})
\end{aligned}
$$

$$
\Rightarrow \quad \frac{\partial p_2}{\partial \theta} = -\frac{1}{G'}\frac{\partial H}{\partial q_2}
$$

$$
\begin{aligned}
\Rightarrow \quad \frac{\partial(q_2,p_2)}{\partial(\theta,I)} &= \frac{\partial q_2}{\partial \theta}\frac{\partial p_2}{\partial I} - \frac{\partial p_2}{\partial \theta}\frac{\partial q_2}{\partial I} \\
&= \frac{1}{G'}\frac{\partial H}{\partial p_2}\frac{\partial p_2}{\partial I} + \frac{1}{G'}\frac{\partial H}{\partial q_2}\frac{\partial q_2}{\partial I}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial(q_2,p_2)}{\partial(\theta,I)} &= \frac{1}{G'}\frac{dH}{dI} \\
&= \frac{1}{G'}\frac{dG}{dI} \\
&= 1.
\end{aligned}
$$

Therefore the Lemma is proved.

Explicitly finding an action_angle transformation is a difficult but essential step in the analytical method in [19].

Substituting (11) into (9) and using the first result in Lemma 2.1, H^γ becomes

$$
\begin{aligned}
H^\gamma(q,p,\theta,I) &= \frac{1}{2}p^2 + 1 - \cos q + G(I) + \\
&\frac{\gamma}{2}\left(q - 2\arcsin\left(\sqrt{\frac{G(I)}{2}}sn(\frac{2}{\pi}K(\sqrt{\frac{G(I)}{2}})\theta,\sqrt{\frac{G(I)}{2}})\right)\right)^2 \qquad (17)
\end{aligned}
$$

where $q = q_1$ and $p = p_1$ for simplicity.

Write

$$
\begin{aligned}
F(q,p) &= \frac{1}{2}p^2 + 1 - \cos q && (18) \\
H^\gamma(q,p,\theta,I) &= F(q,p) + G(I) + \frac{\gamma}{2}(q - q_2(\theta,I))^2 && (19) \\
H^0(q,p,I) &= F(q,p) + G(I) && (20)
\end{aligned}
$$

$$
\begin{aligned}
H^1(q,p,\theta,I) &= \frac{1}{2}(q-q_2(\theta,I))^2 && (21)\\
\Omega(I) &= \frac{\partial G}{\partial I}. && (22)
\end{aligned}
$$

Thus, from (19),

$$\frac{\partial H^\gamma}{\partial I} = \Omega(I) + \gamma\frac{\partial H^1}{\partial I}$$

Referring to (13), we have $\Omega(I) > 0$. Thus on each energy manifold with energy h, H^γ is invertible and can be solved for I if γ is sufficiently small. Then

$$I = L^\gamma(q,p,\theta,I;h) = L^0(q,p;h) + \gamma L^1(q,p,\theta,;h) + O(\gamma^2)$$

where

$$
\begin{aligned}
L^0 &= G^{-1}(h - F(q,p))\\
L^1 &= \frac{H^1(q,p,\theta,L^0(q,p,h))}{\Omega(L^0(q,ph))}\\
H^1(q,p,\theta,I) &= \frac{1}{2}(q-q_2(\theta,I))^2
\end{aligned}
$$

Under the action_angle transformation (11)- (13) , the equation (10) becomes

$$
\begin{aligned}
\dot{q} = \frac{\partial H^\gamma}{\partial p} \quad &, \quad \dot{p} = -\frac{\partial H^\gamma}{\partial q}\\
\dot{\theta} = \frac{\partial H^\gamma}{\partial I} \quad &, \quad \dot{I} = -\frac{\partial H^\gamma}{\partial \theta}
\end{aligned}
\tag{23}
$$

which is as in [19, page 213] with $q = q_1$ and $p = p_1$.

Let

$$
\begin{aligned}
q' &= \frac{\partial q}{\partial \theta} = \frac{\dot{q}}{\dot{\theta}} = \frac{\frac{\partial H^\gamma}{\partial p}}{\frac{\partial H^\gamma}{\partial I}}\\
p' &= \frac{\partial p}{\partial \theta} = \frac{\dot{p}}{\dot{\theta}} = -\frac{\frac{\partial H^\gamma}{\partial q}}{\frac{\partial H^\gamma}{\partial I}}.
\end{aligned}
\tag{24}
$$

Differentiating (19) implicitly gives

$$
\begin{aligned}
\frac{\partial H^\gamma}{\partial q} + \frac{\partial H^\gamma}{\partial I}\frac{\partial L^\gamma}{\partial q} &= 0\\
\frac{\partial H^\gamma}{\partial p} + \frac{\partial H^\gamma}{\partial I}\frac{\partial L^\gamma}{\partial p} &= 0.
\end{aligned}
\tag{25}
$$

The equations (24)-(25) imply

$$\begin{aligned} q' &= -\frac{\partial L^\gamma}{\partial p}(q,p,\theta;h) \\ p' &= -\frac{\partial L^\gamma}{\partial q}(q,p,\theta;h). \end{aligned} \tag{26}$$

These equations are called the reduced Hamiltonian system[19]. Substituting $L^\gamma(q,p,\theta,I;h)$ into (26) yields

$$\begin{aligned} q' &= -\frac{\partial L^0}{\partial p}(q,p;h) - \gamma\frac{\partial L^1}{\partial p}(q,p,\theta;h) + O(\gamma^2) \\ p' &= \frac{\partial L^0}{\partial q}(q,p;h) + \gamma\frac{\partial L^1}{\partial q}(q,p,\theta;h) + O(\gamma^2). \end{aligned} \tag{27}$$

The equations (27) have the form to which Melnikov's method can be applied.

2.2 Chaotic Motion

To study the chaotic motion of the equation(27), we cite Theorem 4.8.4 and Corollary 4.8.5 in [19, page 224].

Theorem 2.1 *Consider a two degree of freedom Hamiltonian of the form (19) and assume that F contains a homoclinic orbit (q^0,p^0) connecting a hyperbolic saddle to itself (or F has a homoclinic cycle). Suppose $\Omega(I) = G'(I) > 0$ for $I > 0$. Let $h^0 = F(q^0,p^0)$ be the energy of the homoclinic orbit and let $h > h^0$ and $I^0 = G^{-1}(h-h^0)$ be constants. Let $\{F,H^1\}(t+\theta_0)$ denote Poisson bracket of $F(q^0,p^0)$ and $H^1(q^0,p^0,\Omega(I^0)t+\theta_0;I^0)$ evaluated at q^0 and p^0. Define*

$$M(\theta_0) = \int_{-\infty}^{+\infty} \{F,H^1\}(t+\theta_0)\,dt$$

and assume that $M(\theta_0)$ is independent of γ. Then for $\gamma > 0$ sufficiently small the Hamiltonian system corresponding to (19) has transverse homoclinic orbits on the energy surface $H^\gamma = h$.

This Theorem implies that

Corollary 2.1 *The system has a hyperbolic invariant set Λ in its dynamics on the energy surface $H = h$; Λ has a dense orbit and thus the system has no global analytic second integral.*

Now we will apply this Theorem to our system (27). It has a homoclinic orbit on the cylinder,

$$F(q^0, p^0) = h^0 = 2$$

namely

$$\begin{aligned} q^0(t) &= \pm 2\arctan(\sinh(t)) \\ p^0(t) &= \pm 2\text{sech}(t). \end{aligned} \tag{28}$$

Let $h > 2$ and $L^0 = G^{-1}(h-2)$. The Poisson bracket of $F(q^0, p^0)$ and $H^1(q^0, p^0, \Omega(l^0)t + \theta^0; l^0)$ is

$$\{F, H^1\} = \frac{\partial F}{\partial q}\frac{\partial H^1}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial H^1}{\partial q}$$

$$H^1(q,p,\theta,I) = \frac{1}{2}(q - q_2(\theta, I))^2 \qquad \frac{\partial H^1}{\partial p} = 0$$

$$\Rightarrow \ \{F, H^1\} = -\frac{\partial F}{\partial p}\frac{\partial H^1}{\partial q} = -p\ (q - q_2(\theta, I))$$

$$\Rightarrow \ \{F, H^1\} = -(q\dot{q} - pq_2(\theta, I))$$

$$\begin{aligned} M(\theta_0) &= \int_{-\infty}^{+\infty} \{F, H^1\}(t + \theta_0)\, dt \\ &= \int_{-\infty}^{+\infty} -(q^0\dot{q}^0 - p^0 q_2(\Omega(l^0)t + \theta_0, l^0))\, dt \\ &= -\left(\frac{1}{2}q^{02}\ |_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} (\pm 2\text{sech}(t))q_2(\Omega(l^0)t + \theta_0, l^0)\, dt\right) \\ &= \pm 2\int_{-\infty}^{+\infty} \text{sech}(t)\ q_2(\Omega(l^0)t + \theta_0, l^0)\, dt \end{aligned}$$

$$\begin{aligned} \frac{\partial M(\theta_0)}{\partial \theta_0} &= \pm 2\int_{-\infty}^{+\infty} \text{sech}(t)\ \frac{\partial q_2(\Omega(l^0)t + \theta_0, l^0)}{\partial \theta_0}\, dt \\ &= \pm 2\int_{-\infty}^{+\infty} \text{sech}(t)\ 2\sqrt{\frac{G(l^0)}{2}}\frac{1}{G'(l^0)} cn\Big(2\frac{K(\sqrt{\frac{G(l^0)}{2}})}{\pi}(\Omega(l^0)t + \theta_0), \sqrt{\frac{G(l^0)}{2}}\Big)\, dt \\ &= \pm 2\sqrt{2G(l^0)}\frac{1}{G'(l^0)}\int_{-\infty}^{+\infty} \text{sech}(t)\, cn\Big(2\frac{K(\sqrt{\frac{G(l^0)}{2}})}{\pi}(\Omega(l^0)t + \theta_0), \sqrt{\frac{G(l^0)}{2}}\Big)\, dt \end{aligned}$$

$$M(0) = \pm 2 \int_{-\infty}^{+\infty} \mathrm{sech}(t)\, q_2(\Omega(I^0)t, I^0)\, dt.$$

By (11), $q_2(-\theta, I) = -q_2(\theta, I)$ and so the integrand is odd in t.

$$M(0) = 0$$

Now we prove that $\theta_0 = 0$ is simple. In general, this is not easy. Writing $m = \sqrt{\frac{G(I^0)}{2}}$; then substituting (14) into $\frac{\partial M(\theta_0)}{\partial \theta_0}$, we have

$$\begin{aligned}
\frac{\partial M(\theta_0)}{\partial \theta_0}\Big|_{\theta_0=0} &= \pm 2\sqrt{2G(I^0)}\frac{1}{G'(I^0)} \\
&\quad \int_{-\infty}^{+\infty} \mathrm{sech}(t)\, \frac{2\pi}{mK(m)} \sum_{s=0}^{\infty} \nu_s \cos(\frac{(2s+1)\pi}{2K(m)} 2\frac{K(m)}{\pi}\Omega(I^0)t)\, dt \\
&= \pm 2\sqrt{2G(I^0)}\frac{1}{G'(I^0)}\frac{2\pi}{mK(m)} \\
&\quad \sum_{s=0}^{\infty} \nu_s \int_{-\infty}^{+\infty} \mathrm{sech}(t)\, \cos(\frac{(2s+1)\pi}{2K(m)} 2\frac{K(m)}{\pi}\Omega(I^0)t)\, dt.
\end{aligned}$$

However,

$$\begin{aligned}
\int_{-\infty}^{+\infty} \mathrm{sech}(t) \cos \frac{(2s+1)\pi}{2K(m)} 2\frac{K(m)}{\pi} \theta\, dt &= \pi \mathrm{sech}(\frac{\pi \frac{(2s+1)\pi}{2K(m)} 2 \frac{K(m)}{\pi} \Omega(I^0)}{2}) \\
&> 0.
\end{aligned}$$

Thus $\frac{\partial M(\theta_0)}{\partial \theta_0}\Big|_{\theta_0=0} > 0$ and $M(\theta_0)$ has a simple zero point $\theta_0 = 0$. From Theorem 2.1 and Corollary 2.1, we obtain

Main Theorem 2.1 *On any energy manifold of energy larger than 2, there exists a Smale horseshoe and there is no global analytic second integral.*

Chapter 3

Symmetry and Linearization

From the Main Theorem 2.1, the system (3) is not integrable when $\gamma \neq 0$. Thus it is very difficult to analyze the bifurcations at a bifurcation point $\gamma_0 \neq 0$ in terms of Poincaré map or averaging method, for both methods are based on integrability of the system. Therefore we will use group theory and singularity theory to explore the bifurcations of the system (3) when $\gamma_0 \neq 0$.

Golubitsky and Schaeffer [32] present a theory to explore bifurcations via singularity theory [34]. First the physical system is reduced to nonlinear algebraic equations via a standard technique such as Liapunov-Schmidt reduction. Then one computes some derivatives at a bifurcation point in order to put the bifurcation problem in one of the normal forms. The bifurcations can then be described using the normal form or its universal unfolding. This theory has also been generalized to dynamical systems with symmetry [33]. Systems with symmetry are common in the physical sciences, and it is well accepted that any reasonable model should yield governing equations that reflect this symmetry. This leads to equations with special invariance properties. It is natural to exploit the invariance, either to deliver special solutions or to reduce the basic problem. Golubitsky and Stewart [10] discuss Hopf bifurcation in the presence of symmetry . They also study some specific applications such as

Taylor-Couette flow [11] and Hopf bifurcation with dihedral group symmetry [12]. Vanderbauwhede studies subharmonic bifurcations in time–reversible systems [14]. Aronson, Golubitsky and Krupa discuss bifurcations in coupled arrays of Josephson junctions. They show that in-phase rotations lose their stability via fixed–point bifurcations and period–doubling bifurcation. In this chapter and in the next chapter, bifurcations from a T-periodic in-phase rotation will be investigated. The study is motivated by two facts. On the one hand, in certain applications such as microwave generators and parametric amplifiers [22], it is desired to operate their SQUID circuits in a stable synchronous (i.e, in-phase) oscillation. Thus it is of interest to determine where in parameter space the synchronous oscillations are stable. On the other hand, it is of interest to determine how the stability of the in-phase solution is lost, i.e., to what types of behaviour these bifurcations lead. Maginu [5] numerically studied bifurcations of the damped SQUID system. His results indicate that the in-phase rotation is unstable for certain range of the coupling strength. Out-phase rotations bifurcate from the in-phase rotation. In the unstable range, there is chaotic motion. Doedel, Aronson and Othmer [4, 6] studied the bifurcations of the system in both the damped and the undamped case. The system undergoes period-doubling bifurcations for suitable coupling strengths. 2T-periodic rotations bifurcate from the T-periodic rotation. In the damped case, the system gets to a chaotic motion via period-doubling cascades.

In Section 3.1, we introduce basic theory and methods used in the rest of the thesis. In Section 3.2, we first define certain function spaces. The bifurcation problems of the equation (3) are then equivalent to those of an equation $\Phi(U,\gamma) = 0$ in these spaces. Then we define actions on the spaces and prove that $\Phi(U,\gamma)$ commutes with these actions. In Section 3.3 we find the linearization of the equation (3) at an in-phase rotation and we describe the distribution of Floquet multipliers as the coupling strength varies.

Introduce variables r, s

$$r = \frac{1}{2}(\phi_1 - \phi_2)$$

$$s = \frac{1}{2}(\phi_1 + \phi_2). \tag{29}$$

Then (3) becomes

$$\begin{aligned} s'' + \sin s \cos r &= 0 \ , \\ r'' + \sin r \cos s &= -2\gamma r \ . \end{aligned} \tag{30}$$

Assume that we have a rotation:

$$\begin{aligned} r = 0 \ , \\ s = s_0(t;h) \ , \ \ s_0(0;h) = 0 \ , \\ s_0(-t;h) = -s_0(t;h) \ , \\ s_0(t+T;h) = s_0(t;h) + 2\pi \ , \end{aligned}$$

where the period $T = 4K(2/h)/\sqrt{2h}$.

For simplicity, we will not explicitly write h in $s_0(t;h)$ throughout the rest of the thesis.

Set

$$s \to s \ + \ s_0(t) \ , \qquad r \to r \ . \tag{31}$$

Then (30) is reduced to

$$\begin{aligned} s'' + \sin(s + s_0(t)) \cos r - \sin s_0(t) &= 0 \ , \\ r'' + \sin r \cos(s + s_0(t)) = -2r\gamma \ , \end{aligned} \tag{32}$$

where

$$s_0'' + \sin s_0 = 0 \ .$$

From (29) and (31), we have

$$\begin{aligned} \phi_1 &= s + r + s_0(t) \ , \\ \phi_2 &= s - r + s_0(t) \ . \end{aligned} \tag{33}$$

If (s, r) is a T-periodic solution of (32), then , from (33), (ϕ_1, ϕ_2) is a T-periodic rotation of the system (3).

3.1 Preliminaries

For completeness we introduce some terminology and results from group theory and singularity theory used in the thesis. The related material can be found in [32, 33].

Let $\mathcal{E}_n$ denote the space of all functions $g : R^n \to R^1$ that are defined and C^∞ on some neighborhood of the origin. Then $\mathcal{E}_n$ is a vector space. An *ideal* $\mathcal{V}$ in $\mathcal{E}_n$ is a linear subspace with the following special property:

if $\phi \in \mathcal{E}_n$ and $f \in \phi$, then $\phi f \in \mathcal{V}$.

If $p_1, \cdots, p_k$ are germs in $\mathcal{E}_n$, then the set of all linear combinations,

$$a_1 p_1 + \cdots + a_k p_k \ ,$$

where $a_i \in \mathcal{E}_n$, is an ideal in $\mathcal{E}_n$. We denote this ideal by $< p_1, \cdots, p_k >$, and we call $p_1, \cdots, p_k$ the *generators* of the ideal. Let

$$\mathcal{U} = \{ f \in \mathcal{E}_n : \ f(0) = 0 \ \} \ .$$

We claim that

$$\mathcal{U} =< x_1, x_2, \cdots, x_n > \ .$$

Lemma 3.1.1 *Let $\mathcal{V}$ and $\mathcal{W}$ be ideals in $\mathcal{E}_n$, and assume that $\mathcal{V} =< p_1, \cdots, p_k >$ is finitely generated. Then $\mathcal{V} \subseteq \mathcal{W}$ if and only if $\mathcal{V} \subseteq \mathcal{W} + \mathcal{U}\mathcal{V}$.*

Definition 1.1 Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. A bounded linear operator $L : \mathcal{X} \to \mathcal{Y}$ is called *Fredholm* if the following two conditions hold.

(i) $KerL$ is a finite-dimensional subspace of $\mathcal{X}$.

(ii) $RangeL$ is a closed subspace of $\mathcal{Y}$ of finite codimension.

Definition 1.2 If L is Fredholm, the *index* of L is the integer

$i(L)$ = dim Ker L - codim Range L.

Lemma 3.1.2 *If $L : \mathcal{X} \to \mathcal{Y}$ is Fredholm, then there exist closed subspaces M and N of $\mathcal{X}$ and $\mathcal{Y}$, respectively, such that*

$$\mathcal{X} = KerL \oplus M \ , \ \mathcal{Y} = N \oplus RangeL$$

3.1.1 Liapunov-Schmidt Reduction

Let

$$\Phi : \mathcal{X} \times R^{k+1} \rightarrow \mathcal{Y} \ , \qquad \Phi(0,0) = 0 \tag{34}$$

be a smooth mapping between Banach spaces. Let L be the differential of Φ at the origin. We assume that L is Fredholm of index zero. Then the Liapunov-Schmidt reduction procedure is as follows:

Step 1. Decompose $\mathcal{X}$ and $\mathcal{Y}$,

$$\begin{aligned} &(a) \quad \mathcal{X} = KerL \oplus M \\ &(b) \quad \mathcal{Y} = N \oplus RangeL \end{aligned} \tag{35}$$

Step 2. Split (34) into an equivalent equation pair,

$$\begin{aligned} (a) \qquad E\Phi(u,\alpha) &= 0 \\ (b) \quad (I-E)\Phi(u,\alpha) &= 0 \end{aligned} \tag{36}$$

where $E : \mathcal{Y} \rightarrow RangeL$ is the projection associated to the splitting (35b).
Step 3. Use (35a) to write $u = v + w$, where $v \in KerL$ and $w \in M$. Then

$$E\Phi(v + w(v,\alpha), \alpha) = 0 \tag{37}$$

Step 4. Define $\phi : KerL \times R^{k+1} \rightarrow N$ by

$$\phi(v,\alpha) = (I - E)\Phi(v + w(v,\alpha), \alpha) \tag{38}$$

Step 5. Choose a basis $v_1, \cdots, v_n$ for $KerL$ and a basis $v_1^*, \cdots, v_n^*$ for N. Define $g : R^n \times R^{k+1} \rightarrow R^n$ by

$$g_i(x,\alpha) = < v_i^*, \phi(x_1 v_1 + \cdots + x_n v_n, \alpha) > \tag{39}$$

Lemma 3.1.3 *If the linearization L is a Fredholm operator of index zero, then solutions of (34) are (locally) in one-to-one correspondence with solutions of the finite system*

$$g_i(x,\alpha)=0 \ , \qquad i=1,\dots,n$$

where g_i is defined by (39).

Let Γ be a Lie group. We say that Γ *acts* on the Banach spaces $\mathcal{Y}$ if for each $\gamma\in\Gamma$ there is an associated invertible linear map $R_\gamma:\mathcal{Y}\to\mathcal{Y}$ with the property that for all $\gamma,\delta\in\Gamma$

$$R_{\gamma\delta}=R_\gamma\circ R_\delta$$

We say that a mapping $\mathbf{\Phi}:\mathcal{X}\to\mathcal{Y}$ *commutes* with the action of Γ on $\mathcal{Y}$ if $\mathcal{X}$ is an invariant subspace of $\mathcal{Y}$ and the following holds for all $\gamma\in\Gamma$, $u\in\mathcal{X}$

$$\mathbf{\Phi}(R_\gamma u)=R_\gamma\mathbf{\Phi}(u)$$

Now we assume that

(a) Γ acts on $\mathcal{Y}$.

(b) $\mathbf{\Phi}:\mathcal{X}\to\mathcal{Y}$ commutes with Γ.

Lemma 3.1.4 *In the Liapunov-Schmidt reduction, if M and N in (35) are invariant subspaces, then the mapping*

$$\phi: KerL\times R^{k+1}\to N$$

defined by (38) commutes with the action of Γ; in symbols

$$\phi(\gamma v,\alpha)=\gamma\phi(v,\alpha) \tag{40}$$

Remark 3.1 Let $w(v,\alpha)$ be the solution in *Step* 3. Then $w(\gamma v,\alpha)=\gamma w(v,\alpha)$.

3.2 Symmetry

Fix the in-phase solution in (32). Define

$$U=\begin{bmatrix} s \\ r \end{bmatrix}$$

$$\Phi(U,\gamma)(t) = \begin{bmatrix} s''(t) + \sin(s(t)+s_0(t))\cos r(t) - \sin s_0(t) \\ r''(t) + \sin r(t)\cos(s(t)+s_0(t)) + 2\gamma r(t) \end{bmatrix} \tag{41}$$

$$\Phi(0,\gamma) = 0. \tag{42}$$

Introduce function spaces:

$$\begin{aligned} C^k_{odd}[0,T] &= \{U(t) \in R^2 \mid t \in R^1, U(t+T) = U(t),\ U(-t) = -U(t)\ , \\ &\quad \textit{and } U(t) \textit{ has } k \textit{ continuous derivatives}\} \\ C^k[0,T] &= \{U(t) \in R^2 \mid t \in R^1, U(t+T) = U(t), \\ &\quad \textit{and } U(t) \textit{ has } k \textit{ continuous derivatives}\}. \end{aligned} \tag{43}$$

Define actions on $C^k[0,T]$:

$$(\alpha U)(t) = U(t+T) \tag{44}$$

$$(\beta U)(t) = BU(t+T) \tag{45}$$

$$(\sigma U)(t) = -U(-t) \tag{46}$$

where

$$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Lemma 3.2.1

$$\Phi(\alpha U,\gamma) = \alpha\Phi(U,\gamma)$$

$$\Phi(\beta U,\gamma) = \beta\Phi(U,\gamma)$$

$$\Phi(\sigma U,\gamma) = \sigma\Phi(U,\gamma).$$

PROOF: Let

$$U = \begin{bmatrix} s \\ r \end{bmatrix} \qquad U_1 = \begin{bmatrix} s_1 \\ r_1 \end{bmatrix} = \alpha U$$

Then $\quad U_1(t) = \alpha U(t) = U(t+T)$, and so $s_1(t) = s(t+T)$, $r_1 = r(t+T)$.

$$
\begin{aligned}
\Phi(\alpha U,\gamma)(t) &= \Phi(U_1,\gamma)(t) \\
&= \begin{bmatrix} \frac{d^2}{dt^2}s_1(t) + \sin(s_1(t)+s_0(t))\cos r_1(t) - \sin s_0(t) \\ \frac{d^2}{dt^2}r_1(t) + \sin r_1(t)\cos(s_1(t)+s_0(t)) + 2\gamma r_1(t) \end{bmatrix} \\
&= \begin{bmatrix} \frac{d^2}{dt^2}s(t+T) + \sin(s(t+T)+s_0(t))\cos r(t+T) - \sin s_0(t) \\ \frac{d^2}{dt^2}r(t+T) + \sin r(t+T)\cos(s(t+T)+s_0(t)) + 2\gamma r(t+T) \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
U \in C^k[0,T] \quad &\Rightarrow \quad U(t+T) = U(t) \\
&\Rightarrow \quad \frac{d^2}{dt^2}U(t+T) = U''(t) \quad U''(t+T) = U''(t) \\
&\Rightarrow \quad \Phi(\alpha U,\gamma)(t) = \Phi(U,\gamma)(t).
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
(\alpha\Phi(U,\gamma))(t) &= \begin{bmatrix} s''(t+T) + \sin(s(t+T)+s_0(t+T))\cos r(t+T) - \sin s_0(t+T) \\ r''(t+T) + \sin r(t+T)\cos(s(t+T)+s_0(t+T)) + 2\gamma r(t+T) \end{bmatrix} \\
&= \begin{bmatrix} s''(t) + \sin(s(t)+s_0(t)+2\pi)\cos r(t) - \sin(s_0(t)+2\pi) \\ r''(t) + \sin r(t)\cos(s(t)+s_0(t)+2\pi) + 2\gamma r(t) \end{bmatrix} \\
&= \Phi(U,\gamma)(t)
\end{aligned}
$$

$$
\begin{aligned}
(\alpha\Phi(U,\gamma))(t) &= (\Phi(\alpha U,\gamma))(t) \ \forall t \in R^1. \\
\Rightarrow \quad \alpha\Phi(U,\gamma) &= \Phi(\alpha U,\gamma).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
(\Phi(\beta U,\gamma))(t) &= \begin{bmatrix} s''(t) + \sin(s(t)+s_0(t))\cos -r(t) - \sin s_0(t) \\ -r''(t) - \sin r(t)\cos(s(t)+s_0(t)) - 2\gamma r(t) \end{bmatrix} \\
&= B(\Phi(U,\gamma)(t)) \\
&= (\beta\Phi(U,\gamma))(t)
\end{aligned}
$$

$$(\Phi(\sigma U,\gamma))(t) = \begin{bmatrix} -\frac{d^2}{dt^2}s(-t) + \sin(-s(-t)+s_0(t))\cos -r(-t) - \sin s_0(t) \\ -\frac{d^2}{dt^2}r(-t) + \sin -r(-t)\cos(-s(-t)+s_0(t)) + 2\gamma(-r(-t)) \end{bmatrix}$$

$$\begin{aligned}(\Phi(\sigma U,\gamma))(t) &= \begin{bmatrix} -s''(-t) - \sin(s(-t)+s_0(-t))\cos r(-t) - (-\sin s_0(-t)) \\ -r''(-t) - \sin r(-t)\cos(s(-t)+s_0(-t)) - 2\gamma r(-t) \end{bmatrix} \\ &= -\Phi(U,\gamma)(-t) \\ &= (\sigma\Phi(U,\gamma))(t).\end{aligned}$$

Furthermore, if

$$U(-t) = -U(t)$$

then

$$U''(-t) = -U''(t).$$

From the above we get $\Phi(U,\gamma)(-t) = -\Phi(U,\gamma)(t)$. Thus

$$\Phi : \quad C^k_{odd}[0,T] \to C^{k-2}_{odd}[0,T] \quad k \geq 2 \tag{47}$$

3.3 Linearization

Linearize (32) about $s = r = 0$:

$$s'' + \cos s_0(t)s = 0 \tag{48}$$

$$r'' + (\cos s_0(t) + 2\gamma)r = 0. \tag{49}$$

From

$$s_0'' + \sin s_0 = 0$$

it follows that

$$s_0''' + \cos s_0 s_0' = 0$$

and

$$s_0'(t) = \sqrt{2h}\, dn(\frac{t}{\sqrt{2h}}, \frac{2}{h}) > 0 \tag{50}$$

Thus $s_0(t)$ is a solution to (48) , and $s_0'(t+T) = s_0'(t)$. Up to a multiplicative constant, $s_0'(t)$ is the only periodic solution of (48).

Lemma 3.3.1 *$s_0'(t)$ is the only periodic solution of the equation (48) and it is even.*

PROOF:

$$s_0(-t) = -s_0(t) \Rightarrow \quad s_0'(-t) = s_0'(t)$$

Substitute $s = s_0'(t)y$ into (48) to get

$$s_0'''(t)y + 2s_0''(t)y' + s_0'(t)y'' + \cos s_0 s_0' y = 0$$

$$s_0'(t)y'' + 2s_0''(t)y' + (s_0''' + \cos s_0)y = 0$$

$$\begin{aligned} y'' &= -\frac{2s_0''}{s_0'}y' \\ y' &= C\exp\left(-\int \frac{2s_0''}{s_0'}dt\right) \\ &= C\exp\left(-\ln s_0'^2\right) \\ &= \frac{C}{s_0'^2(t)} \geq \frac{C}{max s_0'^2} \end{aligned}$$

$$\Rightarrow \quad y(t) \geq \frac{C}{max s_0'^2}t \rightarrow \infty \quad as \ t \ \rightarrow \infty.$$

Thus $y(t)$ is unbounded and hence $y(t)$ can not be periodic. Therefore $s_0'(t)$ is the only periodic solution up to a multiplicative constant.

The equation (49) is a Hill equation. It can be transformed into a standard Hill equation [28] as follows. Let

$$t = \frac{T}{\pi}\theta$$

Then

$$\frac{d}{dt} = \frac{d}{d\theta} = \frac{\pi}{T}\frac{d}{d\theta} \qquad\qquad \frac{d^2}{dt^2} = (\frac{\pi}{T})^2\frac{d^2}{d\theta^2}$$

and (49) becomes

$$(\frac{\pi}{T})^2\frac{d^2}{d\theta^2}r + (\cos s_0(\frac{T}{\pi}\theta) + 2\gamma)r = 0$$

or

$$\frac{d^2}{d\theta^2}r + ((\frac{T}{\pi})^2 2\gamma + (\frac{T}{\pi})^2\cos s_0(\frac{T}{\pi}\theta))r = 0 \qquad (51)$$

For the existence and distribution of the Floquet multipliers of the equation (51), we have the following theorem.

Theorem 3.3.1 *Consider the equation* $\ddot{y}+(a-\phi(t))y=0$, $\phi(t+\pi)=\phi(t)$, *where* a *is constant and* $\phi(t)$ *is assumed real and continuous. Then there exist two sequences* $a_0 < a_1 \le a_2 \le \cdots, a_1^* \le a_2^* \le a_3^* \le \cdots$ *of real numbers,* $a_k, a_k^* \to \infty$ *as* $k \to \infty$,

$$a_0 < a_1^* \le a_2^* < a_1 \le a_2 < a_3^* \le a_4^* < a_3 \le a_4 < \cdots,$$

such that this equation has a periodic solution of least period π *(or* 2π*) if and only if* $a = a_k$ *for some* $k = 0,1,2\ldots$ *(or* a_k^* *for some* $k = 0,1,2\ldots$*). This solution is stable in the intervals*

$$(a_0, a_1^*) \ , \ (a_2^*, a_1) \ , \ (a_2, a_3^*) \ , \ (a_4^*, a_3)$$

and unstable in the intervals

$$(-\infty, a_0) \ , \ (a_1^*, a_2^*) \ , \ (a_1, a_2) \ , (a_3^*, a_4^*) \ , (a_3, a_4) \ldots$$

The solution is stable at a_{2k+1} *or* a_{2k+2} *(or* a_{2k+1}^* *or* a_{2k+2}*) if and only if* $a_{2k+1} = a_{2k+2}$ *(or* $a_{2k+1} = a_{2k+2}$*),* $k \ge 0$.

This Theorem is given in [28, page 128]. The intervals $[a_{2n-1}, a_{2n}]$ and $[a_{2n-1}^*, a_{2n}^*]$ are called finite instability intervals.

Theorem 3.3.2 *With the assumptions in Theorem 3.3.1, there exists just one finite instability interval if and only if*

$$\ddot{\phi}(t) = 3\phi^2(t) + \alpha\phi(t) + \beta$$

where α *and* β *are constants.*

This Theorem is from [21]. Using the two Theorems in the above, we can prove

Corollary 3.3.1 *With the assumptions and notations in Theorem 3.3.1,*

1. $\gamma_0 = 0$,

2. *there exists exactly one finite instability interval.*

PROOF:

1. The T-periodic function $r = s_0'(t)$ is a solution of the equation (49) when $\gamma = 0$ because in this case (49) reduces to the equation (48). Thus $\gamma = 0$ is a eigenvalue of the equation (49) with the boundary condition $r(0) = r(T)$ and $r'(0) = r'(T)$. The corresponding eigenfunction $s_0'(t)$ is strictly positive from (50). From the Sturm-Liouville theory it follows that 0 is the smallest eigenvalue of this equation, and therefore $\gamma_0 = 0$.

2. Let $\phi(t) = -\cos s_0(t)$ for the equation (49). If the condition in Theorem 3.3.2 is satisfied, then the Corollary is proved. In fact,

$$\begin{aligned}
\phi'(t) &= \sin s_0(t) s_0'(t) && (52)\\
\phi''(t) &= \cos s_0(t) s_0'^2(t) + \sin s_0(t) s_0''(t) && (53)\\
&= \cos s_0(t) s_0'^2(t) - \sin^2 s_0(t) && (54)\\
&= \cos s_0(t) s_0'^2(t) - 1 + \cos^2 s_0(t) && (55)\\
&= \cos s_0(t) s_0'^2(t) - 1 + \phi^2(t) && (56)\\
&= -\phi(t) s_0'^2(t) - 1 + \phi^2(t). && (57)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{1}{2} s_0'^2(t) + 1 - \cos s_0(t) &= h && (58)\\
s_0'^2(t) &= 2((h-1) + \cos s_0(t)) && (59)\\
&= 2(h-1) - 2\phi(t). && (60)
\end{aligned}$$

Substitute (60) into (57) to get

$$\begin{aligned}
\phi''(t) &= -2(h-1)\phi(t) + 2\phi^2(t) - 1 + \phi^2(t) && (61)\\
&= 3\phi^2(t) - 2(h-1)\phi(t) - 1. && (62)
\end{aligned}$$

From the numerical computations in [4], it is found that the only finite instability interval is $[a_1^*, a_2^*]$. Therefore the distribution of Floquet multipliers of (49) is as shown in Figure 1.

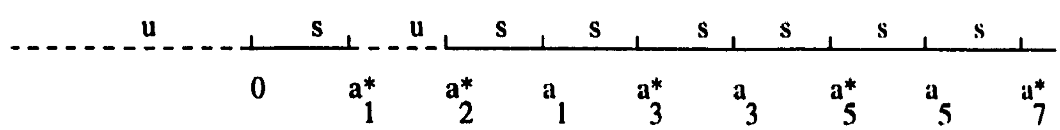

Figure 1: Distribution of Floquet multipliers

Chapter 4

Bifurcations When Strongly Coupled

In this Chapter we deal with bifurcations from the in-phase solutions when γ is not small. Numerical results [6] indicate that there are period-doubling bifurcations and that 2T-periodic rotations bifurcate from the T-periodic rotation. These bifurcations are theoretically confirmed in our analysis.

From the discussion of the distribution of Floquet multipliers in the last Chapter, we classify the investigation into two cases.

- Bifurcations when Floquet multipliers are -1.
- Bifurcations when Floquet multipliers are $+1$.

Bifurcations with Floquet multipliers -1 or $+1$ are called period-doubling and fixed-point bifurcations, respectively. In Section 4.1, some lemmas are presented to show the linearization is Fredholm of index zero. It follows that the Liapunov-Schmidt approach can be applied to our SQUIDs. In Section 4.2, period-doubling bifurcation is discussed. It is found that 2T-periodic out-of-phase rotations bifurcate from the T-periodic in-phase rotations. Depending on the solutions of the linearization, the system exhibits different kinds of bifurcations. The system has Hopf bifurcation if the linearization has odd 2T-periodic solutions. If the linearization has no odd 2T-periodic solutions, the system has a degenerate $\mathbf{Z_2} \oplus \mathbf{Z_2}$ -bifurcation. In this case, the nonlinear $\mathbf{Z_2} \oplus \mathbf{Z_2}$ -symmetry bifurcation equation is shown to be degenerate and under certain conditions it is equivalent to a simplified normal form. The bifurcation diagrams are presented. In Section 4.3, the fixed-point bifurcation is studied. It is shown that T-periodic out-of-phase rotations bifurcate from the T-periodic in-phase rotations. Normal form and bifurcation diagrams are presented. They are similar to the ones in the Section 4.2.

4.1 Some Lemmas

This section presents several lemmas. They will be used to prove that the linearization L is Fredholm of index zero. Consider

$$\ddot{x} + \phi(t)x = 0 \tag{63}$$

where

$$\phi(t+T) = \phi(t) \quad T > 0 \quad \phi(-t) = \phi(t) \quad x \ , \ t \in R^1$$

Lemma 4.1.1 *There exist two solutions $x_1(t)$ and $x_2(t)$ such that*

1. $x_1(-t) = x_1(t) \qquad x_2(-t) = -x_2(t)$
2. $x_1(t)x_2'(t) - x_1'(t)x_2(t) = 1$
3. $x_1(t \pm T) = x_1(T)x_1(t) \pm x_1'(T)x_2(t)$
   $x_2(t \pm T) = \pm x_2(T)x_1(t) + x_2'(T)x_2(t)$
4. $x_1(T) = x_2'(T)$.

PROOF:
If $x(t)$ is a solution, then $\frac{d^2}{dt^2}x(-t) + \phi(-t)x(-t) = 0$. Thus $\ddot{x}(-t) + \phi(t)x(-t) = 0$ because $\phi(t)$ is even and $\frac{d^2}{dt^2}x(-t) = \ddot{x}(-t)$. Hence $x(-t)$ is a solution.

Assume that $x_1(t)$ and $x_2(t)$ are the solutions of (63) with conditions:

$$x_1(0) = 1 \ , \ x_1'(0) = 0 \ , \ x_2(0) = 0 \ , \ x_2'(0) = 1$$

Then we have the following results:

1. Let $z_1(t) = x_1(-t)$, $z_2(t) = -x_2(-t)$, then

$$z_1(0) = x_1(0) = 1 \quad , \quad z_1'(0) = -x_1'(0) = 0$$

$$\Rightarrow \quad z_1(t) = x_1(t) \quad \Rightarrow \quad x_1(-t) = x_1(t)$$

and

$$z_2(0) = -x_2(0) = 0 \quad , \quad z_2'(0) = x_2'(0) = 1$$

$$\Rightarrow \quad z_2(t) = x_2(t) \quad \Rightarrow \quad x_2(-t) = -x_2(t)$$

.

2.

$$\begin{aligned} &\frac{d}{dt}(x_1(t)x_2'(t) - x_1'(t)x_2(t)) \\ &= x_1(t)x_2''(t) - x_2(t)x_1''(t) = x_1(t)(-\phi(t)x_2(t)) - (\phi(t)x_1(t))x_2(T) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} &x_1(t)x_2'(t) - x_1'(t)x_2(t) = x_1(0)x_2'(0) - x_1'(0)x_2(0) \\ &= 1. \end{aligned}$$

3. Both $x_1(t \pm T)$ and $x_2(t \pm T)$ are solutions of (63) because $\phi(t)$ is T-periodic. Thus

$$x_1(t \pm T) = c_1 x_1(t) + c_2 x_2(t)$$

At $t = 0$

$$x_1(\pm T) = c_1 \ , \ \Rightarrow \ c_1 = x_1(T)$$

$$\pm x_1'(T) = x_1'(0 \pm T) = c_1 x_1'(0) + c_2 x_2'(0) = c_2.$$

Similarly

$$x_2(\pm T) = c_1 \ , \ \Rightarrow \ c_1 = \pm x_2(T)$$

$$x_2'(\pm T) = c_2 \ , \ \Rightarrow \ c_2 = x_2'(T).$$

4. From result 3,

$$1 = x_1(T-T) = x_1(T)x_1(T) - x_1'(T)x_2(T) = x_1^2(T) - x_1'(T)x_2(T) \tag{64}$$

$$0 = x_2(T-T) = -x_2(T)x_1(T) + x_2'(T)x_2(T). \tag{65}$$

From result 2,

$$x_1(T)x_2'(T) - x_1'(T)x_2(T) = 1 \tag{66}$$

From (65), $x_2(T)(x_2'(T) - x_1(T)) = 0.$

Thus $x_2(T) = 0$ or $x_1(T) = x_2'(T).$

If $x_2(T) = 0$, then by (64) and (66),

$$x_1^2(T) = 1 \ , \ x_1(T)x_2'(T) = 1$$

$$x_1(T) = \pm 1 \quad \Rightarrow \quad \pm x_2'(T) = 1 \quad x_2'(T) = \pm 1$$

$$x_1(T) = x_2'(T)$$

Now we investigate the Floquet multipliers of the equation (63). Given $x_1(t)$ and $x_2(t)$ in Lemma 4.1.1, the fundamental solution matrix $X(t)$ is

$$X(t) = \begin{bmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{bmatrix}$$

$$X(T) = \begin{bmatrix} x_1(T) & x_2(T) \\ x_1'(T) & x_2'(T) \end{bmatrix}$$

The characteristic equation is

$$\rho^2 - (x_1(T) + x_2'(T))\rho + 1 = 0$$

By (4) in Lemma 4.1.1, $x_1(T) = x_2'(T)$

$$\rho^2 - 2x_1(T)\rho + 1 = 0$$

$$\rho = \frac{2x_1(T) \pm \sqrt{4x_1^2(T) - 4}}{2} = x_1(T) \pm \sqrt{x_1^2(T) - 1}$$

We have four different cases.

(i) If $|\, x_1(T) \,| > 1$, then ρ_1 , ρ_2 are real. One of $|\, \rho_1 \,|$, $|\, \rho_2 \,|$ is smaller than 1, the other is larger than 1.

(ii) If $| x_1(T) | < 1$ then ρ is complex, and $| \rho | = 1$.

(iii) If $\quad x_1(T) = 1 = x_2'(T) \quad$ then $\quad \rho_1 = \rho_2 = 1 \quad$ and

$$X(T) = \begin{bmatrix} 1 & x_2(T) \\ x_1'(T) & 1 \end{bmatrix}$$

and

$$1 - x_2(T)x_1'(T) = 1 \qquad \text{thus} \qquad x_1'(T)x_2(T) = 0$$

and hence $\quad x_1'(T) = 0 \quad$ or $\quad x_2(T) = 0$

If $x_1'(T) = 0$, then $x_1(t+T) = x_1(t)$ because $x_1(T) = 1$. Similarly, $x_2(t+T) = x_2(t)$ if $x_2(T) = 0$. In this case there exists at least one T-periodic solution, which is odd or even.

(iv) If $\quad x_1(T) = -1 = x_2'(T) \quad$ then $\quad \rho_1 = \rho_2 = -1 \quad$ and

$$X(T) = \begin{bmatrix} -1 & x_2(T) \\ x_1'(T) & -1 \end{bmatrix}$$

and

$$(-1)(-1) - x_2(T)x_1'(T) = 1 \qquad \text{thus} \qquad x_1'(T)x_2(T) = 0$$

and hence $\quad x_1'(T) = 0 \quad$ or $\quad x_2(T) = 0$.

If $x_1'(T) = 0$, then $x_1(t+T) = -x_1(t)$ because $x_1(T) = -1$. Similarly, $x_2(t+T) = -x_2(t)$ if $x_2(T) = 0$. Thus $x_1(t+2T) = -x_1(t+T) = x_1(t)$ and $x_2(t+2T) = x_2(t)$. In this case, there exists at least one 2T-periodic solution, which is odd or even.

In summary, we have

Lemma 4.1.2 *1. There exists at least one T-periodic solution , which is even or odd if the Floquet multipliers are* $+1$.
2. There exists at least one 2T-periodic solution , which is even or odd if the Floquet multipliers are -1.

Consider the nonhomogeneous equation corresponding to (63)

$$\ddot{x} + \phi(t)x = f(t) \tag{67}$$

where $f(t)$ is continuous. The solution of this equation is given by

$$x(t) = x_1(t)\left(x_0 - \int_0^t x_2(s)f(s)\,ds\right) + x_2(t)\left(x_0' + \int_0^t x_1(s)f(s)\,ds\right) \qquad (68)$$

where $x_1(s)$ and $x_2(s)$ are the solutions in Lemma 4.1.1.
In fact

$$\begin{aligned} x'(t) &= x_1'(t)\left(x_0 - \int_0^t x_2(s)f(s)\,ds\right) + x_1(t)(-x_2(t)f(t)) + \\ &\quad x_2'(t)\left(x_0' - \int_0^t x_1(s)f(s)\,ds\right) + x_2(t)(x_1(t)f(t)) \\ &= x_1'(t)\left(x_0 - \int_0^t x_2(s)f(s)\,ds\right) + x_2'(t)\left(x_0' + \int_0^t x_1(s)f(s)\,ds\right). \qquad (69) \end{aligned}$$

$$\begin{aligned} x''(t) &= x_1''(t)\left(x_0 - \int_0^t x_2(s)f(s)\,ds\right) + x_1'(t)(-x_2(t)f(t)) + \\ &\quad x_2''(t)\left(x_0' - \int_0^t x_1(s)f(s)\,ds\right) + x_2'(t)(x_1(t)f(t)) + \\ &= -\phi(t)x_1(t)\left(x_0 - \int_0^t x_2(s)f(s)\,ds\right) - \phi(t)x_2(t)\left(x_0' + \int_0^t x_1(s)f(s)\,ds\right) + \\ &\quad f(t)(x_1(t)x_2'(t) - x_2(t)x_1'(t)) \\ &= -\phi(t)x(t) + f(t). \end{aligned}$$

Lemma 4.1.3 *If equation (63) has odd T-(or 2T-) periodic solutions then $x_2(t)$ given in Lemma 4.1.1 is a T-(or 2T-) periodic solution.*

PROOF: Assume that $x(t+T) = x(t)$ or $x(t+2T) = x(t)$, $x(-t) = -x(t)$. Then $x(t) = c_1x_1(t) + c_2x_2(t)$, and $c_1 = x(0) = 0$, so $c_2 = x'(0) \neq 0$ and $x_2 = x(t)/x'(0)$.

Lemma 4.1.4 *Assume that equation (63) has an odd T-periodic solution. Then the equation*

$$\ddot{x} + \phi(t)x = f(t)$$

has an odd T-periodic solution if f(t) satisfies

$$f(-t) = -f(t) \ , \ f(t+T) = f(t)$$

$$< f, x_2 > \stackrel{\text{def}}{=} \frac{1}{T}\int_0^T f(s)x_2(s)\,ds = 0.$$

PROOF: By Lemma (4.1.3), x_2 is T-periodic. Then from Lemma 4.1.1 and $x_1(T) = x_2'(T) = 1$,

$$x_1(t \pm T) = x_1(t) \pm x_1'(T)x_2(t)$$

$$x_2(t \pm T) = \pm x_2(T)x_1(t) + x_2(t).$$

Set $x_0 = 0$ in (68), then

$$x(t) = x_1(t)\left(-\int_0^t x_2(s)f(s)\,ds\right) + x_2(t)\left(x_0' + \int_0^t x_1(s)f(s)\,ds\right).$$

Since $x_1(t)$ is even and both $x_2(t)$ and $f(t)$ are odd, we have

$$\begin{aligned}
x(-t) &= x_1(-t)\left(-\int_0^{-t} x_2(s)f(s)\,ds\right) + x_2(-t)\left(x_0' + \int_0^{-t} x_1(s)f(s)\,ds\right) \\
&= x_1(t)\left(\int_0^t x_2(-s)f(-s)\,ds\right) - x_2(t)\left(x_0' - \int_0^t x_1(-s)f(-s)\,ds\right) \\
&= x_1(t)\left(\int_0^t x_2(s)f(s)\,ds\right) - x_2(t)\left(x_0' + \int_0^t x_1(s)f(s)\,ds\right) \\
&= -x(t)
\end{aligned}$$

which means that $x(t)$ is odd and $x(0) = 0$.

$$\begin{aligned}
x(T) &= x_1(T)\left(-\int_0^T x_2(s)f(s)\,ds\right) + x_2(T)\left(x_0' + \int_0^T x_1(s)f(s)\,ds\right) \\
&= 0 + x_2(0)\left(x_0' + \int_0^T x_1(s)f(s)\,ds\right) \\
&= 0.
\end{aligned}$$

Now from (69),

$$x'(T) = x_1'(T)\left(-\int_0^T x_2(s)f(s)\,ds\right) + x_2'(T)\left(x_0' - \int_0^T x_1(s)f(s)\,ds\right).$$

From the assumption that $\int_0^T x_2(s)f(s)\,ds = 0$ and $x_2'(T) = 1$ we get

$$x'(T) = x_0' - \int_0^T x_1(s)f(s)\,ds$$

$$\int_0^T x_1(s)f(s)\,ds \;=\; \int_0^T x_1(s)f(s)\,ds - x_1'(T)\int_0^T x_2(s)f(s)\,ds$$

$$
\begin{aligned}
&= \int_0^T f(s)(x_1(s) - x_1'(T)x_2(s))\,ds \\
&= \int_0^T f(s)x_1(s-T)\,ds \\
&= \int_{-T}^0 x_1(\tau)f(\tau+T)\,d\tau \\
&= \int_{-T}^0 x_1(\tau)f(\tau)\,d\tau \\
&= -\int_T^0 x_1(-\tau)f(-\tau)\,d\tau \\
&= -\int_0^T x_1(\tau)f(\tau)\,d\tau
\end{aligned}
$$

$$\Rightarrow \qquad \int_0^T x_1(s)f(s)\,ds = 0$$

$$\Rightarrow \quad x'(T) = x_0' \quad \Rightarrow \quad x(t+T) = x(t).$$

Corollary 4.1.1 *Assume that equation (63) has an odd 2T-periodic solution. Then the equation*

$$\ddot{x} + \phi(t)x = f(t)$$

has an odd 2T-periodic solution iff f(t) satisfies

$$f(-t) = -f(t) \quad , \quad f(t+2T) = f(t)$$

$$< f, x_2 > \stackrel{\text{def}}{=} \frac{1}{2T}\int_0^{2T} f(s)x_2(s)\,ds = 0.$$

PROOF: Replace T by 2T in the Lemma (4.1.4).

Lemma 4.1.5 *Suppose that there is an even T-periodic solution $\mu(t)$ of the equation (63) that satisfies $\mu(t) \neq 0$ for all $t \in R$. Then there exists an odd T- (or 2T-) periodic solution for the nonhomogeneous equation if $f(t)$ satisfies*

$$f(-t) = -f(t) \qquad f(t+T) = f(t) \quad (\text{ or } f(t+2T) = f(t)\ ).$$

PROOF: Let $x(t) = \mu(t)y(t)$. Then $\ddot{x} = \ddot{\mu}y + 2\dot{\mu}\dot{y} + \mu\ddot{y}$.

$$\ddot{\mu}y + 2\dot{\mu}\dot{y} + \mu\ddot{y} + \phi(t)\mu(t)y = f(t)$$

$$\mu\ddot{y} + 2\dot{\mu}\dot{y} = f(t)$$

$$\ddot{y} = -\frac{2\dot{\mu}}{\mu}\dot{y} + \frac{f(t)}{\mu}$$

Thus

$$\begin{aligned}
\dot{y} &= \exp\left(-2\int \frac{\dot{\mu}}{\mu}\,dt\right)\left(C + \int_0^t \exp\left(2\int \frac{\dot{\mu}}{\mu}\,dt\right) \frac{f(s)}{\mu(s)}\,ds\right) \\
&= \exp\left(-\ln \mu^2\right)\left(C + \int_0^t \exp\left(\ln \mu^2\right) \frac{f(s)}{\mu(s)}\,ds\right) \\
&= \frac{1}{\mu^2(t)}\left(C + \int_0^t \mu(s) f(s)\,ds\right).
\end{aligned}$$

$$\Rightarrow \quad y(t) = \int_0^t \frac{1}{\mu^2(\tau)}\left(C + \int_0^\tau \mu(s) f(s)\,ds\right) d\tau.$$

If $f(t+T) = f(t)$

$$\begin{aligned}
\dot{y}(t+T) &= \frac{1}{\mu^2(t+T)}\left(C + \int_0^{t+T} \mu(s) f(s)\,ds\right) \\
&= \frac{1}{\mu^2(t)}\left(C + \int_0^{t+T} \mu(s) f(s)\,ds\right)
\end{aligned}$$

$$\begin{aligned}
\dot{y}(t+T) - \dot{y}(t) &= \frac{1}{\mu^2(t)}\int_t^{t+T} \mu(s) f(s)\,ds = \frac{1}{\mu^2(t)}\int_0^{T} \mu(s) f(s)\,ds \\
&= \frac{1}{\mu^2(t)}\int_{-T/2}^{T/2} \mu(s) f(s)\,ds \\
&= 0
\end{aligned}$$

$$\Rightarrow \qquad \dot{y}(t+T) = \dot{y}(t).$$

Now

$$\begin{aligned}
y(-t) &= \int_0^{-t} \frac{1}{\mu^2(\tau)}\left(C + \int_0^\tau \mu(s) f(s)\,ds\right) d\tau \\
&= -\int_0^{t} \frac{1}{\mu^2(-\tau)}\left(C + \int_0^{-\tau} \mu(s) f(s)\,ds\right) d\tau \\
&= -\int_0^{t} \frac{1}{\mu^2(\tau)}\left(C - \int_0^{\tau} \mu(-s) f(-s)\,ds\right) d\tau \\
&= -\int_0^{t} \frac{1}{\mu^2(\tau)}\left(C + \int_0^{\tau} \mu(s) f(s)\,ds\right) d\tau \\
&= -y(t).
\end{aligned}$$

So $y(t)$ is odd. Also

$$\dot{y}(t+T) - \dot{y}(t) = 0$$
$$y(t+T) - y(t) = y(T) - y(0) = y(T)$$
$$y(T) = \int_0^T \frac{1}{\mu^2(\tau)} \left(C + \int_0^\tau \mu(s) f(s)\, ds \right) d\tau.$$

If we set $y(T) = 0$ then we can uniquely solve the constant C. Substitute this value in $y(t)$ to get an odd T-periodic solution. Similarly, if $f(t+2T) = f(t)$ then

$$\begin{aligned} \dot{y}(t+2T) - \dot{y}(t) &= \frac{1}{\mu^2(t)} \int_t^{t+2T} \mu(s) f(s)\, ds = \frac{1}{\mu^2(t)} \int_0^{2T} \mu(s) f(s)\, ds \\ &= \frac{1}{\mu^2(t)} \int_{-T}^{T} \mu(s) f(s)\, ds \\ &= 0 \end{aligned}$$

$$\Rightarrow \qquad \dot{y}(t+2T) = \dot{y}(t).$$

Also

$$y(t+2T) - y(t) = y(2T)$$
$$y(2T) = \int_0^{2T} \frac{1}{\mu^2(\tau)} (C + \int_0^\tau \mu(s) f(s)\, ds)\, d\tau.$$

Solving for C in $y(2T) = 0$, we obtain an odd 2T-periodic solution.

Consider the equations

$$\dot{x} = A(t)x \tag{70}$$
$$\dot{x} = A(t)x + f(t) \tag{71}$$

where $A(t)$ is 2×2 matrix and $f(t)$ $, x \in R^2$; $A(t+T) = A(t)$ and $f(t+T) = f(t)$. We cite the following Lemma from [28, page 146].

Lemma 4.1.6 *The equation (71) has a T-periodic solution if and only if* $\int_0^T y(t) f(t)\, dt = 0$ *for all T-periodic solutions* $y(t)$ *of the adjoint equation*

$$\dot{y} = -A^T(t) y \tag{72}$$

If $X(t)$ is the fundamental matrix of the equation (70) and $X(0) = I$, then $(X^{-1}(t))^T$ is the fundamental matrix of (72). Thus if equation (70) has Floquet multipliers $\rho_i(i = 1, 2, \ldots, m)$ then, from Linear Algebra, the adjoint equation (72) has Floquet multipliers $\rho_i^{-1}(i = 1, 2, \ldots, m)$. Thus we have

Lemma 4.1.7 *If the equation (70) has a simple Floquet multiplier +1(or -1), so does its adjoint equation (72).*

4.2 Period-doubling Bifurcation

Assume that the equation (49) has Floquet multipliers = -1 at $\gamma = \gamma_0 \neq 0$. Then, from Lemma 4.1.2, if it has one linearly independent 2T-periodic solution then this solution is even or odd. If it has two linearly independent 2T-periodic solutions, then one is even and the other is odd. We have two cases.

A1: there exists an odd 2T- periodic solution to equation (49).

A2: there is no odd 2T- periodic solution.

Assumption **A2** means that there exists a unique, even, linearly independent 2T-periodic solution to equation (49).

4.2.1 Hopf Bifurcation

Assume that **A1** holds. Consider the map in (47)

$$\Phi : \ C^2_{odd}[0, 2T] \ \to \ C^0_{odd}[0, 2T] \tag{73}$$

Introduce a norm on $C^k[0, T]$

$$\| U \| = \sum_{j=0}^{k} \max_{t \in [0,T]} | U^{(j)} | \tag{74}$$

where $U(t), V(t) \in R^2$ and $U^{(j)} = \frac{d^j}{dt^j}U(t)$. Linearizing Φ about $U = 0$ and $\gamma = \gamma_0$ gives rise to

$$LU \stackrel{\text{def}}{=} (d\Phi)U = \frac{d}{d\tau}\Phi(0+\tau U)\mid_{\tau=0}$$

$$\Phi(\tau U) = \begin{bmatrix} \frac{d^2}{dt^2}(\tau s) + \sin(\tau s + s_0)\cos(\tau r) - \sin s_0 \\ \frac{d^2}{dt^2}(\tau r) + \sin(\tau r)\cos(\tau s + s_0) + 2\gamma\tau r \end{bmatrix}$$

$$\begin{aligned} \frac{d\Phi(\tau U)}{d\tau}\mid_{\tau=0} &= \begin{bmatrix} \frac{d^2}{d\tau^2}(s) + \cos(\tau s + s_0)s\cos(\tau r) - r\sin(\tau s + s_0)\sin(\tau r) \\ \frac{d^2}{d\tau^2}(r) + r\cos(\tau r)\cos(\tau s + s_0) - \sin(\tau r)\sin(\tau s + s_0)s + 2\gamma r \end{bmatrix}_{\tau=0} \\ &= \begin{bmatrix} s'' + \cos s_0 s \\ r'' + (\cos s_0 + 2\gamma_0)r \end{bmatrix} \end{aligned}$$

$$LU = \begin{bmatrix} s'' + \cos s_0 s \\ r'' + (\cos s_0 + 2\gamma_0)r \end{bmatrix}. \tag{75}$$

Define

$$\begin{aligned} L_1 s &= s'' + \cos s_0 s \\ L_2 r &= r'' + (\cos s_0 + 2\gamma_0)r \\ LU &= \begin{bmatrix} L_1 s \\ L_2 r \end{bmatrix}. \end{aligned} \tag{76}$$

Lemma 4.2.1.1 **1.** *$C^k_{odd}[0,2T]$ is a Banach space with norm defined in (74).*

2. *$dim Ker(L) = 1$*

3. *$C^0_{odd}[0,2T] = Ker(L) \oplus Range(L)$*

4. *Range(L) is closed.*

5. *L is Fredholm of index 0.*

PROOF:

1. Suppose that $\{U_n\} \subseteq C^k_{odd}[0,2T]$ is a Cauchy series. So

$$\mid U^{(j)}_m(t) - U^{(j)}_n(t) \mid \leq \| U_m - U_n \|$$

Then $\{U_n(t)\}$ is a Cauchy series for any $t \in R^1$ and $U_n^{(j)}(t)$ converges uniformly to $U^{(j)}(t)$ $(j = 0, 1, 2, \ldots k)$ in R^1 Hence $U(t)$ has kth derivatives. Also

$$U_n(t + 2T) = U_n(t) \;\; U_n(-t) = -U_n(t)$$

As $n \to \infty$

$$U(t + 2T) = U(t) \;\; U(-t) = -U(t) \quad \Rightarrow \quad U(t) \in C^k_{odd}[0, 2T]$$

2. $L_1 s = 0$ has a linearly independent even T-periodic solution from Lemma 3.3.1. Thus it has only the zero solution in $C^2_{odd}[0, 2T]$. With **A1**, $L_2 r = 0$ has an odd 2T-periodic solution. By Lemma 4.1.3, one of them is $x_2(t)$ with $x_2(0) = 0$ and $x_2'(0) = 1$. It has only one linearly independent odd 2T-periodic solution. Therefore $dimKer(L) = 1$ and $Ker(L) = span(\begin{bmatrix} 0 \\ x_2(t) \end{bmatrix})$.

3. First we claim $\forall U, V \in C^2_{odd}[0, 2T]$

$$< U, LV > = < LU, V >$$

where

$$< U, V > = \frac{1}{2T} \int_0^T (U(s), V(s))\, ds$$

and $(U(s), V(s))$ is the inner product in R^2. Let

$$U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{aligned}
\int_0^{2T} u_1'' v_1\, dt &= \int_0^{2T} v_1\, du_1' = v_1 u_1' \,|_0^{2T} - \int_0^{2T} u_1' v_1'\, dt \\
&= -\int_0^{2T} v_1'\, du_1' = \int_0^{2T} v_1'' u_1\, dt \\
\Rightarrow \; < U, LV > &= \frac{1}{2T} \int_0^{2T} (u_1(v_1'' + \cos s_0 v_1) + u_2(v_2'' + (\cos s_0 + 2\gamma_0) v_2))\, dt \\
&= \frac{1}{2T} \left(\int_0^{2T} u_1 v_1''\, dt + \int_0^{2T} \cos s_0 u_1 v_1\, dt + \right. \\
&\quad \left. \int_0^{2T} u_2 v_2''\, dt + \int_0^{2T} (\cos s_0 + 2\gamma_0) u_2 v_2\, dt \right) \\
&= \frac{1}{2T} \int_0^{2T} (v_1(u_1'' + \cos s_0 u_1) + v_2(u_2'' + (\cos s_0 + 2\gamma_0) u_2))\, dt \\
&= < LU, V > .
\end{aligned}$$

$\forall V \in Ker(L)$, $\forall U \in C^K_{odd}[0,2T]$, we have

$$< LU, V >=< U, LV >=< U, 0 >= 0$$

Thus

$$Ker^T(L) \supseteq Range(L) \tag{77}$$

Now $\forall U \in Ker^T(L)$ we have

$$< U, V_0 >= 0 \tag{78}$$

where $V_0 = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} \in Ker(L)$. Let $U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $W = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$. Consider the equation $LW = U$, or

$$L_1 w_1 = u_1 \qquad L_2 w_2 = u_2$$

We have $u_1(-t) = -u_1(t)$, $u_1(t+2T) = u_1(t)$, because $U \in C^0_{odd}[0,2T]$. $L_1 s = 0$ has an even T-periodic solution $s = s_0'(t)$, and $s_0'(t) > 0$ $t \in R^1$. From Lemma 4.1.5, there exists a $w_1(t)$ such that

$$L_1 w_1(t) = u_1(t) \ , \ \ u_1(t+2T) = u_1(t) \ , \ \ u_1(-t) = -u_1(t) \ .$$

From (78),

$$0 =< U, V_0 >= \frac{1}{2T}\int_0^{2T} (u_1 \cdot 0 + u_2 x_2)\, dt$$

$$\Rightarrow \qquad \int_0^{2T} u_2 x_2 \, dt = 0$$

Thus the requirements of Corollary 4.1.1 are all satisfied. So there exists a solution $w_2(t)$ such that

$$L_2 w_2(t) = u_2(t) \ , \ \ u_2(t+2T) = u_2(t) \ , \ \ u_2(-t) = -u_2(t)$$

Thus $LW = U$ and $U \in Range(L)$. Hence $Ker^T(L) \subseteq Range(L)$. Therefore

$$Ker^T(L) = Range(L)$$

Because $dim Ker(L) = 1 < \infty$, we have

$$C^0_0[0,2T] = Ker(L) \oplus Ker^T(L) = Ker(L) \oplus Range(L)$$

4. $\forall\{U_n\} \subseteq Range(L)$ and

$$U_n \overset{\|\cdot\|}{\rightarrow} U \ , \ U \in C^0_{odd}[0, 2T]$$

we have

$$< U_n, V >= 0 \ \forall V \in Ker(L)$$

As $n \to \infty$ $< U, V >= 0$, thus $U \in Ker^T(L) = Range(L)$. Therefore $Range(L)$ is closed.

5. From the conclusions of 1, 2, 3 and 4, we have

$$dim Ker(L) = 1 \qquad Codim Range(L) = 1$$

$$index = dim Ker(L) - Codim Range(L) = 0$$

and $Range(L)$ is closed. Thus L is Fredholm of index 0.

In terms of Lemma 4.2.1.1. we can employ Liapunov-Schmidt reduction with symmetry to study the bifurcations of the system (see § 3.1).

Define

$$E : \ C^0_{odd}[0, 2T] \to Range(L)$$

Then $\Phi(U, \gamma) = 0$ is equivalent to

$$\begin{aligned} E\Phi(U, \gamma) &= 0 \\ (I - E)\Phi(U, \gamma) &= 0 \end{aligned} \qquad (79)$$

Furthermore the bifurcation equation is reduced to

$$\psi(V, \gamma) = 0 \ \psi \ , V \ \in Ker(L)$$

and it inherits the symmetries from the system (41). Take the basis of $Ker(L)$ to be

$$V_0 = \begin{bmatrix} 0 \\ r_2 \end{bmatrix}$$

Then the bifurcation equation is reduced to a nonlinear algebraic equation:

$$g(x,\gamma) \stackrel{\text{def}}{=} < \psi(xV_0,\gamma), V_0 >= 0 \tag{80}$$

$$\begin{aligned} g(\alpha x,\gamma) &= \alpha g(x,\gamma) \\ g(\beta x,\gamma) &= \beta g(x,\gamma) \\ g(\sigma x,\gamma) &= \sigma g(x,\gamma). \end{aligned} \tag{81}$$

Now we will find representations of the symmetries α , β and σ on R^1.
Let $V = xV_0$,

$$\alpha V = V(t+T) = x \begin{bmatrix} 0 \\ x_2(t+T) \end{bmatrix}$$

$$= x \begin{bmatrix} 0 \\ -x_2(t) \end{bmatrix} = -xV_0$$

$$\Rightarrow \qquad \alpha x = -x$$

$$\beta V = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = -xV_0$$

$$\Rightarrow \qquad \beta x = -x$$

$$\sigma V = -V(-t) = V(t) \qquad \Rightarrow \quad \sigma x = x.$$

Then from (81), we have

$$g(-x,\gamma) = -g(x,\gamma). \tag{82}$$

This means that the bifurcation equation (82) has $\mathbf{Z_2}$-symmetry on R^1.

Theorem 4.2.1.1 *If $g(x,\lambda)$ is C^∞ on some neighborhood of the origin and $g(-x,\lambda) = -g(x,\lambda)$, then there is a smooth coefficient $a(u,\lambda)$ such that $g(x,\lambda) = a(x^2,\lambda)x$.*

This Theorem is from [32, page 249]. Setting $\lambda = \gamma - \gamma_0$ then we have

$$g(x,\gamma) = a(x^2,\gamma)x. \tag{83}$$

Now we compute derivatives of g.

$$
\begin{aligned}
g_x &= a_u 2x^2 + a(x^2,\gamma) && (84)\\
a(0,\gamma_0) &= g_x(0,\gamma_0) && (85)\\
g_{xx} &= a_{uu}(2x)2x^2 + 4a_u x + a_u 2x && (86)\\
g_{xxx}(0,\gamma_0) &= 4a_u(0,\gamma_0) + 2a_u(0,\gamma_0) = 6a_u(0,\gamma_0) && (87)\\
g_{x\gamma}(0,\gamma_0) &= a_\gamma(0,\gamma_0). && (88)
\end{aligned}
$$

Using the formula in [32, page 295], we get

$$
\begin{aligned}
g_x(0,\gamma_0) &= 0 \quad \Rightarrow \quad a(0,\gamma_0) = 0\\
g_{x\lambda}(0,\gamma_0) &= < V_0, (d\Phi_\gamma)V_0 - (d^2\Phi)(V_0, L^{-1}E\Phi_\gamma) >\\
g_{xxx}(0,\gamma_0) &= < V_0, (d^3\Phi)(V_0,V_0,V_0) - 3(d^2\Phi)(V_0, W_{00}) >\\
W_{00} &= L^{-1}E(d^2\Phi)(V_0,V_0).
\end{aligned}
$$

Now

$$
\Phi(U,\gamma) = \begin{bmatrix} s'' + \sin(s+s_0)\cos r - \sin s_0 \\ r'' + \sin r \cos(s+s_0) + 2\gamma r \end{bmatrix}.
$$

Thus

$$
\Phi_\gamma = \begin{bmatrix} 0 \\ 2r \end{bmatrix}
$$

and $\Phi_\gamma(0,\gamma_0) = 0$. Hence

$$
\begin{aligned}
g_{x\lambda}(0,\gamma_0) &= < V_0, (d\Phi_\gamma)V_0 > && (89)\\
(d\Phi_\gamma)V_0 &= \frac{\partial}{\partial t}\begin{bmatrix} 0 \\ 2tx_2 \end{bmatrix}_{t=0} && (90)\\
&= \begin{bmatrix} 0 \\ 2x_2 \end{bmatrix} = 2V_0 && (91)
\end{aligned}
$$

$$
\Rightarrow \qquad g_{x\gamma}(0,\gamma_0) =< V_0, 2V_0 >= 2 < V_0, V_0 > > 0 \qquad (92)
$$

$$
\Rightarrow \qquad a_\gamma(0,\gamma_0) = 2 < V_0, V_0 > > 0. \qquad (93)
$$

From Implicit Function Theorem, there exists a solution $\gamma = \gamma(u)$ such that $a(u, \gamma(u)) \equiv 0$. Thus we have

Main Theorem 4.2.1 *For each in-phase rotation, there is an infinite series of bifurcation points γ at which the Floquet multipliers are -1. Furthermore, an odd 2T-period out-of-phase rotation bifurcates if the linearization at this point has an odd 2T-periodic solution.*

To get normal form of the bifurcation equation, we cite a Theorem from [32, page 256].

Theorem 4.2.1.2 *Let $g(x,\lambda) = a(u,\lambda)x$ be C^∞ in the neighborhood of the origin. Then g is strongly $\mathbf{Z_2}$-equivalent to $(\varepsilon u^k + \delta\lambda)x$ if and only if*

$$a = \frac{\partial a}{\partial u} = \cdots = \frac{\partial^{k-1}}{\partial u^{k-1}} a = 0$$

at $u = \lambda = 0$ and

$$\varepsilon = \text{sgn } \frac{\partial^k}{\partial u^k} a(0,0) \quad \delta = \text{sgn } a_\lambda(0,0)$$

Thus if $\frac{\partial^k}{\partial u^k} a(0,\gamma_0) \neq 0$ for some $k > 0$, we have the normal form $(\varepsilon u^k + \gamma - \gamma_0)x$ because $\delta = \text{sgn } a_\lambda(0,\gamma_0) = 1$. In general, computing the derivative $\frac{\partial^k}{\partial u^k} a(0,\gamma_0)$ is tedious. Here we only give the first order derivative. To do so, we first derive some formulas. Let

$$V_i = \begin{bmatrix} s_i \\ r_i \end{bmatrix} \quad i = 1,2,3$$

At $U = 0$ and $\gamma = \gamma_0$, we have

$$\begin{aligned} d^2\Phi(V_1,V_2) &= \frac{\partial^2}{\partial t_1 \partial t_2} \begin{bmatrix} (t_1 s_1 + t_2 s_2)'' + \sin(t_1 s_1 + t_2 s_2 + s_0) \\ \cos(t_1 r_1 + t_2 r_2) - \sin s_0 \\ (t_1 r_1 + t_2 r_2)'' + \sin(t_1 r_1 + t_2 r_2) \\ \cos(t_1 s_1 + t_2 s_2 + s_0) + 2\gamma_0(t_1 r_1 + t_2 r_2) \end{bmatrix}_{t_i=0} \\ &= \frac{\partial}{\partial t_1} \begin{bmatrix} s_2'' + \cos(t_1 s_1 + t_2 s_2 + s_0)s_2 \cos(t_1 r_1 + t_2 r_2) - \\ \sin(t_1 s_1 + t_2 s_2 + s_0)\sin(t_1 r_1 + t_2 r_2) r_2 \\ r_2'' + \cos(t_1 r_1 + t_2 r_2) r_2 \cos(t_1 s_1 + t_2 s_2 + s_0) \\ - \sin(t_1 r_1 + t_2 r_2)\sin(t_1 s_1 + t_2 s_2 + s_0) s_2 + 2\gamma_0 r_2 \end{bmatrix}_{t_i=0} \end{aligned}$$

$$= \begin{bmatrix} -\sin s_0\ s_1 s_2 - \sin s_0\ r_2 r_1 \\ -\sin s_0\ r_2 s_1 - \sin s_0\ r_1 s_2 \end{bmatrix}$$

$$d^2\Phi(V_1, V_2)(0, \gamma_0) = -\begin{bmatrix} s_1 s_2 + r_1 r_2 \\ s_1 r_2 + r_1 s_2 \end{bmatrix} \sin s_0 \qquad (94)$$

$$\begin{aligned}
d^3\Phi(V_1, V_2, V_3) &= \frac{\partial^3}{\partial t_1 \partial t_2 \partial t_3} \begin{bmatrix} (t_1 s_1 + t_2 s_2 + t_3 s_3)'' + \sin(t_1 s_1 + t_2 s_2 + t_3 s_3 + s_0) \\ \cos(t_1 r_1 + t_2 r_2 + t_3 r_3) - \sin s_0 \\ (t_1 r_1 + t_2 r_2 + t_3 r_3)'' + \sin(t_1 r_1 + t_2 r_2 + t_3 r_3) \\ \cos(t_1 s_1 + t_2 s_2 + t_3 s_3 + s_0) + 2\gamma(t_1 r_1 + t_2 r_2 + t_3 r_3) \end{bmatrix}_{t_i=0} \\
&= \frac{\partial^2}{\partial t_1 \partial t_2} \begin{bmatrix} \cos(t_1 s_1 + t_2 s_2 + t_3 s_3 + s_0) s_3 \cos(t_1 r_1 + t_2 r_2 + t_3 r_3) - \\ \sin(t_1 s_1 + t_2 s_2 + t_3 s_3 + s_0) \sin(t_1 r_1 + t_2 r_2 + t_3 r_3) r_3 \\ \cos(t_1 r_1 + t_2 r_2 + t_3 r_3) r_3 \cos(t_1 s_1 + t_2 s_2 + t_3 s_3 + s_0) \\ - \sin(t_1 r_1 + t_2 r_2 + t_3 r_3) \sin(t_1 s_1 + t_2 s_2 + t_3 s_3 + s_0) s_3 \end{bmatrix}_{t_i=0} \\
&= \frac{\partial}{\partial t_1} \begin{bmatrix} -\sin(t_1 s_1 + t_2 s_2 + t_3 s_3 + s_0) s_3 s_2 \cos(t_1 r_1 + t_2 r_2 + t_3 r_3) - \\ \cos(t_1 s_1 + t_2 s_2 + t_3 s_3 + s_0) s_3 \sin(t_1 r_1 + t_2 r_2 + t_3 r_3) r_2 - \\ \cos(t_1 s_1 + t_2 s_2 + t_3 s_3 + s_0) s_2 \sin(t_1 r_1 + t_2 r_2 + t_3 r_3) r_3 - \\ \sin(t_1 s_1 + t_2 s_2 + t_3 s_3 + s_0) \cos(t_1 r_1 + t_2 r_2 + t_3 r_3) r_2 r_3 \\ -\sin(t_1 r_1 + t_2 r_2 + t_3 r_3) r_3 r_2 \cos(t_1 s_1 + t_2 s_2 + t_3 s_3 + s_0) - \\ \cos(t_1 r_1 + t_2 r_2 + t_3 r_3) r_3 \sin(t_1 s_1 + t_2 s_2 + t_3 s_3 + s_0) s_2 - \\ \cos(t_1 r_1 + t_2 r_2 + t_3 r_3) r_2 \sin(t_1 s_1 + t_2 s_2 + t_3 s_3 + s_0) s_3 \\ -\sin(t_1 r_1 + t_2 r_2 + t_3 r_3) \cos(t_1 s_1 + t_2 s_2 + t_3 s_3 + s_0) s_2 s_3 \end{bmatrix}_{t_i=0} \\
&= \begin{bmatrix} -\cos s_0\ s_1 s_2 s_3 - \cos s_0\ r_1 r_2 s_3 - \cos s_0\ r_1 r_3 s_2 - \cos s_0\ r_2 r_3 s_1 \\ -\cos s_0\ r_1 r_2 r_3 - \cos s_0\ s_1 s_2 r_3 - \cos s_0\ s_1 s_3 r_2 - \cos s_0\ s_2 s_3 r_1 \end{bmatrix}
\end{aligned}$$

$$d^3\Phi(V_1, V_2, V_3)(0, \gamma_0) = -\begin{bmatrix} s_1 s_2 s_3 + r_1 r_2 s_3 + r_1 r_3 s_2 + r_2 r_3 s_1 \\ r_1 r_2 r_3 + s_1 s_2 r_3 + s_1 s_3 r_2 + s_2 s_3 r_1 \end{bmatrix} \cos s_0, \qquad (95)$$

From equations (94) and (95),

$$d^2\Phi(V_0, V_0) = \begin{bmatrix} -x_2^2 \\ 0 \end{bmatrix} \sin s_0$$

$$d^3\Phi(V_0, V_0, V_0) = \begin{bmatrix} 0 \\ -x_2^3 \end{bmatrix} \cos s_0.$$

But

$$< d^2\Phi(V_0, V_0), V_0 >= 0$$
$$\Rightarrow \ d^2\Phi(V_0, V_0) \in Range(L).$$

Thus

$$LW_{00} = Ed^2\Phi(V_0, V_0) = d^2\Phi(V_0, V_0)$$

where W_{00} is to be solved in $Range(L)$.

$$\begin{aligned} L_1(W_{00})_1 &= -x_2^2 \sin s_0(t) \\ L_2(W_{00})_2 &= 0 \\ \Rightarrow \quad (W_{00})_2 &= 0 \quad (W_{00} \in Range(L)). \end{aligned}$$

Thus

$$\begin{aligned} g_{xxx}(0,\gamma_0) &= < V_0, d^3\Phi(V_0, V_0, V_0) > -3 < V_0, d^2\Phi(V_0, W_{00}) > \\ &= -\frac{1}{2T}\int_0^{2T} x_2^4(s)\cos s_0(s)\,ds + 3\frac{1}{2T}\int_0^{2T} x_2^2(s)(W_{00})_1(s)\sin s_0(s)\,ds \end{aligned}$$

Therefore

$$a_u(0,\gamma_0) = \frac{1}{6}\left(-\frac{1}{2T}\int_0^{2T} x_2^4(s)\cos s_0(s)\,ds + 3\frac{1}{2T}\int_0^{2T} x_2^2(s)(W_{00})_1(s)\sin s_0(s)\,ds\right). \tag{96}$$

4.2.2 Degenerate $Z_2 \oplus Z_2$-Bifurcation

Assume that **A2** holds. That means that $L_2 r = 0$ has a unique, linearly independent, even, 2T-periodic solution. In this case, $\mathbf{Z_2 \oplus Z_2}$-bifurcation occurs. Instead of considering the mapping Φ on $C^k_{odd}[0, 2T]$, we now have

$$\Phi: \ C^2[0,2T] \ \rightarrow \ C^0[0,2T] \tag{97}$$

Similar to Lemma 4.2.1.1, we have

Lemma 4.2.2.1 *1. $C^k[0,2T]$ is a Banach space with the norm defined in (74).*

2. $dim Ker(L) = 2$

3. $C^0[0,2T] = Ker(L) \oplus Range(L)$

4. $Range(L)$ is closed.

5. L is Fredholm of index 0.

PROOF:

1. Proved in Lemma 4.2.1.1 1.
2. $L_1 s = 0$ has an even T-periodic solution $s_0'(t)$. It is also 2T-periodic. With **A2**, $L_2 r = 0$ has an even 2T-periodic solution, which is $x_1(t)$ with $x_1(0) = 1,\ x_1'(0) = 0$. Therefore $dim Ker(L) = 2$ and $Ker(L) = span(\begin{bmatrix} s_0'(t) \\ 0 \end{bmatrix}),\ \begin{bmatrix} 0 \\ x_1(t) \end{bmatrix})$.

3. Similar to Lemma 4.2.1.1,

$$Ker^T(L) \supseteq Range(L)$$

Now $\forall U \in Ker^T(L)$, we have $< U, V >= 0 \quad \forall V \in Ker(L)$
Therefore $< s_0', u_1 >= 0$, $< x_1, u_2 >= 0$.

On the other hand, the adjoint equation of $\ddot{x} + \phi(t)x = 0$ is itself. According to Lemma 4.1.6, both equations $L_1 s = u_1$ and $L_2 r = u_2$ have 2T-periodic solutions. Thus $U \in Range(L)$ and $Ker(L)^T \subseteq Range(L)$. So $Range(L) = Ker(L)^T$. And we have $C^0[0,2T] = Ker(L) \oplus Range(L)$
The proofs of 4 and 5 are completely similar to the corresponding parts of Lemma 4.2.1.1.

Remark 4.1 Let $C_i^k[0,2T]$ $(i = 1,2)$ be the *ith* components of elements $U(t)$ in $C^k[0,2T]$. $\Phi_1(s) = s'' + \sin(s + s_0) - \sin s_0$ and $\Phi_1 : C_1^2[0,2T] \rightarrow C_1^0[0,2T]$. $L_1 s = (d\Phi_1)s = s'' + \sin s_0\ s$. Similar to the proof of the last Lemma, we conclude that L_1 is a Fredholm of index zero.

Having proved Lemma 4.2.2.1, we can use the Liapunov-Schmidt reduction to investigate the bifurcations in this case. Take as basis of $Ker(L)$

$$V_1 = \begin{bmatrix} s_0' \\ 0 \end{bmatrix} \qquad V_2 = \begin{bmatrix} 0 \\ x_1 \end{bmatrix}$$

Then the bifurcation equation becomes

$$\begin{aligned} g_1(x,y,\gamma) &\stackrel{\text{def}}{=} < V_1, \Phi(xV_1 + yV_2 + W(x,y,\gamma),\gamma) > \\ g_2(x,y,\gamma) &\stackrel{\text{def}}{=} < V_2, \Phi(xV_1 + yV_2 + W(x,y,\gamma),\gamma) > . \end{aligned} \tag{98}$$

These equations commute with the symmetries α, β and σ. Representations of those symmetries on R^2 are derived as follows. Let $V = xV_1 + yV_2$

$$\alpha V = V(t+T) = xV_1(t+T) + yV_2(t+T) = xV_1(t) - yV_2(t).$$

The last equation is from $s_0'(t+T) = s_0'(t)$ and $x_1(t+T) = -x_1(t)$. Thus

$$\alpha \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} \tag{99}$$

$$\beta V = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} xs_0' \\ yx_1 \end{bmatrix} = xV_1 - yV_2$$

$$\beta \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} \tag{100}$$

$$\sigma V = -V(-t) = -xV_1(-t) - yV_2(-t) = -xV_1(t) - yV_2(t)$$

$$\sigma \begin{bmatrix} x \\ y \end{bmatrix} = - \begin{bmatrix} x \\ y \end{bmatrix}. \tag{101}$$

With the equations (99)-(101) and the fact that the bifurcation equation (98) commutes with the actions, we obtain

$$\begin{aligned} g_1(x,-y,\gamma) &= g_1(x,y,\gamma) \\ g_2(x,-y,\gamma) &= -g_2(x,y,\gamma) \end{aligned} \tag{102}$$

$$\begin{aligned} g_1(-x,-y,\gamma) &= -g_1(x,y,\gamma) \\ g_2(-x,-y,\gamma) &= -g_2(x,y,\gamma). \end{aligned} \tag{103}$$

From (102) and (102),

$$g_1(-x,y,\gamma) = g_1(-x,-(-y),\gamma) = -g_1(x,-y,\gamma) = -g_1(x,y,\gamma) \quad (104)$$

$$g_2(-x,y,\gamma) = g_2(-x,-(-y),\gamma) = -g_2(x,-y,\gamma) = g_2(x,y,\gamma) \quad (105)$$

The equations (102)-(105) justify that the bifurcation equation has $\mathbf{Z_2} \oplus \mathbf{Z_2}$ -symmetry [32, page 419].

Now we will reduce the bifurcation equation to its normal form. First we will compute derivatives to see what should be in the normal form.

$$A \stackrel{\text{def}}{=} \frac{1}{3!}g_{1xxx} = \frac{1}{3!}(< V_1, d^3\Phi(V_1,V_1,V_1) - 3d^2\Phi(V_1,W_{11}) >) \quad (106)$$

$$B \stackrel{\text{def}}{=} \frac{1}{2}g_{1xyy} = \frac{1}{2}(< V_1, d^3\Phi(V_1,V_2,V_2) - d^2\Phi(V_1,W_{22}) - 2d^2\Phi(V_2,W_{21} >) \quad (107)$$

$$C \stackrel{\text{def}}{=} \frac{1}{2}g_{2xxy} = \frac{1}{2}(< V_2, d^3\Phi(V_1,V_1,V_2) - d^2\Phi(V_2,W_{11}) - 2d^2\Phi(V_1,W_{21} >) \quad (108)$$

$$D \stackrel{\text{def}}{=} \frac{1}{3!}g_{2yyy} = \frac{1}{3!}(< V_2, d^3\Phi(V_2,V_2,V_2) - 3d^2\Phi(V_2,W_{22}) >) \quad (109)$$

$$\delta \stackrel{\text{def}}{=} g_{2y\gamma} = < V_2, (d\Phi_\gamma)V_2 - d^2\Phi(V_2, L^{-1}E\Phi_\gamma > \quad (110)$$

$$W_{st} = L^{-1}Ed^2\Phi(V_s,V_t) \quad (111)$$

$$LW_{11} = Ed^2\Phi(V_1,V_1) = E\begin{bmatrix} -s_0'^2 \sin s_0 \\ 0 \end{bmatrix} = \begin{bmatrix} -s_0'^2 \sin s_0 \\ 0 \end{bmatrix} \quad (112)$$

$$LW_{21} = Ed^2\Phi(V_2,V_1) = E\begin{bmatrix} 0 \\ -x_1 s_0' \sin s_0 \end{bmatrix} = \begin{bmatrix} 0 \\ -x_1 s_0' \sin s_0 \end{bmatrix} \quad (113)$$

$$LW_{22} = Ed^2\Phi(V_2,V_2) = E\begin{bmatrix} -x_1^2 \sin s_0 \\ 0 \end{bmatrix} = \begin{bmatrix} -x_1^2 \sin s_0 \\ 0 \end{bmatrix}. \quad (114)$$

The last three equations come from the fact that the right hand sides are all orthogonal to $Ker(L)$. Thus they are in $Range(L)$.

$$W_{st} \in Range(L)$$

$$
\begin{aligned}
L_2(W_{ii})_2 = 0 \quad (i = 1, 2) \\
\Rightarrow \qquad (W_{ii})_2 = 0 \\
L_1(W_{21})_1 = 0 \\
\Rightarrow \qquad (W_{21})_1 = 0,
\end{aligned}
$$

From (94) and (95),

$$d^2\Phi(V_1, W_{11}) = -\begin{bmatrix} s_0'(W_{11})_1 \\ 0 \end{bmatrix} \sin s_0 \tag{115}$$

$$d^2\Phi(V_2, W_{22}) = -\begin{bmatrix} 0 \\ x_1(W_{22})_1 \end{bmatrix} \sin s_0 \tag{116}$$

$$d^2\Phi(V_1, W_{22}) = -\begin{bmatrix} s_0'(W_{22})_1 \\ 0 \end{bmatrix} \sin s_0 \tag{117}$$

$$d^2\Phi(V_1, W_{21}) = -\begin{bmatrix} 0 \\ s_0'(W_{21})_2 \end{bmatrix} \sin s_0 \tag{118}$$

$$d^2\Phi(V_2, W_{21}) = -\begin{bmatrix} x_1(W_{21})_2 \\ 0 \end{bmatrix} \sin s_0 \tag{119}$$

$$d^2\Phi(V_2, W_{11}) = -\begin{bmatrix} 0 \\ x_1(W_{11})_1 \end{bmatrix} \sin s_0 \tag{120}$$

$$d^3\Phi(V_1, V_1, V_1) = -\begin{bmatrix} s_0'^3 \\ 0 \end{bmatrix} \cos s_0 \tag{121}$$

$$d^3\Phi(V_2, V_2, V_2) = -\begin{bmatrix} 0 \\ x_1^3 \end{bmatrix} \cos s_0 \tag{122}$$

$$d^3\Phi(V_1, V_2, V_2) = -\begin{bmatrix} x_1^2 s_0' \\ 0 \end{bmatrix} \cos s_0 \tag{123}$$

$$d^3\Phi(V_1, V_1, V_2) = -\begin{bmatrix} 0 \\ s_0'^2 x_1 \end{bmatrix} \cos s_0 \tag{124}$$

$$(d\Phi_\gamma)V_2 = 2V_2, \tag{125}$$

Therefore

$$\delta = 2 < V_2, V_2 > \tag{126}$$

$$A = -\frac{1}{6}\left(\frac{1}{2T}\int_0^{2T} s_0'^4(t)\cos s_0(t)\,dt - 3\frac{1}{2T}\int_0^{2T} s_0'^2(t)(W_{11})_1(t)\sin s_0\,dt\right) \quad (127)$$

$$B = -\frac{1}{2}\left(\frac{1}{2T}\int_0^{2T} x_1^2(t)s_0'^2(t)\cos s_0(t)\,dt - \frac{1}{2T}\int_0^{2T} s_0'^2(t)(W_{22})_1(t)\sin s_0(t)\,dt - 2\frac{1}{2T}\int_0^{2T} s_0'(t)x_1(t)(W_{21})_2(t)\sin s_0(t)\,dt\right) \quad (128)$$

$$C = -\frac{1}{2}\left(\frac{1}{2T}\int_0^{2T} x_1^2(t)s_0'^2(t)\cos s_0(t)\,dt - \frac{1}{2T}\int_0^{2T} x_1^2(t)(W_{11})_1(t)\sin s_0(t)\,dt - 2\frac{1}{2T}\int_0^{2T} s_0'(t)x_1(t)(W_{21})_2(t)\sin s_0(t)\,dt\right) \quad (129)$$

$$D = -\frac{1}{6}\left(\frac{1}{2T}\int_0^{2T} x_1^4(t)\cos s_0(t)\,dt - 3\frac{1}{2T}\int_0^{2T} x_1^2(t)(W_{22})_1(t)\sin s_0\,dt\right). \quad (130)$$

Lemma 4.2.2.2 *There are smooth coefficients $p(u,v,\lambda)$ and $q(u,v,\lambda)$ such that*

$$g_1(x,y,\gamma) = p(u,v,\lambda)x \qquad g_2(x,y,\gamma) = q(u,v,\lambda)y$$

where $u = x^2$, $v = y^2$, $\lambda = \gamma - \gamma_0$ and $p(0,0,0) = 0$ $q(0,0,0) = 0$.

PROOF: This is a direct application of Theorem 4.2.1.1 and

$$g_{1x}(0,0,\gamma_0) = 0 \quad g_{2y}(0,0,\gamma_0) = 0$$

Lemma 4.2.2.3 *$p(u,0,\lambda)$ is independent of λ.*

PROOF:
In L-S reduction, there exists a unique solution $W(x,y,\gamma)$ such that

$$E\Phi(xV_1 + yV_2 + W(x,y,\gamma)\ ,\ \gamma) = 0 \quad (131)$$

From Remark 3.1, $W(\beta V,\gamma) = \beta W(V,\gamma)$. Thus

$$(W)_2(x,-y,\lambda) = -(W)_2(x,y,\lambda)$$

Thus $(W)_2(x,0,\gamma) = 0$. Therefore at $y = 0$, the equation (131) becomes

$$E\begin{bmatrix} s'' + \varepsilon s' + \sin(s+s_0) - \sin s_0 \\ 0 \end{bmatrix} = 0$$

where $s = xs_0' + (W)_1(x, 0, \gamma)$, i.e. ,

$$E \begin{bmatrix} \Phi_1(xs_0' + (W)_1(x,0,\gamma)) \\ 0 \end{bmatrix} = 0 \tag{132}$$

where Φ_1 is defined in Remark 4.1.

On the other hand, applying the L-S reduction to Φ_1 on $C_1^0[0, 2T]$ in the Remark 4.1, we have a unique solution $w = w(x)$ such that

$$E_1\Phi_1(xs_0' + w(x)) = 0 \tag{133}$$

where $E_1 : C_1^0[0, 2T] \rightarrow Range(L_1)$ is a projection. Now

$$< E \begin{bmatrix} \Phi_1(xs_0' + (W)_1(x,0,\gamma)) \\ 0 \end{bmatrix}, V_2 >= 0$$

Then

$$\begin{bmatrix} \Phi_1(xs_0' + (W)_1(x,0,\gamma)) \\ 0 \end{bmatrix} = E \begin{bmatrix} \Phi_1(xs_0' + (W)_1(x,0,\gamma)) \\ 0 \end{bmatrix} + x^* V_1 \tag{134}$$

where

$$x^* = \frac{< \begin{bmatrix} \Phi_1 \\ 0 \end{bmatrix}, V_1 >}{< V_1, V_1 >} = \frac{< \Phi_1, s_0' >}{< s_0', s_0' >}$$

So $< \Phi_1 - x^* s_0', s_0' >= 0.$ Thus $E\Phi_1 = \Phi_1 - x^* s_0'$. From (132) and (134),

$$E \begin{bmatrix} \Phi_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \Phi_1 - x^* s_0' \\ 0 \end{bmatrix} = \begin{bmatrix} E_1\Phi_1 \\ 0 \end{bmatrix} = 0 \tag{135}$$

or

$$E_1\Phi_1(xs_0' + (W)_1(x,0,\gamma)) = 0 \tag{136}$$

By the uniqueness of solution in equation (133) and (136), we have

$$(W)_1(x,0,\gamma) = w(x)$$

Now

$$\begin{aligned} g_1(x,0,\gamma) &= < V_1, \Phi(xV_1 + W(x,0,\gamma),\gamma) > \\ &= < s_0', \Phi_1(xs_0' + (W)_1(x,0,\gamma)) > \\ &= < s_0', \Phi_1(xs_0' + w(x)) > \end{aligned}$$

This implies that $g_1(x, 0, \gamma) = p(u, 0, \lambda)x$ is independent on γ. Thus $p(u, 0, \lambda)$ does not depend on λ.

Lemma 4.2.2.4

$$p(u, v, \lambda) = h_1(u, v) + \eta_1(u, v, \lambda) \tag{137}$$

$$q(u, v, \lambda) = h_2(u, v, \lambda) + \eta_2(u, v, \lambda) \tag{138}$$

where

$$h_1(u, v) = Au + Bv$$

$$h_2(u, v, \lambda) = Cu + Dv + \delta\lambda$$

$$\eta_1(u. v, \lambda) = p_1(u)u^2 + p_{12}(u, v, \lambda)uv + p_{22}(u, v, \lambda)v^2 + p_{23}(u, v, \lambda)\lambda v$$

$$\stackrel{\text{def}}{=} p_1(u)u^2 + p_2(u, v, \lambda)v$$

$$\eta_2(u, v, \lambda) = q_{11}(u, v, \lambda)u^2 + q_{12}(u, v, \lambda)uv + q_{13}(u, v, \lambda)u\lambda + q_{22}(u, v, \lambda)v^2 +$$

$$q_{23}(u, v, \lambda)v\lambda + q_{33}(u, v, \lambda)\lambda^2$$

and A, B, C , D and δ are constants.
PROOF:

By Lemma 4.2.2.3, we write $p(u, 0, \lambda) = p_0(u)$. From Taylor's Theorem[32, pages 60] and $p_0(0) = p(0, 0, 0) = 0$, we have

$$p_0(u) = Au + P_1(u)u^2$$

$$p(u, v, \lambda) - p_0(u) = p_2(u, v, \lambda)v$$

and

$$p_2(u, v, \lambda) = p_2(0, 0, 0) + p_{12}(u, v, \lambda)u + p_{22}(u, v, \lambda)v + p_{23}(u, v, \lambda)\lambda$$

The first formula is obtained by taking $B = p_2(0, 0, 0)$. Similarly, applying Taylor's Theorem to $q(u, v, \lambda)$ at $(0, 0, 0)$, the second formula follows.

From this Lemma, $\frac{\partial p(u,v,\lambda)}{\partial \lambda} = 0$ if $v = 0$. Therefore the bifurcation equation is degenerate and not discussed in [32].

Write

$$h(u,v,\lambda) \stackrel{\text{def}}{=} (h_1(u,v), h_2(u,v,\lambda)) \tag{139}$$

$$\eta(u,v,\lambda) \stackrel{\text{def}}{=} (\eta_1(u,v,\lambda), \eta_2(u,v,\lambda)). \tag{140}$$

Main Theorem 4.2.2 *If $ABCD\delta \neq 0$ and $AD - BC \neq 0$, then*

$$RT(h + t\eta, \mathbf{Z_2} \oplus \mathbf{Z_2}) = RT(h, \mathbf{Z_2} \oplus \mathbf{Z_2}).$$

PROOF:

According to Table 3.1 in [33, page 177], $RT(h + t\eta, \mathbf{Z_2} \oplus \mathbf{Z_2})$ is generated by

$$[p,0] \ , \ [0,q] \ , \ [0,up] \ , \ [vq,0] \ , \ [up_u, uq_u] \ , \ [vp_v, vq_v]$$

where $p = h_1 + t\eta_1$,$q = h_2 + t\eta_2$ and $t \in [0,1]$.

Let Ω_t be generated by

$$z[p,0] \ , \ z[0,q] \ , \ z[up_u, uq_u] \ , \ z[vp_v, vq_v] \ , \ [0,up]$$

where $z = u$, v or λ. Obviously $\Omega_t \subseteq RT(h + t\eta, \mathbf{Z_2} \oplus \mathbf{Z_2})$.

Ω is generated by

$$[u^2,0] \ , \ [uv,0] \ , \ [u\lambda,0] \ , \ [v^2,0] \ , \ [v\lambda,0]$$

$$[0,u^2] \ , \ [0,uv] \ , \ [0,u\lambda] \ , \ [0,v^2] \ , \ [0,v\lambda] \ , \ [0,\lambda^2].$$

We claim

$$\Omega_t = \Omega. \tag{141}$$

If (141) holds, then

$$\Omega \subseteq RT(h + t\eta, \mathbf{Z_2} \oplus \mathbf{Z_2}) \ , \tag{142}$$

and

$$RT(h + t\eta, \mathbf{Z_2} \oplus \mathbf{Z_2}) = \Omega + < [p,0] \ , \ [0,q] \ , \ [up_u, uq_u] \ , \ [vp_v, vq_v] > \tag{143}$$

where $< f_1, \cdots, f_k >$ stands for the ideal generated by $f_1, \cdots, f_k$. In fact, up and vq are quadratic for any $t \in R_1$. Thus $[0, up]$, $[vq, 0] \in \Omega$. Now we establish (141). Write the generators of $RT(h + t\eta, \mathbf{Z_2} \oplus \mathbf{Z_2})$ as

$$
\begin{aligned}
[p, 0] &= [h_1, 0] + [t\eta_1, 0] \\
[0, q] &= [0, h_2] + [0, t\eta_2] \\
[0, up] &= [0, uh_1] + [0, tu\eta_1] \\
[vq, 0] &= [vh_2, 0] + [tv\eta_2, 0] \\
[up_u, uq_u] &= [uh_{1u}, uh_{2u}] + [tu\eta_{1u}, tu\eta_{2u}] \\
[vp_v, vq_v] &= [vh_{1v}, vh_{2v}] + [tv\eta_{1v}, tv\eta_{2v}]
\end{aligned} \tag{144}
$$

and

$$
\begin{aligned}
[h_1, 0] &= A[u, 0] + B[v, 0] \\
[0, h_2] &= C[0, u] + D[0, v] + \delta[0, \lambda] \\
[0, uh_1] &= A[0, u^2] + B[0, uv] \\
[vh_2, 0] &= C[uv, 0] + D[v^2, 0] + \delta[u\lambda, 0] \\
[uh_{1u}, uh_{2u}] &= A[u, 0] + C[0, u] \\
[vh_{1v}, vh_{2v}] &= B[v, 0] + D[0, v]
\end{aligned} \tag{145}
$$

$$
\begin{aligned}
[t\eta_1, 0] &= tp_1(u)[u^2, 0] + tp_2(u, v, \lambda)[v, 0] \\
[0, tu\eta_1] &= [0, tp_1(u)u^3 + tp_2(u, v\lambda)uv] \\
\eta_{1u} &= p_1' u^2 + 2p_1 u + p_{2u}(u, v, \lambda)v.
\end{aligned}
$$

Thus

$$
[t\eta_1, 0] \ , \ [0, t\eta_2] \ , \ [0, tu\eta_1] \ , \ [tv\eta_2, 0] \ , \ [tu\eta_{1u}, tu\eta_{2u}] \ , \ [tv\eta_{1v}, tv\eta_{2v}] \in \Omega. \tag{146}
$$

Expanding the generators of Ω_t as

$$
u[p, 0] = u[h_1, 0] + u[t\eta_1, 0]
$$

$$
\begin{aligned}
v[p,0] &= v[h_1,0] + v[t\eta_1,0] \\
\lambda[p,0] &= \lambda[h_1,0] + \lambda[t\eta_1,0] \\
u[0,q] &= u[0,h_2] + u[0,t\eta_2] \\
v[0,q] &= v[0,h_2] + v[0,t\eta_2] \\
\lambda[0,q] &= \lambda[0,h_2] + \lambda[0,t\eta_2] \\
u[up_u,uq_u] &= u[uh_{1u},uh_{2u}] + u[tu\eta_{1u},tu\eta_{2u}] \\
v[up_u,uq_u] &= v[uh_{1u},uh_{2u}] + v[tu\eta_{1u},tu\eta_{2u}] \\
\lambda[up_u,uq_u] &= \lambda[uh_{1u},uh_{2u}] + \lambda[tu\eta_{1u},tu\eta_{2u}] \\
u[vp_v,vq_v] &= u[vh_{1v},vh_{2v}] + u[tv\eta_{1v},tv\eta_{2v}] \\
v[vp_v,vq_v] &= v[vh_{1v},vh_{2v}] + v[tv\eta_{1v},tv\eta_{2v}] \\
\lambda[vp_v,vq_v] &= \lambda[vh_{1v},vh_{2v}] + \lambda[tv\eta_{1v},tv\eta_{2v}] \\
[0,up] &= [0,uh_1] + [0,tu\eta_1]. \qquad (147)
\end{aligned}
$$

From (145) - (147), we have

$$\Omega_t \subseteq \Omega. \qquad (148)$$

If

$$\Omega \subseteq \Omega_t + \mathcal{U}_{u,v,\lambda}\Omega \qquad (149)$$

where $\mathcal{U}_{u,v,\lambda} = < u,v,\lambda >$. From Nakayama's lemma (see § 3.1), we have

$$\Omega \subseteq \Omega_t. \qquad (150)$$

By (148) and (150), we get $\Omega = \Omega_t$.

Now we claim

$$\Omega_t + \mathcal{U}_{u,v,\lambda}\Omega = \Omega_0 + \mathcal{U}_{u,v,\lambda}\Omega \qquad (151)$$

Ω_0 is the Ω_t when $t = 0$. By (146), all t-terms in the right hand side of (147) belong to $\mathcal{U}_{u,v,\lambda}\Omega$. Therefore all the generators of Ω_t are in $\Omega_0 + \mathcal{U}_{u,v,\lambda}\Omega$.

$$\Rightarrow \qquad \Omega_t \subseteq \Omega_0 + \mathcal{U}_{u,v,\lambda}\Omega$$

$$\Omega_t + \mathcal{U}_{u,v,\lambda}\Omega \subseteq \Omega_0 + \mathcal{U}_{u,v,\lambda}\Omega. \qquad (152)$$

Moving all t-terms in (147) to the left hand side , we conclude that all generators of Ω_0 are in $\Omega_t + \mathcal{U}_{u,v,\lambda}\Omega$. Thus

$$\Omega_0 + \mathcal{U}_{u,v,\lambda}\Omega \subseteq \Omega_t + \mathcal{U}_{u,v,\lambda}\Omega. \tag{153}$$

From (152) and (153), (151) is established. From (149) and (151), we have to prove

$$\Omega \subseteq \Omega_0 + \mathcal{U}_{u,v,\lambda}\Omega. \tag{154}$$

Define

$$X = [\ [u^2,0]\ ,\ [uv,0]\ ,\ [v^2,0]\ ,\ [u\lambda,0]\ ,\ [v\lambda,0]\ ,$$
$$[0,u^2]\ ,\ [0,uv]\ ,\ [0,v^2]\ ,\ [0,u\lambda]\ ,\ [0,v\lambda]\ ,\ [0,\lambda^2]\ ]^T$$

$$b = [\ u[h_1,0]\ ,\ v[h_1,0]\ ,\ v^2[h_{1v},h_{2v}]\ ,\ \lambda[h_1,0]\ ,\ v\lambda[h_{1v},h_{2v}]$$
$$[0,uh_1]\ ,\ uv[h_{1u},h_{2u}]\ ,\ v[0,h_2]\ ,\ u[0,h_2]\ ,\ u\lambda[h_{1u},h_{2u}]\ ,\ \lambda[0,h_2]\ ]^T.$$

Expanding the generator vector b of Ω_0 as the linear combination of the one X of Ω, we have

$$\Theta X = b$$

where Θ is given by

$$\begin{bmatrix}
A & B & & & & & & & & & \\
& A & B & & & & & & & & \\
& & B & & & & & D & & & \\
& & & A & B & & & & & & \\
& & & & B & & & & & D & \\
& & & & & A & B & & & & \\
& A & & & & & C & & & & \\
& & & & & & C & D & & \delta & \\
& & & & & C & D & & \delta & & \\
& & & A & & & & & C & & \\
& & & & & & & & C & D & \delta
\end{bmatrix}$$

The determinant of this matrix is $A^3B^2C\delta^2D(BC-AD)$, which is not zero by assumption. Thus X is a linear combination of b. Hence

$$\Omega \subseteq \Omega_0 \tag{155}$$

and (154) is proved. By (144) and (145),

$$[p,0]\ ,\ [0,q]\ ,\ [up_u,uq_u]\ ,\ [vp_v,vq_v] \in \Omega + < [h_1,0]\ ,\ [0,h_2]\ ,\ [uh_{1u},uh_{2u}]\ ,\ [vh_{1v},vh_{2v}] > .$$

From (143),

$$RT(h+t\eta,\mathbf{Z_2}\oplus\mathbf{Z_2}) \subseteq \Omega + < [h_1,0]\ ,\ [0,h_2]\ ,\ [uh_{1u},uh_{2u}]\ ,\ [vh_{1v},vh_{2v}] >,$$

But

$$[h_1,0]\ ,\ [0,h_2]\ ,\ [uh_{1u},uh_{2u}]\ ,\ [vh_{1v},vh_{2v}] \in \Omega + < [p,0]\ ,\ [0,q]\ ,\ [up_u,uq_u]\ ,\ [vp_v,vq_v] >.$$

So

$$RT(h+t\eta,\mathbf{Z_2}\oplus\mathbf{Z_2}) \supseteq \Omega + < [h_1,0]\ ,\ [0,h_2]\ ,\ [uh_{1u},uh_{2u}]\ ,\ [vh_{1v},vh_{2v}] >,$$

Thus

$$RT(h+t\eta,\mathbf{Z_2}\oplus\mathbf{Z_2}) = \Omega + < [h_1,0]\ ,\ [0,h_2]\ ,\ [uh_{1u},uh_{2u}]\ ,\ [vh_{1v},vh_{2v}] >$$

which is independent on t.

Main Theorem 4.2.3 *If $ABCD\delta \neq 0$ and $AD-BC \neq 0$, then the bifurcation equation(98) is strongly $\mathbf{Z_2}\oplus\mathbf{Z_2}$-equivalent to $(h_1(u,v,\lambda)x, h_2(u,v,\lambda)y)$.*

PROOF: This is from applying Theorem 1.3 [33, page 168] to the result of last Theorem.

Main Theorem 4.2.4 *If $ABCD\delta \neq 0$ and $AD-BC \neq 0$, then the bifurcation equation(98) is strongly $\mathbf{Z_2}\oplus\mathbf{Z_2}$-equivalent to*

$$(\varepsilon_1 x^3 + \varepsilon_2 xy^2, nx^2y + \varepsilon_3 y^3 + \varepsilon_4 \lambda y)$$

where

$$\varepsilon_1 = sgn(A) \ , \ \varepsilon_2 = sgn(B)$$

$$\varepsilon_3 = sgn(D) \ , \ \varepsilon_4 = sgn(\delta)$$

and

$$n = \mid \frac{B}{AD} \mid C \ , n \neq \varepsilon_1\varepsilon_2\varepsilon_3.$$

PROOF:

Take

$$Z(x, y, \lambda) = (ax, by) \ , \ \Lambda(\lambda) = \sigma\lambda$$

$$S(x, y, \lambda) = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$$

where a , b , c and d are positive constants. In terms of the definition of equivalence [32, page 398], we have

$$\begin{aligned} S(x, y, \lambda)\,(h_1(ax, by)\ a\ x, h_2(ax, by, \sigma\lambda)\ b\ y)^T &= \\ \left(\varepsilon_1 x^3 + \varepsilon_2 xy^2, nx^2y + \varepsilon_3 y^3 + \varepsilon_4 \lambda y\right)^T & \\ \Rightarrow \quad (ca^3Ax^3 + cab^2Bxy^2, da^2bCx^2y + db^3Dy^3 + db\sigma\delta\lambda y) & \\ = (\varepsilon_1 x^3 + \varepsilon_2 xy^2, nx^2y + \varepsilon_3 y^3 + \varepsilon_4 \lambda y). & \end{aligned}$$

So

$$\begin{aligned} ca^3 \mid A \mid &= 1 \\ cab^2 \mid B \mid &= 1 \\ db^3 \mid D \mid &= 1 \\ db\sigma \mid \delta \mid &= 1 \end{aligned}$$

Thus

$$a = \sqrt{\frac{\mid B \mid}{\mid A \mid}} b$$

$$
\begin{aligned}
c &= \frac{1}{a^3 \mid A \mid} = \frac{1}{\frac{|B|}{|A|}\sqrt{\frac{|B|}{|A|}}b^3 \mid A \mid} \\
&= \frac{\sqrt{\mid A \mid}}{\mid B \mid^{\frac{3}{2}} b^3} \\
d &= \frac{1}{\mid D \mid b^3} \\
\sigma &= \frac{\mid D \mid}{\mid \delta \mid} b^2 \\
n &= da^2bC = \frac{1}{\mid D \mid b^3}\frac{\mid B \mid}{\mid A \mid} b^2 bC \\
&= \frac{\mid B \mid}{\mid DA \mid} C.
\end{aligned}
$$

Define

$$g(x, y, \lambda) = (p_1(u, v, \lambda)x, q_2(u, v, \lambda)y).$$

Then $g(x, y, \lambda)$ has different types of solutions.

(1) Trivial solution: $x = 0$, $y = 0$

(2) x-mode solutions: $p(u, 0, \lambda) = 0$, $y = 0$, $x \neq 0$.

(3) y-mode solutions: $x = 0$, $q(0, v, \lambda) = 0$, $y \neq 0$.

(4) mixed mode solutions: $p(u, v, \lambda) = 0$, $q(u, v, \lambda) = 0$, $x \neq 0$, $y \neq 0$.

Define:

$$p = \varepsilon_1 u + \varepsilon_2 v \quad q = nu + \varepsilon_3 v + \varepsilon_4 \lambda$$

where

$$u = x^2 \ , v = y^2 \ , \varepsilon_i = \pm 1 \ \ (i = 1, 2, 3) \quad n \neq \varepsilon_1 \varepsilon_2 \varepsilon_3$$

From (126), we have $\delta > 0$. So $\varepsilon_4 = 1$. Hence we have

(1) Trivial solution: $x = 0$, $y = 0$

(2)There is no x-mode solution.

(3) y-mode solutions $x = 0$, $\lambda = -\varepsilon_3 y^2$.

(4) Mixed-mode solutions $x \neq 0$ $y \neq 0$

Solving the equations $p(u, v) = 0$ and $q(u, v, \lambda) = 0$, we have

$$\lambda = \varepsilon_2(\varepsilon_1 \varepsilon_3 - \varepsilon_2 n)u \tag{156}$$

Figure 2: Bifurcation diagrams $\varepsilon = \pm 1$

$$\lambda = -\varepsilon_1(\varepsilon_1\varepsilon_3 - \varepsilon_2 n)v \tag{157}$$

$$u = -\varepsilon_1\varepsilon_2 v \tag{158}$$

4.2.3 Bifurcation Diagrams

The bifurcation diagrams of normal form $(\varepsilon u + \gamma - \gamma_0)x = 0$ for the case **A1** are given in Figure 2.

The equations of the bifurcation diagrams in the case **A2** are presented below:

(1) $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$ and $n \neq 1$.
y-mode: $\lambda = -y^2$
mixed-mode: $x^2 + y^2 = 0$, no non-zero solutions.
(2) $\varepsilon_1 = \varepsilon_2 = 1$, $\varepsilon_3 = -1$ and $n \neq -1$.
y-mode: $\lambda = y^2$
mixed-mode: $x^2 + y^2 = 0$, no non-zero solutions.
(3) $\varepsilon_1 = \varepsilon_2 = -1$, $\varepsilon_3 = 1$ and $n \neq 1$.
y-mode: $\lambda = -y^2$
mixed-mode: $x^2 + y^2 = 0$, no non-zero solutions.
(4) $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1$ and $n \neq -1$.
y-mode: $\lambda = y^2$
mixed-mode: $x^2 + y^2 = 0$, no non-zero solutions.
(5) $\varepsilon_1 = 1$, $\varepsilon_2 = -1$, $\varepsilon_3 = 1$ and $n \neq -1$.
y-mode: $\lambda = -y^2$
mixed-mode: $\lambda = -(1+n)x^2$ and $\lambda = -(1+n)y^2$

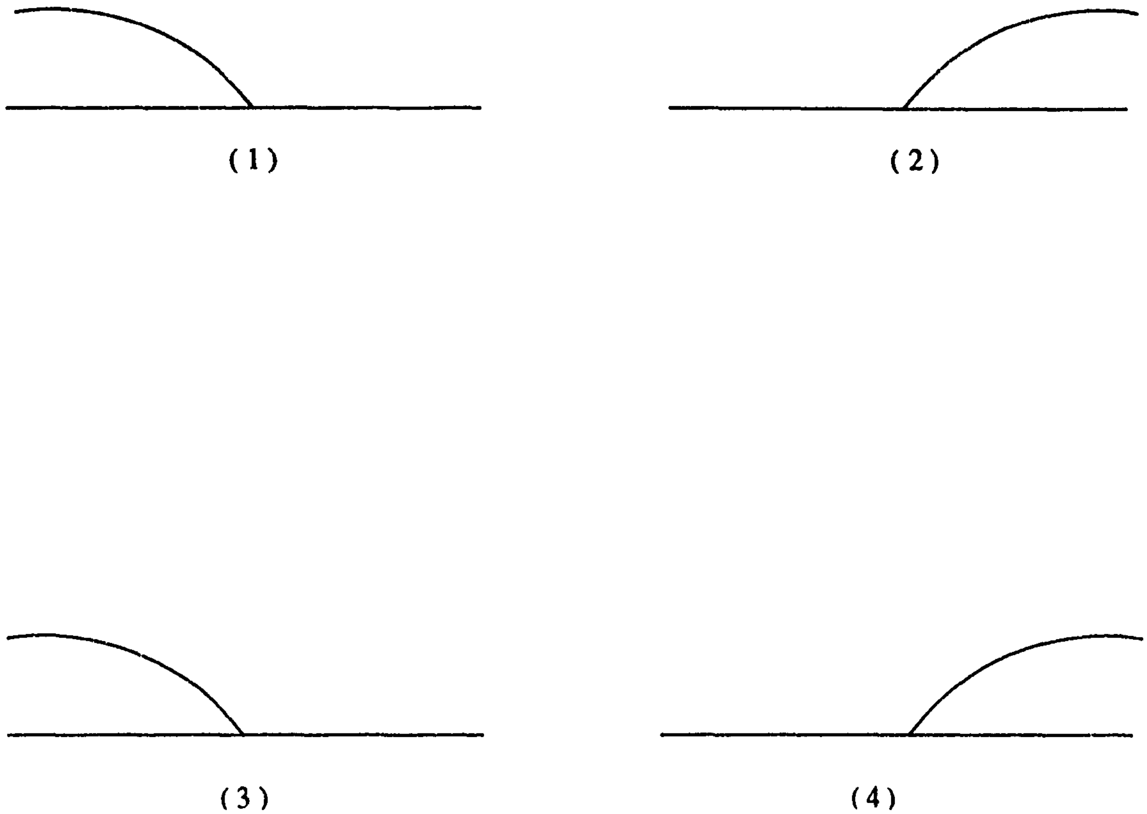

Figure 3: Bifurcation diagrams $\varepsilon_1 = \varepsilon_2 = \pm 1$, $\varepsilon_3 = \pm 1$

(6) $\varepsilon_1 = 1$, $\varepsilon_2 = -1$, $\varepsilon_3 = -1$ and $n \neq 1$.

y-mode: $\lambda \; = \; y^2$

mixed-mode: $\lambda = (n-1)x^2$ and $\lambda = (n-1)y^2$

(7) $\varepsilon_1 = -1$, $\varepsilon_2 = 1$, $\varepsilon_3 = 1$ and $n \neq -1$.

y-mode: $\lambda \; = \; -y^2$

mixed-mode: $\lambda = -(1+n)x^2$ and $\lambda = -(1+n)y^2$

(8) $\varepsilon_1 = -1$, $\varepsilon_2 = 1$, $\varepsilon_3 = -1$ and $n \neq 1$.

y-mode: $\lambda \; = \; y^2$

mixed-mode: $\lambda = (1-n)x^2$ and $\lambda = (1-n)y^2$

The corresponding bifurcation diagrams are shown in Figure 3 and Figure 4.

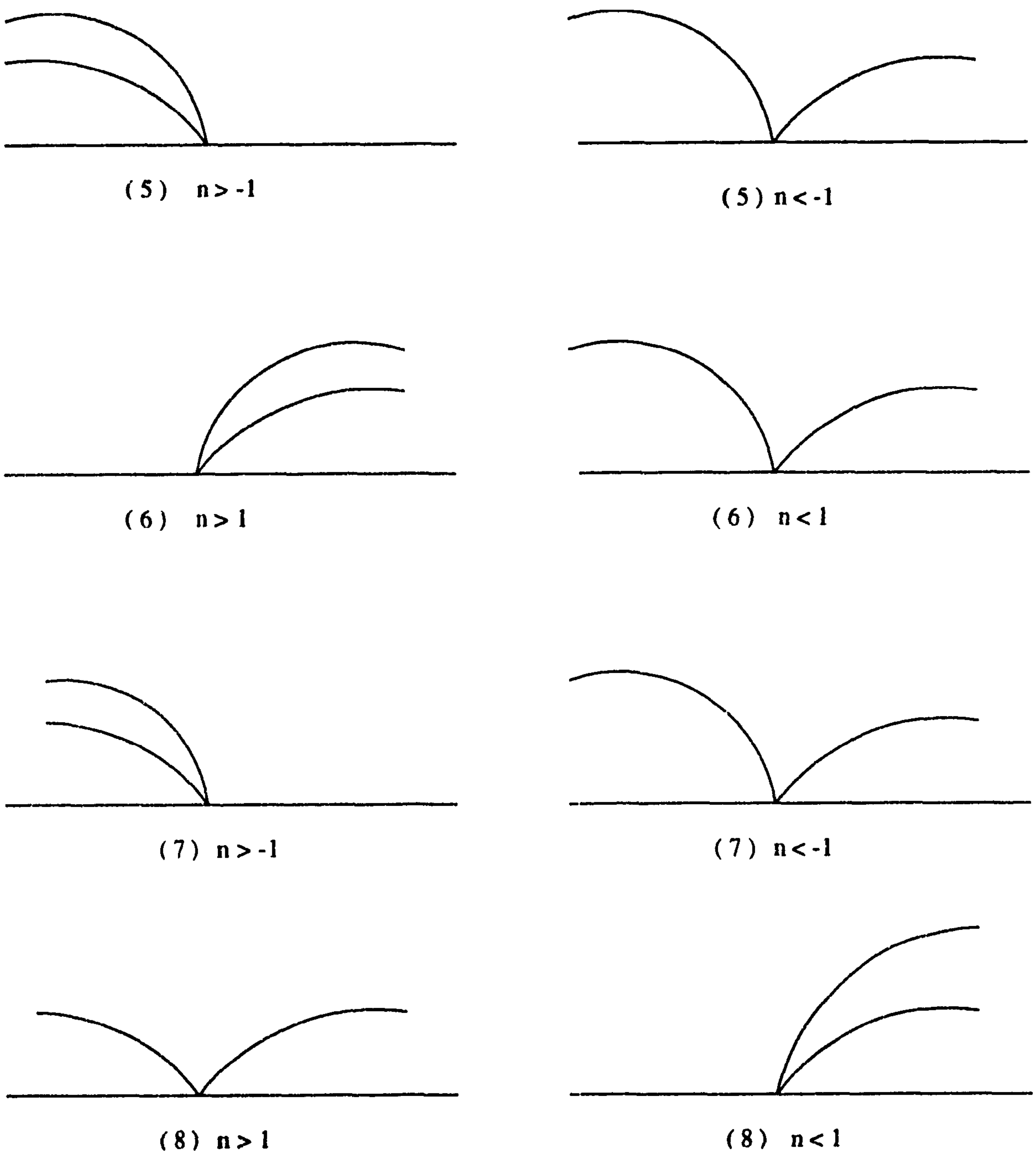

Figure 4: Bifurcation diagrams $\varepsilon_1 \neq \varepsilon_2$, $\varepsilon_3 = \pm 1$

4.3 Fixed-point Bifurcation

Assume that equation (49) has Floquet multipliers $= +1$ at $\gamma = \gamma_0 \neq 0$. Then, from Lemma (4.1.2), if it has one linearly independent T-periodic solution then this solution is even or odd. If it has two linearly independent T-periodic solutions then one is even and the other is odd. Therefore we have two cases.

B1: there exists an odd T- periodic solution of the equation (49).

B2: there are no odd T- periodic solutions.

The assumption **B2** implies that there exists a unique, even, linearly independent, T- periodic solution of equation (49).

4.3.1 Hopf Bifurcation

Assume that **B1** holds. Consider the map in (47)

$$\Phi : \; C^2_{odd}[0,T] \;\rightarrow\; C^0_{odd}[0,T] \tag{159}$$

Similar to the case **A1** , we can get a Lemma like Lemma 4.2.1.1 and apply the L-S method to get the reduced bifurcation equation and finally derive representations of the symmetries on R^1. In summary, the results are given below.

Lemma 4.3.1.1 *1. $C^k_{odd}[0,T]$ is a Banach space with norm defined in (74).*

2. $dim Ker(L) = 1$

3. $C^0_{odd}[0,T] = Ker(L) \oplus Range(L)$

4. $Range(L)$ is closed.

5. L is Fredholm of index 0.

The reduced bifurcation equation is

$$g(x,\gamma) = 0 \tag{160}$$

and $Ker(L) = span(V_0)$, $V_0 = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$.

Let $V = xV_0$

$$\alpha V = V(t+T) = V(t) \quad \Rightarrow \quad \alpha x = x$$

$$\sigma V = -V(-t) = V(t) \quad \Rightarrow \quad \sigma x = x$$

$$\beta V = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = -xV_0$$

Thus

$$\beta x = -x$$

Therefore

$$g(-x, \gamma) = -g(x, \gamma) \tag{161}$$

which implies that $g(x, \gamma)$ has $\mathbf{Z_2}$-symmetry on R^1. Then similar to the calculations and analysis in the case **A1**, we have

Main Theorem 4.3.1 *For each in-phase rotation, there is an infinite series of bifurcation points γ at which the Floquet multipliers are +1. Furthermore, an odd T-periodic out-of-phase solution occurs if the linearization at this point has an odd T-periodic solution.*

4.3.2 Degenerate $Z_2 \oplus Z_2$-Bifurcation

Assume that **B2** holds. This means that $L_2 r = 0$ has a unique, linearly independent, even, T-periodic solution. In this case, $\mathbf{Z_2} \oplus \mathbf{Z_2}$-bifurcation occurs. Instead of considering the mapping Φ on $C^k_{odd}[0, T]$, we now have

$$\Phi : \; C^2[0,T] \rightarrow C^0[0,T] \tag{162}$$

Then we have

Lemma 4.3.2.1 *1. $C^K[0,T]$ is a Banach space with norm defined in (74).*

2. $dim Ker(L) = 2$

3. $C^0[0,T] = Ker(L) \oplus Range(L)$

4. $Range(L)$ is closed.

5. L is Fredholm of index 0.

Take a basis of $Ker(L)$ as

$$V_1 = \begin{bmatrix} s_0' \\ 0 \end{bmatrix} \qquad V_2 = \begin{bmatrix} 0 \\ x_1 \end{bmatrix}$$

Then the bifurcation equations become

$$\begin{aligned} g_1(x,y,\gamma) &\stackrel{\text{def}}{=} < V_1, \Phi(xV_1 + yV_2 + W(x,y,\gamma),\gamma) > \\ g_2(x,y,\gamma) &\stackrel{\text{def}}{=} < V_2, \Phi(xV_1 + yV_2 + W(x,y,\gamma),\gamma) > \end{aligned} \tag{163}$$

These equations commute with the symmetries α, β and σ.

$$\alpha \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \tag{164}$$

$$\beta \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} \tag{165}$$

$$\sigma \begin{bmatrix} x \\ y \end{bmatrix} = - \begin{bmatrix} x \\ y \end{bmatrix}. \tag{166}$$

The bifurcation equations have $\mathbf{Z_2} \oplus \mathbf{Z_2}$-symmetry and we have similar conclusions about normal form and bifurcation diagram as those corresponding to the case **A2**.

Chapter 5

Bifurcations When Damped and Forced

In the preceding two chapters, we studied the bifurcations in SQUIDs without bias current and damping. What will happen when these are present? This is the subject of this chapter.

Consider the system

$$\begin{aligned} \ddot{\phi}_1 + \varepsilon\dot{\phi}_1 + \sin(\phi_1) &= I + \gamma(\phi_2 - \phi_1) \\ \ddot{\phi}_2 + \varepsilon\dot{\phi}_1 + \sin(\phi_2) &= I + \gamma(\phi_1 - \phi_2). \end{aligned} \tag{167}$$

When $\gamma = 0$, this equation is reduced to that of a single pendulum with constant external force,

$$\ddot{\phi} + \varepsilon\dot{\phi} + \sin(\phi) = I \ , \tag{168}$$

for which the dynamics are well understood [5]. The system (168) possesses an asymptotically stable rotation i.e., $\phi(t+T) = \phi(t) + 2\pi$, when $I > I_0(\varepsilon)$ and $\varepsilon > 0$. This rotation is also a monotonically increasing function of t. Throughout this chapter, we assume $I > I_0(\varepsilon)$ and $\varepsilon > 0$.

For the system (167), Maginu [5] observed that the in-phase rotation $\phi_1(t) = \phi_2(t) = \phi(t)$ is asymptotically stable when γ is small and large, but unstable for an intermediate range of γ-values, for suitable values of ε and I. His numerical results indicate the existence of chaotic motion in the unstable range, but he did not study the transitions through chaotic behavior as γ passes through this range. Doedel, Aronson and Othmer [4] numerically obtained that the in-phase rotation is unstable in $(-\infty, 0) \cup (\gamma_1(\varepsilon), \gamma_2(\varepsilon))$ for some nonnegative $\gamma_1(\varepsilon)$ and $\gamma_2(\varepsilon)$, and stable otherwise. They also numerically investigated the transition to chaotic motion. They found that there are period-doubling cascades.

In this Chapter, we theoretically investigate bifurcations from the T-periodic in-phase rotation. In Section 5.2 we discuss the period-doubling bifurcations. It is shown that the bifurcation equation has $\mathbf{Z}_2$ symmetry on R^2 Under certain conditions it is equivalent to a simplified normal form. Bifurcation diagrams are presented. There are 2T-periodic out-of-phase rotations bifurcating from the in-phase rotation. In Section 5.3 we study the fixed-point bifurcations. There are T-periodic out-of-phase rotations bifurcating from the in-phase rotation under certain conditions. The normal form and bifurcation diagrams are qualitatively the same as in Section 5.2.

5.1 Linearization and Symmetry

With the transformation (29), (167) becomes

$$\begin{aligned} s'' + \varepsilon s' + \sin s \cos r &= 0 \\ r'' + \varepsilon r' + \sin r \cos s &= -2\gamma r. \end{aligned} \tag{169}$$

The in-phase solution is given by

$$s = s_0(t) \ , \ r = 0 \ \ s_0(t+T) = s_0(t) + 2\pi \ .$$

Transform

$$s \longrightarrow s \ + \ s_0(t) \, , \qquad r \longrightarrow r \ .$$

Then (169) becomes

$$\begin{aligned} s'' + \varepsilon s' + \sin(s + s_0(t))\cos r - \sin s_0(t) &= 0 \\ r'' + \varepsilon r' + \sin r \cos(s + s_0(t)) + 2r\gamma &= 0 \end{aligned} \tag{170}$$

where $s_0'' + \varepsilon s_0' + \sin s_0 = I$. The 2T-periodic solutions of (170) are 2T-periodic rotations of the equation (167).

Define

$$\Phi(U,\gamma)(t) = \begin{bmatrix} s''(t) + \varepsilon s'(t) + \sin(s(t) + s_0(t))\cos r(t) - \sin s_0(t) \\ r''(t) + \varepsilon s'(t) + \sin r(t)\cos(s(t) + s_0(t)) + 2\gamma r(t) \end{bmatrix} \tag{171}$$

Then

$$\Phi(0,\gamma) = 0 \tag{172}$$

Lemma 5.1

$$\Phi(\alpha U,\gamma) = \alpha\Phi(U,\gamma)$$
$$\Phi(\beta U,\gamma) = \beta\Phi(U,\gamma)$$

where α and β are defined in Chapter 3.
PROOF: Similar to Lemma 3.2.1.

Linearize (170) at $s = r = 0$ to get

$$s'' + \varepsilon s' + \cos s_0(t) s = 0 \tag{173}$$

$$r'' + \varepsilon s' + (\cos s_0(t) + 2\gamma) r = 0 \tag{174}$$

The Floquet multipliers of the equation (173) are

$$\rho_1 = 1 \ , \ \rho_2 = e^{-\varepsilon} < 1$$

The Floquet multipliers of the equation (174) are

$$\rho_3\rho_4 = e^{-\varepsilon} < 1$$

Possible bifurcation occurs when $|\rho_3| = 1$ or $|\rho_4| = 1$. Thus, without loss of generality, we have two cases:

- Period-doubling bifurcation (Floquet multiplier = -1) .
- Fixed-point bifurcation (Floquet multiplier = +1).

The linearization of $\Phi(U,\gamma)$ at $s=0$, $r=0$, $\gamma=\gamma_0$ is given by

$$LU = \begin{bmatrix} s'' + \varepsilon s' + \cos s_0 s \\ r'' + \varepsilon r' + (\cos s_0 + 2\gamma_0) r \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} L_1 s \\ L_2 r \end{bmatrix}. \tag{175}$$

5.2 Period-doubling Bifurcation

In this section we assume that one of Floquet multipliers of the equation (174) is -1 at $\gamma=\gamma_0$.

Consider the mapping

$$\Phi: \; C^2[0,2T] \;\rightarrow\; C^0[0,2T] \tag{176}$$

Then L is Fredholm of index zero.

Lemma 5.2.1 *1. $dim Ker(L) = 2$, $dim Ker(L^*) = 2$*

2. $C^0[0,2T] = Ker(L^*) \dotplus Range(L)$

3. $Range(L)$ is closed.

4. L is a Fredholm of index 0.

Above, L^* is the adjoint operator of L, which is defined as

$$L^*U = \begin{bmatrix} s'' - \varepsilon s' + \cos s_0 s \\ r'' - \varepsilon r' + (\cos s_0 + 2\gamma_0) r \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} L_1^* s \\ L_2^* r \end{bmatrix}.$$

Proof:

1. $L_1 s = 0$ has a T-periodic solution s_0'. According to the assumption, $L_2 s = 0$ has a 2T-periodic solution, say, $r_2(t)$. Then

$$Ker(L) = span(V_1, V_2) \qquad V_1 = \begin{bmatrix} s_0' \\ 0 \end{bmatrix} \quad V_2 = \begin{bmatrix} 0 \\ r_2 \end{bmatrix}.$$

Both L_1 and L_2 have simple Floquet multipliers. So do L_1^* and L_2^* from Lemma 4.1.7. Therefore $L_1^* s = 0$ has a T-periodic solution $s^*(t)$ and $L_2^* r = 0$ has a 2T-periodic solution $r^*(t)$, and

$$Ker(L^*) = span(V_1^*, V_2^*) \qquad V_1^* = \begin{bmatrix} s^* \\ 0 \end{bmatrix} \quad V_2^* = \begin{bmatrix} 0 \\ r^* \end{bmatrix}$$

Thus $dim Ker(L^*) = 2$.

2. From integration by parts, we have $< U, L^*V >=< LU, V >$. Thus $\forall LU \in Range(L)$, we have $< LU, V >=< U, L^*V >= 0 \quad \forall V \in Ker(L^*)$ and hence

$$Range(L) \subseteq Ker^T(L^*)$$

On the other hand

$$\forall W \in Ker^T(L^*) \quad < W, V >= 0 \quad \forall V \in Ker(L^*).$$

Thus $< (W)_1, s^* >= 0 \qquad < (W)_2, r^* >= 0$.

From Lemma 4.1.6, there exists a 2T-periodic solution for the equations $L_1 s = (W)_1$ and $L_2 r = (W)_2$, respectively. So $W \in Range(L)$ and $Range(L) \supseteq Ker^T(L^*)$. Thus

$$Range(L) = Ker^T(L^*)$$

Now $C^0[0, 2T] = Ker(L^*) \oplus Ker^T(L^*)$ and so

$$C^0[0, 2T] = Ker(L^*) \oplus Range(L)$$

3. and 4. are straightfoward.

Remark 5.1 $Ker(L^*)$ is an invariant subspace of the symmetries α and β. In fact, $\beta V_1^* = V_1^*$, $\beta V_2^* = -V_2^*$ and $\alpha V_1^*(t) = V_1^*(t+T) = V_1^*(t)$. Then βV_1^*, βV_2^* and αV_1^* are all in $Ker(L^*)$. $\alpha V_2^*(t) = V_2^*(t+T) = \begin{bmatrix} 0 \\ r^*(t+T) \end{bmatrix}$. $L^* r^*(t+T) = 0$ and $r^*(t+T)$ is 2T-periodic. Thus, for some constant c, $r^*(t+T) = cr^*(t)$. So $\alpha V_2^* = cV_2^* \in Ker(L^*)$.

With this Lemma and Remark 5.1, we can use the L-S reduction with symmetry to study the bifurcations. The bifurcation equations are reduced to

$$g_1(x,y,\gamma) \stackrel{\text{def}}{=} < V_1^*, \Phi(xV_1 + yV_2 + W(x,y,\gamma),\gamma) >$$
$$g_2(x,y,\gamma) \stackrel{\text{def}}{=} < V_2^*, \Phi(xV_1 + yV_2 + W(x,y,\gamma),\gamma) > \tag{177}$$

where V_1^*, $V_2^* \in Ker(L^*)$.

Let $V = xV_1 + yV_2$, then

$$\beta V = xV_1 - yV_2$$

$$\beta \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} \tag{178}$$

To get the representation of α, we do the following. $r_2(t+T)$ is a 2T-periodic solution of $L_2 r = 0$. So $r_2(t+T) = cr_2(t)$, where c is a constant. Then

$$r_2(t) = r_2(t+2T) = cr_2(t+T) \qquad r_2(T) = cr_2(0)$$

and

$$r_2(0) = cr_2(T) = c(cr_2(0)) \qquad r_2'(0) = cr_2'(T) = c(cr_2'(0))$$

At least one of $r_2(0)$ and $r_2'(0)$ can not be zero; otherwise $r_2(t) = 0$. Therefore $c^2 = 1$. And c must be -1; otherwise $r_2(t+T) = r_2(t)$, which is a contradiction. So

$$r_2(t+T) = -r_2(t)$$

and

$$\alpha V = V(t+T) = xV_1(t+T) + yV_2(t+T) = xV_1(t) - yV_2(t)$$
$$\alpha \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} \tag{179}$$

Thus

$$g_1(x,-y,\gamma) = g_1(x,y,\gamma)$$
$$g_2(x,-y,\gamma) = -g_2(x,y,\gamma) \tag{180}$$

which means that $g(x,y,\gamma)$ has Z_2-symmetry on R^2 [33, page 417].

5.2.1 Normal Form

Let $\lambda = \gamma - \gamma_0$

Lemma 5.2.1.1 *There are smooth coefficients $p(u,v,\lambda)$ and $q(u,v,\lambda)$ such that*

$$g_1(x,y,\gamma) = p(u,v,\lambda) \qquad g_2(x,y,\gamma) = q(u,v,\lambda)y$$

where $u = x$, $v = y^2$, $\lambda = \gamma - \gamma_0$ and $p(0,0,0) = 0$, $q(0,0,0) = 0$, $p_u(0,0,0) = 0$

PROOF: This is a direct application of Theorem 4.2.1.1 and

$$g_{1x}(0,0,\gamma_0) = 0 \ , \ g_{2y}(0,0,\gamma_0) = 0$$

Lemma 5.2.1.2 *$p(u,0,\lambda)$ is independent of λ.*

Proof: Similar to Lemma 4.2.2.3.

Lemma 5.2.1.3

$$p(u,v,\lambda) = h_1(u,v) + p_1(u)u^3 + p_2(u,v,\lambda)v \tag{181}$$

$$q(u,v,\lambda) = h_2(u,\lambda) + q_2(u,v,\lambda) \tag{182}$$

where

$$h_1(u,v) = Au^2 + Bv$$
$$h_2(u,\lambda) = Cu + \delta\lambda$$
$$p_2(u,v,\lambda) = p_{21}(u,v,\lambda)u + p_{22}(u,v,\lambda)v + p_{23}(u,v,\lambda)\lambda$$
$$q_2(u,v,\lambda) = Dv + q_{11}(u,v,\lambda)u^2 + q_{12}(u,v,\lambda)uv +$$
$$q_{13}(u,v,\lambda)u\lambda + q_{22}(u,v,\lambda)v^2 + q_{23}(u,v,\lambda)v\lambda + q_{33}(u,v,\lambda)\lambda^2$$

and A, B, C , D and δ are constants.
Proof:
By Lemma 5.2.1.2, one can write $p(u,0,\lambda) = p_0(u)$. From Lemma 5.2.1.2, $0 = p(0,0,0) = p_0(0)$ and $0 = p_u(0,0,0) = p_0'(0)$. By Taylor Theorem [32, page 60]

$$p_0(u) = Au^2 + p_1(u)u^3$$

$$p(u,v,\lambda) - p_0(u) = p_2(u,v,\lambda)v$$

and

$$p_2(u,v,\lambda) = p_2(0,0,0) + p_{22}(u,v,\lambda)v + p_{23}(u,v,\lambda)\lambda$$

Setting $B = p_2(0,0,0)$, we derive the first formula. Applying Taylor's Theorem to $q(u,v,\lambda)$ at (0,0,0) again, we get the second formula.

Define

$$\begin{aligned}
h(u,v,\lambda) &= (h_1(u,v), h_2(u,v,\lambda)) && (183)\\
\eta(u,v,\lambda) &= (\eta_1(u,v,\lambda), \eta_2(u,v,\lambda)) && (184)\\
\eta_1(u,v,\lambda) &= p_1(u)u^3 + p_2(u,v,\lambda)v && (185)\\
\eta_2(u,v,\lambda) &= q_2(u,v,\lambda). && (186)
\end{aligned}$$

Now we compute their derivatives to obtain the normal form. Straightforward computation leads to

$$\begin{aligned}
A &= \frac{1}{2}g_{1xx}(0,0,0) && (187)\\
B &= \frac{1}{2}g_{1yy}(0,0,0) && (188)\\
C &= g_{2xy}(0,0,0) && (189)\\
\delta &= g_{2y\lambda}(0,0,0). && (190)
\end{aligned}$$

Using the formulas in [32, page 295], we find

$$\begin{aligned}
g_{1xx} &= < V_1, d^2\Phi(V_1,V_1) > && (191)\\
g_{1yy} &= < V_1, d^2\Phi(V_2,V_2) > && (192)\\
g_{2xy} &= < V_2, d^2\Phi(V_1,V_2) > && (193)\\
g_{2y\lambda} &= < V_2, (d\Phi_\lambda)V_2 > . && (194)
\end{aligned}$$

The damping terms have no effect on the second order derivatives , so we can still use the formulas (94).

$$d^2\Phi(V_1,V_1) = \begin{bmatrix} -s_0'^2 \sin s_0 \\ 0 \end{bmatrix}$$

$$
\begin{aligned}
d^2\Phi(V_1, V_2) &= \begin{bmatrix} 0 \\ -r_2 s_0' \sin s_0 \end{bmatrix} \\
d^2\Phi(V_2, V_2) &= \begin{bmatrix} -r_2^2 \sin s_0 \\ 0 \end{bmatrix} \\
(d\Phi_\lambda)V_2 &= 2V_2
\end{aligned}
$$

So

$$A = -\frac{1}{2} < s_0', s_0'^2 \sin s_0 > \tag{195}$$

$$B = -\frac{1}{2} < s_0', r_2^2 \sin s_0 > \tag{196}$$

$$C = - < r_2, r_2 s_0' \sin s_0 > \tag{197}$$

$$\delta = 2 < r_2, r_2 > \; > \; 0. \tag{198}$$

Main Theorem 5.1 *If $A \neq 0$, $B \neq 0$, $C \neq 0$ and $\delta \neq 0$, then*

$$RT(h + t\eta, \mathbf{Z_2}) = RT(h, \mathbf{Z_2})$$

PROOF:

According to Table 3.1 in [33, page 177], $RT(h + t\eta, \mathbf{Z_2})$ is generated by

$$[p, 0] \;,\; [vq, 0] \;,\; [0, p] \;,\; [0, q] \;,\; [vp_v, vq_v] \;,\; [up_u, uq_u] \;,\; [vp_u, vq_u] \;,\; \lambda[p_u, q_u]$$

where $p = h_1 + t\eta_1$, $q = h_2 + t\eta_2$ and $t \in [0, 1]$.

Let Ω_t be generated by

$$[p, 0] \;,\; [vq, 0] \;,\; [0, p] \;,\; [0, q] \;,\; [vp_v, vq_v] \;,\; u[vp_v, vq_v] \;,\; [up_u, uq_u] \;,\; [vp_u, vq_u]$$

Obviously, $\Omega_t \subseteq RT(h + t\eta, \mathbf{Z_2})$.

Ω is generated by

$$[v, 0] \;,\; [uv, 0] \;,\; [u^2, 0] \;,\; [v\lambda, 0] \;,\; [0, u] \;,\; [0, v] \;,\; [0, \lambda] \;,\; [0, u^2].$$

We claim

$$\Omega_t = \Omega \quad \forall t \in [0, 1]. \tag{199}$$

If (199) holds, then

$$RT(h + t\eta, \mathbf{Z_2}) = \Omega + < [u\lambda, 0] > \tag{200}$$

which is independent of t and the Theorem is proved. In fact, from the generators of Ω_t,

$$RT(h + t\eta, \mathbf{Z_2}) = \Omega_t + < \lambda[p_u, q_u] > . \tag{201}$$

From Lemma 5.2.1.3,

$$\begin{aligned}
\lambda[p_u, q_u] &= \lambda[p_u, 0] + q_u[0, \lambda] && (202)\\
&= \lambda[2Au + 3p_1(u)u^2 + \tfrac{\partial p_2}{\partial u}v, 0] + q_u(u, v, \lambda)[0, \lambda] && (203)\\
&= 2A[u\lambda, 0] + 3p_1(u)\lambda[u^2, 0] + \lambda\tfrac{\partial p_2}{\partial u}[v, 0] + q_u(u, v, \lambda)[0, \lambda]. && (204)
\end{aligned}$$

All the terms except $2A[u\lambda, 0]$ in the right hand side of (204) are in Ω. So

$$\lambda[p_u, q_u] \in \Omega + < [u\lambda, 0] > . \tag{205}$$

If $A \neq 0$, from (204),

$$[u\lambda, 0] \in \Omega + < \lambda[p_u, q_u] > \tag{206}$$

By (205),

$$\Omega + < \lambda[p_u, q_u] > \subseteq \Omega + < [u\lambda, 0] > \tag{207}$$

By (206),

$$\Omega + < [u\lambda, 0] > \subseteq \Omega + < \lambda[p_u, q_u] > \tag{208}$$

Thus

$$\Omega + < [u\lambda, 0] > = \Omega + < \lambda[p_u, q_u] > . \tag{209}$$

From (199), (201) and (209), (200) is true. To finish the proof, we must establish (199). Write the generators of Ω_t as

$$\begin{aligned}
[p, 0] &= [h_1, 0] + [t\eta_1, 0]\\
[vq, 0] &= [vh_2, 0] + [tv\eta_2, 0]\\
[0, p] &= [0, h_1] + [0, t\eta_1]\\
[0, q] &= [0, h_2] + [0, t\eta_2]
\end{aligned}$$

$$
\begin{aligned}
[vp_v, vq_v] &= [vh_{1v}, vh_{2v}] + [tv\eta_{1v}, tv\eta_{2v}] \\
u[vp_v, vq_v] &= u[vh_{1v}, vh_{2v}] + u[tv\eta_{1v}, tv\eta_{2v}] \\
[up_u, uq_u] &= [uh_{1u}, uh_{2u}] + [tu\eta_{1u}, tu\eta_{2u}] \\
[vp_u, vq_u] &= [vh_{1u}, vh_{2u}] + [tv\eta_{1u}, tv\eta_{2u}] \qquad (210)
\end{aligned}
$$

and

$$
\begin{aligned}
[vh_{1v}, vh_{2v}] &= B[v, 0] \\
u[vh_{1v}, vh_{2v}] &= B[uv, 0] \\
[h_1, 0] &= A[u^2, 0] + B[v, 0] \\
[vh_2, 0] &= C[uv, 0] + \delta[v\lambda, 0] \\
[0, h_1] &= A[0, u^2] + B[0, v] \\
[0, h_2] &= C[0. u] + \delta[0, \lambda] \\
[uh_{1u}, uh_{2u}] &= 2A[u^2, 0] + C[0, u] \\
[vh_{1u}, vh_{2u}] &= 2A[uv, 0] + C[0, v]. \qquad (211)
\end{aligned}
$$

From Lemma 5.2.1.3, we have

$$
\begin{aligned}
[t\eta_1, 0] &= tp_1(u)u[u^2, 0] + tp_2(u, v, \lambda)[v, 0] \\
[tv\eta_2, 0] &= t\eta_2[v, 0] \\
&= t(Dv + q_{11}u^2 + q_{12}uv + q_{13}u\lambda + q_{22}v^2 + q_{23}v\lambda + q_{33}\lambda^2)[v, 0] \\
[0, t\eta_1] &= tp_1(u)u[0, u^2] + t(p_{21}u + p_{22}v + p_{23}\lambda)[0, v] \\
[0, t\eta_2] &= tD[0, v] + tq_{11}u[0, u] + tq_{12}u[0, v] \\
&\quad + tq_{13}u[0, \lambda] + tq_{22}v[0, v] + tq_{23}v[0, \lambda] + tq_{33}\lambda[0, \lambda] \\
[tv\eta_{1v}, tv\eta_{2v}] &= t\eta_{1v}[v, 0] + t\eta_{2v}[0, v] \\
u[tv\eta_{1v}, tv\eta_{2v}] &= tu\eta_{1v}[v, 0] + tu\eta_{2v}[0, v] \\
u[t\eta_{1u}, t\eta_{2u}] &= tu[3p_1(u)u^2 + \tfrac{\partial p_2}{\partial u}v, 0] + t\eta_{2u}[0, u] \\
v[t\eta_{1u}, t\eta_{2u}] &= t\eta_{1u}[v, 0] + t\eta_{2u}[0, v]. \qquad (212)
\end{aligned}
$$

By (211) and (212), all terms in the right hand side of (210) are in Ω. So

$$\Omega \supseteq \Omega_t. \qquad (213)$$

On the other hand, if

$$\Omega \subseteq \Omega_t + \mathcal{U}_{u,v,\lambda}\Omega \tag{214}$$

then, from Nakayama's Lemma, $\Omega \subseteq \Omega_t$, and the claim is true. Now we claim

$$\Omega_t + \mathcal{U}_{u,v,\lambda}\Omega = \Omega_0 + \mathcal{U}_{u,v,\lambda}\Omega \tag{215}$$

Actually, by (212)

$$\begin{gathered} [t\eta_1, 0]\ ,\ [tv\eta_2, 0]\ ,\ [0, t\eta_1]\ ,\ u[tv\eta_{1v}, tv\eta_{2v}]\ ,\ u[t\eta_{1u}, t\eta_{2u}] \\ [0, t\eta_2] - tD[0,v]\ ,\ [tv\eta_{1v}, tv\eta_{2v}] - tD[0,v]\ ,\ v[t\eta_{1u}, t\eta_{2u}] \in \mathcal{U}_{u,v,\lambda}\Omega. \end{gathered} \tag{216}$$

From the second and last equations in (211),

$$[0,v] = \frac{1}{C}(v[h_{1u}, h_{2u}] - \frac{2A}{B}u[vh_{1v}, vh_{2v}]) \in \Omega_0$$

Therefore,

$$[tv\eta_{1v}, tv\eta_{2v}]\ ,\ [0, t\eta_2] \in \Omega_0 + \mathcal{U}_{u,v,\lambda}\Omega$$

And all the generators of Ω_t are in $\Omega_0 + \mathcal{U}_{u,v,\lambda}\Omega$. So

$$\Omega_t + \mathcal{U}_{u,v,\lambda}\Omega \subseteq \Omega_0 + \mathcal{U}_{u,v,\lambda}\Omega \tag{217}$$

However, from (210) and (216),

$$\begin{gathered} [h_1, 0]\ ,\ [vh_2, 0]\ ,\ [0, h_1]\ ,\ u[vh_{1v}, vh_{2v}] \\ u[h_{1u}, h_{2u}]\ ,\ v[h_{1u}, h_{2u}] \in \Omega_t + \mathcal{U}_{u,v,\lambda}\Omega \end{gathered} \tag{218}$$

The fourth and fifth equations in (210) give

$$\begin{aligned} [0, h_2] &= [0,q] - t[0, \eta_2 - Dv] - tD[0,v] \\ [vh_{1v}, vh_{2v}] &= [vp_v, vq_v] - ([tv\eta_{1v}, tv\eta_{2v}] - tD[0,v]) - tD[0,v] \end{aligned} \tag{219}$$

The sixth and last equations in (210) give

$$u[vp_v, vq_v] = u[vB, 0] + u[tv\eta_{1v}, tv\eta_{2v}] \tag{220}$$

$$v[p_u, q_u] = 2A[uv, 0] + C[0,v] + v[t\eta_{1u}, t\eta_{2u}] \tag{221}$$

By (220) and (221)

$$[0,v] = \frac{1}{C}\left(v[p_u,q_u] - \frac{2A}{B}(u[vp_v,vq_v] - u[tv\eta_{1v},tv\eta_{2v}]) - v[t\eta_{1u},t\eta_{2u}]\right) \ ,$$

and, from (216),

$$[0,v] \in \Omega_t + \mathcal{U}_{u,v,\lambda}\Omega \tag{222}$$

From (216) , (219) and (222)

$$\begin{aligned} [0,h_2]\ ,\ [vh_{1v},vh_{2v}] &\in \Omega_t + \mathcal{U}_{u,v,\lambda}\Omega \\ \Omega_0 &\subseteq \Omega_t + \mathcal{U}_{u,v,\lambda}\Omega \\ \Rightarrow \qquad \Omega_0 + \mathcal{U}_{u,v,\lambda}\Omega &\subseteq \Omega_t + \mathcal{U}_{u,v,\lambda}\Omega \end{aligned} \tag{223}$$

By (217) and (223), we have (215).
From (214) and (215), if

$$\Omega \subseteq \Omega_t + \mathcal{U}_{u,v,\lambda}\Omega = \Omega_0 + \mathcal{U}_{u,v,\lambda}\Omega \tag{224}$$

then (199) is true. Define

$$\begin{aligned} X &= [\ [v,0]\ ,\ [uv,0]\ ,\ [u^2,0]\ ,\ [v\lambda,0]\ ,\ [0,u]\ ,\ [0,v]\ ,\ [0,\lambda]\ ,\ [0,u^2]\ ]^T \\ b &= [\ v[h_{1v},h_{2v}]\ ,\ uv[h_{1v},h_{2v}]\ ,\ [h_1,0]\ ,\ [vh_2,0]\ ,\ [0,h_1]\ ,\ [0,h_2]\ , \\ &\qquad u[h_{1u},h_{2u}]\ ,\ v[h_{1u},h_{2u}]\ ]^T. \end{aligned}$$

By (211)

$$\Theta X = b$$

where Θ is given by

$$\begin{bmatrix} B & & & & & & & \\ & B & & & & & & \\ B & & A & & & & & \\ & C & & \delta & & & & \\ & & & & & B & & A \\ & & & & C & & \delta & \\ & & 2A & & C & & & \\ & 2A & & & & C & & \end{bmatrix}$$

The determinant of this matrix is $-A^2B^2C^2\delta^2 \neq 0$; X is also the linear combination of b. Thus

$$\Omega \subseteq \Omega_0 \tag{225}$$

Main Theorem 5.2 *If $A \neq 0$, $B \neq 0$, $C \neq 0$ and $\delta \neq 0$, then the bifurcation equation(180) is strongly $\mathbf{Z_2}$-equivalent to $(h_1(u,v), h_2(u,\lambda)y)$.*

PROOF: This is from applying Theorem 1.3 [33, page 168] to the result of the last Theorem.

5.2.2 Bifurcation Diagrams

The solutions of the normal form can be solved from the following equations.

$$h_1 = Ax^2 + By^2 = 0 \quad h_2 = Cxy + \delta\lambda y = 0$$

$$\det dh = h_{1u}h_{2v} - h_{1v}h_{2u} = -BC$$

Notice that $\delta > 0$ from (198), we have

(1) Steady-state solution: $x = 0$, $y = 0$;

(2) Periodic solutions: $\lambda = -\frac{1}{\delta}Cx$ and $\lambda = \pm\frac{1}{\delta}\sqrt{\frac{|B|}{|A|}}Cy$ if $AB < 0$.

Thus

Main Theorem 5.3 *If $AB < 0$, there exists a 2T-periodic rotation bifurcating from the T-periodic rotation.*

The bifurcation diagrams of $\lambda = \frac{1}{\delta}Cx$ are thus shown in Figure 5

5.3 Fixed-point Bifurcation

In this section we assume that one of Floquet multipliers of the equation (174) is +1 at $\gamma = \gamma_0$.

Consider the mapping

$$\Phi : \quad C^2[0,T] \rightarrow C^0[0,T] \tag{226}$$

Then

Figure 5: Period-doubling Bifurcation $AB < 0$

Lemma 5.3.1 *1. $dimKer(L) = 2$, $dimKer(L^*) = 2$*

2. $C^0[0,T] = Ker(L^*) \oplus Range(L)$

3. $Range(L)$ is closed.

4. L is Fredholm of index 0.

Also $Ker(L^*)$ is invariant under the symmetries α and β. Thus the bifurcation equation is reduced to

$$g_1(x, y, \gamma) = 0$$
$$g_2(x, y, \gamma) = 0 \tag{227}$$

which commutes with the symmetries α and β.

Now $\alpha V = V(t + T) = V(t)$, then the representations become

$$\alpha \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \tag{228}$$

$$\beta \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} . \tag{229}$$

Thus the bifurcation equation (227) is Z_2-symmetry on R^2, and the analysis, normal form and bifurcation diagrams are qualitatively the same as the ones in the last section.

References

[1] Ben-Jacob, E., Bergman, D.J., Imry, Y., Matkowsky, B.J. Schuss,Z. 1983. Thermal activation from the fluxoid and the voltage states of dc SQUIDs. *J. Appl. Phys.* **54**(11), 6533–6542.

[2] Hahlbohm, H.D., Lübbig,H. (eds) 1985. SQUIDS'85 - *Superconducting Quantum Interference Devices and their Applications.* Walter de Gruyter. New York.

[3] Riitta Hari & Olli V. Lounasmaa. 1989. Recording an Interpretation of Cerebral Magnetic Fields. *Science.* Vol. 244. 432-436.

[4] Doedel, E,J., Aronson, D.G. , Othmer, H.G. 1988. The dynamics of coupled current-biased Josephson junctions I. *IEEE Trans. Circuits and Systems.* **35**(7). 810–817.

[5] K.Maginu. 1983. Spatially homogeneous and inhomogeneous oscillations and chaotic motion in the active Josephson Junction line. *SIAM J. Appl. Math.* No.2. 225–243.

[6] Aronson, D.G. Doedel, E,J., Othmer, H.G. 1991. The dynamics of coupled current-biased Josephson junctions *Int. J. Bifurcation and Chaos.* Vol. 1, No. 1. 51–66.

[7] Henderson, M., Levi, M. & Odeh,F. 1991. The geometry and computation of the dynamics of coupled pendula. *Int. J. Bifurcation and Chaos.* **1**(1), 27–50.

[8] Holmes, P.J., and Marsden, J.E. 1982. Perturbations of n degree of freedom Hamiltonian systems with symmetry. *Comm. Math. Phys.*, **82**, 523–544.

[9] Holmes, P.J., and Marsden, J.E. 1982. Horseshoes in perturbations of Hamiltonians with two degrees of freedom. *J. Math. Phys.*, **23**(4), 669 675.

[10] Golubitsky, M. and Stewart, I.N. 1985. Hopf bifurcation in the presence of symmetry. *Arch. Rational Mech. Anal.* **87**, No.2, 107–165.

[11] Golubitsky, M. and Stewart, I.N. 1986. Symmetry and stability in Taylor- Couette flow. *SIAM J. Math. Anal.* **17**, No.2, 249–288.

[12] Golubitsky, M. and Stewart, I.N. Hopf bifurcation with dihedral group symmetry: coupled nonlinear nonlinear oscillators. Multiple parameter bifurcation Theory. *Contemporary Mathematics* **56**, A.M.S., Providence.

[13] Imry,Y. & Schulman, L. 1978. Qualitative theory of the nonlinear behavior of coupled Josephson junctions. *J. Appl. Phys.* **49**(2), 749 -758.

[14] A. Vanderbauwhede. 1986. Bifurcation of subharmonic solutions in time reversible systems. *J. Applied Math. Physics*(ZAMP). Vol. 37.

[15] Zimmerman, J.E. & Sullivan, D.B. 1977. High-frequency limitations of the double-junction SQUID amplifier. *Appl. Phys. Lett.* **31** (5), 360 362.

[16] Doedel,E.J., Jepson, A.D. &Keller, H.B. 1984. Numerical methods for Hopf bifurcation and continuation of periodic solution paths. in *computing Methods in Applied Sciences and Engineering* VI. eds Glowinski. R. & Lions. J. L. North-Holland.

[17] Holodniok,M. & Kubicěk, M. 1984. **DERPER** - An algorithm for the continuation of periodic solutions in ordinary differential equations. *J. Comp. Phys.* **55**, 254–267.

[18] Keller, H.B. & Jepson, A.D. 1984 Steady state and periodic solution paths: Their bifurcations and computations, in *Bifurcation: Analysis, Algorithms and Applications*, eds. Küpper, T., Mittelmann, H.D. & Weber,H. Birkhäuser ISNM series, no. 70, 219–246.

[19] J. Guckenheimer. and P. Holmes. 1983. *Nonlinear oscillations, dynamical system, and bifurcations of vector fields.* Spring-Verlag,New York.

[20] S. Wiggins. 1990. *Introduction to Applied Nonlinear Dynamical Systems and Chaos.* Springer-Verlag. New York.

[21] W. Goldberg. 1976. Necessary and sufficient conditions for determing a Hill's equation from its spectrum. *J. Math. Anal. Appl.*, Vol. 55, 549–554.

[22] D.G. Aronson, M. Golubitsky and M. Krupa. 1991. Coupled arrays of Josephson junctions and bifurcation of maps with S_N symmetry. *Nonlinearity.* **4**, 861–902.

[23] A.A. Abidi and L.O. Chua. 1979. On the dynamics of Josephson-junction circuits. *Electronic circuits and systems.* Vol. 3, No. 4, 186–200.

[24] G. Grüner & A. Zettl. 1985. CDW conduction: a novel collective transport phenomenon in solids. *Phys. Rep.* **119**, 119–232.

[25] M.Levi. 1984. Beating modes in the Josephson junction. In *Chaos in Nonlinear Dynamical Systems*, ed. J. Chandra. SIAM, Philadelphia, 56–73.

[26] M.Levi. 1988. Caterpillar solutions in coupled pendula. *Ergod. Th. and Dynam. Sys.* **8**, 153-174.

[27] Minorsky.N. 1983. *Nonlinear Oscillations.* Robert E. Krieger Publishing Co.. Malabar. Fla.

[28] J.K.Hale 1969 *Ordinary Differential Equation.* Wiley-Interscience. New York.

[29] M.G. Crandall and P.H. Rabinowitz. 1971. Bifurcation from simple eigenvalue. *J. Funct. Anal.*, **8**, 321–340.

[30] M.G. Crandall and P.H. Rabinowitz. 1973. Bifurcation, perturbation of simple eigenvalues and linearized stability. *Arch. Rat. Mech. Anal.*, **52**, 161–180.

[31] P.Hartman. 1964. *Ordinary Differential Equations.* Wiley, New York.

[32] M.Golubitsky. and D.G. Schaeffer. 1985. *Singularities and Groups in Bifurcation Theory.* Vol 1.

[33] M.Golubitsky. I.N. Stewart. and D.G. Schaeffer. 1988. *Singularities and Groups in Bifurcation Theory.* Vol 2.

[34] Arnold, V. I. 1981. *Singularity Theory.* London Mathematical Society Lecture Notes Series, 53, Cambridge University Press, Cambridge.

[35] Arnold, V. I. 1983. *Geometrical Methods in the Theory of Ordinary Differential Equations.* Springer.

[36] Aubry, S. & LeDaeron. 1983. The discrete Frenkel-Koutorova model and its extensions., *Physica* **8D**, 381–422.

[37] Mather,J., 1982. Existence of quasi-periodic orbits for twist homomorphisms. *Topology* **21**, 457–467.

[38] Moser, J.K., 1973. Stable and Random Motions in Dynamical System. *Study* 77. Princeton University Press.

[39] L.M. Milne-Thomson. 1950. *Jacobian Elliptic Function Tables.* Dover Publications Inc.

[40] Lichtenberg,A.J., and Lieberman, M.A. 1982. *Regular and Stochastic Motion.* Spring–Verlag: New Work, Heidelberg, Berlin.

[41] Salam,F.M.A. & Sastry, S.S., 1984. The complete dynamics of the forced Josephson junction circuit: The region of chaos. *Chaos in Nonlinear Dynamical Systems*, ed. Chandra, J. (SIAM).

[42] Salam, F.M.A. 1987. The Melnikov technique for highly dissipative systems, *SIAM J. Appl. Math.* **47**, 232–243.