# Dynamic Analysis of Sandwich Beam and Frame Systems

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A Thesis

In

The Department

of

Building, Civil and Environmental Engineering

Presented in Partial Fulfilment of the Requirements
for the Degree of Master of Applied Science (Civil Engineering) at
Concordia University
Montreal, Quebec, Canada

July 2003

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*Your file Votre référence*
*ISBN: 0-612-83863-3*
*Our file Notre référence*
*ISBN: 0-612-83863-3*

ABSTRACT

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The dynamic analysis of a general class of sandwich beam and frame systems is presented. The beam element employed in the analysis has three degrees of freedom per node, corresponding to the deflection $v$, the rotation of the beam $\phi$ and the slope $\partial v / \partial x$. The stiffness and consistent mass matrices of the sandwich beam and frame systems are derived by using the inhomogeneous solution to the fourth order differential equation as the interpolating function between the degrees of freedom of each node in the element. The consistent mass matrix of the sandwich beam element is developed by means of a special shape function consistent with the displacement functions used in the derivation of the stiffness matrix. Superposing the element matrices of the individual elements, the total stiffness and mass matrices of the entire structure can be constructed by a method known as the direct stiffness method which presents the results in a format ideally suited for computer implementation. Static and dynamic problems are included for demonstration of the consistency of the theory; the results are comparable with results published in the literature and the effects of certain important parameters on the frequency results are evaluated. With the introduction of the uniform axial displacement, the element is applicable to frame analysis.

**Key words:** Sandwich Construction; Beam and Frame Systems; Governing Equation; Stiffness Matrix; Consistent Mass Matrix; Dynamic Analysis; Frequency; Eigen-value; Finite Element Methods; Structural Analysis; Structural Design.

## ACKNOWLEDGMENTS

Thanks to God Almighty for making this task a possibility at this time of my life.

I wish to thank the International Council for Canadian studies for the funding they provided, through the length of this program under Canadian Commonwealth Fellowship and Scholarship program. The research grant by Concordia University is thankfully acknowledged too.

I am sincerely grateful to my thesis supervisor Dr. Kinh H. Ha for his invaluable suggestions and criticisms during the preparation of this work. His prompt response to my requests is a major factor towards the speedy completion of this work.

I owe a bond of love to the Dura-conteh family of which Dura, Isata, Abdul, Yayah, Aminata, Musa, Gibrilla, Abu and Mohamed are in the nucleus.

Finally, I wish to express my deep gratitude to my wife Mariatu Ballay Conteh for her unrelentless support and encouragement during the preparation of this work.

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The following symbols and notations are used throughout the work;

| Symbol | Description | Unit |
|---|---|---|
| $A_1$, $A_2$ | Cross-sectional areas of faces | $mm^2$ |
| $(AE)_1$, $(AE)_2$ | In-plane stiffness of faces | $N$ |
| $\mathbf{B}$ | Force transformation matrix | - |
| $b$ | Effective width | $mm$ |
| $c$ | Thickness of core | $mm$ |
| $d$ | Distance between centroids of faces | $mm$ |
| $d_1$, $d_2$ | Distances from the centroids of the faces to the centroid of the Sandwich beam | $mm$ |
| $E_1$, $E_2$ | Elastic modulus of the faces | $N/mm^2$, $MPa$ |
| $(EI)_{f1}$, $(EI)_{f2}$ | Local bending stiffness of the faces | $Nmm^2$ |
| $F$ | Axial force in faces | $N$ |
| $\mathbf{f}$ | Element flexibility matrix | - |
| $G$ | Shear modulus of core | $N/mm^2$, $MPa$ |
| $I_1$, $I_2$ | First moment of areas of the faces | $mm^4$ |
| $\mathbf{K}$ | Element stiffness matrix | - |
| $L$ | Length of Element | $mm$ |
| $\mathbf{M}_a$ | Element Mass contribution due to axial action | $kg$ |

| | | |
|---|---|---|
| $M_d$ | Section moment due to membrane action in faces | $Nmm$ |
| $M_{f1}$, $M_{f2}$ | Moment due to local bending in faces | $Nmm$ |
| $M_o$ | End moment | $Nmm$ |
| $\mathbf{M}_r$ | Element Mass contribution due to rotation | $kg$ |
| $\mathbf{M}_T$ | Element Mass contribution due to translation | $kg$ |
| $\mathbf{N}$ | Shape function matrix from displacement function | - |
| $\mathbf{N}_a$ | Axial shape function matrix | - |
| $\mathbf{N}_x$ | First differential of $N$ with respect to $x$ | - |
| $\mathbf{N}_\phi$ | Shape function matrix from rotation function | - |
| $S$ | Shear stiffness of sandwich beam | $N/mm$ |
| **R, Q** | Nodal force vector | $N$ |
| **r, q** | Nodal displacement vector | $mm$ |
| **T** | Coordinate transformation matrix | - |
| $t_1$, $t_2$ | Facing thicknesses | $mm$ |
| $u_1$, $u_2$ | In-plane displacements (due to bending) of points in the center of the faces | $mm$ |
| $V_c$ | Shear force in core | $N$ |
| $V_{f1}$, $V_{f2}$ | Shear forces in Faces | $N$ |
| $v$ | Deflection in beam | $mm$ |

| | | |
|---|---|---|
| $\omega$ | Natural frequency | $rad/\sec$, $Hz$ |
| $x$, $y$ | Rectangular coordinate system | $mm$ |
| $\rho$ | Density (Mass per unit volume) | $kg/mm^3$ |
| $\lambda$ | Frequency parameter | - |
| $\gamma$ | Shear strain in core | - |
| $\tau$ | Shear stress in core | $N/mm^2$, MPa |
| $\phi$ | Section rotation of face | $rad$ |
| $(AE)_f$ | Parameter defined by (2.5) | - |
| $(EI)_f$ | Parameter defined by (2.7) | - |
| $\alpha$ | Parameter defined by (2.17) | - |
| $\mu$ | Parameter defined by (2.35) | - |
| $M_f$ | Parameter defined by (2.6) | - |

Subscript

| | |
|---|---|
| $c$ | Core |
| $f$ | Facing |
| $i$, $j$ | Left and right end of member |
| $1,2$ or $f1, f2$ | Property referring to dissimilar faces |

Operators

| | |
|---|---|
| ${}^{d}\!/_{dx}$ | Ordinary differential with respect to $x$ |
| $\int dx$ | Ordinary integral with respect to $x$ |

# CHAPTER 1

# INTRODUCTION (GENERAL OVERVIEW)

## 1.1 BACKGROUND (DEFINITION)

Beams and frames systems made of more than one material are often used in structural systems to utilize the advantages of the different materials in the composite. As an illustration, a reinforced concrete structure comprises of two principal materials having specific functions; the concrete is excellent in compression but performs badly in tension whilst the steel can resist tensile forces that may produce bending in the system. The inclusion of a material in a composite should take into account its function within the composite. The increased use of advanced composites as high performance structural

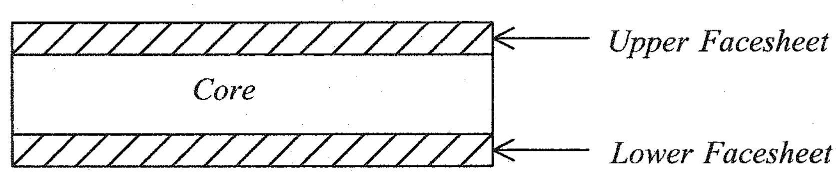


Figure 1.1 Sandwich Panel Model

components necessitates an accurate prediction method that reflects their multi-layered an-isotropic behavior.

A typical configuration of a sandwich panel model for use in general construction is shown in Figure 1.1. According to the American Society for Testing Materials (ASTM), "*a structural sandwich construction can be defined as a special form of a laminated composite, comprising of a combination of different materials that are bonded together to each other so as to utilise the properties of each separate element to the structural advantage of the whole assembly*".

Sandwich construction is a special class of laminates where the inner layers are composed of more flexible materials. It usually consists of three layers of which the two outer layers are of high-strength material while the core is often of low strength. It may also be configured as multiple cores with multiple facings. In any efficient sandwich construction, the face sheets, which are of high strength, act principally in direct tension and compression. The core serves to keep the facial layers at the correct distance apart, must not allow one face to slide over the other, must be able to take transverse shear and may also act as thermal barrier. Transforming the composite to an equivalent cross section containing the same material will serve to emphasize the difference in material properties and functions. The behaviour of a sandwich beam in terms of material properties is comparable to the I-beam, which is an efficient structural shape because the stiffer material is placed in the flange situated farthest from the neutral axis[1]. In this comparison, the core is replaced with the equivalent cross-section of the face sheet material.

[1] Centre of bending of a structure; The midpoint of a symmetric structure

Structural engineers use sandwich construction to achieve a stiff, lightweight structure. The use of rectangular Sandwich plates in many branches of engineering such as civil, mechanical, aeronautical and marine engineering has been extensive.

Sandwich panels are available today for a wide variety of applications in building structures. Their use ranges from simple walls to applications in the refrigeration and air-conditioning sector. A number of attractive solutions are offered by sandwich panels in building, be it roofs or walls. They live up to every architectural and structural challenge in the building industry. Sandwich panel construction is suitable for all load-bearing systems and is a viable option for virtually all applications. Modern building materials must meet a lot of demands. Airports, climate-controlled rooms, cold stores, exhibition halls, hotels, power plants, sports facilities to mention few are some of the facilities in building structures that make use of sandwich panels. Sandwich panels are easy and fast to work with under all weather conditions at the construction site; they therefore allow accelerated project schedule.

In the foreseeable future, sandwich construction will be used extensively in weight sensitive structures since it offers the possibility of achieving high bending stiffness with small weight penalty. One of the unique features of sandwich structures is that by adjusting the material and geometric parameters of the face-sheets and core, various sandwich structures can be optimally created for special applications.

## 1.2 BEHAVIORAL CHARACTERISTICS OF SANDWICH BEAM SYSTEMS

When sandwich construction is subjected to transverse loading, facial materials tend to resist bending and the core resists the shear stresses as well as compressive stresses normal to the panel. Membrane action[2] is the primary carrier of end moment leaving only a small amount of the end moment to be resisted by the faces particularly when the end condition is rigid insert. The most distinct difference in the behavioural characteristics of the Sandwich beam construction to a homogenous isotropic beam is that the former has a low resistance to shear parallel to the beams length.

The deformation in Sandwich systems (beams, frames, etc.) is of two parts as shown in Figure 1.2;

i. deformation due to bending, $\Delta_b$

ii. deformation due to shear, $\Delta_s$

Consider the length $dx$ of the sandwich beam section, Figure 1.2, $\Delta_b^{'}$ and $\Delta_b^{''}$ are the first and second derivatives of the bending deflection respectively and $\Delta_s^{'}$ is the first derivative of the shear deformation. By superposing the deformation due to bending and that due to shear, the complete deformation of the sandwich beam is obtained. The contribution of the bending deformation to the total deflection is more pronounced in slender beams, almost all the deflection of the beam is due to the bending deformation in this case. As the depth of the beam is increased, the contribution of the shear deformation to the total deformation increases. Stresses are developed along the profiles of the faces due to bending of the faces about their own centroidal axis. The upper portion of the

[2] Membrane action is analogous to axial stresses of the faces, resist the stretching and shrinking of fibers in a given cross-section

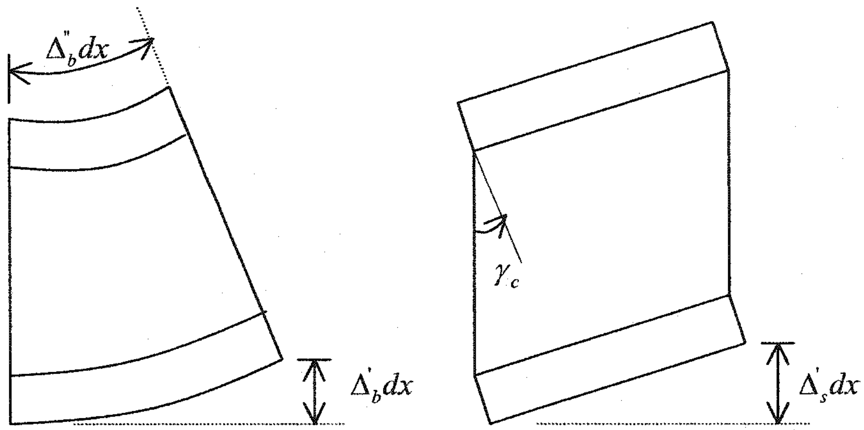


Figure 1.2 Deformation in a Sandwich Element (a) Bending deformation
(b) Shear deformation

profiles is in compression, while the lower portions are in tension. The study of the behaviours of structures under the influence of external excitation proved that the structure can take various shapes which differ in frequencies; thus the study of the static case alone is an under-estimation of the analysis.

Vibration differs from static behaviours in two important respects.

i. First, the vibration of the system with an external force with respect to time.
ii. Secondly, the motion of the structure gives rise to inertia forces.

Inertia forces correspond to the changing momentum and are distributed along the structure in proportion to its mass. The applied loads, the inertia forces and the elastic resistance are in a continually changing state of dynamic equilibrium. Structural calculations for static loads are generally much easier than for dynamic excitation and that is why structural engineers prefer to adopt equivalent static forces as far as possible for the analysis. However, most forms of loading have dynamic components and some forms of structure, especially if they are slender are susceptible to the dynamic effect.

A very important consideration in structural design is the prevention and/or control of vibration in the structure. Within the general framework of random vibration theory, three principle problems can be identified depending on whether attention is focused on the response, the excitation, or the dynamic system. Most commonly, the system and its excitation are to be known and the problem is to predict the statistical information about the dynamic response or reliability of the system. The most critical state in a structural system is when the frequency of excitation of the system approaches that of the natural frequency of the structure (resonance) in which case there is the possibility of structural failure.

The determination of the natural frequency of the structural system by the use of mathematical models of the system is of great significance. An approach on the dynamic mechanical response of a sandwich structure involves the computation of its natural frequencies corresponding to the different mode shapes. Every snapshot of the system during motion is a mode shape. This in general emphasizes the difference in analysis of the static situation of a sandwich construction and that, which involves vibration.

## 1.3 MATERIALS AND MATERIAL PROPERTIES

The choice of materials is vast and since the introduction of fibre composites the choice of face materials has increased to an infinite number of different materials, all with different properties. Choice of material in a sandwich structure depends on the material requirements (high strength, environmental resistance, surface finish, etc). The number of available cores has increased dramatically in recent years since the introduction of more and more competitive cellular plastic. Combination options of the face sheet materials with different core materials for roofs and walls enable new ideas to be integrated in a wide range of applications. Hence the design of sandwich structures is just as much a materials selection problem as a sizing problem.

It is incumbent on the engineer/designer to have reliable information about the strength and stiffness of the materials used in the design for efficient analysis and design of sandwich structures. The best bet is to resort to tests for obtaining adequate material properties. The vast number of material choices may appear as an additional complexity but is really one of the main features of using sandwich constructions; the materials best suited for a specific application may be utilized and some drawbacks can be overcome by geometrical sizing. By increasing the thickness of the core some reinforced plastics can assume high stiffness comparable to that of metals. Thus, the primary objective of the designer is to achieve an efficient design that will utilize each material component to its ultimate limit. The design process typical with sandwich structures is parametric design[3].

Various possibilities exist in the design process; the first and most common is that which requires the determination of the thickness of the core given that the material

[3] This type of design concerns with modification of dimensions; If the dimension used in the design does not meet the design then member sizing is required for a repeat design

properties and thickness of the facial materials are provided. The second type requires the determination of the thickness of the faces and core given that the materials are clearly specified. The design procedure for the above design processes and others are clearly outlined in [2].

### 1.3.1 FACE MATERIALS

A face sheet material can be obtained from any structural material that is available in the form of a thin sheet and can serve its purpose. Two main groups exist for facial materials. The first, which is considered the largest, contains materials such as plywood, cement, reinforced plastic, and fibre components. The latter group contains steel, stainless steel and aluminium alloys. Following are the functions of the face material;

- High tensile and compressive strength
- High stiffness giving high flexural rigidity
- Impact resistance
- Surface finish
- Environmental resistance (chemical, UV, heat, etc.)
- Wear resistance

Fibre composites has since its introduction been used extensively in sandwich construction since most composites offers strength properties similar to or even higher than those of metals, though the stiffness is often lower in magnitude. Thus, with a light core, the composites produce high rigidity. A second reason is that the manufacturing process is easier than that of metal. The anisotropic behavior of the composite posed some complexity for the engineer but in reality it offers the possibility for the materials to be placed in the best way possible to support applied loads. A detailed description of fibre-reinforced composite materials is outlined in [3]. "*Almost any structural material which is available in the form of a thin sheet may be used to form the faces of a sandwich panel* [2]". This statement is a pointer to the wide variety of materials ranging from wood, plastics to metals. Typical mechanical examples of some commonly used face materials can be found in [2] and [5].

### 1.3.2 CORE MATERIALS

A core must be chosen such that the least load possible is added to the total weight of the sandwich structure. In serving its key functions, the core must not change in thickness, thus requiring a fairly high modulus of elasticity perpendicular to the faces. The core is exposed to shear so that global deformations and core shear stresses are produced by the shear strains in the core. The selection of a core that will not fail under the applied load is a primary objective of the designer. The thermal and acoustical insulation properties of the sandwich structure depend on the core material used and also the core thickness. Following are the immediate characteristics of the core material;

- Low density
- Stiffness perpendicular to the faces
- Thermal insulation
- Shear modulus and shear strength

In load carrying sandwich construction, the core used can fall into one of the following groups; corrugated, honeycomb, balsa wood and cellular foams.

A Corrugated material is one that is been bend into ridges, folded to take various shapes to produce stronger material. Several corrugated and flat materials can be glued together to increase the effective thickness of the core. The shear stress in the direction of the corrugations can be so great that it can be taken as infinite.

Honeycomb cores are commonly used in aerospace applications and it exists in a variety of shapes that depends primarily on the application. Honeycombs have the excellent mechanical properties, very high stiffness perpendicular to the faces and the highest shear stiffness and strength to weight ratios of all available core materials. For any given density a honeycomb core is expected to be stiffer than a corrugated core (in the plane perpendicular to the corrugations). High cost is the main drawback in the use of honeycomb as core material.

Balsa was the first material used as core in load carrying sandwich structures. It is a wood but can be seen under a microscope as a high aspect ratio closed-cell structure. The sensitivity of Balsa to humidity is high such that there is rapid decline in its properties with the water content. By utilizing Balsa in its "end-grain" shape (it is cut up in cubic pieces and bonded together edge wise so that its fibre direction is always in the direction of the face), this problem can be overcome. The drawback is that all the blocks have different densities and the limit must be from the piece having the lowest properties.

Cellular foams do not offer the same high stiffness and strength to weight ratios as honeycombs but have other very important advantages. Cellular foams are in general less expensive than honeycomb but more importantly, foam is a solid on a macroscopic level making the manufacturing of sandwich elements easier; the foam surface is easier to bond to, surface preparation and shaping is simple and connection of core blocks are easily performed by adhesive bonding. Cellular foams offer high thermal insulation, acoustical damping, and the closed cell structure of most foams ensure that the structure will become buoyant and resistant to water penetration. However, there is the existence of different foams with different advantages and disadvantages [1] and typical mechanical and thermal properties of some core materials [12].

## 1.4 LITERATURE REVIEW

### 1.4.1 OVERVIEW

Much research has been done on the bending, vibration and buckling problems of sandwich construction. Different physical characteristics exists for the face sheets and core components of a sandwich construction, and as such the analysis of a sandwich structure (plate/shell) requires a degree of sophistication that is greater than that of classical or Reissner-Mindlin plate/shell theory.

As an enforcement of the fact that the structural properties of the composite improves for the better as the thickness of the core material is doubled Dan Zenkert [1] estimated the corresponding increase in strength stiffness and strengths and their relative properties derived. In his study, it was confirmed that when the distance between the centroids of the facings is doubled, the bending strength is doubled and the flexural rigidity is increased by four times the existing. Potter, K [4] pointed out that with only a 6% increase in weight of the beam due to the introduction of a honeycomb core material now sandwich beam, such that the thickness is increased by a factor of four, the bending stiffness and strengths are increased by 37 and 9 respectively. Thus sandwich construction has attracted the attention of many researchers and the number of papers that has been published on this topic in recent years can reflect its importance.

### 1.4.2 REVIEW OF PREVIOUS STUDIES

There is considerable body of literature on the vibrations of rectangular plates. Iguchi (1937, 1940) studied plates with different boundary conditions by means of analytical methods, and in particular the free vibrations of the completely free plate are examined in Iguchi (1953). In his latter paper, exact solutions are given for the eigen-functions by writing them in a sum of two different Levy expansions and by considering the different symmetries and asymmetries. An approach similar to that mentioned above was respected by Gorman and Sharma (1976) [6], Gorman (1978) and in the book of Gorman (1982) [8], which all deal exclusively with the free vibrations of rectangular plates. Leissa (1973) [7] used similar conditions and referred to Iguchi's results in his book, "free vibration of rectangular plates for different boundary conditions". His approach is related to that of Gorman and Iguchi and is also analogous to procedures used in the static plate problems. General assumption for deformation considered by most of the papers mentioned above is that of Vlasov's rigid contour assumption, which states that, "*the cross-section remains undisturbed during deformation or plane cross section assumption, that is to say, the original plane cross-section remains plane during bending*".

According to Bernoulli, straight lines normal to the middle surface before deformation remain straight, normal to the middle surface and unchanged in length after deformation. This theory is comparable to the classical plate theory, which assumes that "*plane sections normal to the mid-plane before deformation remain plane and normal to the mid-plane after deformation*". In effect the initial and final positions of all points on the surface are known based on the fact that the initial and final positions of points on the middle surface are known. The strains at any point on the surface can be calculated in terms of the displacement of the middle surface alone. This helps in converting a three-dimensional problem to two-dimensional one, and a two-dimensional problem to a one-dimensional one. In the paper, Stiffness Matrix analysis for exact solution of Sandwich beam and Frame systems [25], it was stated that, "*not withstanding the complication in the formulation, the many degrees of freedom, many of the existing finite elements for Sandwich plates can be specialized or even used directly for the analysis of beams though the solutions provided to the problem will be approximate because of the dependence of accuracy on the number of elements used in the model*". Therefore, conclusions arrived at for plate analysis can be applied to a beam. Love and Kirchoff first applied this theory to plates and shells.

Although the idea of a strong, durable and lightweight design is widely accepted, many of the existing Sandwich models ignore transverse normal and/or shear stresses even though these stresses are crucial in failure analysis. In the Love-Kirchhoff approximation, the effects of deformation of both transverse strains and/or transverse stresses are neglected. This results in natural frequencies that are too high due to the fact that the classical theory postulates an infinite rigidity in transverse shear. It was seen that corrections are needed especially for the case of built-up structures such as the "sandwiched" structure in which the central parts are lightened and have a relatively low resistance to transverse shear. Corrections are also needed for homogenous structures in which the wavelength of the deflection is of the order of the magnitude of the thickness (for instance for thick, stocky structures).

Due to the special properties exhibited by the composite materials, such as high degrees of anisotropy and weak rigidities in transverse shears, the method of analysis based on the classical theories become inadequate. It has been shown by Wu, C. and Vinson, J. R. [29] that the classical beam theory is inadequate for the analysis of thick laminated plates since it over predicts the natural frequencies. It is important to take into consideration the effect of shear deformation in the study of relatively thick laminated plates and therefore the need to use some appropriate shear deformation theories.

Yang et al [30] developed the first-order shear deformation theory. Mindlin plate theory reported by Reddy [18] include transverse shear and rotary inertia effects by assuming a shear profile resulting in a closer approximation to the natural frequencies of the laminate. A linear distribution of the in-plane normal and shear stresses through the thickness is given which yields non-zero transverse shear stresses on the plate boundary planes, and therefore shear correction factors are required. In [17], a higher order theory is used to develop the stiffness analysis of beams and Bernard Nayroles [21] employed the higher order finite element method for sandwich plate's analysis. Reddy [19] presented the high-order shear deformation theory that leads to a non-linear distribution of the shear stresses through the thickness. This provides the parabolic distribution by which the conditions on the boundary plane are fulfilled and the need for shear correction factors is removed. Reddy, J. N. and Khdeir, A. A. [34] and Reddy, J. N. and Phan, N. D. [20] used the first-order shear deformation theory and the high-order deformation theory extensively for the study of free vibration of laminated plates. These theories have been applied to some extent to study the behaviour of sandwich material.

M. Meunier and R. A. Shenoi [31] presented a closed form solution for natural frequencies of sandwich plate panels. S. Mirza and Ni Li [32] presented an analytical approach based on the reciprocal theory for free vibration of sandwich panels. Masoud Rais-Rohani and Pierre Marcellier [33] presented a method that provides approximated analytical solutions for the free vibration and buckling of sandwich plates.

Y. Frostig and G. J. Simitses [23] developed a totally analytical approach to study the bending behaviour of sandwich beams subject to transverse loading. By means of the variational methods, Vladimir S. Sokolinsky, Steven R. Nutt and Yeoshua Frostig [28] derived the equation describing the free vibrations of sandwich beam with soft and stiff core. Different boundary conditions are imposed and finite differences are used to approximate the governing equations. The frequency equations for vibrating homogeneous beams, including the effects due to shear, corresponding to different types of support are developed in [46]. M. E. Raville and En-Shiuh Ueng Ming-Min Lei [44] employed an energy approach with the use of the Langragian Multiplier to determine the natural frequencies of a fixed-fixed sandwich beam. T. Sakiyama, H. Matsuda and C. Morita [52] employed the Green function in an analytical approach to analyze the free vibration of sandwich beam. By means of displacement functions, I. K. Silverman [42] obtained approximate eigen-values for sandwich beams using the Galerkin type solution. S. Oskooei and J. S. Hansein [24] and A. Barut, E. Madenci, J. Heinrich, A, Tessler [22] employed a higher order finite element models for the analysis of sandwich plates. K. M. Ahmed [47] and K. M. Ahmed [48] presented a displacement based finite element methods for the dynamic analysis of curved and straight sandwich beams respectively. K. M. Ahmed [49] presented a displacement based finite element methods for static and dynamic analysis of sandwich structures. Most of the authors mentioned in this work, however, assumed that the elastic modulus of the core in the vertical direction was infinite (incompressible core).

The papers mentioned above describe various approaches to the mathematical formulation of sandwich construction problems. Most complex engineering problems cannot be solved by these methods. The reason for this is that most structures may have certain structural irregularities or complex boundary conditions, which may cause these methods to fail; or at least become impractical, such that there use is limited because of the complexity of the governing equation. Analytical solutions are possible only for simple structures with simple boundary conditions. For complex problems, close form solutions do not exist and consequently, the use of numerical methods must be resorted to. In this class, the finite element method emerges as an elegant, simple and extremely powerful method, which virtually removes all mentioned limitations. During the last decades, this new methods have been extensively developed and now reserved a unique position in the field of structural analysis.

The dynamic behavior of a distributed-parameter beam or beam system described in a continuous system is obtained with the aid of the dynamic stiffness matrix by numerical means in [45]. I. Baychev [39], in his finite element for frames with variable characteristics presented a numerical formulation of the stiffness matrix, loads vector and mass matrix for frame elements with smoothly varying geometrical and physical characteristics. In the study of the dynamic analysis of multistory frame, the dynamic stiffness matrix is presented in [41] by means of the force method.

Ha (1991) [25] put forward the formulation of the stiffness matrix and the procedure for the analysis of a general class of sandwich beam and frame structures subjected to arbitrary loading and boundary conditions. In this paper, the exact solution of deflections and stresses are computed such that the governing differential equation, all boundary conditions, and inter-element compatibility are satisfied. The formulation of the stiffness matrix and the fixed end moment for common loading were represented with explicit form expressions. The application of the theory is to sandwich construction of both thick and thin facings, with or without edge reinforcement. In the analysis, an equation was derived based on which other essential equations were generated from. The application of the formulation to the general configuration of sandwich beams with corrugated or honeycombed cores as well as to the class of beams coupled by elastomeric medium is enhanced by the way the basic parameters were defined.

Amongst other considerations made in this work is the fact that in establishing the governing differential equation, the total bending moment at any section of the beam is due to the contribution of bending coming from the couple caused by the in-plane forces acting on the face sheet layers and that of the facial moments. In order to establish the fact that there is compatibility between connected members, the following must hold;

1. Continuity of the deflection, $v$
2. Continuity of the first derivative of the deflection, $v'$
3. Continuity in the rotation, $\phi$ or the shear strain, $\gamma$

Ha [25] and so many other researchers used the conditions mentioned above to establish the fact that a typical member must have six degrees of freedom i.e. three-degree of freedom per node[4]. In his finite element displacement method, Ahmed [47] modelled displacements $v$ and $w$ in three ways in an increasing order to accommodate three, four and five degrees of freedom per node with continuity in $v$, $w$ and $\partial w/\partial y$, $v$, $\partial v/\partial y$, $w$ and $\partial w/\partial y$, and $v$, $\partial v/\partial y$, $w$, $\partial w/\partial y$, and $\partial^2 w/\partial y^2$ respectively. K. M. Ahmed [48] includes the effect of transverse shear with honeycomb core in the modelling with six degree of freedom per node in his beam analysis corresponding to $v$, $\partial v/\partial y$, $\phi$, $\partial \phi/\partial y$, $w$, and $\partial w/\partial y$, including three rigid body modes. K. M. Ahmed [49] employed a model with five and seven degrees of freedom per node. Finite element models with

[4] The most simple model is that consisting of three-degree of freedom per node

additional degrees of freedom such as curvatures, higher order derivatives of displacements or shear strains are particularly necessary when inter-lamina shear stresses are of special interest.

The focus of the paper, "stiffness matrix for exact solution of sandwich beam and frame systems" is for the efficient optimum design of sandwich beam systems. The results presented so far are useful for application to the static case only but as indicated earlier on by the author, there is room for extension. The author emphasized the fact that quantitative assessment can be easily carried out with his theory, “Exact stiffness analysis of beam and frame systems”, with provision for extension for the case of a large deflection, overall buckling analysis, and vibration analysis. Kinh H. Ha and Luis S. Salvador (1992) [26] put forward the stiffness matrices for exact analysis of sandwich beam systems, which is an extension of the Ha (1991) [25]. The paper by K. H. Ha and L. S. Salvador incorporates buckling analysis, which serves as the prime difference from the paper put forward by Ha [25].

The formulation and synthesis of the mass matrix has not enjoyed the same degree of investigation as the stiffness matrix, primarily because the mass matrix is required for limited but not less important classes of problems. The consistent mass matrix of the distributed system was developed systematically in [11] and Consistent stiffness and mass matrices for fixed-hinged beam element taking into account only shear deformations are developed in explicit expressions in [40]. By using the homogeneous solution to the fourth order governing differential equation as interpolation function between the degrees of freedom at each node of the element, the stiffness matrix and consistent mass matrix of a thin-walled beam element with an asymmetric cross-section is established in [38]. John, S. Archer [16], in an attempt to improve the accuracy of the dynamic analysis as it is affected by the mass matrix, a consistent mass matrix contribution is investigated that accounts for actual distribution of mass throughout the structure in a manner similar to Rayleigh Ritz formulation. According to this Author, the natural frequencies obtained by the use of the consistent mass matrix are upper bound to the exact solution. By using a general solution for the Bernoulli-Euler differential equation, Toshiro Hayashikawa and Noboro Watanabe [37] developed an analytical method for determining eigen-values of continuous beams. In all the approximate methods based on finite element method presented, frequency values produced by the consistent mass matrix are upper bound. K. H. Ha and T. M. Tan [50] incorporated the consistent mass matrix in an efficient dynamic analysis of the continuum shear wall system.

The purpose of this investigation is to present a numerical approach based on the displacement finite element methods for the free vibration of sandwich beam and frame systems applicable to a more general arbitrary boundary conditions. The already derived governing differential equation, displacement function and the development of the element stiffness matrix in Ha [25] form a foundation for this study. Whilst the solution of approximate theories making use of the conventional finite element method depends on the assumed displacement field, the displacement functions used in this work is exact such that the governing differential equation satisfies the boundary conditions and inter-element compatibility. Markus [43], Ahmed [47], Kimel [51] and Mead [53] considered vibrational effects coming only from the flexural behavior of the sandwich beam whilst this work includes rotation and longitudinal vibration effects. By means of special shape function developed here in, a consistent mass matrix for sandwich beam and frame systems is presented. The present theory is preferred based on the fact that the governing differential equation is derived from the model presented and that it is simple and less complicated. Convergence of results is shown by numerical examples. It will be seen later on in this work that the theory is consistent while the solution obtained from other finite element methods will depend primarily on the assumed displacement field used in the model that produced the approximate results. By comparing computed results with those of earlier references, the accuracy of the theory included in this work is verified.

## 1.5 OBJECTIVE OF STUDY

The main objective of the present study is to present an efficient dynamic analysis such that the mass matrix is consistent with the shape functions[5] used in the derivation of the stiffness matrix. The difficulty is that the stiffness matrix in [25] was established without the use of any shape functions. To achieve the goal in this work, the following objectives must be satisfied:

1. Develop the shape function required for deriving the mass matrix from the displacement used in the development of the stiffness matrix in a step by step manner (numerically).
2. Derivation of the mass matrix with the aid of the shape functions.
3. Verification of the theory by comparing the numerical results with related studies.

---

[5] Shape functions are used to convert nodal displacement to displacement anywhere along the beam.

# CHAPTER 2

# ELEMENT MATRICES FOR A STRUCTURAL SANDWICH BEAM

This chapter presents the derivation of element matrices for the proposed sandwich beam element. For the sake of completeness, the derivation of the governing differential equation and of the stiffness matrix [25] is also presented.

In conventional finite element, shape functions are formerly represented in the form of polynomials. The accuracy of the finite element methods for the solution of structural problems depends mainly on the selection of the displacement field. Unlike the conventional finite element methods, where the solution is an approximate one because the shape functions employed are approximate. Here, the governing differential equation is used to derive the unit-load displacement functions that lead to the element matrices.

First, the displacement functions so developed are used to derive the flexibility matrix of the simply supported sandwich beam, and subsequently a transformation matrix will be developed that converts the flexibility into the stiffness matrix. Since the flexibility matrix is exact, the stiffness matrix obtained by transformation is expected to be exact. In this process, no conventional shape functions will be introduced.

## 2.1 GOVERNING DIFFERENTIAL EQUATION

The governing equation to be derived in the following will be used in the other sections that follow. Most distinctive is the fact that the basic parameters are so defined such that the applicability of the formulation is extended to the general configuration of sandwich construction as well as to the class of beams coupled with elastomeric medium[6]. The derivation of the governing differential equation is a very important requirement in the development of the theory. The stiffness and mass matrices that are needed to describe the elastic property and the resistance of the system are derived from the deflection function obtained from the governing differential equation. The accuracy of the entire theory relies very much on the correctness of this equation.

### 2.1.1 GENERAL ASSUMPTIONS

In the development of the governing equations the following basic assumptions are taken into consideration;

1. Linear elastic material

[6] Material capable of being deformed

2. The core resists only shear (i.e. in normal stresses are neglected, thus the entire bending moment at any section is carried by the faces whose own shear deformations are neglected)
3. The modulus of elasticity of the core in the direction perpendicular to the axis of the beam is infinite; thus, the core is assumed to be inextensible in the thickness / transverse direction;
4. All points in a given section deflect the same amount;
5. Bonding between the layers is perfect to provide continuity/compatibility in the deformations at the interfaces
6. The face sheet materials are homogenous and isotropic while the core may be homogenous

### 2.1.2 GOVERNING BEAM EQUATION

The differential equation that governs the behaviour of sandwich beams subject to arbitrary loading and support conditions is derived in this section. This equation will be used to derive the displacement functions for different unit-load conditions in the static case.

Accepting the stated assumptions, the differential equation leads to exact solution of displacement for different loading conditions in the static case. Figure 2.1(a) shows the forces acting at a section of the sandwich beam; the dimensions of the sandwich beam components are also shown in this figure. In figure 2.1(b), the variation of the stress through a section of the sandwich beam is shown. The displacements due to shear deformation of the sandwich beam section as explained in the previous chapter are represented by Figure 2.1(c).

The in-plane force acting on the top and bottom face sheet of the sandwich beam

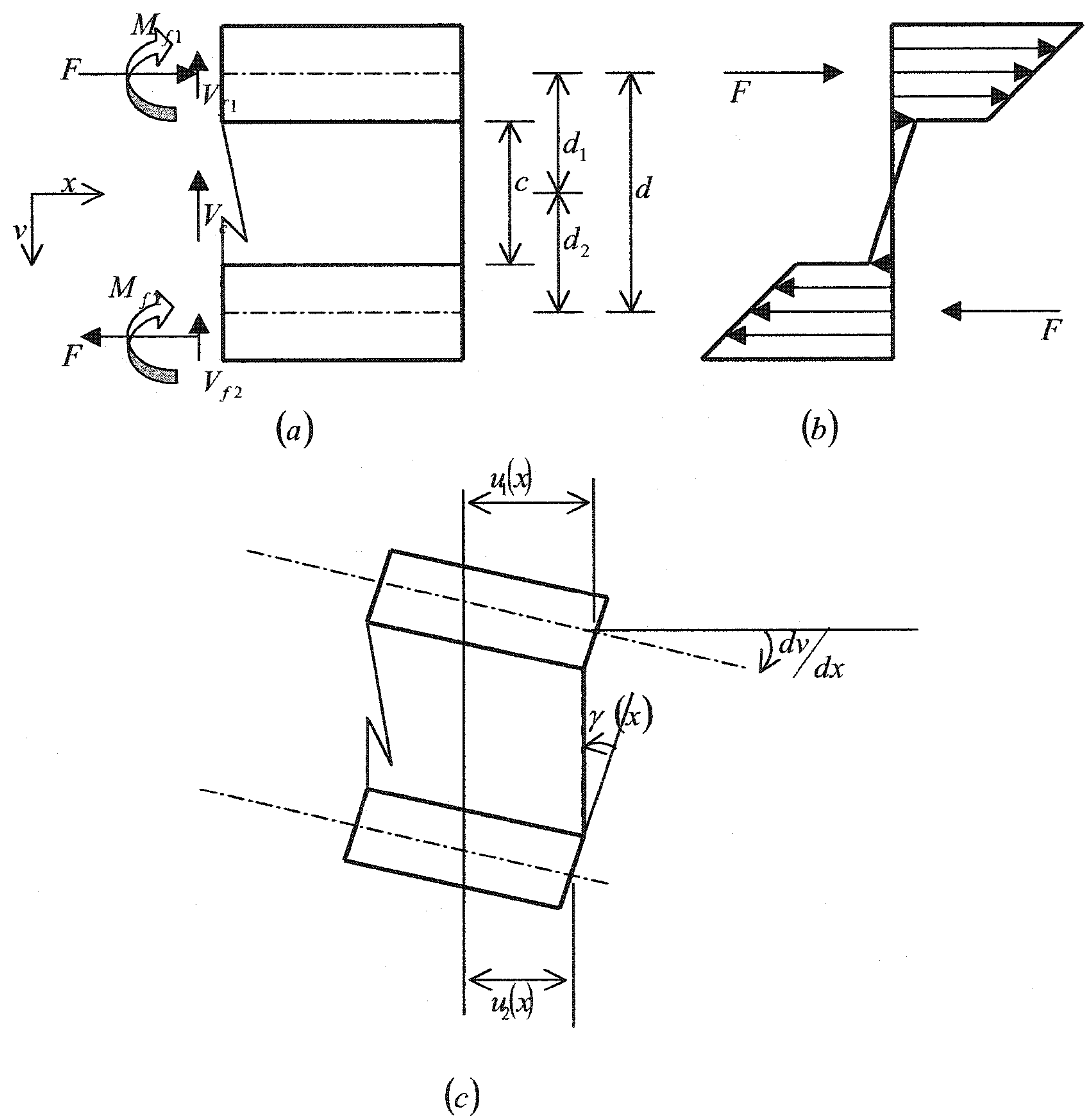


Figure 2.1 Deformations and Displacements in Sandwich Beam Section

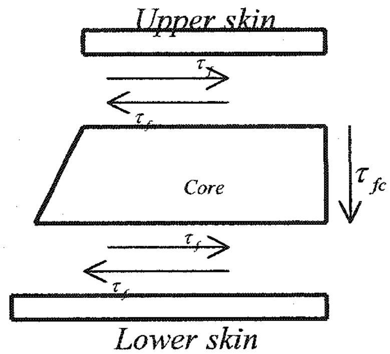


Figure 2.2 Core Shear in Sandwich Beam

section are $F_1$ and $F_2$ respectively. In general, the commonly used core materials, honeycomb and foam are considered to have low in-plane stiffness as compared to the transverse stiffness. In the following analysis the face sheet will be the component responsible for resisting the axial load.

Let $u_1(x)$ and $u_2(x)$ be the $x$-displacements of generic points in the centroidal axes of the top and bottom faces, respectively due to the stretching of the faces caused by the in-plane forces (Fig. 2.1(c)). The moment caused by the stretching of the faces due to the in-plane forces $F(x)$ is written as

$$M_d = F(x)d \quad \text{.....(2.1)}$$

where $F(x)$ is the in-plane force

$d$ is the distance between the centroids of the two facings.

Although it is assumed that shear in the sandwich beam takes place only in the core, it does not imply that shear stresses in the skins of the face sheet materials are neglected. The shear stress $\tau$ of the core for a sandwich beam is as shown in Figure 2.2. In this figure, it is also indicated that the shear strain is constant across the depth of the core

When the net axial force on the cross-section is zero, the same force component acts on the top and bottom faces and in opposite direction. The condition $F_1 = -F_2 = F$ holds such that;

$$F_1 = (AE)_1 u_1' \quad \text{.......(2.2a)}$$

$$F_2 = (AE)_2 u_2' \quad \text{.......(2.2b)}$$

where $(AE)_1$ and $(AE)_2$ are the in-plane stiffness of the faces;

$u_1'$ and $u_2'$ are the first derivatives with respect to $x$ for the $x$-displacements of generic points in the centroidal axis of the top and bottom faces respectively.

Equation 2.2 then leads to

$$-u_1' + u_2' = \frac{F}{(AE)_1} + \frac{F}{(AE)_2} \quad \text{....... (2.3a)}$$

$$u_1' - u_2' = -\frac{F}{(AE)_f} \quad \text{.......(2.3b)}$$

where

$$\frac{1}{(AE)_f} = \frac{1}{(AE)_1} + \frac{1}{(AE)_2} \quad \text{.......... (2.4)}$$

$$(AE)_f = \frac{(AE)_1 (AE)_2}{(AE)_1 + (AE)_2} \quad \text{.......... (2.5)}$$

Local bending deformation due to the facial moments $M_{f1}$ and $M_{f2}$ will be experienced by the faces in addition to the already established uniform stretching of the faces. The local bending is related to the overall beams curvature as follows;

$$M_f = M_{f1} + M_{f2} = -(EI)_{f1} v'' - (EI)_{f2} v'' = -(EI)_f v'' \quad \text{.......... (2.6)}$$

where $v''$ = the second derivative of the deflection $v(x)$ with respect to $x$; and

$$(EI)_f = (EI)_{f1} + (EI)_{f2} \quad \text{.......... (2.7)}$$

is the sum of the local bending stiffness of the faces.

The total deflection of a sandwich beam is the sum of the deformation due to bending and that due to shear. Thus, the total bending moment at a beam section is the sum of the in plane bending forces and the facial bending as follows

$$M(x) = M_d + M_f = Fd - (EI)_f v'' \quad \text{.......... (2.8)}$$

The preceding equation will become the governing equation once the axial force $F$ is expressed in terms of the deflection $v(x)$. It is assumed here that the eccentricity of the net axial thrust is zero. The Eccentricity is zero when the axial thrust is located at the neutral axis (Fig. 2.1(a)). Its location is at distances $d_1$ and $d_2$ as shown in Figure 2.1. For equilibrium of the net force, the distances $d_1$ and $d_2$ may be defined as

$$d_1 = \frac{A_2 E_2}{(AE)_f} d \qquad \text{.......(2.9a)}$$

$$d_2 = \frac{A_1 E_1}{(AE)_f} d \qquad \text{.......(2.9a)}$$

The core is mainly subjected to shear so that the core shear strain produce global deformations and core shear strains. "*Since inter-laminar shear stresses are continuous across layers, they should not be evaluated using an individual layer's elastic constants, but rather by other means, such as equilibrium consideration* [15]". The faces must share this deformation. Axial equilibrium of an isolated face shows that the interlayer shear stress shown in Figure 2.2 is

$$\tau = \frac{1}{b}\frac{dF}{dx} \qquad \text{.......(2.10)}$$

The shear strain in the core is obtained by substitution of $F$ from (2.8) into (2.10)

$$\gamma(x) = \frac{\tau}{G} = \frac{1}{bdG}\left[M'(x) + (EI)_f v'''(x)\right] \quad (2.11a)$$

where $G =$ shear modulus of the core material. The total shear in the core is then

$$V_c = \tau bc = \frac{c}{d}\left[M'(x) + (EI)_f v'''(x)\right] \quad (2.11b)$$

Note that the shear $V_c$ can also be found as the difference between the total shear $V = M'$ and the facing shears, which are determined by moment equilibrium of the faces

$$\left.\begin{aligned} V_{f1} &= \frac{dM_{f1}}{dx} + \tau b\frac{t_1}{2} = -(EI)_{f1}v''' + \tau b\frac{t_1}{2} \\ V_{f2} &= -(EI)_{f2}v''' + \tau b\frac{t_2}{2} \end{aligned}\right\} \quad (2.11c)$$

in which $t_1$ and $t_2$ are the facing thicknesses.

To account for shear deformations, the beam rotation degree of freedom $\phi$ must be independent of the slope $v'$ of the beam. Considering the geometry of the deformed section shown in Figure 2.1(c), the shear strain can be expressed as

$$\gamma(x) = \frac{d}{c}(v' - \phi) \quad (2.12)$$

where

$$\phi = \frac{u_{1(x)} - u_{2(x)}}{d} \qquad (2.13)$$

and $c =$ thickness of the core. The angle $\phi(x)$ can be viewed as the average rotation of the cross section. Note that positive values of the parameters in (2.12) and (2.13) are as shown in Figure 2.1. Elimination of $\gamma(x)$ from (2.11a) and (2.12) yields

$$\phi(x) = v'(x) - \frac{c}{bd^2G}\left[M'(x) + (EI)_f v'''(x)\right] \qquad (2.14)$$

The axial force $F$ can be obtained by substituting the preceding equation into equation 2.3 and is given as

$$F = -d(AE)_f \phi' \qquad (2.15a)$$

$$F = -d(AE)_f \left\{ v'' - \frac{c}{bd^2G}\left[M'' + (EI)_f v^{iv}\right]\right\} \qquad (2.15b)$$

Substituting for $F$ into Eq. 2.8, the following is obtained

$$M(x) = -d^2(AE)_f \left\{ v'' - \frac{c}{bd^2G}\left[M'' + (EI)_f v^{iv}\right]\right\} + (EI)v'' \qquad (2.16a)$$

After some simplification, the governing equation can be written as

$$\frac{d^4v(x)}{dx^4} - \alpha^2\frac{d^2v(x)}{dx^2} = \frac{\alpha^2 M(x)}{EI} - \frac{1}{(EI)_f}\frac{d^2M(x)}{dx^2} \quad \text{(2.16b)}$$

where

$$\alpha^2 = \frac{bG}{c}\left[\frac{1}{(AE)_f} + \frac{d^{\;2}}{(EI)_f}\right] \quad \text{(2.17)}$$

and

$$(EI) = (EI)_f + d^2(AE)_f \quad \text{(2.18)}$$

Betancourt-Angel (1972) [9] derived a similar equation by using the variational formulation, which also yields the following natural boundary conditions;

First, either the shear stress in the core is zero at the end, i.e. from Eq. 2.10,

$$0 = \frac{1}{b}\frac{d}{dx}\left\{M(x) + (EI)_f v''\right\} \quad \text{(2.19a)}$$

Thus

$$M' + (EI)_f v''' = 0 \quad \text{(2.19b)}$$

or, second, the curvature $v''$ is specified, meaning that the local bending moment in the faces is prescribed.

Note that the factor $(bG)/c$ in (2.17) can be replaced by the term $S/d^2$, where $S =$ beam's shear stiffness, which can be found by simple experiment. This substitution effectively removes the parameter $b$ from the formulation, thus generalizing the theory for application to sandwich beams with corrugated or honeycombed cores and beams coupled by an elastomeric layer.

## 2.2 FLEXIBILITY MATRIX OF A BEAM ELEMENT

Our intention is first of all, to establish the flexibility matrix, which will be used to derive the stiffness matrix later on in this chapter. The flexibility coefficients are obtained directly from the displacement function to be derived from the governing differential equation. The displacement at point $i$ due to a unit force or moment at $j$ is the flexibility coefficient $f_{ij}$. This requires that the element must be supported. Among many stable configurations, the simply supported element is chosen as shown in Figure 2.3.

Careful examination of Eq. 2.12 and Figure 2.1 shows that in order to have compatibility between connected elements, we need continuity of the deflection $v$, its first derivative $v'$ and either the rotation $\phi$ or the shear strain[7] $\gamma$. These parameters may serve as the nodal degrees of freedom in the sandwich beam element. Thus by definition, the element's flexibility relation is

$$\mathbf{q} = \mathbf{f}\,\mathbf{Q} \qquad \text{.......(2.20)}$$

[7] Caused by the ability of the material to slide apart

where the each degree of freedom of a system represents a nodal displacement vector **q** and a corresponding nodal force **Q** defined as

$$\mathbf{Q} = \{M_{di}\ M_{fi}\ M_{dj}\ M_{fj}\} \quad \text{.......(2.21)}$$

$$\mathbf{q} = \{\phi_i\ v_i'\ \phi_j\ v_j'\} \quad \text{.......(2.22)}$$

The stiffness matrix can simply be obtained by transformation of the element flexibility matrix, as will be shown in the upcoming sections.

### 2.2.1 DISPLACEMENT FUNCTION

The displacement function that is needed for the derivation of the element matrices to be derived in this section is different from the conventional displacement function. Consider the simply supported sandwich beam element (Figure 2.3), subjected to the end moment $M_0$ defined by

$$M_0 = M_{di} + M_{fi} \quad \text{.......(2.23)}$$

This moment $M_0$ applied at the end $i$ gives rise to the following moment at section $x$

$$M(x) = \frac{M_0}{L}(L - x) \quad \text{.......(2.24)}$$

Solution of the governing equation (i.e. Eq. 2.16) for the loading $M_0$ can be found as

$$v(x) = \frac{M_0 x}{6(EI)L}\left(x^2 - 3Lx + 2L^2\right) + A\left[\frac{x}{L} - 1 + \frac{\sinh\alpha(L-x)}{\sinh\alpha L}\right] \quad \text{.......................................(2.25)}$$

The validity of the preceding equation is checked to see if the geometric boundary conditions are satisfied as follows:

At node $i$, the following boundary conditions holds

$$v(0) = 0, \quad \text{.......................................................................(2.26)}$$

At node $j$, the following boundary conditions holds

$$v(L) = v''(L) = 0, \quad \text{.......................................................................(2.27)}$$

where $v''(x)$ is the curvature at abscissa $x$. Note that the abscissa 0 is for parameters at node $i$ and abscissa L is the arbitrary coordinate at node $j$. The implication of the second boundary condition for node $j$ of the sandwich beam is that shear strain is not prevented at this end.

To evaluate the integration constant, $A$, the skins are assumed to deflect the same way the whole beam does and as long as the moment $M_{fi}$ exists, curvature is non-zero and is given by the following boundary condition.

$$v''(0) = -\frac{M_{fi}}{(EI)_f} \qquad \text{.......................................................(2.28)}$$

where $M_{fi}$ and $(EI)_f$ are the local bending moment and the local bending stiffness of the face skin material at node $i$ as in expressions 2.6 and 2.7 respectively. The negative sign is to maintain consistency with the sign convention used throughout this work. The preceding equation is identical to the well known bending moment equation for a homogenous isotropic beam. From Eq. 2.25 we write

$$v''(x) = -\frac{M_0}{(EI)L}\{x - L\} + A\alpha^2 \frac{\sinh \alpha(L - x)}{\sinh \alpha L} \qquad \text{.........................................(2.29)}$$

At node $i$

$$v''(0) = -\frac{M_0}{(EI)L} + A\alpha^2 \qquad \text{.......................................................(2.30)}$$

Elimination of $v''(0)$ from Eq. 2.25 and Eq. 2.30 yields the integration constant $A$ as

$$A = \frac{1}{\alpha^2}\left[\frac{M_0}{(EI)} - \frac{M_{fi}}{(EI)_f}\right] \qquad \text{.......................................................(2.31)}$$

Alternatively the integration constant $A$ can be evaluated in a different way as follows. When rigid edge inserts[8] prevent relative in-plane displacement of the facings at the left edge, the core's shear strain is zero. The whole externally applied shear force is taken by the skins and Eq. 2.19(b) is written as

$$v'''(x) = \frac{M'(x)}{L(EI)_f} \quad \text{.......(2.32)}$$

Thus

$$v'''(0) = \frac{M_0}{L(EI)_f} \quad \text{.......(2.33)}$$

the integration constant can be written as

$$A = -\frac{M_0}{L\mu\alpha} \tanh \alpha L \quad \text{.......(2.34)}$$

in which

$$\mu = \frac{c\alpha^4 (EI)_f^2}{Gbd^2} \quad \text{.......(2.35)}$$

The preceding solution will lead to the flexibility coefficients as shown is the next section.

[8] No shear deformation at built-in ends

### 2.2.2 ELEMENTS OF FLEXIBILITY MATRIX

In this section, the individual nodal displacements $q_i$ are derived for unit moment $M_d$ or $M_f$. Consider the simply supported sandwich beam element of Figure 2.3 subjected to end actions **R** in which $M_d$ and $M_f$ are the end moments. All signs of forces depicted in this figure are taken as positive. We are to establish to establish the flexibility matrix in the presence of the existing forces of the simply supported sandwich beam element (constrained element, Figure 2.3).

The rotation of the sandwich beam section, Eq. 2.14, can now be written as

$$\phi(x) = \frac{dv}{dx} + \frac{c}{db^2G}\left[\frac{M_0}{L} - (EI)_f \frac{d^3v}{dx^3}\right] \qquad \text{.......(2.36)}$$

The above basic solution provides the flexibility coefficients of the first two columns of

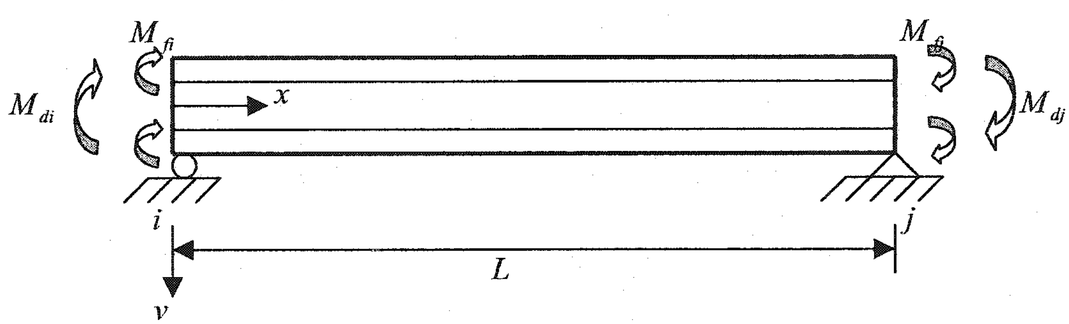


Figure 2.3 Simply SupportedSandwich Beam Configuration

the flexibility matrix **f**, and the remaining coefficients can be obtained by symmetry of the system. The flexibility coefficients $f_{ij}$ is the displacement at the direction $i$ due to a unit nodal force applied at the direction $j$. As an illustration, by substituting into the preceding solutions:

$$M_0 = M_{di} = 1 \text{ and } M_{fi} = 0, \qquad \text{.......(2.37)}$$

the following flexibility coefficients are obtained:

$$\left.\begin{aligned} f_{11} &= f_{33} = \phi(0) \\ f_{21} &= f_{43} = v'(0) \\ f_{31} &= f_{13} = \phi(L) \\ f_{41} &= f_{23} = v'(L) \end{aligned}\right\} \qquad \text{.......(2.38)}$$

Similarly, by substituting

$$M_0 = M_{fi} = 1 \text{ and } M_{di} = 0, \qquad \text{.......(2.39)}$$

the coefficients of the second and fourth columns are obtained:

$$\left.\begin{aligned} f_{12} &= f_{34} = \phi(0) \\ f_{22} &= f_{44} = v'(0) \\ f_{32} &= f_{14} = \phi(L) \\ f_{42} &= f_{24} = v'(L) \end{aligned}\right\} \qquad \text{.......(2.40)}$$

Maxwell's law of reciprocity states that, "*as long as the material of the structure is elastic and follows hook's law the member flexibility matrix will be symmetric*". The complete flexibility matrix given in Appendix A is found to be symmetric as expected.

## 2.3 STIFFNESS MATRIX

The vibration of an elastic body is a function of the stiffness $EI$ as well as its mass. As the stiffness is increased, in general it may be stated that the frequency of vibration increases. The stiffness of a structural system is a measure of its resistance. A typical member *ij* has six degrees of freedom, which are chosen as in Figure 2.4. A transformation matrix that includes rigid body motion in the degree of freedom of a system will be introduced in this section. The stiffness matrix can simply be obtained by transformation of the element flexibility matrix obtained in the previous section.

### 2.3.1 STIFFNESS MATRIX OF A BEAM ELEMENT

The degrees of freedom for a sandwich beam element should be selected to allow for flexural deformations and rigid body motion. The six-degree of freedom member *ij* as shown in Figure 2.4 of nodal load forces, **R** that includes the rigid body motions has the following relationship with **Q**

$$\mathbf{R}_{6\times1} = \mathbf{B}_{6\times4}\,\mathbf{Q}_{4\times1} \qquad \text{.........(2.41)}$$

where **B** is the force transformation matrix that can be derived by consideration of equilibrium.

$$\mathbf{B} = \begin{bmatrix} 1/L & 1/L & 1/L & 1/L \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1/L & -1/L & -1/L & -1/L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{.........(2.42)}$$

$$\mathbf{Q} = \{M_{di}\ M_{fi}\ M_{dj}\ M_{fj}\} \qquad \text{.........(2.43)}$$

$$\mathbf{R} = \{V_i\ M_{di}\ M_{fi}\ V_j\ M_{dj}\ M_{fj}\} \qquad \text{.........(2.44)}$$

where $V$ is the total shear at the end $i$ and $i$ and all other notations carry their usual meanings.

$$V = V_c + V_{f1} + V_{f2} \qquad \text{.........(2.45)}$$

$$M_f = M_{f1} + M_{f2} \qquad \text{.........(2.46)}$$

The nodal displacements corresponding to $\{R\}$ are

$$\mathbf{r} = \{v_i\ \phi_i\ v_i'\ v_j\ \phi_j\ v_j'\} \qquad \text{.........(2.47)}$$

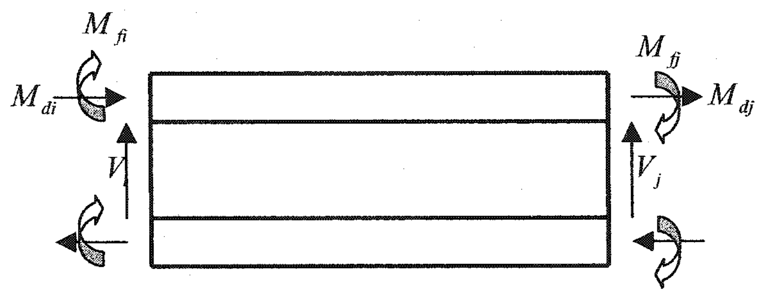


Figure 2.4 Sandwich Beam Element with no Restraint

By definition, the stiffness matrix **K** relates the nodal displacements to the nodal forces **R**. The following relation can be used to transform the already derived flexibility matrix of Eq. 2.38 and Eq. 2.40 to the $6 \times 6$ stiffness matrix **K** and $\mathbf{R} = \mathbf{Kr}$ corresponding to the degrees of freedom of the beam element.

$$\mathbf{K} = \mathbf{B}\mathbf{f}^{-1}\mathbf{B}^{T} \qquad \text{.........(2.48)}$$

Like the flexibility matrix, the stiffness matrix will result as a non-diagonal one and thus referred to as Static coupling. The stiffness matrix **K** has coefficients $\mathbf{k}_{ij}$ that represent the elastic restraining force at $i$ developed by a unit displacement $\mathbf{x}_j$ given that all other coordinate displacements are zero. From Eq. 2.21 and Eq. 2.22, the displacement vector **q** corresponding to **Q** is

$$\mathbf{q} = \mathbf{B}^{T}\mathbf{r} \qquad \text{.........(2.49)}$$

## 2.3.2 STIFFNESS MATRIX INCLUDING AXIAL ACTION

It has been assumed in the previous sections that the resultant axial force is zero. Nevertheless, the possibility still exists for a non-zero resultant axial force and the necessary modifications to include this effect to the established expressions are included in this section. Figure 2.5 shows the unrestrained sandwich beam element that includes axial displacement.

The forces and displacement equations that includes axial action, corresponding to the degrees of freedom for a sandwich beam simply supported at its ends can be written as

$$\mathbf{Q} = \{M_{di}\ M_{fi}\ M_{dj}\ M_{fj}\ N\} \qquad \text{.......(2.50)}$$

$$\mathbf{q} = \{\ \phi_i\ v_i'\ \phi_j\ v_j'\ u\} \qquad \text{.......(2.51)}$$

where $N$ = the resultant axial force

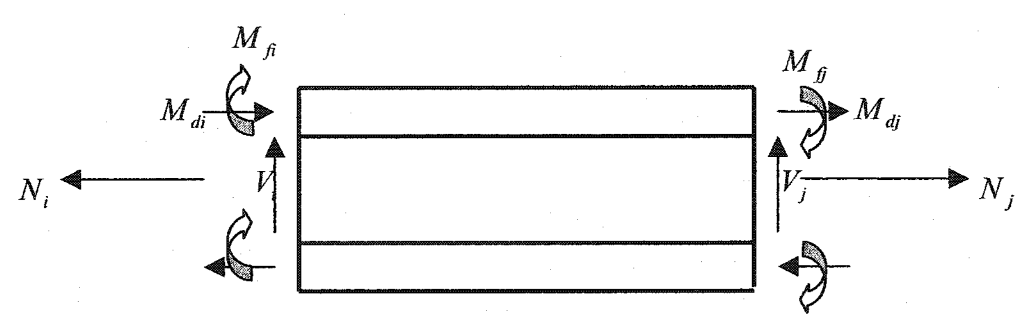


Figure 2.5 Sandwich Beam Element with no Restraint including axial effect

$u$ = the axial deformation of the section

The flexibility matrix is refined to $5 \times 5$ and the last element is known as the axial flexibility coefficient and is written as

$$f_{55} = \frac{L}{E(A_1 + A_2)} \qquad \text{.......(2.52)}$$

The set of forces corresponding to the degrees of freedom of the sandwich beam element including axial effect are

$$\mathbf{R} = \{V_i\ N_i\ M_{di}\ M_{fi}\ V_j\ N_j\ M_{dj}\ M_{fj}\} \qquad \text{.......(2.53)}$$

and the corresponding nodal displacements corresponding to these degrees of freedom are

$$\mathbf{r} = \{v_i\ u_i\ \phi_i\ v_i'\ v_j\ u_j\ \phi_j\ v_j'\} \qquad \text{.......(2.54)}$$

To include the axial effect, the force transformation matrix that transforms the flexibility matrix to an $8 \times 8$ stiffness matrix is

$$\mathbf{B} = \begin{bmatrix} 1/L & 1/L & 1/L & 1/L & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1/L & -1/L & -1/L & -1/L & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad \text{.....(2.55)}$$

The stiffness matrix can now be established with the aid of Eq. 2.48.

### 2.3.3 DEGREES OF FREEDOM

A member has to be free of supports within its span in order to avoid the introduction of unknown reaction forces; every member must be continuous since each member has its element matrix associated to their material properties and dimensions. Members and nodes together form the complete system and the support conditions are specified by restraining the nodes. The deformed shape of the sandwich beam system has to be consistent with the support conditions as well as the member connection types. Once the members are connected together, they will be matched with the nodal displacements.

The isolated members, Figures 2.3 and 2.4 are members of a complete system whose internal forces depend on their loadings. The member-end shears and moments may be thought of as external forces acting on the member ends. The displaced positions are represented by the end displacements and the corresponding forces of shear restrained

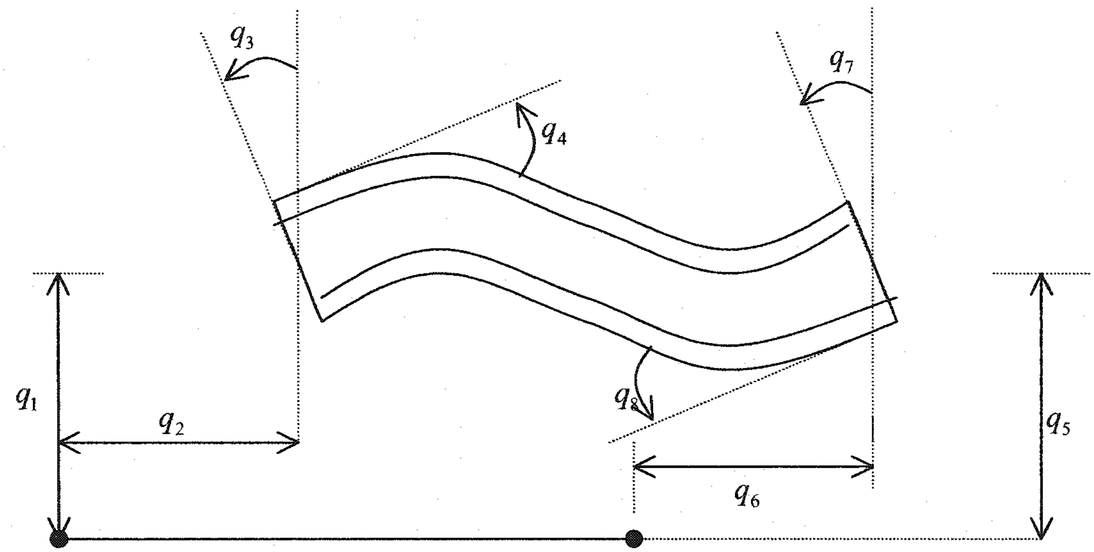


Figure 2.6 Sandwich Beam Element's degrees of freedom

element Figure 2.3 are as indicated by Eq. 2.21 and Eq. 2.22 and the unrestrained element model Figure 2.4 are as indicated by Eq. 2.44 and Eq. 2.47; Figure 2.5 includes axial vibration effects and its degrees of freedom and corresponding force are as indicated by Eq. 2.54 and Eq. 2.53.

To ensure that the solution satisfies compatibility as well as equilibrium and material properties throughout the system, it is seen convenient for the system model to be an assembly of members interconnected by nodes. The degrees of freedom of the system must be sufficient to describe the deformed shape of the system as well as the deformations in the members. The degree of freedom corresponding to the end actions for the sandwich beam element is as shown in Figure 2.6.

### 2.3.4 SYSTEM STIFFNESS MATRIX

In finite element methods, a system is modeled as an assemblage of finite elements and hence, the stiffness matrix for the system is assembled from that of the elements' stiffness matrices. For this assemblage of individual elements to represent the structure adequately, there must be geometric compatibility at the elements nodes, i.e., the displacements at the nodes shared by several elements must be the same for every such element. Moreover, the corresponding nodal forces must be statically equivalent to the applied forces.

The equation of motion of the complete system can be obtained by an assembling process that amounts to expressing the potential energy (i.e., for the stiffness) and the kinetic energy (i.e., for the mass) in terms of contributions from the individual elements. The potential energy of a system consists of the strain energy stored in elastic elements, and the energy, which is a function of the distances between system masses and some arbitrary datum. Thus, the potential energy which is a function of the system's resistance, accounts for the system's stiffness.

The potential energy can be written in the form

$$V(t) = \frac{1}{2}\sum_{e=1}^{N}\left\{\bar{q}\right\}_e^T\left[\bar{k}\right]_e\left\{\bar{q}\right\}_e = \frac{1}{2}\sum_{e=1}^{N}\left\{\bar{r}\right\}_e^T\left[\bar{K}\right]_e\left\{\bar{r}\right\}_e \qquad \Big) \; r = v_m \quad \text{...........(2.56)}$$

$$= \frac{1}{2}\left\{\bar{r}\right\}^T\left[\bar{K}\right]\left\{\bar{r}\right\}$$

where

$$\left[\bar{K}\right] = \sum_{e=1}^{N}\left[\bar{K}\right]_e \quad \text{......(2.57)}$$

is the symmetric stiffness matrix for the complete system. Because of static coupling[9], the final stiffness matrix is expected to be non-diagonal.

The direct stiffness method is employed in the analysis as this has gained popularity for the solution of structural problems due to its simplicity and amenability to computer analysis. In the direct stiffness method, the element matrices of individual elements are transformed from their local coordinate to global coordinates. The element matrices are then superimposed in the global coordinate to obtain the system matrix.

### 2.3.5 DISPLACEMENTS AND STRESSES

Consideration in this section will be made for the case of a sandwich beam subjected to bending only. The flexibility equation for a general element is written as

$$\mathbf{q}_{4\times1} = \mathbf{f}_{4\times4}\mathbf{Q}_{4\times1} + \mathbf{q}^{o}_{4\times1} \quad \text{......(2.58)}$$

where the vector $\mathbf{q}^{o}_{4\times1}$ contains the nodal displacements due to member loading on a simply supported beam with no end-shear restraint (Fig. 2.3),

9 the principle coordinate is not the only coordinate system used to describe such system

$$\mathbf{q}^o = \{\phi_i^{o'}, v_i^{o'}, \phi_j^{o'}, v_j^{o'}\} \quad \text{.......(2.59)}$$

The vector of Fixed-end forces $\mathbf{Q}^o_{4\times1}$ is found by equating Eq. 2.52 to zero such that

$$\mathbf{Q}^o_{4\times1} = -\mathbf{f}^{-1}_{4\times4}\,\mathbf{q}^o_{4\times1} \quad \text{.......(2.60)}$$

The number of component displacements and forces can be adjusted accordingly to include axial translational effects as discussed in Section 2.3.2. For use in the direct stiffness matrix the vector of fixed end forces due to member loading is obtained by transformation as;

$$\mathbf{R}^o = \mathbf{B}\mathbf{Q}^o \quad \text{.......(2.61)}$$

The final member end forces are given by the relation

$$\mathbf{R} = \mathbf{K}\mathbf{r} + \mathbf{R}^o \quad \text{.......(2.62)}$$

The stresses can be calculated at a nodal point i, directly from **R** and **r** as,

$$F = \frac{M_{di}}{d} \quad \text{.......(2.1)}$$

$$M_{f1} = \frac{(EI)_{f1}}{(EI)_f} M_{fi} \quad \text{.......(2.63a)}$$

and

$$M_{f2} = \frac{(EI)_{f2}}{(EI)_f} M_{fi} \quad \text{.......... (2.63b)}$$

Solutions for sections not at the nodal points can be obtained by using earlier equations in Section 2.1.2 in conjunction with the influence functions generated from the previously presented basic solution function in Section 2.2.1. Alternatively, these solutions can also be obtained by interpolation of the foregoing solutions of the known nodal points.

## 2.4 MASS MATRIX

As mentioned in earlier chapters, the vibration of a body may be classified as either forced or free. Initial conditions or disturbances on a structural system can cause the structure to vibrate. The conditions generally manifest themselves as an energy input such as velocity imparted to the mass of a structure and thus associated to its kinetic energy. As a result of the system's distributed mass, inertia forces are developed in the system; the mass of a Sandwich beam is a measure of its inertia[10]. The mass matrix is a very important requirement in the dynamic analysis of a structural system.

In the case of free vibration, the resulting structural vibration occurs in the absence of any externally applied forces. A body vibrating freely (that is, without impressed vibration from an outside source) does so at one or more of its natural

---

[10] The resistance to a change in motion

frequencies. Since, there is no external excitation acting on the structure, the vibration diminishes with time as the energy input of the structure from the initial conditions eventually is dissipated.

The formulation of a consistent mass matrix needed for dynamic analysis will be presented in this section. The procedure is similar to that used by Ha and Tan [27] for continuum shear walls. The mass matrix is consistent when both the mass matrix and the stiffness matrix are derivable from the same shape functions. To derive the mass matrix, we need shape functions, which however were not directly used in the derivation of the stiffness matrix. The shape functions will be established in such a way that they facilitate the numerical evaluation of the mass matrix by numerical integration.

In an attempt to improve the accuracy of the dynamic analysis as it is affected by the mass matrix, a consistent mass matrix construction is investigated, that accounts for the actual distribution of mass throughout the structure. The total kinetic energy $T$ of the beam element is a contribution of the kinetic energy due to the vertical translation, rotation and axial translation of the Sandwich beam. The third part of the kinetic energy has been neglected in most of the existing sandwich beam theories. In order that this theory is appropriate for the general class of sandwich beam element, the axial translation energy is taken into consideration in the vibration analysis. This is true since the axial kinetic energy increases as the possibility of the extreme layers sliding apart increases, which will be evident from the formulation. Thus the total kinetic energy is formulated as

$$T = \frac{1}{2}\int_0^L \rho(A_1 + A_2)\dot{v}^2\, dx + \frac{1}{2}\int_0^L \rho(I_1 + I_2)\dot{v}_x^2\, dx + \frac{1}{2}\int_0^L \rho(A_1 \dot{u}_1^2 + A_2 \dot{u}_2^2)dx \quad \text{.............(2.64)}$$

where, the three integrals represent the contribution due to, respectively, the transverse translation, rotation and axial action. The term $\dot{v}$ denotes the time derivative of $v(x)$; $v_x$ denotes the derivative of $v(x)$ with respect to $x$; $\rho$ is the mass density and with the area $A = A_1 + A_2$; and the area moment of inertia about the neutral axis $I_f = I_1 + I_2$, Eq. 2.64 can be written as

$$T = \frac{1}{2}\int_0^L \rho A \dot{v}^2 \, dx + \frac{1}{2}\int_0^L \rho I_f \dot{v}_x^2 \, dx + \frac{1}{2}\int_0^L \rho (A_1 \dot{u}_1^2 + A_2 \dot{u}_2^2) dx \quad \text{.........................(2.65a)}$$

Eq. 2.65(a) can be written in matrix form as

$$T = \frac{1}{2}\int_0^L \begin{Bmatrix} \dot{v} \\ \dot{v}_x \\ \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix}^T \begin{bmatrix} \rho A & 0 & 0 & 0 \\ 0 & \rho I_f & 0 & 0 \\ 0 & 0 & \rho A_1 & 0 \\ 0 & 0 & 0 & \rho A_2 \end{bmatrix} \begin{Bmatrix} \dot{v} \\ \dot{v}_x \\ \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} \quad \text{........................................(2.65b)}$$

The above equation will lead to mass matrix once the displacements are expressed in terms of nodal displacements via shape functions.

### 2.4.1 ELEMENT MASS MATRIX DUE TO TRANSLATIONAL KINETIC ENERGY

Consider the sandwich beam element displaced from the initial position as shown in Figure 2.7. The displacement function $P$ is due to the nodal forces $\{Q\}$. It can be found as the sum of the product of the nodal forces and the displacement functions due to unit forces:

$$v(x) = \sum_{i}^{n} Q_i g_i = \mathbf{g}_v \mathbf{Q} \qquad \text{.........(2.66)}$$

where

$Q_i$, force at degrees of freedom for a simply supported element as shown in Figure 2.3

$g_i$, displacement function due to unit load applied $Q_i$ as shown in Figure 2.3

the subscript $v$ represents quantities associated only with the restrained Sandwich beam

From the basic equation for displacement or stiffness method analysis:

$$\mathbf{Q} = \mathbf{f}^{-1}\mathbf{q} \qquad \text{.........(2.67)}$$

From Eq. 2.41, the force transformation matrix can be incorporated as

$$\mathbf{R} = \mathbf{B}\mathbf{Q} \qquad \text{.........(2.68)}$$

or simply written in the form

$$\mathbf{q} = \mathbf{B}^T\mathbf{r} \qquad \text{.........(2.69)}$$

Substituting for **q** in Eq. 2.67

$$\mathbf{Q} = \mathbf{f}^{-1}\mathbf{B}^T\mathbf{r} \quad \text{.....(2.70)}$$

Substituting for **Q** in Eq. 2.66

$$v(x) = \mathbf{g}_v\mathbf{f}^{-1}\mathbf{B}^T\mathbf{r} = \mathbf{N}^f\mathbf{r} \quad \text{.....(2.71)}$$

in which

$$\mathbf{N}^f = \mathbf{g}_v\mathbf{f}^{-1}\mathbf{B}^T \quad \text{.....(2.72)}$$

Eq. 2.72 defines the shape function in a manner consistent with the displacement

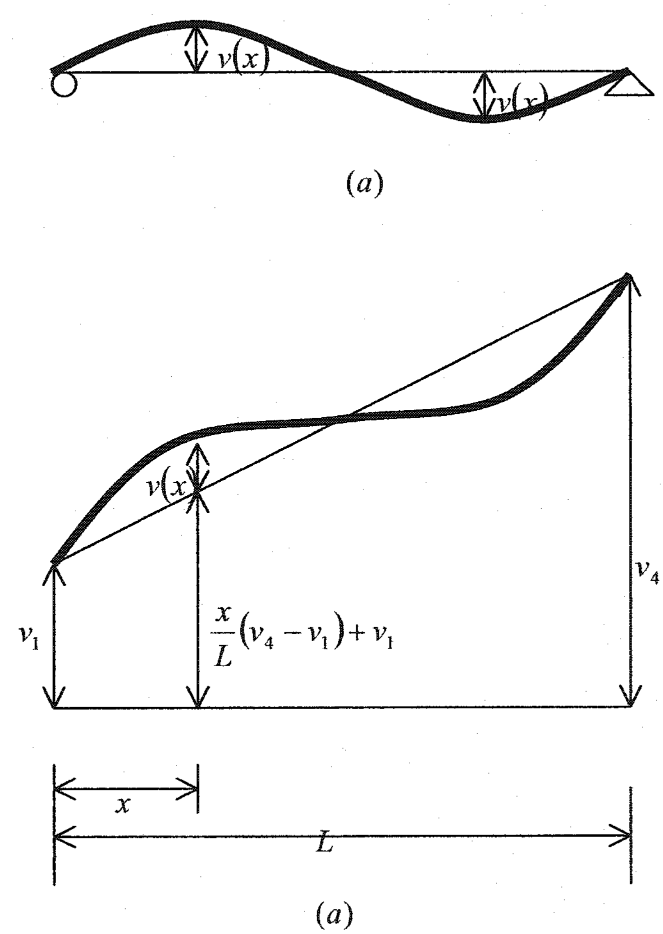


Figure 2.7 Sandwich Beam Element's displacement (a) Restrained
(b) Unrestrained

functions used in the derivation of the flexibility matrix. It is very important to impress the fact that the rigid body motion $v_1$ and $v_4$ (Fig. 2.7) are not yet included in this equation. Note also that the transformation matrix in the above equation can be represented as shown in Eq. 2.42 and 2.55 depending on whether the axial vibration effect is included in the analysis.

The unit load deflection function $g_v(x)$ can be derived numerically by using the displacement function $v(x)$ (i.e. Eq. 2.25). The function $g_v(x)$ can be written in the following way

$$g_v(x) = g_{di}(x) + g_{fi}(x) + g_{dj}(x) + g_{fj}(x) + g_0(x) \quad \text{.................................... (2.73)}$$

where the influence function $g_{di}(x)$ correspond to the loading condition specified by

$$M_0 = M_{di} = 1 \text{ and } M_{fi} = 0, \quad \text{..........................................................(2.74)}$$

$g_{fi}(x)$ correspond to the loading condition specified by

$$M_0 = M_{fi} = 1 \text{ and } M_{di} = 0, \quad \text{..........................................................(2.75)}$$

and $g_0(x)$ is solution due to member loading acting over the simply supported element of Figure 2.3. The influence functions for the actions at $j$ are obtained by symmetry, i.e by change of variable;

$$z = L - x \qquad \text{.....(2.76)}$$

$$\frac{d^k v}{dz^k} = (-1)^k \frac{d^k v}{dx^k}, \qquad k = 1,2,3, \ \dots \ , n. \qquad \text{.....(2.77)}$$

The displacement function $g(x)$ due to unit load applied at the degrees of freedom of a simply supported beam is as shown in Appendix B.

The shape function $\mathbf{N}^f$ derived for the sandwich beam so far corresponds to the restrained element in Figure 2.7(a). The final shape function matrix **N** must be corrected for rigid body motion $v_1$ and $v_4$ corresponding to the first and fourth degrees of freedom respectively, as shown in Figure 2.7(b). Two degree of freedom adjusted for because of rigid body motion at both ends (for a uniform bar element undergoing uniform deformation) to the already existing shape function matrix.

$$v(x) = \mathbf{N}^f \mathbf{r} + \mathbf{N}_{v1} v_1 + \mathbf{N}_{v4} v_4 \qquad \text{.....(2.78)}$$

The shape functions needed to correct for rigid body motion are linear and are given as

$$\mathbf{N}_{v1} = 1 - x/L \qquad \text{.....(2.79a)}$$

$$\mathbf{N}_{v4} = x/L \qquad \text{.....(2.79b)}$$

Eq. 2.66 can therefore be represented as

$$v = \mathbf{N}\, r \qquad \text{.......(2.80)}$$

From the preceding equation, the following equations can easily be obtained;

$$v_x = \mathbf{N}_x\, r \qquad \text{.......(2.81a)}$$

$$\dot{v} = \mathbf{N}\, \dot{r} \qquad \text{.......(2.81b)}$$

$$\dot{v}_x = \mathbf{N}_x\, \dot{r} \qquad \text{.......(2.81c)}$$

where $\mathbf{N}_x$ is the first derivative of the shape function $\mathbf{N}$ with respect to $x$.

Using the preceding equations, the first term in Eq. 2.65, the translation mass term $\mathbf{M}_t$ can be extracted as indicated below:

$$T_1 = \frac{1}{2}\int_0^L \dot{v}\rho A \dot{v}\, dx = \frac{1}{2}\int_0^L \rho A \dot{r}^T \mathbf{N}^T \mathbf{N} \dot{r}\, dx = \frac{1}{2}\dot{r}^T \mathbf{M}_t \dot{r} \qquad \text{.......(2.82)}$$

The Translational mass matrix can be written as

$$\mathbf{M}_t = \rho A \int_0^L \mathbf{N}^T \mathbf{N}\, dx \qquad \text{.......(2.83)}$$

### 2.4.2 ELEMENT MASS MATRIX DUE TO ROTATIONAL KINETIC ENERGY

In a similar way, the rotational mass inertia of the beam element, $\mathbf{M}_r$ can be extracted as indicated below:

$$T_2 = \frac{1}{2}\int_0^L \dot{v}_x \rho I_f \dot{v}_x \, dx = \frac{1}{2}\int_0^L \rho I_f \dot{r}^T \mathbf{N}_x^T \mathbf{N}_x \dot{r} \, dx = \frac{1}{2}\dot{r}^T \mathbf{M}_r \dot{r} \quad \text{.................(2.84)}$$

The Rotational mass matrix is written as

$$\mathbf{M}_r = \rho I_f \int_0^L \mathbf{N}_x^T \mathbf{N}_x \, dx \quad \text{.........................................(2.85)}$$

### 2.4.3 ELEMENT MASS MATRIX DUE TO AXIAL KINETIC ENERGY

For axial deformation, in-plane displacement occurs at the mid-plane of the individual layers. Starting from the bottom layer, these displacements can be eliminated layer by layer thereby leaving those associated with the top and bottom to represent a degree of freedom. The velocity components $\dot{u}_1$ and $\dot{u}_2$ of Eq. 2.66 can be found by relating $u_1$ and $u_2$ to the nodal displacements vector **r** through the use of the rotation function $\phi(x)$. It can be easily shown from Figure 2.1© that:

$$u_1 = u_0 + d_1\phi \quad \text{.........................................................(2.86a)}$$

$$\gamma = 0 \quad \text{.........................................................(2.86b)}$$

where the term in $u_0$ contains the uniform stretching of the element at both ends and $d_1$ and $d_2$ are the distances from the centroid of the Sandwich beam element to the centroids of the top and bottom face sheets.

By following the same procedure used in the derivation of the shape function **N**, a similar set of shape functions $\mathbf{N}_\phi$ can be established:

$$\phi(x) = \mathbf{g}_\phi \mathbf{f}^{-1} \mathbf{B}^T \mathbf{r} = \mathbf{N}_\phi \mathbf{r} \qquad \text{.......(2.87)}$$

in which

$$\mathbf{N}_\phi = \mathbf{g}_\phi \mathbf{f}^{-1} \mathbf{B}^T \qquad \text{.......(2.88)}$$

Similarly $g_\phi$, the rotation function can be generated numerically by using the following relation

$$g_\phi(x) = g_{\phi di}(x) + g_{\phi fi}(x) + g_{\phi dj}(x) + g_{\phi fj}(x) + g_{\phi 0}(x) \qquad \text{.......(3.89)}$$

The fifth term, which is the influence due to member loading, is not needed for deriving the mass matrix. Applying the loading conditions given by Eq. 2.74 and Eq. 2.75, these terms can be obtained as given in Appendix C.

Using Eq. 2.87, Eq. 2.86 can be written as

$$u_1 = u_0 + d_1 \mathbf{N}_\phi r \qquad \text{.......(2.90a)}$$

$$u_2 = u_0 - d_2 \mathbf{N}_\phi r \qquad \text{(2.90b)}$$

The uniform stretching $u_0$ can be expressed in terms of the axial displacement as

$$u_0 = \left(1 - \frac{x}{L}\right) u_1 + \frac{x}{L} u_4 \qquad \text{(2.91)}$$

where

$$\mathbf{N}_a = \{0, 1 - x/L, 0, 0, 0, x/L, 0, 0\} \qquad \text{(2.92)}$$

$$u_0 = \mathbf{N}_a \mathbf{r} \qquad \text{(2.93)}$$

Eq. 3.90 then becomes

$$u_1 = \mathbf{N}_a r + d_1 \mathbf{N}_\phi r \qquad \text{(2.94a)}$$

$$u_2 = \mathbf{N}_a r - d_2 \mathbf{N}_\phi r \qquad \text{(2.94b)}$$

Taking the time derivative of the previous equation:

$$\dot{u}_1 = \mathbf{N}_a \dot{r} + d_1 \mathbf{N}_\phi \dot{r} \qquad \text{(2.95a)}$$

$$\dot{u}_2 = \mathbf{N}_a \dot{r} - d_2 \mathbf{N}_\phi \dot{r} \qquad \text{(2.95b)}$$

Extracting the last term of Eq. 2.66 and then making use of the preceding equations, the term $T_3$ can be expressed as

$$T_3 = \frac{\rho}{2}\int_0^L (A_1 \dot{u}_1^2 + A_2 \dot{u}_2^2)dx$$

$$= \frac{\rho}{2}\int_0^L [A_1(\mathbf{N}_a \dot{r} + d_1 \mathbf{N}_\phi \dot{r})^2 + A_2(\mathbf{N}_a \dot{r} - d_2 \mathbf{N}_\phi \dot{r})^2]dx \quad \text{.....................(2.96a)}$$

$$T_3 = \frac{\rho}{2}\int_0^L [A\dot{r}^T \mathbf{N}_a^T \mathbf{N}_a \dot{r}$$

$$+ \quad (A_1 d_1^2 + A_2 d_2^2)\dot{r}^T \mathbf{N}_\phi^T \mathbf{N}_\phi \dot{r}$$

$$+ \quad 2(A_1 d_1 - A_2 d_2)\dot{r}^T \mathbf{N}_a^T \mathbf{N}_\phi \dot{r}]dx \quad \text{....................................(2.96b)}$$

$$T_3 = \frac{1}{2}\dot{r}^T \mathbf{M}_{a1}\dot{r} + \frac{1}{2}\dot{r}^T \mathbf{M}_{a2}\dot{r} + \frac{1}{2}\dot{r}^T \mathbf{M}_{a3}\dot{r} \quad \text{....................................(2.96c)}$$

$$\mathbf{M}_{a1} = \rho A \int_0^L \mathbf{N}_a^T \mathbf{N}_a dx \quad \text{....................................................(2.97)}$$

$$\mathbf{M}_{a2} = \rho(A_1 d_1^2 + A_2 d_2^2)\int_0^L \mathbf{N}_\phi^T \mathbf{N}_\phi dx \quad \text{............................................(2.98)}$$

$$\mathbf{M}_{a3} = 2\rho(A_1 d_1 - A_2 d_2)\int_0^L \mathbf{N}_a^T \mathbf{N}_\phi dx \quad \text{............................................(2.99)}$$

The Axial mass matrix can be written as

$$\mathbf{M}_a = \mathbf{M}_{a1} + \mathbf{M}_{a2} + \mathbf{M}_{a3} \quad \text{...........................................(2.100)}$$

By summing masses from translational, rotational and axial kinetic energy, the final mass for the element can be computed from the following equation as;

$$\mathbf{M} = \mathbf{M}_t + \mathbf{M}_r + \mathbf{M}_a \qquad \text{.........(2.101)}$$

## 2.4.4 SYSTEM MASS MATRIX

Since the degree of freedom for the element is a subset of the degree of freedom of the system, the consistent mass matrix for the system is obtained in a way similar to mentioned for the stiffness matrix. Whilst in the stiffness matrix, the potential energy accounts for the system's stiffness, here the kinetic energy accounts for the system's mass. The kinetic energy of the system is a function of the system masses, thus the kinetic energy can be written in the form

$$T(t) = \frac{1}{2}\sum_{e=1}^{N}\left\{\dot{\bar{q}}\right\}_e^T\left[\bar{m}\right]_e\left\{\dot{\bar{q}}\right\}_e = \frac{1}{2}\sum_{e=1}^{N}\left\{\dot{\bar{r}}\right\}_e^T\left[\bar{M}\right]_e\left\{\dot{\bar{r}}\right\}_e \qquad r = 0 \qquad \text{.........(2.102)}$$

$$= \frac{1}{2}\left\{\dot{\bar{r}}\right\}^T\left[\bar{M}\right]\left\{\dot{\bar{r}}\right\}$$

where

$$\left[\bar{M}\right] = \sum_{e=1}^{N}\left[\bar{M}\right]_e \qquad \text{.........(2.103)}$$

is the symmetric mass matrix for the complete system, which is obtained by a simple addition of the extended element mass matrices. The resulting mass matrix **M** has coefficients $m_{ij}$ representing the mass inertial load at coordinate $i$ developed by unit acceleration $\ddot{x}_j = 1$. The final mass matrix will be non-diagonal therefore referred to as dynamic coupling[11]. As before, the direct stiffness method is employed in this section to obtain the system mass matrix.

---

[11] Non-diagonal mass matrix due to $\ddot{z}$, $\ddot{\phi}$ and/or $\ddot{x}$

# CHAPTER 3

# SOLUTIONS FOR SANDWICH BEAM AND FRAME SYSTEM

This chapter has two sections; one section presents the standard eigen-value formulation for free vibration, and the other concerns with the implementation of the present theory to the general class of sandwich beam and frame systems.

## 3.1 FREE VIBRATION FREQUENCY FOR SANDWICH BEAMS

Any structure possessing mass and elastic properties is subject to vibration. When a system is displaced from its static equilibrium position and then released, it vibrates freely with a frequency that depends upon the mass and stiffness of the system. For an ideal elastic structure (one with no internal damping force), the structure may vibrate for an indefinitely long period of time after excitation. In areas of the world where earthquake activity is a matter of record and may reasonably be anticipated in the future, natural frequencies of vibration of new structures must be considered as a major factor in the design procedure.

An elastic body may have infinite number of modes of vibration. In a free vibration mode all particles are vibrating at the same frequency. Since each frequency requires a coordinate to define the particle's position at any instant, it follows that the system has infinite degrees of freedom. A dynamic system can take several mode shapes that are associated with different frequencies. Each mode shape can be described as a fundamental set of special deflection form by means of which any general deflection of the structure may be expressed. The mode shape depends primarily on the end condition. The configurations for the first three modes of a sandwich beam, simply supported at both ends are as shown in Figure 3.1. The number of independent coordinates required to

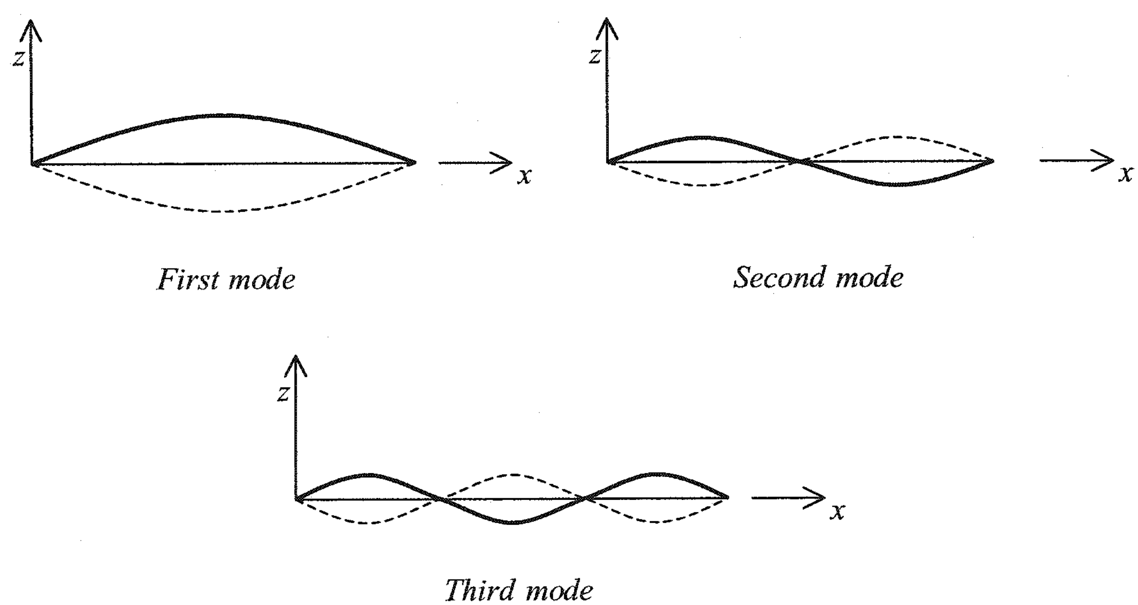


Figure 3.1 First three mode configuration of a simply supported sandwich beam

describe the configurations of a system during its vibration depends upon the number of degrees of freedom of the model used to represent the system, which corresponds to the various natural frequencies.

The lowest frequency of vibration is called the fundamental frequency and the highest frequencies are termed harmonics. In many problems, especially those dealing with structures, the fundamental mode is of particular importance because the amplitudes of vibration are the largest.

### 3.1.1 SYMMETRIC AND ANTISYMMETRIC MODES IN SANDWICH CONSTRUCTION

In the case of sandwich construction, the solution of the eigen-value problem falls into one of two types, namely, symmetric (a displacement pattern that is symmetric with respect to the beam's center line i.e. the two faces moves 180 degrees out of phase), Figure 3.2(a) and the antisymmetric mode (a displacement pattern that is antisymmetric with respect to the beam's center line i.e. the two face sheets moves in phase) Figure 3.2(b). Since, the assumption in our theory is that of a perfect bond between the core and the face sheets thus the type of mode shape expected is one that is global in behavior (antisymmetric).

### 3.1.2 EIGEN VALUE PROBLEM

Any linearly elastic continuum will have natural frequencies and modes ofvibration that can be investigated by considering the mass of the body as well as the stiffness. The differential equation of motion for a system in which structural damping or viscous damping is either not present or considered insignificant and also the applied force being zero is written in a compact form as

$$\mathbf{M}\left\{\ddot{x}\right\} + \mathbf{K}\{x\} = \{0\}; \qquad \text{.........(3.1)}$$

where $\mathbf{M}$ = the mass matrix of the system

$\mathbf{K}$ = the stiffness matrix of the system

$\{x\}$ = the elastic displacement of the beam/frame system

$\left\{\ddot{x}\right\}$ = the acceleration of the beam/frame system

$\{0\}$ = the null column matrix containing only zero elements

The preceding equation represents the general formulation of a wide range of problems in dynamics of structures for free un-damped system. In what follows, Eq. 3.1 serves as the fundamental equation for the formulation of the equation pertinent to determining the natural frequencies of the n-degree of freedom sandwich beam systems.

A very unique characteristic of the mass and stiffness matrices of linear elastic systems is that they are symmetric, that is the elements of the mass matrix should indicate that $m_{ij} = m_{ji}$ and the elements of the stiffness matrix indicates $k_{ij} = k_{ji}$. Given that the mass matrix is non-diagonal and symmetric and further assuming that the motion is harmonic[12], so that $x_i = X_i e^{j\omega t}$, and that the relationship existing for the amplitude $\{\ddot{x}\} = -\omega^2 \{x\}$, obtained from the equation of motion is written as

$$\mathbf{K}\{x\} = \omega^2 \mathbf{M}\{x\} \qquad \text{.......................................................(3.1)}$$

or

$$\mathbf{M}^{-1}\mathbf{K}\{x\} = \omega^2 \{x\} \qquad \text{.......................................................(3.3)}$$

where $\omega$ = the circular frequency of vibration, $rad/\sec$

$f = \omega/2\pi$, frequency in $Hz$

$X$ = the amplitude of the motion

All other terms carry their usual meaning

The relationships among the amplitudes $x$ are called eigen-functions or characteristic functions. The squares of the natural frequencies and corresponding sets of coordinate $\{x\}$ values describing the normal mode shapes are referred to as eigen-values and eigen-vectors (characteristic vector, or modal column) respectively, and they are one of the fundamental importance in the analysis of free vibration of multiple degree of freedom system.

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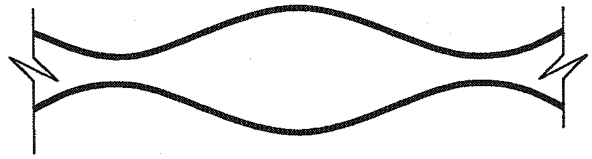
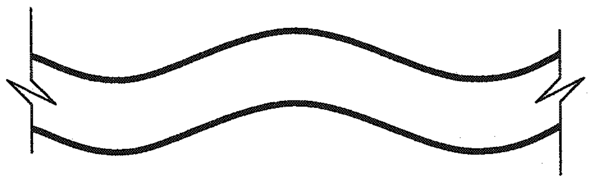
[12] A motion is harmonic when the acceleration of the mass is proportional to the displacement, with damping neglected



*a*) *Symmetric*



*a*) *Antisymmetric*

Figure 3.2 Vibration modes of Sandwich beam

## 3.2 IMPLEMENTATION OF THE THEORY TO THE GENERAL CLASS OF SANDWICH BEAM AND FRAME SYSTEMS

In this section, it will be shown that the developed Sandwich beam element is applicable to the analysis of a general class of Sandwich beam and frame systems. In order to extend the beam element to apply to frame systems the overall axial action is superposed with the bending action resulting in an 8 by 8 element stiffness matrix that can be transformed into global coordinates.

### 3.2.1 ELEMENT MATRICES FOR SANDWICH BEAM AND FRAME SYSTEMS

The preceding chapter was centered on the typical beam element; this section put forward adjustments required for the theory to be good for the general class of beam and frame systems. Various types of elements are used in the finite element analysis of different types of systems. The elements of the element matrices are written such that the axial and bending deformations are accounted for. The task at hand is the need for the derivation of the stiffness and mass matrices such that the boundary conditions, axial thrust and the inclination of the sandwich beam and frame element are taken into consideration.

#### 3.2.1.1 AXIAL THRUST

In a bid to generalize the theory to that of general class of sandwich beam and frame systems, the theory must incorporate the uniform stretching of the entire frame section. The following modifications in the formulation are deemed necessary.

The contribution of the axial thrust will be significant and will appear in the flexibility matrix **f** and geometric stiffness matrix $\mathbf{K}_G$ of the following stiffness relation;

$$\mathbf{K} = \mathbf{B}\mathrm{f}^{-1}\mathbf{B}^T + \mathbf{K}_G \qquad \text{....(3.5)}$$

where $\mathbf{K}_G$, the geometric stiffness matrix, accounts for the secondary moments produced by the thrust $P$ and is written as;

$$\mathbf{K}_G = \frac{P}{L}\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{.......................................(3.6)}$$

$$\mathbf{B} = \begin{bmatrix} 1/L & 1/L & 1/L & 1/L & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1/L & -1/L & -1/L & -1/L & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{.......................................(2.49)}$$

The eight degree of freedom element is as shown in Figure 2.5. From the previous equations, it is evident that the degree of freedom has increased by two. The vectors of nodal displacements and nodal forces are, respectively;

$$\mathbf{r} = \{v_i \; u_i \; \phi_i \; v_i' \; v_j \, u_j \; \phi_j \; v_j'\} \quad \text{.......................................(2.48)}$$

$$\mathbf{R} = \{V_i\ N_i\ M_{di}\ M_{fi}\ V_j\ N_j\ M_{dj}\ M_{fj}\} \qquad \text{.....(2.47)}$$

### 3.2.1.2 COORDINATE TRANSFORMATION

The second issue is that of the relationship between the local and global axis. It can be recalled that according to the finite element method, the dynamic system is regarded as an assemblage of individual discrete elements. The displacement components at the joints of any individual elements are chosen in a direction that depends on the nature of the element under consideration. The individual elements can be part of structural members and the structural members in turn can be part of a more complex structure, such as a frame. The individual elements have to be treated such that they can have different orientations in space necessitating the transformation of local coordinates into global coordinates.

The orientation of the global axis is arbitrary but is generally selected to be parallel to as many as possible of the local axes of the system elements. To make the displacements and forces compatible at a node so that they can be matched or added up, a transformation of the element stiffness matrix has to be performed. The following relations transform the element's displacements and forces from global axes to local axes;

$$\mathbf{q}^m = \mathbf{T}\mathbf{r}^m \qquad \text{.....(3.8a)}$$

and in explicit form;

$$\begin{Bmatrix} \bar{v}_i \\ \bar{u}_i \\ \bar{\phi}_i \\ \bar{v}'_i \\ \bar{v}_j \\ \bar{u}_j \\ \bar{\phi}_j \\ \bar{v}'_j \end{Bmatrix} = [T] \begin{Bmatrix} v_i \\ u_i \\ \phi_i \\ v'_i \\ v_j \\ u_j \\ \phi_j \\ v'_j \end{Bmatrix} \qquad \text{.........(3.8b)}$$

$$\mathbf{Q}^m = \mathbf{TR}^m \qquad \text{.........(3.8c)}$$

From Figure 3.3, it can be seen that the local nodal displacements $\bar{v}_i$, $\bar{u}_i$, $\bar{\phi}_i$, $\bar{v}'_i$, $\bar{v}_j$, $\bar{u}_j$, $\bar{\phi}_j$ and $\bar{v}'_j$ and global joint displacement $v_i$, $u_i$, $\phi_i$, $v'_i$, $v_j$, $u_j$, $\phi_j$ and $v'_j$, for nodes $i$ and $j$ are related by;

$$\left.\begin{aligned} \bar{v}_i &= v_i \cos\alpha - u_i \sin\alpha \\ \bar{u}_i &= v_i \sin\alpha + u_i \cos\alpha \\ \bar{\phi}_i &= \phi_i \\ \bar{v}'_i &= v'_i \end{aligned}\right\} \qquad \text{.........(3.9a)}$$

and

$$\left.\begin{aligned} \bar{v}_j &= v_j \cos\alpha - u_j \sin\alpha \\ \bar{u}_j &= v_j \sin\alpha + u_j \cos\alpha \\ \bar{\phi}_j &= \phi_j \\ \bar{v}'_j &= v'_j \end{aligned}\right\} \qquad \text{.........(3.9b)}$$

respectively.

The member global stiffness and mass matrices is obtained from the following relation;

$$\mathbf{K}^m = \mathbf{T}^{T\,L}\mathbf{K}^m\mathbf{T} \qquad \text{.......(3.10)}$$

$$\mathbf{M}^m = \mathbf{T}^{T\,L}\mathbf{M}^m\mathbf{T} \qquad \text{.......(3.11)}$$

where $\mathbf{K}^m$ and $\mathbf{M}^m$ are the element's global stiffness and mass matrices, ${}^L\mathbf{K}^m$ and ${}^L\mathbf{M}^m$ are the element's local stiffness and mass matrices and $\mathbf{T}$ is the matrix of direction cosines and is written as;

$$\begin{array}{c} \\ \\ \mathbf{T} = \end{array} \begin{array}{c} \\ \bar{v}_i \\ \bar{u}_i \\ \bar{\phi}_i \\ \bar{v}_i' \\ \bar{v}_j \\ \bar{u}_j \\ \bar{\phi}_j \\ \bar{v}_j' \end{array} \begin{array}{c} \begin{array}{cccccccc} v_i & u_i & \phi_i & v_i' & v_j & u_j & \phi_j & v_j' \end{array} \\ \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\alpha & -\sin\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \qquad \text{.......(3.12a)}$$

where $\alpha$ is the angle that the element makes with the global horizontal axis as shown on the Figure 3.3. When uniform axial stretching is excluded, the transformation matrix for the six degree of freedom element becomes

$$\mathbf{T} = \begin{array}{c} \\ \bar{v}_i \\ \bar{\phi}_i \\ \bar{v}'_i \\ \bar{v}_j \\ \bar{\phi}_j \\ \bar{v}'_j \end{array} \begin{array}{c} \begin{matrix} v_i & \phi_i & v'_i & v_j & \phi_j & v'_j \end{matrix} \\ \begin{bmatrix} \cos\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \qquad \text{.......................................(3.12b)}$$

### 3.2.2 BOUNDARY CONDITION

Boundary condition affects the natural frequencies of a structural system vibrating about positions of equilibrium. There is some form of joint supports of some type in a structural system that prevents movement/displacement at one of its degrees of freedom, which must be accounted for in the analysis. So far, the joints of the structure have been treated with the assumption that they are free in all direction and can be referred to as a system composing of natural joints. Boundary conditions for sandwich structure may be seen to differ from that of ordinary structures particularly because of the pronounced shear behavior of sandwich structures. Sandwich beam and frame systems boundary conditions can be classified as either soft or hard depending on whether shear is allowed or prevented at a support.

Boundary conditions can now be imposed on the system to specialize the problem and the resulting element matrices will be an appropriate one to be used in vibration analysis. The introduction of constraints to a structural system reduces the degrees of

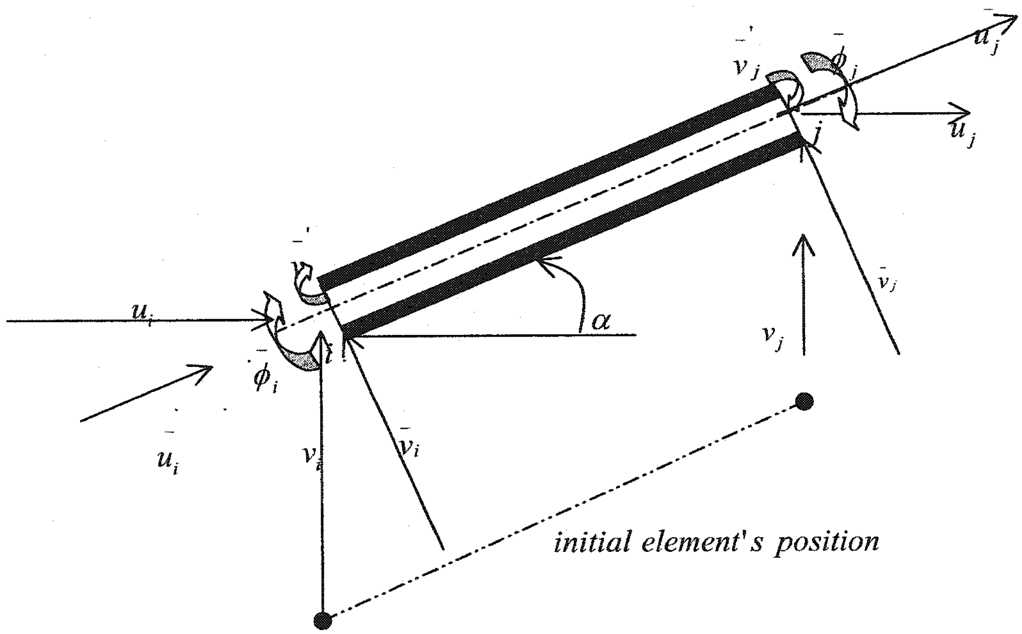



Figure 3.3 Sandwich Beam Element's Local Coordinate

freedom of such system, which in turn increases the stability of the system. It is true in this case that one, all or none of the four-degree of freedom i.e. deflection or vertical translation, longitudinal or axial translation, slope or first derivative of the deflection and the rotation of the section can be restrained depending on the constraint condition. It was clearly shown in previous chapters that each node comprises of three/four degree of freedom and these displacements are either allowed or prevented depending on whether the boundary condition is free, simply supported and/or clamped/fixed. As the stability of the Sandwich beam is increased by an increase in the restrained directions, the energy is transferred to the system thereby causing an increase in the natural frequency of vibration.

#### 3.2.2.1 FREE END

At the free end, the node is free to displace. However, if the end has a rigid-insert the transverse shear strain becomes zero. This type of boundary condition imposes the equality of the rotation $\phi$ and the slope $v'$, in according to the following:

$$v' = \phi \qquad \text{.........(3.13)}$$

#### 3.2.2.2 SIMPLE SUPPORT

Consider a beam with a simply supported end where there is no restraint against deformation due to shear; the condition here is that there is zero transverse displacement (deflection) as shown by the following boundary condition;

$$v = 0 \qquad \text{.........(3.14)}$$

The preceding boundary condition is illustrated in Figure 3.4(a) by allocating zero (i.e. no displacement) to the restrained direction. In the case where in an additional condition is imposed such as that where the shear strain is non-existent ($\gamma = 0$). Such a boundary condition is imposed by, in according to Eq. 3.13, imposing the equality of the rotation $\phi$ and the slope $v'$. Figure 3.4(b) illustrates this type of boundary condition (i.e. degrees of freedom 1 and 2 happens to be equal). In this Figure, the two rotations are assigned with the same degree of freedom as an indication of coupling of the two nodal degrees of

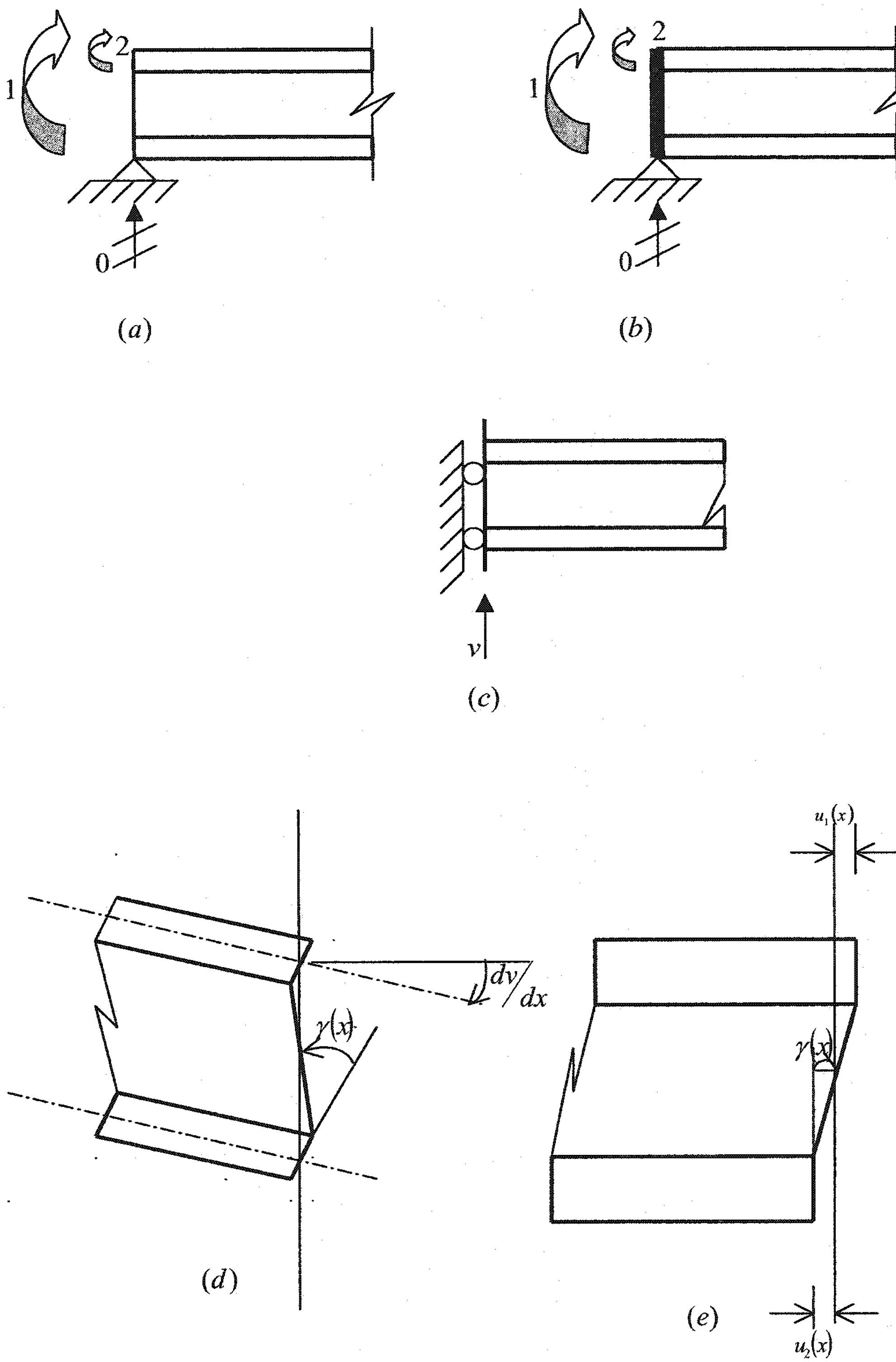



Figure 3.4 Practical and Theoretical Boundary Conditions for Sandwich Beam

freedom. Note that a condition of zero moment can be directly imposed by means of the nodal moments $M_d$ and $M_f$.

3.2.2.3 FIXED SUPPORT

Given the condition that the end is clamped, one can say that the node has been restrained in all direction.

$v = 0$, $v' = 0$ and $\phi = 0$. ......................................................(3.15)

In the case of a clamped support on roller (Fig. 3.4©) the slope $v'$ and the rotation $\phi$ are constrained (i.e. $v \neq 0$, $v' = 0$ and $\phi = 0$). Note that apart from the practical boundary conditions that exist, it is possible theoretically, to impose only one of the degrees of freedom corresponding to the sandwich beam's end moment. (only one of the last two conditions of Expression 3.15 can be imposed), that is either the faces are allowed to rotate without translation (Fig. 3.4(d)) or the faces translate without rotation (Fig.3.4(e)).

There are theoretical possibilities to impose various/different support conditions to the different element at a support and this may cause a change in the sandwich beam's response. One would expect that as more restraint stiffens the system, the natural frequency is increased.

CHAPTER 4

# NUMERICAL EXAMPLE PROBLEMS OF SANDWICH BEAMS

This chapter presents numerical examples of sandwich beam analysis using the developed element matrices from the previous chapters. Numerical solutions are computed using CMAP (version 6.6.8) programming package run on an IBM-compatible PC under Windows Operating System [14]. CMAP provides many built-in functions for quick and easy implementation.

Several numerical examples will be presented later on in this chapter on Sandwich beam and frame systems incorporating the ideas explained in previous chapters. The reason for the numerical investigation is to verify the validity of the element matrices, the convergence characteristics in both static and dynamic problems

## 4.1 STATICS

The validity of the stiffness matrix and its ability to produce exact solution for the static case has been established [25]. For the present implementation, it is of interest to verify its correctness. For this purpose, test cases have been introduced to verify that;

1. The solution is exact using one or more elements
2. The result approaches those predicted by Engineering Beam theory when the core's shear modulus increases.

It is not surprising that the results produced by the element were found to be exact because the element stiffness matrix is exact and all the boundary conditions and inter-element compatibility are satisfied. Solutions by this theory were compared with results from other available exact analytical solutions by [1], [10] and [13]; the solutions were identical. Even though one member element is enough to produce exact results as indicated in case one, as a further test for consistency of the theory and its implementation, the number of elements was increased up to three and identical results were obtained for all the different tests. As a final test, the modulus of elasticity of the core material is increased to a very high value, up to six times the elastic modulus of the faces. The solutions produced for deflection and stresses approaches that by a homogenous isotropic beam since the sandwich beam with a stiff core behave as a single unit. The solution to the deflection and stresses can therefore be referred to as exact. The stiffness matrix employed in this work has been proved to be accurate, what remains to be investigated is the validity of the mass matrix.

## 4.2 DYNAMICS

The natural frequencies of vibration will be determined numerically using the developed theory for sandwich beams and frame systems having different combinations of thickness and physical properties of the material components and also variable boundary conditions. For practical applications, the higher frequencies are usually not of prime interest since the lower frequencies carry most of the system energy. Numerical results in this section will be presented for the lower modes.

### 4.2.1 NUMERICAL RESULTS FOR SANDWICH BEAM

Numerical verification of the present theory "Dynamic analysis of sandwich beam and frame systems" will be presented for the following

- Consistency and Convergence of Solution
- Accuracy of the Theory
- Axial action effect
- Effects of Core shear modulus
- Effects of Dimensional factors

The Cmap program with its program listings written for numerical testing of the various possibilities outlined above can be found in Appendices E, F and G.

#### 4.2.1.1 CONSISTENCY AND CONVERGENCE OF SOLUTION

To investigate the consistency of the theory, the program written for sandwich beam in appendix E is used to produce results for comparison with the homogeneous beam

Table 4.1 Frequency Parameter $\lambda$, for sandwich beam with a core of high shear modulus (modeling a homogenous beam) for different support conditions

| | | Number of Elements | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | n | 6 | 7 | 10 | 12 | 14 | 18 | 24 | 30 | Exact |
| Sandwich Beam with simply supported edges | 1st | 3.176 | 3.16 | 3.155 | 3.150 | 3.148 | 3.145 | 3.144 | 3.143 | 3.1416 |
| | 2nd | 6.525 | 6.471 | 6.382 | 6.353 | 6.335 | 6.314 | 6.301 | 6.295 | 6.2832 |
| | 3rd | 9.956 | 9.925 | 9.731 | 9.647 | 9.592 | 9.528 | 9.484 | 9.463 | 9.4248 |
| | 4th | 12.64 | 13.16 | 13.19 | 13.05 | 12.94 | 12.80 | 12.70 | 12.65 | 12.566 |
| | 5th | 13.87 | 15.33 | 16.61 | 16.52 | 16.38 | 16.15 | 15.97 | 15.88 | 15.708 |
| | 6th | 16.32 | 16.30 | 19.61 | 19.93 | 19.86 | 19.57 | 19.29 | 19.14 | 18.850 |
| | 7th | | 19.05 | 21.73 | 23.00 | 23.26 | 23.04 | 22.67 | 22.44 | 21.991 |
| | 8th | | | 22.93 | 25.38 | 26.38 | 26.52 | 26.09 | 25.79 | 25.133 |
| | 9th | | | 23.50 | 26.94 | 28.94 | 29.91 | 29.56 | 29.18 | 28.274 |
| | 10th | | | 27.23 | 27.84 | 30.79 | 33.08 | 33.04 | 32.61 | 31.416 |
| Cantilever Sandwich Beam | 1st | 1.874 | 1.874 | 1.874 | 1.875 | 1.875 | 1.875 | 1.875 | 1.875 | 1.8751 |
| | 2nd | 4.775 | 4.753 | 4.723 | 4.714 | 4.708 | 4.703 | 4.699 | 4.697 | 4.6941 |
| | 3rd | 8.223 | 8.143 | 8.006 | 7.961 | 7.933 | 7.902 | 7.881 | 7.871 | 7.8548 |
| | 4th | 11.62 | 11.64 | 11.40 | 11.29 | 11.22 | 11.13 | 11.07 | 11.05 | 10.996 |
| | 5th | 13.75 | 14.69 | 14.90 | 14.74 | 14.61 | 14.43 | 14.31 | 14.25 | 14.137 |
| | 6th | 15.12 | 16.23 | 18.28 | 18.23 | 18.07 | 17.81 | 17.59 | 17.48 | 17.279 |
| | 7th | | 17.64 | 21.07 | 21.61 | 21.57 | 21.26 | 20.93 | 20.76 | 20.420 |
| | 8th | | | 22.76 | 24.53 | 24.94 | 24.75 | 24.33 | 24.07 | 23.562 |
| | 9th | | | 23.48 | 26.60 | 27.94 | 28.22 | 27.77 | 27.44 | 26.704 |
| | 10th | | | 25.20 | 27.75 | 30.26 | 31.59 | 31.26 | 30.84 | 29.845 |
| Fixed-fixed Sandwich Beam | 1st | 4.781 | 4.771 | 4.752 | 4.746 | 4.742 | 4.737 | 4.734 | 4.733 | 4.7300 |
| | 2nd | 8.058 | 8.053 | 7.981 | 7.947 | 7.924 | 7.897 | 7.878 | 7.869 | 7.8532 |
| | 3rd | 11.01 | 11.29 | 11.32 | 11.25 | 11.20 | 11.12 | 11.07 | 11.04 | 10.996 |
| | 4th | 13.01 | 13.91 | 14.67 | 14.63 | 14.54 | 14.41 | 14.30 | 14.24 | 14.137 |
| | 5th | 13.97 | 15.60 | 17.76 | 17.98 | 17.94 | 17.76 | 17.57 | 17.47 | 17.279 |
| | 6th | 18.61 | 16.38 | 20.27 | 21.10 | 21.29 | 21.16 | 20.90 | 20.74 | 20.420 |
| | 7th | | 21.71 | 22.04 | 23.76 | 24.45 | 24.57 | 24.27 | 24.05 | 23.562 |
| | 8th | | | 23.08 | 25.80 | 27.21 | 27.93 | 27.68 | 27.40 | 26.704 |
| | 9th | | | 23.54 | 27.16 | 29.44 | 31.12 | 31.11 | 30.78 | 29.845 |
| | 10th | | | 31.04 | 27.94 | 31.08 | 34.02 | 34.53 | 34.20 | 32.987 |

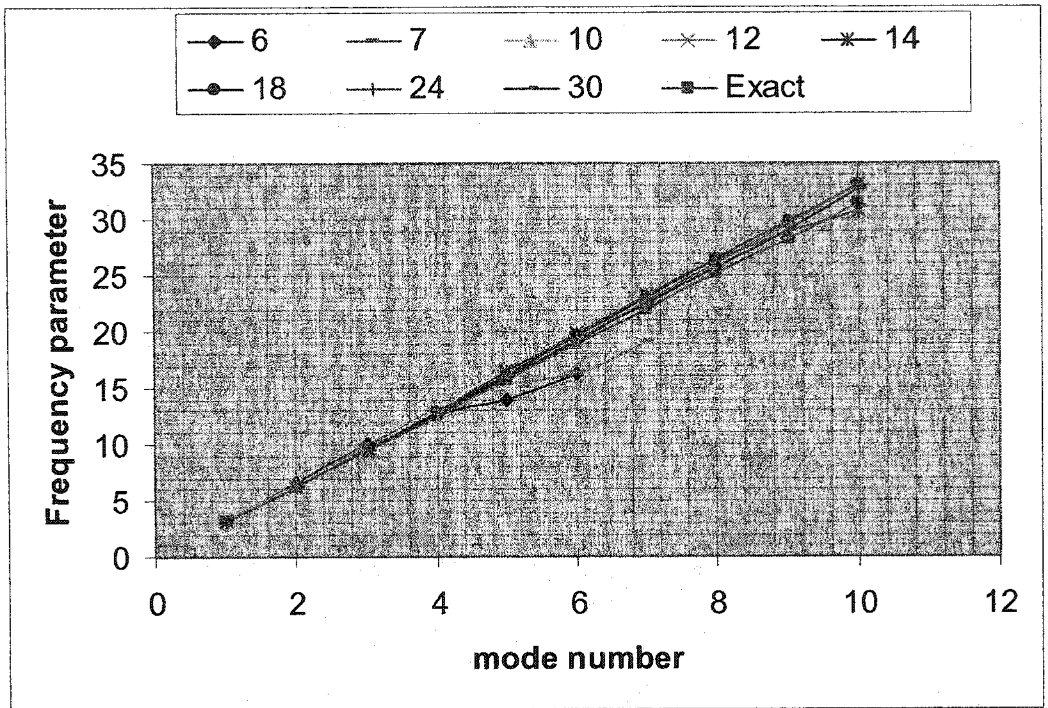


Figure 4.1 Convergence of the natural frequency parameter to the exact Value (Homogenous beam) for a Simply supported sandwich

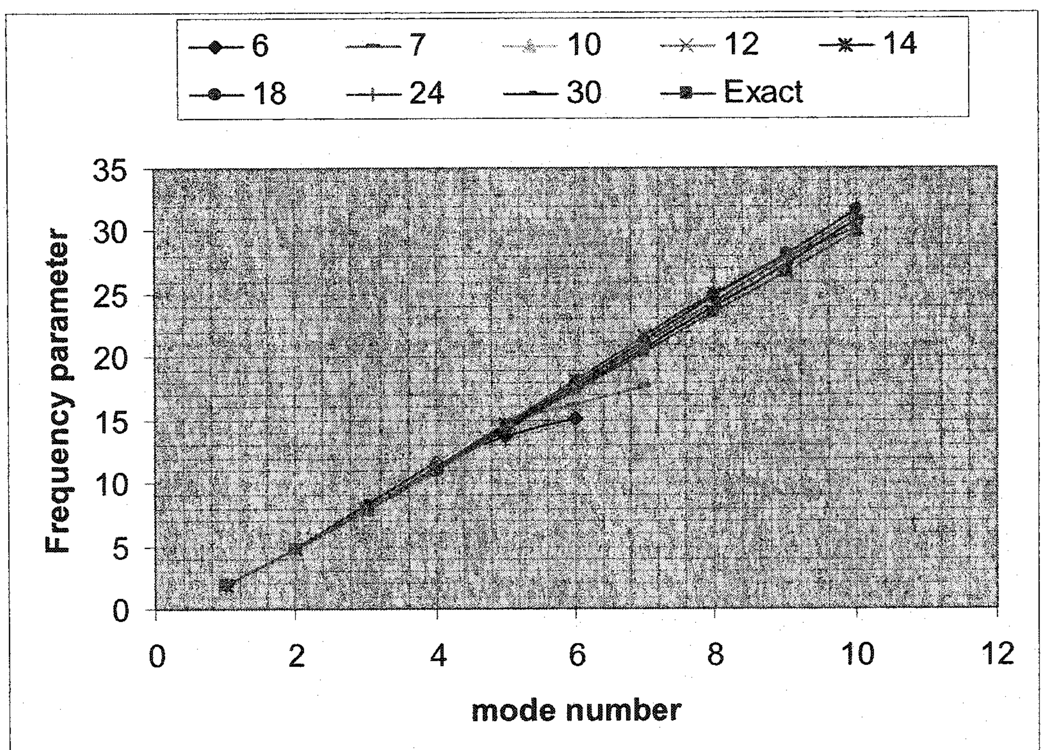


Figure 4.2 Convergence of the natural frequency parameter to the exact Value (homogenous beam) for a Cantilever sandwich beam

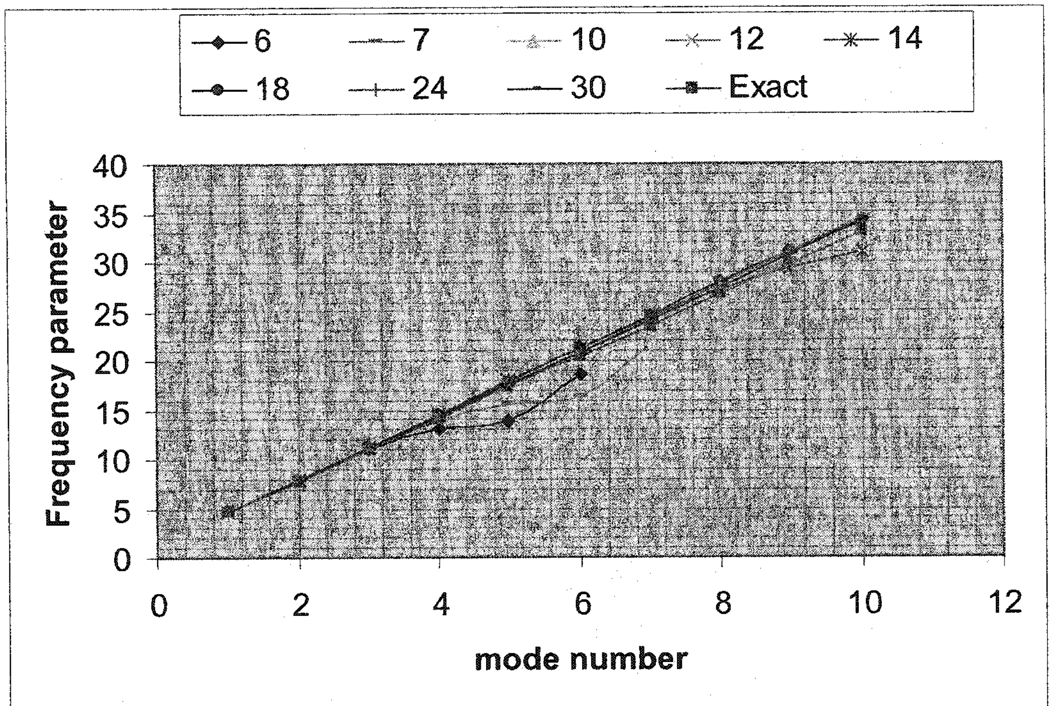



Figure 4.3 Convergence of the natural frequency parameter to the exact Value (Homogenous beam) for a Fixed-fixed sandwich beam

situation. To achieve this goal, the core is made very stiff by increasing the shear modulus to a very high value (up to the Elastic modulus of the face sheet material).

Following are the dimensions and material properties of the sandwich beam, selected arbitrarily; Length of beam, $l = 2.5m$, thickness of core $c = 0.0127m$, thickness of face $t1 = t2 = 0.0004572m$, Elastic modulus of faces $E_1 = E_2 = E = 6.8 \times 10\ GPa$, Core shear modulus $G = E$ and the density of the facial material is $\rho = 2.680 \times 10^3$ $kg/m^3$.

The natural frequency parameter for the sandwich beam shown in Table 4.1 for comparison with the homogenous isotropic beam can be calculated by means of the following equation.

$$\lambda^4 = \frac{\rho\omega^2 L^4}{EI} \qquad \text{.....(4.1)}$$

Figures 4.1, 4.2 and 4.3 show the convergence of the natural frequency parameter $\lambda$ for the simply supported, cantilever and fixed-fixed sandwich beams respectively to the exact values for a homogenous isotropic beam. Each graph of Figures 4.1, 4.2 and 4.3

Table 4.2 Convergence of the solution [natural frequencies $f(Hz)$] due to Transverse and Rotational vibration effects for simply supported sandwich beam

| Number of Elements | Modes | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1st | 2nd | 3rd | 4th | 5th | 6th | 7th | 8th | 9th | 10th |
| 2 | 68.84 | | | | | | | | | |
| 3 | 66.33 | 250.8 | | | | | | | | |
| 4 | 64.37 | 268.7 | 517.2 | | | | | | | |
| 5 | 63.34 | 262.0 | 576.9 | 860.0 | | | | | | |
| 6 | 62.76 | 255.3 | 574.5 | 957.6 | 1264 | | | | | |
| 7 | 62.42 | 250.6 | 561.1 | 966.5 | 1388 | 1710 | | | | |
| 8 | 62.17 | 247.3 | 549.3 | 949.9 | 1408 | 1851 | 2180 | | | |
| 9 | 62.01 | 245.0 | 539.7 | 930.3 | 1391 | 1878 | 2330 | 2661 | | |
| 10 | 61.88 | 243.3 | 532.6 | 913.2 | 1365 | 1861 | 2364 | 2817 | 3142 | |
| 11 | 61.80 | 242.1 | 527.1 | 897.8 | 1340 | 1831 | 2347 | 2855 | 3303 | 3920 |
| 12 | 61.74 | 241.1 | 522.8 | 888.1 | 1319 | 1799 | 2108 | 2839 | 3347 | 3785 |
| 24 | 61.47 | 237.2 | 505.6 | 841.4 | 1223 | 1635 | 2068 | 2516 | 2979 | 3454 |
| 40 | 61.43 | 236.4 | 501.9 | 830.7 | 1201 | 1596 | 2006 | 2424 | 2847 | 3275 |
| 56 | 61.41 | 236.2 | 500.9 | 828.4 | 1195 | 1585 | 1988 | 2397 | 2810 | 3225 |
| 64 | 61.41 | 236.1 | 500.7 | 827.8 | 1194 | 1583 | 1984 | 2391 | 2801 | 3213 |
| 72 | 61.41 | 236.1 | 500.5 | 827.3 | 1193 | 1581 | 1981 | 2387 | 2795 | 3205 |

represents a sample model for a certain number of elements in the sandwich beam and one corresponding to that of the exact results for the homogenous isotropic beam.

To study the convergence characteristics of the present theory, the simply supported sandwich beam is used as the model for analysis. The dimensions and material properties of the sandwich beam arbitrarily selected as follows; Length of beam, $l = 0.9144m$, thickness of core $c = 0.0127m$, thickness of face $t1 = t2 = 0.0004572m$, Elastic modulus of faces $E_1 = E_2 = E = 6.8 \times 10\ GPa$, Core shear modulus $G = 0.0012E$ and the density of the facial material is $\rho = 2.680 \times 10^3\ {kg}/{m^3}$. This model is analyzed with increasing number of elements in a single span sandwich beam and the results observed to check the rate of convergence.

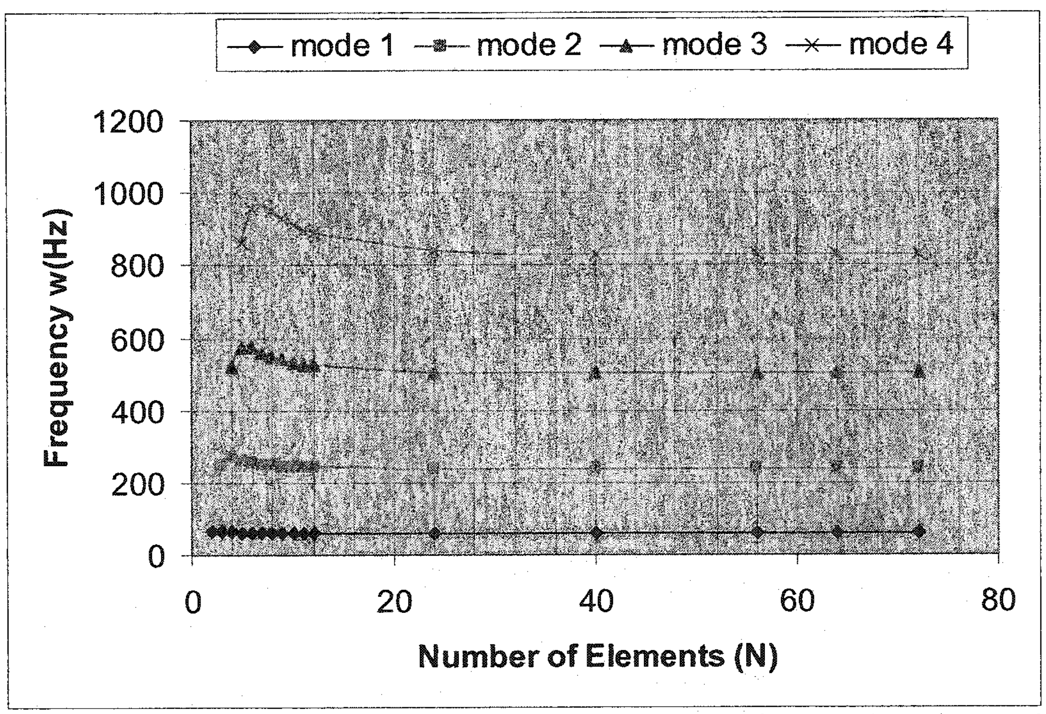



Figure 4.4 Convergence Study for the lower Modes for simply supported sandwich beam due to translational and rotational effects

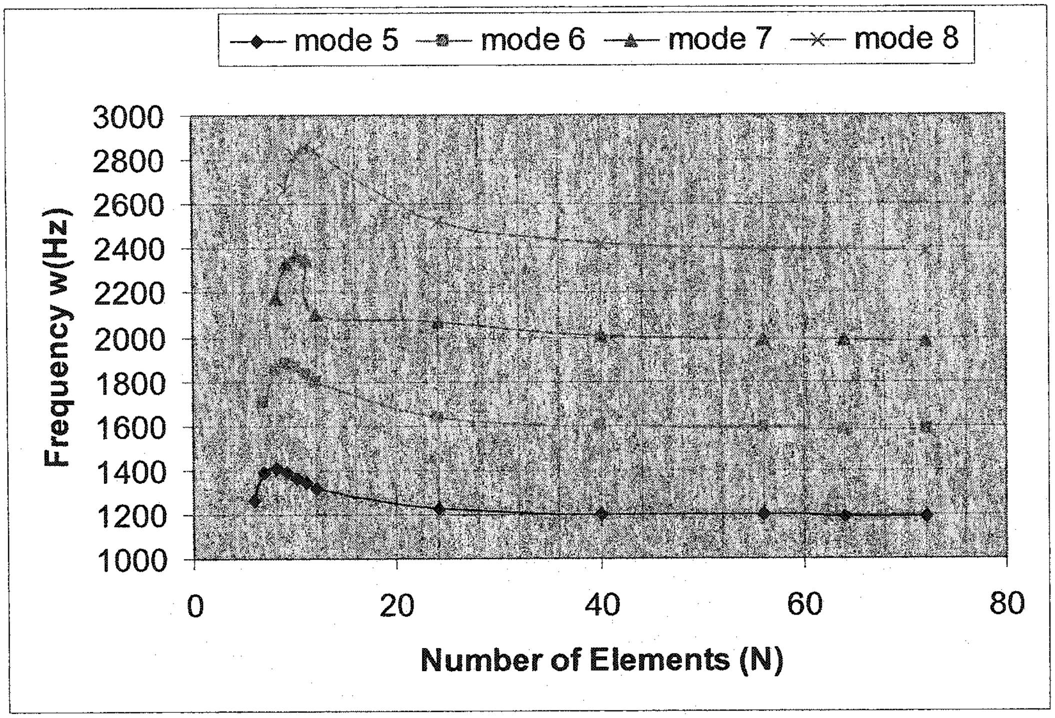



Figure 4.5 Convergence Study for the upper Modes for simply supported sandwich beam due to translational and rotational effects

Numerical values for the vibration frequency $\omega$, obtained from the simply supported single span sandwich beam, starting from two members per span with successive increase in the number of elements is shown in Table 4.2. Figures 4.3 and 4.4 shows the graphs of frequencies plotted against the number of elements used in the sandwich beam for the lowest 8 modes. These figures are very appropriate for convergence study of the theory.

4.2.1.1.1 DISCUSSION OF RESULTS ON CONSISTENCY AND CONVERGENCE

The values of the 10 lowest natural frequency parameter $\lambda$, of simply supported, cantilever and fixed-fixed sandwich beams produced by the program for sandwich beam when the core is made stiff is similar to that of a homogeneous isotropic beam as shown in Table 4.1. As the number of elements is increased, the results approach that of the exact solution of a homogeneous isotropic beam as indicated in Figures 4.1, 4.2 and 4.3. The theory is therefore consistent. The sandwich beam can be modeled to behave in a way similar to the homogenous isotropic beam by modeling the core stiff (as stiff as the face sheet materials).

It is important to note also that the natural frequency of vibration of the sandwich beam system depends on the end condition; as the degree of stability of sandwich beam is increased by changing the end support condition, the energy of vibration in turn is transferred to the system causing an increase in the natural frequency. Boundary condition therefore continues to be a dependent factor for the natural frequency of the sandwich beam as explained in the previous chapter.

The results shown in Table 4.2 are the natural frequency of a sandwich beam taking into consideration only translational and rotational effects. The reduction in mesh size tends to reduce the element stiffness while the mass remains unaltered; thus the natural frequency is expected to decrease subsequently. From Figure 4.4, the fundamental mode decreases monotonically while the other modes are a bit complicated. Figure 4.5 shows the natural frequency for the higher modes. The higher modes increases briefly as the mesh size is reduced until a local maximum is reached, followed by a steady decline until the mesh size is finest. The higher the particular mode, the higher the number of element for which the corresponding frequencies curve reaches its local maximum. The phenomena behind the complication in the natural frequency results are explained in terms of the ratio of the depth of the sandwich beam to the depth of the core $(d/c)$ by reference [42]. The author asserted that an increase in face thickness did not necessarily lead to an increase in frequency, irrespective of the mode. Higher vibration modes of Sandwich beam with less pronounced or complete neglect to axial vibration effects are disrupted.

In the paper by Toshiro Hayashikara and Noboro Watanabe [37], it was shown that the consistent mass matrix method can produce results to within one percent of

Table 4.3 Convergence of the solution [natural frequencies $f(Hz)$] due to Transverse and Rotation and Axial vibration effects for simply supported sandwich beam

| Number of Elements | Modes | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1st | 2nd | 3rd | 4th | 5th | 6th | 7th | 8th | 9th | 10th |
| 2 | 68.79 | | | | | | | | | |
| 3 | 66.31 | 250.2 | | | | | | | | |
| 4 | 64.37 | 267.9 | 504.3 | | | | | | | |
| 5 | 63.34 | 261.4 | 572.3 | 848.3 | | | | | | |
| 6 | 62.76 | 254.9 | 570.3 | 940.8 | 1228 | | | | | |
| 7 | 62.40 | 250.3 | 558.0 | 947.7 | 1339 | 1619 | | | | |
| 8 | 62.17 | 247.1 | 546.5 | 935.1 | 1357 | 1735 | 1992 | | | |
| 9 | 62.00 | 244.8 | 537.6 | 917.3 | 1343 | 1758 | 2101 | 2324 | | |
| 10 | 61.89 | 243.1 | 530.7 | 901.2 | 1322 | 1745 | 2125 | 2421 | 2609 | |
| 11 | 61.80 | 241.9 | 525.5 | 888.8 | 1301 | 1722 | 2113 | 2443 | 2691 | 2764 |
| 12 | 61.74 | 241.0 | 521.3 | 878.7 | 1282 | 1698 | 2090 | 2433 | 2711 | 2762 |
| 24 | 61.48 | 237.2 | 504.8 | 835.1 | 1198 | 1565 | 1913 | 2229 | 2505 | 2744 |
| 40 | 61.43 | 236.4 | 501.1 | 825.5 | 1179 | 1533 | 1866 | 2165 | 2427 | 2651 |
| 56 | 61.41 | 236.2 | 500.1 | 822.8 | 1173 | 1524 | 1853 | 2148 | 2404 | 2622 |
| 64 | 61.41 | 236.1 | 499.9 | 822.2 | 1172 | 1521 | 1849 | 2143 | 2398 | 2616 |
| 72 | 61.41 | 236.1 | 499.7 | 821.8 | 1171 | 1520 | 1847 | 2140 | 2394 | 2611 |

Table 4.4 Convergence of the theory [natural frequencies $f(Hz)$] for Transverse, Rotational and Axial vibration effects for sandwich beam with soft core

| Number of Elements | Modes | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1st | 2nd | 3rd | 4th | 5th | 6th | 7th | 8th | 9th | 10th |
| 2 | 35.45 | | | | | | | | | |
| 3 | 33.42 | 81.22 | | | | | | | | |
| 4 | 32.67 | 77.09 | 124.2 | | | | | | | |
| 5 | 32.31 | 74.65 | 119.8 | 163.5 | | | | | | |
| 6 | 32.11 | 73.23 | 115.9 | 159.9 | 198.5 | | | | | |
| 7 | 32.00 | 72.36 | 113.4 | 155.6 | 196.8 | 229.7 | | | | |
| 8 | 31.91 | 71.78 | 111.6 | 152.2 | 192.7 | 229.9 | 257.4 | | | |
| 9 | 31.86 | 71.39 | 110.4 | 149.6 | 188.9 | 226.6 | 259.3 | 282.0 | | |
| 10 | 31.82 | 71.10 | 109.5 | 147.7 | 185.8 | 223.0 | 257.2 | 285.3 | 303.9 | |
| 11 | 31.79 | 70.89 | 108.8 | 146.2 | 183.3 | 219.7 | 254.0 | 284.4 | 308.3 | 323.6 |
| 12 | 31.77 | 70.73 | 108.3 | 145.1 | 181.3 | 216.8 | 250.8 | 282.0 | 308.6 | 328.9 |
| 24 | 31.69 | 70.10 | 106.2 | 140.5 | 172.8 | 203.4 | 232.4 | 259.8 | 285.7 | 310.5 |
| 40 | 31.67 | 69.96 | 105.8 | 139.4 | 170.9 | 200.1 | 227.3 | 252.4 | 275.8 | 297.8 |
| 56 | 31.66 | 69.92 | 105.7 | 139.1 | 170.3 | 199.1 | 225.6 | 250.1 | 272.7 | 293.6 |
| 64 | 31.66 | 69.90 | 105.6 | 139.0 | 170.1 | 198.8 | 225.2 | 249.5 | 271.8 | 292.5 |
| 72 | 31.66 | 69.90 | 105.6 | 139.0 | 170.0 | 198.6 | 224.9 | 249.1 | 271.2 | 291.7 |

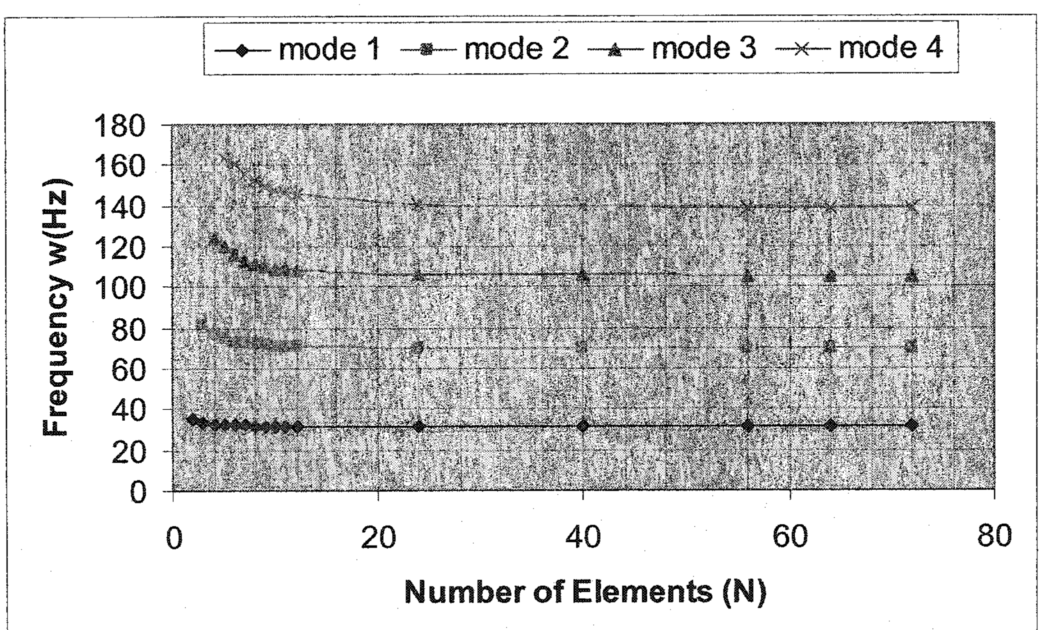



Figure 4.6 Convergence Study for the lower Modes for a simply supported Sandwich beam with soft core due to the three vibration effects

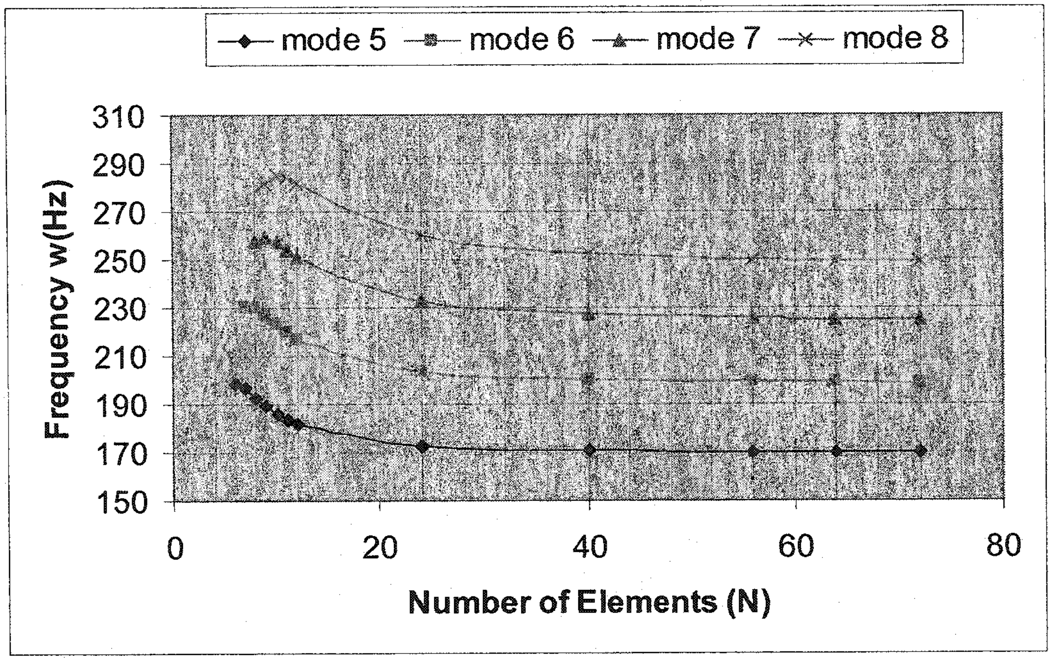



Figure 4.7 Convergence Study for the upper Modes for a simply supported Sandwich beam with soft core due to the three vibration effects

accuracy if the number of beam segments used in the analysis is twice the number of eigen-values required. In Table 4.2, frequency values for higher vibration modes for smaller number of elements are not represented because they don't correspond to the flexural behavior of the sandwich beam. It is the number of available translational degrees of freedom that will dictate how many degrees of freedom modes will be considered. However, further steps can be taken to investigate the convergence characteristics of the sandwich beam.

The developed theory assumes perfect bonding, and therefore it's always antisymmetric (global) response. The discrepancy in the eigen-values for the higher modes is reduced with the introduction of axial effects in Table 4.3. It is important to note that the convergence characteristics can still be improved particularly for the higher modes.

Table 4.4 shows the convergence of frequency values when the shear modulus of the core is replaced with the value $G = 0.000012E$. Here, the disruption in the sandwich beam behavior is shifted to very high modes whilst the modal frequencies considered in this section are seen to converge monolithically. Figure 4.6 and 4.7 shows the rate of convergence of the sandwich beam element with soft core.

The fundamental mode carries over 90% of the total energy of the sandwich beam which is demonstrated by its very high amplitude, thus will not be expected to deviate easily from its expected shape. It is for this reason that the fundamental modal frequency will be used in the study of the sandwich beam's behavior in the sections that follows. The convergence characteristic of the model considered in this work is good.

#### 4.2.1.2 ACCURACY OF THEORY

Natural frequencies of sandwich beams with various boundary conditions will be presented in this section. Results produced by this theory will be compared with available results from existing theories in the literature.

##### 4.2.1.2.1 SIMPLY SUPPORTED SANDWICH BEAM (S-S)

Numerical solutions for a simply supported sandwich beam are compared as given in Table 4.5, along with other theoretical results by other authors. Following are the dimensions and material properties of the sandwich beam used by other references in the literature; Length of beam, $l = 0.9144m$, thickness of core $c = 0.0127m$, thickness of face $t1 = t2 = 0.0004572m$, Elastic modulus of faces $E_1 = E_2 = E = 6.89 \times 10\ GPa$, Core shear modulus $G = 0.0012E$ and the density of the facial material is $\rho = 2.680 \times 10^3$ $kg/m^3$.

Table 4.5 Comparisons with other theoretical natural frequencies $f(Hz)$ for simply supported sandwich beam

| Mode N | Present Theory 72 Elements | | Ref. [52] | Mead [53] | | | Ref. [48] | Ref. [47] |
|---|---|---|---|---|---|---|---|---|
| | T & R | T, R & A | | Simpson | Trapezoidal | Cubic | | |
| 1st | 61.805 | 61.805 | 56.159 | 55.996 | 56.023 | 55.996 | 55.50 | 57.5 |
| 2nd | 237.67 | 237.62 | 215.82 | ---- | ---- | ---- | ---- | ---- |
| 3rd | 503.79 | 502.99 | 457.22 | 456.89 | 459.14 | 456.86 | 451.0 | 467.0 |
| 4th | 832.64 | 827.15 | 755.05 | ---- | ---- | ---- | ---- | ---- |
| 5th | 1200.9 | 1178.8 | 1087.9 | 1090.1 | 1107.7 | 1089.9 | 1073 | 1111 |
| 6th | 1591.5 | 1523.0 | 1440.3 | ---- | ---- | ---- | ---- | ---- |
| 7th | 1994.2 | 1859.5 | 1802.7 | 1811.6 | 1876.3 | 1811.6 | 1779 | 1842 |
| 8th | 2402.7 | 2154.2 | 2169.8 | ---- | ---- | ---- | ---- | ---- |
| 9th | 2813.9 | 2409.5 | 2538.2 | 2555.6 | 2723.0 | 2561.3 | 2510 | 2594 |
| 10th | 3225.8 | 2627.7 | 2906.2 | ---- | ---- | ---- | ---- | ---- |
| 11th | 3637.7 | 2772.7 | ---- | 3290.7 | 3655.5 | 3329.5 | ---- | ---- |
| 12th | 4049.5 | 2813.0 | ---- | ---- | ---- | ---- | ---- | ---- |
| 13th | 4461.4 | 2970.3 | ---- | 3993.6 | 4707.8 | 4138.3 | ---- | ---- |
| 14th | 4873.9 | 3104.1 | ---- | ---- | ---- | ---- | ---- | ---- |
| 15th | 5285.8 | 3219.5 | ---- | 5738.1 | 5937.8 | 5042.7 | ---- | ---- |

Table 4.6 Comparisons with other theoretical natural frequencies $f(Hz)$ for Cantilever beams

| Mode N | Present Theory 72 Elements | | T. Sakiyama H. Matsuda and C. Morita [52] | Mead [82] | Ahmed [48] | Ahmed [47] |
|---|---|---|---|---|---|---|
| | T & R | T, R & A | | | | |
| 1st | 36.509 | 36.5085 | 33.146 | 34.242 | 32.79 | 33.97 |
| 2nd | 215.30 | 215.241 | 195.96 | 201.85 | 193.5 | 200.5 |
| 3rd | 554.53 | 553.162 | 503.43 | 520.85 | 499 | 517 |
| 4th | 982.64 | 971.365 | 893.28 | 920.4 | 886 | 918 |
| 5th | 1462.92 | 1415.76 | 1328.5 | 1381.3 | 1320 | 1368 |
| 6th | 1969.92 | 1782.25 | 1790.7 | 1867 | 1779 | 1844 |
| 7th | 2490.15 | 1838.91 | 2260.2 | 2374 | 2249 | 23331 |
| 8th | 3014.71 | 2211.62 | 2738.9 | 2905 | 2723 | 2824 |
| 9th | 3540.61 | 2521.98 | 3212.8 | ---- | ---- | ---- |
| 10th | 4066.08 | 2774.04 | 3691.6 | ---- | ---- | ---- |

#### 4.2.1.2.2 CANTILEVERED SANDWICH BEAM (C-F)

Numerical solutions for Cantilever sandwich beam are compared as given in Table 4.6, along with other theoretical results by other authors. Following are the dimensions and material properties of the sandwich beam; Length of beam, $l = 0.7112m$, thickness of core $c = 0.0127m$, thickness of face $t1 = t2 = 0.0004572m$, Elastic modulus of faces $E_1 = E_2 = E = 6.89 \times 10\ GPa$, Core shear modulus $G = 0.0012E$ and the density of the facial material is $\rho = 2.680 \times 10^3\ {kg}/{m^3}$. The agreement between the present theory and other theories can be studied from the results of table 4.6 for a cantilever beam.

Table 4.7(a) Comparisons with other theoretical and experimental natural frequencies $f(Hz)$, of Fixed-fixed Sandwich beams

| Mode n | Present Theory 72 Elements | | T. Sakiyama H. Matsuda and C. Morita [52] | Raville's Exp. Results [44] |
|---|---|---|---|---|
| | T & R | T, R & A | | |
| 1st | 10.202 | 10.202 | 8.5228 | ---- |
| 2nd | 27.959 | 27.959 | 23.359 | ---- |
| 3rd | 54.416 | 54.416 | 45.454 | ---- |
| 4th | 89.141 | 89.141 | 74.460 | ---- |
| 5th | 131.78 | 131.78 | 110.06 | 112.3 |
| 6th | 181.88 | 181.88 | 151.89 | 154.3 |
| 7th | 239.00 | 239.00 | 199.52 | 202.1 |
| 8th | 302.69 | 302.56 | 252.58 | 254.9 |
| 9th | 372.37 | 372.16 | 310.66 | 312.5 |
| 10th | 447.65 | 447.25 | 373.34 | 376.0 |

4.2.1.2.3 CLAMPED/FIXED SANDWICH BEAM (C-C)

Numerical solutions for fixed sandwich beam are compared as given in table 4.7(a), 4.7(b) and 4.7(c), along with other theoretical and experimental results by other authors. Following are the dimensions and material properties of the sandwich beam; Length of beam, $l = 2.43744m, \quad 1.82808m, \quad 1.21872m$ for tables 4.7(a), 4.7(b) and 4.7(c) respectively, thickness of core, $c = 0.0063475m$, thickness of face $t1 = t2 = 0.00040624$ $m$, Elastic modulus of faces $E_1 = E_2 = E = 6.89 \times 10\ GPa$, Core shear modulus $G = 0.001E$ and the density of the facial material is $\rho = 2.6873 \times 10^3\ {}^{kg}/_{m^3}$.

Table 4.7(b) Comparisons with other theoretical and experimental natural frequencies $f(Hz)$, of Fixed-fixed sandwich beams

| Mode n | Present Theory 72 Elements | | T. Sakiyama H. Matsuda and C. Morita [52] | Raville's Exp. Results [44] |
|---|---|---|---|---|
| | T & R | T, R & A | | |
| 1st | 18.063 | 18.063 | 15.090 | ---- |
| 2nd | 49.266 | 49.266 | 41.159 | ---- |
| 3rd | 95.307 | 95.294 | 79.614 | ---- |
| 4th | 155.02 | 155.01 | 129.48 | 134.8 |
| 5th | 227.32 | 227.26 | 189.83 | 196.3 |
| 6th | 310.94 | 310.86 | 259.67 | 269.5 |
| 7th | 404.77 | 404.52 | 337.91 | 349.6 |
| 8th | 507.55 | 507.05 | 423.61 | 431.6 |
| 9th | 618.25 | 617.02 | 515.86 | 519.5 |
| 10th | 735.91 | 733.67 | 613.70 | 632.8 |

Table 4.7(c) Comparisons with other theoretical and experimental natural frequencies $f(Hz)$, of Fixed beams

| Mode N | Present Theory 72 Elements | | T. Sakiyama H. Matsuda and C. Morita [52] | V.S.Sokolinsky Steven R. Nutt And Y. Frostig [28] | Raville's Exp. Results [44] |
|---|---|---|---|---|---|
| | T & R | T, R & A | | | |
| 1st | 40.175 | 40.175 | 33.563 | 34.6 | ---- |
| 2nd | 108.17 | 108.17 | 90.364 | 93.1 | ---- |
| 3rd | 205.98 | 205.98 | 172.07 | 177.2 | 185.5 |
| 4th | 329.12 | 328.99 | 274.91 | 282.8 | 280.3 |
| 5th | 473.47 | 473.01 | 395.42 | 406.3 | 399.4 |
| 6th | 635.02 | 633.83 | 530.34 | 544.3 | 535.2 |
| 7th | 810.75 | 807.46 | 676.85 | 693.7 | 680.7 |
| 8th | 997.48 | 990.60 | 832.43 | 852.0 | 867.2 |
| 9th | 1192.92 | 1180.80 | 995.36 | 1017.1 | 1120.0 |
| 10th | 1395.31 | 1372.98 | 1163.9 | 1187.3 | 1201.0 |

#### 4.2.1.2.4 DISCUSSION OF RESULTS ON ACCURACY TESTS

Natural frequency results for various boundary conditions were presented above to check the accuracy of the theory. The deviations of the results by the other authors to the present theory for effects due to translation and rotation are summarized below;

1. For the Sandwich beam, simply supported at both ends, the percentage difference in the results varies from 7% to 10%
2. For the case of the Cantilever Sandwich beam, the percentage difference in the results varies from 5% to 11%
3. For the case of the Fixed-fixed Sandwich beam, the percentage difference in the results varies from 9% to 17%

From the deviation results, one can conclude that there is good agreement between the present theory and those from other references.

One feature of the present theory is its ability to simulate the natural frequency values for all the modes of a sandwich beam element with various boundary conditions. There is a consistent increase in frequency of subsequent modes as the modes increases. The slight discrepancies between the results of the present work and those from other authors may be due to the differences mentioned below.

According to Toshiro Hayashikara and Noboro Watanabe [37], eigen values obtained by consistent mass matrix method are upper bound to the exact values whilst that from lump mass method is lower bound. In the paper by John S. Acher [16], it is shown that the natural frequencies obtained by the use of the consistent mass matrix are upper bound to the exact values. A large number of beam segments were needed for the lump mass method to obtain the same degree of accuracy. The procedure is applicable to the general dynamic response analysis and is demonstrably superior to the usual procedure of physical mass lumping by application to frequency analysis for sandwich beam with different boundary conditions with uniformly distributed masses.

The prediction of the higher order frequencies with equal accuracy to the lower ones constitutes a significant challenge for other analytical approaches, including that of finite element. D. J. Mead and S. Sivakumuran [53] applied the Stodola method to study vibration of sandwich beam. This method involves a four-fold integration of an initial approximate method to obtain a better approximation to the mode. Convergence of this theory can only be assured for the fundamental mode. The authors continued by stating that the integration process by the Simpson rule may sometimes give accurate results for the lower modes and the cubic rule gives accurate results for higher modes. Results produced by [16] and other references making use of the Rayleigh's method are believed to be limited to the fundamental natural frequency since a different shape function must be used for each higher modes. The higher order frequencies are predicted with the same degree of accuracy as the lower ones by the present theory. The other papers, except for T. Sakiyama, H. Matsuda and C. Morita [52] and V. S. Sokolinsky, Steven R. Nutt and Y. Frostig [28] could not provide frequency values for all the modes.

Most of the papers mentioned in this section have similar assumptions that are comparable to the ones used in this work. However, additional assumptions made by the papers mentioned above limit their applicability.

Further assumptions made by K. M. Ahmed [48] in his displacement based finite element technique are that;

1. The thicknesses of the face sheets, however, are assumed to be equal and small compared with the overall thickness of the sandwich section.
2. The faces have the same material

In an unsymmetrically laminated multi-core sandwich beam, a bending stretching coupling occurs that is not present in a symmetrically laminated structure. In general, most of the papers in the literature are good for sandwich beam with symmetric configuration. Although the main qualitative features of vibration response are more readily perceived using beams with symmetric section, it would have been a better idea for the theory to be general, incorporating the non-identical face sheet sandwich beam. By way of including uniform stretching in the face sheet materials, the present theory is capable of treating sandwich beam with anti-symmetric configuration. This feature renders the theory fit for application to the general class of beams and frame system.

In addition to the limitations caused by these assumptions, the theory itself is completely analytical making use of assumed displacement fields that dictates the accuracy of the generalized mass and stiffness matrices. If the displacement functions are not properly selected, the results obtained may not converge to the correct answers. The major difference between this paper and [47] which is by the same author is the number of degrees of freedom considered per node. Higher degrees of freedom are considered in [48], which makes the element less stiff. More importantly, [48] considered in-plane action in the formulation of the sandwich beam energy relation.

T. Sakiyama, H. Matsuda and C. Morita [52] presented a completely analytical solution based on the discrete green function. The Green function is an assumed displacement function, which depends on the mode shape and length of the beam; it is obtained as a discrete solution of differential equation governing the flexural behavior of the sandwich beam under the action of a concentrated load. Without the introduction of a characteristic equation that is derived from the discrete Green function, the determination of the natural frequency would have been by trial and error. There are lots of complex equations in this method, which increases the possibility of error accumulation.

In the paper by V. S. Sokolinsky, Steven R. Nutt and Y. Frostig [28], the use of assumed displacement techniques was avoided, instead, employed finite differences to approximate the governing equation for different support condition. According to the author, the highest order of derivatives entering the mathematical formulation was reduced to two by the introduction of new functions. However, the exclusion of certain terms in the approximation of governing equations could affect the results produced by this means. They consider shear in the core by using the higher order theory, and that the core is transversely flexible.

Note also that the equations describing free vibrations of sandwich beams with a soft core used in [28] were derived by the application of Hamilton's principle. Among several assumptions made in deriving these equations were the following;

1. Acceleration fields of the face sheets have the same shape as their static deformation fields.
2. The effects of rotatory inertia of the face sheets is negligible

It was mentioned by several authors, M. E. Raville, En Shinh Ueng and Ming-Min Lei [44] in particular that ideal fixed ends can never be achieved experimentally. The lower frequency results by [44] may be due to the lack of complete fixity.

The references mentioned above considered the contribution due to flexural and shearing motion in the course of the formulation of the overall mass matrix and frequency relation although the primary concern is flexure except that the paper by Ahmed [47] and Mead [53], which considered only the flexural characteristics of the sandwich beam. In the present work, equal attention is given to all the motions of the sandwich beam during vibration. The dynamic analysis presented is quite general in the sense that the mass matrix due to flexural mode, uniform stretching and shearing (rotation) can be readily obtained as well as its applicability to the general frame element and to a much more general boundary condition.

#### 4.2.1.3 AXIAL ACTION EFFECTS

While the author of this definition realized that axial action had some effects on the behavior of the sandwich beam, it is necessary to illustrate such effects by means of example problems. Material properties and dimensions for the sandwich beam in the example problem for the study on consistency of theory are used here, only that the shear modulus is been replaced by $G = 0.0012E$. Numerical solutions for a sandwich beam with different support conditions are analyzed for transverse and radial effects and these results are compared with those that include axial effects as given in Table 4.8.

#### 4.2.1.3.1 DISCUSSION OF RESULTS ON AXIAL ACTION

Table 4.8 gives values of natural frequency of sandwich beam with various boundary conditions due to different vibration effects. The few papers considered in the previous section provided natural frequency results coming from translational and rotational effects. Somehow, as in many cases, some amount of the axial vibration effects can be incorporated in the analysis. Some of the references are convinced that axial vibration effect was completely included in their analysis. This is explained by the idea that the slope and curvature incorporates stretching of the sandwich beam's faces.

Most part of the natural frequency is due to the translational mass effect and the contribution from the rotational inertia is almost negligible. This work includes the contribution due to axial (longitudinal) inertia effects, which has made significant difference to the total frequency. The introduction of the axial vibration effects brings the frequency values for subsequent modes more closely. The axial vibration effect becomes more pronounced as the vibration mode is increased from the first going upwards; there is considerable decrease in the numerical values for the eigen-values of higher vibration modes. The effect on the natural frequency of the sandwich beam due to the rotation mass is almost negligible for the various support conditions considered for numerical analysis. Thus, most approximate theories would give zero effect to the mass matrix due to rotation on the natural frequency of a sandwich beam. A complete vibration analysis must take into consideration the natural frequency coming from the axial action effect.

As mentioned in Section 4.2.1.1.1, because of its high amplitude the fundamental mode is expected to possess most of the energy of the sandwich beam, thus it is barely affected by changes made to the vibration effects. The fundamental mode will be considered for behavioral study of the sandwich beam in the upcoming sections.

Table 4.8 Comparing natural frequencies $f(Hz)$ due to various vibration effects, for different support conditions

| 30 Elem. | Mode n | Translation and Rotation Effect | Translation Rotation and Axial Effect | Translation and Axial Effect | Translation Effect |
|---|---|---|---|---|---|
| Sandwich Beam with simply supported edges | 1st | 8.322 | 8.322 | 8.322 | 8.322 |
| | 2nd | 33.187 | 33.187 | 33.187 | 33.187 |
| | 3rd | 74.310 | 74.310 | 74.310 | 74.310 |
| | 4th | 131.223 | 131.204 | 131.204 | 131.223 |
| | 5th | 203.320 | 203.257 | 203.257 | 203.320 |
| | 6th | 289.862 | 289.600 | 289.600 | 289.862 |
| | 7th | 390.010 | 389.392 | 389.392 | 390.01 |
| | 8th | 502.939 | 501.476 | 501.476 | 502.939 |
| | 9th | 627.804 | 624.568 | 624.568 | 627.804 |
| | 10th | 763.444 | 757.449 | 757.449 | 763.444 |
| Cantilever Sandwich Beam | 1st | 2.965 | 2.965 | 2.965 | 2.965 |
| | 2nd | 18.506 | 18.506 | 18.506 | 18.506 |
| | 3rd | 51.547 | 51.523 | 51.523 | 51.547 |
| | 4th | 100.255 | 100.242 | 100.242 | 100.255 |
| | 5th | 164.246 | 164.246 | 164.246 | 164.246 |
| | 6th | 242.835 | 242.678 | 242.678 | 242.835 |
| | 7th | 335.171 | 334.755 | 334.755 | 335.171 |
| | 8th | 440.487 | 439.509 | 439.509 | 440.487 |
| | 9th | 557.950 | 503.794 | 503.794 | 557.950 |
| | 10th | 686.582 | 555.903 | 555.903 | 686.582 |
| Fixed-fixed Sandwich Beam | 1st | 18.724 | 18.724 | 18.724 | 18.724 |
| | 2nd | 51.202 | 51.202 | 51.202 | 51.202 |
| | 3rd | 99.405 | 99.392 | 99.392 | 99.405 |
| | 4th | 162.462 | 162.462 | 162.462 | 162.462 |
| | 5th | 239.632 | 239.526 | 239.526 | 239.632 |
| | 6th | 330.107 | 329.684 | 329.684 | 330.107 |
| | 7th | 432.947 | 431.981 | 431.981 | 432.947 |
| | 8th | 547.408 | 545.322 | 545.322 | 547.408 |
| | 9th | 672.416 | 668.260 | 668.260 | 672.416 |
| | 10th | 807.464 | 799.900 | 799.900 | 807.464 |

Table 4.9 Frequency Parameter $\lambda$, for a single span sandwich beam with varying core stiffness for different support conditions

| | | $G/E$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 40 elem. | Mode n | $1/10^7$ | $1/10^6$ | $1/10^5$ | $1/10^4$ | $1/10^3$ | $1/10^2$ | 1/10 | 1 | Exact |
| Sandwich Beam with simply supported edges | 1st | 1.206 | 2.045 | 2.859 | 3.107 | 3.139 | 3.142 | 3.142 | 3.142 | 3.1416 |
| | 2nd | 1.736 | 3.001 | 4.845 | 6.030 | 6.261 | 6.287 | 6.289 | 6.290 | 6.2832 |
| | 3rd | 2.171 | 3.709 | 6.265 | 8.660 | 9.351 | 9.437 | 9.445 | 9.446 | 9.4248 |
| | 4th | 2.574 | 4.305 | 7.393 | 10.98 | 12.39 | 12.59 | 12.61 | 12.62 | 12.566 |
| | 5th | 2.965 | 4.835 | 8.351 | 13.01 | 15.38 | 15.76 | 15.80 | 15.81 | 15.708 |
| | 6th | 3.354 | 5.320 | 9.197 | 14.81 | 18.29 | 18.94 | 19.01 | 19.02 | 18.850 |
| | 7th | 3.745 | 5.775 | 9.960 | 16.41 | 21.13 | 22.13 | 22.24 | 22.25 | 21.991 |
| | 8th | 4.139 | 6.205 | 10.66 | 17.85 | 23.88 | 25.33 | 25.49 | 25.51 | 25.133 |
| | 9th | 4.536 | 6.619 | 11.31 | 19.17 | 26.53 | 28.54 | 28.77 | 28.80 | 28.274 |
| | 10th | 4.937 | 7.019 | 11.92 | 20.37 | 29.09 | 31.77 | 32.08 | 32.11 | 31.416 |
| Cantilever Sandwich Beam | 1st | 0.864 | 1.400 | 1.785 | 1.865 | 1.874 | 1.875 | 1.875 | 1.875 | 1.8751 |
| | 2nd | 1.513 | 2.489 | 3.747 | 4.534 | 4.678 | 4.694 | 4.696 | 4.695 | 4.6941 |
| | 3rd | 1.996 | 3.369 | 5.470 | 7.286 | 7.794 | 7.858 | 7.863 | 7.865 | 7.8548 |
| | 4th | 2.414 | 4.020 | 6.755 | 9.713 | 10.85 | 11.01 | 11.02 | 11.02 | 10.996 |
| | 5th | 2.809 | 4.592 | 7.824 | 11.86 | 13.84 | 14.16 | 14.20 | 14.20 | 14.137 |
| | 6th | 3.196 | 5.102 | 8.740 | 13.77 | 16.78 | 17.33 | 17.39 | 17.40 | 17.279 |
| | 7th | 3.581 | 5.576 | 9.558 | 15.47 | 19.63 | 20.51 | 20.60 | 20.61 | 20.420 |
| | 8th | 3.968 | 6.019 | 10.30 | 17.00 | 22.40 | 23.69 | 23.84 | 23.85 | 23.562 |
| | 9th | 4.359 | 6.443 | 10.98 | 18.40 | 24.43 | 24.43 | 24.43 | 24.43 | 26.704 |
| | 10th | 4.752 | 6.850 | 11.62 | 19.66 | 25.08 | 26.89 | 27.10 | 27.12 | 29.845 |
| Fixed-fixed Sandwich Beam | 1st | 1.267 | 2.155 | 3.510 | 4.493 | 4.705 | 4.729 | 4.731 | 4.731 | 4.7300 |
| | 2nd | 1.816 | 3.048 | 5.049 | 7.095 | 7.766 | 7.852 | 7.861 | 7.863 | 7.8532 |
| | 3rd | 2.271 | 3.764 | 6.361 | 9.430 | 10.80 | 11.00 | 11.02 | 11.02 | 10.996 |
| | 4th | 2.691 | 4.366 | 7.453 | 11.51 | 13.76 | 14.15 | 14.19 | 14.20 | 14.137 |
| | 5th | 3.097 | 4.903 | 8.401 | 13.37 | 16.66 | 17.31 | 17.38 | 17.39 | 17.279 |
| | 6th | 3.498 | 5.394 | 9.243 | 15.05 | 19.47 | 20.48 | 20.59 | 20.61 | 20.420 |
| | 7th | 3.900 | 5.855 | 10.01 | 16.58 | 22.20 | 23.66 | 23.83 | 23.84 | 23.562 |
| | 8th | 4.302 | 6.291 | 10.71 | 17.98 | 24.84 | 26.85 | 27.08 | 27.11 | 26.704 |
| | 9th | 4.707 | 6.709 | 11.36 | 19.26 | 27.39 | 30.04 | 30.37 | 30.40 | 29.845 |
| | 10th | 5.112 | 7.114 | 11.97 | 20.44 | 29.84 | 33.25 | 33.67 | 33.71 | 32.987 |

#### 4.2.1.4 EFFECTS OF CORE SHEAR MODULUS

The shear modulus of the core material has significant impact on the response of sandwich beams. A very stiff core would make the sandwich beam behave in a manner

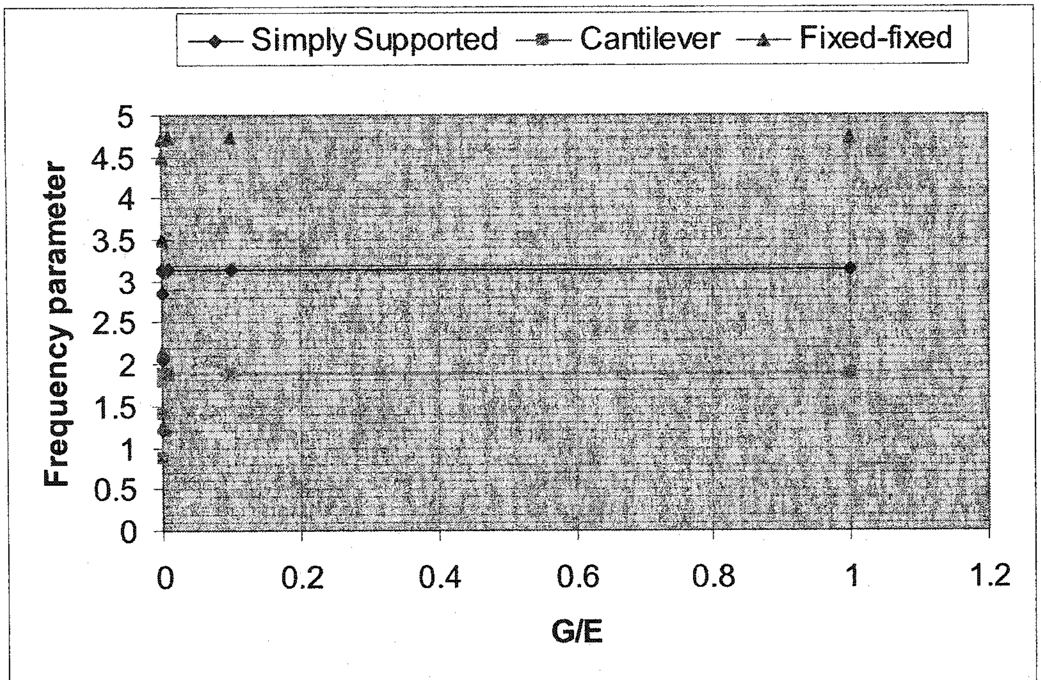



Figure 4.8 Fundamental frequency parameter $\lambda$, plotted against the ratio $G/E$ for different support conditions

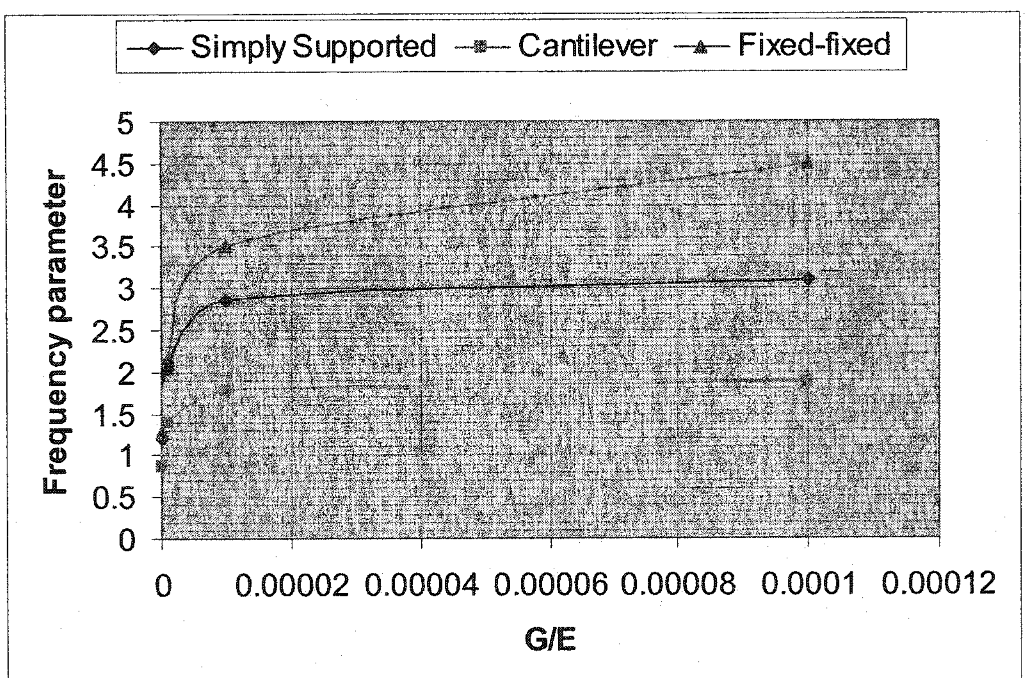



Figure 4.9 Fundamental frequency parameter $\lambda$, for the early values of $G/E$ (magnified) for different support conditions

Table 4.10 Natural frequency $\omega$, of sandwich beam with varying core thickness for different support conditions $f(Hz)$

| 40 elem. | Mode n | c/t = 1 | 2 | 5 | 10 | 20 | 30 | 40 | 50 |
|---|---|---|---|---|---|---|---|---|---|
| Sandwich Beam with simply supported edges | 1st | 4.503 | 6.606 | 13.01 | 23.72 | 45.01 | 66.10 | 86.97 | 107.7 |
| | 2nd | 18.01 | 26.38 | 51.74 | 93.60 | 175.1 | 253.6 | 329.2 | 402.1 |
| | 3rd | 40.54 | 59.23 | 115.3 | 206.1 | 377.1 | 535.2 | 680.8 | 814.2 |
| | 4th | 72.10 | 104.9 | 202.2 | 356.0 | 633.8 | 876.4 | 1085 | 1258 |
| | 5th | 112.7 | 163.3 | 310.7 | 537.1 | 927.1 | 1243 | 1485 | 1656 |
| | 6th | 162.3 | 234.0 | 438.8 | 743.6 | 1239 | 1602 | 1839 | 1967 |
| | 7th | 220.9 | 316.6 | 584.3 | 969.0 | 1556 | 1933 | 2129 | 2198 |
| | 8th | 288.6 | 410.7 | 745.0 | 1208 | 1865 | 2222 | 2359 | 2369 |
| | 9th | 365.3 | 516.0 | 918.8 | 1457 | 2157 | 2470 | 2540 | 2496 |
| | 10th | 450.8 | 631.6 | 1104 | 1710 | 2428 | 2677 | 2683 | 2594 |
| Cantilever Sandwich Beam | 1st | 1.603 | 2.353 | 4.640 | 8.468 | 16.11 | 23.72 | 31.30 | 38.84 |
| | 2nd | 10.05 | 14.72 | 28.89 | 52.38 | 98.28 | 142.8 | 185.9 | 227.9 |
| | 3rd | 28.12 | 41.10 | 80.13 | 143.7 | 264.3 | 377.0 | 482.5 | 581.1 |
| | 4th | 55.09 | 80.2 | 155.0 | 273.8 | 490.8 | 683.8 | 855.3 | 1005 |
| | 5th | 91.02 | 132.1 | 252.0 | 437.6 | 761.6 | 1033 | 1255 | 1377 |
| | 6th | 136.0 | 196.2 | 369.3 | 629.0 | 1060 | 1377 | 1377 | 1428 |
| | 7th | 189.9 | 272.4 | 504.8 | 842.0 | 1372 | 1393 | 1634 | 1790 |
| | 8th | 252.7 | 360.2 | 656.4 | 1072 | 1377 | 1739 | 1962 | 2069 |
| | 9th | 324.5 | 459.2 | 821.9 | 1313 | 1683 | 2053 | 2229 | 2275 |
| | 10th | 405.2 | 568.7 | 999.6 | 1377 | 1985 | 2327 | 2439 | 2428 |
| Fixed-fixed Sandwich Beam | 1st | 10.19 | 14.92 | 29.21 | 52.71 | 98.16 | 141.6 | 183.1 | 223.0 |
| | 2nd | 28.06 | 40.96 | 79.48 | 141.5 | 256.9 | 362.3 | 459.1 | 548.1 |
| | 3rd | 54.95 | 79.90 | 153.4 | 268.5 | 474.1 | 652.7 | 808.6 | 943.5 |
| | 4th | 90.77 | 131.3 | 248.8 | 427.4 | 732.5 | 982.3 | 1185 | 1344 |
| | 5th | 135.5 | 194.9 | 363.7 | 612.5 | 1017 | 1326 | 1551 | 1701 |
| | 6th | 189.0 | 270.3 | 496.2 | 818.5 | 1316 | 1662 | 1877 | 1990 |
| | 7th | 251.5 | 357.0 | 644.1 | 1040 | 1618 | 1972 | 2151 | 2210 |
| | 8th | 322.7 | 454.6 | 805.7 | 1274 | 1912 | 2248 | 2372 | 2376 |
| | 9th | 402.6 | 562.7 | 978.9 | 1516 | 2193 | 2486 | 2548 | 2501 |
| | 10th | 491.2 | 680.7 | 1162 | 1762 | 2455 | 2689 | 2689 | 2599 |

similar to that of a homogenous isotropic beam. This was indicated in the early sections of this chapter. One may be curious to ascertain it's impact on sandwich beam when the shear modulus in that material is varied. This section presents the behavior of the

Table 4.11 Natural frequency $\omega$, of sandwich beam with varying face sheet thickness for different support conditions $f(Hz)$

| | | $t/c$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 40 elem. | Mode n | 1 | 2 | 5 | 10 | 20 | 30 | 40 | 50 |
| Sandwich Beam with simply supported edges | 1st | 95.97 | 130.1 | 217.2 | 244.5 | 172.9 | 141.2 | 122.2 | 109.3 |
| | 2nd | 264.3 | 348.9 | 345.8 | 299.4 | 341.8 | 348.5 | 349.6 | 349.4 |
| | 3rd | 446.3 | 546.7 | 531.9 | 630.0 | 668.1 | 676.7 | 679.9 | 681.6 |
| | 4th | 637.4 | 597.0 | 840.5 | 953.1 | 997.7 | 1010 | 1016 | 1019 |
| | 5th | 773.3 | 858.0 | 846.8 | 996.6 | 1330 | 1346 | 1354 | 1359 |
| | 6th | 835.2 | 891.1 | 1158 | 1276 | 1528 | 1686 | 1696 | 1701 |
| | 7th | 1037.4 | 1123 | 1467 | 1601 | 1665 | 2029 | 2041 | 2048 |
| | 8th | 1039 | 1389 | 1777 | 1929 | 2004 | 2085 | 2389 | 2398 |
| | 9th | 1243 | 1636 | 2063 | 2260 | 2347 | 2375 | 2591 | 2753 |
| | 10th | 1451 | 1656 | 2089 | 2595 | 2693 | 2726 | 2743 | 2755 |
| Cantilever Sandwich Beam | 1st | 39.07 | 56.29 | 103.1 | 161.4 | 201.6 | 202.7 | 199.0 | 195.4 |
| | 2nd | 169.5 | 233.9 | 366.4 | 387.6 | 440.8 | 481.4 | 497.3 | 504.3 |
| | 3rd | 359.0 | 474.7 | 528.3 | 584.3 | 684.4 | 815.0 | 841.1 | 846.4 |
| | 4th | 546.7 | 646.5 | 658.7 | 761.6 | 825.8 | 865.9 | 1024 | 1163 |
| | 5th | 736.8 | 724.6 | 981.5 | 1104 | 1163 | 1180 | 1192 | 1235 |
| | 6th | 847.4 | 960.0 | 1246 | 1377 | 1377 | 1377 | 1377 | 1377 |
| | 7th | 930.1 | 1226 | 1377 | 1413 | 1491 | 1513 | 1524 | 1530 |
| | 8th | 1120 | 1289 | 1492 | 1748 | 1831 | 1855 | 1868 | 1874 |
| | 9th | 1316 | 1377 | 1605 | 1943 | 2169 | 2200 | 2213 | 2222 |
| | 10th | 1355 | 1515 | 1929 | 2123 | 2515 | 2548 | 2565 | 2574 |
| Fixed-fixed Sandwich Beam | 1st | 149.6 | 209.0 | 381.2 | 545.3 | 640.2 | 663.5 | 672.4 | 675.4 |
| | 2nd | 325.0 | 463.3 | 695.9 | 700.5 | 686.0 | 680.3 | 677.3 | 676.7 |
| | 3rd | 525.5 | 740.4 | 843.2 | 959.0 | 1020 | 1036 | 1042 | 1046 |
| | 4th | 732.8 | 882.7 | 1067 | 1230 | 1316 | 1340 | 1350 | 1356 |
| | 5th | 943.7 | 1028 | 1417 | 1586 | 1665 | 1688. | 1699 | 1705 |
| | 6th | 1038 | 1289 | 1608 | 1867 | 1992 | 2024 | 2038 | 2046 |
| | 7th | 1158 | 1576 | 1835 | 2190 | 2343 | 2375 | 2390 | 2398 |
| | 8th | 1369 | 1578 | 2045 | 2246 | 2680 | 2722 | 2741 | 2751 |
| | 9th | 1587 | 1872 | 2390 | 2608 | 2755 | 2755 | 2755 | 2755 |
| | 10th | 1622 | 2113 | 2672 | 2755 | 3042 | 3081 | 3101 | 3112 |

sandwich beam, represented in the natural frequency parameter $\lambda$ for easy comparison with the case of the homogeneous beam shown in Table 4.9. Material properties and dimensions for the sandwich beam in the example problem for the axial action effect is used here, only that the shear modulus of the core is varied. Eq. 4.1 is used to calculate the frequency parameter, $\lambda$.

#### 4.2.1.4.1 DISCUSSION OF RESULTS ON EFFECTS OF CORE SHEAR MODULUS

The values of the 10 lowest natural frequency parameter $\lambda$, of simply supported, cantilever and fixed sandwich beams produced by the program for sandwich beam when the shear in the core is varied is as shown in Table 4.9. The effect of increasing the stiffness of the core material on the fundamental frequency of sandwich beams under various boundary conditions is shown on Figure 4.8. It is seen that as the ratio $G/E$ increases, the natural frequency of a sandwich beam approaches that of the homogenous isotropic beam. As expected for a consistent mass matrix, the frequency parameters are upper bound to the exact values of homogeneous isotropic beam. It can also be seen from Table 4.9 that the effect of core rigidity on the higher frequencies is very pronounced, the frequencies being increased appreciably with increasing shear rigidity. Therefore theories that neglects shear deformation might give acceptable answers only for the first few modes, provided that the core used is stiff.

For clarity, the curves representing the variation of fundamental frequency parameter $\lambda$ with the ratio $G/E$ are shown on Figure 4.9. This region completely describes the behavior of the sandwich beam with soft core.

#### 4.2.1.5 EFFECTS OF DIMENSIONAL FACTORS

Free vibrations of sandwich beams for different supporting cases are studied assuming different configurations and sets of material properties. More importantly, this section outlines the observations for the natural frequency when certain important parameters are varied.

Numerical solutions for the natural frequency of a symmetric rectangular sandwich beam with various boundary conditions are presented for the following possibilities

i) Table 4.10 shows frequencies when the core thickness is varied, all the remaining parameters fixed

ii) Table 4.11 shows frequencies when the thickness of the face sheets is varied, all the remaining parameters remain fixed

iii) Table 4.12 shows frequencies when the length of the beam is varied, all the remaining parameters remain fixed.

Following are the dimensions and material properties of the sandwich beam selected arbitrarily; Length of beam, $l = 0.9144m$, thickness of core $c = 0.0127m$, thickness of face $t1 = t2 = 0.0004572m$, Elastic modulus of faces $E_1 = E_2 = E = 6.8 \times 10\ GPa$, Core shear modulus $G = 0.0012E$ and the density of the facial material is $\rho = 2.680 \times 10^3\ {kg}/{m^3}$

Table 4.12 Natural frequency $\omega$, of sandwich beam with varying span length for different support conditions $f(Hz)$

| 40 elem. | Mode n | Aspect ratio $L/(c+t1+t2)$: 1 | 2 | 5 | 10 | 20 | 30 | 40 | 50 |
|---|---|---|---|---|---|---|---|---|---|
| Sandwich Beam with simply supported edges | 1st | 4075 | 3896 | 3099 | 1750 | 609.4 | 292.0 | 169.0 | 109.6 |
| | 2nd | 4125 | 4075 | 3786 | 3102 | 1752 | 986.1 | 610.2 | 409.6 |
| | 3rd | 4467 | 4128 | 3975 | 3581 | 2611 | 1755 | 1185 | 833.2 |
| | 4th | 4938 | 4290 | 4074 | 3797 | 3111 | 2378 | 1759 | 1306 |
| | 5th | 5529 | 4480 | 4075 | 3919 | 3409 | 2818 | 2247 | 1764 |
| | 6th | 6211 | 4708 | 4150 | 4000 | 3601 | 3124 | 2628 | 2165 |
| | 7th | 6959 | 4972 | 4223 | 4062 | 3733 | 3342 | 2918 | 2498 |
| | 8th | 7761 | 5269 | 4299 | 4075 | 3830 | 3502 | 3140 | 2767 |
| | 9th | 8605 | 5598 | 4382 | 4116 | 3905 | 3626 | 3313 | 2983 |
| | 10th | 9483 | 5951 | 4473 | 4166 | 3966 | 3723 | 3450 | 3159 |
| Cantilever Sandwich Beam | 1st | 3976 | 3471 | 1949 | 781.6 | 233.0 | 107.6 | 61.39 | 39.55 |
| | 2nd | 4240 | 4049 | 3583 | 2436 | 1078 | 573.4 | 348.5 | 232.1 |
| | 3rd | 4673 | 4191 | 3918 | 3393 | 2154 | 1317 | 856.8 | 593.2 |
| | 4th | 5208 | 4366 | 4035 | 3715 | 2857 | 2027 | 1428 | 1035 |
| | 5th | 5854 | 4579 | 4108 | 3875 | 3269 | 2580 | 1967 | 1499 |
| | 6th | 6572 | 4827 | 4178 | 3970 | 3514 | 2965 | 2312 | 1850 |
| | 7th | 7350 | 5110 | 4252 | 4035 | 3676 | 3083 | 2414 | 1932 |
| | 8th | 8174 | 5423 | 4333 | 4088 | 3790 | 3233 | 2760 | 2307 |
| | 9th | 9037 | 5765 | 4420 | 4137 | 3874 | 3425 | 3022 | 2616 |
| | 10th | 9930 | 6131 | 4516 | 4185 | 3941 | 3568 | 3224 | 2865 |
| Fixed-fixed Sandwich Beam | 1st | 4437 | 4015 | 3205 | 2061 | 967.2 | 538.1 | 335.7 | 227.1 |
| | 2nd | 4859 | 4247 | 3837 | 3147 | 1897 | 1187 | 792.7 | 559.8 |
| | 3rd | 5520 | 4458 | 4019 | 3616 | 2659 | 1860 | 1323 | 971.6 |
| | 4th | 6150 | 4681 | 4118 | 3827 | 3137 | 2424 | 1839 | 1408 |
| | 5th | 6954 | 4960 | 4200 | 3947 | 3431 | 2843 | 2289 | 1829 |
| | 6th | 7710 | 5247 | 4282 | 4027 | 3621 | 3142 | 2652 | 2204 |
| | 7th | 8600 | 5589 | 4367 | 4090 | 3752 | 3357 | 2935 | 2522 |
| | 8th | 9440 | 5932 | 4460 | 4146 | 3849 | 3517 | 3154 | 2783 |
| | 9th | 10389 | 6322 | 4561 | 4198 | 3924 | 3640 | 3325 | 2996 |
| | 10th | 11292 | 6711 | 4669 | 4250 | 3985 | 3737 | 3461 | 3170 |

### 4.2.1.5.1 DISCUSSION OF RESULTS ON DIMENSIONAL FACTORS

The effect of changing the core thickness with constant face thickness on the frequency is shown in Table 4.10 and Figure 4.10 shows the graph of the fundamental frequency plotted against the core to face sheet thickness ratio, $c/t$, $(t = t1 = t2)$, for the different should be focused on the behavior of the curves; a growth in frequency values is shown by all the curves as the ratio $c/t$ is increased. This is because an increase in this ratio has the effect of increasing the stiffness of the sandwich beam.

The wide range of values considered did not show any sign of abnormality as no extremum points were detected. The curves for the various boundary conditions considered show some form of linear dependence between the frequency and core to face sheet thickness ratio. The thicker the core, the higher the frequency and the sandwich beam behavior is improved when the core thickness is increased, provided that local instability does not occur.

The effect of increasing the face thickness on the vibration response of sandwich beam appears on Table 4.11 and Figure 4.11 shows the graph of fundamental frequency plotted against the face sheet to core thickness ratio $t/c$ for the different boundary conditions. An increase in the thickness of the face sheet leads to an increase in the stiffness of the section and at the same time increases the mass density. The combined effect is such that the natural frequency increases with thickness.

The cantilever type sandwich beam frequency increases rapidly at the start of the curve and continues to increase albeit slowly. The behaviour of the fixed-fixed type sandwich beam is similar to that of the cantilever only that it is of higher frequencies. In the case of the simply supported sandwich beam, the curve continues to rise until a local

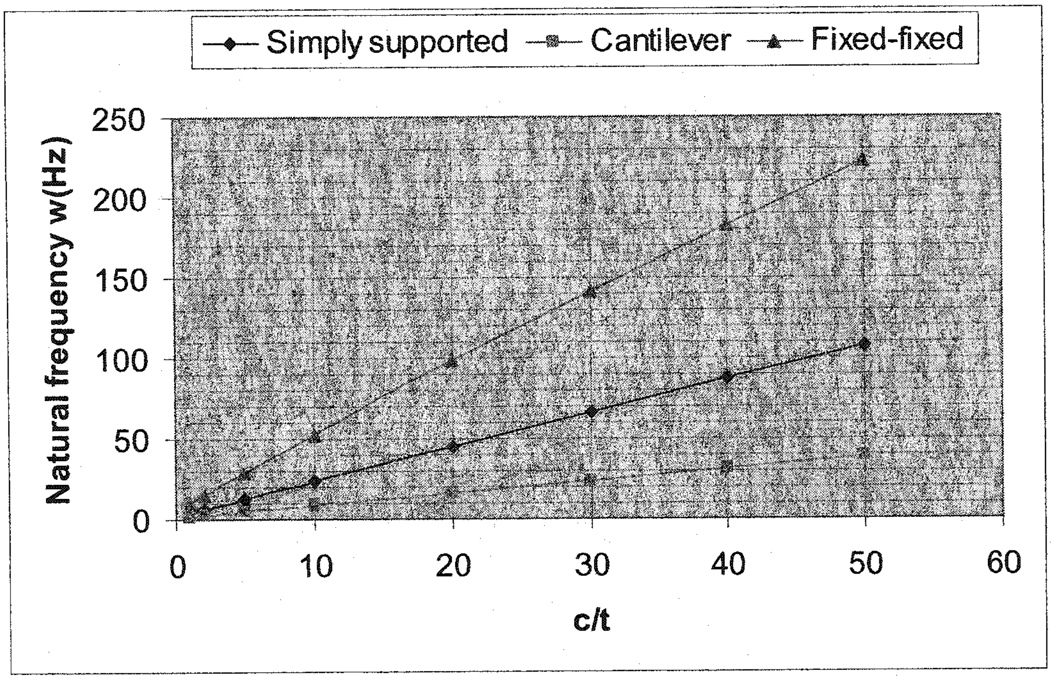



Figure 4.10 Fundamental frequency $(Hz)$ plotted against the ratio $c/t$ (core thickness to face thickness) for different support conditions

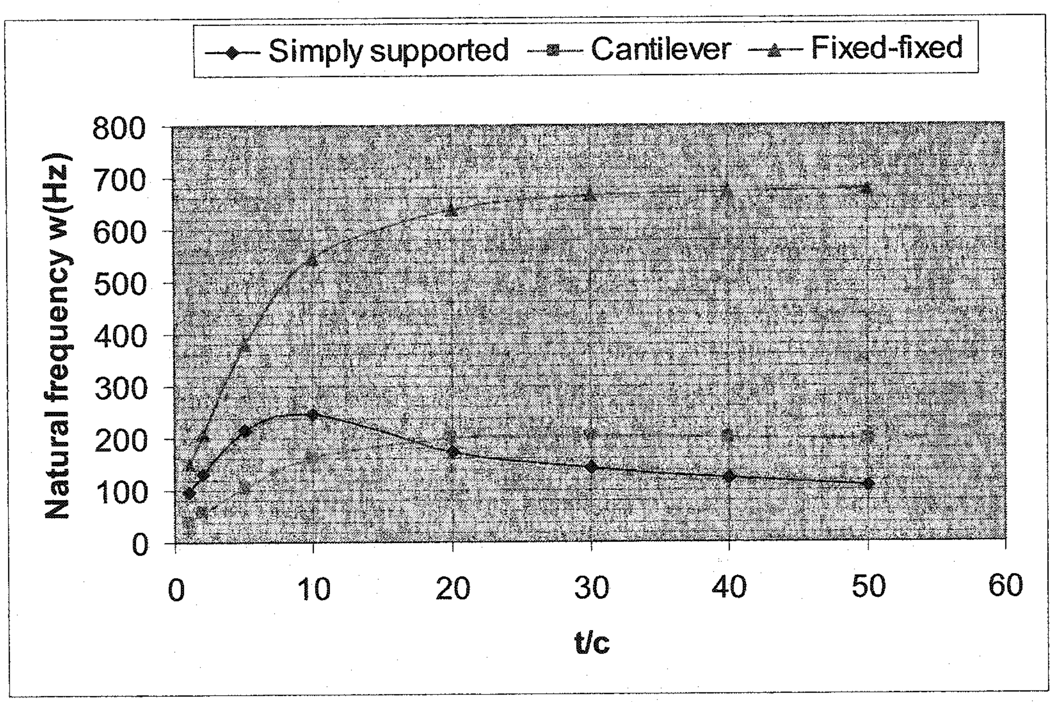



Figure 4.11 Fundamental frequency $(Hz)$ plotted against the ratio $t/c$ (face thickness to core thickness) for different support conditions

maximum is reached and then begins to fall. This disruption in the behavior of the sandwich beam is due to the fact that the simply supported sandwich beam is not restrained along the longitudinal direction. As the face sheet thickness increases the axial vibration effects are promoted thus the system becomes unstable. At this stage, there is the tendency for the face sheet materials to vibrate independently (i.e. in a local manner) as shown in Figure 3.2(a). The antisymmetric vibration modes give way to the symmetric vibration modes as the thickness ratio $t/c$ is increased to a high value.

Figure 4.12 shows the frequency curves for the fundamental frequency for the different support conditions given that the axial vibration effect is neglected. Although some changes is observed in the dependence of the frequency and the face sheet to core

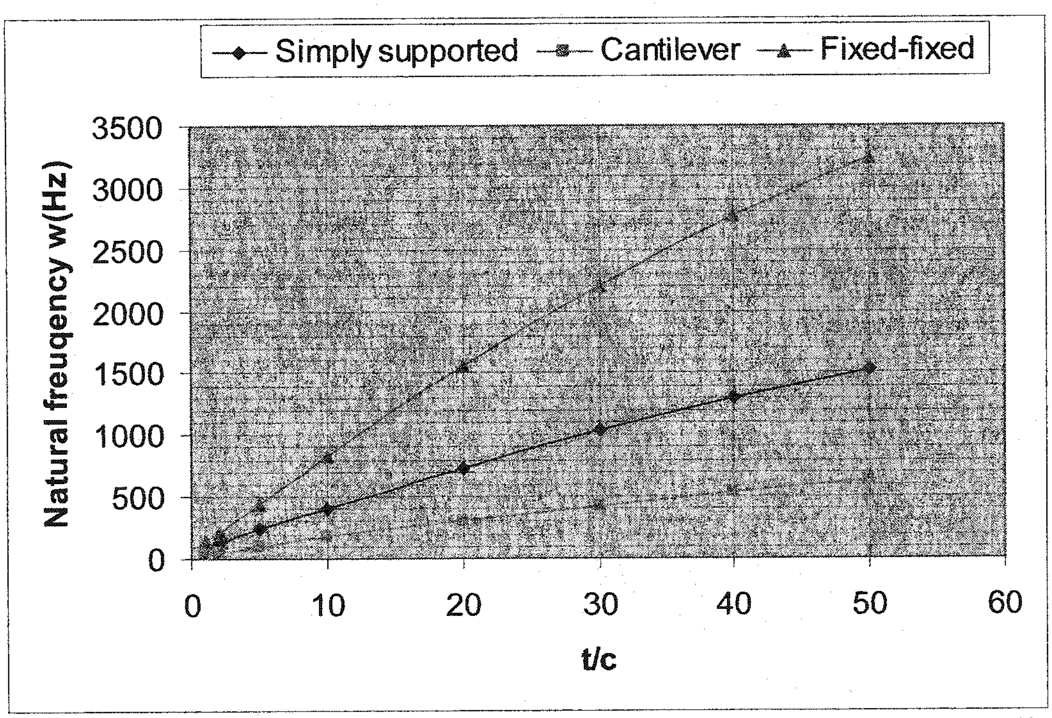



Figure 4.12 Fundamental frequency ($Hz$) plotted against the ratio $t/c$ (face thickness to core thickness) for different support conditions neglecting axial vibration effects

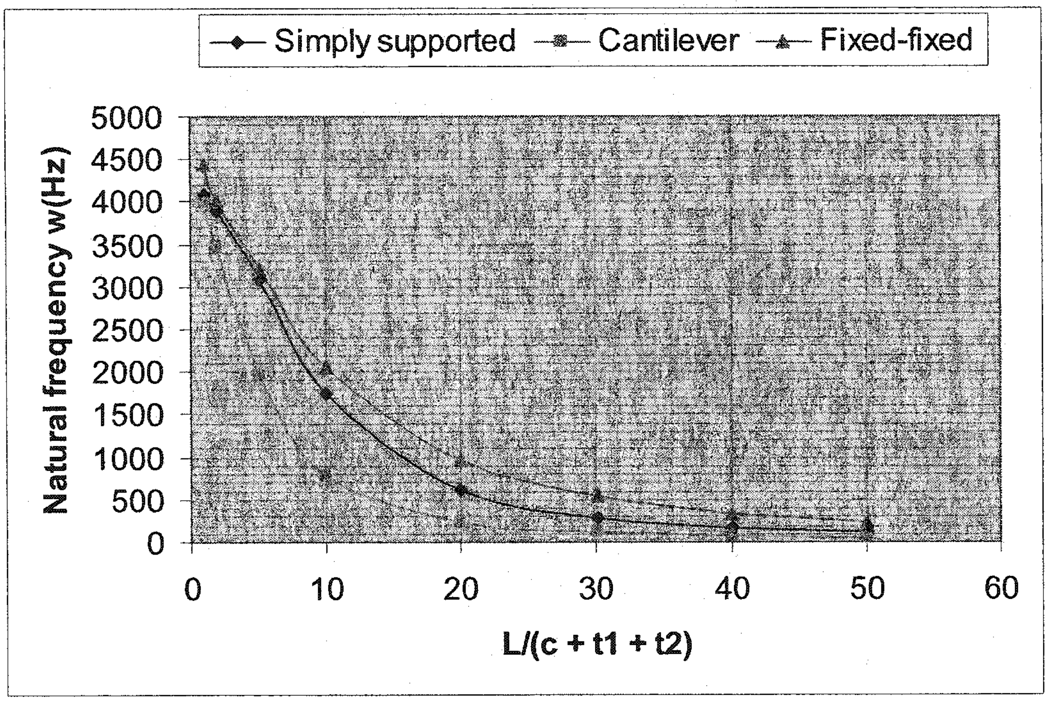



Figure 4.13 Fundamental frequency $(Hz)$ plotted against the ratio $L/(c+t1+t2)$ (Aspect ratio) for different support conditions

thickness ratio for the cantilever and fixed-fixed sandwich beam, dramatic change is seen for the curve in question which now increases with the ratio face sheet to core thickness ratio $(t/c)$ in a continuous manner. The study shows that increasing the thickness of the faces does not necessarily lead to an increase in the natural frequencies of end supported sandwich beams.

This characteristic behavior of the sandwich beam re-emphasizes the importance of uniform axial stretching effects in the free vibration of sandwich beams. A complete study on the behavior of sandwich beam must take into consideration the contribution of the uniform axial stretching in the formulations.

The effect of increasing the length of the sandwich beam on the vibration effect is shown on Table 4.12 and Figure 4.13 shows the graph of the fundamental frequencies plotted against the aspect ratio, $L/(c+t1+t2)$. Increasing the length will dramatically decrease the bending stiffness of the beam, thus leading to a decline in frequency value. The frequency curves for all support conditions drops rapidly within a short range and then continues to fall albeit slowly. In general, increasing the aspect ratio in turn reduces the bending stiffness of the sandwich beam, thus the frequency is expected to fall.

# CHAPTER 5

# SUMMARY AND CONCLUSION

The increasing use of sandwich construction and the need to predict accurately their static and dynamic behavior requires the use of a more general theory. Since the displacement function was derived from the governing differential equation for the general class of sandwich beam and frame systems, it can easily be applied to the static and dynamic analysis of such structures. While an exact stiffness matrix for sandwich beam element is available, its application to vibration analysis is enhanced by the use of a consistent mass matrix. The derivation of a special shape functions required for obtaining the mass matrix was presented.

The natural frequencies of free vibration of sandwich beam and frame systems with different boundary conditions have been investigated by the displacement based finite element method in which an element having six and eight-degrees of freedom i.e. three and four degrees of freedom per node is used. The range of application for this tool varies from specially designed materials for aerospace and marine applications to thick sandwich structures used in civil engineering structures. Computer programs capable of handling any type of sandwich beam and frame structures were developed.

Application of the six and eight-degrees of freedom element to the general class of sandwich beam and frame systems leads to the following conclusions;

1. The deflection functions exhibit good convergence characteristics, and enable the medium frequency regime to be explored at minimum computational expense.

2. The solution for free vibration approaches that of a homogeneous isotropic beam when the shear modulus of the core material is increased. The theory can be referred to as consistent.

3. The results produced by this theory are in good agreement with results from the literature. The higher order frequencies are predicted with the same accuracy as the lower ones, a task that constitutes a significant challenge for most of the theories in the literature.

4. The model considered so far is the simplest for a sandwich beam since continuity is maintained in deflection $v$, slope $v'$ and rotatation $\phi$. It takes into account shear deformations without the introduction of any degree of freedom other than the nodal deflection, edge rotations and slope. Generally, it is indicated by some of the references, [47] and [48] to mention a few, that the higher the degree of freedom per node the softer the element. K.M. Ahmed [49] who modeled the element with five and seven degrees of freedom per node confirmed that the element incorporating the seven degrees of freedom per node can be relied upon to give reasonably accurate results for both static and dynamic analysis of

sandwich structures. The seven degrees of freedom model requires fewer elements for convergence.

5. Boundary support condition is another factor that influences the natural frequencies of a sandwich beam element. It was also agreed upon by many authors M. E. Raville, En Shinh Ueng and Ming-Min Lei [44] in particular that ideal fixed ends can never be exactly achieved experimentally and thus the lack of complete fixity results in lower frequencies. Theories that are in excellent agreement with experimental results assumes similar support conditions. The fact that the theoretical support condition considered in this theory cannot be easily achieved experimentally is a clear manifestation that there is the possibility that the frequencies predicted by this theory could be higher.

6. The inclusion of axial action in the dynamic analysis of sandwich beam and frame systems is advancement in the theory. It is evident from the results that the frequency contribution due to axial action is significant. A systematic decline in values of the ratio of the frequency that includes axial effect to that excluding axial effect indicates that the axial effect is more pronounced at higher modes because the amplitude of the mode shapes decreases with increase in modal values.

7. The frequency of vibration of the sandwich beam depends on its dimensional factors. The frequency of a sandwich beam increases with increase in thickness of the core and decreases when the length is increased. The study also showed that increasing the thickness of the face does not necessarily lead to an increase in the

natural frequencies of end supported sandwich beam. Thus the frequency decreases with increase in the aspect ratio.

8. The vibration modes of sandwich beams with thin and thick face sheets and constrained longitudinal displacements corresponding to low and higher frequencies, consists predominantly of antisymmetric vertical displacement of the face sheets. For a simply supported sandwich beam, at certain range of frequencies, there is a localized vibration pattern consisting of a longitudinal displacements and small symmetrical vertical displacements between the supports.

Above all that needs mention about in this work is that of the problem of modeling and it was evident in the results that the primary concern of a Structural Civil Engineer must be that of how a problem should be modeled. Over-Simplification of a problem could change the problem, thus any physical system to be solved has to be reduced such that the material taken to the lab for analysis provides a good representation of the true system.

The main advantage of the developed element is that it facilitates the analysis of complex systems that becomes intractable to analytical treatment. The procedure included is straightforward and easy to understand when compared with theories in the literature. The model presented herein is very efficient and versatile for the dynamic analysis of sandwich beam and frame systems.

## RECOMMENDATION FOR FUTURE WORK

Regarding future work the following should be considered;

1. The theory and analysis presented in this thesis can easily be extended for solution of a class of fourth order differential equation. The technique is efficient and provides exact solution to static or steady state problems.
2. Part of the energy dissipated during vibration is absorbed by the vibrating system. This is accompanied by a decrease in amplitude of the modes which indicates that there is some amount of damping in the system. An extension of the present theory to include the damping must be considered for further studies.
3. The theory and analysis in this thesis is applicable to sandwich construction of both single core and multiple cores with multiple facings. An extension of the present theory to solutions of problems involving multi-layer system of elements as in soil is recommended.

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# APPENDICES

Appendix A **Elements of Flexibility Matrix**

$$f_{11} = f_{33} = \frac{L}{3(EI)}\left[1 + \frac{3}{L^2\alpha^2} + \frac{3c(AE)_f}{bGL^2}\right] + \frac{1}{(EI)\alpha\tanh\alpha L}\left[\frac{\mu}{\alpha^2(EI)_f} - 1\right]$$

$$f_{12} = f_{21} = f_{34} = f_{43} = \frac{L}{3(EI)} + \frac{1}{\alpha^2(EI)}\left(\frac{1}{L} - \frac{\alpha}{\tanh\alpha L}\right)$$

$$f_{22} = f_{44} = \frac{L}{3(EI)} - \frac{1}{\mu}\left(\frac{1}{L} - \frac{\alpha}{\tanh\alpha L}\right)$$

$$f_{23} = f_{32} = f_{14} = f_{41} = \frac{1}{(EI)}\left[-\frac{L}{6} + \frac{1}{\alpha^2}\left(\frac{1}{L} - \frac{\alpha}{\sinh\alpha L}\right)\right]$$

$$f_{24} = f_{42} = -\frac{L}{6(EI)} - \frac{1}{\mu}\left(\frac{1}{L} - \frac{\alpha}{\sinh\alpha L}\right)$$

$$f_{13} = f_{31} = \frac{L}{(EI)}\left(-\frac{1}{6} + \frac{1}{\alpha^2L^2}\right) + \frac{c(AE)_f}{bGL} + \frac{1}{\alpha(EI)\sinh\alpha L}\left[\frac{\mu}{\alpha^2(EI)_f} - 1\right]$$

Appendix B **Deflection function elements for unit load ($g$)**

$$g_1(x)=\frac{1}{6(EI)L}\left(x^3-3Lx^2+2L^2x\right)+\frac{1}{\alpha^2(EI)}\left[\frac{x}{L}-1+\frac{\sinh\alpha(L-x)}{\sinh\alpha L}\right]$$

$$g_2(x)=\frac{1}{6(EI)L}\left(x^3-3Lx^2+2L^2x\right)+\frac{1}{\alpha^2}\left(\frac{1}{(EI)}-\frac{1}{(EI)_f}\right)\left[\frac{x}{L}-1+\frac{\sinh\alpha(L-x)}{\sinh\alpha L}\right]$$

$$g_3(x)=-\frac{1}{6(EI)L}\left(x^3-L^2x\right)+\frac{1}{\alpha^2(EI)}\left[-\frac{x}{L}+\frac{\sinh\alpha x}{\sinh\alpha L}\right]$$

$$g_4(x)=-\frac{1}{6(EI)L}\left(x^3-L^2x\right)+\frac{1}{\alpha^2}\left(\frac{1}{(EI)}-\frac{1}{(EI)_f}\right)\left[-\frac{x}{L}+\frac{\sinh\alpha x}{\sinh\alpha L}\right]$$

(Axial load deflection function is assumed to be zero as shown below)

$$g_5(x)=0$$

Appendix C **Rotation function elements for unit load ($g_\phi$)**

$$g_{\phi1}(x)=\frac{1}{6(EI)L}\left(3x^2-6Lx+2L^2\right)+\frac{1}{\alpha^2(EI)}\left[\frac{1}{L}-\alpha\frac{\cosh\alpha(L-x)}{\sinh\alpha L}\right]$$
$$-\frac{c}{bd^2G}\left[-\frac{1}{L}+(EI)_f\left\{\frac{1}{(EI)L}-\frac{\alpha\cosh\alpha(L-x)}{(EI)\sinh\alpha L}\right\}\right]$$

$$g_{\phi2}(x)=\frac{1}{6(EI)L}\left(3x^2-6Lx+2L^2\right)+\frac{1}{\alpha^2}\left(\frac{1}{(EI)}-\frac{1}{(EI)_f}\right)\left[\frac{1}{L}-\alpha\frac{\cosh\alpha(L-x)}{\sinh\alpha L}\right]$$
$$-\frac{c}{bd^2G}\left[-\frac{1}{L}+(EI)_f\left\{\frac{1}{(EI)L}-\alpha\left(\frac{1}{(EI)}-\frac{1}{(EI)_f}\right)\frac{\cosh\alpha(L-x)}{\sinh\alpha L}\right\}\right]$$

$$g_{\phi3}(x)=\frac{1}{6(EI)L}\left(3x^2-L^2\right)+\frac{1}{\alpha^2(EI)}\left[\frac{1}{L}-\alpha\frac{\cosh\alpha x}{\sinh\alpha L}\right]$$
$$-\frac{c}{bd^2G}\left[-\frac{1}{L}+(EI)_f\left\{\frac{1}{(EI)L}-\frac{\alpha\cosh\alpha x}{(EI)\sinh\alpha L}\right\}\right]$$

$$g_{\phi4}(x)=\frac{1}{6(EI)L}\left(3x^2-L^2\right)+\frac{1}{\alpha^2}\left(\frac{1}{(EI)}-\frac{1}{(EI)_f}\right)\left[\frac{1}{L}-\alpha\frac{\cosh\alpha x}{\sinh\alpha L}\right]$$
$$-\frac{c}{bd^2G}\left[-\frac{1}{L}+(EI)_f\left\{\frac{1}{(EI)L}-\alpha\left(\frac{1}{(EI)}-\frac{1}{(EI)_f}\right)\frac{\cosh\alpha x}{\sinh\alpha L}\right\}\right]$$

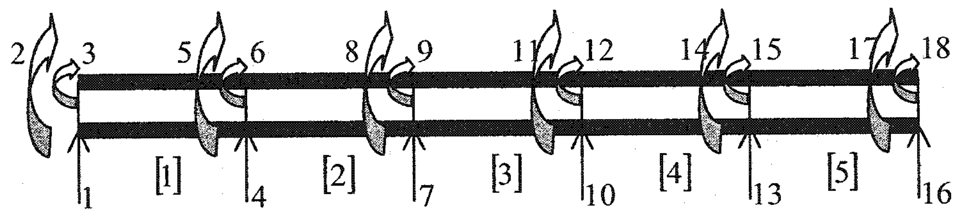
(Axial load rotation function is assumed to be zero as shown below)

$$g_{\phi5}(x)=0$$

Appendix D **Deflection function for a simply supported beam subjected to uniform load q over the entire span of length L (no shear restraint at the ends )**

$$v(x) = \frac{qx}{24(EI)}\left(x^3 - 2Lx^2 + L^3\right) + \frac{q}{\mu\alpha^2}\left[\frac{\cosh\alpha\left(\frac{L}{2} - x\right)}{\cosh\frac{\alpha L}{2}} - 1\right] + \frac{x}{2}(L - x)$$

Appendix E **Program for Dynamic Analysis of continuous Sandwich beam and frame Systems Due to translational and Rotational Effect. (This program calculates Eigenvalues),**




The model corresponds to the data given in this program for a continuous sandwich beam divided into equal elements with numbered system nodal coordinates. The program could be written to accept data input during runtime; however, it is more convenient to include the data within the program itself.

```
float sDOF,              // Number of free nodal displacements: System DOF
    nDOF  = 3,           // Number of degrees of freedom/node
    eNODE = 2,           // Number of nodes/element
    eDOF = nDOF * eNODE; // Number of element dof
main()
{ /*
   PROGRAM FOR SYMMETRIC AND ASSYMMETRIC SANDWICH BEAM
   AND FRAME ANALYSIS. DIRECT STIFFNESS METHOD FOR
   CONSTRAINED SYSTEM; i.e. EIGENVALUES ARE OBTAINED FOR ALL
```

```
  NODAL DISPLACEMENTS EXCLUDING RESTRAINED DIRECTIONS.
  PROGRAM CONSIDERS ONLY TRANSLATIONAL AND ROTATIONAL
  EFFECTS
 */
 GetData();                 // Define all data for beam, truss or frame
 NodeID();                  // Generate id matrix for nodal DOF
 SystemStiff();             // Setup system stiffness matrix
  Systemmass();             // Setup system mass matrix
  Constraint();             // Impose constraint conditions on [M] and [K]
  print(K2,M2);
  EigenValue();             // Compute squares of natural frequencies
}

GetData()
{ // This function defines ALL data required for problem.
  // --- General data ---
  sEl = getnum("Enter number of element", 5);
  sNodes = sEl + 1;                // number of nodal points
  sPreDispls  = 0;                 // number of non-zero prescribed displacements

  // -------- Define coordinates --------
  Totallength = 0.9144;
  Member = Totallength/sEl;
```

```
defmat(x[sNodes], 0 :: Totallength :: Member );

defmat(y[sNodes], sNodes : 0);

print("Nodal coordinates",x,y);


// -- Define element connectivity and end-condition ----

 defmat(ElCon[eNODE,sEl],

      1 :: sEl:: 1,                         // i: first node of each element

   2 :: sNodes :: 1);                       // j: second node

print(^"Element connectivity",ElCon);

// -- Define element properties --

b = 0.03;  c = 0.0127;  t1 = 0.0004572;  t2 = 0.0004572;  // meters

E1 = 6.89e+10;  E2 = 6.89e+10;   G = 0.0012*E1;        //  Pa


// --- Define member axial forces ---

defmat(P[sEl], sEl:0);


// Convenient data

A1=b*t1;  A2=b*t2;

EIf1=b*E1*t1^3/12;  EIf2=b*E2*t2^3/12;


// 5 Basic parameters: c,d,bG,EIf,AEf

d = c+(t1+t2)/2;

bG = b*G;
```

```
EIf=EIf1+EIf2;

//EIf = E1*((c+t1+t2)^3)*b/12;

AEf = (A1*E1 * A2*E2) / (A1*E1 + A2*E2);


// Derived parameters

al2=bG*(1/AEf+d^2/EIf)/c;

al=sqrt(al2);

AG=d^2*bG/c;

Mu=al2^2*EIf^2/AG; EI=EIf+d^2*AEf;

If1 = EIf1/E1;  If2 = EIf2/E2;

rho1 = 2680;                    // kilogram per cubic meter

rho2 = 2680;                    // kilogram per cubic meter

A = A1 + A2; If = If1 + If2;

d1 = (A2*E2/AEf)*d;

d2 = (A1*E1/AEf)*d;


// -- Define nodal restraints by inserting 1 into id matrix --

zero(id[nDOF,sNodes]);          // clear before filling with 1


  /* 1:shear   2: global moment   3: local facing moment

 ________________________________________________________ */

 /* Condition for Node 1: the first node.

 Place selected condition last  */
```

```
id[1,1] = 0;  id[2,1] = 0;  id[3,1] = 0;        // Free
id[1,1] = 1;  id[2,1] = 1;  id[3,1] = 1;        // Clamped
id[1,1] = 1;  id[2,1] = 0;  id[3,1] = 2;        // Smpl. and no shear def.
id[1,1] = 1;  id[2,1] = 0;  id[3,1] = 0;        // Smpl. with shear def.

/* Condition for Node 2: the second node.
 Place selected condition last  */
id[1,2] = 1;  id[2,2] = 1;  id[3,2] = 1;        // Clamped
id[1,2] = 1;  id[2,2] = 0;  id[3,2] = 2;        // Smpl. and no shear def.
id[1,2] = 1;  id[2,2] = 0;  id[3,2] = 0;        // Smpl. with shear def.
id[1,2] = 0;  id[2,2] = 0;  id[3,2] = 0;        // Free

/* Condition for Node 3: the second node.
 Place selected condition last  */
id[1,3] = 1;  id[2,3] = 0;  id[3,3] = 0;        // Smpl. with shear def.
id[1,3] = 1;  id[2,3] = 1;  id[3,3] = 1;        // Clamped
id[1,3] = 1;  id[2,3] = 0;  id[3,3] = 2;        // Smpl. and no shear def.
id[1,3] = 0;  id[2,3] = 0;  id[3,3] = 0;        // Free

/* Condition for Node 4: the second node.
 Place selected condition last  */
id[1,4] = 1;  id[2,4] = 1;  id[3,4] = 1;        // Clamped
id[1,4] = 1;  id[2,4] = 0;  id[3,4] = 2;        // Smpl. and no shear def.
```

```
 id[1,4] = 1;  id[2,4] = 0;  id[3,4] = 0;        // Smpl. with shear def.
 id[1,4] = 0;  id[2,4] = 0;  id[3,4] = 0;        // Free

  /* Condition for Node 5: the second node.
  Place selected condition last  */
 id[1,5] = 1;  id[2,5] = 1;  id[3,5] = 1;        // Clamped
 id[1,5] = 1;  id[2,5] = 0;  id[3,5] = 2;        // Smpl and no shear def.
 id[1,5] = 1;  id[2,5] = 0;  id[3,5] = 0;        // Smpl with shear def.
 id[1,5] = 0;  id[2,5] = 0;  id[3,5] = 0;        // Free

  /* Condition for Node 6: the first node.
  Place selected condition last  */
 id[1,6] = 0;  id[2,6] = 0;  id[3,6] = 0;        // Free
 id[1,6] = 1;  id[2,6] = 1;  id[3,6] = 1;        // Clamped
 id[1,6] = 1;  id[2,6] = 0;  id[3,6] = 2;        // Smpl. and no shear def.
 id[1,6] = 1;  id[2,6] = 0;  id[3,6] = 0;        // Smpl. with shear def.
 print(^"Nodal restraint list",id);
}

SystemStiff()
{ // Setup system stiffness matrix [K] and impose boundary conditions
 float m;
 mat el_id[eDOF], Km[eDOF,eDOF];
```

```
  zero(K[sDOF,sDOF]);                         // Initialize system stiffness [K]
  for(m=1;m<=sEl;m=m+1) {                     // For each member m
    ElementID(m, el_id);                      // Get member end-displ labels
    ElementStiff(m,Km);                       // Member stiffness [Km] in global axes
    subop(Km+K[el_id;el_id]);                 // Assembled into [K]
  }
  !K2=K;                                      // Save copy in [K2]
}


Constraint()
{ // Provide cnstraint for the system mass and stiffness matices
  float i, j;
  zero(Fo[sDOF]);
  for(n=1;n<=sDOF;n=n+1) {
      c = constr[n];
      switch (1) {                            // Inspect each direction
        case c == 0 : break;                  // No constraint here
        case c == i#: c = 0;                  // Fully restrained: displ = 0
        default:
          subop(Fo > K2[n;1::sDOF]);
          subop(Fo > K2[1::sDOF;n]);
          subop(Fo > M2[n;1::sDOF]);
          subop(Fo > M2[1::sDOF;n]);
```

```
            for(i=1; i<=sDOF; i=i+1){
              for(j=1; j<=sDOF; j=j+1){
              if(i==j){
                if(K2[i,j]==0) {K2[i,j]=1;}
                if(M2[i,j]==0) {M2[i,j]=1;}
              }
            }
          }
        }
      }
    }

ElementID(float m, mat d)
{  // Return nodal displacement numbers of element m  // 3 DOF per node
   float i=ElCon[1,m], j=ElCon[2,m];                        // element nodes
  d[1] = id[1,i];  d[2] = id[2,i];  d[3] = id[3,i];
  d[4] = id[1,j];  d[5] = id[2,j];  d[6] = id[3,j];
}

Length1(float m)
{ // Returns the horizontal distance of element m
  float i, j, L;
  i = ElCon[1,m];  j = ElCon[2,m];                          // element nodes
```

```
 L1 = x[j]-x[i];
 return L1;
}

Length2(float m)
{ // Returns the vertical distance of element m
  float i, j, L;
  i = ElCon[1,m];  j = ElCon[2,m];                    // element nodes
  L2 = y[j]-y[i];
  return L2;
}

Length(float m)
{ // Returns length of the z-x plane of element m
  float i, j, L;
  Length1(m);
  Length2(m);
  L = sqrt(L2^2+L1^2);
  AL=al*L; AL2=AL/2;
  if(L<=0) { print(^^"Incorrect nodal x and y-coordinates",j,i); end;}
  return L;
}
```

```
Transformation(float m)
{ // Transforms member axis to that of the global
  Length1(m);
  Length2(m);
  alpha = L1/L;
  defmat(T[6,6], alpha, 0, 0, 0, 0, 0,
                 0, 1, 0, 0, 0, 0,
                 0, 0, 1, 0, 0, 0,
                 0, 0, 0, alpha, 0, 0,
                 0, 0, 0, 0, 1, 0,
                 0, 0, 0, 0, 0, 1);
}

NodeID()
{ /* . Save constraint info into {constr} for later use;
    . Fill id[nDOF,sNodes] with displ. numbers even at restraints;
    . Compute number of system nodal displacements sDOF.
  Input: id[nDOF,sNodes]: nodal constraint information
  output:id[nDOF,sNodes]: nodal displacement numbers
       constr[nDOF*sNodes]: assigns a value at each nodal direction:
          0       : no restraint.
          i# = 1.e30: infinite restraint (i.e. zero displacement)
          a value less than i#: prescribed non-zero displacement.
```

```
    sDOF: number of system DOF  */
float i, j, k, n; // local vars for loop control
zero (constr[nDOF*sNodes]);                        // initialized to zero
sDOF=0;
for(n=1;n<=sNodes;n=n+1) {                         // each node starting from 1
  for(j=1;j<=nDOF;j=j+1) {                         // each DOF: x, y, rotation
    i = id[j,n];
    if(i==0 || i==1) { sDOF=sDOF+1;}         /* count number of displcmts
                                              at but exclude coupled directions  */
    switch(TRUE=1) {
      case i==1:                                   // fully restrained direction
        constr[sDOF]=i#;                           // mark direction
      case i==0:                                   // free to displace
        id[j,n]=sDOF;                              // assign DOF even at restraint
        break;
      case i>1 && n>=i:                            // dof coupled to node i>1
        id[j,n]=id[j-1,i];                         // pick up previous DOF at node i
        break;
      default: print(^"Invalid data in id[j,n],j,n =",i,j,n);
        end;
    }
  }
}
```

```
print(^"Nodal displacement numbers",id);
for(i=1; i<= sPreDispls; i=i+1) {          // Prescribed non-zero displacement
  n = Ndis[i,1];                           // Node number where there is prescribed displ.
  for(j=1;j<=nDOF;j=j+1) {                 // each DOF: x, y, rotation
    k = id[j,n];                           // Nodal displacemnt number
    if(Ndis[i,j+1]) {
      if(constr[k]==i#) { print(^"Bad prescribed displ data: Node",
        n," Direction",k," was fully restrained"); end; }
      constr[k] = Ndis[i,j+1];                        // save value
    }
  }
}
} // end NodeID()

B_matrix(float m)
{ // produces rotation matrix [T] for element m
  float i=ElCon[1,m], j=ElCon[2,m];                   // element nodes
  L=Length(m);
  defmat(B[6,4],4:1/L,1,4:0,1,0,0,4:-1/L,0,0,1,4:0,1);
}

ElementStiff(float m, mat Km)
{ // Compute element stiffness matrix [Km] of element m
```

```
L=Length(m);
zero(F[4,4]);
Bfive(Mo=1,Mf=0);
F[3,3]=F[1,1]=phi(0);
F[1,3]=F[3,1]=phi(L);
F[4,3]=F[2,1]=v1(0);
F[2,3]=F[4,1]=v1(L);
Bfive(Mo=1,Mf=1);
F[1,2]=F[3,4]=phi(0);
F[3,2]=F[1,4]=phi(L);
F[2,2]=F[4,4]=v1(0);
F[4,2]=F[2,4]=v1(L);
!k4=F^-1;                                  // Stiffness matrix obtained from
                                           // transformation of Flexibility matrix

defmat(KG[6,6], -1, 0, 0, 1, 0, 0,         // Geometric stiffness matrix
                 0, 0, 0, 0, 0, 0,
                 0, 0, 0, 0, 0, 0,
                 1, 0, 0,-1, 0, 0,
                 0, 0, 0, 0, 0, 0,
                 0, 0, 0, 0, 0, 0);
B_matrix(m);
Transformation(m);
if(P[m] == 0)
```

```
    {
      !Km=T~*(B*k4*B~)*T;
      }
    else{
      !Km=(T~*(B*k4*B~)*T) + P[m]*KG/L;
  }
}

Bfive(float Mo, float Mf)
{
  B5=(Mo/EI-Mf/EIf)/al2;
}

// Deflection function for sandwich beam with end moment
v(float x) {
 return Mo*(2*L^2-3*L*x+x^2)*x/(6*EI*L) + B5*(x/L-1+hsos(al*(L-x),AL));
}

// Derivatives of deflection function for sandwich beam with end moment
v1(float x) {
 return Mo*(2*L^2-6*L*x+3*x^2)/(6*EI*L) + B5*(1/L-al*hcos(al*(L-x),AL));
}
```

```
v2(float x) {
 return Mo*(6*x-6*L)/(6*EI*L) + B5*al2*hsos(al*(L-x),AL);
}

v3(float x) {
 return Mo/(EI*L) - B5*al^3*hcos(al*(L-x),AL);
}

v4(float x) {
  return B5*al2^2*hsos(al*(L-x),AL);
}

v5(float x) {
  return -B5*al^5*hcos(al*(L-x),AL);
}

// Skin rotation (u1-u2)/d for sandwich beam section
phi(float x) {
  return  v1(x) - c*(-Mo/L+EIf*v3(x))/(bG*d^2);
}

// Influence displacement functions for sandwich beam with end moments
g1(float x) /* Influence displacement function for the first DOF*/
```

```
{
  return (x^3-3*L*x^2+2*L^2*x)/(6*EI*L) +
        (x/L-1+hsos(al*(L-x),AL))/(al2*EI);
}

g2(float x)
{
  return (x^3-3*L*x^2+2*L^2*x)/(6*EI*L) +
        (x/L-1+hsos(al*(L-x),AL))*(1/EI-1/EIf)/al2;
}

g3(float x)
{
  return -(x^3-L^2*x)/(6*EI*L) +
        (-x/L+hsos(al*x,AL))/(al2*EI);
}

g4(float x)
{
  return -(x^3-L^2*x)/(6*EI*L) +
        (-x/L+hsos(al*x,AL))*(1/EI-1/EIf)/al2;
}
```

```
// Derivatives of influence displ. Functions for sandwich beam with end moment

g11(float x) /*First derivative of Influence displacement function for the first DOF*/

{

  return (3*x^2-6*L*x+2*L^2)/(6*EI*L) +

        (1/L-al*hcos(al*(L-x),AL))/(al2*EI);

}


g12(float x)

{

  return (3*x^2-6*L*x+2*L^2)/(6*EI*L) +

        (1/L-     al*hcos(al*(L-x),AL))*(1/EI-1/EIf)/al2;

}


g13(float x)

{

  return -(3*x^2-L^2)/(6*EI*L) +

        (-1/L+al*hcos(al*x,AL))/(al2*EI);

}


g14(float x)

{

  return -(3*x^2-L^2)/(6*EI*L) +

        (-1/L+al*hcos(al*x,AL))*(1/EI-1/EIf)/al2;
```

```
}

Shapes(float x)
{
  // Compute shape function matrices for element m
  zero(N[1,6], Nx[1,6], Nphi[1,6]);
  defmat(N1[6], 1-x/L, 0, 0, 0, 0, 0);          // rigid body motion in node i
  defmat(N2[6], 0, 0, 0, x/L, 0, 0);            // rigid body motion in node j
  defmat(N3[1,6], -1/L, 0, 0, 0, 0, 0);
  defmat(N4[1,6], 0, 0, 0, 1/L, 0, 0);
  defmat(g[1,4], g1(x), g2(x), g3(x), g4(x));
  defmat(gd[1,4], g11(x), g12(x), g13(x), g14(x));
  Bfive(Mo=1,Mf=0);
  F[3,3]=F[1,1]=phi(0);
  F[1,3]=F[3,1]=phi(L);
  F[4,3]=F[2,1]=v1(0);
  F[2,3]=F[4,1]=v1(L);
  Bfive(Mo=1,Mf=1);
  F[1,2]=F[3,4]=phi(0);
  F[3,2]=F[1,4]=phi(L);
  F[2,2]=F[4,4]=v1(0);
  F[4,2]=F[2,4]=v1(L);
```

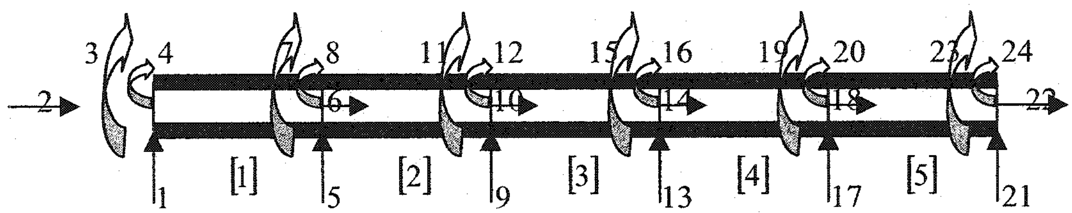
```
  // Final shape function matrices
  !N5 = g * F^-1 * B~;
  !N = N5 + N1 + N2;              // Sandwich beam deflection shape function matrix
  !N6 = gd * F^-1 * B~;
  !Nx = N6 + N3 + N4;             // First derivative of sandwich beam deflection matrix
}

Elementmasses(float m, mat Mm)
{ // Compute element mass matrix [Mm] of element m due to translational [Mt]
  // and rotation [Mr] effects
  float x;
  mat Mt[6,6], Mr[6,6], Ma1[6,6], Ma2[6,6], Ma3[6,6];
  zero(Mm[6,6]);
  B_matrix(m);
  gausspt(n = 4, XG, XW, 0, L);          // Exact weights
  for(i=1; i<=n; i=i+1){
  Shapes(XG[i]);
  !Mt = Mt + (XW[i]*(rho1*A1+rho2*A2)* N~*N);
  !Mr = Mr + (XW[i]*(rho1*If1+rho2*If2)*Nx~* Nx);
  }
  !Mm = Mt + Mr;
}
```

```
Systemmass()
{ // Setup system mass matrix [M] and impose boundary conditions
  float m;
  mat el_id[eDOF], Mm[eDOF,eDOF];
  zero(M[sDOF,sDOF]);                   // Initialize system mass [M]
  for(m=1;m<=sEl;m=m+1) {               // For each member m
    ElementID(m, el_id);                // Get member end-displ labels
    Elementmasses(m,Mm);                // Member mass [Mm] in global axes
    subop(Mm+M[el_id;el_id]);           // Assembled into [M]
  }
  !M2=M;                                // Save copy in [M2]
}


EigenValue()
{     // Computes the squares of the natural frequencies
      // eigen(!v = K2, ev);            // Standard form
         eigen(!V = K2, M2, EV);        // Generalized form
         //print(V);
         print(EV);
}
```

Appendix F **Program for Dynamic Analysis of continuous Sandwich beam and frame Systems Due to Translational, Rotational and Uniform Axial stretching Effects (This program calculates Eigenvalues).**




The model corresponds to the data given in this program for a continuous sandwich beam divided into equal elements with numbered system nodal coordinates. The program could be written to accept data input during runtime; however, it is more convenient to include the data within the program itself.

```
float sDOF,                    // Number of free nodal displacements: System DOF
    nDOF  = 4,                 // Number of degrees of freedom/node
    eNODE = 2,                 // Number of nodes/element
    eDOF = nDOF * eNODE;       // Number of element dof
main()
{ /*
   PROGRAM FOR SYMMETRIC AND ASSYMMETRIC SANDWICH BEAM
   AND FRAME ANALYSIS. DIRECT STIFFNESS METHOD FOR
   CONSTRAINED SYSTEM; i.e. EIGENVALUES ARE OBTAINED FOR ALL
```

```
  NODAL DISPLACEMENTS EXCLUDING RESTRAINED DIRECTIONS.
  PROGRAM CONSIDERS TRANSLATIONAL, ROTATIONAL AND AXIAL
EFFECTS
 */
 GetData();                    // Define all data for beam, truss or frame
 NodeID();                     // Generate id matrix for nodal DOF
 SystemStiff();                // Setup system stiffness matrix
 Systemmass();                 // Setup system mass matrix
 Constraint();                 // Impose constraint conditions on [M] and [K]
 //print(K2,M2);
 EigenValue();                 // Computes the squares of the natural frequencies
}

GetData()
{ // This function defines ALL data required for problem.
  // --- General data ---
  sEl = getnum("Enter number of element", 5);
  sNodes = sEl + 1;              // number of nodal points
  sPreDispls  = 0;               // number of non-zero prescribed displacements

  // -------- Define coordinates --------
  Totallength = 0.9144;
  Member = Totallength/sEl;
```

```
defmat(x[sNodes], 0 :: Totallength :: Member );

defmat(y[sNodes], sNodes : 0);

print("Nodal coordinates", x, y);


// -- Define element connectivity and end-condition ----

 defmat(ElCon[eNODE,sEl],

      1 :: sEl:: 1,                                  // i: first node of each element

      2 :: sNodes :: 1);                             // j: second node

print(^"Element connectivity",ElCon);

// -- Define element properties --

b = 0.03;  c = 0.0127;  t1 = 0.0004572;  t2 = 0.0004572;      // meters

E1 = 6.89e+10;  E2 = 6.89e+10;   G = 0.0012*E1;               // Pa


// --- Define member axial forces ---

defmat(P[sEl], sEl:0);


// Convenient data

A1=b*t1;   A2=b*t2;

EIf1=b*E1*t1^3/12;   EIf2=b*E2*t2^3/12;


// 5 Basic parameters: c,d,bG,EIf,AEf

d = c+(t1+t2)/2;

bG = b*G;
```

```
EIf=EIf1+EIf2;

//EIf = E1*((c+t1+t2)^3)*b/12;

AEf = (A1*E1 * A2*E2) / (A1*E1 + A2*E2);


// Derived parameters

al2=bG*(1/AEf+d^2/EIf)/c;

al=sqrt(al2);

AG=d^2*bG/c;

Mu=al2^2*EIf^2/AG; EI=EIf+d^2*AEf;

If1 = EIf1/E1;  If2 = EIf2/E2;

rho1 = 2680;                    // kilogram per cubic meter

rho2 = 2680;                    // kilogram per cubic meter

A = A1 + A2; If = If1 + If2;

d1 = (A2*E2/AEf)*d;

d2 = (A1*E1/AEf)*d;


// -- Define nodal restraints by inserting 1 into id matrix --

zero(id[nDOF,sNodes]);          // clear before filling with 1


  /* 1:shear   2: global moment   3: local facing moment

_______________________________________________________ */

 /* Condition for Node 1: the first node.

 Place selected condition last  */
```

```
id[1,1] = 0; id[2,1] = 0; id[3,1] = 0; id[4,1] = 0;        // Free
id[1,1] = 1; id[2,1] = 1; id[3,1] = 1; id[4,1] = 1;        // Clamped
id[1,1] = 1; id[2,1] = 1; id[3,1] = 0; id[4,1] = 2;        // Smpl. and no shear def.
id[1,1] = 1; id[2,1] = 1; id[3,1] = 0; id[4,1] = 0;        // Smpl. with shear def.

 /* Condition for Node 2: the second node.
 Place selected condition last  */
id[1,2] = 1; id[2,2] = 1; id[3,2] = 1; id[4,2] = 1;        // Clamped
id[1,2] = 1; id[2,2] = 1; id[3,2] = 0; id[4,2] = 2;        // Smpl. and no shear def.
id[1,2] = 1; id[2,2] = 1; id[3,2] = 0; id[4,2] = 0;        // Smpl. with shear def.
id[1,2] = 0; id[2,2] = 0; id[3,2] = 0; id[4,2] = 0;        // Free

 /* Condition for Node 3: the second node.
  Place selected condition last  */
id[1,3] = 1; id[2,3] = 1; id[3,3] = 0; id[4,3] = 0;        // Smpl. with shear def.
id[1,3] = 1; id[2,3] = 1; id[3,3] = 1; id[4,3] = 1;        // Clamped
id[1,3] = 1; id[2,3] = 1; id[3,3] = 0; id[4,3] = 2;        // Smpl. and no shear def.
id[1,3] = 0; id[2,3] = 0; id[3,3] = 0; id[4,3] = 0;        // Free

 /* Condition for Node 4: the second node.
 Place selected condition last  */
id[1,4] = 1; id[2,4] = 1; id[3,4] = 1; id[4,4] = 1;        // Clamped
id[1,4] = 1; id[2,4] = 1; id[3,4] = 0; id[4,4] = 2;        // Smpl. and no shear def.
```

```
  id[1,4] = 1; id[2,4] = 1; id[3,4] = 0; id[4,4] = 0;        // Smpl. with shear def.
  id[1,4] = 0; id[2,4] = 0; id[3,4] = 0; id[4,4] = 0;        // Free

  /* Condition for Node 5: the second node.
  Place selected condition last  */
  id[1,5] = 1; id[2,5] = 1; id[3,5] = 1; id[4,5] = 1;        // Clamped
  id[1,5] = 1; id[2,5] = 1; id[3,5] = 0; id[4,5] = 2;        // Smpl. and no shear def.
  id[1,5] = 1; id[2,5] = 1; id[3,5] = 0; id[4,5] = 0;        // Smpl. with shear def.
  id[1,5] = 0; id[2,5] = 0; id[3,5] = 0; id[4,5] = 0;        // Free

  /* Condition for Node 6: the first node.
  Place selected condition last  */
  id[1,6] = 0; id[2,6] = 0; id[3,6] = 0; id[4,6] = 0;        // Free
  id[1,6] = 1; id[2,6] = 1; id[3,6] = 1; id[4,6] = 1;        // Clamped
  id[1,6] = 1; id[2,6] = 1; id[3,6] = 0; id[4,6] = 2;        // Smpl. and no shear def.
  id[1,6] = 1; id[2,6] = 1; id[3,6] = 0; id[4,6] = 0;        // Smpl. with shear def.
  print(^"Nodal restraint list",id);
}

SystemStiff()
{ // Setup system stiffness matrix [K] and impose boundary conditions
  float m;
  mat el_id[eDOF], Km[eDOF,eDOF];
```

```
  zero(K[sDOF,sDOF]);                    // Initialize system stiffness [K]
  for(m=1;m<=sEl;m=m+1) {                // For each member m
    ElementID(m, el_id);                 // Get member end-displ labels
    ElementStiff(m,Km);                  // Member stiffness [Km] in global axes
    subop(Km+K[el_id;el_id]);            // Assembled into [K]
  }
  !K2=K;                                 // Save copy in [K2]
}

Constraint()
{ // Provide cnstraint for the system mass and stiffness matices
  float i, j;
  zero(Fo[sDOF]);
  for(n=1;n<=sDOF;n=n+1) {
      c = constr[n];
      switch (1) {                       // Inspect each direction
        case c == 0 : break;             // No constraint here
        case c == i#: c = 0;             // Fully restrained: displ = 0
        default:
          subop(Fo > K2[n;1::sDOF]);
          subop(Fo > K2[1::sDOF;n]);
          subop(Fo > M2[n;1::sDOF]);
          subop(Fo > M2[1::sDOF;n]);
```

```
            for(i=1; i<=sDOF; i=i+1){
              for(j=1; j<=sDOF; j=j+1){
              if(i==j){
                if(K2[i,j]==0) {K2[i,j]=1;}
                if(M2[i,j]==0) {M2[i,j]=1;}
              }
            }
          }
        }
      }
    }

ElementID(float m, mat d)
{ // Return nodal displ numbers of element m          // 4 DOF per node
 float i=ElCon[1,m], j=ElCon[2,m];                    // element nodes
  d[1] = id[1,i];  d[2] = id[2,i];  d[3] = id[3,i]; d[4] = id[4,i];
  d[5] = id[1,j];  d[6] = id[2,j];  d[7] = id[3,j]; d[8] = id[4,j];
}

Length1(float m)
{ // Returns the horizontal distance of element m
  float i, j, L;
  i = ElCon[1,m];  j = ElCon[2,m];                     // element nodes
```

```
 L1 = x[j]-x[i];
 return L1;
}

Length2(float m)
{ // Returns the vertical distance of element m
  float i, j, L;
  i = ElCon[1,m];  j = ElCon[2,m]; // element nodes
  L2 = y[j]-y[i];
  return L2;
}

Length(float m)
{ // Returns length of the z-x plane of element m
  float i, j, L;
  Length1(m);
  Length2(m);
  L = sqrt(L2^2+L1^2);
  AL=al*L; AL2=AL/2;
  if(L<=0) { print(^^"Incorrect nodal x and y-coordinates",j,i); end;}
  return L;
}
```

```
Transformation(float m)
{ // Transforms member axias to that of the global
  Length1(m);
  Length2(m);
  Alpha = L1/L;
  Beta = L2/L;
  defmat(T[8,8], Alpha, -Beta, 0, 0, 0, 0, 0, 0,
                 Beta, Alpha, 0, 0, 0, 0, 0, 0,
                 0, 0, 1, 0, 0, 0, 0, 0,
                 0, 0, 0, 1, 0, 0, 0, 0,
                 0, 0, 0, 0, Alpha, -Beta, 0, 0,
                 0, 0, 0, 0, Beta, Alpha, 0, 0,
                 0, 0, 0, 0, 0, 0, 1, 0,
                 0, 0, 0, 0, 0, 0, 0, 1);
}

NodeID()
{ /* . Save constraint info into {constr} for later use;
    . Fill id[nDOF,sNodes] with displ. numbers even at restraints;
    . Compute number of system nodal displacements sDOF.
 Input: id[nDOF,sNodes]: nodal constraint information
 output:id[nDOF,sNodes]: nodal displacement numbers
      constr[nDOF*sNodes]: assigns a value at each nodal direction:
```

```
    0       : no restraint.
    i# = 1.e30: infinite restraint (i.e. zero displacement)
    a value less than i#: prescribed non-zero displacement.
  sDOF: number of system DOF  */
float i, j, k, n; // local vars for loop control
zero (constr[nDOF*sNodes]);                        // initialized to zero
sDOF=0;
for(n=1;n<=sNodes;n=n+1) {                         // each node starting from 1
  for(j=1;j<=nDOF;j=j+1) {                         // each DOF: x, y, rotation
    i = id[j,n];
    if(i==0 || i==1) { sDOF=sDOF+1;}           /* count number of displcmts
                                                  at but exclude coupled directions  */
    switch(TRUE=1) {
      case i==1:                                // fully restrained direction
        constr[sDOF]=i#;                        // mark direction
      case i==0:                                // free to displace
        id[j,n]=sDOF;                           // assign DOF even at restraint
        break;
      case i>1 && n>=i:                         // dof coupled to node i>1
        id[j,n]=id[j-1,i];                      // pick up previous DOF at node i
        break;
      default: print(^"Invalid data in id[j,n],j,n =",i,j,n);
        end;
```

```
      }
    }
  }
  print(^"Nodal displacement numbers",id);
  for(i=1; i<= sPreDispls; i=i+1) {          // Prescribed non-zero displacement
    n = Ndis[i,1];                            // Node number where there is prescribed displ.
    for(j=1;j<=nDOF;j=j+1) {                  // each DOF: x, y, rotation
      k = id[j,n];                            // Nodal displacemnt number
      if(Ndis[i,j+1]) {
        if(constr[k]==i#) { print(^"Bad prescribed displ data: Node",
          n," Direction",k," was fully restrained"); end; }
        constr[k] = Ndis[i,j+1];                          // save value
      }
    }
  }
} // end NodeID()


B_matrix(float m)
{ // produces rotation matrix [T] for element m
 float i=ElCon[1,m], j=ElCon[2,m];                      // element nodes
 L=Length(m);
 defmat(B[8,5],4:1/L,0,     4:0,-1,
          1, 4:0,      0,1,3:0,
```

```
            4:-1/L,0,     4:0,1,

            2:0,1,2:0,    3:0,1,0);

}


ElementStiff(float m, mat Km)

{ // Compute element stiffness matrix [Km] of sandwich beam element, m

  L=Length(m);

  zero(F[5,5]);

  Bfive(Mo=1,Mf=0);

  F[3,3]=F[1,1]=phi(0);

  F[1,3]=F[3,1]=phi(L);

  F[4,3]=F[2,1]=v1(0);

  F[2,3]=F[4,1]=v1(L);

  Bfive(Mo=1,Mf=1);

  F[1,2]=F[3,4]=phi(0);

  F[3,2]=F[1,4]=phi(L);

  F[2,2]=F[4,4]=v1(0);

  F[4,2]=F[2,4]=v1(L);

  F[5,5]= L/((E1*A1)+(E2*A2));             // Axial stiffness coefficient

  !k4=F^-1;                                // Stiffness matrix obtained from

                                           // transformation of Flexibility matrix

  defmat(KG[8,8], -1, 0, 0, 0, 1, 0, 0, 0,  // Geometric stiffness matrix

                  0, 0, 0, 0, 0, 0, 0, 0,
```

```
                    0, 0, 0, 0, 0, 0, 0, 0,
                    0, 0, 0, 0, 0, 0, 0, 0,
                    1, 0, 0, 0,-1, 0, 0, 0,
                    0, 0, 0, 0, 0, 0, 0, 0,
                    0, 0, 0, 0, 0, 0, 0, 0,
                    0, 0, 0, 0, 0, 0, 0, 0);
  B_matrix(m);
  Transformation(m);
  if(P[m] == 0)
    {
      !Km=T~*(B*k4*B~)*T;
    }
    else{
      !Km=(T~*(B*k4*B~)*T) + P[m]*KG/L;
   }
}

Bfive(float Mo, float Mf)
{
  B5=(Mo/EI-Mf/EIf)/al2;
}

// Deflection function for sandwich beam with end moment
```

```
v(float x) {

 return Mo*(2*L^2-3*L*x+x^2)*x/(6*EI*L) + B5*(x/L-1+hsos(al*(L-x),AL));

}


// Derivatives of deflection function for sandwich beam with end moment

v1(float x) {

 return Mo*(2*L^2-6*L*x+3*x^2)/(6*EI*L) + B5*(1/L-al*hcos(al*(L-x),AL));

}


v2(float x) {

 return Mo*(6*x-6*L)/(6*EI*L) + B5*al2*hsos(al*(L-x),AL);

}


v3(float x) {

 return Mo/(EI*L) - B5*al^3*hcos(al*(L-x),AL);

}


v4(float x) {

   return B5*al2^2*hsos(al*(L-x),AL);

}


v5(float x) {

   return -B5*al^5*hcos(al*(L-x),AL);
```

```
}

// Skin rotation (u1-u2)/d for a sandwich beam section
phi(float x) {
  return  v1(x) - c*(-Mo/L+EIf*v3(x))/(bG*d^2);
}

// Influence displacement functions for sandwich beam with end moment
g1(float x) /* Influence displacement function for the first DOF*/
{
  return (x^3-3*L*x^2+2*L^2*x)/(6*EI*L) +
         (x/L-1+hsos(al*(L-x),AL))/(al2*EI);
}

g2(float x)
{
  return (x^3-3*L*x^2+2*L^2*x)/(6*EI*L) +
         (x/L-1+hsos(al*(L-x),AL))*(1/EI-1/EIf)/al2;
}

g3(float x)
{
  return -(x^3-L^2*x)/(6*EI*L) +
```

```
        (-x/L+hsos(al*x,AL))/(al2*EI);
}

g4(float x)
{
  return -(x^3-L^2*x)/(6*EI*L) +
        (-x/L+hsos(al*x,AL))*(1/EI-1/EIf)/al2;
}

// Derivatives of influence displ. functions for sandwich beam with end moment
g11(float x) /*First derivative of Influence displacement function for the first DOF*/
{
  return (3*x^2-6*L*x+2*L^2)/(6*EI*L) +
        (1/L-al*hcos(al*(L-x),AL))/(al2*EI);
}

g12(float x)
{
  return (3*x^2-6*L*x+2*L^2)/(6*EI*L) +
        (1/L-  al*hcos(al*(L-x),AL))*(1/EI-1/EIf)/al2;
}

g13(float x)
```

```
{
  return -(3*x^2-L^2)/(6*EI*L) +
          (-1/L+al*hcos(al*x,AL))/(al2*EI);
}

g14(float x)
{
  return -(3*x^2-L^2)/(6*EI*L) +
          (-1/L+al*hcos(al*x,AL))*(1/EI-1/EIf)/al2;
}

// Influence rotation functions for a sandwich beam with end moment
gphi1(float x) /* Influence rotation function for the first DOF*/
{
  return (3*x^2-6*L*x+2*L^2)/(6*EI*L) +
          (1/L-al*hcos(al*(L-x),AL))/(al2*EI) -
          (-1/L+EIf*(1/(EI*L)-al*(1/EI)*hcos(al*(L-x),AL)))*c/(b*d^2*G);
}

gphi2(float x)
{
  return (3*x^2-6*L*x+2*L^2)/(6*EI*L) +
          (1/L-al*hcos(al*(L-x),AL))*(1/EI-1/EIf)/al2 -
```

```
        (-1/L+EIf*(1/(EI*L)-al*(1/EI-1/EIf)*hcos(al*(L-x),AL)))*c/(b*d^2*G);
}

gphi3(float x)
{
  return (3*x^2-L^2)/(6*EI*L) +
        (1/L-al*hcos(al*x,AL))/(al2*EI) -
        (-1/L+EIf*(1/(EI*L)-al*(1/EI)*hcos(al*x,AL)))*c/(b*d^2*G);
}

gphi4(float x)
{
  return (3*x^2-L^2)/(6*EI*L) +
        (1/L-al*hcos(al*x,AL))*(1/EI-1/EIf)/al2 -
        (-1/L+EIf*(1/(EI*L)-al*(1/EI-1/EIf)*hcos(al*x,AL)))*c/(b*d^2*G);
}

Shapes(float x)
{
  // Compute shape function matrices for sandwich beam element, m
  zero(N[1,8], Nx[1,8], Nphi[1,8]);
  defmat(Na[1,8], 0, 1-x/L, 0, 0, 0, x/L, 0, 0);          /* Uniform stretching shape
                                                            */ function matrix
```

```
defmat(N1[8], 1-x/L, 0, 0, 0, 0, 0, 0, 0);          // rigid body motion in node i

defmat(N2[8], 0, 0, 0, 0, x/L, 0, 0, 0);            // rigid body motion in node j

defmat(N3[1,8], -1/L, 0, 0, 0, 0, 0, 0, 0);

defmat(N4[1,8], 0, 0, 0, 0, 1/L, 0, 0, 0);

defmat(g[1,5], g1(x), g2(x), g3(x), g4(x), 0);

defmat(gd[1,5], g11(x), g12(x), g13(x), g14(x), 0);

defmat(gphi[1,5], gphi1(x), gphi2(x), gphi3(x), gphi4(x), 0);

Bfive(Mo=1,Mf=0);

F[3,3]=F[1,1]=phi(0);

F[1,3]=F[3,1]=phi(L);

F[4,3]=F[2,1]=v1(0);

F[2,3]=F[4,1]=v1(L);

Bfive(Mo=1,Mf=1);

F[1,2]=F[3,4]=phi(0);

F[3,2]=F[1,4]=phi(L);

F[2,2]=F[4,4]=v1(0);

F[4,2]=F[2,4]=v1(L);

F[5,5]= L/((E1*A1)+(E2*A2));


// Final shape function matrices

!N5 = g * F^-1 * B~;

!N = N5 + N1 + N2;              // Sandwich beam deflection shape function matrix

!N6 = gd * F^-1 * B~;
```

```
  !Nx = N6 + N3 + N4;               // First derivative of sandwich beam deflection matrix

  !Nphi= gphi*F^-1 * B~;            // Sandwich beam rotation shape function matrix

}


Elementmasses(float m, mat Mm)

{     // Compute element mass matrix [Mm] of element m due to translation [Mt],

      //  rotation [Mr] and uniform axial stretching [Ma] effects

  float x;

  mat Mt[8,8], Mr[8,8], Ma[8,8], Ma1[8,8], Ma2[8,8], Ma3[8,8];

  zero(Mm[8,8]);

  B_matrix(m);

  gausspt(n = 4, XG, XW, 0, L);                     // Exact weights

  for(i=1; i<=n; i=i+1){

  Shapes(XG[i]);

  !Mt = Mt + (XW[i]*(rho1*A1+rho2*A2)* N~*N);

  !Mr = Mr + (XW[i]*(rho1*If1+rho2*If2)*Nx~* Nx);

  !Ma1 = Ma1 + (XW[i]*(rho1*A1+rho2*A2)*Na~*Na);

  !Ma2 = Ma2 + (XW[i]*(rho1*A1*d1^2+rho2*A2*d2^2)*Nphi~*Nphi);

  !Ma3 = Ma3 + (XW[i]*2*(rho1*A1*d1-rho2*A2*d2)*Na~*Nphi);

  }

  !Ma = Ma1 +Ma2 + Ma3;

  !Mm = Mt + Mr + Ma;

}
```

```
Systemmass()

{ // setup system mass matrix [M] and impose boundary conditions

 float m;

 mat el_id[eDOF], Mm[eDOF,eDOF];

  zero(M[sDOF,sDOF]);                    // Initialize system mass [M]

  for(m=1;m<=sEl;m=m+1) {                // For each member m

    ElementID(m, el_id);                 // Get member end-displ labels

    Elementmasses(m,Mm);                 // Member mass [Mm] in global axes

    subop(Mm+M[el_id;el_id]);            // Assembled into [M]

  }

  !M2=M;                                 // Save copy in [M2]

}


EigenValue()

{    // Compute the squares of the natural frequencies

    // eigen(!v = K2, ev);                // Standard form

       eigen(!V = K2, M2, EV);           // Generalized form

       print(^,EV, V);

}
```

Appendix G **Program for calculating frequency parameter for a homogeneous beam.**

```
main()
{
  b = 0.01;   t1 = 0.0004572;  t2 = 0.0004572; c = 0.0127;    // meters
  E1 = E2 = 6.8e+10;                                            // Pa
  rho1 = rho2 = 2680; rhoc = 119.69;
  d = c+(t1+t2)/2;
  A1=b*t1;  A2=b*t2;
  EIf1=b*E1*t1^3/12;  EIf2=b*E2*t2^3/12;
  EIf=EIf1+EIf2;
  AEf = (A1*E1 * A2*E2) / (A1*E1 + A2*E2);
  EI=EIf + d^2*AEf;
  w2 = 347;                                    // Square of the natural frequency
  w = (sqrt(w2));///(2*pi#);
  L = 2.5;
  lambda = ((w*(L^2))*sqrt((2*rho1*A1) / EI))^0.5;
  //lambda = (w2*(L^4)*(rho1*A1+rho2*A2) / EI)^0.25;
  print(^,lambda);
}
```