Central Limit Theorem
for some
Classes of Dynamical Systems

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Abstract

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We consider a transformation T of the unit interval $[0,1]$ into itself which is piecewise C^2 and expanding. Using the spectral decomposition of the Frobenius-Perron operator of T, we give a proof of the Central Limit Theorem for

$$\left(\frac{1}{n}\right)\sum_{i=0}^{n-1} f \circ T^i,$$

where f is a function of bounded variation. It is also shown that the speed of covergence in the Central Limit Theorem is of the order $\frac{1}{\sqrt{n}}$.

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Chapter 1

Introduction

1.1 Aim of the Thesis

This thesis is a study of limit theorems for piecewise expanding transformation. The study is motivated by the importance of the limit theorems in the present day nonlinear dynamical systems. In many practical problems, we would like to know how fast these limit theorems converge.

1.2 Outline of the Thesis

For a class of expanding transformations of the unit interval $[0,1]$ into itself, we will prove the Central Limit Theorem for the process $(f \circ T^n)_{n\in N}$, where f is a real-valued function of bounded variation. We will also prove that the speed of convergence in the Central Limit Theorem is of order $1/\sqrt{n}$.

It has been proven by Lasota and Yorke (1973) that if $T : [0,1] \to [0,1]$ is piecewise C^2 transformation with $\inf_{x\in[0,1]} \left| T' \right| > 1$, then there exist a T-invariant measure μ which is absolutely continuous and has density h of bounded variation. Wong [13] has proven that, if (T,μ) is weakly mixing then for some positive σ and for any fixed $z \in R$,

$$\lim_{n\to\infty} \mu \left\{ \left(\frac{1}{\sigma\sqrt{n}} \right) \left(\sum_{k=0}^{n-1} f \circ T^k - n\mu(f) \right) < z \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left(-u^2/2\right)\, du.$$

where f is a function of bounded variation.

For the class of T's considered in Lasota and Yorke [10], the spectral decomposition of Frobenius-Perron operator Φ can be found in Keller [7], from which a Central Limit Theorem for stationary process $f \circ T^n$ on $([0,1],\mu)$ is deduced, if μ is weak mixing.

In Chapter 2 we give the preliminary definitions, notations and results from functional analysis and probability theory relevant to this work.

In Chapter 3, we will introduce operator $P_f(i\theta)$ and we will study the spectrum of $P_f(i\theta)$ for the values of θ in the neighborhood of 0. The basic result of this chapter is due to Rellich (see [2]), who described how the isolated point spectrum of an operator varies when this operator depends analytically on a parameter.

In Chapter 4, we will prove a version of functional Central Limit Theorem for transformation T and estimate the speed of convergence.

Chapter 2

Background Material in Functional Analysis and Probability Theory

In this chapter we will briefly review some well-known notions of functional analysis and probability theory. We have selected only the concepts that we will need in the following Chapters.

2.1 Measure space and Integration

Definition 2.1.1 Let X be a set. A $\sigma-$algebra of subsets of X is a collection $\sum$ of subsets of X satisfying:

1) $X \in \sum$;

2) $A \in \sum \Rightarrow X \backslash A \in \sum$;

and

3) $A_n \in \sum,\ n = 0, 1, ... \Rightarrow \cup_{n=0}^{\infty} A_n \in \sum$.

Definition 2.1.2 A real valued function m on a $\sigma-$algebra $\sum$ is a measure if:

1) $m(\emptyset) = 0$

2) $m(A) \geq 0$ for all $A \in \sum$;

and

3) $m(\cup_{n=0}^{\infty} A_n) = \sum_{n=0}^{\infty} m(A_n)$ if $\{A_n\}$ is a finite or infinite sequence of pairwise disjoint sets from $\sum$, that is $A_i \cap A_j = \emptyset$ for $i \neq j$.

Definition 2.1.3 If $\sum$ is a $\sigma-$algebra of subsets of X and if m is a measure on $\sum$, then the triple $(X, \sum, m)$ is called a measure space. The sets that belong to $\sum$ are called measurable sets.

Definition 2.1.4 A measure $(X, \sum, m)$ is called finite if $m(X) < \infty$. In particular, if $m(X) = 1$, then the measure space is said to be normalized or probabilistic.

Definition 2.1.5 Let $(X, \sum, m)$ be a measure space. A real-valued function $f : X \to R$ is measurable if $f^{-1}(I) \in \sum$ for every interval $I \subset R$.

Theorem 2.1.1 (Radon-Nikodym theorem) Let $(X, \sum, m)$ be a measure space and let ν be a another finite measure on $\sum$ with the property that $\nu(A) = 0$ for all $A \in \sum$ such that $m(A) = 0$. Then there exists a non-negative integrable function $f : X \to R$ such that

$$\nu(A) = \int_A f(x) \; m(dx)$$

for all $A \in \sum$.

Proposition 2.1.1 If f_1 and f_2 are integrable functions such that

$$\int_A f_1(x) \, m(dx) = \int_A f_2(x) \; m(dx)$$

for all $A \in \sum$, then $f_1 = f_2$ a.e.(almost everywhere, i.e., $m(\{x : f_1(x) \neq f_2(x)\} = 0)$).

Let $(X, \sum, m)$ be a measure space and let $L^1(X, \sum, m)$ be the space of integrable functions on $(X, \sum, m)$. For simplicity of notation we will use L^1_m instead of $L^1(X, \sum, m)$.

Corollary 2.1.1 If $(X, \sum, m)$ is a measure space and ν is a second finite measure on $\sum$ such that $\nu(A) = 0$ whenever $m(A) = 0$, then there exists a unique element $f \in L^1_m$ such that

$$\nu(A) = \int_A f(x) \; m(dx)$$

for all $A \in \sum$.

Definition 2.1.5 Let f be a function defined on $[0, 1]$. The support of a function f is

$$\text{Supp} f = \text{cl}\{x : f(x) \neq 0\},$$

where cls(A) denotes the closure of a set A.

Definition 2.1.6 Let $(X, \sum, m)$ be a measure space and let the set $D(X, \sum, m)$ be defined by

$$D\left(X, \sum, m\right) = \left\{f \in L^1_m : f \geq 0 \text{ and } \|f\|_1 = 1, \text{ where } \|f\|_1 \text{ denotes the } L^1_m \text{ norm}\right\}.$$

Any function $f \in D(X, \sum, m)$ is called a density.

Definition 2.1.7 If $f \in L_m^1$ and $f \geq 0$, then the measure

$$m_f(A) = \int_A f(x) \; m(dx)$$

is said to be absolutely continuous with respect to m and f is called the Radon-Nikodym derivative of m_f with respect to m. In the special case when $f \in D(X, \sum, m)$, we also say that f is the density of m_f and that m_f is the normalized measure.

Definition 2.1.8 Let $(X, \sum, m)$ be a measure space. A transformation $T{:}X \to X$ is measurable if $T^{-1}(A) \in \sum$ for all $A \in \sum$.

Definition 2.1.9 A measurable transformation $T : X \to X$ on a measure space $(X, \sum, m)$ is non-singular if

$$m\left(T^{-1}(A)\right) = 0$$

for all $A \in \sum$ such that $m(A) = 0$.

2.2 Frobenius-Perron operator

Consider a non-singular transformation $T : X \to X$ on a measure space

(X, Σ, m). We define $\Phi : L^1_m \to L^1_m$ in two steps:

1. Let $f \in L^1_m$ and $f \geq 0$. Consider

$$\int_{T^{-1}(A)} f(x) \; m(dx). \tag{2.2.1}$$

Since

$$T^{-1}(\cup_i A_i) = \cup_i T^{-1}(A_i),$$

it follows from the property of Lebesgue integral that the integral (2.2.1) defines a finite measure. Thus, by Corollary 2.1.1, there exists a unique element in L^1_m, which we denote by Φf, such that

$$\int_A \Phi f(x) \; m(dx) = \int_{T^{-1}(A)} f(x) \; m(dx)$$

for any $A \in \Sigma$.

2. Now let $f \in L^1_m$ be arbitrary, that is, not necessarily non-negative. Write $f = f^+ - f^-$, where $f^+ = \max(0, f)$ and $f^- = (0, -f)$, and define

$$\Phi f = \Phi f^+ - \Phi f^-.$$

Then

$$\int_A \Phi f(x)\, m(dx) = \int_{T^{-1}(A)} f^+(x)\, m(dx) - \int_{T^{-1}(A)} f^-(x)\, m(dx),$$

that is

$$\int_A \Phi f(x)\ m(dx) = \int_{T^{-1}(A)} f(x)\ m(dx), \tag{2.2.2}$$

for any $A \in \sum$. Then from Proposition 2.1.1 and the non-singularity of T, it follows that equation (2.2.2) uniquely defines Φ.

Definition 2.2.1 Let $(X, \sum, m)$ be a measure space. If $T : X \to X$ is a non-singular transformation, the unique operator $\Phi : L_m^1 \to L_m^1$ defined by the equation (2.2.2) is called the Frobenius-Perron operator corresponding to T.

It is straightforward to show from (2.2.2) that Φ has the following properties:

FP1) $\Phi(\alpha f_1 + \beta f_2) = \alpha \Phi f_1 + \beta \Phi f_2$

for all $f_1, f_2 \in L_m^1, \alpha, \beta \in R$. Thus, Φ is a linear operator;

FP2) $\Phi f \geq 0$ if $f \geq 0$;

FP3) $\int_X \Phi f(x)\ m(dx) = \int_X f m(dx)$, $f \in L_m^1$;

FP4) If $T^n = \underbrace{T \circ \cdots \circ T}_{n}$ then $\Phi_{T^n} = \Phi_T^n$, where Φ_T is the is the F-P operator corresponding to T.

Let us consider the transformation T on $[0, 1]$, which is differentiable and invertible. Then T must be monotone. Suppose T is an increasing function and

T^{-1} has a continuous derivative. Then

$$T^{-1}([a,x]) = \left[T^{-1}(a), T^{-1}(x)\right]$$

and we have

$$\begin{aligned} \Phi f(x) &= \frac{d}{dx}\int_{T^{-1}(a)}^{T^{-1}(x)} f(s)\,ds \\ &= f\left(T^{-1}(x)\right)\frac{d}{dx}\left[T^{-1}(x)\right]. \end{aligned}$$

If T is decreasing, then the sign of right hand side is reversed. Thus, in the general one-dimensional case, for T which is differentiable and invertible with continuous $\frac{dT^{-1}}{dx}$,

$$\Phi f(x) = f\left(T^{-1}(x)\right)\frac{d}{dx}\left|\left[T^{-1}(x)\right]\right|.$$

Example Let

$$T(x) = \begin{cases} -2x+1 & \text{if } 0 \le x \le \frac{1}{2} \\ 2x-1 & \text{if } \frac{1}{2} \le x \le 1 \end{cases}.$$

For any interval $[0,x]\subset[0,1]$, we have

$$T^{-1}([0,x])=\left[\frac{1-x}{2},\frac{1}{2}\right]\cup\left[\frac{1}{2},\frac{1+x}{2}\right].$$

Then

$$\begin{aligned}(\Phi_T f)(x) &= -\frac{d}{dx}\int_0^{\frac{1-x}{2}} f dx+\frac{d}{dx}\int_{\frac{1}{2}}^{\frac{1+x}{2}} f dx\\ &= \frac{1}{2}f\left(\frac{1-x}{2}\right)+\frac{1}{2}f\left(\frac{1+x}{2}\right)\\ &= \frac{1}{2}\left[f\left(\frac{1-x}{2}\right)+f\left(\frac{1+x}{2}\right)\right].\end{aligned}$$

Theorem 2.2.1 Let $T\mid_{I_j}\in C^1[a_{j-1},a_j]$ (first derivative of T exists and continuous) be monotone, $j=1,2,...,n$, where $0=a_0<a_1<\cdots<a_n<1$. Then we have

$$\Phi f(x)=\sum_j f(\sigma_j x)\,\psi_j(x)\,\chi_{J_j}(x)$$

where σ_j is the inverse of T over $J_j=T(I_j)$; $\psi_j(x)=\left|\sigma_j'(x)\right|$; χ_j is the indicator function of J_j.

Proof: Let $A_j(x) = \sigma_j([0,x]) \cap I_j$. Then

$$\int_{A_j(x)} f(s)\ ds = \pm \int_{\sigma_j(0)}^{\sigma_j(x)} f(s)\, \chi_{I_j}(s)\ ds. \tag{2.2.3}$$

We want $\int_{A_j(x)} f \geq 0$ when $f \geq 0$. $T\mid_{I_j}$ is monotone, σ_j is monotone and $T\mid_{I_j}$ and σ_j are either both increasing or both decreasing. Therefore

$$\frac{\sigma_j'(x)}{\left|\sigma_j'(x)\right|} = \frac{\sigma_j'(y)}{\left|\sigma_j'(y)\right|}$$

for all $x, y \in [0,1]$. We use this to set the sign in (2.2.3), thus

$$\int_{A_j(x)} f(s)\ ds = \frac{\sigma_j'(x)}{\left|\sigma_j'(x)\right|} \int_{\sigma_j(0)}^{\sigma_j(x)} f(s)\, \chi_{I_j}(s)\ ds.$$

This implies

$$\begin{aligned} \frac{d}{dx} \int_{A_j(x)} f(s)\, ds &= \frac{\sigma_j'(x)}{\left|\sigma_j'(x)\right|} \frac{d}{dx} \int_{\sigma_j(0)}^{\sigma_j(x)} f(s)\, \chi_{I_j}(s)\, ds \\ &= \frac{\sigma_j'(x)}{\left|\sigma_j'(x)\right|} f(\sigma_j x)\, \chi_{I_j}(\sigma_j(x))\, \sigma_j'(x) \end{aligned}$$

$$= \frac{\left|\sigma_j'(x)\right|^2}{\left|\sigma_j'(x)\right|} f(\sigma_j x)\, \chi_{I_j}(\sigma_j(x))$$
$$= f(\sigma_j x)\, \psi_j(x)\, \chi_{I_j}(\sigma_j(x)).$$

Note that

$$\begin{aligned} \chi_{I_j}(\sigma_j x) &= 1 \Leftrightarrow \sigma_j x \in I_j \\ &\Leftrightarrow x \in T(I_j) = J_j \\ &\Leftrightarrow \chi_{J_j} = 1. \end{aligned}$$

Therefore

$$\chi_{J_j}(x) = \chi_{I_j}(\sigma_j x),$$

and we have

$$\frac{d}{dx}\int_{A_j(x)} f(s)\, ds = \sum_j f(\sigma_j x)\, \Psi_j(x)\, \chi_{J_j}(x).$$

For

$$T = \sum_{i=1}^{n} T\mid_{I_j} \chi_{I_j},$$

$$\sigma([0,x]) = \cup_{j=0}^{n-1} A_j(x),$$

where $A_j's$ are disjoint since $I_j's$ are disjoint. Thus

$$\begin{aligned}\Phi f(x) &= \frac{d}{dx}\int_{\sigma([0,x])} f(s)\ ds \\ &= \frac{d}{dx}\sum_{j=0}^{n}\int_{A_j(x)} f(s)\ ds \\ &= \sum_j f(\sigma_j x)\,\psi_j(x)\,\chi_{J_j}(x).\end{aligned}$$

□

2.3 Functions of bounded variations:

In this section, we will briefly discuss some properties of an important class of functions: functions of bounded variation, which are intimately connected with monotonic functions.

Let the function $f(x)$ be defined and finite on the interval $I = [a, b]$. Subdivide $[a, b]$ into subintervals by means of points $a = x_1 < \cdots < x_n = b$ and consider the sum

$$s_n(f) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|.$$

We define the variation of f by

$$V(f) = \sup s_n(f),$$

where the supremum is taken over all finite partitions of I. If $f \in L^1_m$ then we define $V(f)$ as the

$$\inf\{V(g) : g = f \text{ a.e.}\}.$$

Some properties of functions of bounded variations on I:

BV 1) If $f_1, \cdots, f_n$ are of bounded variations on I, then

$$V(f_1 + f_2 + \cdots + f_n) \leq V(f_1) + V(f_2) + \cdots + V(f_n).$$

BV 2) If $g : [\alpha, \beta] \to I$ and $f : I \to R$, then

$$V(f \circ g) \leq V(f).$$

BV 3) If f is of bounded variation on $I = [a, b]$ and g is C^1 on $[a, b]$, then

$$V(fg) \leq (\sup|g|)\, V(f) + \int_a^b \left|f(x)\, g'(x)\right| dx.$$

BV 4) If f is a function of bounded variation on I and $[a, b] \subset I$ then

$$V\left(f\chi_{[a,b]}\right) \leq 2V(f) + \frac{2}{b-a}\int_a^b |f(x)|\, dx.$$

Proof of BV 4) Without loss of generality, assume that the partitions of the interval $[0,1]$ will always contain the points a and b. Then

$$s_n\left(f\chi_{[a,b]}\right) \le s_n(f) + |f(a)| + |f(b)| \tag{2.3.1}$$

Let c be an arbitrary point in $[a,b]$. Then from (2.3.1),

$$\begin{aligned} s_n\left(f\chi_{[a,b]}\right) &\le s_n(f) + |f(a) - f(c)| + |f(b) - f(c)| + 2|f(c)| \\ &\le 2V(f) + 2|f(c)|. \end{aligned}$$

It is always possible to choose the point c such that

$$|f(c)| \le \frac{1}{b-a}\int_a^b |f(x)|\,dx$$

so that

$$s_n\left(f\chi_{[a,b]}\right) \le 2V(f) + \frac{2}{b-a}\int_a^b |f(x)|\,dx,$$

which gives

$$V\left(f\chi_{[a,b]}\right) \le 2V(f) + \frac{2}{b-a}\int_a^b |f(x)|\,dx.$$

□

2.4 Invariant Measures, Measure Preserving Transformation, Ergodicity

Definition 2.4.1 Let $(X, \sum, \mu)$ be a measure space and $T : X \to X$ a measurable transformation. Then T is said to be measure preserving if

$$\mu\left(T^{-1}(A)\right) = \mu(A)$$

for all $A \in \sum$.

Since the property of being measure preserving is dependent on T as well as μ, we will alternatively say that the measure μ is invariant under T if T is measure preserving. Note that every measure preserving transformation is necessarily non-singular with respect to its invariant measure.

Theorem 2.4.1 Let $(X, \sum, m)$ be a measure space, let T be a non-singular transformation, and let Φ be the Frobenius-Perron operator associated with T. Consider a non-negative $f \in L^1_m$. Then a measure μ given by

$$\mu(A) = \int_A f(x)\ m(dx) \text{ for all } A \in \sum$$

is T-invariant if and only if f is a fixed point of Φ.

Proof. Assume μ is T-invariant. Then by the definition of invariant measure,

$$\mu(A) = \mu\left(T^{-1}(A)\right),$$

for all $A \in \sum$, or

$$\int_A f(x)\ m(dx) = \int_{T^{-1}(A)} f(x)\ m(dx), \tag{2.4.1}$$

for $A \in \sum$. However by the definition of F-P operator, we have

$$\int_A \Phi f(x)\, m(dx) = \int_{T^{-1}(A)} f(x)\, m(dx), \tag{2.4.2}$$

for $A \in \sum$. Comparing (2.4.1) and (2.4.2) we have by Proposition 2.1.1

$$\Phi f = f,\ m\text{-a.e.}$$

Conversely, if $\Phi f = f$ for some $f \in L^1_m$, $f \geq 0$, then the definition of the F-P operator implies equation (2.4.1) and thus μ is T-invariant.

□

Example 2.4.1 Let $(X, \sum, m)$ be a probability space, where $X = [0,1]$, $\sum$ is

Borel σ-algebra, and m is Lebesgue measure. Let $T : X \to X$ be a map defined by $T(x) = rx \,(\text{mod } 1)$, where $r \geq 2$ is an integer. Then T is measure preserving.

For any interval $[a, b] \subset [0, 1]$,

$$T^{-1}[a, b] = \cup_{i=0}^{r-1} \left[\frac{i+a}{r}, \frac{i+b}{r} \right].$$

Thus, we get

$$\begin{aligned} m\left(T^{-1}[a, b]\right) &= m\left(\cup_{i=0}^{r-1} \left[\frac{i+a}{r}, \frac{i+b}{r}\right]\right) \\ &= \sum_{i=0}^{r-1} m\left[\frac{i+a}{r}, \frac{i+b}{r}\right] \\ &= \sum_{i=0}^{r-1} \frac{b-a}{r} \\ &= m[a, b]. \end{aligned}$$

Definition 2.4.2 A transformation T is said to be ergodic if there exists no non-trivial subset of A which is invariant under T. More precisely, T is ergodic if for all $A \in \sum$ for which $T^{-1}(A) = A$, $\mu(A) = 0$ or $\mu(X \backslash A) = 0$.

As an example let us consider the rotation F on the unit circle S^1, where $F(x) = x + \theta$ and $\theta \in [0, 2\pi]$ is constant. Obviously the measure induced by the arc length is invariant under F. Depending on whether θ is rational or irrational,

F is not ergodic or ergodic respectively.

Definition 2.4.3 A transformation T is called mixing if

$$\lim_{n \to \infty} \mu\left(A \cap T^{-n}(B)\right) = \mu(A)\,\mu(B),$$

for all $A, B \in \sum$.

Roughly speaking, this condition means that if one starts with a set A of initials conditions, then after many iterations the fraction of solutions points lying in some (arbitrary given) set B equals the product of the measure of the sets A and B. Mixing is loosely called irregular or chaotic behavior.

Definition 2.4.4 A transformation T is called weakly mixing if

$$\lim_{k \to \infty} \frac{1}{k} \sum_{n=0}^{k-1} \left|\mu\left(A \cap T^{-n}(B)\right) - \mu(A)\,\mu(B)\right| = 0$$

for all $A, B \in \sum$.

Note: T is weak mixing $\Rightarrow$ T is ergodic.

2.5 Piecewise Monotonic Mapping

Definition 2.5.1 A transformation $T : [0,1] \to R$ will be called piecewise C^2 (second derivative of T exists and continuous) if there is partition of $[0,1]$, $\mathcal{P} =$

$\{(0,a_1),\cdots,(a_{r-1},1)\}$ where (a_{j-1},a_j) is an open interval, such that, for each $0<a_1<...<a_{r-1}<1$, $i=1,2,\cdots,r$, $T_{I_j}=T\mid_{(a_{j-1},a_j)}$ can be extended to the closed interval $[a_{j-1},a_j]$ as a C^2 function. T need not be continuous at the point a_j.

Theorem 2.5.1 Let $T:[0,1]\to[0,1]$ be a piecewise C^2 transformation such that $\inf|T'|>1$. Then for any $f\in L^1_m$ the sequence

$$\frac{1}{k}\sum_{n=0}^{k-1}\Phi^n f$$

is convergent in L^1_m−norm to a function $h_f\in L^1_m$. The limit function has the following properties:

1) $f\geq 0\Rightarrow h_f\geq 0$;

2) $\int_0^1 h_f dm=\int_0^1 f dm$;

3) $\Phi_T h_f=h_f$ and consequently the measure $d\mu=h_f\ dm$ is T-invariant.

4) The function h_f is of bounded variation, moreover, there exists a constant C independent of choice of initial f such that the variation of the limiting function h_f satisfies the inequality

$$V(h_f)\leq C\left\|f\right\|_1. \tag{2.5.1}$$

Proof. See Lasota and Yorke [10].

Now we will give definition and some properties of T that we will use in Chapter 3. We will consider a transformation $T : I \to I$, where $I = [0,1]$. Let m be the Lebesgue measure and L^1_m, the space of integrable functions. Consider a finite or countable sequence $\{a_j\}$ of points in I and let $I_j = (a_{j-1}, a_j)$. We assume that

(1) $T\mid_{I_j}$ is strictly monotone and can be extended on $\overline{I}_j$, where $I_j = (a_{j-1}, a_j)$ as a C^2 function;

(2) $\{T(I_j)\}$ is composed of finite number of disjoint intervals;

(3) There exists an $n \geq 1$ such that

$$\gamma = \inf_{x \in I} \left| (T^n)'(x) \right| \geq 1$$

Condition (1) allows the existence of local inverse of T and condition (3) means that T is expanding. F-P operator associated with T is equal to the operator $\Phi : L^1_m \to L^1_m$ defined by

$$\int_0^1 \Phi f \cdot g \, dm = \int_0^1 f \cdot g \circ T \, dm$$

where $f \in L^1_m$, $g \in L^\infty_m$.

This operator is a positive contraction and we have $\Phi f = f \Leftrightarrow$ measure $\mu = fm$ is invariant under T. Hypothesis (1) of T gives us an explicit form of Φ :

$$\Phi f(x) = \sum_j f(\sigma_j x)\, \psi_j(x)\, \chi_j(x),$$

where σ_j is the inverse of T over $J_j = T(I_j)$; $\psi_j(x) = \left|\sigma_j'(x)\right|$; χ_j is the indicator function J_j.

2.6 Banach space and spectral theory

In this section we will give some standard results of analysis which can be found in any standard book of Functional Analysis (see [2]).

Definition 2.6.1 A norm on a linear space X is a function $\|\cdot\| : X \to R; x \to \|x\|$ satisfying the following properties for all x, $y \in X$ and $\alpha \in R$ or C:

1.$\|x\| \geq 0$;

2.$\|x\| = 0 \Longleftrightarrow x = 0$;

3.$\|\alpha x\| = |\alpha|\, \|x\|$;

4.$\|x + y\| \leq \|x\| + \|y\|$.

The pair $(X, \|.\|)$ is then called a normed linear space.

A normed linear space $(X, \|.\|)$ is a metric space if one defines the metric $\rho(x,y) = \|x - y\|$. Thus the notions of convergent sequences and Cauchy sequences are naturally carried over to $(X, \|.\|)$.

Definition 2.6.2 A sequence $\langle x_n \rangle$ in a normed linear space X is said to converge to an element $x \in X$ given an $\varepsilon > 0$, there is an N such that $\|x_n - x\| < \varepsilon, \forall n > N$.

We write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Definition 2.6.3 A sequence $\langle x_n \rangle$ in a normed linear space X is called Cauchy sequences, if for any given $\varepsilon > 0$, there is an N such that satisfies the condition $\|x_n - x_m\| < \varepsilon$, $\forall\ m, n > N$.

Definition 2.6.4 A normed linear space X is called complete if every Cauchy sequence $\langle x_n \rangle$ in X is convergent, i.e., for every Cauchy sequence $\langle x_n \rangle$ in X, there is an element $x \in X$ such that $x_n \to x$.

Definition 2.6.5 A subset A of a Banach space is called relatively compact if its closure $\overline{K}$ is compact.

Definition 2.6.6 Let K and L be Banach spaces. An operator $\Phi : K \to L$ is called a compact linear operator if Φ is linear and if for every bounded subset K' of K, the image $\Phi\left(K'\right)$ is relatively compact, that is, the closure $\overline{\Phi\left(K'\right)}$ is compact.

From now on, we will consider bounded linear operator Φ in a Banach space K. We exclude the trivial case $K = \{0\}$.

Definition 2.6.7 The resolvent set $r(\Phi)$ of Φ is the set of complex numbers z, for which $(zId - \Phi)^{-1}$ exists (where Id denotes the identity operator on K) and is bounded on K. The spectrum $\sigma(\Phi)$ of Φ, is the complement of $r(\Phi)$. The function $R(z) = (zI - \Phi)^{-1}$, defined on $r(\Phi)$, is called the resolvent function of Φ.

Lemma 2.6.1 The resolvent set $r(\Phi)$ is open and the resolvent function $R(z)$ is analytic in $r(\Phi)$.

Corollary 2.6.1 If $d(z)$ is the distance from z to the spectrum $\sigma(\Phi)$, then $\|R(z)\| \geq \frac{1}{d(z)}$. Thus $R(z) \to \infty$ as $d(z) \to 0$, and the resolvent set is the natural domain of analyticity of $R(z)$.

Definition 2.6.8 The spectral radius $\rho(\Phi)$ of a bounded operator Φ is the radius

$$\rho(\Phi) = \sup_{z \in \sigma(\Phi)} |z| = \lim_{n \to \infty} \sqrt[n]{\|\Phi\|^n}$$

of the smallest closed disc centered at the origin of complex z-plane and containing $\sigma(\Phi)$.

Theorem 2.6.1 Let Φ, Φ_1 be two bounded operators from K to L, z be in

$r(\Phi)$ and $\|\Phi - \Phi_1\| < |R(z)|^{-1}$. Then z is in $r(\Phi_1)$ and

$$R_1(z) = R(z) \sum_{n=0}^{\infty} [(\Phi - \Phi_1) R(z)]^n ,$$

where $R_1(z)$ is the resolvent of Φ_1.

2.7 Some Properties of Characteristic functions

In this section we will briefly review some results of characteristic functions which, we will need later, can be found in any standard book of Probability Theory (see [4]).

Let $X = I$ be the unit interval, $\sum = \mathcal{B}_I$ the Borel σ-algebra and m be the Lebesgue measure. Then $(X, \sum, m)$ is a probability space. A random variable f on the probability space $(X, \sum, m)$ is a Borel measurable function $f : X \to R$ and f is said to be extended random variable iff f is a Borel measurable function from X to $\overline{R}$. If f is a measurable function on $(X, \sum, m)$, the probability measure induced by f is the measure m_f on $\sum$ given by

$$m_f(B) = m\{x : f(x) \in B\},$$

$B \in \sum$. The distribution function of a measurable function f, is the function F_f

from R to $[0,1]$ given by

$$F_f(t) = m\{x : f(x) \leq t\},$$

$t \in R$. Since, for $a < b$,

$$\begin{aligned} F_f(b) - F_f(a) &= m\{x : a < f(x) \leq b\} \\ &= m_f((a,b]), \end{aligned}$$

An important tool in the study of the measurable functions or distribution functions is the characteristic function. For any measurable function f with measure m_f and distribution function F_f, the characteristic function is defined to be h on R as follows, $\forall t \in R$,

$$h(t) = \int_X \exp(itf(x))\, dm(x).$$

$\forall t \in R$. The characteristic function has the following simple properties:

i) $\forall t \in R : |h(t)| \leq 1 = h(0); h(-t) = \overline{h(-t)}$;

ii) h is uniformly continuous on R;

iii) If $g = af + b$, where a and b are constants, then the characteristic function of f and g are connected by the equation $h_g(t) = h_f(at)\exp(ibt)$;

iv) If $\{h_n, n \geq 1\}$ are characteristic functions, $\alpha_n \geq 0$, $\sum_{n=1}^{\infty} \alpha_n = 1$, then $\sum_{n=1}^{\infty} \alpha_n h_n$ is a characteristic function;

v) If $\{h_n, 1 \leq j \leq n\}$ are characteristic functions, then $\Pi_{j=1}^{n} h_j$ is a characteristic function.

Characteristic functions are uniquely appropriate in the study of the measurable functions because of the following result.

Theorem 2.7.1 Let $f_1, f_2, ..., f_n$ be the independent measurable functions, and let $S_n = f_1 + f_2 + \cdots + f_n$. Then the characteristic function of S_n is the product of the characteristic function of the f_i's, where $i = 1, 2, ..., n$.

Note: The above theorem allows us to compute the characteristic function of S_n, knowing only the measure of the individual $f_i's$. In fact, if the characteristic function is known the distribution is determined (see [2]).

We shall now give the following theorem that makes a one-to-one correspondence between one dimensional distributions and characteristic functions. In the following we will write $F_f = F$.

Theorem 2.7.2 Let $h(t)$ and $F(x)$ be the characteristic function and distribution function of a measurable function f. If a and b are the continuity points of F , then

$$F(b) - F(a) = \frac{1}{2\pi} \int_{-c}^{c} \frac{\exp(-ita) - \exp(-itb)}{it} h(t)\, dt.$$

Theorem 2.7.3 (uniqueness) A distribution function is uniquely determined by its characteristic function.

Note: The following theorem is an important consequence of the above theorem.

Theorem 2.7.4 A necessary and sufficient condition for the convergence of a sequences $\{F_n(x)\}$ of d.f.'s to a d.f. $F(x)$ is that the sequence of corresponding c.f.'s $\{h_n(t)\}$ converges for all values of t to a function $h(t)$, continuous at $t = 0$. The limit $h(t)$ is then identical with the c.f.'s of $F(x)$ and $\{F_n(t)\}$ converges to $F(t)$ uniformly in every finite interval.

By virtue of the uniqueness theorem the values of the characteristic function

$$h(t) = \int_X \exp(itf(x))\, dm(x) = \int_X \exp(itx)\, dF(x)$$

for all t determines the distribution function $F(x)$. It is natural to expect that all other numerical characteristics of one dimensional distribution (distribution) function can be expressed in terms of its characteristic function. The most important numerical characteristics of one dimensional distributions are its moments. The first moment is given by

$$E(f) = \int_X f dm$$

is usually referred as expectation or mean. The second moment is given by

$$E(f - E(f))^2 = \sigma^2$$

and is called the variance.

The connection between the characteristic function and the moments is given by the following :

Lemma 2.7.1 If a measurable function. f has a moment of order k. then its characteristic function $h(t)$ has continuous derivatives upto and including the k-th order. Moreover

$$\alpha_s = \frac{1}{i^s}\left[\frac{d^s}{dt^s}h(t)\right]_{t=0} \quad (s = 1, 2, \cdots, k).$$

where α_s denotes the moment of order s.

Chapter 3

Frobenius-Perron Operator and its Perturbations

In this chapter we will use the powerful Ionescu-Tulcea and Marinescu Theorem discussed in [6] to obtain a useful spectral decomposition for the Frobenius-Perron operator Φ.

3.1.Operator Φ and its Spectrum

Here we will study the spectrum of Φ, where Φ is an operator on a subspace of L^1_m. Let $BV = \{f \in L^1_m : V(f) < \infty\}$, which is a linear subspace of L^1_m, but it is not closed with respect to norm $\|\cdot\|_1$. We define for $f \in V$, $\|f\|_{BV} = V(f) + \int |f|\, dm = V(f) + \|f\|_1$. It can be proved that $\|\cdot\|_{BV}$ is a norm on BV and $(BV, \|\cdot\|_{BV})$ is a Banach space and BV is dense in $(L^1_m, \|\cdot\|_1)$. The spectrum of Φ is described by the theorem of Ionescu-Tulcea and Marinescu ([6]).

Theorem 3.1.1 Given BV and L, two complex Banach spaces $BV \subset L$, with respective norms $\|\cdot\|_{BV}$ and $\|\cdot\|_L$. Suppose

(a) If $f_n \in BV, f \in L, \lim_{n\to\infty} \|f_n - f\|_L = 0$ and $\|f_n\| \leq M$, for all n,

$$\text{then } f \in BV \text{ and } \|f\|_{BV} \leq M < \infty$$

Let Φ be a bounded operator $\Phi : BV \to L$ with respect to $\|\cdot\|_{BV}$. Then, in addition, we assume that

(b) $\sup_{n\geq 0}\{\|\Phi^n f\|_L, f\in BV, \|f\|_{BV}\leq 1\}<\infty$

(c) $\exists\ n_0, 0<\alpha<1$ and $\beta<\infty$ such that

$$\|\Phi^{n_o} f\|_{BV}<\alpha\|f\|_{BV}+\beta\|f\|_L, \forall f\in K.$$

(d) If BV' is a bounded set of $(BV, \|\cdot\|_{BV})$ then $\Phi^{n_0}BV'$ is relatively compact in $(L, \|\cdot\|_L)$.

Then Φ has only finite number of eigenvalues of modulus 1: $\lambda_1, \cdots, \lambda_p$ and the corresponding eigen subspaces :

$$E_i=\{f\in L:\Phi f=\lambda_i f\},$$

$i=1,2,...,p$ are finite dimensional and contained in BV.

Operator Φ^n may be written as:

$$\Phi^n=\sum_{i=1}^{p}\lambda_i^n\Phi_i+\Psi^n,\ n\geq 1$$

where Φ_i are projections onto the proper subspace E_i, $\|\Phi_i\|_L\leq 1$ and $\Psi: L_m^1\rightarrow L_m^1$ such that $\sup_{n\geq 1}\|\Psi^n\|_L<\infty$. Also $\Phi_i\Phi_j=\Phi_j\Phi_i=0$ if $i\neq j$, $\Phi_i^2=\Phi_i$, $\Phi_i\Psi=\Psi\Phi_i=0$. Finally $\Psi(BV)\subset BV$ and Ψ has spectral radius $\rho(\Psi)<1$ in $(BV, \|\cdot\|_{BV})$.

Proposition 3.1.1 If the mapping T satisfies $(1),(2)$ and (3) (see page 22

) then Φ satisfies hypothesis of Theorem 3.1.1.

Proof (a) It follows since for every $C > 0$ the set

$$\{f \in L_m^1 : \|f\|_{BV} < C\},$$

is relatively compact in L_m^1.

(b) It follows since $\|\Phi^n f\|_1 = \|\Phi(\Phi^{n-1} f)\|_1 \le \|\Phi^{n-1} f\|_1 \le \|f\|_1$.

(c) Lasota and Yorke [10] has proved that for $f \in K$, $\exists\, n_0$ such that

$$V(\Phi^{n_0} f) \le \alpha V(f) + \beta \|f\|_1$$

where $0 < \alpha < 1, 0 < \beta < \infty$ are independent of f. First of all if T satisfies (1) and (2) then for all n, T^n satisfies (1) and (2). Let $f \in BV$ and let us go back to the proof of Lasota and Yorke [10]. Let us choose N such that $\gamma^N > 2$. Also let $S = T^{nN}$ satisfy (1) and (2). The F-P operator associated with S is Φ^{nN}, which is given by

$$\Phi^{nN} f(x) = \textstyle\sum_j f(\sigma_j x)\, \psi_j(x)\, \chi_j(x),\ |\psi_j(x)| \le \gamma^{-N}.$$

We have,

$$V\left(\Phi^{nN} f\right) \;\le\; \sum_j V\left((f \circ \sigma_j)\, \psi_j \chi_j\right)$$

$$\leq \sum_j V_{J_j}\left(\left(f\circ\sigma_j\right)\psi_j\right)$$
$$+ \sum_j \left(\left|\left(f\circ\sigma_j\right)\left(Ta_{j-1}\right)\psi_j\left(Ta_{j-1}\right)\right| + \left|\left(f\circ\sigma_j\right)\left(Ta\right)\psi_j\left(Ta_j\right)\right|\right)$$

Now, let $g = \left(f\circ\sigma_j\right)\psi_j$ be a function of bounded variation. Then we have

$$|g(x)| + |g(y)| \leq V_{(x,y)}(g) + \left(\frac{2}{(y-x)}\right)\int_x^y |g|\,dm$$

Therefore,

$$V\left(\Phi^{nN} f\right) \leq 2\sum_j V_{J_j}\left(\left(f\circ\sigma_j\right)\psi_j\right) + \sum_j \left(2/m\left(J_j\right)\right)\int_{I_j}|f|\,dm.$$

Using (2), there exists a $\delta > 0$, $\delta = \min_j m\left(J_j\right)$such that

$$\sum_j \left(2/m\left(J_j\right)\right)\int_{I_j}|f|\,dm \leq (2/\delta)\,\|f\|_1$$

and we have

$$V_{J_j}\left(\left(f\circ\sigma_j\right)\psi_j\right) \quad = \quad \int_{J_j}\left|d\left(f\circ\sigma_j\right)\psi_j\right|$$

$$\leq \int_{J_j} |f \circ \sigma_j| \left|\psi_j'\right| dm + \int_{J_j} \psi_j \left|d\left(f \circ \sigma_j\right)\right|$$
$$\leq K \int_{J_j} |f \circ \sigma_j| \psi_j \, dm + \gamma^{-N} \int_{J_j} \left|d\left(f \circ \sigma_j\right)\right|,$$

where $K = \left(\max\left|\psi_j'(x)\right| / \min \psi_j(x)\right)$.

The constant K is finite which is obvious when the partition associated with T is finite where T is piecewise C^2. When the partition is countable, condition (2) allows us to come to the same result. Therefore by changing the variable we have,

$$V_{J_j}\left(\left(f \circ \sigma_j\right) \psi_j\right) \leq K \int_{I_j} |f| \, dm + \gamma^{-N} \int_{I_j} |df| \, .$$

Then we obtain,

$$V\left(\Phi^{nN} f\right) \leq \left(2/\gamma^N\right) V(f) + (K + 2/\delta) \|f\|_1$$

Hence

$$V\left(\Phi^{nN} f\right) \leq \alpha V(f) + \beta \|f\|_1,$$

where $\beta = K + 2/\delta$ and $\alpha = 2\gamma^{-N} < 1$.

By definition,

$$\begin{aligned}\left\|\Phi^{nN}f\right\|_{BV} &= V\left(\Phi^{nN}f\right)+\left\|\Phi^{nN}f\right\|_1\\ &\leq \left(2/\gamma^N\right)V(f)+(k+2/\delta)\|f\|_1+\left\|\Phi^{nN}f\right\|_1\\ &\leq \left(2/\gamma^N\right)\|f\|_{BV}+(k+2/\delta)\|f\|_1+\|f\|_1\\ &= \left(2/\gamma^N\right)\|f\|_{BV}+(1+k+2/\delta)\|f\|_1\end{aligned}$$

and since $\gamma^N > 2$, the result follows.

(d) It follows from (a) and since $\left\|\Phi^{nN}f\right\|_{BV} \leq \left(2/\gamma^N\right)\|f\|_{BV}+(1+k+2/\delta)\|f\|_1$.

□

3.2 Existence of Invariant Measure and the description of the operator P induced by T

Set $h_n = (1/n)\sum_{k=0}^{n-1}\Phi^k(1)$, where 1 denotes here the constant function equal everywhere to 1. Then $h_n \geq 0$, since Φ is a positive operator and $\int h_n dm = 1$ because of $m(\Phi f) = m(f)$. Using Theorem 3.1.1 and evaluating the geometric series it follows that h_n converges in L_m^1. The limit function is invariant under Φ and has integral 1. Thus 1 is the eigen-value of Φ, say $\lambda_1 = 1$, and it follows from Theorem 3.1.1, that h_n converges to $\Phi_1(1) = h$. Now we will assume T is weakly mixing. Then μ is T-invariant, which follows from the following equalities, because $\Phi h = h$:

$$\mu(f \circ T) = m(h \cdot (f \circ T)) = m(\Phi(h \cdot (f \circ T))) = m(f \cdot \Phi h) = m(fh),$$

$$\mu(f) = m(fh).$$

Therefore $\mu = hm$ is a probability measure, which is invariant under T and $\lambda_1 = 1$, is the eigenvalue of Φ. Then 1 is the only eigen-value and T ergodic. Indeed,

$$\begin{aligned}
\Phi^n f &= \sum_{i=1}^{p} \lambda_i^n \Phi_i f + \Psi^n f \\
&= \lambda_1^n \Phi_1 f + \Psi^n f \\
&= (1)^n \Phi_1 f + \Psi^n f \\
&= \Phi_1 f + \Psi^n f
\end{aligned}$$

where $\Phi_1 f = m(f)\, h$

Notice that (T, μ) is weakly mixing $\Leftrightarrow$ (T^n, μ) is weakly mixing $\forall\, n$.

Example 3.2.1. The continuous fraction transformation (or Gauss transformation): $T : [0, 1) \longrightarrow [0, 1)$ is given by

$$T(x) = \begin{cases} \left\{\frac{1}{x}\right\}, x \neq 0 \\ 0, \ x = 0 \end{cases} .$$

where $\left\{\frac{1}{x}\right\}$ denotes the fractional part of $\frac{1}{x}$. Then F-P operator is given by

$$\Phi f(x) = \sum_{j=1}^{\infty} f(1/(j+x))(1/(j+x))^2$$

The condition (3) for $n = 2$ is given by:

$$\inf \left| \left(T^2\right)'(x) \right| = \frac{9}{4}.$$

Transformation T possesses an invariant measure μ (see [10]), that is,

$$\mu(E) = \int_E h(x)\,dx$$

where

$$h(x) = \frac{1}{\log 2 \ (1+x)}$$

and

$$\mu\left(T^{-1}(E)\right) = \mu(E).$$

Since $\gamma = 9/4 > \sqrt{2}$, (T, μ) is weakly mixing (see [1]).

In the following we will assume that T satisfies $(1), (2)$, and (3) (see page 22) and that (T, μ) is weakly mixing. Now we will introduce a lemma which will give a description P induced by T.

Proposition 3.2.1 The operator P, defined by

$$Pf = \Phi(fh)/h$$

is the operator in BV and it is a positive contraction L^1_μ.

Proof Since Φ is a positive contraction in L^1_m,

$$\begin{aligned} \|Pf\|_{1,\mu} &= \int_0^1 |\Phi(fh)|\, dm \\ &\leq \int_0^1 \Phi(|f|\, h)\ dm \\ &= \int_0^1 |f|\, h\ dm \\ &= \|f\|_{1,\mu}. \end{aligned}$$

□

Remark: As $\Phi^n(fh) = m(fh)\,h + \Psi^n(fh)$, we have

$$P^n = \mu + Q^n, \ \forall\, n \geq 1$$

where the spectral radius of Q in BV, $\rho(Q) < 1$

Proposition 3.2.2 The operator P, defined by Proposition 3.2.1 satisfies the hypothesis of Theorem 3.1.1. In particular, there exists an n_0 such that

$$\|P^{n_0} f\|_K \leq \alpha \|f\|_{BV} + \beta \|f\|_{1,\mu}$$

where $0 < \alpha < 1, \beta < \infty$ and

$$\|f\|_{1,\mu} = \int_0^1 |f|\, d\mu.$$

Proof: (a) It follows since for every $c > 0$ the set $\{f \in L_m^1 : \|f\|_{BV} < c\}$ is relatively compact in L_m^1.

(b) For $1/h \in BV$,

$$V(1/h) \leq \frac{1}{D^2} V(h),$$

where $D > 0$ and $D \leq h(x) \leq 1/D$ (see [8]) and Φ is a bounded operator on

BV. Thus P is also a bounded operator on BV.

(c) Using the proof of Proposition 3.1.1 and $\|fg\|_{BV} \leq 2\|f\|_{BV}\|g\|_{BV}$ we have

$$\begin{aligned}
\left\|P^{nN} f\right\|_{BV} &= \left\|\Phi^{nN}(fh)/h\right\|_{BV} \\
&\leq 2\|1/h\|_{BV}\left\|\Phi^{nN}(fh)\right\|_{BV} \\
&\leq \left(8/\gamma^{N}\right)\|1/h\|_{BV}\|h\|_{BV}\|f\|_{BV} \\
&\quad +2\|1/h\|_{BV}(K+2/\delta+1)\|f\|_{1,\mu}.
\end{aligned}$$

(d) It follows from (a) and (c).

□

3.3 Spectrum of $\mathbf{P}_f(i\theta)$

Now we introduce an operator $P_f(i\theta)$. Suppose $f \in BV$ has real values and $\theta \in R$. Define

$$P_f(i\theta)(g) = P(\exp(i\theta f)g).$$

Let

$$S_0 f = 0.$$

and

$$S_n f = \sum_{k=0}^{n-1} f \circ T^k, n \geq 1.$$

Now we have the following lemma which will help us in the study of the characteristic function of $S_n f$ (see Chapter 4) through iteration of $P_f\,(i\theta)$ and the spectrum of $P_f\,(i\theta)$.

Lemma 3.3.1 For $\theta \in R$, $P_f^n\,(i\theta)\,(g) = P^n\,(\exp\,(i\theta S_n f)\,g)$, where $f \in L_m^1, g \in L_m^\infty$ and $n \geq 0$.

Proof We have,

$$\begin{aligned} P^n(\exp\,(i\theta S_n f)\,g) &= P\left(P^{n-1}\left(\exp(i\theta f \circ T^{n-1})\exp\,(i\theta S_{n-1} f)\,g\right)\right) \\ &= P\left(\exp\,(i\theta f) \cdot P^{n-1}\,(\exp\,(i\theta S_{n-1} f)\,g)\right) \\ &= P_f\,(i\theta)\left(P^{n-1}\,(\exp\,(i\theta S_{n-1} f)\,g)\right) \\ &= P_f^n\,(i\theta)\,(g)\,. \end{aligned}$$

since $P^n\,(f \circ T^n \cdot g) = f \cdot P^n g$, $\forall\; n \geq 1$.

□

In the following, we will study the spectrum of $P_f(i\theta)$ for all values of θ in the neighborhood of 0. One type of result is due to Rellich (see [2]) who described how the isolated point spectrum of an operator varies when this operator depends analytically on a parameter.

Proposition 3.3.1 For all $\theta \in R$, the operator $P_f(i\theta)$ is a continuous operator on BV (as well as on L^1_μ) and the function $\theta \to P_f(i\theta)$ is analytic.

Proof Using definition of P and since $\|fg\|_{BV} \leq 2\|f\|_{BV}\|g\|_{BV}$., we have $\|P_f(i\theta)(g)\|_{BV} = \|P(\exp(i\theta f)g)\|_{BV} \leq 2\|P\|_{BV}\|\exp(i\theta f)\|_{BV}\|g\|_{BV}$.

We have

$$\begin{aligned}
\|\exp(i\theta f)\|_{BV} &= V(\exp(i\theta f)) + 1 \\
&\leq V(\cos\theta f) + V(\sin\theta f) + 1 \\
&\leq 2|\theta| V(f) + 1.
\end{aligned}$$

Thus, we have

$$\|P_f(i\theta)(g)\|_{BV} \leq C(\theta)\|g\|_{BV}.$$

Also

$$\|P_f(i\theta)(g)\|_{1,\mu} \leq \|\exp(i\theta f)g\|_{1,\mu} = \|g\|_{1,\mu}.$$

We have $P_f(i\theta) = P(\exp(i\theta f)g) = \sum_{n=0}^{\infty} ((i\theta)^n/n!)\ P(f^n \cdot g)$.

Then the series $\sum_{n=0}^{\infty} ((i\theta)^n/n!)\ P(f^n \cdot g)$ is absolutely convergent in BV since

$$(|\theta|^n/n!)\,\|P(f^n \cdot g)\|_{BV} \leq ((2|\theta|)^n/n!)\,\|P\|_{BV}\,\|f\|_{BV}^n\,\|g\|_{BV},$$

and therefore $\theta \mapsto P_f(i\theta)$ is analytic.

□

Proposition 3.3.2 There exists a real number $a > 0$ such that for $|\theta| < a$ we have

1) For $g \in K$ and $n \geq 1$

$$P_f^n(i\theta)(g) = \lambda^n(i\theta)\,N_1(i\theta)(g) + P_2^n(i\theta)(g)$$

where $\lambda(i\theta)$ is the unique eigenvalue of biggest modulus of $P_f(i\theta)$ and $|\lambda(i\theta)| > (2+\rho(Q))/3$. $N_1(i\theta)$ is the projection onto the subspace E_θ of dimension 1, corresponding to $\lambda(i\theta)$. $P_2^n(i\theta)$ is an operator on BV of spectral radius

$$\rho(P_2^n(i\theta)) \leq ((1+\rho(Q))/3)^n$$

and

$$P_2^n(i\theta)\,E_\theta = 0.$$

2) Mappings $\theta \mapsto \lambda(i\theta)$, $\theta \mapsto N_1(i\theta)$, $\theta \mapsto P_2(i\theta)$ are analytic

3)

$$\|P_2^n(i\theta)(1)\|_{BV} \leq C\,|\theta|\,((1+\rho(Q))/3)^n,$$

where C is a positive constant.

Before we prove the proposition, we need the following.

Lemma 3.3.2 $R(z)$ is the resolvent of P in BV defined by

$$\begin{aligned} R(z) &= 1/(zId - P) \\ &= \mu/(z-1) + \sum_{n=0}^{1} Q^n/z^{n-1} \end{aligned}$$

which is defined if $|z| > \rho(Q)$ and $z \neq 1$ (Id denotes the identity generator on BV).

Proof.

$$\begin{aligned} R(z) &= 1/(zI-P) = \frac{(zI-P)+P}{z}R(z) \\ &= \frac{I}{z} + \frac{P}{z}R(z) = \frac{I}{z} + \frac{P}{z}\cdot\frac{1}{zI-P} \end{aligned}$$

$$
\begin{aligned}
&= \frac{I}{z} + \frac{P}{z} \cdot \frac{1}{z} \cdot \frac{1}{\left(I - \frac{P}{z}\right)} \\
&= \frac{I}{z} + \frac{P}{z^2}\left[I + \frac{P}{z} + \frac{P^2}{z^2} + \frac{P^3}{z^3} + \cdots\right] \\
&= \frac{I}{z} + \frac{P}{z^2} + \frac{P^2}{z^3} + \frac{P^3}{z^4} + \cdots \\
&= \frac{\mu + I}{z} + \frac{\mu + Q}{z^2} + \frac{\mu + Q^2}{z^3} + \frac{\mu + Q^3}{z^4} + \cdots \\
&= \frac{\mu}{z} + \frac{\mu}{z^2} + \frac{\mu}{z^3} + \frac{\mu}{z^4} + \cdots \\
&\quad + \frac{I}{z} + \frac{Q}{z^2} + \frac{Q^2}{z^3} + \frac{Q^3}{z^4} + \cdots \\
&= \frac{\mu}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right] + \sum_{n=0}^{\infty} \frac{Q^n}{z^{n+1}} \\
&= \frac{\mu}{z}\left(1 - \frac{1}{z}\right)^{-1} + \sum_{n=0}^{\infty} \frac{Q^n}{z^{n+1}} \\
&= \frac{\mu}{z-1} + \sum_{n=0}^{\infty} \frac{Q^n}{z^{n+1}}.
\end{aligned}
$$

Proof of Proposition 3.3.2

Here is brief reminder of certain facts of spectral theory:

(1) Let us define

$$R_{i\theta}(z) = \sum_{n=0}^{\infty} \left((P_f(i\theta) - P)R(z)\right)^n .$$

If $\|P_f(i\theta) - P\|_{BV} < 1/\|R(z)\|_{BV}$, then above series converges absolutely in BV and defined the resolvent of $P_f(i\theta)$.

(2) Let I_1 and I_2 be circles with centres 1 and 0 and radii $\rho_1 = (1 - \rho(Q))/3$ and $\rho_2 = (1 + 2\rho(Q))/3$ respectively. Let $\delta > 0$ be such that $\delta < \rho_1$ and $\rho(Q) + \delta < \rho_2$. Let us define $M_\delta = \sup \|R(z)\|_{BV}$, where the supremum is taken over $|z| > \rho(Q) + \delta$ and $|z - 1| < \delta$. If $\|P_f(i\theta) - P\|_{BV} < 1/M_\delta$ then circles I_1 and I_2 are in the resolvent set of $P_f(i\theta)$. Then the projections are:

$$
\begin{aligned}
N_1(i\theta) &= (1/2\pi i) \int_{I_1} R_{i\theta}(z)\, dz \\
N_2(i\theta) &= (1/2\pi i) \int_{I_2} R_{i\theta}(z)\, dz
\end{aligned}
$$

For $\|N_1(i\theta) - \mu\|_{BV} < 1$, the image E_θ of N_1 is of dimension 1 and we have

$$
P_f(i\theta) N_1(i\theta)(g_\theta) = N_1(i\theta) P_f(i\theta)(g_\theta) = \lambda(i\theta)(g_\theta),
$$

where $g_\theta \in K$ generates E_θ. Therefore, for all $n \geq 1$, we have

$$
\begin{aligned}
P_f^n(i\theta) &= P_f^n(i\theta) N_1(i\theta) + P_f^n(i\theta) N_2(i\theta) \\
&= \lambda^n(i\theta) N_1(i\theta) + P_2^n(i\theta),
\end{aligned}
$$

where

$$P_2^n(i\theta) = (1/2\pi i)\int_{I_2} z^n R_{i\theta}(z)\,dz.$$

(3) For $|\theta| < a$, we have:

$$R_{i\theta}(z) = R(z) + i\theta R_{i\theta}^{(1)}(z)$$

where

$$\begin{aligned} P_2^n(i\theta)(1) &= (1/2\pi i)\int_{I_2} z^n R(z)(1)\,dz + (\theta/2\pi)\int_{I_2} z^n R_{i\theta}^{(1)}(z)(1)\,dz \\ &= (\theta/2\pi)\int_{I_2} z^n R_{i\theta}^{(1)}(z)(1)\,dz. \end{aligned}$$

Now,

$$\|P_2^n(i\theta)(1)\|_{BV} \leq C|\theta|\rho_2^n;\ \rho_2 = \frac{1+\rho(Q)}{3}$$

where

$$C = (1/2\pi)\sup_{|z|=\rho_2, |\theta|<a}\left\|R_{i\theta}^{(1)}(z)\right\|_{BV}.$$

□

Proposition 3.3.3 The operator $P_f(i\theta)$ defined by Lemma 3.3.1 satisfies the hypothesis of Theorem 3.1.1.

Proof

(a) It follows from the proof (a) of Proposition 3.2.1.

(b) It follows from the proof (b) of Proposition 3.2.1.

(c) Using Lemma 3.1.1, we have

$$\begin{aligned}
\left\|P_f^{nN}(i\theta)(g)\right\|_{BV} &= \left\|P^{nN}(\exp i\theta S_{nN}f)(g)\right\|_{BV} = \left\|\Phi^{nN}\left((\exp i\theta S_n f)\, gh\right)/h\right\|_{BV} \\
&\leq 2\,\|1/h\|_{BV}\left\|\Phi^{nN}\left((\exp i\theta S_n f)\, gh\right)\right\|_{BV} \\
&\leq \left(16/\gamma^N\right)\|1/h\|_{BV}\,\|h\|_{BV}\,\|\exp(i\theta S_n f)\|_{BV}\,\|g\|_{BV} \\
&\quad +2\,\|1/h\|_{BV}\,(K+2/\delta+1)\,\|g\|_{1,\mu},
\end{aligned}$$

where $\gamma = \inf_x (T^n)'(x)$ and

$$\begin{aligned}
\|\exp i\theta S_{nN}f\|_{BV} &= V(\exp i\theta S_{nN}f)+1 \\
&\leq 2\,|\theta|\,V(S_{nN}f)+1
\end{aligned}$$

$$\leq 2|\theta| \sum_{k=0}^{nN-1} V(f \circ T^n) + 1$$
$$\leq 2nN|\theta| V(f) + 1.$$

Therefore, $\forall \theta \in R$, $\exists\, n_0 = nN_0$ such that

$$\left(16/\gamma^{N_0}\right) \|1/h\|_{BV} \|h\|_{BV} (2nN_0 |\theta| V(f) + 1) < 1.$$

Then the result follows.

(d) It follows from (a) and (c).

$\square$

Chapter 4

Central Limit Theorem for piecewise Expanding Transformation

4.1 Central Limit Theorem

We will consider only functions $f \in BV$, such that: (4.1.1) the functional equation

$$f = k + \phi \circ T - \phi$$

has no solution $\phi \in BV$, $k \in R$. We consider such assumptions unless otherwise $\sigma^2 = 0$ (see **Remark 4.1.1** for more detail).

Theorem 4.1.1 Suppose $T : I \to I$ satisfies $(1), (2), (3)$ [see page 22] and that the dynamical system (T, μ) is weakly mixing. Suppose the condition (4.1.1) is satisfied. Then, we have

$$\sigma^2 = \lim_{n\to\infty} \int_0^1 \left(\frac{(S_n f - n\mu(f))}{\sqrt{n}} \right)^2 d\mu > 0$$

and $\forall z \in R$,

$$\lim_{n\to\infty} \left\{ \mu \left(\frac{(S_n f - n\mu(f))}{\sigma\sqrt{n}} \right) \leq z \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left(-u^2/2\right) du$$

where

$$S_0 f = 0.$$

$$S_n f = \sum_{k=0}^{n-1} f \circ T^k, n \geq 1$$

The proof of the theorem is given in a series of lemmas. Using Lemma 3.3.1 and Proposition 3.3.2, for $|\theta| < a$, we have

$$\begin{aligned} \int_0^1 \exp(i\theta S_n f)\, d\mu &= \int_0^1 P_f^n(i\theta)(1)\, d\mu \\ &= \lambda^n(i\theta) \int_0^1 N_1(i\theta)(1)\, d\mu + \int_0^1 P_2^n(i\theta)(1)\, d\mu. \end{aligned}$$

Lemma 4.1.1 Under the assumptions of Theorem 4.1.1:

$$\lambda'(0) = \mu(f).$$

Proof We have

$$\int_0^1 \exp((it/n) S_n f)\, d\mu = \int_0^1 P_f^n(it/n)(1)\, d\mu$$

Using Proposition 3.3.2, for n sufficiently large, we have

$$\int_0^1 \exp\left(\left(it/n\right) S_n f\right) d\mu = \left(\lambda\left(it/n\right)\right)^n \int_0^1 N_1\left(it/n\right)(1)\, d\mu + \int_0^1 P_2^n\left(it/n\right)(1)\, d\mu$$

and

$$\left|\int_0^1 P_2^n\left(it/n\right)(1)\, d\mu\right| \leq \left\|P_2^n\left(it/n\right)(1)\right\|_{BV} \leq C\left(\left|t\right|/n\right) \rho_2^n.$$

We also have

$$N_1\left(it/n\right) = \mu + \left(it/n\right) N_1^{(1)} - \left(t^2/2n^2\right) N_1^{(2)} + t^2/n^2 \overline{N_1}\left(it/n\right)$$

where $N_1^{(1)}, N_1^{(2)}$ and $\overline{N_1}\left(it/n\right)$ are bounded in BV, with

$$\lim_{n\to\infty} \left\|\overline{N_1}\left(it/n\right)\right\|_{BV} = 0.$$

Then, we have

$$\lim_{n\to\infty} \int_0^1 N_1\left(it/n\right)(1)\, d\mu = 1$$

and

$$\lambda\left(it/n\right) = 1 + \left(it/n\right)\lambda'(0) - \left(t^2/2n^2\right)\lambda''(0) + \left(t^2/n^2\right)\overline{\lambda}\left(it/n\right),$$

where

$$\lim_{n\to\infty} \overline{\lambda}(it/n) = 0.$$

Then,

$$\lim_{n\to\infty} (\lambda(it/n))^n = \exp\left(it\lambda'(0)\right).$$

On the other hand,

$$\lim_{n\to\infty} (1/n) S_n f = \mu(f)$$

almost everywhere. So for all $t \in R$, we have

$$\exp\left(it\lambda'(0)\right) = \exp(it\mu(f)).$$

□

Without loss of generality and to simplify the calculations, from now on, we will assume $\mu(f) = 0$.

Lemma 4.1.2 Under the assumptions of Theorem 4.1.1:

$$\lambda''(0) = \lim_{n\to\infty} \int_0^1 \left(\frac{S_n f}{\sqrt{n}}\right)^2 d\mu.$$

Proof Notice that we have

$$\partial^2/\partial t^2 \left\{ \int_0^1 \exp\left(\left(\frac{it}{\sqrt{n}} \right) S_n f \right) d\mu \right\}_{t=0} = - \int_0^1 \left(\frac{S_n f}{\sqrt{n}} \right)^2 d\mu.$$

Using Proposition 3.3.2, we have

$$\int_0^1 \exp\left(\left(\frac{it}{\sqrt{n}} \right) S_n f \right) \; d\mu = \lambda^n \left(\frac{it}{\sqrt{n}} \right) \int_0^1 N_1 \left(\frac{it}{\sqrt{n}} \right) (1) \; d\mu + \int_0^1 P_2^n \left(\frac{it}{\sqrt{n}} \right) (1) \; d\mu.$$

We also have

$$P_2^n \left(\frac{it}{\sqrt{n}} \right) (1) = (1/2\pi i) \int_{I_2} z^n R_{it/\sqrt{n}} (z) \, (1) \, dz.$$

Then, for sufficiently large n and $|z| = \rho_2$, we can write $R_{it/\sqrt{n}}(z)$ in the form:

$$R_{it/\sqrt{n}} (z) = R(z) + \left(\frac{it}{\sqrt{n}} \right) R^{(1)}(z) - (t^2/2n) R^{(2)}(z) + (t^2/n) \overline{R}_{it/\sqrt{n}}(z)$$

where $R^{(1)}(z)$, $R^{(2)}(z)$ and $\overline{R}_{it/\sqrt{n}}(z)$ are bounded operators of BV and

$$\lim_{n \to \infty} \left\| \overline{R}_{it/\sqrt{n}}(z) \right\|_{BV} = 0.$$

Therefore, we have:

$$P_2^n\left(\frac{it}{\sqrt{n}}\right)\ (1) = \left(\frac{t}{2\pi i\sqrt{n}}\right)\int_{I_2} z^n R^{(1)}(z)(1)\,dz - \left(t^2/4\pi in\right)\int_{I_2} z^n R^{(2)}(z)(1)\ dz$$
$$- \left(t^2/2\pi in\right)\int_{I_2} z^n \overline{R}_{it/\sqrt{n}}(z)\ dz,$$

where

$$\partial^2/\partial t^2\left\{\left(\int_0^1 P_2^n\left(\frac{it}{\sqrt{n}}\right)(1)\right)\ d\mu\right\}_{|t=0} = (-1/2\pi in)\int_{I_2} z^n R^{(2)}(z)(1)\ dz.$$

Using the definition of $\lambda^n(it/n)$ and $N_1(it/\sqrt{n})$, we obtain

$$\left(\partial^2/\partial t^2\right)\left(\lambda^n\left(\frac{it}{\sqrt{n}}\right)\right)\ \int_0^1 N_1\left(\frac{it}{\sqrt{n}}\right)(1)\,d\mu_{|t=0} = \lambda''(0) - (1/n)\,N_1^{(2)}(1).$$

Therefore the limit of

$$\int_0^1 (S_n f/n)^2 d\mu$$

exists and equals to $\lambda''(0)$.

□

Lemma 4.1.3 Let

$$\sigma^2 = \lim_{n\to\infty}\int_0^1\left(\frac{S_n f}{\sqrt{n}}\right)^2\ d\mu.$$

Then we have the following representation of σ^2:

$$\sigma^2 = \int_0^1 \left(P\left(g^2\right) - (Pg)^2\right) d\mu, \text{ where } g = (I-P)^{-1} f.$$

Proof We have

$$\begin{aligned}
\sigma^2 &= \sum_{k=-\infty}^{\infty} \int_0^1 f \cdot f \circ T^{|k|} \, d\mu \\
&= \sum_{k=-\infty}^{\infty} \int_0^1 P^{|k|} f \cdot f \, d\mu \\
&= \sum_{k=-\infty}^{\infty} \int_0^1 Q^{|k|} f \cdot f \, d\mu \\
&= \int_0^1 (2g - f) \, f \, d\mu,
\end{aligned}$$

where $g = \sum_{k=0}^{\infty} Q^k f$.

If we define

$$g = \sum_{k=0}^{\infty} Q^k f \; = \sum_{k=0}^{\infty} P^k f \; = (I-P)^{-1} f,$$

then we have

$$\sigma^2 = \int_0^1 (g + Pg)(g - Pg) \, d\mu$$

$$= \int_0^1 P\left(g^2\right) - (Pg)^2 \, d\mu$$

□

Lemma 4.1.4 We have

$$\lim_{n\to\infty} \int_0^1 P_f^n\left(\frac{it}{\sqrt{n}}\right)(1)\ d\mu = \exp\left(-t^2\sigma^2/2\right).$$

Proof It follows from the proof of Lemma 4.1.1 with $\lambda'(0) = 0$ and where it/n replaced by $it/\sqrt{n}$.

□

Lemma 4.1.5 (Fortet and Leonov) $\sigma^2 > 0$ iff f is not of the form of $f = \phi \circ T - \phi$ where $\phi \in K$.

Proof The constant k of Theorem 4.1.1 is equal to $\mu(f)$. Here we have $k = 0$. By Lemma 4.2.3, $\sigma^2 = 0$ iff $Pg^2 = (Pg)^2$ almost everywhere either

$$\Phi\left(g^2 h\right)\Phi(h) = (\Phi(gh))^2$$

or,

$$\left(\sum_j g^2(\sigma_j(x))\, h(\sigma_j(x))\, \Phi_j(x)\, \chi_j(x)\right)\left(\sum_j h(\sigma_j(x))\, \Phi_j(x)\, \chi_j(x)\right)$$

$$= \left(\sum_j \left(g(\sigma_j(x))\, h^{1/2}(\sigma_j(x))\, \Phi_j^{1/2}(x)\, \chi_j(x)\right)\left(h^{1/2}(\sigma_j(x))\, \Phi_j^{1/2}(x)\, \chi_j(x)\right)\right)^2$$

where $g(\sigma_j(x)) = u(x)$ almost everywhere in J_j and u is a function independent of j. Indeed, if we have equality in the Cauchy's inequality then the terms are proportional and we have

$$f(x) = g(x) - Pg(x) = g(x) - g(\sigma_j(x))$$

for all j.

Now for fixed j, there exists one $y \in I_j$ such that $Ty = x$. As $g(\sigma_j(x))$ is independent of j , we have

$$f(Ty) = g(Ty) - g(y)$$

in BV. But also we have

$$f = (g - f) \circ T - (g - f) = \phi \circ T - \phi.$$

□

Proof of Theorem 4.1.1 Using Lemma 4.1.1 and 4.1.4, we have

$$\lim_{n \to \infty} \int_0^1 \exp\left(\frac{it}{\sqrt{n}} S_n f\right) d\mu = \exp\left(\frac{-t^2\sigma^2}{2}\right).$$

Let $t = \frac{t'}{\sigma}$, then

$$\lim_{n \to \infty} \int_0^1 \exp\left(\frac{it'}{\sigma\sqrt{n}} S_n f\right) d\mu = \exp\left(\frac{-t'^2}{2}\right).$$

However, this limit is the characteristic function of $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left(-u^2/2\right) du$ and then

$$\lim_{n \to \infty} \left\{\mu\left(\frac{(S_n f - n\mu(f))}{\sigma\sqrt{n}}\right) \leq z\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left(-u^2/2\right) du.$$

□

Remark 4.1.1 If ϕ is a measurable solution of the functional equation

$$f = \phi \circ T - \phi$$

Then we have

$$S_n f = \phi \circ T^n - \phi.$$

Given $C > 0$, then

$$\mu\left(\left|\phi \circ T^n / \sqrt{n}\right| > C\right) = \mu\left(\left|\phi / \sqrt{n}\right| > C\right)$$

and therefore $S_n f / \sqrt{n} = (\phi \circ T^n - \phi) / \sqrt{n}$ tends to zero in probability. As $S_n f / \sqrt{n} \to 0$ in probability is equivalent to $\sigma^2 = 0$ if exists, by Lemma 4.1.5, $\phi_1 \in BV$ such that $f = \phi_1 \circ T - \phi_1$. The transformation is ergodic, $\phi - \phi_1$ is constant and is equivalent saying that the functional equation has no measurable solution or solution in BV.

4.2 The speed of convergence in the Central Limit Theorem:

The preceding method allows us to find the speed of convergence in the Central Limit Theorem. We can prove that the speed of convergence in the Central Limit Theorem is $1/\sqrt{n}$. We will do this using the inequality of Esseen [3].

Lemma 4.2.1 There exists a real number $a > 0$ such that for all $|u| < a\sqrt{n}$, we have

$$\left| \int_0^1 \exp\left(\left(\frac{iu}{\sigma\sqrt{n}}\right) S_n f\right) - \exp\left(-u^2/2\right) \, d\mu \right|$$

$$\leq \exp\left(-u^2/4\right)\left(2A\left|u\right|^3/\sigma^3\sqrt{n} + B\frac{|u|}{\sigma\sqrt{n}}\right)$$

$$+\left(\frac{C\left|u\right|}{\sigma\sqrt{n}}\right)\rho_2^n,$$

where A, B and C are positive constants.

Proof: We know that

$$\left| \int_0^1 \exp\left(\left(\frac{iu}{\sigma\sqrt{n}}\right) S_n f\right) - \exp\left(-u^2/2\right) \, d\mu \right|$$

$$\leq \int_0^1 \left| P_f^n\left(\frac{iu}{\sigma\sqrt{n}}\right)(1) - \exp\left(-u^2/2\right)\right| d\mu.$$

We will estimate the last integral using the technique of the proof of the limit theorem. It is sufficient to use Proposition 3.3.2, putting $\theta = u/\sigma\sqrt{n}$:

$$
\begin{aligned}
P_f^n(i\theta) &= \lambda^n(i\theta)\, N_1(i\theta) + P_2^n(i\theta) \\
&= \left(1 + i\theta\lambda'(0) - \left(\theta^2/2\right)\lambda''(0) - \left(i\theta^3/6\right)\lambda'''(0) + \theta^3\overline{\lambda}\ (i\theta)\right)^n \times \\
&\quad \times \left(\mu + i\theta N_1^{(1)} - \left(\theta^2/2\right) N_1^{(2)} + \theta^2\overline{N}_1(i\theta)\right) + P_2^n(i\theta) \\
&= \exp\left(n\left(-\theta^2/2\right)\sigma^2 + iA_1\theta^3 + \theta^3\varepsilon(\theta)\right) \times \\
&\quad \times \left(\mu + i\theta N_1^{(1)} - \left(\theta^2/2\right) N_1^{(2)} + \theta^2\overline{N}_1(i\theta)\right) + P_2^n(i\theta),
\end{aligned}
$$

where A_1 is a constant and $\lim_{\theta\to 0}\varepsilon(\theta) = 0$. We therefore have

$$
\begin{aligned}
&\int_C^1 \left| P_f^n\left(\frac{iu}{\sigma\sqrt{n}}\right)(1) - \exp\left(-u^2/2\right)\right| d\mu \\
&\leq A_n(u) + B_n(u) + (C\,|u|\,\sigma\sqrt{n})\rho_2^n,
\end{aligned}
$$

where

$$A_n(u) = \exp\left(-u^2/2\right)\left|\exp\left[i_1 A_1 u^3/\sigma^3\sqrt{n} + u^3/\sigma^3\sqrt{n}\,\varepsilon\left(\frac{u}{\sigma\sqrt{n}}\right)\right] - 1\right|$$

and

$$B_n(u) = \exp\left(-u^2/2\right)\exp\left[iA_1 u^3/\sigma^3\sqrt{n} + u^3/\sigma^3\sqrt{n}\,\varepsilon\left(\frac{u}{\sigma\sqrt{n}}\right)\right] \times$$

$$\times \int_0^1 \left|\frac{u}{\sigma\sqrt{n}}\, iN_1^{(1)}(1) - \left(\frac{u}{2\sigma\sqrt{n}}\right) N_1^{(2)}(1) - \left(\frac{u}{2\sigma\sqrt{n}}\right)\;)\;\overline{N}_1\left(\frac{iu}{\sigma\sqrt{n}}\right)(1)\right| d\mu.$$

We can find a real number $a > 0$ such that $|u| < a\sqrt{n}$, where we have, $2Aa/\sigma^3 < 1/4$. where $A = |A_1|$. Then we have

$$\left|iAu^3/\sigma^3\sqrt{n} + u^3/\sigma^3\sqrt{n}\,\varepsilon\left(\frac{u}{\sigma\sqrt{n}}\right)\right| \le |u|\,2Au^2/\sigma^3\sqrt{n}$$

$$\le u^2/4$$

and

$$\left\| iN_1^{(1)}(1) - \left(\frac{u}{\sigma\sqrt{n}}\right) N_1^{(2)}(1) - \left(\frac{u}{2\sigma\sqrt{n}}\right) \overline{N_1}\left(\frac{iu}{\sigma\sqrt{n}}\right)(1) \right\|_V \leq B.$$

Therefore,

$$\begin{aligned} A_n(u) &\leq \exp\left(-u^2/2\right) \left| i_1 A_1 u^3/\sigma^3\sqrt{n} + u^3/\sigma^3\sqrt{n}\ \varepsilon\left(\frac{u}{\sigma\sqrt{n}}\right)\right| \times \\ & \qquad \times \exp\left|\left[i_1 A_1 u^3/\sigma^3\sqrt{n} + u^3/\sigma^3\sqrt{n}\ \varepsilon\left(\frac{u}{\sigma\sqrt{n}}\right)\right]\right| \\ \text{since } |e^z - 1| &\leq |z| \exp|z|. \\ &\leq \exp\left(-u^2/2\right)\ |u|\, 2Au^2/\sigma^3\sqrt{n}\ \exp\left(u^2/4\right). \\ &\leq \exp\left(-u^2/4\right)\ \frac{|u|^3\, 2A}{\sigma^3\sqrt{n}}. \end{aligned}$$

Similarly we can show,

$$B_n(u) \leq \exp\left(-u^2/4\right) \frac{|u|}{\sigma\sqrt{n}} B.$$

This completes the proof.

□

Theorem 4.2.1 Assume that the hypothesis are the same as for the Theorem 4.1.1. Then $\exists$ a constant $C > 0$ such that $\forall z \in R$, we have

$$\left|\mu\left\{\left(\frac{S_n f - n\mu(f)}{\sigma\sqrt{n}}\right) \leq z\right\} - \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{z} \exp\left(-u^2/2\right) du\right| \leq \frac{C}{\sqrt{n}}$$

Proof Using inequality of the Esseen [3], for all $U > 0$ and $n \geq 1$, we have $\sup_{z\in R}\left|\mu\left\{\left(\frac{S_n f}{\sigma\sqrt{n}}\right) \leq z\right\} - \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{z} \exp\left(-u^2/2\right)\ du\right|$

$\leq K/U + (1/\pi)\int_{-U}^{U}(1/|u|)\left|\int_0^1 \exp\ \left[\left(\frac{iu}{\sigma\sqrt{n}}\right)S_n f\right] d\mu - \exp\left(-u^2/2\right)\ du\right|,$

where $K = 24/\pi\sqrt{2\pi}$.

Using Lemma 4.1.1,

$\left|\ \int_0^1 \exp\left(\left(\frac{iu}{\sigma\sqrt{n}}\right)S_n f\right) - \exp\left(-u^2/2\right)\ d\mu\right| \leq$

$\leq \frac{k}{a\sqrt{n}} + \left(\frac{1}{\pi}\right)\int_{-U}^{U}\left(\frac{1}{|u|}\right)[\ \exp\left(-u^2/4\right)\left(2A\,|u|^3/\sigma^3\sqrt{n} + B\frac{|u|}{\sigma\sqrt{n}}\right)$

$+\left(\frac{C|u|}{\sigma\sqrt{n}}\right)\rho_2^n\ ].$

If we put $U = a\sqrt{n}$, then

$$\sup_{z\in R}\left|\mu\left\{\frac{S_n f}{\sigma\sqrt{n}} \leq z\right\} - \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{z} \exp\left(-u^2/2\right)\ du\right|$$

$$\leq \frac{K}{a\sqrt{n}} + \left(\frac{1}{\pi}\right)\int_{-a\sqrt{n}}^{a\sqrt{n}}\left(\frac{1}{|u|}\right)[\exp\left(-u^2/4\right)\left(2A\frac{|u|^3}{\sigma^3\sqrt{n}} + B\frac{|u|}{\sigma\sqrt{n}}\right)$$

$$+ C\left(\frac{|u|}{\sigma\sqrt{n}}\right)\rho_2^n\ ]\ du.$$

$$\leq \frac{K}{a\sqrt{n}} + \left(\frac{1}{\pi}\right)\int_{-a\sqrt{n}}^{a\sqrt{n}}[\exp\left(-u^2/4\right)2A\frac{|u|^2}{\sigma^3\sqrt{n}} + B\frac{1}{\sigma\sqrt{n}} + C\left(\frac{|u|}{\sigma\sqrt{n}}\right)\rho_2^n\ ]\,du$$

$$= \frac{K}{a\sqrt{n}} + R_n,$$

where

$$\begin{aligned} R_n &= \left(\frac{1}{\pi}\right)\int_{-a\sqrt{n}}^{a\sqrt{n}}[\exp\left(-u^2/4\right)2A\frac{|u|^2}{\sigma^3\sqrt{n}} + B\frac{1}{\sigma\sqrt{n}} + C\left(\frac{|u|}{\sigma\sqrt{n}}\right)\rho_2^n\,]\,du \\ &= \left(\frac{1}{\pi}\right)\int_{-a\sqrt{n}}^{a\sqrt{n}}\exp\left(-u^2/4\right)2A\frac{|u|^2}{\sigma^3\sqrt{n}}du + \int_{-a\sqrt{n}}^{a\sqrt{n}}\exp\left(-u^2/4\right)B\frac{1}{\sigma\sqrt{n}}du \\ &\quad + \int_{-a\sqrt{n}}^{a\sqrt{n}}C\left(\frac{|u|}{\sigma\sqrt{n}}\right)\rho_2^n du \end{aligned}$$

$$= I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \left(\frac{1}{\pi}\right)\int_{-a\sqrt{n}}^{a\sqrt{n}} \exp\left(-u^2/4\right) 2A\frac{|u|^2}{\sigma^3\sqrt{n}}du \\ &= \left(\frac{2A}{\pi\sigma^3\sqrt{n}}\right)\int_{-a\sqrt{n}}^{a\sqrt{n}} u^2\exp\left(-u^2/4\right)\, du. \end{aligned}$$

Let $\frac{u^2}{4} = y$ then,

$$\begin{aligned} I_1 &= \left(\frac{4A}{\pi\sigma^3\sqrt{n}}\right)\int_0^{\frac{a\sqrt{n}}{2}} y\exp\left(-y\right)\frac{2}{\sqrt{y}}\, dy \\ &= \left(\frac{8A}{\pi\sigma^3\sqrt{n}}\right)\int_0^{\frac{a\sqrt{n}}{2}} y^{\frac{1}{2}}\exp\left(-y\right) dy \end{aligned}$$

and then as $n \to \infty$, $I_1 = O\left(\frac{1}{\sqrt{n}}\right)$.

Also

$$I_2 = \left(\frac{1}{\pi}\right)\int_{-a\sqrt{n}}^{a\sqrt{n}} \exp\left(-u^2/4\right) B\frac{1}{\sigma\sqrt{n}}\, du$$

and as $n \to \infty$, $I_2 = O\left(\frac{1}{\sqrt{n}}\right)$.

Also

$$\begin{aligned} I_3 &= \left(\frac{1}{\pi}\right)\int_{-a\sqrt{n}}^{a\sqrt{n}} C\left(\frac{|u|}{\sigma\sqrt{n}}\right)\rho_2^n du \\ &= \frac{C\rho_2^n}{\pi\sigma\sqrt{n}}\int_{-a\sqrt{n}}^{a\sqrt{n}} du \end{aligned}$$

$$= \frac{C\rho_2^n}{\pi\sigma\sqrt{n}}\left(2a\sqrt{n}\right)$$
$$= \frac{2C}{\pi}\rho_2^n$$

and since $\rho_2 < 1$ then $I_3 = O\left(\frac{1}{\sqrt{n}}\right)$as $n \to \infty$. Hence the whole some $I_1 + I_2 + I_3$ behaves as $O\left(\frac{1}{\sqrt{n}}\right)$.

□

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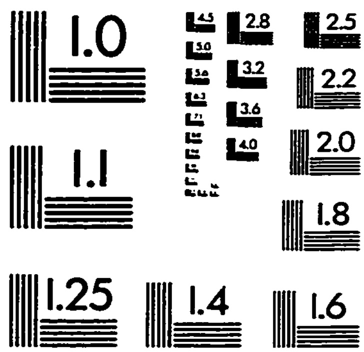
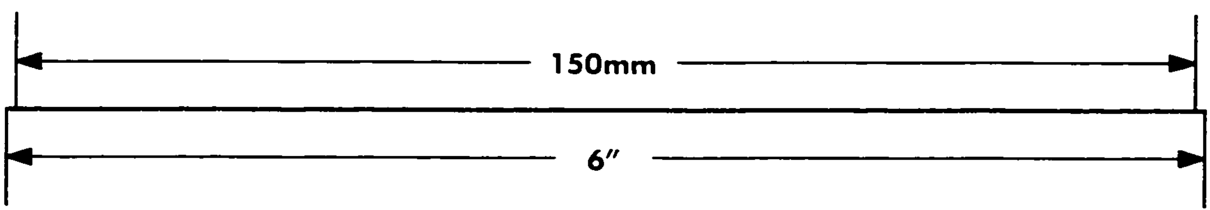
TEST TARGET (QA-3)





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