A New Approach in the Transient Analysis of ATM Multiplexers with Bursty Sources

Faouzi Kamoun

A Thesis
in
the Department
of Electrical and Computer Engineering

Presented in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy at
Concordia University
Montreal, Quebec, Canada

June1995

ISBN 0-612-05072-6

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ABSTRACT

A New Approach in the Transient Analysis of ATM Multiplexers with Bursty Sources

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Concordia University, 1995.

In this dissertation, we propose a new approach for the queueing analysis of discrete-time queues with correlated arrivals, arising in the ATM environment. In the first part of this work, we focus on the discrete-time transient analysis of a single server ATM multiplexer, where the arrival process consists of the superposition of the traffic generated by a homogeneous as well as by a heterogeneous set of independent binary Markov sources. We propose a new approach in the derivation of the transient joint probability generating function of the buffer content and the number of active sources. From this, time-dependent performance measures such as mean, variance and distribution of the queue length can be derived. Further, the transient analysis allows us to derive closed form expressions for the steady-state probability generating functions of the queue length, packet delay, as well as their corresponding first moments. We also present the idle and busy period analysis of the system. In the second part of this dissertation, we extend the approach to the transient and steady-state analysis of a multiserver ATM multiplexer and finally, in the third part, we demonstrate the applicability of the proposed approach in the steady-state analysis of a tandem queuing network with correlated arrivals. First we derive the steady-state joint generating function of the contents of the queues and the number of active sources. From this any moment of the queue length at each node can be extracted. In addition we derive explicit expressions for the average delay at each node as well as for the total average delay in the network.The main contribution of the first two parts of this work is to show how to extend the queueing

analysis of the GI/D/c queue in order to handle the correlation in the arrival process. The advantage behind the proposed approach is that it places the ATM multiplexer analysis on the same platform as that of the GI/D/c queue. The main contribution of the third part of this work is to establish a general framework, under which an exact performance analysis can be carried out, at the network level, in an ATM environment.

Acknowledgments

I would like to express my deepest appreciation to my thesis supervisor, Dr. M. Mehmet Ali, whose support, guidance and constructive criticism have been of considerable help throughout the course of this work. He introduced me to the various aspects in the performance analysis of communications networks. I am also indebted to professors Jeremiah. F. Hayes, E. Plotkin and M. A. Comeau for stimulating my interest in stochastic processes, queueing theory and computer communications.

Words fall short when I attempt to express my love and gratitude to my parents and to my wife, Thauraya. This thesis would have never been completed without their love, moral support and encouragement.

Lastly, I would like to thank all my friends at Concordia University, for sharing with me a challenging experience. I would like also to acknowledge the financial assistance of Concordia University through the Concordia University Graduate fellowship.

God Grant me the serenity

To accept the problems that I cannot solve

The persistence to solve the problems that I can

And the wisdom to know the difference

I Dedicate this Work to my Parents:

Mohamed and Emna

and to my Wife:

Thauraya

for their Sacrifices, Love and Support.

Table of Contents

Table of Contents vii

List of Figures xiii

List of Tables xv

List of Abbreviations xvi

List of Symbols xviii

CHAPTER I Introduction 1

1.1 The Evolution Towards ATM 2

1.2 The Advantages of ATM 3

1.2.1 The Integration Capability 3

1.2.2 The Bandwidth Scalabilty 4

1.3 Statistical Multiplexing in ATM 5

1.4 The Binary Markov On/Off Model and its Time Scales 8

1.5 Transient Behavior and Buffer Dimensioning 12

1.6 Transient Behavior and Congestion Control 13

1.6.1 Reactive Congestion Control 13

1.6.2 Preventive Congestion Control 14

1.6.2.1 Admission Control 14

1.6.2.2 Call Acceptance Decision Criteria and Transient Behavior 15

1.7 ATM Multiplexer Transient Analysis 18

1.8 ATM Multiplexer Steady-State Analysis 19

1.8.1 The GI/D/1 Approximation 20

1.8.2 The Fluid Approximation Model 20

1.8.3 The Matrix-Analytic Approach 21

1.8.4 The Markovian Arrival Approximation 22

1.8.5 The Diffusion Approximation Method 23

1.9 ATM Tandem Queuing Networks 25

1.10 Outline and Organization of the Dissertation 27

CHAPTER II The Discrete-Time GI/D/1 Queue Revisited 31

2.1 Introduction 31

2.2 The Transient Analysis 34

2.2.1 Example 1: The M/D/1 Case 36

2.2.2 Example 2: The Geo/D/1 Case 37

2.3 The Steady-State Analysis 39

2.4 Conclusion 39

CHAPTER III Transient and Steady-State Analysis of a Single Server ATM Multiplexer 41

3.1 Introduction 41

3.2 The Single Type of Traffic Case 43

3.2.1 Model Description and Notations 43

3.2.2 The Imbedded Markov Chain Analysis 46

3.2.3 Proposition 3.1 47

3.2.4 Proposition 3.2 48

3.2.5 The Solution Method 50

3.2.5.1 Theorem 3.1 50

3.2.5.2 Transient and Steady State Analysis of the Number of Active Users 52

3.2.5.3 Transient and Steady State Analysis of the Buffer Occupancy Distribution 54

3.2.5.4 Theorem 3.2 54

3.2.5.5 Computing the Transient Probabilities 's $p_k(0)$'s when $f(z) = z$ 60

3.2.6 Transient Mean of the Queue Length Distribution 64

3.2.7 Transient Variance of the Queue Length Distribution 65

3.2.8 Steady State PGF of the Buffer Occupancy Distribution 67

3.2.9 Steady State Mean and Variance of the Buffer Length 69

3.2.10 Asymptotic Analysis 71

3.3 The Multiple Type of Traffic Case 73

3.3.1 The Imbedded Markov Chain Analysis 74

3.3.2 Theorem 3.3: 77

3.3.3 Transient/Steady-State Analysis of the Number of Active Users 78

3.3.4 Transient and Steady-State Analysis of the Buffer Occupancy Distribution 80

3.3.4.1 Theorem 3.4 80

3.3.4.2 Transient Mean of the Queue Length Distribution 88

3.3.4.3 Steady-State PGF of the Queue Occupancy Distribution 89

3.3.4.4 Steady-State Mean of the Queue Length Distribution 90

3.3.5 The Infinite Source Model 91

3.4 Steady-State Distribution of the Packet Delay 92

3.5 Idle and Busy Period Analysis 93

3.5.1 The Idle Period 94

3.5.2 The Busy Period 95

3.5.2.1 Busy Period Analysis of the GI/D/1 Queue 95

3.5.2.2 Busy Period Analysis of the ATM Multiplexer 99

3.6 Numerical Results 101

CHAPTER IV Transient and Steady-State Analysis of a Multi-Server ATM Multiplexer 110

4.1 Introduction 110

4.2 The Single Type of Traffic Case 111

4.2.1 The Imbedded Markov Chain Analysis 111

4.2.2 The Solution Method 113

4.2.2.1 Theorem 4.1: 113

4.2.2.2 Transient/Steady-State Analysis of the Buffer Occupancy Distribution 115

4.2.2.3 Theorem 4.2: 116

4.2.3 Transient Mean of the Queue Length Distribution 121

4.2.4 Transient Variance of the Queue Length Distribution 122

4.2.5 Steady-State PGF of the Buffer Occupancy Distribution 122

4.2.6 Steady-State Mean and Variance of the Buffer length 125

4.3 The Multiple Type of traffic case 127

4.3.1 The Imbedded Markov Chain Analysis 127

4.3.2 Theorem 4.3: 129

4.3.3 Transient/Steady-State Analysis of the Buffer Occupancy Distribution 131

4.3.3.1 Theorem 4.4 132

4.3.3.2 Transient Mean of the Queue Length Distribution 138

4.3.3.3 Steady-State PGF of the Queue Occupancy Distribution 138

4.3.3.4 Steady-State Mean of the Queue Length Distribution 140

4.4 Steady-State Distribution of the Packet Delay 141

4.5 Numerical Results 141

CHAPTER V Queuing Analysis of ATM Tandem Networks 144

5.1 Preliminary 144

5.2 General Model Description and Notations 148

5.3 A Two-Nodes Tandem Network 150

5.3.1 The Solution Method 155

5.3.1.1 Theorem 5.1: 155

5.3.1.2 Steady State Analysis of the System: 159

5.4 The General N-Node Tandem Network 171

5.4.1 Numerical Results 176

CHAPTER VI Conclusion and Suggestions for Future Research 178

6.1 Conclusion 178

6.2 Main Contributions 179

6.3 Suggestions for Future Research 180

REFERENCES 183

Appendix A1 190

Appendix A2 190

Appendix A3 191

Appendix A4 191

Appendix A5 191

Appendix A6 192

Appendix A7 193

Appendix A8 194

Appendix A9 194

Appendix A10 196

Appendix A11 197

Appendix A12 198

Appendix A13 199

List of Figures

FIGURE.1.1. ATM Source Activity at Three Different Time Scales 10

FIGURE.1.2. Effect of Different Time Scales on the Cell Loss Probabilities 11

FIGURE.3.1 An ATM Multiplexer Loaded with *m* Homogeneous Bursty Sources 43

FIGURE.3.2 The Single Source Model.. 44

FIGURE.3.3 An Illustrating Example 96

FIGURE.3.4 Transient Probability of an Empty Buffer for a Fixed Offered Load and Different Number of Sources (m) 102

FIGURE.3.5 Transient Mean of the Queue Length for Different Values of Number of Sources (m).. 102

FIGURE.3.6 Transient Variance of the Queue Length for Different Values of m 103

FIGURE.3.7 Transient Probability of Overflow Versus Buffer Size................. 104

FIGURE.3.8 Transient Probability of an Empty Buffer for a Fixed Offered Load and Different Number of Sources (m) 105

FIGURE.3.9 Transient Mean of the Queue Length, for Different Values of Number of Sources (m).. 106

FIGURE.3.10 Transient Variance of the Queue Length for Different Values of m 106

FIGURE.3.11 Transient Probabilities of Overflow Versus Buffer Size 107

FIGURE.3.12 Transient Versus Steady State Probabilities of Overflow For Small Buffer Size 107

FIGURE.3.13 Mean Packet Delay for Different Number of Sources (m) 108

FIGURE.3.14 Variance of the Packet Delay for Different Values of *m* *109*

FIGURE.4.1 Steady-State Mean Queue Length as a Function of Load, ρ 142

FIGURE.4.2 Steady-State Variance of the Queue Length as a Function of Load, ρ 143

FIGURE.4.3 Mean Packet Delay as a Function of Load, ρ. 143

FIGURE.5.1 The Four Nodes Tandem Queuing Model Considered in [81].. 45

FIGURE.5.2 The General Model for the ATM Tandem Network 148

FIGURE.5.3 A Two-Node Tandem Network......................... 150

FIGURE.5.4 Partitioning the State Space of the Queue Lengths into 4 Non-Overlapping Regions 153

FIGURE.5.5 Average Queue Length at Nodes 1 and 2 as a Function of $\rho = \rho_1$ 176

FIGURE.5.6 Average Packet Delay as a Function of $\rho = \rho_1$ 177

List of Tables

TABLE 1.1 Parameter Values for Typical VBR Traffic Sources, as Proposed by CCITT (inter-burst and burst length, exponentially distributed) 9

List of Abbreviations

ATDM : Asynchronous Time Division Multiplexing.

ATM : Asynchronous Transfer Mode.

BER : Bit Error Ratio.

B-ISDN : Broadband Integrated Services Digital Network.

bps : Bit per second.

CAC : Call Acceptance Control.

CBR : Constant Bit Rate.

CCITT : Consultative Committee on International Telephone and Telegraphy.

D-MAP : Discrete-time Batch Markovian Arrival Process.

FCFS : First Come First Served.

HDTV : High Definition Television.

IC : Initial Condition.

IPP : Interrupted Poisson Process.

ISDN : Integrated Services Digital Network.

Kbps : Kilobits per second.

LAN : Local Area Network.

LIFO : Last in First Out

Mbps	: Megabits per second.
MMBP	: Markov Modulated Bernoulli Process.
MMPP	: Markov Modulated Poisson Process.
N-ISDN	: Narrowband Integrated Services Digital Network.
PBX	: Private Branch Exchange.
PGF	: Probability Generating Function.
PGM	: Probability Generating Matrix.
QoS	: Quality of Service.
RHS	: Right Hand Side
SBPP	: Switched Batch Poisson Process.
TDM	: Time division Multiplexing.
VBR	: Variable Bit Rate.
VCI	: Virtual Circuit Identifier.
VPI	: Virtual Path Identifier.
WAN	: Wide Area Network.

List of Symbols

$V(z)$: Steady-state PGF of the number of arrivals during a slot.

$Q_k(z)$: PGF of the queue length distribution at the end of the k^{th} slot.

$Q(z)$: Steady-state PGF of the queue length distribution.

$p_k(0)$: Probability that the buffer is empty at the end of the k^{th} slot.

$\frac{d^i}{dz^i}$: The i^{th} derivative operator.

H_g : The generalized Hypergeometric function.

Γ : The Gamma Function.

$\begin{bmatrix} a \\ b \end{bmatrix}$: The Binomial coefficient.

α : Probability that a source is active, given that it was active during previous slot.

β : Probability that a source is idle, given it was idle during previous slot.

Δ : Correlation index.

m : Total number of sources feeding the multiplexer.

i_k : Queue length at the end of slot k.

a_k : Number of active sources during slot k.

b_k : Number of packet arrivals during slot k.

$f_{j,k}$: Number of packets generated by the j^{th} active user during slot k.

$f(z)$: PGF of the number of packets generated by the j^{th} active user during slot k.

$p_k(i,j)$: joint probability of the queue length at the end of slot k and the number of active sources during slot k.

ρ_s : Single source average number of arrivals per slot.

$\hat{A}(z)$: Probability Generating Matrix of the an individual process.

$\hat{A}^T(z)$: Probability Generating Matrix of the superposition process.

$A_k(y)$: Marginal PGF of the number of active sources during slot k.

$A(y)$: Steady-state Marginal PGF of the number of active sources.

$\pi_0(k)$:Probability of a source being Off during slot k.

$\pi_1(k)$:Probability of a source being On during slot k.

π_0 : Steady-state probability of a source being Off.

π_1 : Steady-state probability of a source being On.

ρ : Steady-state load of the system.

$P_k(z)$: Marginal PGF of the queue length distribution at the end of the k^{th} slot.

$[x,y]^+$: max (x,y).

$[x,y]^-$: min (x,y).

$\overline{N}_k$: Average queue length at the end of the k^{th} slot.

$\sigma^2_{N_k}$: Variance of the queue length distribution at the end of the k^{th} slot.

$P(z)$: Steady-state PGF of the queue length.

$\bar{N}$	: Steady-state average queue length.
σ^2_N	:Steady-state variance of the queue length distribution.
$\bar{A}_k(y)$	: Transient PGF of the number of active sources in the infinite source model.
τ	: Number of sources' types feeding the multiplexer.
α_i	: Probability that a source, of type *i*, is active, given that it was active during previous slot.
β_i	: Probability that a source, of type *i*, is idle, given it was idle during previous slot.
a^i_k	: Number of active sources, of type *i*, during slot *k*.
b^i_k	: Number of packet arrivals, from type *i* sources, during slot *k*.
$f^{(i)}_{j,k}$	: Number of packets generated by the jth active user, of type *i*, during slot *k*.
$f_i(z)$	: PGF of the number of packets generated by the jth active user, of type *i*, during slot *k*.
$Q_k(z, \vec{y})$	: Transient joint PGF of the system under the multiple type of traffic case.
$A_k(\vec{y})$	: Transient PGF of the number of active sources under the multiple type of traffic case.
$D(z)$	: Steady-state PGF of the packet delay.
$\bar{d}$	: Mean packet delay.

σ_d^2 : Variance of packet delay.

I^* : Length of an idle period.

$I(z)$: PGF of the idle period.

$\overline{I^*}$: Mean idle period.

$\sigma_{I^*}^2$: Variance of the idle period.

B^* : Length of busy period.

$B^*(z)$: PGF of the busy period.

$\overline{B^*}$: Mean busy period.

m_i : Number of sources feeding node *i*.

$a_{i,k}$: Number of active sources feeding node *i* during slot *k*.

$b_{i,k}$: Number of packets generated by the m_i sources during slot *k*.

$f^i_{j,k}$: Number of packets generated by the j^{th} active source during slot *k* and which is destined to node *i*.

$Q_k(z_1, z_2, y_1, y_2)$: Joint PGF of the two nodes tandem network at the end of the k^{th} slot.

$Q(z_1, z_2, y_1, y_2)$: Joint steady-state PGF of the two nodes tandem network.

$\bar{N}_i$: Mean queue length of node *i*.

$\bar{T}_i$: Average delay at node *i*.

$\bar{T}$: Total average delay.

ρ_i : Total arrival rate to node *i*.

$V_i(z_i)$: Marginal PGF of the buffer length at node *i*.

CHAPTER I

Introduction

Today's telecommunication networks are evolving very fast in response to increasing numbers of users and the emergence of new telecommunication services like High Definition TV (HDTV), high quality video-phony, high speed data transfer and multimedia. To support these new services, the network must acquire remarkable networking capabilities so as to deliver multi-gigabit bandwidth in an efficient, integrated and cost-effective manner.

To respond to these new challenges, the telecommunication industry has opted for the Asynchronous Transfer Mode (ATM) as the potential transfer technique which will support the multimedia applications of today and tomorrow. ATM replaces the basic variable-length packet units, used in many current operating networks and which are difficult and slow to transport, with fixed-length packets, known also as cells. In this sense, ATM provides a means of developing a single, very fast network that will enable the multiplexing, transport, and switching of all types of traffic, end-to-end, at very high speeds. This will also reduce the total cost since there will not be separate telecommunication networks for different types of traffic.

Although ATM is being viewed today as a very promising solution to the recent issues of network bandwidth and service integration, it has also created new challenges for network designers, who must ensure that the industry will satisfy its customers' requirements for a good quality of service.

This thesis examines some of the important aspects in the design and performance analysis of ATM systems. In particular, this thesis is concerned with the performance analysis of discrete-time queues whose arrival process consists of the traffic generated by a special type of bursty sources, which is frequently encountered in the source characterization in the ATM environment. But first, we give a brief survey on how ATM originated and on the main features which made it possible for the ATM to be the transfer mode of choice for future telecommunications networks.

1.1 The Evolution Towards ATM

During the early 1980s, the Consultative Committee on International Telephone and Telegraphy (CCITT) has developed standards for the Narrowband Integrated Services Digital Network (N-ISDN), which will give public networks some capabilities to carry digital data traffic. The standard specified two types of interfaces:

- *A Basic rate* access at 144 Kbps, which consists of two 64 Kbps channels and one 16 Kbps signaling channel.

- *A Primary rate* access at 1.544 Mbps (T1 bandwidth) and 2.048 Mbps (E1 bandwidth), which includes a 64 Kbps signaling channel. The remaining bandwidth of each of these two primary rate interfaces is partitioned into many combinations of the basic 64 Kbps channels [1].

In the mid-1980's, and in response to higher bandwidth demands from the public telecommunication industry, the CCITT has begun examining the Broadband ISDN (B-ISDN) as the ultimate choice which will allow all types of traffic to be carried on the same all-digital network. In 1988, the CCITT adopted ATM as the transport mechanism for future Broadband traffic.

The private communication sector was also interested in the application and standardization of ATM and this led to the creation of the *ATM forum* in 1991.

Today, the ATM forum includes more than 160 major public and private telecommunication companies and its growing activities have contributed, so far, in the specification of many standards related to ATM network interfaces and signaling, among others.

1.2 The Advantages of ATM

ATM came as the solution to two new emerging telecommunication problems, namely the requirement for an integrated interface for the support of multiple types of traffic, including data, voice and video and the need for higher speed networks.

1.2.1 The Integration Capability

Until very recently, voice and data have been carried through almost two separate networks:

Typically, voice traffic is carried on synchronous links, using Time Division Multiplexing (TDM) techniques, which switch messages in accordance to their position in the frame. TDM is well suited for applications which deal with traffic that is being generated regularly, at a fixed rate and which is time sensitive. However, TDM is not appropriate for bursty traffic since bandwidth may be allocated to a channel, even though the channel does not need it.

On the other hand, data traffic is often carried on asynchronous links, using packet switching techniques. The user's information is segmented into *variable-length* packets, which are identified either by an address or a connection identifier. Even though packet switching has the advantage of using bandwidth on demand, it is not suitable for carrying delay-sensitive traffic such as voice. This is due to the fact that a node which generates very long packets might adversely affect the operation of the other nodes in the network.

In addition, dedicating one TDM-based network for voice traffic (ex. a Private Branch Exchange (PBX)) and another packet-switched based network for data traffic (ex. a Wide Area Network (WAN)) can be very costly. Hence one of the objectives of using ATM is to integrate all types of traffic in the same switching and transmission facility. In particular, ATM was designed to carry four classes of traffic:

- Class A: Constant Bit Rate (CBR), connection oriented, synchronous traffic (ex. uncompressed voice or video).

- Class B: Variable Bit Rate (VBR), connection oriented, synchronous traffic (ex. compressed voice and video).

- Class C: variable bit rate, connection oriented, asynchronous traffic (ex. X.25, frame relay services).

- Class D: connectionless packet data (ex. LAN traffic).

One of the key factors behind the success of ATM in supporting integrated traffic is the use of *fixed-length* packets (each consisting of a 48-octet payload and a 5-octet header), which are very suitable for delay sensitive traffic. In fact, the fixed size of ATM cells reduces the uncertainty of delay, a problem which is encountered in packet-switched networks, due to the variable length packets.

1.2.2 The Bandwidth Scalabilty

One of the advantages of ATM is that it is capable of scaling to the huge bandwidth demand which has been created by the introduction of new multimedia applications and by the increase in the number of users. This capability of ATM to scale to high speeds makes the implementation of B-ISDNs a reality and this is mainly due to the fact that ATM is a *connection-oriented* technology. In fact, in an ATM network, a connection has to be established between the users before the cells are routed to their destinations. To identify the connection to which it belongs, each ATM cell contains a Virtual Path Identifier (VPI) and a Virtual Circuit Identifier (VCI) which reside in the ATM header. The key advantage of ATM is that it is based on the concept *virtual circuit*, where bandwidth reservation is both flexible and dynamic, unlike TDM which uses static bandwidth reservation. In fact in an ATM network, an based on the connection VPI/VCI information, bandwidth is reserved over a certain duration but, unlike TDM, there is no specific reservation for particular slots in the frame. In addition ATM incorporates the advantage of packet switching in allocating bandwidth on demand and whenever needed. It exploits the statistical variations (bit rate fluctuations) in the users' traffic to perform statistical multiplexing, which eventually leads to a better use of the network resources. In this sense, sources may share a link capacity whose value is less than the sum of their individual peak bit rates.

Recent technological developments in electronics and optics have also helped ATM networks to scale to high speeds, at a reasonable cost. For instance, during the past few years, there has been a lot of progress in upgrading optical transmission systems. Today, single mode and multimode fibers are available for multi-gigabit rates, very long distances and very low Bit Error Ratios (BERs).

1.3 Statistical Multiplexing in ATM

One can view an ATM network as a collection of nodes which are connected by a set of transmission links. Based on their VPI/VCI, ATM cells are routed from a source node to a destination node, following the store and forward principle. When a cell reaches the nearby node, it is temporarily stored there until the transmission channel to the next node becomes available. For this purpose, and at each node, switching elements are installed to route the incoming cells to the appropriate output link. For those cells which cannot be transmitted immediately, buffer space has been provisioned at each switching element. Therefore, in an ATM network, several sources will be accessing a single link, such as a trunk line, carrying hundreds of connections. As mentioned before, ATM achieves high bandwidth gain by performing statistical multiplexing on the incoming packet streams. However, this multiplexing gain is often counterbalanced by the very stringent Quality of Service (QoS) requirements of users, which are often expressed in terms of packet loss, delay and delay jitter. For instance, voice traffic has a transmission rate of several kilobits per second and is delay sensitive, while high speed data traffic, used for instance in file transfers or LAN interconnections, is of hundreds of megabits per second and is loss sensitive. Therefore, in order to provide and maintain the QoS, not only does the ATM network have to be designed with the correct buffer sizes, but it should also monitor the traffic sources through the implementation of efficient admission, bandwidth allocation and flow control policies. The goal is to ensure that for each new accepted call, there will be enough bandwidth available along the corresponding virtual path. This will guarantee not only the QoS of the accepted call but also the QoS of all the virtual connections already established along parts of the virtual path.

It turns out, however, that in order to implement efficient admission and flow control strategies, one needs to acquire a very good understanding of the statistical multiplexing of the aggregate traffic generated by multimedia sources (with possibly different characteristics) on the ATM links.

From a modeling point of view, the choice of a connection-oriented fast-packet switching (with fixed-length packets) technique in a B-ISDN leads naturally to the choice of a slotted time axis with synchronized message transmission in the modeling of an ATM system [2]. In addition, the multiplexing of voice, data and video sources on high capacity ATM links gives rise to a very interesting discrete-time queuing problem, at the multiplexer's level, which involves a deterministic server and a special correlated discrete-time arrival process. Most often, the quantities of interest are the buffer occupancy (number of packets stored in the system) and the cell delays (or waiting times) experienced by the packets in the buffer. In discrete-time models, cell delay at a multiplexer is defined as the number of slots between the end of the slot during which the cell arrives and the end of the slot when the cell leaves the system.

The performance evaluation of statistical multiplexers with correlated arrivals has been a major field of research and investigation for the past few years. In addition, the performance analysis of ATM multiplexers has introduced a significant change in the way uncorrelated traffic (such as Poisson or Bernoulli) dominated the traditional performance evaluation methods. In fact when dealing with the traffic generated by multimedia sources like Variable Bit Rate (VBR) video codecs or with the traffic volume emitted from a sporadic data transfer between two computer terminals, the uncorrelated random arrival process assumption becomes inadequate because of the dependency which characterizes the cell stream. In particular the traffic generated by the superposition of VBR sources is characterized by:

- A positive correlation (characterized by a positive auto-correlation coefficient in the number of arrivals) at the burst scale: this means that if many cells arrive during the time interval $(t, t+\Delta t)$,where Δt is greater than a source cell inter-arrival time, then most likely many cells will follow during the time interval $(t+\Delta t, t+2\Delta t)$ due to the slow change in the traffic intensity within a burst.

- A negative correlation at the cell scale: this means that if many cells arrive during a time interval which is shorter than the source inter-cell time, then most likely fewer and fewer cells will arrive during the next interval due to the periodic emission of cells within each burst [3].

For these reasons, source characterization in ATM networks has also been a major field of research during the past years due to its direct impact on the performance evaluation of ATM systems. In addition while the traffic characteristics of some services (ex. voice) are generally well understood, the source behavior of other services (ex. VBR video) is still being investigated. For an excellent review on the various traffic source models for ATM networks, the reader is referred to [4].

1.4 The Binary Markov On/Off Model and its Time Scales

Among the most versatile traffic models which has been used for the characterization of ATM sources, is the binary (On/Off) Markov model. This model consists of a sporadic source (also known as a Burst-Silence source) which alternates between periods of active bursts and periods of silences, where no cells are generated.

Even though more complicated sources, such as the three-state model [5], have been proposed for more accurate modeling of services like video conference, these have rarely been applied since they are not very suitable for use in analytical studies.

On the other hand, the On/Off model is very popular and has often been used for the modeling of ATM traffic. For instance a binary Markov source has been successfully applied for the modeling of a voice source (ex. [6],[7]). In addition, in [8], a video source is modeled as a birth-death process, which consists of the superposition of a number of independent and identical On/Off sources. Nguyen and Mark [9] have also proposed an analytical source model for VBR coded video sources, where the output bit stream from each video source is modeled as the superposition of L independent and identical two-state Markov processes. The

CCITT has also provided parameter values for the On/Off sources that are to be used as traffic models for typical ATM sources. This is illustrated in Table 1.1 [10].

TABLE 1.1 Parameter Values for Typical VBR Traffic Sources, as Proposed by CCITT (Inter-burst and burst length, exponentially distributed)

Representative service	Mean burst length in number of cells	Average cell arrival rate
Connectionless Data	200	700 Kbits/sec
VBR Video	2	25 Mbits/sec
Connection-oriented Data	20	25 Mbits/sec
Background data/video	3	1 Mbit/sec
VBR video/data	30	21 Mbits/sec
Slow video	3	6 Mbits/sec

Because of its versatility and flexibility, the binary Markov source has been chosen, throughout this dissertation, as the basic model for the characterization of ATM traffic sources. Hence this thesis will be mainly concerned with the analysis of statistical ATM multiplexers whose input processes consist of the superposition of many independent traffic streams, each being modeled by a sporadic source, with its own characteristic.

In addition, and as shown in figure 1.1, the activity of an ATM source can be characterized at three time scales, namely call, burst and cell scales [11].

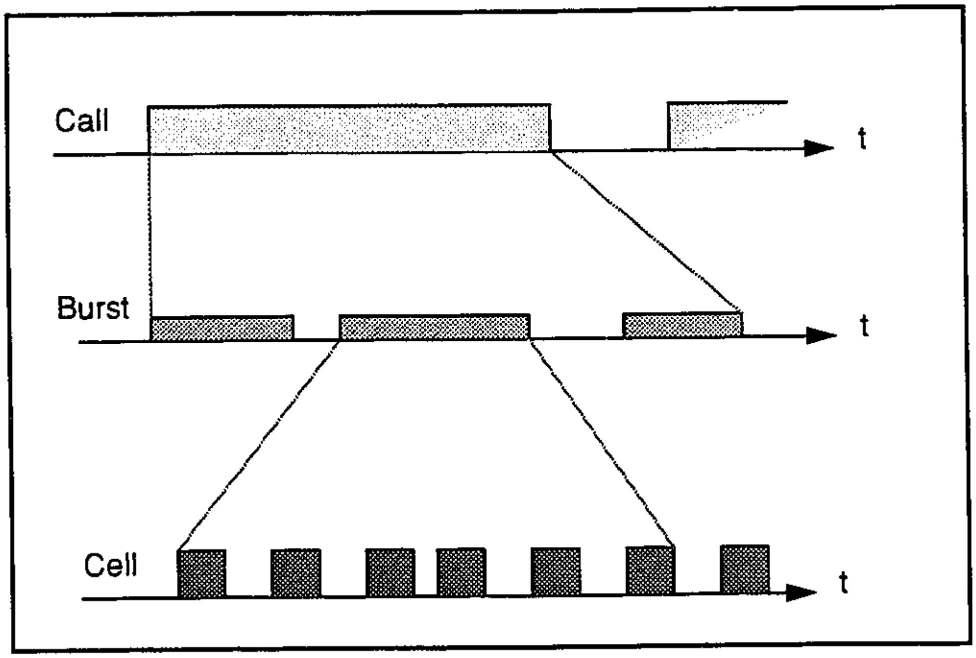

FIGURE.1.1. ATM Source Activity at Three Different Time Scales

A call, once set up between two users, is maintained during the entire connection, which typically can range from few minutes to many hours. Each call is segmented into an alternate sequence of burst and silences. Each burst, in turn, is partitioned into a stream of fixed length cells. The cell inter-arrival times within a burst can have an arbitrary distribution, though, most often, deterministic inter-arrival times of a fixed number of slots are used.

In an ATM network, the sources access the buffer through statistical multiplexing and when the buffer size is finite, cells can be discarded if the buffer becomes full. Therefore the probability of cell losses due to buffer overflow is among the most important performance measures in an ATM multiplexer, especially when dealing with loss-sensitive traffic. The effect of the different time scales on the performance of an ATM buffer is illustrated in figure 1.2, in the context of cell loss probabilities.

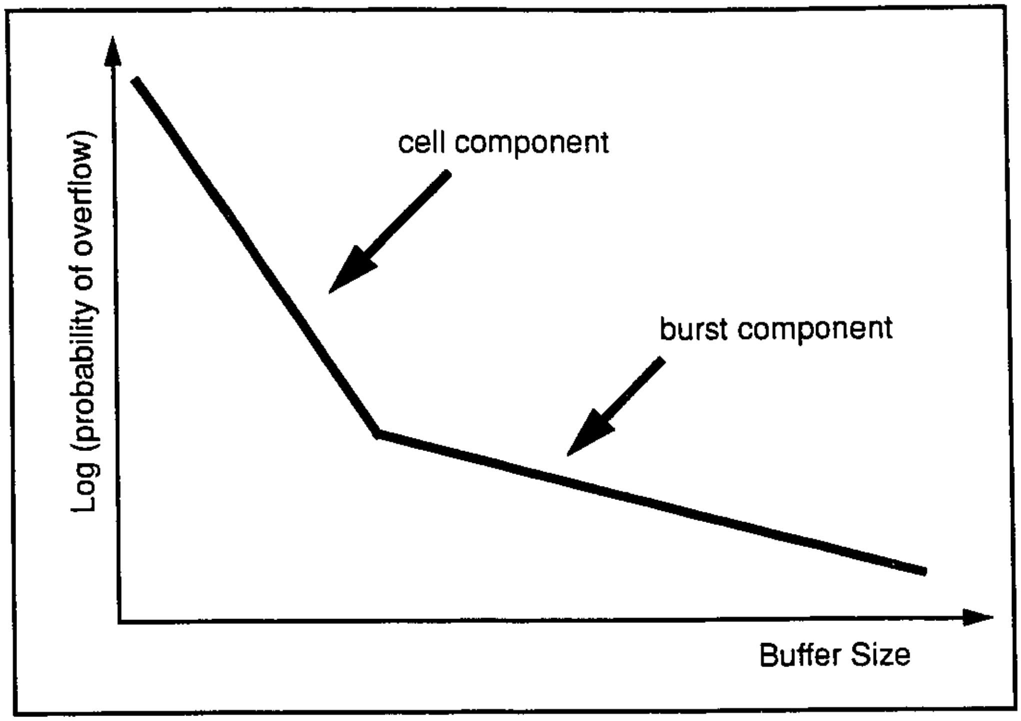

FIGURE.1.2. Effect of Different Time Scales on the Cell Loss Probabilities

As shown above, at small buffer sizes, cell loss probabilities are caused by the random fluctuations at the cell level. As the buffer size gets larger, smoothing of the cell-scale fluctuations takes place due to buffering and cell losses are mainly due to higher level traffic characteristics, such as traffic fluctuations at the burst scale. In response to this multi-level description in the activity of the sources, the performance analysis of ATM multiplexers can be carried out at any of the three time scales depicted in figure 1.2, though burst-scale and cell-scale models are the most encountered in the literature, since a call-scale model can be substituted by a burst-scale model with long bursts [2].

After this introductory background material on ATM networks and their traffic characterization, we are ready now to focus more on some of the main ATM performance analysis issues which will be dealt with in this dissertation.

First, we highlight the importance of taking into account the transient dynamics of the ATM multiplexer by looking into two major areas in the design and performance analysis of ATM systems, namely buffer dimensioning and resource management. Following that, we give a brief survey on the various analytical methods which have been developed so far for the performance analysis of single ATM multiplexers. We will then highlight the urgent need to carry the performance analysis of ATM systems at the network's level and present some of the approaches which have been proposed for this purpose. We should also note that for a detailed discussion of each of the methods, described in the survey which follows, the reader is advised to consult the referenced documents, since such a discussion is beyond the scope of this dissertation.

1.5 Transient Behavior and Buffer Dimensioning

In an ATM network, the use of very high speed channels has led to a very common situation, where a large number of cells are in transit between two ATM switching nodes. Further, ATM sources are bursty in the sense that a source may generate cells at nearly peak rate for some period of time and then suddenly becomes inactive. Therefore, when many ATM sources are simultaneously active, severe congestion and hence undesirable cell losses and delay may occur at the network nodes. For these reasons, buffer sizes in the ATM environment, should be determined properly, taking into account the fact that transient overload may result in large cell losses. It is unfortunate, however, that most of the buffer dimensioning problems in ATM have been investigated through mathematical analysis techniques, which are based on steady-state results. In particular, the steady-state probability of cell losses has been used as the main criterion for the choice of the appropriate buffer size. However, in actual situations, the dynamics of the network traffic can lead to temporal congestion and hence cell losses, even though the long-term time-averaged value of the cell loss rate is acceptable.

1.6 Transient Behavior and Congestion Control

Another area where the understanding of the multiplexer non-stationary or transient behavior becomes crucial is the congestion control problem in high speed networks. In ATM networks, congestion control is a real challenge due to the use of high speed channels and the bursty nature of the traffic sources feeding the network. First, we briefly review the congestion control problem in ATM networks, then we will highlight the importance of taking into account the time-dependent behavior of the ATM multiplexer in order to implement efficient congestion control strategies.

There are two types of congestion control schemes, which have been developed for ATM networks, namely *reactive* congestion control and *preventive* congestion control [12].

1.6.1 Reactive Congestion Control

Reactive congestion control is responsible for taking the appropriate actions to bring the degree of network congestion back to an acceptable level. Once congestion is detected, reactive control instructs the responsible node to slow down its traffic through a feedback mechanism. However, the high transmission speed of ATM networks makes the implementation of any type of reactive congestion control inefficient. In fact, in a feed-back type congestion control, and because of the high values in the ratio of the propagation delay to the cell transmission time, the time it will take to inform the originating source that the network is overloaded will be so long that corrective measures cannot be taken on time ([13],[14]). Hence, most reactive congestion control schemes are effective over short distances only. To overcome this problem, preventive congestion control has been proposed.

1.6.2 Preventive Congestion Control

Preventive congestion control attempts to prevent congestion before it occurs. The goal is to ensure, in advance, that the network traffic volume will not reach a critical level which will cause an unacceptable congestion state. Most often, preventive congestion control is implemented at the access nodes of the ATM network. There are also two ways to implement preventive control, namely *bandwidth enforcement* (policing) and *admission control* [12]. *Admission control* decides whether to accept or reject a new connection at the time of call set up. This decision is often based on the current traffic descriptors of the new connection and on the current network utilization. The goal of *bandwidth enforcement* is to ensure that, once a connection is set up between two users, any change (violation) in the declared user's traffic characteristics will not deteriorate the overall network performance. Bandwidth enforcement schemes will not be discussed here and we refer the interested reader to [15] for a good survey on this subject. We will rather focus on admission control, since it is one of the areas where the understanding of the time-dependent behavior of ATM multiplexers is crucial.

1.6.2.1 Admission Control

Admission control can be thought of as a resource allocation scheme which attempts to maintain a balance between QoS and network utilization by limiting the number of connections in the network. The goal of admission control is to ensure that a new call is accepted if it is guaranteed that it will not degrade the overall network performance. To guarantee this, and when a new connection is requested at a particular node, the network first checks for the service requirements of this call, which can be expressed in terms of:

-The acceptable cell transmission delay.

- The acceptable cell loss probability.

The network also examines the traffic characteristics of the new connection, in order to predict whether the network performance will be maintained, once the call is accepted. Most often, these traffic characteristics are specified by the user in terms of some parameters, known also as traffic descriptors. The most used traffic descriptors are peak and average bit rates, burstiness factor (ratio of peak bit rate to average bit rate), bit rate variance, average burst length, squared coefficient of variation of the cell inter-arrival times (i.e. ratio of the variance to the square mean of cell inter-arrival times), among many others. Which traffic descriptors are best suited to describe the traffic characteristics of a new call is still an open question. Another major research area in admission control deals with the decision criteria that the ATM network should adopt in deciding whether to accept or reject a new connection. In the following, we elaborate more on this issue since it is closely related to one of the major topics investigated in this thesis and which deals with the investigation of the transient behavior of ATM multiplexers.

1.6.2.2 Call Acceptance Decision Criteria and Transient Behavior

The best known decision criterion in admission control are the cell transmission delays and the cell loss probabilities, since these are very good indicators of the degree of network's congestion. Most often, in admission control, cell transmission delays and cell loss probabilities are expressed in terms of their long-term, time-averaged values (ex. [16],[17]). However, and as mentioned in [12], because of the bursty nature of the ATM sources, the network traffic will fluctuate dynamically, taking the overall network from one degree of congestion to another, even when the number of calls is constant. As a result, long-term, time-averaged values may not be sufficient to take the appropriate call acceptance/rejection decision. Figure 1.3 [18] illustrates, as an example, how cell loss probabilities change with time in an ATM node.

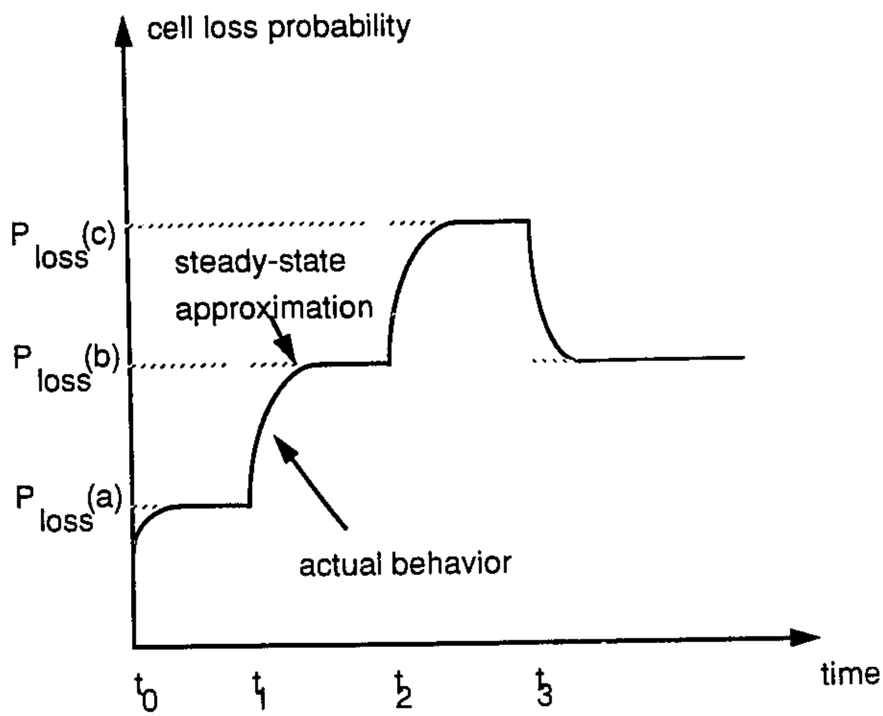

Figure 1.3. Time Dependent Behavior of Cell Loss Probability [18].

In the above figure, the number of active calls changes from *a*, at time t_0, to *b*, at time t_1, to *c* at time t_2 and back to *b*, at time t_3.The solid curve shows the actual (time-dependent) behavior of the cell loss probability. As may be seen from the above figure, when the number of active calls jumps to *b* (steady-state value) at time t_1, the node starts losing many cells and progressively moves towards the next level of congestion, where the cell loss probability reaches the steady-state value, $P_{loss}(b)$. When at time t_2, the number of active calls increases again, the node enters a new congestion level, which corresponds to cell loss probabilities that exceed the corresponding steady-state value, $P_{loss}(b)$. Eventually, at time t_3, the node gradually moves towards its steady-state congestion level, and so on..

The above example shows that the bursty and dynamic behavior of ATM traffic may lead to situations of temporal congestion where unacceptable cell losses occur, even though the steady-state value of cell losses is very small. In [18], the inefficiency of using long-term, time-averaged cell loss probability as a decision criterion for call acceptance is demonstrated through extensive numerical examples. It has been found that even if the long-term, time-averaged cell loss probability is kept small, severe congestion periods can occur and last for few hundred milliseconds. In a voice conversation, these congestion periods may lead to the loss of many bursts. In [18], it has also been shown that the use of *instantaneous* cell loss probability as a decision criterion in admission control can be a very effective remedy against these burst cell losses.

In [19], the transient behavior of voice cell loss probability is further investigated. It has been shown that, once congestion occurs, the cell loss probability becomes large and may remain so for a long period of time, leading to a noticeable voice quality degradation at the destination. Further, it has been shown that the cell loss probability during a blocking period (i.e. when the buffer is full) exceeds the corresponding long-term, time-averaged value. Therefore, it has been concluded [19] that the long-term, time-averaged cell loss probabilities are not appropriate in the measurement of voice distortion since they ignore the temporal behavior of voice cell losses.

Besides the instantaneous cell loss probability, there are also some other time -dependent performance measures which are very important in admission control. Among these, is the estimation of the time it takes for a node to come back to a desired level of congestion, once it has entered an undesired congestion level [18].

Having highlighted the importance of taking into account the transient behavior of ATM multiplexers, when implementing efficient congestion control and buffer dimensioning strategies, we move to the next important issue, namely how to derive time-dependent performance measures in an ATM multiplexer.

Because of the dynamic nature of ATM traffic, exact analytical results concerning time-dependent performance measures are generally difficult to obtain. One of the main goals of this dissertation to investigate this problem in the discrete-time domain. In addition, and as we will show in subsequent chapters, not only does a transient analysis enable us to derive time dependent performance measures, but it also allows us to derive the corresponding steady-state results as well. Next, we give a brief survey on what has been done, so far, in the transient as well as in the steady-state performance analysis of ATM multiplexers.

1.7 ATM Multiplexer Transient Analysis

The transient analysis of statistical multiplexers has been the subject of intensive research for a very long time. The most classical example is perhaps the transient solution of the M/M/1 queue ([20],[21],[22],[23],[24]). Transient solutions of other variants of this queue have also appeared in the literature ([25],[26]). For example, in the discrete domain, Jenq [27] proposed an *approximate* algorithm for the computation of the transient mean and variance of the queue length in a discrete-time queuing system with independent general arrivals and geometric output process.

For queues with correlated arrivals, analytical results concerning transient behavior are somehow limited, especially for the discrete-time case. For continuous time transient analysis, the theory of linear operators and spectral analysis is used in [13] to derive the transient solution of the joint probability distribution of the number of active sources and the content of an ATM multiplexer. The computational complexity of the approach involves numerical solution for boundary conditions and inverse Laplace transforms. In [28] the aggregate packet arrival process to an ATM multiplexer is approximated by a Markov Modulated Poisson Process (MMPP) and the resulting queue is described in terms of a two-dimensional continuous-time Markov chain. By embedding at the arrival epochs and using a matrix-geometric approach, an iterative solution for the probability distribution of the queue size at any arrival epoch is derived, based on the solution of the Chapman-Kolmogorov equations.

For the general discrete-time batch Markovian arrival process, a solution for the transient probabilities of an empty buffer and for the mean queue length (in the transform domain) is proposed in ([29],[30]). The computational complexity of the spectral decomposition approach involves the determination of the eigenvalues and the eigenvectors of the probability generating matrix of the system, as well as performing a numerical transform inversion, based on the Cauchy's integral formula. In addition the matrix form of the transform-domain expressions, derived in ([29],[30]), makes these latters not very easy to handle.

Recently Lucantoni et al [31] derived matrix equations for the two-dimensional transforms of the transient workload and queue length distributions in a single-server queue with a continuous-time batch Markovian arrival process. They applied a two-dimensional transform inversion algorithm, based on the Fourier series method and an iterative solution technique to solve for a matrix equation, in order to derive numerical results for the transient probability distributions.

1.8 ATM Multiplexer Steady-State Analysis

There is a considerable amount of literature on the steady-state analysis of ATM multiplexers. Different approaches, both exact and approximate, have been proposed and each of them has its own advantages and disadvantages. Further, we can identify five main approaches in the steady-state analysis of a single ATM multiplexer loaded with bursty sources:

1.8.1 The GI/D/1 Approximation

Renewal arrival processes such as Poisson, geometric, and hypergeometric processes have been primarily used in the analysis of voice multiplexers [6] and ATM switching fabrics (ex.[32]). Even though the traffic of a voice source, for instance, can be modeled by a renewal process, the usage of renewal processes for the superposition traffic violates the basic fact that the superposition of a number of renewal processes generally does not result in a renewal process. Therefore to capture some of the burstiness of the actual arrival process, the parameters of packet inter-arrival distribution are chosen so as to ensure a large coefficient of variation. An efficient algorithm for the performance analysis of the Geo/D/1/K queue can be found in [33]. These models make queuing analysis very simple and provide good approximation for the superposition of ATM traffic under heavy load situations [6]. However these models ignore the correlation at the packet level and hence they lead to a significant underestimation of the packet loss probabilities, highlighting the difficulty reported by many researchers when attempting to use the renewal approximation.

1.8.2 The Fluid Approximation Model ([8],[34],[35],[36],[37])

The fluid approximation model is suitable for the analysis of ATM multiplexers, at the burst scale and it often uses the average burst period as the unit of time. In this model, the discrete packet stream feeding the multiplexer is approximated by a continuous flow of information and the deterministic server can be thought of as a "sink" that would allow the incoming flow "drain" at a constant rate equal to the mean processing rate of the server. Typically, the packet generation process consists of number of sources which alternate between burst and silence periods.

The main assumption governing the operation of the fluid model is that each active source generates information at a uniform rate of one unit of information per unit of time. In addition the server removes information from the buffer at a uniform rate of C units per unit of time. The fluid approximation method was applied by Anick et al [36] to analyze an infinite buffer ATM multiplexer which is loaded with the superposition of statistically independent and identical On/Off sources. They assumed that each active source generates one unit of information in an active period and they took the average duration of the active period as the unit of time.Stern and Elwalid [38] extended the analysis to the case of non-identical fluid sources. The finite buffer case was also studied by Tucker [39]. This approach accurately captures the correlation behavior of the superposition traffic, but does not account for the stochastics (cell fluctuations) in flow. For this reason, it has been found that the fluid approach gives approximate results which are accurate only for very large buffer sizes [8], since for small buffers the stochastics become more important than the correlation. Recall that the fluid model does not take into account the cell fluctuations within bursts. The model also suffers from numerical problems for large systems, due to state space explosion.

1.8.3 The Matrix-Analytic Approach (ex. [40],[41],[42])

This, we believe, has been the most widely used approach and it makes extensive use of spectral decomposition theory and the properties of Kronecker product of matrices. Let i_k and J_k denote the buffer occupancy and the phase of the arrival process (ex. number of arrivals) immediately after the kth departure from the queue, respectively. The matrix-analytic approach is based on the fact that the transition probability matrix of the pair (i_k, J_k) in two successive departures has a block-partitioned structure which is similar to that of an M/G/1 queueing model. Further the application of the matrix analytic technique assumes that the Markov chain governing the arrival process is finite. This assumption is generally accepted as, in actual situations, the number of packets which can arrive during a slot is often bounded. For excellent description of the major steps of this approach, the reader is referred to [41].

The main advantage of the matrix-analytic technique is that, unlike the previous two methods, the approach is exact and takes into consideration the individual contribution of each source. However most of the performance measures which are derived from this method are given in general matrix forms whose evaluation require extensive numerical computations. In addition the computational complexity of this approach increases rapidly with increasing number of sources. As an example, consider a statistical multiplexer which is loaded with traffic generated by N non-homogeneous On/Off sources, each being described by a two-state Markov chain. Such a system will be described by a 2^N - state Markov chain, where each state corresponds to the number of each type of active sources. This exponential growth in the size of the state space can put some constraints on the size of the system that can be studied with matrix-analytic techniques.

1.8.4 The Markovian Arrival Approximation

The method consists of approximating the traffic generated by the superposition of ATM sources by a generic source model, consisting of a Markov modulated process. This is a doubly stochastic process which is characterized by a multi-state source whose state transitions are governed by a Markov chain. In each state (i) , i cells are generated with a state dependent rate, ρ_i. The most popular Markovian process is the Markov Modulated Poisson Process (MMPP). The discrete-time equivalent to the MMPP is the Markov Modulated Bernoulli Process (MMBP) [33] or the Switched Batch Poisson Process (SBPP) [34]. In the simplest 2-state MMPP, four parameters have to be estimated through statistical moment matching methods between the actual process and the surrogate MMPP model.

Most often, one has to select the four MMPP parameters so that the following characteristics are matched:

- The mean arrival rate.

- The variance to mean ratio for the number of arrivals in a short period.

- The variance to mean ratio for the number of arrivals in a long interval, and

- The third moment of the number of arrivals in a short interval.

The main advantage of the approach is that there are algorithms, based on the matrix analytical approach, which are available for the analysis of the MMPP/G/1/K queue ([43],[44],[45],[46]). However the applicability of this method is limited by the complexity of the solution method which requires two steps. First, a set of nonlinear equations has to be solved to estimate the parameters of the model. Then the queuing algorithm has to be run to analyze the model. More importantly the Markovian Arrival approximation is limited by the fact that the accuracy of the estimation procedure, for the surrogate model parameters, is critical for the success of the approximation. Further, the accuracy of the MMPP model decreases when the sources have long burst durations and the characteristic knee, shown in figure 1.2, could not be reproduced from the model [2].

1.8.5 The Diffusion Approximation Method ([47],[48])

In this method, the aggregate traffic of ATM sources is approximated by a diffusion process, consisting of the Ornstein-Uhlenbeck process. The buffer occupancy is also approximated by a diffusion process. Kobayashi and Ren ([47],[48]) have shown that this process provides a good approximation for the superposition of On/Off sources. Very recently, it has been shown that the diffusion process is also applicable for the modeling of more general sources, with arbitrary number of states and general distributions for the states' duration.

Besides these five approaches, some other methods have also been proposed (ex. [49],[50]). In particular, a model with binary Markov sources was analyzed by Bruneel [51], using a generating function approach. Bruneel considered the case where each active source generates at least one packet in each slot. He derived a functional equation for the PGF of the buffer occupancy. From the functional equation, Bruneel was able to extract explicit expressions for the steady-state mean queue length. The reader is also referred to [2] for some further readings related to traffic modeling and queueing techniques which have been developed for ATM multiplexer models.

In the first two parts of this dissertation, we introduce a new approach for the queueing analysis of ATM multiplexers with bursty sources. These sources are of the On/Off Markovian type and they have been widely used in the modeling of Broadband traffic. The proposed approach handles the single server, as well as the multiserver queueing problem, for both homogeneous as well as heterogeneous traffic sources. Our approach is basically an extension of the classical GI/D/1 analysis to the ATM multiplexer case. Through some elegant mathematical manipulations, we show how to rewrite the transient joint generating function of the ATM system into a suitable form. We then take transform, with respect to discrete-time, and determine sufficient number of linear equations to solve for all the unknowns in this generating function. By direct application of Abel's theorem, we show how it becomes possible to extract the steady-state joint generating function with a remarkable ease. It turns out that our steady-state results for the joint generating function of the ATM system are the indirect solutions to some, rather complicated, "nonlinear" functional equations.

So far, we have focused our attention on the ATM multiplexing problem by looking to a single queue in isolation. A more interesting case arises when we deal with the performance analysis of multistage queues. In an ATM environment, and because of the single path (virtual circuit) routing of packets, the queuing analysis of tandem queues can be considered today as one of the major research area in the performance characterization of ATM systems. As we will explain in the subsequent section, the transition from a single ATM queue analysis to a tandem queue analysis is by no means trivial. As the traffic streams within the ATM network interfere with one another, additional complexity in the analysis is introduced and it gets worst as the network size (i.e. the number of queues) increases.

1.9 ATM Tandem Queuing Networks

Despite the flexibility of the asynchronous transfer mode in supporting a wide range of multimedia services, each with its own bandwidth and QoS requirement, and despite the availability of ample bandwidth within an ATM network, there are many network design and performance analysis problems which still remain unsolved. For instance, today there is a strong need to develop the performance analysis of an ATM *network* since, so far, most of the queueing models related to an ATM environment have been confined to a single multiplexer or to a switch analysis.

An ATM network that consists of multiplexers and switching elements with output buffers can be modeled exclusively as a network of queues where queues correspond to the network's buffers and servers correspond to the network's links. The understanding of the change of traffic characteristics as packets pass through a number of switching nodes is of a crucial importance in ATM networks especially in the context of call acceptance control (CAC) (which decides wether or not a new call set up request can be accommodated). As mentioned earlier, a CAC is often based on a set of traffic descriptors that partially characterizes the new incoming packet streams. However because of the inevitable interference with other packet streams in the network nodes, these traffic descriptors might be the subject of substantial change as the packet stream traverses the network. This may lead to severe network congestion, which is not anticipated by the CAC.

One of the most powerful results in queuing network research is the theory of the so-called product-form networks. By extending the pioneering work of Jackson ([52],[53])] some networks have been found to retain the product form solution. Unfortunately, ATM tandem networks do not enjoy the practical application of the product-form solution results. In addition there are many other factors which complicate the exact analysis of these networks. Among these we can cite the followings:

- The arrival process to each node is often complicated and exhibits strong correlation. This correlation among arrivals makes the corresponding analysis far more complicated than that of uncorrelated case.

- The interaction among the traffic streams in the network gives rise to some, rather complicated, statistical dependence among the nodes. This results in some boundary functions in the expression of the joint generating function of the system, whose number grows exponentially with the number of nodes. The determination of these boundary unknowns is generally very complex, even for a four-node network with simple uncorrelated arrivals and partial interference (see for example ([54]],[55],[56]).

- Since, as mentioned earlier, the joint generating function of an ATM tandem network does not posses a product-form solution then the direct application of the well known combined iterative/decomposition methods ([57],[58]) become hard to justify. Recall that the combined iterative/decomposition techniques attempt to decompose a product-form queuing network into smaller subnetworks which are easier to analyze than the original network. By doing so it is found that the sum of the costs of solving each subnetwork times the number of iterations is usually far less than the cost of solving the whole network.

For these reasons, and in the lack of exact methods, most of the previous work in the analysis of tandem networks with correlated arrivals has either focused on simulation experiments (ex. [59],[60]) or on some *approximate* models, whereby each node is analyzed in isolation, after fitting an approximate model to the departure process of each node (ex.[61],[62],[63]). This last approach, which is based on decomposition techniques [64] has been applied extensively in the approximate analysis of computer communications networks [65]. We will further elaborate on this approach later on, when we will discuss the analysis of ATM tandem queues in chapter 5. In particular, we will also demonstrate the applicability of our proposed approach (originally developed for a single multiplexer analysis) to the steady-state performance evaluation of a tandem queueing network. Our goal is to establish a general framework for the exact performance analysis of some tandem configurations which arise in an ATM environment.

1.10 Outline and Organization of the Dissertation

The intent of this dissertation is to offer an alternate simple and efficient approach for the transient and steady-state analysis of ATM multiplexers with correlated arrivals. In addition, as we will show shortly, the proposed approach can also be applied to analyze ATM tandem queues.

In the first part of this thesis, we consider a discrete-time single server ATM multiplexer whose arrival process consists of the superposition of the traffic generated by a number of independent binary Markov sources. This arrival process belongs to the class of discrete-time batch Markovian arrival processes (D-MAP) and is extensively used in the modeling of voice, video and file transfer. A new approach for the transient analysis of the resulting queue at the cell level is proposed for the single type and for the multiple type of traffic cases. The approach uses an embedded Markov chain approach and is an extension of the classical method used in the transient analysis of single server queues with uncorrelated arrivals [66]. We derive closed-form expressions for the *transient/steady-state* marginal probability generating functions (PGFs) of the queue length and the number of active sources. From these, *time-dependent/steady-state* performance measures, such as the mean and variance of the queue length, are derived. In addition, from the steady-state PGF of the queue length, derived here, and using the results of [67], closed form expressions for the steady-state PGF of the packet delay and its corresponding first two moments are presented.

In the second part of this thesis, we extend our approach to the general multi-server case. The results obtained in the first two parts of this dissertation can be used to answer some significant questions which arise in the design and performance analysis of ATM systems, such as:

- What is the right buffer size required for a predetermined number of sources and grade of service?.

- For a given buffer size and grade of service, what is the maximum number of sources which can be accommodated by the system?.

- What is the significance of the transient analysis results in the context of congestion control?.

In the third part of this thesis, we show how the proposed approach can also be applied in the performance analysis of a tandem queuing network with correlated arrivals. The main thrust of this part of our research was motivated by the need to carry an *exact* performance analysis of an ATM system at the network's level since, so far, most of the queuing models related to an ATM environment have focused on a single multiplexer or on a switch analysis. By deriving the steady-state joint generating function of the system, we were able to extract closed form expressions for the moments of the queue lengths, the average delay at each node as well as for the total average delay in terms of the parameters of the network.

We have organized this thesis as follows:

The next chapter gives an overview of the transient and steady-state analysis of a single-server queue with uncorrelated arrivals and introduces the general framework of the approach, that will be used in the subsequent chapters, to handle the correlation in the arrival process. In chapter 3, an ATM single server queue loaded with m homogeneous binary Markov sources is considered and a functional equation relating the joint PGF of the system between two consecutive slots is given. The functional equation is put into a suitable form which makes it possible to derive transient and steady-state expressions for the marginal PGFs of the queue length and the number of active sources. From these, time-dependent and stationary performance measures are obtained. We also show how to recover the mean queue length formula derived in [51], despite the unavailability of the PGF there. We then present an asymptotic analysis for the infinite source model, which will be followed by the investigation of the Idle and Busy periods of the system. Next, we allow the sources to have different statistical characteristics and hence generalize the single server queueing analysis to the multiple types of traffic case. Chapter 4 generalizes the analysis to the multi-server case. In chapter 5, we extend our work to the exact steady-state analysis of ATM tandem configurations, with a special emphasis on a two-node tandem queueing network. We assume that each node is fed with the traffic generated by the superposition of identical binary Markov sources and model the system by a discrete-time multidimensional Markov chain. A functional equation relating the joint PGF of this system between two consecutive slots is derived and then rewritten in a suitable form which allows us to derive the marginal PGF of the buffer occupancy distribution at each node. Some results for the mean queue lengths, mean packet delay as well as for the total delay in the network are provided. Finally, in chapter 6, we give a conclusion, followed by a summary of the main contributions of the thesis and some suggestions for future research.

CHAPTER II

The Discrete-Time GI/D/1 Queue Revisited

2.1 Introduction

The discrete-time GI/D/1 system is a single server queue, with infinite capacity. Time is divided into fixed-length intervals, called slots and packets which arrive during a slot cannot be served until the beginning of the next slot. The queue is characterized by a deterministic server, whose service time equals to a single slot and by a renewal arrival process, whose inter-arrival times are general independent (GI) and identically distributed random variables. More specifically the numbers of packets arriving at the queue during consecutive slots are independent and identically distributed (i.i.d) positive discrete random variables with a general probability distribution, with generating function $V(z)$. Further it is assumed that the equilibrium condition, $V'(1) < 1$, is satisfied.

The discrete-time GI/D/1 model has many applications in communications systems, and appears in the context of many polling and multiplexing (mainly Asynchronous Time Division Multiplexing (ATDM)) problems. Among the main constraints which limit the applicability of this model is the independence assumption in the input packet stream, which is violated in many practical communications systems as arrival processes often exhibit a high degree of correlation which significantly affects the queue length behavior.

In this chapter, we focus on the number of packets in the GI/D/1 system, which thereafter will be referred to as buffer content. Our main goal is to familiarize the reader with the unifying approach which will be subsequently used in the analysis of the transient and limiting behavior of the ATM multiplexer. In fact, as we will see in subsequent chapters, our approach to the queueing analysis of ATM multiplexers will be based on an extension of some of the techniques presented in this chapter. Interestingly, it turns out that there are some results (ex. transient probabilities of an empty system) in the GI/D/1 case which are almost identical to those corresponding to the ATM single-server queue.

Throughout this chapter, and as well as for the remaining of this thesis, we adopt the length of a slot as the unit of time. In the time axis, and following the assumptions in [51], slots will be sequentially numbered in ascending order of positive integers such that the j^{th} slot is located in time $(j\text{-}1,j]$ where $(j=1,2...)$. Let j^- and j^+ be the time epochs immediately before and after time j. Then, throughout our analysis, we assume that a packet which has completed service in slot j, is considered to have left the system sometimes in (j^-,j). Further, a packet whose service time is assumed to have taken place in slot $(j+1)$, starts his service in the time interval (j,j^+). Although the arrival times of packets within a time slot are arbitrary, we will assume that packet arrivals during slot j take place at time j^-. This assumption is introduced because, in discrete models, changes in the system state typically take place at the slots' boundaries. Hence a message which completes service at the end of the j^{th} slot leaves behind it all those messages that have arrived in slot j as well as those that have been waiting at the beginning of the slot.

An imbedded Markov chain analysis of the GI/D/1 queue yields the following well known equation relating the probability generating function (PGF) of the queue length between two consecutive slots:

$$Q_k(z) = V(z)\left[\frac{Q_{k-1}(z) - Q_{k-1}(0)}{z} + Q_{k-1}(0)\right] \qquad , k \geq 1 \qquad (2.1)$$

where $Q_k(z)$ is the PGF of the buffer occupancy distribution at the end of the k^{th} slot and $V(z)$ is the PGF of the number of arrivals during a slot. In the sequel, (2.1) will be referred to as a "linear" (as opposed to a "non-linear") functional equation.

Traditionally the steady-state PGF, $Q(z)$, of the queue length is derived from (2.1) by argueing that if a steady-state solution exists then we must have:

$$\lim_{k \to \infty} Q_{k+1}(z) = \lim_{k \to \infty} Q_k(z) = Q(z) \qquad (2.2)$$

From which we get the well known formula:

$$Q(z) = \frac{(1-\rho)(z-1)V(z)}{z-V(z)} \qquad (2.3)$$

where $\rho = V'(1)$ is the load of the system and is determined from the normalization condition, $Q(1) = 1$.

Alternatively, it is possible to derive $Q(z)$, using a well known approach in queueing theory, which can be found for instance in [66] and [68]. First let $Q(z, w)$ and $P(w)$ be the one-dimensional transforms, defined by:

$$Q(z, w) = \sum_{k=0}^{\infty} Q_k(z) w^k \qquad (|w| < 1) \qquad (2.4a)$$

$$P(w) = \sum_{k=0}^{\infty} p_k(0) w^k \qquad (|w| < 1) \qquad (2.4b)$$

where $p_k(0) = Q_k(0)$. Then by substituting (2.1) into (2.4a), and using the shifting property of this w transform, we obtain:

$$Q(z, w) - Q_0(z) = wV(z)\left[\frac{Q(z, w) - P(w)}{z} + P(w)\right]$$

or equivalently:

$$Q(z, w) = \frac{Q_0(z)z + (z-1)P(w)wV(z)}{z - wV(z)} \qquad (2.5)$$

Next, by applying Abel's theorem (Appendix A4) to $Q(z, w)$,as given above, we can write $Q(z) = \lim_{w \to 1^-} (1-w)Q(z, w)$, or equivalently:

$$Q(z) = \frac{(1-\rho)(z-1)V(z)}{z-V(z)} \qquad (2.6)$$

where we used the fact that $p(0) = \lim_{k \to \infty} p_k(0) = \lim_{w \to 1^-} (1-w) Q(w)$.

However this approach does not work if the PGFs of the buffer occupancy distribution, between two consecutive slots, are related through nonlinear functional equations, such is the case for the ATM multiplexers considered in the next chapter. In such cases, the non-linearity can be handled by expressing the PGF of the system at the end of the kth slot in terms of the initial PGF and the probabilities that the system was empty during all previous slots. Such an approach, thereafter referred to "*the modified transform technique* "is illustrated below in the context of the GI/D/1 queue analysis. We first start with the full characterization of the transient PGF of the GI/D/1 queue length distribution.

2.2 The Transient Analysis

The main idea is that by considering the first few values of k in (2.1), one might easily express $Q_k(z)$ as follows:

$$Q_k(z) = \left[\frac{V(z)}{z}\right]^k Q_0(z) + (z-1) \sum_{j=1}^{k} \left[\frac{V(z)}{z}\right]^j p_{k-j}(0) \qquad (2.7)$$

where the transient probabilities of empty buffer, $p_k(0) = Q_k(0)$, are the only unknowns which remain to be determined in order to fully characterize $Q_k(z)$. Note that, in (2.7),$Q_0(z)$ denotes the initial PGF of the buffer content and it is presumably known. To evaluate $p_k(0)$'s in (2.7) we proceed as follows:

First, by substituting (2.7) into (2.4a), we get:

$$\begin{aligned} Q(z,w) &= Q_0(z) \frac{z}{z - wV(z)} + (z-1) \sum_{k=0}^{\infty} \sum_{j=1}^{k} \left[\frac{V(z)}{z}\right]^j p_{k-j}(0)\, w^k \\ &= Q_0(z) \frac{z}{z - wV(z)} + (z-1) \sum_{j=1}^{\infty} \left[\frac{V(z)\,w}{z}\right]^j \sum_{k=j}^{\infty} p_{k-j}(0)\, w^{k-j} \\ &= Q_0(z) \frac{z}{z - wV(z)} + (z-1) \sum_{j=1}^{\infty} \left[\frac{V(z)\,w}{z}\right]^j P(w) \end{aligned}$$

or equivalently:

$$Q(z,w) = \frac{Q_0(z)z + (z-1)P(w)wV(z)}{z - wV(z)} \tag{2.8}$$

The above expression is identical to the one previously derived in (2.5) and which can be found for instance in [66].

Next, from Rouchés theorem (Appendix A1), it can be shown that if the system is stable, then the equation:

$$z - wV(z) = 0 \tag{2.9}$$

regarded as an equation in z, has a unique root, z^*, inside the unit circle. In addition since $Q(z,w)$ is analytical inside the poly-disc $\langle |z| \le 1; |w| < 1 \rangle$ [66] then the numerator of (2.8) must also be zero at $z = z^*$. Hence we have:

$$Q_0(z^*)z^* + (z^*-1)P(w)wz^* = 0$$

or equivalently:

$$P(w) = \frac{Q_0(z^*)}{1 - z^*} \tag{2.10}$$

The above suggests that in order to find $p_k(0)$, one may solve for the unique root of (2.9) inside the unit disk and then invert $P(w)$.

Alternatively, from Lagrange's theorem (Appendix A2) and with $a = 0$, $\psi(z) = \dfrac{Q_0(z)}{1-z}$; $g(z) = V(z)$, equation (2.10) can be expressed as:

$$P(w) = \frac{Q_0(z^*)}{1-z^*} = p_0(0) + \sum_{k=1}^{\infty} \frac{w^k}{k!} \frac{d^{k-1}}{dz^{k-1}} \left[V(z)^k \left[\frac{Q_0(z)}{(1-z)^2} + \frac{Q'_0(z)}{1-z} \right] \right] \Bigg|_{z=0} \tag{2.11}$$

and therefore the transient probabilities, $p_k(0)$'s are given by:

$$p_k(0) = \frac{1}{k!} \frac{d^{(k-1)}}{dz^{k-1}} \left[V(z)^k \left[\frac{Q_0(z)}{(1-z)^2} + \frac{Q'_0(z)}{1-z} \right] \right] \Bigg|_{z=0} \qquad \forall (k \ge 1) \tag{2.12}$$

In the special case, where the system is initially empty (i.e. $Q_0(z) = 1$) the above reduces to:

$$p_k(0) = \frac{1}{k!} \frac{d^{(k-1)}}{dz^{k-1}} \left[\frac{V(z)^k}{(1-z)^2} \right] \Bigg|_{z=0} \qquad \forall (k \ge 1) \tag{2.13}$$

Using Leibniz's rule for the kthderivative of a product, and as shown in Appendix A3, the above expression can also be written as:

$$p_k(0) = \frac{1}{k}\sum_{i=0}^{k-1}\frac{(k-i)}{i!}\frac{d^i}{dz^i}[V(z)^k]\Big|_{z=0} \qquad \forall(k\geq 1) \qquad (2.14)$$

Once the transient probabilities of empty buffer, $p_k(0)$'s, are found, the transient PGF of the queue length at the end of any particular slot is completely defined as given in (2.7). This also enables the derivation of time-dependent performance measures, such as the transient mean and variance of the queue length at the end of any particular slot. In addition, and as illustrated in the next two examples, there are situations were the transient probabilities, $p_k(0)$'s, are easier to derive by inverting (2.10) with respect to *w*, rather than by using (2.12) and vice versa.

2.2.1 Example 1: The M/D/1 Case

Since for the M/D/1 queue, the PGF of the number of arrivals during any particular slot is $V(z) = e^{-\rho(1-z)}$ then, with an initially empty buffer, we have:

$$Q_k(z) = \left[\frac{e^{-\rho(1-z)}}{z}\right]^k + (z-1)\sum_{j=1}^{k}\left[\frac{e^{-\rho(1-z)}}{z}\right]^j p_{k-j}(0) \qquad (2.15)$$

From (2.14) and using the fact that:

$$\frac{d^i}{dz^i}[V(z)^k]\Big|_{z=0} = \frac{d^i}{dz^i}[e^{-\rho k(1-z)}]\Big|_{z=0} = (\rho k)^i e^{-\rho k}$$

we get the following closed form expression for the transient probabilities $p_k(0)$'s:

$$\begin{aligned} p_k(0) &= \frac{e^{-\rho k}}{k}\cdot\sum_{i=0}^{k-1}\frac{(k-i)}{i!}(\rho k)^i \qquad \forall(k\geq 1) \\ &= \frac{e^{-\rho k}}{k}\left[ke^{\rho k}(1-\rho) + \frac{(\rho k)^{k+1}H_g(2,k+2,\rho k)}{(k+1)!}\right] \\ &= (1-\rho) + \frac{e^{-\rho k}\rho(\rho k)^k H_g(2,k+2,\rho k)}{(k+1)!} \end{aligned} \qquad (2.16)$$

where H_g is the generalized (known also as the Barnes's extended) Hypergeometric function, which, for integers *n* and *d*, is defined by:

$$H_g(n,d,z) = \sum_{i=0}^{\infty} \frac{1}{i!} \frac{(n+i-1)!}{(n-1)!} \cdot \frac{(d-1)!}{(d+i-1)!} z^i \qquad (2.17)$$

Equations (2.15)-(2.17) completely determine the transient PGF of the M/D/1 queue length distribution at the end of any particular slot.

2.2.2 Example 2: The Geo/D/1 Case

This example illustrates the case where, sometimes, it becomes more convenient to solve for the unique root of (2.9) inside the unit circle and then invert $P(w)$, as in (2.10), in order to compute the transient probabilities, $p_k(0)$'s (which are the only remaining unknowns for the full characterization of the transient PGF of the buffer content).

In the Geo/D/1 system, the number of arrivals during consecutive slots are modeled as independent geometrically distributed random variables which are characterized by the PGF, $V(z) = \frac{1-p}{1-pz}$. For convenience, we express this PGF in terms of the load, $\rho = V'(1) = \frac{p}{1-p}$, of the system and, hence, we can write:

$$V(z) = \frac{1}{(1+\rho)-\rho z}$$

If the system is initially empty, then from (2.7), the PGF of the buffer content at the end of the kth slot is given by:

$$Q_k(z) = \left[\frac{1}{z(1+\rho-\rho z)}\right]^k + (z-1)\sum_{j=1}^{k}\left[\frac{1}{z(1+\rho-\rho z)}\right]^j p_{k-j}(0) \qquad (2.18)$$

Next, it is straight-forward to verify that the solution of the equation $z = wV(z)$, which lies in the unit circle is given by:

$$z^* = \frac{1+\rho-\sqrt{(1+\rho)^2-4w\rho}}{2\rho} \qquad (2.19)$$

and hence from(2.10), we have:

$$P(w) = \frac{(1-\rho) + \sqrt{(1+\rho)^2 - 4w\rho}}{2(1-w)} \tag{2.20}$$

Expressing the radical term in (2.20) in terms of the infinite Binomial series yields:

$$\sqrt{(1+\rho)^2 - 4w\rho} = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{2} \\ k \end{bmatrix} (1+\rho)^{1-2k} (-4\rho w)^k \tag{2.21}$$

where the Binomial coefficients are given by the general formula:

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)} \tag{2.22}$$

with $\Gamma(x)$ being the Gamma function defined by:

$$\Gamma(x) = \int_0^{\infty} z^{x-1} e^{-z} dz$$

and which satisfies $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(x) = (x-1)\Gamma(x-1)$; $(x>0)$, among others [69].

Finally, substituting (2.21) into (2.20) and taking the inverse w transform of the resulting expression yields the following closed-form expression for the transient probabilities of an empty buffer:

$$p_k(0) = \frac{1}{2}\left[1 - \rho + \sum_{i=0}^{k} \begin{bmatrix} \frac{1}{2} \\ i \end{bmatrix} (-4\rho)^i (1+\rho)^{1-2i}\right] \tag{2.23}$$

To the best of ou. .nowledge, the expressions for the transient probabilities of an empty buffer as given in (2.16) and (2.23) are new results, though, we should point out that their derivation in this thesis arises as the result of an illustration, rather than being an objective by itself.

2.3 The Steady-State Analysis

The steady-state PGF, $Q(z) = \lim_{k \to \infty} Q_k(z)$, of the GI/D/1 queue length distribution can be found by applying Abel's theorem to $Q(z, w)$ as given in (2.8). Hence we can write $Q(z) = \lim_{w \to 1^-} (1 - w) Q(z, w)$, or equivalently:

$$Q(z) = \frac{(1-\rho)(z-1)V(z)}{z - V(z)} \tag{2.24}$$

As expected, and because of the Markovian property of the model, the steady-state PGF is independent of the initial queue behavior which is embedded in the $Q_0(z)$ term in (2.8). In addition, in (2.24) we have recovered the steady-state result which was previously derived in (2.3) using the argument of (2.2).

2.4 Conclusion

In this chapter we have presented the transient and steady-state analysis of a basic discrete time model with infinite capacity, independent arrivals and deterministic server. The main simplifying assumption in the analysis is the uncorrelation in the arrival process, whereby the numbers of packets entering the system during consecutive slots have been modeled as i.i.d. discrete random variables. This slot to slot independence assumption in the arrival process has made the analysis fairly easy, especially when dealing with the steady-state queue behavior where the use of the classical argument given in equation (2.2) enables quick extraction of the steady-state PGF of the buffer length from the general formula presented in (2.1). However there are many occasions where the independence assumption governing the packet arrival process becomes unrealistic. In particular, in an ATM environment, the packet arrival process to the buffers exhibits a high degree of correlation as the users' packet generation process is often bursty. In order to handle this correlation in the activity of the users, more complicated queueing models have been introduced. For these models, and as we will show shortly, the classical argument (2.2) in the steady-state analysis of the queue length distribution does not allow the extraction of the steady-state PGF, mainly because of the presence of "non-linear" functional equations relating the joint PGF of the system between two consecutive slots. We believe that the failure of argument (2.2) in deriving the steady-state PGF of the buffer occupancy distribution is among the main reasons why the queuing analysis tool in the Broadband context has shifted towards matrix geometric and spectral decomposition approaches, among others.

In the sequel, we will show how the use of the transform technique, described in section (2.2), becomes a very powerful tool for the transient as well as for the steady-state analysis of the queue length behavior of an ATM system. In addition although this work deals exclusively with the superposition of binary Markov sources in the modeling of the packet arrival process to the ATM multiplexer, other correlated arrival processes might also be envisaged.

CHAPTER III

Transient and Steady-State Analysis of a Single Server ATM Multiplexer

3.1 Introduction

We have seen in chapter 1, that an ATM network can be viewed as a collection of nodes which are connected by a set of transmission links. Further, at each node, switching elements are installed to route the incoming cells to the appropriate output link of the node. For those cells which cannot be transmitted immediately, buffer space has been provisioned at each switching element. Generally speaking, buffering can be provided at the input side, at the output side or at some intermediate level of each switching element. A combination of buffering (such as input/output) has also been envisaged [2].

In response to the ATM approach of statistical multiplexing, there has been a tremendous interest among telecommunications researchers in studying the queuing performance of ATM systems. Most often, the performance evaluation has been carried out by considering ATM *switching elements* and *statistical multiplexers*. This chapter investigates the performance evaluation of ATM statistical multiplexers in terms of the buffer occupancy and cell delay, among others. To capture the bursty behavior of ATM sources, we have chosen the well known On/Off bursty source model as the basic building block for the modeling of the input traffic to the multiplexer.

Throughout this chapter, we model an ATM multiplexer as a discrete-time queueing system with infinite storage capacity and a single deterministic server.

Except for the arrival process, the modeling assumptions are the same as those previously described for GI/D/1 queue and can be briefly summarized as follows:

- The time axis is slotted and the transmission of a packet starts at the beginning of a slot and ends at the end of the same slot.

- Packets cannot leave the queue at the end of the slot during which they arrived. In addition all packets have the same fixed size and the service discipline is FCFS with no priority.

-The average number of packet arrivals during any slot is always less than one so that the system is stable and a steady-state exists.

This chapter is divided into two main parts. In the first part, we focus on the transient/steady-state queuing analysis of a single ATM multiplexer when the arrival process consists of the traffic generated by the superposition of m homogeneous and mutually independent binary Markov sources, of the type described in the next section. We develop a new transform technique and show how by transforming the functional equation relating the joint PGF of the system between two consecutive slots into a suitable form, it becomes possible to derive transient and steady state expressions for the marginal PGFs of the queue length and the number of active sources. In the second part, we generalize the analysis to the heterogenous case, where we consider the case of multiple types of sources, each with its own statistical characteristics. It is also interesting to note that the results of this chapter can also be applied in the analysis of ATM switching elements with output queuing [70]. More specifically each output queue of the switch can be modeled by an ATM buffer, of the type described here, with the logical input lines to each output queue being replaced by the Markov sources.

3.2 The Single Type of Traffic Case

3.2.1 Model Description and Notations

As shown in figure 3.1, we consider an ATM multiplexer which is fed with *m* mutually independent and identical binary Markov sources, each alternating between an *On* and an *Off* state. The multiplexer contains a buffer, shared by all the sources, and which is used for the temporarily storage of the packets before they are transmitted on the common output link.

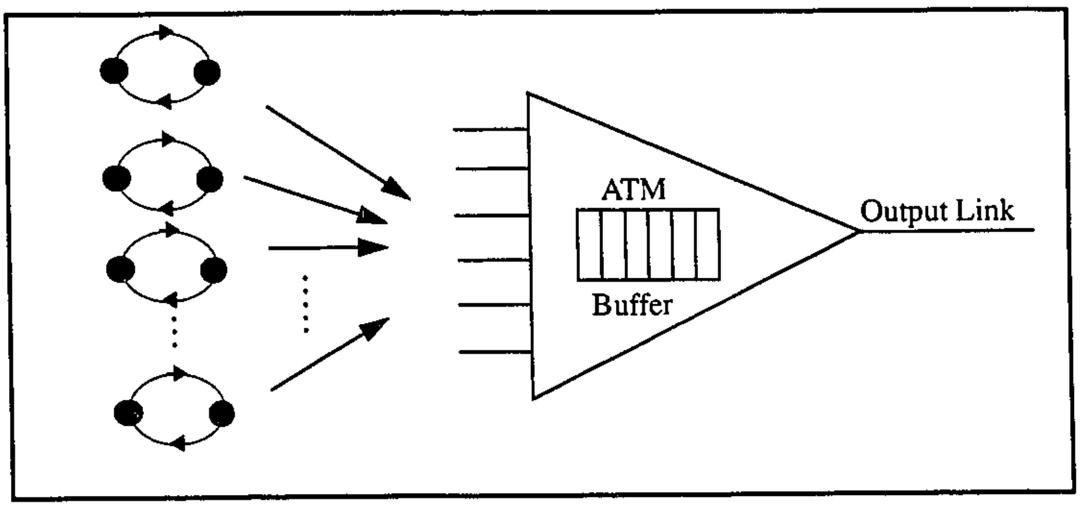

FIGURE.3.1 An ATM Multiplexer Loaded with *m* Homogeneous Bursty Sources

We assume that during an "active" slot each source generates at least one packet, with a PGF $f(z)$, while during a "passive" slot no packet is generated. This implies that State transitions are synchronized to occur at the slots' boundaries according to a two-state aperiodic (for each state, the probability of returning to that state is positive for all steps) and irreducible (all states are reachable from all other states) discrete-time Markov chain. A transition from an idle to an active state occurs with probability $(1-\beta)$, while the probability of a transition from an active to an idle state is $(1-\alpha)$. As a result, the lengths of the *On* and *Off* periods are geometrically distributed with means $\frac{1}{1-\alpha}$ and $\frac{1}{1-\beta}$, respectively (fig.3.2). Next let:

$$t_{10} = Pr[\text{ a } source \text{ is active} | \text{ it was idle during previous slot}] = 1-\beta$$

$$t_{11} = Pr[\text{a source is active} | \text{ it was active during previous slot}] = \alpha$$

and define the correlation index, $\Delta = t_{11} - t_{10} = \alpha + \beta - 1$, which is zero for a Bernoulli process. By choosing $\Delta \neq 0$, we can incorporate some slot-to-slot dependency in the activity of a single source and hence in the packet arrival process to the multiplexer. In fact when α and β are high $(0 < \Delta \leq 1)$ we have a "positive correlation" whereby packets have tendency to arrive in clusters. Alternatively when α and β are low $(-1 \leq \Delta < 0)$ we have a negative correlation where the arrivals are more dispersed in time. Because of their simplicity and capability to capture some of the correlation behavior which characterizes the ATM traffic, Binary Markov sources have been widely used as basic building blocks to model Broadband traffic, including voice (ex.[6]) and video [37].

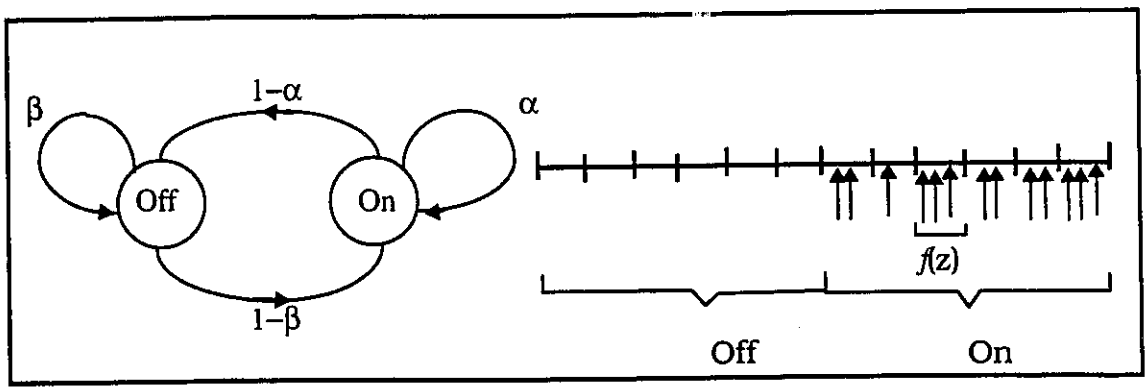

FIGURE.3.2 The Single Source Model

The queuing model under consideration can be formulated as a discrete-time two-dimensional Markov chain. The state of the system is defined by the pair (i_k, a_k) where i_k is the queue size at the end of slot *k* and a_k is the number of active sources during slot *k*. From [51], a_{k+1} and a_k are related through the relationship:

$$a_{k+1} = \sum_{j=1}^{a_k} c_j + \sum_{j=1}^{m-a_k} d_j \tag{3.1a}$$

where c_j and d_j are two sets of i.i.d Bernoulli random variables with corresponding PGFs:

$$c(z) = 1 - \alpha + \alpha z \tag{3.1b}$$

$$d(z) = \beta + (1 - \beta) z \tag{3.1c}$$

In (3.1a), the first term represents the number of users which were active during slot *k* and which remain active during slot *k*+1, while the second term represents the number of users which were idle during slot *k* and which become active at the next slot.

The number of packets, b_k, which arrive at the multiplexer during slot *k* is then:

$$b_k = \sum_{j=1}^{a_k} f_{j,k} \tag{3.1d}$$

where $f_{j,k}$ is the number of packets generated by the j^{th} active user during slot *k*. All the $f_{j,k}$'s are assumed to be i.i.d with PGF *f(z)*. We also note that since, in this case, each source generates at least one packet per active slot then $f(0) = 0$. This also implies that if the random variable i_k is zero then a_k must also be zero, since packets cannot also leave the buffer at the end of the slot during which they arrived [51]. Thus if the buffer is empty at the end of slot *k*, then all the sources should have been Off during slot *k*.

Bruneel [51] has done some initial work on the steady-state analysis of the above model, using a generating function approach. He derived a *functional equation* for the steady-state joint PGF of the system, from which he was able to extract the first moment of the queue length distribution.

3.2.2 The Imbedded Markov Chain Analysis

Let:

$$Q_k(z,y) = E[z^{i_k}y^{a_k}] = \sum_{i=0}^{\infty}\sum_{j=0}^{m} z^i y^j p_k(i,j)$$

denote the joint PGF of the pair (i_k, a_k), where $p_k(i,j) = Pr(i_k = i, a_k = j)$. In [51],a functional equation relating the joint PGF of the system between two consecutive slots is derived and is given by:

$$Q_{k+1}(z,y) = [d(y\cdot f(z))]^m\left[\frac{Q_k(z,Y) - Q_k(0,0)}{z} + Q_k(0,0)\right] \tag{3.2}$$

where:

$$Y = \frac{c(y\cdot f(z))}{d(y\cdot f(z))} = \frac{1-\alpha+\alpha y f(z)}{\beta+(1-\beta)yf(z)}$$

Because on the right hand side (RHS) of the above equation, $Q_k(z,y)$ appears with the substitution, $y = Y$, equation (3.2) will be identified as being a "non-linear" functional equation.

Further, using the classical argument that for large values of k, the functions $Q_k(z,y)$ and $Q_{k+1}(z,y)$ converge to the same limiting function, namely $Q(z,y)$, yields the following functional equation relating the steady-state joint PGF of a_k and i_k [51]:

$$zQ(z,y) = [d(y\cdot f(z))]^m[Q(z,Y) + p_0(z-1)] \tag{3.3}$$

where p_0 is the steady-state probability of an empty buffer. However, unlike in the uncorrelated case, the above approach does not allow here the derivation of $Q(z,y)$ since the functional equation (3.3) cannot be solved because of the presence of the Y term on the RHS of (3.3). By considering those values of y and z for which the second arguments of Q on both sides of (3.3) are equal, Bruneel [51] was able to derive a closed form expression for the steady-state mean queue length, $\overline{N}$, but the steady state PGF of the queue length could not be determined. Under the more general assumption where zero packet arrivals are allowed during an "active" slot, Daigle *et.al* [71] used a Matrix Analytic approach ([41],[72]) to derive a general form for the solution of the PGF of the queue size, which involves the inverse of an $(m+1)\times(m+1)$ matrix and the computation of the one step transition probabilities. An expression for the mean queue length, which also requires the inversion of an $(m+1)\times(m+1)$ matrix and the computation of the transition probabilities was derived. The solution for $\overline{N}$, as derived in [71], could not be reduced to the more explicit expression found in [51].

In the sequel, we propose a new approach which is an extension of the transform technique previously applied to the GI/D/1 queue (sections: 2.2-2.3). The main contribution of this chapter is to show how the new approach enables the derivation of explicit closed-form expressions for the transient/steady state joint and marginal PGFs of the system, as well as for some transient and steady state performance measures. In addition our analysis assumes an arbitrary, but a priori known, initial condition, $Q_0(z, y)$. But first we need to prove the following 2 important intermediate results:

3.2.3 Proposition 3.1

Let $\Phi(k)$ be the function defined by the recurrence relation:

$$\Phi(k+1) = \Phi(k)\big|_{y=Y}$$

$$\Phi(0) = y$$

Then:

$$\Phi(k) = \frac{U(k)}{X(k)} \tag{3.4}$$

where $U(k)$ and $X(k)$ satisfy the following recurrence relationships:

$$U(0) = y\ ;\ U(1) = (1-\alpha) + \alpha y f(z)\ ;\ U(k) = [\beta + \alpha f(z)]\,U(k-1) + [1-\alpha-\beta] f(z)\,U(k-2) \tag{3.5a}$$

$$X(0) = 1\ ;\ X(1) = \beta + (1-\beta) y f(z)\ ;\ X(k) = [\beta + \alpha f(z)]\,X(k-1) + [1-\alpha-\beta] f(z)\,X(k-2) \tag{3.5b}$$

PROOF

For k=0,1 (3.4) is obviously true. For k=2, and since $\Phi(1) = Y$ then:

$$\Phi(2) = \Phi(1)|_{y=Y} = \frac{1-\alpha+\alpha Yf(z)}{\beta+(1-\beta)Yf(z)} = \frac{1-\alpha+\alpha\frac{1-\alpha+\alpha yf(z)}{\beta+(1-\beta)yf(z)}f(z)}{\beta+(1-\beta)\frac{1-\alpha+\alpha yf(z)}{\beta+(1-\beta)yf(z)}f(z)}$$

$$= \frac{[\beta+\alpha f(z)]\{(1-\alpha)+\alpha yf(z)\}+[1-\alpha-\beta]f(z)\{y\}}{[\beta+\alpha f(z)]\{\beta+(1-\beta)yf(z)\}+[1-\alpha-\beta]f(z)\{1\}} = \frac{U(2)}{X(2)}$$

Hence (3.4) is verified. Next let us suppose that (3.4) is true for the order k, let us prove that it is also true for the order k+1, i.e.: $\Phi(k+1) = \frac{U(k+1)}{X(k+1)}$.

$\Phi(k+1)$ is obtained by substituting y=Y, into $\Phi(k)$, giving:

$$\Phi(k+1) = \Phi(k)|_{y=Y} = \frac{U(k)|_{y=Y}}{X(k)|_{y=Y}} \tag{3.6}$$

Next it is easy to prove (see Appendix A5 and A6) that:

$$U(k)|_{y=Y} = \frac{U(k+1)}{X(1)} \quad \text{and} \quad X(k)|_{y=Y} = \frac{X(k+1)}{X(1)}$$

and therefore:

$$\Phi(k+1) = \frac{U(k+1)}{X(1)} \cdot \frac{X(1)}{X(k+1)} = \frac{U(k+1)}{X(k+1)}$$

which completes the proof.□

3.2.4 Proposition 3.2

The functions $X(k)$ and $U(k)$, as defined by the recurrence formulas (3.5a,b), are given by:

$$\begin{aligned} X(k) &= C_1\lambda_1^k + C_2\lambda_2^k \\ U(k) &= D_1\lambda_1^k + D_2\lambda_2^k \end{aligned} \tag{3.7}$$

where $\lambda_{1,2}$, $C_{1,2}$ and $D_{1,2}$ are given by:

$$\lambda_{1,2} = \frac{\beta+\alpha f(z) \mp \sqrt{(\beta+\alpha f(z))^2+4(1-\alpha-\beta)f(z)}}{2} \tag{3.8a}$$

$$C_{1,2} = \frac{1}{2} \mp \frac{2(y-y\beta-\alpha)f(z)+(\beta+\alpha f(z))}{2\sqrt{(\beta+\alpha f(z))^2+4(1-\alpha-\beta)f(z)}} \tag{3.8b}$$

$$D_{1,2} = \frac{y}{2} \mp \frac{2(1-\alpha+\alpha y f(z))-(\beta+\alpha f(z))y}{2\sqrt{(\beta+\alpha f(z))^2+4(1-\alpha-\beta)f(z)}} \tag{3.8c}$$

PROOF

Equations (3.5a) and (3.5b) are homogeneous linear difference equations, with constant coefficients, and they both have the same characteristic equation, namely:

$$\lambda^2-(\beta+\alpha f(z))\lambda-(1-\alpha-\beta)f(z) = 0 \tag{3.9}$$

Assuming two distinct roots, $\lambda_{1,2}$, and using the corresponding initial conditions, specified in (3.5a,b), yields (3.7), where $C_{1,2}$ and $D_{1,2}$, as given in (3.8b,c) are the "constants" which are found from the corresponding initial conditions.

It is interesting to note that the roots, $\lambda_{1,2}$, of the characteristics equation (3.9) turn out to be ([41],[73]) the eigenvalues of the Probability Generating Matrix (PGM):

$$\hat{A}(z) = \begin{bmatrix} \beta & 1-\beta \\ (1-\alpha)f(z) & \alpha f(z) \end{bmatrix}$$

of the arrival process originating from a single Markov source. In particular λ_2 corresponds to the Perron-Frobeneous eigenvalue (or spectral radius) of $\hat{A}(z)$. This eigenvalue satisfies the following relations [41]:

$$\lambda_2\big|_{z=1} = 1 \quad , \quad |\lambda_1| \le |\lambda_2| \le 1 \quad , \quad \left.\frac{d\lambda_2}{dz}\right|_{z=1} = \rho_s \tag{3.10}$$

where ρ_s denotes the single source average number of packet arrivals (per-slot).

In [41] it was shown that the PGM of the superposition process, denoted by $\hat{A}^T(z)$, can be expressed as the kronecker product of the PGM's of the individual processes; i.e:

$$\hat{A}^T(z) = \underbrace{\hat{A}(z) \otimes \hat{A}(z) \otimes \ldots \otimes \hat{A}(z)}_{m}$$

which has a dimension of $2^m \times 2^m$. In addition the Perron-Frobeneous eigenvalue of the PGM of the superposition traffic is the product of the individual Perron-Frobeneous eigenvalues of the Markov sources.

With the preliminary results of propositions 3.1 and 3.2 in hand, we are ready now to tackle the non-linear functional equation (3.2), in a way which does not differ much (though less trivial!) from the ideas discussed in sections 2.2-2.3. The next theorem presents the key to one of the main contributions of this dissertation. As we will show in subsequent chapters, the same principle will be applied to deal with ATM multiserver queues as well as with ATM tandem queueing networks.

3.2.5 The Solution Method

By expanding $Q_{k+1}(z, y)$ in (3.2) for the first few values of *k*, we can prove by recurrence the following major result:

3.2.5.1 Theorem 3.1

The joint PGF of the queueing system under consideration, as given by the functional equation (3.2), can be written as follows:

$$Q_k(z, y) = \frac{B(k)}{z^k} Q_0(z, \Phi(k)) + (z-1) \sum_{j=1}^{k} \frac{B(j)}{z^j} p_{k-j}(0) \qquad (3.11)$$

where $p_k(0) = Q_k(0, 0)$ is the probability of an empty buffer at the end of the kth slot, $B(k) = [X(k)]^m$ and $\Phi(k)$ is as defined in (3.4).

PROOF

Throughout this proof, we make use of the fact that if $B^*(k) = B(k)|_{y=Y}$ then $B^*(k) = \frac{B(k+1)}{B(1)}$. This follows directly from the fact that $X(k)|_{y=Y} = \frac{X(k+1)}{X(1)}$, as shown in Appendix A5.

Hence for k=0, the functional equation (3.2) yields:

$$Q_1(z,y) = B(1)\cdot\{\frac{Q_0(z,\Phi(1))-p_0(0)}{z}+p_0(0)\}$$

$$= \frac{B(1)}{z}Q_0(z,\Phi(1)) + \frac{B(1)}{z}(z-1)p_0(0) \tag{3.12}$$

For k=1:

$$Q_2(z,y) = B(1)\cdot\{\frac{Q_1(z,\Phi(1))-p_1(0)}{z}+p_1(0)\}$$

Substituting (3.12) in the above gives:

$$Q_2(z,y) = \frac{B(1)}{z}\{\frac{B^*(1)}{z}Q_0(z,\Phi(2)) + \frac{B^*(1)}{z}(z-1)p_0(0)\} - \frac{B(1)}{z}p_1(0) + B(1)p_1(0)$$

$$= \frac{B(2)}{z^2}Q_0(z,\Phi(2)) + (z-1)\sum_{j=1}^{2}\frac{B(j)}{z^j}p_{2-j}(0) \tag{3.13}$$

and therefore (3.11) is verified for k=1,2 and obviously for k=0. Next let us suppose that (3.11) is true for the order (k), i.e.:

$$Q_k(z,y) = \frac{B(k)}{z^k}Q_0(z,\Phi(k)) + (z-1)\sum_{j=1}^{k}\frac{B(j)}{z^j}p_{k-j}(0) \tag{3.14}$$

Let us prove that is also true for the order (k+1), i.e:

$$Q_{k+1}(z,y) = \frac{B(k+1)}{z^{k+1}}Q_0(z,\Phi(k+1)) + (z-1)\sum_{j=1}^{k+1}\frac{B(j)}{z^j}p_{k+1-j}(0) \tag{3.15}$$

By substituting (3.14) into (3.2) we get:

$$Q_{k+1}(z,y) = B(1)\left\{\frac{\frac{B^*(k)}{z^k}Q_0(z,\Phi(k+1)) + (z-1)\sum_{j=1}^{k}\frac{B^*(j)}{z^j}p_{k-j}(0) - p_k(0)}{z} + p_k(0)\right\}$$

$$= \frac{B(k+1)}{z^{k+1}}Q_0(z,\Phi(k+1)) + (z-1)\sum_{j=1}^{k}\frac{B(j+1)}{z^{j+1}}p_{k-j}(0) - \frac{B(1)}{z}p_k(0) + B(1)p_k(0)$$

$$= \frac{B(k+1)}{z^{k+1}}Q_0(z,\Phi(k+1)) + (z-1)\sum_{j=1}^{k+1}\frac{B(j)}{z^j}p_{k+1-j}(0)$$

Hence (3.15) is proved and this completes the proof of the theorem.□

A comparison between the general forms of expressions (2.7) and (3.11) reveals a striking resemblance, with the only difference being that, in the GI/D/1 case, $V(z)^j$ (which has the interpretation of the PGF of the number of arrivals in j service times) becomes simply $B(j)$ in the correlated case. Next we show how the new expression (3.11) for the transient joint PGF simplifies considerably the transient analysis of the queuing system under consideration. We start our analysis by investigating the dynamics of the binary Markov sources.

3.2.5.2 Transient and Steady State Analysis of the Number of Active Users

Let $A_k(y)$ denote the marginal PGF of the number of active users during slot k. Then from (3.11):

$$A_k(y) = Q_k(1,y) = B(k)Q_0(1,\Phi(k))|_{z=1} = B(k)A_0(\Phi(k))|_{z=1}. \tag{3.16}$$

If we further assume the same probabilistic initial state for all sources and denote by $\pi_0(0)$ and $\pi_1(0)$ the probabilities of a source being initially *Off* and *On*, respectively, then $A_0(y) = [\pi_0(0) + \pi_1(0)y]^m$.

Hence, using (3.16) and substituting for $\Phi(k) = \dfrac{U(k)}{X(k)}$ and $B(k) = [X(k)]^m$ yields:

$$A_k(y) = B(k)\,[\pi_0(0) + \pi_1(0)\,\Phi(k)]^m\Big|_{z=1} = [\pi_0(0)X(k) + \pi_1(0)\,U(k)]^m\Big|_{z=1}$$

From the above and by substituting z=1 in (3.7-3.8) we get:

$$A_k(y) = [\pi_0(k) + \pi_1(k)\,y]^m \qquad (3.17)$$

where:

$$\pi_1(k) = \pi_1(0)\,(\alpha+\beta-1)^k + \frac{1-\beta}{2-\alpha-\beta}\,(1-(\alpha+\beta-1)^k)$$

$$\pi_0(k) = 1-\pi_1(k)$$

Therefore the number of active users during slot k follows a Binomial distribution with mean $m\pi_1(k)$ and variance $m\pi_1(k)\,\pi_0(k)$. The steady-state PGF of the number of active sources is obtained by letting $k \to \infty$ in (3.17), giving:

$$A(y) = A_\infty(y) = [\pi_0 + \pi_1 y]^m$$

where: $\pi_1 = \lim_{k\to\infty} \pi_1(k) = \frac{1-\beta}{2-\alpha-\beta}$ and $\pi_0 = \lim_{k\to\infty} \pi_0(k) = 1-\pi_1$. Therefore, in steady-state, the number of active users follows a Binomial distribution with mean $m\pi_1$ and variance $m\pi_1\pi_0$. Since each source generates, on the average, $\bar{f} = \frac{d}{dz}f(z)\Big|_{z=1}$ packets per active slot, then the load of the system is given by:

$$\rho = m\pi_1\bar{f} = m\frac{1-\beta}{2-\alpha-\beta}\bar{f}$$

In order for the ATM multiplexer to be stable and hence for a stochastic equilibrium to exist, we require that the average number of arrivals per slot, ρ, is strictly less than the average number of packets that can be transmitted within a time slot, or equivalently:

$$m\frac{1-\beta}{2-\alpha-\beta}\bar{f} < 1.$$

We next focus on the transient/steady-state behavior of the buffer occupancy.

3.2.5.3 Transient and Steady State Analysis of the Buffer Occupancy Distribution

Let $P_k(z) = Q_k(z, 1)$ denote the marginal PGF of the queue length at the end of the k^{th} slot. Then from (3.11), we have:

$$P_k(z) = \frac{\tilde{B}(k)}{z^k} Q_0(z, \tilde{\Phi}(k)) + (z-1) \sum_{j=1}^{k} \frac{\tilde{B}(j)}{z^j} p_{k-j}(0) \qquad (3.18)$$

where:

$$\tilde{B}(k) = B(k)|_{y=1} = [\tilde{X}(k)]^m = (\tilde{C}_1 \lambda_1^k + \tilde{C}_2 \lambda_2^k)^m \qquad (3.19a)$$

$$\tilde{\Phi}(k) = \Phi(k)|_{y=1} = \frac{\tilde{U}(k)}{\tilde{X}(k)} = \frac{\tilde{D}_1 \lambda_1^k + \tilde{D}_2 \lambda_2^k}{\tilde{C}_1 \lambda_1^k + \tilde{C}_2 \lambda_2^k} \qquad (3.19b)$$

$$\tilde{U}(k) = U(k)|_{y=1} \quad , \quad \tilde{X}(k) = X(k)|_{y=1}$$

$$\tilde{C}_i = C_i|_{y=1} \quad ; \quad \tilde{D}_i = D_i|_{y=1} \qquad \forall i \in \{1, 2\} \qquad (3.20)$$

with C_i, D_i and λ_i as given in (3.8) and $U(k)$ and $X(k)$ as in (3.7).

From (3.18) we see that the $p_k(0)$'s are the only terms which remain to be evaluated in order to fully characterize the transient PGF of the queue length. The following theorem provides a means to compute them.

3.2.5.4 Theorem 3.2

Let $P(z, w)$ and $P(w)$ be the one-dimensional transforms, defined by:

$$P(z, w) = \sum_{k=0}^{\infty} P_k(z) w^k \qquad (|w| < 1) \qquad (3.21)$$

and:

$$P(w) = \sum_{k=0}^{\infty} p_k(0) w^k \qquad (|w| < 1) \qquad (3.22)$$

Then:

$$P(z,w) = \sum_{i=0}^{\infty}\sum_{j=0}^{m}\sum_{\kappa=0}^{m} \frac{\displaystyle\sum_{l=[\kappa-j,0]^{+}}^{[m-j,\kappa]^{-}} \begin{bmatrix} m-j \\ l \end{bmatrix} \begin{bmatrix} j \\ \kappa-l \end{bmatrix} z^{i} p_0(i,j)\, \tilde{D}_1^{\kappa-l}\, \tilde{D}_2^{j-\kappa+l}\, \tilde{C}_1^{l}\, \tilde{C}_2^{m-j-l}\, z}{z - w\lambda_1^{\kappa}\lambda_2^{m-\kappa}}$$

$$+ (z-1)\sum_{\kappa=0}^{m} \begin{bmatrix} m \\ \kappa \end{bmatrix} \frac{\tilde{C}_1^{\kappa}\, \tilde{C}_2^{m-\kappa}\, P(w)\, w\lambda_1^{\kappa}\lambda_2^{m-\kappa}}{z - w\lambda_1^{\kappa}\lambda_2^{m-\kappa}} \qquad (3.23)$$

where the notations $[x,y]^{+}$ and $[x,y]^{-}$ denote $max(x,y)$ and $min(x,y)$, respectively.

In addition with, $H(z) = \lambda_2^{m}$, we also have:

$$P(w) = \frac{Q_0(z^*, r(z^*))}{1-z^*} \qquad (3.24)$$

where z^* is the unique solution of the equation $z = wH(z)$ inside the unit circle and $r(z) = \dfrac{\lambda_2 - \beta}{(1-\beta)f(z)}$.

PROOF

From (3.18), the w transform, $P(z,w) = \sum_{k=0}^{\infty} P_k(z)\, w^k$ where ($|w| < 1$) is given by:

$$P(z,w) = \sum_{k=0}^{\infty} \tilde{B}(k)\, Q_0(z, \tilde{\Phi}(k)) \left[\frac{w}{z}\right]^{k} + (z-1)\sum_{k=0}^{\infty}\sum_{j=1}^{k} \frac{\tilde{B}(j)}{z^{j}} p_{k-j}(0)\, w^{k} \qquad (3.25)$$

We first look to the first term in (3.25) namely, $I = \sum_{k=0}^{\infty} \tilde{B}(k)\, Q_0(z, \tilde{\Phi}(k)) \left[\frac{w}{z}\right]^{k}$. Since:

$$Q_0(z, \tilde{\Phi}(k)) = \sum_{i=0}^{\infty}\sum_{j=0}^{m} z^{i} [\tilde{\Phi}(k)]^{j} p_0(i,j)$$

then, first by substituting for $\tilde{B}(k)$ and $\tilde{\Phi}(k)$ as in (3.19, 3.20) and, afterwards, by applying the Binomial theorem we get:

$$I = \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{m} z^i \tilde{U}(k)^j \tilde{X}(k)^{m-j} p_0(i,j) \left[\frac{w}{z}\right]^k$$

$$= \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{m}\sum_{r=0}^{m-j}\sum_{s=0}^{j} \begin{bmatrix} m-j \\ r \end{bmatrix} \begin{bmatrix} j \\ s \end{bmatrix} z^i p_0(i,j) (\tilde{D}_1\lambda_1^k)^s (\tilde{D}_2\lambda_2^k)^{j-s} (\tilde{C}_1\lambda_1^k)^r (\tilde{C}_2\lambda_2^k)^{m-j-r} \left[\frac{w}{z}\right]^k$$

Interchanging the order of summation gives:

$$I = \sum_{i=0}^{\infty}\sum_{j=0}^{m}\sum_{r=0}^{m-j}\sum_{s=0}^{j} \begin{bmatrix} m-j \\ r \end{bmatrix} \begin{bmatrix} j \\ s \end{bmatrix} z^i p_0(i,j) \tilde{D}_1^s \tilde{D}_2^{j-s} \tilde{C}_1^r \tilde{C}_2^{m-j-r} \sum_{k=0}^{\infty} \left[\frac{\lambda_1^{r+s}\lambda_2^{m-(r+s)} w}{z}\right]^k$$

Finally the last term in the above expression can be further simplified to yield:

$$I = \sum_{i=0}^{\infty}\sum_{j=0}^{m}\sum_{r=0}^{m-j}\sum_{s=0}^{j} \begin{bmatrix} m-j \\ r \end{bmatrix} \begin{bmatrix} j \\ s \end{bmatrix} \frac{z^i p_0(i,j) \tilde{D}_1^s \tilde{D}_2^{j-s} \tilde{C}_1^r \tilde{C}_2^{m-j-r} z}{z - w\lambda_1^{r+s}\lambda_2^{m-(r+s)}}$$

Next we consider the second term in (3.25) which can be expanded as follows:

$$II = (z-1)\sum_{k=0}^{\infty}\sum_{j=1}^{k} \frac{\tilde{B}(j)}{z^j} p_{k-j}(0) w^k \quad = (z-1)\left[\sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{\tilde{B}(j)}{z^j} p_{k-j}(0) w^k - \sum_{k=0}^{\infty} p_k(0) w^k\right]$$

$$= (z-1)\left[P(w)\cdot\sum_{k=0}^{\infty}\frac{\tilde{B}(k)}{z^k}w^k - P(w)\right] \quad = (z-1)P(w)\cdot\sum_{k=1}^{\infty}\frac{\tilde{B}(k)}{z^k}w^k$$

Once again, substituting for $\tilde{B}(k)$ as in (3.19a) and using the Binomial theorem gives:

$$II = (z-1)P(w)\cdot\sum_{k=1}^{\infty}(\tilde{C}_1\lambda_1^k + \tilde{C}_2\lambda_2^k)^m \left[\frac{w}{z}\right]^k$$

$$= (z-1)P(w)\cdot\sum_{k=1}^{\infty}\sum_{\kappa=0}^{m}\begin{bmatrix} m \\ \kappa \end{bmatrix}(\tilde{C}_1\lambda_1^k)^{\kappa}(\tilde{C}_2\lambda_2^k)^{m-\kappa}\left[\frac{w}{z}\right]^k$$

$$= (z-1)P(w)\sum_{\kappa=0}^{m}\begin{bmatrix} m \\ \kappa \end{bmatrix}\tilde{C}_1^{\kappa}\tilde{C}_2^{m-\kappa}\sum_{k=1}^{\infty}\left[\frac{\lambda_1^{\kappa}\lambda_2^{m-\kappa}w}{z}\right]^k$$

$$= (z-1)P(w)\,w\sum_{\kappa=0}^{m}\begin{bmatrix} m \\ \kappa \end{bmatrix}\frac{(\tilde{C}_1\lambda_1)^{\kappa}(\tilde{C}_2\lambda_2)^{m-\kappa}}{z - w\lambda_1^{\kappa}\lambda_2^{m-\kappa}}$$

Therefore, combining both terms, we obtain:

$$P(z,w) = \sum_{i=0}^{\infty}\sum_{j=0}^{m}\sum_{r=0}^{m-j}\sum_{s=0}^{j}\begin{bmatrix} m-j \\ r \end{bmatrix}\begin{bmatrix} j \\ s \end{bmatrix}\frac{z^i p_0(i,j)\tilde{D}_1^{s}\tilde{D}_2^{j-s}\tilde{C}_1^{r}\tilde{C}_2^{m-j-r}z}{z - w\lambda_1^{r+s}\lambda_2^{m-(r+s)}}$$

$$+ (z-1)P(w)\,w\sum_{\kappa=0}^{m}\begin{bmatrix} m \\ \kappa \end{bmatrix}\frac{(\tilde{C}_1\lambda_1)^{\kappa}(\tilde{C}_2\lambda_2)^{m-\kappa}}{z - w\lambda_1^{\kappa}\lambda_2^{m-\kappa}}$$

or equivalently, with the change of variables, $\kappa = r+s$, we get:

$$P(z,w) = \sum_{i=0}^{\infty}\sum_{j=0}^{m}\sum_{\kappa=0}^{m}\frac{\sum_{l=[\kappa-j,0]^{+}}^{[m-j,\kappa]^{-}}\begin{bmatrix} m-j \\ l \end{bmatrix}\begin{bmatrix} j \\ \kappa-l \end{bmatrix}z^i p_0(i,j)\tilde{D}_1^{\kappa-l}\tilde{D}_2^{j-\kappa+l}\tilde{C}_1^{l}\tilde{C}_2^{m-j-l}z}{z - w\lambda_1^{\kappa}\lambda_2^{m-\kappa}}$$

$$+ (z-1)\sum_{\kappa=0}^{m}\begin{bmatrix} m \\ \kappa \end{bmatrix}\frac{\tilde{C}_1^{\kappa}\tilde{C}_2^{m-\kappa}P(w)\,w\lambda_1^{\kappa}\lambda_2^{m-\kappa}}{z - w\lambda_1^{\kappa}\lambda_2^{m-\kappa}} \qquad (3.26)$$

where the notations $[x,y]^{+}$ and $[x,y]^{-}$ denote $max(x,y)$ and $min(x,y)$, respectively. This ends the proof of the first part of the theorem.

Next we determine $P(w)$ by invoking the analytical property of $P(z,w)$ inside the poly-disc ($|z| \le 1; |w| < 1$) as follows:

First let $\Re = \{0,1,2,\ldots,m\}$. Then for each $\kappa \in \Re$, let us consider the roots of the equation:

$$z = V_{\kappa}(z) = w\lambda_1^{\kappa}\lambda_2^{m-\kappa}. \qquad (3.27)$$

Let $h(z) = z$ and $g_{\kappa}(z) = -V_{\kappa}(z)$. Since $|\lambda_1| \le |\lambda_2| \le 1$ and $|w| < 1$, then for each $\kappa \in \Re$:

$$|g_{\kappa}(z)| = |w\lambda_1^{\kappa}\lambda_2^{m-\kappa}| \le |\lambda_2^{m-\kappa}|$$

Further $\lambda_2^{m-\kappa}$ is a valid generating function (GF), and for a small $\varepsilon > 0$ and on $|z| = 1+\varepsilon$, a GF, *G(z),* satisfies: $|G(z)| \le 1 + \varepsilon G'(1)$ [68].

Using this bound, we get $|g_\kappa(z)| \le 1+\varepsilon\frac{(m-\kappa)(1-\beta)}{2-\alpha-\beta}\bar{f}$. On $|z| = 1+\varepsilon$ we also have $|h(z)| = (1+\varepsilon)$ and therefore if the system is stable, i.e. ($\rho = \frac{m(1-\beta)}{2-\alpha-\beta}\bar{f} < 1$), then for each $\kappa \in \Re$, $|h(z)| > |g_\kappa(z)|$ on $|z| = 1+\varepsilon$. From Rouchè's theorem $h(z)$ and $g_\kappa(z)+h(z)$ have the same number of zeros inside $|z| = 1+\varepsilon$. Evidently $h(z)$ has one zero inside $|z| = 1+\varepsilon$ and therefore (3.27) has also 1 root inside $|z| = 1+\varepsilon$. Moreover, since $f(0) = 0$ then $\lambda_1|_{z=0} = 0$ and hence for any $\kappa \in \Re - \{0\}$, the unique root of (3.27) inside the unit disk is $z^* = 0$ which also appears in the numerator of (3.26) since $\tilde{C}_1|_{z=0}$ is also zero. For κ =0, the corresponding term in (3.26) is given by:

$$\tilde{P}(z,w) = \frac{z\sum_{i=0}^{\infty}\sum_{j=0}^{m} z^i p_0(i,j)\,\tilde{C}_2^{m-j}\tilde{D}_2^{j} + (z-1)\,\tilde{C}_2^{m}P(w)\,w\lambda_2^m}{z-w\lambda_2^m} \tag{3.28}$$

Denote by z^* the unique root of $z = wH(z) = w\lambda_2^m$ inside $|z| \le 1$. Since $P(z,w)$ is bounded on ($|z| \le 1; |w| < 1$), then the numerator of (3.28) must also be zero at z^*, which implies that:

$$(z^*-1)P(w) + \sum_{i=0}^{\infty}\sum_{j=0}^{m} z^{*i}\left(\frac{\hat{D}_2}{\hat{C}_2}\right)^{j} p_0(i,j) = 0$$

or equivalently:

$$P(w) = \frac{Q_0\left(z^*, \frac{\hat{D}_2}{\hat{C}_2}\right)}{1-z^*} \tag{3.29}$$

where $\frac{\hat{D}_2}{\hat{C}_2} = \left.\frac{\tilde{D}_2}{\tilde{C}_2}\right|_{z=z^*}$.

Next, if we let $r(z) = \frac{\tilde{D}_2}{\tilde{C}_2}$ then, after some algebra, it is easy to verify that:

$$r(z) = \frac{\lambda_2-\beta}{(1-\beta)f(z)}$$

and therefore (3.29) becomes:

$$P(w) = \frac{Q_0(z^*, r(z^*))}{1-z^*} \tag{3.30}$$

and this completes the proof of the theorem.□

Through the result given in (3.24), theorem 3.2 enables us to completely determine the transient PGF of the queue length distribution. It is also worthwhile to note the similarity between the expression of $P(w)$ as given in (3.30) and the corresponding expression (2.10) in the uncorrelated case, for the GI/D/1 queue.

Next, we show how to compute the unknown probabilities, $p_k(0)$'s, by application of Lagrange's theorem (Appendix A2) to (3.30). Therefore, with *a*=0, *g(z)*=*H(z)* and:

$$\psi(z) = \frac{Q_0(z, r(z))}{1-z}$$

Lagrange's theorem yields:

$$P(w) = \frac{Q_0(z^*, r(z^*))}{1-z^*} = p_0(0) + \sum_{k=1}^{\infty} \frac{w^k}{k!} \frac{d^{k-1}}{dz^{k-1}} \left(H(z)^k \left[\frac{Q_0(z, r(z))}{(1-z)^2} + \frac{Q'_0(z, r(z))}{1-z} \right] \right)_{z=0}$$

This implies that:

$$p_k(0) = \frac{1}{k!} \frac{d^{k-1}}{dz^{k-1}} \left(H(z)^k \left[\frac{Q_0(z, r(z))}{(1-z)^2} + \frac{Q'_0(z, r(z))}{1-z} \right] \right)_{z=0} \qquad \forall (k \geq 1)$$

In the special case, where the system is initially empty, with all sources being in the Off state (i.e. $Q_0(z, y) = 1$), the above equation reduces to:

$$p_k(0) = \frac{1}{k!} \frac{d^{k-1}}{dz^{k-1}} \left(\frac{H(z)^k}{(1-z)^2} \right) \Bigg|_{z=0} \qquad \forall (k \geq 1) \qquad (3.31)$$

or equivalently:

$$p_k(0) = \frac{1}{k} \sum_{i=0}^{k-1} \frac{(k-i)}{i!} \frac{d^i}{dz^i} [H(z)^k] \Bigg|_{z=0} \qquad \forall (k \geq 1) \qquad (3.32)$$

It is interesting to observe that the above for the transient probability of an empty buffer is identical to that of the GI/D/1 queue (2.14), with $V(z) = H(z)$ .In this case, finding a closed form expression for the ith derivative of $H(z)^k = \lambda_2^{mk}$ is not an easy task, though a recursive solution can be developed. We therefore propose a new approach, which we *illustrate* for the case where each active user generates one packet per slot (i.e. with $f(z) = z$).

3.2.5.5 Computing the Transient Probabilities $p_k(0)$'s when $f(z) = z$

Let us assume a deterministic initial condition, whereby initially the buffer contains i_0 packets with a_0 sources ($a_0 \le m$) being in the *ON* state. This implies that $Q_0(z, y) = z^{i_0} y^{a_0}$.

The idea of our approach is based on the fact that the equation, $z = wH(z)$, implies that:

$$2\left(\frac{z}{w}\right)^{1/m} - (\beta + \alpha f(z)) = \sqrt{(\beta + \alpha f(z))^2 + 4(1 - \alpha - \beta) f(z)} \qquad (3.33)$$

With the change of variables, $x = \left(\frac{z}{w}\right)^{1/m}$, and after squaring both sides of the above equation, we can rewrite (3.33) as follows:

$$x^2 = \beta x + \alpha x f(wx^m) + (1 - \alpha - \beta) f(z)$$

or equivalently:

$$x = \beta + \left(\alpha + \frac{1 - \alpha - \beta}{x}\right) f(wx^m)$$

When $f(z) = z$, this becomes equivalent to:

$$x = \beta + wg(x)$$

where:

$$g(x) = \alpha x^m + (1 - \alpha - \beta) x^{m-1}$$

It is easy to verify that $|wg(x)| \le |x - \beta|$ is satisfied at all points *x* on the perimeter of the unit circle. Then, from Lagrange theorem, the equation $x = \beta + wg(x)$, regarded as an equation in *x* has exactly one root, x^*, inside the unit circle.

In addition, from (3.30):

$$P(w) = \frac{(z^*)^{i_0}(r(z^*))^{a_0}}{1-z^*}$$

Substituting for $r(z)$ and taking into account the fact that $z = wx^m$ implies that $x = \lambda_2$, we obtain:

$$P(w) = \frac{(wx^{*m})^{i_0}\left[\frac{x^*-\beta}{(1-\beta)wx^{*m}}\right]^{a_0}}{1-wx^{*m}} = \frac{(wx^{*m})^{i_0-a_0}(x^*-\beta)^{a_0}}{(1-\beta)^{a_0}(1-wx^{*m})} \tag{3.34}$$

From Lagrange's theorem (Appendix A2), with $a = \beta$ and:

$$\psi(x) = \frac{(wx^m)^{i_0-a_0}(x-\beta)^{a_0}}{(1-\beta)^{a_0}(1-wx^m)} \tag{3.35}$$

we have:

$$P(w) = \frac{(wx^{*m})^{i_0-a_0}(x^*-\beta)^{a_0}}{(1-\beta)^{a_0}(1-wx^{*m})} = \psi(\beta) + \sum_{k=1}^{\infty}\frac{w^k}{k!}\left[\frac{d^{k-1}[\psi'(x)g(x)^k]}{dx^{k-1}}\right]_{x=\beta} \tag{3.36}$$

Our next goal is to develop the term, $\psi'(x)g(x)^k$, which appears above. To accomplish this, we proceed as follows:

First, we expand $\psi(x)$ in (3.35)as an infinite sum. To do so, we apply the Binomial theorem to expand the term, $(x-\beta)^{a_0}$, and we express the rational, $\frac{1}{1-wx^m}$ in terms of a geometric series. Hence we get:

$$\psi(x) = \frac{(wx^m)^{i_0-a_0}\left[\sum_{l=0}^{a_0}\begin{bmatrix}a_0\\l\end{bmatrix}x^l(-\beta)^{a_0-l}\right]}{(1-\beta)^{a_0}}\sum_{i=0}^{\infty}(wx^m)^i$$

$$= \frac{1}{(1-\beta)^{a_0}}\sum_{i=0}^{\infty}\sum_{l=0}^{a_0}\begin{bmatrix}a_0\\l\end{bmatrix}(-\beta)^{a_0-l}w^{i+i_0-a_0}\cdot x^{m(i+i_0-a_0)+l}$$

Similarly, from the Binomial theorem, we have:

$$g(x)^k = [\alpha x^m + (1-\alpha-\beta)x^{m-1}]^k = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \alpha^{k-j}(1-\alpha-\beta)^j x^{mk-j}$$

and therefore we can write:

$$\psi'(x)g(x)^k = \frac{1}{(1-\beta)^{a_0}} \sum_{i=0}^{\infty}\sum_{l=0}^{a_0}\sum_{j=0}^{k} \begin{bmatrix} a_0 \\ l \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} \alpha^{k-j}(1-\alpha-\beta)^j(-\beta)^{a_0-l} w^{i+i_0-a_0} \cdot (m(i+i_0-a_0)+l)\, x^{m(k+i+i_0-a_0)+l-1-j}$$

Next, it can be shown (Appendix A.7) that $\forall (N \geq 0)$, $\frac{d^k}{dx^k}(x^N) = \begin{bmatrix} N \\ k \end{bmatrix} k! x^{N-k}$ and hence, from the above, we have:

$$\left[\frac{d^{k-1}[\psi'(x)g(x)^k]}{dx^{k-1}}\right]_{x=\beta} = \frac{1}{(1-\beta)^{a_0}} \sum_{i=0}^{\infty}\sum_{l=0}^{a_0}\sum_{j=0}^{k} \begin{bmatrix} a_0 \\ l \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} \begin{bmatrix} m(k+i+i_0-a_0)+l-1-j \\ k-1 \end{bmatrix} (k-1)!\,\alpha^{k-j}(1-\alpha-\beta)^j \cdot$$

$$\cdot (-\beta)^{a_0-l} w^{i+i_0-a_0} \cdot \ (m(i+i_0-a_0)+l)\,\beta^{m(k+i+i_0-a_0)+l-j-k} \qquad (3.37)$$

In addition, from (3.35) we note that:

$$\psi(\beta) = \begin{cases} 0 & (a_0 \neq 0) \\ \frac{(w\beta^m)^{i_0}}{1-w\beta^m} & (a_0 = 0) \end{cases} \qquad (3.38)$$

Hence $\forall (0 < a_0 \leq m)$ (3.36) and (3.37-3.38) yields:

$$P(w) = \frac{1}{(1-\beta)^{a_0}} \sum_{k=1}^{\infty} \sum_{i=0}^{\infty}\sum_{l=0}^{a_0}\sum_{j=0}^{k} \frac{w^{i+i_0-a_0+k}}{k} \begin{bmatrix} a_0 \\ l \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} \begin{bmatrix} m(k+i+i_0-a_0)+l-1-j \\ k-1 \end{bmatrix} \alpha^{k-j}(1-\alpha-\beta)^j \cdot$$

$$\cdot (-\beta)^{a_0-l}(m(i+i_0-a_0)+l)\,\beta^{m(k+i+i_0-a_0)+l-j-k} \qquad (3.39)$$

Let $\varsigma_0 = i_0 - a_0$. Then with the change of variables: $\kappa = k+i+\varsigma_0$ we obtain:

$$P(w) = \frac{1}{(1-\beta)^{a_0}} \sum_{i=0}^{\infty} \sum_{\kappa=1+i+\varsigma_0}^{\infty} \sum_{l=0}^{a_0} \sum_{j=0}^{\kappa-i-\varsigma_0} \frac{w^{\kappa}}{\kappa-i-\varsigma_0} \begin{bmatrix} a_0 \\ l \end{bmatrix} \begin{bmatrix} \kappa-i-\varsigma_0 \\ j \end{bmatrix} \begin{bmatrix} m\kappa+l-1-j \\ \kappa-i-\varsigma_0-1 \end{bmatrix} \alpha^{\kappa-i-\varsigma_0-j}(1-\alpha-\beta)^j \cdot$$

$$\cdot (-1)^{a_0-l}(m(i+\varsigma_0)+l)\,\beta^{(m-1)\kappa-j+i+i_0}$$

or equivalently:

$$P(w) = \frac{1}{(1-\beta)^{a_0}} \sum_{\kappa=1+\varsigma_0}^{\infty} \sum_{i=0}^{\kappa-1-\varsigma_0} \sum_{l=0}^{a_0} \sum_{j=0}^{\kappa-i-\varsigma_0} \frac{w^{\kappa}}{\kappa-i-\varsigma_0} \begin{bmatrix} a_0 \\ l \end{bmatrix} \begin{bmatrix} \kappa-i-\varsigma_0 \\ j \end{bmatrix} \begin{bmatrix} m\kappa+l-1-j \\ \kappa-i-\varsigma_0-1 \end{bmatrix} \alpha^{\kappa-i-\varsigma_0-j}(1-\alpha-\beta)^j$$

$$\cdot (-1)^{a_0-l}(m(i+\varsigma_0)+l)\,\beta^{(m-1)\kappa-j+i+i_0}$$

If we further let $\iota = \kappa - i - \varsigma_0$ then we obtain:

$$P(w) = \frac{1}{(1-\beta)^{a_0}} \sum_{\kappa = 1+\varsigma_0}^{\infty} \sum_{\iota=1}^{\kappa-\varsigma_0} \sum_{l=0}^{a_0} \sum_{j=0}^{\iota} \frac{w^{\kappa}}{\iota} \begin{bmatrix} a_0 \\ l \end{bmatrix} \begin{bmatrix} \iota \\ j \end{bmatrix} \begin{bmatrix} m\kappa + l - 1 - j \\ \iota - 1 \end{bmatrix} \alpha^{\iota - j} (1-\alpha-\beta)^j \cdot$$

$$\cdot (-1)^{a_0 - l} (m(k-\iota)+l) \beta^{m\kappa - j + a_0 - \iota} \qquad (3.40)$$

Finally, identifying the coefficients of w^{κ} in (3.40) yields the following closed -form expression for the transient probabilities of an empty buffer: $\forall (0 < a_0 \le m)$:

$$p_k(0) = 0 \qquad (0 \le k < 1 + i_0 - a_0)$$

and $\forall (k \ge 1 + i_0 - a_0)$:

$$p_k(0) = \frac{1}{(1-\beta)^{a_0}} \sum_{\iota=1}^{k-\varsigma_0} \sum_{l=0}^{a_0} \sum_{j=0}^{\iota} \begin{bmatrix} a_0 \\ l \end{bmatrix} \begin{bmatrix} \iota \\ j \end{bmatrix} \begin{bmatrix} mk + l - 1 - j \\ \iota - 1 \end{bmatrix} \alpha^{\iota - j} (1-\alpha-\beta)^j (-1)^{a_0 - l} \frac{(m(k-\iota)+l)}{\iota} \beta^{mk - j + a_0} \qquad (3.41)$$

For the case, where all sources are initially *OFF* $(a_0 = 0)$, and using ((3.36)-(3.38)), the transient probabilities $p_k(0)$'s become:

$$p_k(0) = \begin{cases} \beta^{mk} + \sum_{\iota=1}^{k-i_0} \sum_{j=0}^{\iota} \begin{bmatrix} \iota \\ j \end{bmatrix} \begin{bmatrix} mk - j - 1 \\ \iota - 1 \end{bmatrix} \frac{m(k-\iota)}{\iota} \alpha^{\iota - j} (1-\alpha-\beta)^j \beta^{mk-j-\iota} & (k \ge i_0) \\ 0 & (k < i_0) \end{cases} \qquad (3.42)$$

Another interesting special case arises if the system is initially empty will all sources being in the *OFF* state. Under this initial condition, and by substituting $i_0 = 0$ in the last expression, we obtain:

$$p_k(0) = \beta^{mk} + \sum_{\iota=1}^{k-1} \sum_{j=0}^{\iota} \begin{bmatrix} \iota \\ j \end{bmatrix} \begin{bmatrix} mk - j - 1 \\ \iota - 1 \end{bmatrix} \frac{m(k-\iota)}{\iota} \alpha^{\iota - j} (1-\alpha-\beta)^j \beta^{mk-j-\iota} \qquad (3.43)$$

In this case, equation ((3.41)-(3.43)) along with (3.18) give a closed-form expression for the transient PGF of the queue length. From this PGF, time-dependent performance measures for the ATM buffer can be derived under various initial states.

3.2.6 Transient Mean of the Queue Length Distribution

Let $\bar{N}_k = \left.\frac{dP_k(z)}{dz}\right|_{z=1}$ denote the average queue length of the ATM multiplexer at the end of the k^{th} slot. Differentiating (3.18) with respect to z, substituting z=1 in the resulting expression and by taking into account the fact that:

$$\tilde{C}_1|_{z=1} = \tilde{D}_1|_{z=1} = 0, \qquad \tilde{C}_2|_{z=1} = \tilde{D}_2|_{z=1} = 1, \quad \lambda_1|_{z=1} = \beta+\alpha-1, \qquad \lambda_2|_{z=1} = 1$$

$$\left.\frac{d\lambda_2}{dz}\right|_{z=1} = \frac{1-\beta}{2-\alpha-\beta}\bar{f}, \qquad \left.\frac{d\tilde{C}_2}{dz}\right|_{z=1} = -\left.\frac{d\tilde{C}_1}{dz}\right|_{z=1} = \frac{(1-\beta)(1-\alpha-\beta)}{(2-\alpha-\beta)^2}\bar{f}$$

$$\left.\frac{d\tilde{D}_2}{dz}\right|_{z=1} = -\left.\frac{d\tilde{D}_1}{dz}\right|_{z=1} = \frac{(\alpha-1)(1-\alpha-\beta)}{(2-\alpha-\beta)^2}\bar{f}$$

we get:

$$\bar{N}_k = \left.\frac{d\tilde{B}(k)}{dz}\right|_{z=1} - k + \frac{d}{dz}Q_0(z, \tilde{\Phi}(k))|_{z=1} + \sum_{j=1}^{k} p_{k-j}(0) \tag{3.44}$$

Next let:

$$\chi(z) = Q_0(z, \tilde{\Phi}(k)) = \sum_{i=0}^{\infty}\sum_{j=0}^{m} z^i \tilde{\Phi}(k)^j p_0(i,j).$$

Then $\chi'(1) = \bar{N}_0 + \tilde{\Phi}'(k)|_{z=1} \cdot \bar{A}_0$, where $\bar{A}_0 = \left.\frac{dA_0(y)}{dy}\right|_{y=1}$, with $A_0(y)$ being the initial PGF of the number of active sources, as defined as in (3.16). One can easily verify that:

$$\tilde{\Phi}'(k)|_{z=1} = [\tilde{D}'_2(1) - \tilde{C}'_2(1)](1-(\alpha+\beta-1)^k) = \frac{\alpha+\beta-1}{2-\alpha-\beta}\bar{f}(1-(\alpha+\beta-1)^k)$$

and therefore:

$$\bar{N}_k = \frac{m(1-\beta)(1-\alpha-\beta)}{(2-\alpha-\beta)^2}\bar{f}(1-(\alpha+\beta-1)^k) - k(1-\rho) + \bar{N}_0$$
$$+ \frac{\alpha+\beta-1}{2-\alpha-\beta}\bar{f}(1-(\alpha+\beta-1)^k)\bar{A}_0 + \sum_{j=1}^{k} p_{k-j}(0) \tag{3.45}$$

or equivalently:

$$\bar{N}_k = \bar{N}_0 + \frac{1-\alpha-\beta}{2-\alpha-\beta}\dot{f}\left(1-(\beta+\alpha-1)^k\right)\left[\frac{m(1-\beta)}{2-\alpha-\beta} - \bar{A}_0\right] - k(1-\rho) + \sum_{j=0}^{k-1} p_j(0) \qquad (3.46)$$

Using the results of the previous section, we can compute the transient probabilities $p_j(0)$'s and hence, from the above equation, we can evaluate the average queue length at the end of any particular slot.

We also note that the above expression (in the "time" domain) for the transient mean of the queue length is much simpler and more handy to use than the corresponding result (in the transform domain) obtained in ([29],[30]) using the spectral decomposition approach.

3.2.7 Transient Variance of the Queue Length Distribution

By differentiating (3.18) twice with respect to z and evaluating the resulting expression at z=1, we get:

$$\left.\frac{d^2 P_k(z)}{dz^2}\right|_{z=1} = \left.\frac{d^2 \tilde{B}(k)}{dz^2}\right|_{z=1} + k(k+1) - 2k\left.\frac{d\tilde{B}(k)}{dz}\right|_{z=1} + 2\left[\left.\frac{d\tilde{B}(k)}{dz}\right|_{z=1} - k\right]\left.\frac{dQ_0(z,\tilde{\Phi}(k))}{dz}\right|_{z=1}$$

$$+ \frac{d^2}{dz^2}Q_0(z,\tilde{\Phi}(k))\Big|_{z=1} + 2\sum_{j=1}^{k}\left[\left.\frac{d\tilde{B}(j)}{dz}\right|_{z=1} - j\right]p_{k-j}(0) \qquad (3.47)$$

To evaluate the above expression, we also need the following intermediate results:

$$\left.\frac{d\lambda_1}{dz}\right|_{z=1} = \frac{(1-\alpha)(1-\alpha-\beta)}{\alpha+\beta-2}\dot{f}\ ,\quad \left.\frac{d^2\lambda_2}{dz^2}\right|_{z=1} = \frac{2(1-\alpha)(1-\beta)(1-\alpha-\beta)}{(\beta+\alpha-2)^3}[\dot{f}]^2 + \frac{1-\beta}{2-\alpha-\beta}f''(1)$$

$$\left.\frac{d^2\tilde{C}_2}{dz^2}\right|_{z=1} = -\left.\frac{d^2\tilde{C}_1}{dz^2}\right|_{z=1} = \frac{2(1-\beta)(\alpha\beta-2\beta+1-\alpha+\alpha^2)(\alpha+\beta-1)}{(\beta+\alpha-2)^4}[\dot{f}]^2 + \frac{(1-\beta)(1-\beta-\alpha)}{(2-\beta-\alpha)^2}f''(1)$$

$$\left.\frac{d^2\tilde{D}_2}{dz^2}\right|_{z=1} = -\left.\frac{d^2\tilde{D}_1}{dz^2}\right|_{z=1} = \frac{2(1-\alpha)(\alpha^2-2\alpha+3+\alpha\beta-3\beta)(1-\alpha-\beta)}{(2-\alpha-\beta)^4}[\dot{f}]^2 - \frac{(1-\alpha)(1-\alpha-\beta)}{(2-\alpha-\beta)^2}f''(1)$$

From the above, and after some algebraic manipulations, $\left.\frac{d^2\tilde{B}(k)}{dz^2}\right|_{z=1}$ and $\left.\frac{d\tilde{B}(k)}{dz}\right|_{z=1}$ are readily obtained as follows:

$$\left.\frac{d\tilde{B}(k)}{dz}\right|_{z=1} = m\frac{(1-\beta)(1-\beta-\alpha)}{(2-\beta-\alpha)^2}\bar{f}(1-(\beta+\alpha-1)^k)+k\rho$$

$$\left.\frac{d^2\tilde{B}(k)}{dz^2}\right|_{z=1} = m(m-1)\left[\frac{(1-\beta)(1-\beta-\alpha)}{(2-\beta-\alpha)^2}\bar{f}(1-(\beta+\alpha-1)^k)+k\frac{1-\beta}{2-\alpha-\beta}\bar{f}\right]^2$$

$$+m\left(\left[\frac{2(1-\beta)(\alpha\beta-2\beta+1-\alpha+\alpha^2)(\alpha+\beta-1)}{(\beta+\alpha-2)^4}[\bar{f}]^2+\frac{(1-\beta)(1-\beta-\alpha)}{(2-\beta-\alpha)^2}f''(1)\right](1-(\alpha+\beta-1)^k)\right.$$

$$+\frac{2k(1-\beta)(1-\alpha)(1-\beta-\alpha)}{(\alpha+\beta-2)^3}[\bar{f}]^2(\alpha+\beta-1)^k+\frac{2k(1-\beta)^2(1-\beta-\alpha)[\bar{f}]^2}{(2-\alpha-\beta)^3}$$

$$\left.+\frac{k(k-1)(1-\beta)^2}{(2-\alpha-\beta)^2}[\bar{f}]^2+\frac{2k(1-\alpha)(1-\beta)(1-\alpha-\beta)}{(\alpha+\beta-2)^3}[\bar{f}]^2+\frac{k(1-\beta)}{2-\alpha-\beta}f''(1)\right)$$

In addition, $\chi''(1) = \left.\frac{d^2}{dz^2}Q_0(z,\tilde{\Phi}(k))\right|_{z=1}$ is given by:

$$\chi''(1) = \sum_{i=0}^{\infty}\sum_{j=0}^{m}\left[i(i-1)+2ij\tilde{\Phi}'(k)|_{z=1}+j(j-1)[\tilde{\Phi}'(k)|_{z=1}]^2+j\tilde{\Phi}''(k)|_{z=1}\right]p_0(i,j)$$

$$= P''_0(1)+[\tilde{\Phi}'(k)|_{z=1}]^2\cdot A''_0(1)+\tilde{\Phi}''(k)|_{z=1}\cdot A'_0(1)+2\tilde{\Phi}'(k)|_{z=1}\cdot\left.\frac{\partial^2}{\partial z\partial y}Q_0(z,y)\right|_{z=y=1}$$

where:

$$\tilde{\Phi}''(k)|_{z=1} = [\tilde{C}''_1(1)-\tilde{D}''_1(1)][1-(\alpha+\beta-1)^k]+2k[\tilde{D}'_1(1)-\tilde{C}'_1(1)](\alpha+\beta-1)^{k-1}\lambda'_1(1)$$

$$+2\tilde{C}'_1(1)[\tilde{C}'_1(1)-\tilde{D}'_1(1)](\alpha+\beta-1)^{2k}+4\tilde{D}'_1(1)\tilde{C}'_1(1)(\alpha+\beta-1)^k-4[\tilde{C}'_1(1)]^2(\alpha+\beta-1)^k$$

$$+2k[\tilde{C}'_1(1)-\tilde{D}'_1(1)](\alpha+\beta-1)^k\lambda'_2(1)+2\tilde{C}'_1(1)[\tilde{C}'_1(1)-\tilde{D}'_1(1)]$$

with:

$$\tilde{C}''_1(1) = \left.\frac{d^2\tilde{C}_1}{dz^2}\right|_{z=1}, \tilde{D}''_1(1) = \left.\frac{d^2\tilde{D}_1}{dz^2}\right|_{z=1}, \tilde{C}'_1(1) = \left.\frac{d\tilde{C}_1}{dz}\right|_{z=1}, \tilde{D}'_1(1) = \left.\frac{d\tilde{D}_1}{dz}\right|_{z=1}$$

$$\lambda'_{1,2}(1) = \left.\frac{d\lambda_{1,2}}{dz}\right|_{z=1}.$$

From the above, the variance of the queue length distribution at the end of the k^{th} slot can be then computed from the general formula:

$$\sigma^2_{N_k} = \left.\frac{d^2P_k(z)}{dz^2}\right|_{z=1} + \overline{N}_k(1-\overline{N}_k)$$

3.2.8 Steady State PGF of the Buffer Occupancy Distribution

In this section, we show how from the transient PGF of the buffer content (3.18), we can derive the exact analytical expression for the corresponding steady-state result. From this PGF we show how to recover the computational formula for the mean buffer occupancy which was previously derived in [51] despite the unavailability of the PGF there. We will also give some further results related to the steady-state queue length behavior. We should note, at this stage, that the application of other solution techniques to this type of queueing model (ex. multivariate Markov Chain analysis, Matrix Analytic and spectral decomposition techniques) has often ended up with *general expressions* for the steady-state PGF and for the first moment of the buffer size, which are not very explicit (ex. [40], [74],[71]). In particular the combination of Matrix Analytic and spectral decomposition approaches yields a general matrix expression for the *marginal* PGF of the buffer content, which, most of the time, is expressed in terms of the probability generating matrix (p.g.m) of the superposition arrival process (or its corresponding eigenvalues, right and left eigenvectors). In addition, the exponential size of the p.g.m adds to the computational complexity of the approach. We should also remind the reader that *approximate solution techniques*, which are based on fluid approximation models, have also been proposed to deal with the type of problem we are envisaging here (see for example [35], [36]).

The steady-state PGF, $P(z) = \lim_{k \to \infty} P_k(z)$, of the queue length distribution can be found by applying Abel's theorem (Appendix A.4) to (3.23). Hence we can write $P(z) = \lim_{w \to 1^-} (1-w) P(z, w)$ or, equivalently:

$$P(z) = \lim_{w \to 1^-} (1-w) \sum_{i=0}^{\infty} \sum_{j=0}^{m} \sum_{\kappa=0}^{m} \frac{\sum_{l=[\kappa-j,0]^+}^{[m-j,\kappa]^-} \begin{bmatrix} m-j \\ l \end{bmatrix} \begin{bmatrix} j \\ \kappa-l \end{bmatrix} z^i p_0(i,j) \tilde{D}_1^{\kappa-l} \tilde{D}_2^{j-\kappa+l} \tilde{C}_1^{l} \tilde{C}_2^{m-j-l} z}{z - w\lambda_1^{\kappa}\lambda_2^{m-\kappa}}$$

$$+ \lim_{w \to 1^-} (1-w)(z-1) \sum_{\kappa=0}^{m} \begin{bmatrix} m \\ \kappa \end{bmatrix} \frac{\tilde{C}_1^{\kappa} \tilde{C}_2^{m-\kappa} P(w) w\lambda_1^{\kappa}\lambda_2^{m-\kappa}}{z - w\lambda_1^{\kappa}\lambda_2^{m-\kappa}}$$

Since the first limit converges to zero and since from Abel's theorem $\lim_{w \to 1^-} (1-w) P(w) = p_\infty(0)$ then the last equation reduces to:

$$P(z) = (1-\rho)(z-1) \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix} \frac{(\tilde{C}_1\lambda_1)^k (\tilde{C}_2\lambda_2)^{m-k}}{z - \lambda_1^k \lambda_2^{m-k}} \qquad (3.48)$$

Though ρ was already obtained in section 3.2.5.2, it can also be derived from (3.48) through the normalization condition, $P(1) = 1$, as follows:

First we note that since $\tilde{C}_1|_{z=1} = 0$ then, except for the first term, all the terms under the summation in (3.48) become zero when evaluated at z=1. Therefore it is convenient to rewrite the steady-state PGF of the buffer length as follows:

$$P(z) = (1-\rho)(z-1)\left[F(z) + \frac{G(z)}{z - H(z)}\right] \qquad (3.49)$$

where:

$$F(z) = \sum_{k=1}^{m} \binom{m}{k} \frac{(\tilde{C}_1\lambda_1)^k (\tilde{C}_2\lambda_2)^{m-k}}{z - \lambda_1^k \lambda_2^{m-k}}, \quad G(z) = (\tilde{C}_2\lambda_2)^m, \quad H(z) = \lambda_2^m$$

or equivalently:

$$P(z)(z-H(z)) = (1-\rho)(z-1)\left[((z-H(z))F(z)+G(z)\right] \tag{3.50}$$

Differentiating both sides of the above equation with respect to z yields:

$$P'(z)(z-H(z))+P(z)(1-H'(z)) = (1-\rho)[(z-H(z))F(z)+G(z)$$
$$+(z-1)(1-H'(z))F(z)+(z-H(z))F'(z)+G'(z)] \tag{3.51}$$

Since $F(1)=0$ and $G(1)=H(1)=1$ then substituting z=1 in the above yields:

$$\rho = H'(1) = \frac{m(1-\beta)}{2-\alpha-\beta}\bar{f}$$

in accordance with the result of section 3.2.5.2.

It is also interesting to note that we could have derived the steady-state load of the system, $\rho = 1-p_{\infty}(0)$, from the transient analysis results by determining the steady-state probability, $p_{\infty}(0)$, through the application of Abel's theorem to $P(w)$ in (3.30), i.e.:

$$p_{\infty}(0) = \lim_{w\to 1^-}(1-w)P(w)$$

Equation (3.48) is a fundamental result which has a significant impact on the steady state analysis of the ATM queue length behavior, as many performance measures can readily be derived from it, as illustrated below.

3.2.9 Steady State Mean and Variance of the Buffer Length

Let $\bar{N}$ denote the steady-state mean buffer length. Then by differentiating (3.51) with respect to z and substituting z=1 in the resulting expression we get:

$$\bar{N} = P'(1) = \frac{H''(1)}{2(1-H'(1))}+G'(1) \tag{3.52}$$

where:

$$G'(1) = \frac{m(1-\beta)(3-2\alpha-2\beta)}{(2-\alpha-\beta)^2}\bar{f} \tag{3.53a}$$

$$H''(1) = m(m-1)\left[\frac{(1-\beta)}{2-\alpha-\beta}\right]^2 [\bar{f}]^2 + m\left[\frac{2(1-\alpha)(1-\beta)(\alpha+\beta-1)}{(2-\alpha-\beta)^3}[\bar{f}]^2 + \frac{1-\beta}{2-\alpha-\beta}f''(1)\right] \tag{3.53b}$$

The expression of the average queue length, as given above, is remarkable in the sense that it depends only on the term of $P(z)$ in (3.48), which corresponds to the case $k = 0$. This expression is also equivalent to the corresponding result which was derived in [51]. Next we focus on the variance of the buffer occupancy distribution.

By differentiating (3.51) twice with respect to *z* and by substituting *z*=1 in the resulting expression we get:

$$P''(1) = \frac{H''(1)}{1-H'(1)}P'(1) + \frac{H'''(1)}{3[1-H'(1)]} + 2[1-H'(1)]F'(1) + G''(1) \tag{3.54}$$

where:

$$\begin{aligned} H'''(1) = {} & m(m-1)(m-2)\left[\frac{1-\beta}{2-\alpha-\beta}\bar{f}\right]^3 \\ & + 3m(m-1)\left[\frac{1-\beta}{2-\alpha-\beta}\bar{f}\right]\left[\frac{2(1-\alpha)(1-\beta)(\alpha+\beta-1)}{(2-\alpha-\beta)^3}[\bar{f}]^2 + \frac{1-\beta}{2-\alpha-\beta}f''(1)\right] \\ & + m\left[\frac{6(1-\beta)(1-\alpha)(\alpha\beta-2\beta+\alpha^2-2\alpha+2)(\alpha+\beta-1)}{(\alpha+\beta-2)^5}\bar{f}^3 + \frac{1-\beta}{2-\alpha-\beta}f'''(1)\right. \\ & \left. + \frac{6(1-\alpha)(1-\beta)(\alpha+\beta-1)}{(2-\alpha-\beta)^3}\bar{f}f''(1)\right] \end{aligned} \tag{3.55a}$$

$$F'(1) = \frac{m(1-\beta)(1-\alpha-\beta)^2}{(2-\alpha-\beta)^3}\bar{f} \tag{3.55b}$$

$$\begin{aligned} G''(1) = {} & m(m-1)\left[\frac{(1-\beta)(3-2\alpha-2\beta)}{(2-\alpha-\beta)^2}\right]^2 [\bar{f}]^2 + m\left[\frac{(1-\beta)(1-\alpha-\beta)}{(2-\alpha-\beta)^2}f''(1)\right. \\ & + \frac{2(1-\beta)(\alpha\beta-2\beta+1-\alpha+\alpha^2)(\alpha+\beta-1)}{(2-\alpha-\beta)^4}[\bar{f}]^2 + \frac{1-\beta}{2-\alpha-\beta}f''(1) \\ & \left. + \frac{2(1-\alpha)(1-\beta)(\alpha+\beta-1)}{(2-\alpha-\beta)^3}[\bar{f}]^2 + \frac{2(1-\beta)^2(1-\alpha-\beta)}{(2-\alpha-\beta)^3}[\bar{f}]^2\right] \end{aligned} \tag{3.56}$$

The variance of the queue length:

$$\sigma^2_N = P''(1) + P'(1)\,[1 - P'(1)] \tag{3.57}$$

can be then easily computed using (3.52)-(3.56).

3.2.10 Asymptotic Analysis

Let us consider the infinite source model which corresponds to the limiting case where $m \to \infty$ and $\beta \to 1$ such that $m(1-\beta) \to \Lambda$, in which case the load of the system becomes $\rho = \frac{\Lambda}{1-\alpha}\bar{f}$ [13]. The asymptotic results derived herein are applicable to the case where the number of sources feeding the multiplexer grows rapidly and the fraction of time spent by each source in the active state decreases so as to keep the traffic intensity approaching the constant ρ.

Without any loss of generality, let us assume that the system is initially empty, with all sources being in the *Off* state. First we focus on the transient distribution of the number of active users, whose PGF (3.17) can, in this case, be written as:

$$A_k(y) = \left[1 - (1-\beta)(1-y)\sum_{i=0}^{k-1}(\alpha+\beta-1)^i\right]^m$$

Let $\tilde{A}_k(y)$ be the corresponding PGF under the above limiting conditions. Then:

$$\tilde{A}_k(y) = \lim_{(m\to\infty,\,\beta\to 1)} A_k(y) = \lim_{m\to\infty}\left[1 - \frac{\Lambda}{m}(1-y)\sum_{i=0}^{k-1}\alpha^i\right]^m = e^{\Lambda\frac{1-\alpha^k}{1-\alpha}(y-1)}$$

The last expression shows that the number of active users at any particular slot, k, follows a Poisson process with a rate $\Lambda\left[\frac{1-\alpha^k}{1-\alpha}\right]$. In particular the corresponding steady-state distribution is also Poisson with a rate $\frac{\Lambda}{1-\alpha}$.

To derive the steady-state behavior of the buffer length under the above limiting case, we proceed as follows:

First by expanding $(\tilde{B}(k) = \tilde{C}_1\lambda_1^k + \tilde{C}_2\lambda_2^k)$ as a Taylor series around $\beta = 1$ we get:

$$\tilde{B}(k) = \tilde{C}_1\lambda_1^k + \tilde{C}_2\lambda_2^k = 1 - \frac{(1-\beta)(f(z)-1)}{1-\alpha f(z)}\left[\alpha f(z)\frac{1-[\alpha f(z)]^k}{1-\alpha f(z)} - k\right] + O((\beta-1)^2)$$

Therefore:

$$\lim_{(m\to\infty,\,\beta\to 1)} \tilde{B}(k) = \lim_{m\to\infty}\left(1 - \frac{\Lambda}{m}\frac{(f(z)-1)}{1-\alpha f(z)}\left[\alpha f(z)\frac{1-[\alpha f(z)]^k}{1-\alpha f(z)} - k\right]\right)^m$$

$$= e^{\frac{\Lambda(f(z)-1)}{1-\alpha f(z)}\left[k - \alpha f(z)\frac{1-[\alpha f(z)]^k}{1-\alpha f(z)}\right]}$$

Next let $\tilde{P}(z)$ denote the steady-state PGF of the buffer length, under the infinite source model. From the expansion of the second term, *II*, in (3.25) (see the proof of theorem 3.2), we note that the steady state PGF of the buffer content, as derived in (3.48), originally comes from the series:

$$P(z) = (1-\rho)(z-1)\sum_{k=1}^{\infty}\frac{\tilde{B}(k)}{z^k}$$

Hence, under our limiting case, with $\rho = \frac{\Lambda}{1-\alpha}\bar{f}$ we have:

$$\tilde{P}(z) = (1-\rho)(z-1)\sum_{k=1}^{\infty}\frac{e^{\frac{\Lambda(f(z)-1)}{1-\alpha f(z)}\left[k-\alpha f(z)\frac{1-[\alpha f(z)]^k}{1-\alpha f(z)}\right]}}{z^k} \tag{3.58}$$

The last equation expresses the steady-state PGF of the buffer occupancy distribution in terms of an infinite series. Finding a closed form expression for this series does not seem to be trivial, mainly because of the presence of the second expression in the exponential term. We might, however, approximate (3.58) as follows:

$$\tilde{P}(z) \cong (1-\rho)(z-1)\sum_{k=1}^{\infty}\left[\frac{e^{\frac{\Lambda(f(z)-1)}{1-\alpha f(z)}}}{z}\right]^k = (1-\rho)(z-1)\frac{\hat{H}(z)}{z-\hat{H}(z)}$$

where:

$$\hat{H}(z) = e^{\frac{\Lambda(f(z)-1)}{1-\alpha f(z)}}$$

Note that since $\hat{H}'(1) = \rho$, then the above approximation still gives a valid PGF which has the same ρ as the original function (3.48). In this case, the first few moments of the queue length can be easily derived. In particular, the mean of the buffer occupancy distribution is given by:

$$\overline{N} = \tilde{P}'(1) = \frac{\hat{H}''(1)}{2[1-\rho]} + \rho$$

with:

$$\hat{H}''(1) = \rho^2 + \frac{\Lambda f''(1)}{1-\alpha} + 2\alpha\Lambda\left[\frac{f'(1)}{1-\alpha}\right]^2$$

3.3 The Multiple Type of Traffic Case

Motivated by the fact that future broadband networks are expected to support multiple types of communication media (and hence of information sources), we devote this section to the generalization of the single server model of section 3.2, to the more general case where the sources are not necessarily identical. More specifically we assume that the multiplexer is fed with m_i sources of type *i*, where $i \in \{1, 2, \ldots \tau\}$. Each source of type *i* alternates between an *On* state where it generates at least one packet per active slot, with a PGF $f_i(z)$,and an *Off* state where no packets are generated. The lengths of the *On* and *Off* periods of a type *i* source are assumed to be geometrically distributed with means $\frac{1}{1-\alpha_i}$ and $\frac{1}{1-\beta_i}$, respectively. The rest of the model's assumptions are the same as those outlined at the beginning of section 3.1. Let a^i_k denote the number of active sources of type *i* during slot *k*. Then:

$$a^i_{k+1} = \sum_{j=1}^{a^i_k} c^i_j + \sum_{j=1}^{m_i - a^i_k} d^i_j \qquad (3.59a)$$

where c^i_j and d^i_j are i.i.d. Bernoulli random variables with corresponding PGFs:

$$c^i(z) = (1-\alpha_i) + \alpha_i z \tag{3.59b}$$

$$d^i(z) = \beta_i + (1-\beta_i) z \tag{3.59c}$$

In (3.59a), the first term represents the number of sources, of type i, which were active during slot k and which remain active during slot k+1, while the second term represents the number of sources, of type i, which were idle during slot k and which become active at the next slot.

Let b^i_k denote the number of packets which arrive at the multiplexer, from type i sources, during slot k. Then:

$$b^i_k = \sum_{j=1}^{a^i_k} f^{(i)}_{j,k}$$

where $f^{(i)}_{j,k}$ is the number of packets generated by the j^{th} active source of type i during slot k. All the $f^{(i)}_{j,k}$'s are assumed to be i.i.d with PGF $f_i(z)$. Next let b_k be the total number of packet arrivals to the multiplexer during slot k. Then:

$$b_k = \sum_{i=1}^{\tau} \sum_{j=1}^{a^i_k} f^{(i)}_{j,k} \tag{3.60}$$

Our first goal is to derive a functional equation which relates the joint PGF of the system between two consecutive slots. Once this is done, the rest of the analysis will follow the approach outlined previously for the single type of traffic case.

3.3.1 The Imbedded Markov Chain Analysis

The queueing model under consideration, here, can be formulated as a discrete-time multidimensional Markov chain. The state of the system is defined by $(i_k, a^1_k, a^2_k, \ldots, a^\tau_k)$, where i_k is the queue length at the end of slot k. The evolution of the queue length is determined by the equation:

$$i_{k+1} = i_k - U(i_k) + b_{k+1} \tag{3.61}$$

where $U(x)$ is a binary-valued random variable, which takes the value 1 if $x > 0$ and 0 otherwise. Next let:

$$Q_k(z, y_1, y_2, \ldots y_\tau) = E\left[z^{i_k} \cdot \prod_{i=1}^{\tau} y_i^{a_k^i}\right] = \sum_{\iota=0}^{\infty} \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} \cdots \sum_{j_\tau=0}^{m_\tau} z^{\iota}\left[\prod_{i=1}^{\tau} y_i^{j_i}\right] p_k(\iota, j_1, j_2, \ldots, j_\tau) \tag{3.62}$$

denote the joint PGF of $i_k, a^1{}_k, a^2{}_k \ldots a^\tau{}_k$. Then:

$$Q_{k+1}(z, y_1, y_2, \ldots y_\tau) = E\left[z^{i_{k+1}} \cdot \prod_{i=1}^{\tau} y_i^{a^i_{k+1}}\right] = E\left[z^{i_k - U(i_k)} \cdot z^{\sum_{j=1}^{\tau} \sum_{j=1}^{a^i_{k+1}} f^{(i)}_{j,k+1}} \cdot \prod_{i=1}^{\tau} y_i^{a^i_{k+1}}\right]$$

From the above, using (3.59) and averaging over the distribution of the $f^{(i)}_{j,k}$'s, the $c^i{}_j$ and the $d^i{}_j$ yields:

$$Q_{k+1}(z, y_1, y_2, \ldots y_\tau) = E\left[z^{i_k - U(i_k)} \cdot z^{\sum_{i=1}^{\tau} \sum_{j=1}^{a^i_{k+1}} f^{(i)}_{j,k+1}} \cdot \prod_{i=1}^{\tau} y_i^{a^i_{k+1}}\right]$$

$$= E\left[E\left[z^{i_k - U(i_k)} \cdot z^{\sum_{i=1}^{\tau} \sum_{j=1}^{a^i_{k+1}} f^{(i)}_{j,k+1}} \cdot \prod_{i=1}^{\tau} y_i^{a^i_{k+1}} \middle| i_k, a^1{}_{k+1}, a^2{}_{k+1} \ldots a^\tau{}_{k+1}\right]\right]$$

$$= E\left[z^{i_k - U(i_k)} \prod_{i=1}^{\tau} (y_i f_i(z))^{a^i_{k+1}}\right]$$

$$= E\left[E\left[z^{i_k - U(i_k)} \prod_{i=1}^{\tau} (y_i f_i(z))^{\sum_{j=1}^{a^i_k} c^i{}_j + \sum_{j=1}^{m_i - a^i_k} d^i{}_j} \middle| i_k, a^1{}_k, a^2{}_k \ldots a^\tau{}_k\right]\right]$$

$$= E\left[z^{i_k - U(i_k)} \cdot \prod_{i=1}^{\tau} [d^i(y_i f_i(z))]^{m_i} \left[\frac{c^i(y_i f_i(z))}{d^i(y_i f_i(z))}\right]^{a^i_k}\right] \tag{3.63}$$

or equivalently:

$$Q_{k+1}(z,y_1,y_2,\ldots,y_\tau) = \prod_{i=1}^{\tau}\left[d^i(y_if_i(z))\right]^{m_i}\cdot E\left[z^{i_k-U(i_k)}\cdot\prod_{i=1}^{\tau}Y_i^{a_k^i}\right] \tag{3.64}$$

where:

$$Y_i = \frac{c^i(y_if_i(z))}{d^i(y_if_i(z))} = \frac{1-\alpha_i+\alpha_iy_if_i(z)}{\beta_i+(1-\beta_i)y_if_i(z)}$$

In the sequel, the notation $\sum_{\bar{j}=\bar{0}}^{\bar{M}}$ will be used to refer to the multidimensional summation given by $\sum_{j_1=0}^{m_1}\sum_{j_2=0}^{m_2}\cdots\sum_{j_\tau=0}^{m_\tau}$. Further, $p_k(\iota,j_1,j_2,\ldots,j_\tau)$ and $Q_k(z,y_1,y_2,\ldots,y_\tau)$ will be simply referred by $p_k(\iota,\bar{J})$ and $Q_k(z,\bar{y})$ respectively. Then from (3.64), by taking into account the fact that when $i_k = 0$ the random variables $a^i{}_k$'s must also be zero (in other words, if the buffer is empty at the end of slot *k*, then all the sources should have been Off during slot *k*), we can remove the *U(x)*'s by noting that since:

$$\begin{aligned} E\left[z^{i_k-U(i_k)}\cdot\prod_{i=1}^{\tau}Y_i^{a_k^i}\right] &= \sum_{\iota=0}^{\infty}\sum_{\bar{j}=\bar{0}}^{\bar{M}} z^{i_k-U(i_k)}\cdot\left[\prod_{i=1}^{\tau}Y_i^{j_i}\right]p_k(\iota,\bar{J}) \\ &= p_k(0,\bar{0}) + \sum_{\iota=1}^{\infty}\sum_{\bar{j}=\bar{0}}^{\bar{M}} z^{(i-1)}\cdot\left[\prod_{i=1}^{\tau}Y_i^{j_i}\right]p_k(\iota,\bar{J}) \\ &= p_k(0,\bar{0}) + \frac{1}{z}\left[\sum_{\iota=0}^{\infty}\sum_{\bar{j}=\bar{0}}^{\bar{M}} z^{i}\cdot\left[\prod_{i=1}^{\tau}Y^{j}{}_i\right]p_k(\iota,\bar{J}) - p_k(0,\bar{0})\right] \end{aligned}$$

then:

$$Q_{k+1}(z,\bar{y}) = \prod_{i=1}^{\tau}\left[d^i(y_if_i(z))\right]^{m_i}\cdot\left[\frac{Q_k(z,Y_1,Y_2,\ldots,Y_\tau)-p_k(0)}{z}+p_k(0)\right] \tag{3.65}$$

where $p_k(0)=Q_k(0,\bar{0})$ is the probability of an empty system. Expanding the above equation for the first few values of *k*, enables us to prove by recurrence the following result:

3.3.2 Theorem 3.3:

The joint PGF of the system, as described by the functional equation (3.65), is given by:

$$Q_k(z,\vec{y}) = \frac{B(k)}{z^k} Q_0(z, \Phi_1(k), \Phi_2(k), \ldots \Phi_\tau(k)) + (z-1)\sum_{j=1}^{k} \frac{B(j)}{z^j} p_{k-j}(0) \quad (3.66)$$

where:

$$\Phi_i(k) = \frac{U_i(k)}{X_i(k)} = \frac{D_{1i}\lambda_{1i}^k + D_{2i}\lambda_{2i}^k}{C_{1i}\lambda_{1i}^k + C_{2i}\lambda_{2i}^k} \quad (3.67)$$

with:

$$\lambda_{1i,2i} = \frac{\beta_i + \alpha_i f_i(z) \mp \sqrt{(\beta_i + \alpha_i f_i(z))^2 + 4(1-\alpha_i - \beta_i) f_i(z)}}{2} \quad (3.68a)$$

$$C_{1i,2i} = \frac{1}{2} \mp \frac{2(y_i - y_i\beta_i - \alpha_i) f_i(z) + (\beta_i + \alpha_i f_i(z))}{2\sqrt{(\beta_i + \alpha_i f_i(z))^2 + 4(1-\alpha_i - \beta_i) f_i(z)}} \quad (3.68b)$$

$$D_{1i,2i} = \frac{y_i}{2} \mp \frac{2(1-\alpha_i + \alpha_i y_i f_i(z)) - (\beta_i + \alpha_i f_i(z)) y_i}{2\sqrt{(\beta_i + \alpha_i f_i(z))^2 + 4(1-\alpha_i - \beta_i) f_i(z)}} \quad (3.68c)$$

and:

$$B(k) = \prod_{i=1}^{\tau} [X_i(k)]^{m_i} \quad (3.68d)$$

PROOF

Substituting k=0 then k=1 in the functional equation (3.65) and, once again, using the fact that if $B^*(k) = B(k)|_{y_1 = Y_1,\, y_2 = Y_2 \ldots y_\tau = Y_\tau}$ then $B^*(k) = \frac{B(k+1)}{B(1)}$ (which is an immediate result of having: $X_i^*(k) = X_i(k)|_{y_i = Y_i} = \frac{X_i(k+1)}{X_i(1)}$), we get:

$$Q_1(z,\vec{y}) = B(1)\left[\frac{Q_0(z, Y_1, Y_2, \ldots, Y_\tau) - p_0(0)}{z} + p_0(0)\right]$$

$$= \frac{B(1)}{z} Q_0(z, \Phi_1(1), \Phi_2(1), \ldots \Phi_\tau(1)) + (z-1)\frac{B(1)}{z} p_0(0)$$

$$Q_2(z,\vec{y}) = B(1)\left[\frac{Q_1(z,Y_1,Y_2,\ldots,Y_\tau)-p_1(0)}{z}+p_1(0)\right]$$

$$=\frac{B(1)}{z}\left[\frac{B^*(1)}{z}Q_0(z,\Phi_1(2),\Phi_2(2),\ldots\Phi_\tau(2))+(z-1)\frac{B^*(1)}{z}p_0(0)\right]-\frac{B(1)}{z}p_1(0)+B(1)p_1(0)$$

$$=\frac{B(2)}{z^2}Q_0(z,\Phi_1(2),\Phi_2(2),\ldots\Phi_\tau(2))+(z-1)\sum_{j=1}^{2}\frac{B(j)}{z^j}p_{2-j}(0)$$

Therefore (3.66) is true for k=1,2 and also for k=0. Next let us suppose that (3.66)is true for the order k, i.e.:

$$Q_k(z,\vec{y}) = \frac{B(k)}{z^k}Q_0(z,\Phi_1(k),\Phi_2(k),\ldots\Phi_\tau(k))+(z-1)\sum_{j=1}^{k}\frac{B(j)}{z^j}p_{k-j}(0) \qquad (3.69)$$

Let us prove that it is also true for the order (k+1), i.e.:

$$Q_{k+1}(z,\vec{y}) = \frac{B(k+1)}{z^{k+1}}Q_0(z,\Phi_1(k+1),\Phi_2(k+1),\ldots\Phi_\tau(k+1))+(z-1)\sum_{j=1}^{k+1}\frac{B(j)}{z^j}p_{k+1-j}(0)$$

By substituting (3.69) into the functional equation (3.65) and using the fact that $B^*(k) = \dfrac{B(k+1)}{B(1)}$ we have:

$$Q_{k+1}(z,\vec{y})=\frac{B(1)}{z}\{\frac{B^*(k)}{z^k}Q_0(z,\Phi_1(k+1),\Phi_2(k+1),\ldots\Phi_\tau(k+1))+(z-1)\sum_{j=1}^{k}\frac{B^*(j)}{z^j}p_{k-j}(0)\}$$

$$-\frac{B(1)}{z}p_k(0)+B(1)p_k(0)$$

$$=\frac{B(k+1)}{z^{(k+1)}}Q_0(z,\Phi_1(k+1),\Phi_2(k+1),\ldots\Phi_\tau(k+1))+(z-1)\sum_{j=1}^{k+1}\frac{B(j)}{z^j}p_{k+1-j}(0)$$

which completes the proof of the theorem.□

3.3.3 Transient/Steady-State Analysis of the Number of Active Users

Let $A_k(\vec{y})=Q_k(1,\vec{y})$ denote the marginal PGF of the number of active users during slot k. Then from (3.66):

$$A_k(\vec{y})=B(k)Q_0(1,\Phi_1(k),\Phi_2(k),\ldots\Phi_\tau(k))\big|_{z=1}=B(k)A_0(\Phi_1(k),\ldots,\Phi_\tau(k))\big|_{z=1}$$

Further, if we assume that $A_0(\hat{y}) = \prod_{i=1}^{\tau} [\pi_{i0}(0) + \pi_{i1}(0) y_i]^{m_i}$, where $\pi_{i0}(0)$ and $\pi_{i1}(0)$ are the probabilities of a type *i* source being initially *Off* and *On* respectively, then we obtain:

$$A_k(\hat{y}) = B(k) \prod_{i=1}^{\tau} [\pi_{i0}(0) + \pi_{i1}(0) \Phi_i(k)]^{m_i} \Big|_{z=1}$$

Substituting $B(k) = \prod_{i=1}^{\tau} [X_i(k)]^{m_i}$ and $\Phi_i(k) = \frac{U_i(k)}{X_i(k)}$ in the above equation gives:

$$A_k(\hat{y}) = \prod_{i=1}^{\tau} [\pi_{i0}(0) X_i(k) + \pi_{i1}(0) U_i(k)]^{m_i} \Big|_{z=1}$$

or equivalently:

$$A_k(\hat{y}) = \prod_{i=1}^{\tau} [\pi_{i0}(k) + \pi_{i1}(k) y_i]^{m_i} \tag{3.70}$$

where:

$$\pi_{i1}(k) = \pi_{i1}(0)\,(\alpha_i + \beta_i - 1)^k + \frac{1-\beta_i}{2-\alpha_i-\beta_i}(1-(\alpha_i+\beta_i-1)^k)$$

$$\pi_{i0}(k) = 1 - \pi_{i1}(k)$$

The steady-state PGF of the number of active sources is obtained by letting $k \to \infty$ in (3.70), giving:

$$A_\infty(\hat{y}) = \prod_{i=1}^{\tau} \left[\frac{1-\alpha_i}{2-\alpha_i-\beta_i} + \frac{1-\beta_i}{2-\alpha_i-\beta_i} y_i\right]^{m_i}$$

In addition since each source of type *i* generates, on the average, $\bar{f}_i = \frac{d}{dz} f_i(z) \Big|_{z=1}$ packets per active slot, then the load of the system is given by:

$$\rho = \sum_{i=1}^{\tau} m_i \frac{1-\beta_i}{2-\alpha_i-\beta_i} \bar{f}_i \tag{3.71}$$

We next focus on the transient/steady-state behavior of the buffer occupancy.

3.3.4 Transient and Steady-State Analysis of the Buffer Occupancy Distribution

Let $P_k(z) = Q_k(z, 1, 1, \ldots, 1)$ denote the marginal PGF of the buffer occupancy distribution at the end of the k^{th} slot. Then from (3.66) we have:

$$P_k(z) = \frac{\tilde{B}(k)}{z^k} Q_0(z, \tilde{\Phi}_1(k), \tilde{\Phi}_2(k), \ldots \tilde{\Phi}_\tau(k)) + (z-1) \sum_{j=1}^{k} \frac{\tilde{B}(j)}{z^j} p_{k-j}(0) \qquad (3.72)$$

where:

$$\tilde{B}(k) = B(k)\big|_{y_1 = y_2 = \ldots = y_\tau = 1} = \prod_{i=1}^{\tau} [\tilde{X}_i(k)]^{m_i} = \prod_{i=1}^{\tau} (\tilde{C}_{1i}\lambda_{1i}^k + \tilde{C}_{2i}\lambda_{2i}^k)^{m_i} \qquad (3.73a)$$

$$\tilde{\Phi}_i(k) = \Phi_i(k)\big|_{y_i = 1} = \frac{\tilde{U}_i(k)}{\tilde{X}_i(k)} = \frac{\tilde{D}_{1i}\lambda_{1i}^k + \tilde{D}_{2i}\lambda_{2i}^k}{\tilde{C}_{1i}\lambda_{1i}^k + \tilde{C}_{2i}\lambda_{2i}^k} \qquad (3.73b)$$

with $\tilde{C}_{ri} = C_{ri}\big|_{y_i = 1}$ and $\tilde{D}_{ri} = D_{ri}\big|_{y_i = 1}$ $\forall (r \in \{1, 2\})$, while λ_{ri}, C_{ri} and D_{ri} are as given in (3.68).

From (3.72) we note that the transient probabilities, $p_k(0)$'s, are the only terms which remain to be evaluated in order to fully characterize the transient PGF of the queue size. The following theorem provides a means to compute them.

3.3.4.1 Theorem 3.4

Let $P(z, w)$ and $P(w)$ be the one-dimensional transforms, defined by:

$$P(z, w) = \sum_{k=0}^{\infty} P_k(z) w^k \qquad (|w| < 1) \qquad (3.74)$$

and

$$P(w) = \sum_{k=0}^{\infty} p_k(0) w^k \qquad (|w| < 1) \qquad (3.75)$$

respectively. Then:

$$P(z,w)=\sum_{\iota=0}^{\infty}\sum_{\bar{J}=\bar{0}}^{\bar{M}}\sum_{\bar{K}=\bar{0}}^{\bar{M}}\frac{\displaystyle\sum_{\bar{l}=[\bar{K}-\bar{J},\bar{0}]^{+}}^{[\bar{M}-\bar{J},\bar{K}]^{-}}\left[\prod_{i=1}^{\tau}\begin{bmatrix}j_i\\k_i-l_i\end{bmatrix}\begin{bmatrix}m_i-j_i\\l_i\end{bmatrix}\tilde{D}_{1i}^{k_i-l_i}\tilde{D}_{2i}^{j_i-k_i+l_i}\tilde{C}_{1i}^{l_i}\tilde{C}_{2i}^{m_i-j_i-l_i}\right]z^{\iota}p_0(\iota,\bar{J})\,z}{z-w\displaystyle\prod_{i=1}^{\tau}\lambda_{1i}^{k_i}\lambda_{2i}^{m_i-k_i}}$$

$$+(z-1)P(w)\sum_{\bar{K}=\bar{0}}^{\bar{M}}\frac{\left[\displaystyle\prod_{i=1}^{\tau}\begin{bmatrix}m_i\\k_i\end{bmatrix}\tilde{C}_{1i}^{k_i}\tilde{C}_{2i}^{m_i-k_i}\right]w\displaystyle\prod_{i=1}^{\tau}\lambda_{1i}^{k_i}\lambda_{2i}^{m_i-k_i}}{z-w\displaystyle\prod_{i=1}^{\tau}\lambda_{1i}^{k_i}\lambda_{2i}^{m_i-k_i}} \qquad (3.76)$$

where the notations $\sum_{\bar{K}=\bar{0}}^{\bar{M}}$ and $\sum_{\bar{l}=[\bar{K}-\bar{J},\bar{0}]^{+}}^{[\bar{M}-\bar{J},\bar{K}]^{-}}$ refer to the multidimensional summations:

$$\sum_{k_1=0}^{m_1}\sum_{k_2=0}^{m_2}\sum_{k_3=0}^{m_3}\cdot\cdot\sum_{k_\tau=0}^{m_\tau}$$

and

$$\sum_{l_1=[k_1-j_1,0]^{+}}^{[m_1-j_1,k_1]^{-}}\sum_{l_2=[k_2-j_2,0]^{+}}^{[m_2-j_2,k_2]^{-}}\sum_{l_3=[k_3-j_3,0]^{+}}^{[m_3-j_3,k_3]^{-}}\cdot\cdot\sum_{l_\tau=[k_\tau-j_\tau,0]^{+}}^{[m_\tau-j_\tau,k_\tau]^{-}}$$

respectively.

In addition, with $H(z)=\prod_{i=1}^{\tau}\lambda_{2i}^{m_i}$, we have:

$$P(w)=\frac{Q_0(z^{*},r_1(z^{*}),r_2(z^{*}),\cdot\quad\cdot\ r_\tau(z^{*}))}{1-z^{*}} \qquad (3.77)$$

where z^* is the unique solution of the equation, $z=wH(z)$, inside the unit circle and $r_i(z)=\dfrac{\lambda_{2i}-\beta_i}{(1-\beta_i)f_i(z)}$.

PROOF

Let $P(z, w)$ be the one-dimensional transform, defined by:

$$P(z, w) = \sum_{k=0}^{\infty} P_k(z)\, w^k \qquad (|w| < 1) \tag{3.78}$$

Then from (3.72):

$$P(z, w) = \sum_{k=0}^{\infty} \tilde{B}(k)\, Q_0(z, \tilde{\Phi}_1(k), \tilde{\Phi}_2(k), \ldots \tilde{\Phi}_\tau(k)) \left[\frac{w}{z}\right]^k + (z-1) \sum_{k=0}^{\infty} \sum_{j=1}^{k} \frac{\tilde{B}(j)}{z^j} p_{k-j}(0)\, w^k \tag{3.79}$$

We first look to the first term, $I = \sum_{k=0}^{\infty} \tilde{B}(k)\, Q_0(z, \tilde{\Phi}_1(k), \tilde{\Phi}_2(k), \ldots \tilde{\Phi}_\tau(k)) \left[\frac{w}{z}\right]^k$. Since:

$$Q_0(z, \tilde{\Phi}_1(k), \tilde{\Phi}_2(k), \ldots \tilde{\Phi}_\tau(k)) = \sum_{\iota=0}^{\infty} \sum_{\bar{J}=\bar{0}}^{\bar{M}} z^{\iota} \left[\prod_{i=1}^{\tau} [\tilde{\Phi}_i(k)]^{j_i}\right] p_0(\iota, \bar{J})$$

then by substituting for $\tilde{B}(k)$ and $\tilde{\Phi}_i(k)$ as in (3.73) and by applying the Binomial theorem, we get:

$$I = \sum_{k=0}^{\infty} \sum_{\iota=0}^{\infty} \sum_{\bar{J}=\bar{0}}^{\bar{M}} z^{\iota} \left[\prod_{i=1}^{\tau} U_i(k)^{j_i} X_i(k)^{m_i - j_i}\right] p_0(\iota, \bar{J}) \left[\frac{w}{z}\right]^k$$

$$= \sum_{k=0}^{\infty} \sum_{\iota=0}^{\infty} \sum_{\bar{J}=\bar{0}}^{\bar{M}} z^{\iota} p_0(\iota, \bar{J}) \left[\prod_{i=1}^{\tau} \sum_{r_i=0}^{m_i - j_i} \sum_{s_i=0}^{j_i} \begin{bmatrix} j_i \\ s_i \end{bmatrix} \begin{bmatrix} m_i - j_i \\ r_i \end{bmatrix} (\bar{D}_{1i}\lambda_{1i}^k)^{s_i} (\bar{D}_{2i}\lambda_{2i}^k)^{j_i - s_i} (\bar{C}_{1i}\lambda_{1i}^k)^{r_i} (\bar{C}_{2i}\lambda_{2i}^k)^{m_i - j_i - r_i}\right] \left[\frac{w}{z}\right]^k$$

Interchanging the order of summations gives:

$$I = \sum_{\iota=0}^{\infty} \sum_{\bar{J}=\bar{0}}^{\bar{M}} \sum_{\bar{R}=\bar{0}}^{\bar{M}-\bar{J}} \sum_{\bar{S}=\bar{0}}^{\bar{J}} z^{\iota} p_0(\iota, \bar{J}) \left[\prod_{i=1}^{\tau} \begin{bmatrix} j_i \\ s_i \end{bmatrix} \begin{bmatrix} m_i - j_i \\ r_i \end{bmatrix} \bar{D}_{1i}^{s_i} \bar{D}_{2i}^{j_i - s_i} \bar{C}_{1i}^{r_i} \bar{C}_{2i}^{m_i - j_i - r_i}\right] \sum_{k=0}^{\infty} \left[\frac{w \prod_{i=1}^{\tau} \lambda_{1i}^{(r_i + s_i)} \lambda_{2i}^{m_i - ((r_i + s_i))}}{z}\right]^k \tag{3.80}$$

where we used the notations $\sum_{\bar{S}=\bar{0}}^{\bar{J}}$ and $\sum_{\bar{R}=\bar{0}}^{\bar{M}-\bar{J}}$ to refer to the multidimensional summations, $\sum_{s_1=0}^{j_1} \sum_{s_2=0}^{j_2} \cdots \sum_{s_\tau=0}^{j_\tau}$ and $\sum_{r_1=0}^{m_1-j_1} \sum_{r_2=0}^{m_2-j_2} \cdots \sum_{r_\tau=0}^{m_\tau-j_\tau}$, respectively.

Finally the last term in (3.80) can be further simplified to yield:

$$I = \sum_{\iota=0}^{\infty} \sum_{\bar{J}=\bar{0}}^{\bar{M}} \sum_{\bar{R}=\bar{0}}^{\bar{M}-\bar{J}} \sum_{\bar{S}=\bar{0}}^{\bar{J}} \frac{\left[\prod_{i=1}^{\tau} \begin{bmatrix} j_i \\ s_i \end{bmatrix} \begin{bmatrix} m_i - j_i \\ r_i \end{bmatrix} \tilde{D}_{1i}^{s_i} \tilde{D}_{2i}^{j_i - s_i} \tilde{C}_{1i}^{r_i} \tilde{C}_{2i}^{m_i - j_i - r_i}\right] z^{\iota} p_0(\iota, \bar{J})\, z}{z - w \prod_{i=1}^{\tau} \lambda_{1i}^{(r_i + s_i)} \lambda_{2i}^{m_i - ((r_i + s_i))}}$$

Next we consider the second term in (3.79) which can be expanded as follows:

$$II = (z-1) \sum_{k=0}^{\infty} \sum_{j=1}^{k} \frac{\tilde{B}(j)}{z^j} p_{k-j}(0)\, w^k \quad = (z-1) \left[\sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{\tilde{B}(j)}{z^j} p_{k-j}(0)\, w^k - \sum_{k=0}^{\infty} p_k(0)\, w^k \right]$$

$$= (z-1) \left[P(w) \cdot \sum_{k=0}^{\infty} \frac{\tilde{B}(k)}{z^k} w^k - P(w) \right] \quad = (z-1) P(w) \cdot \sum_{k=1}^{\infty} \frac{\tilde{B}(k)}{z^k} w^k$$

Once again, substituting for $\tilde{B}(k)$ as in (3.73a), using the Binomial theorem and interchanging the order of summations gives:

$$II = (z-1) P(w) \cdot \sum_{k=1}^{\infty} \left[\prod_{i=1}^{\tau} (\tilde{C}_{1i} \lambda_{1i}^k + \tilde{C}_{2i} \lambda_{2i}^k)^{m_i} \right] \left[\frac{w}{z} \right]^k$$

$$= (z-1) P(w) \cdot \sum_{k=1}^{\infty} \left[\prod_{i=1}^{\tau} \sum_{j_i=0}^{m_i} \begin{bmatrix} m_i \\ j_i \end{bmatrix} (\tilde{C}_{1i} \lambda_{1i}^k)^{j_i} \cdot (\tilde{C}_{2i} \lambda_{2i}^k)^{m_i - j_i} \right] \left[\frac{w}{z} \right]^k$$

$$= (z-1) P(w) \sum_{\bar{J}=\bar{0}}^{\bar{M}} \left[\prod_{i=1}^{\tau} \begin{bmatrix} m_i \\ j_i \end{bmatrix} (\tilde{C}_{1i})^{j_i} \cdot (\tilde{C}_{2i})^{m_i - j_i} \right] \sum_{k=1}^{\infty} \left[\frac{w \prod_{i=1}^{\tau} \lambda_{1i}^{j_i} \lambda_{2i}^{m_i - j_i}}{z} \right]^k$$

$$= (z-1) P(w) \sum_{\bar{J}=\bar{0}}^{\bar{M}} \frac{\left[\prod_{i=1}^{\tau} \begin{bmatrix} m_i \\ j_i \end{bmatrix} (\tilde{C}_{1i})^{j_i} \cdot (\tilde{C}_{2i})^{m_i - j_i} \right] w \prod_{i=1}^{\tau} \lambda_{1i}^{j_i} \lambda_{2i}^{m_i - j_i}}{z - w \prod_{i=1}^{\tau} \lambda_{1i}^{j_i} \lambda_{2i}^{m_i - j_i}}$$

Combining I and II yields:

$$P(z,w)=\sum_{\iota=0}^{\infty}\sum_{\bar{J}=\bar{0}}^{\bar{M}}\sum_{\bar{R}=\bar{0}}^{\bar{M}-\bar{J}}\sum_{\bar{S}=\bar{0}}^{\bar{J}}\frac{\left[\prod_{i=1}^{\tau}\begin{bmatrix}j_i\\s_i\end{bmatrix}\begin{bmatrix}m_i-j_i\\r_i\end{bmatrix}\tilde{D}_{1i}^{s_i}\tilde{D}_{2i}^{j_i-s_i}\tilde{C}_{1i}^{r_i}\tilde{C}_{2i}^{m_i-j_i-r_i}\right]z^{\iota}p_0(\iota,\bar{J})\,z}{z-w\prod_{i=1}^{\tau}\lambda_{1i}^{(r_i+s_i)}\lambda_{2i}^{m_i-((r_i+s_i))}}$$

$$+\,(z-1)\,P(w)\sum_{\bar{J}=\bar{0}}^{\bar{M}}\frac{\left[\prod_{i=1}^{\tau}\begin{bmatrix}m_i\\j_i\end{bmatrix}(\tilde{C}_{1i})^{j_i}\cdot(\tilde{C}_{2i})^{m_i-j_i}\right]w\prod_{i=1}^{\tau}\lambda_{1i}^{j_i}\lambda_{2i}^{m_i-j_i}}{z-w\prod_{i=1}^{\tau}\lambda_{1i}^{j_i}\lambda_{2i}^{m_i-j_i}}$$

or equivalently, with the change of variables: $\bar{K}=\bar{R}+\bar{S}$:

$$P(z,w)=\sum_{\iota=0}^{\infty}\sum_{\bar{J}=\bar{0}}^{\bar{M}}\sum_{\bar{K}=\bar{0}}^{\bar{M}}\frac{\sum_{\bar{l}=[\bar{K}-\bar{J},\bar{0}]^{+}}^{[\bar{M}-\bar{J},\bar{K}]^{-}}\left[\prod_{i=1}^{\tau}\begin{bmatrix}j_i\\k_i-l_i\end{bmatrix}\begin{bmatrix}m_i-j_i\\l_i\end{bmatrix}\tilde{D}_{1i}^{k_i-l_i}\tilde{D}_{2i}^{j_i-k_i+l_i}\tilde{C}_{1i}^{l_i}\tilde{C}_{2i}^{m_i-j_i-l_i}\right]z^{\iota}p_0(\iota,\bar{J})\,z}{z-w\prod_{i=1}^{\tau}\lambda_{1i}^{k_i}\lambda_{2i}^{m_i-k_i}}$$

$$+\,(z-1)\,P(w)\sum_{\bar{K}=\bar{0}}^{\bar{M}}\frac{\left[\prod_{i=1}^{\tau}\begin{bmatrix}m_i\\k_i\end{bmatrix}\tilde{C}_{1i}^{k_i}\tilde{C}_{2i}^{m_i-k_i}\right]w\prod_{i=1}^{\tau}\lambda_{1i}^{k_i}\lambda_{2i}^{m_i-k_i}}{z-w\prod_{i=1}^{\tau}\lambda_{1i}^{k_i}\lambda_{2i}^{m_i-k_i}} \qquad (3.81)$$

where the notation $\sum_{\bar{l}=[\bar{K}-\bar{J},\bar{0}]^{+}}^{[\bar{M}-\bar{J},\bar{K}]^{-}}$ refers to the multidimensional summation:

$$\sum_{l_1=[k_1-j_1,0]^{+}}^{[m_1-j_1,k_1]^{-}}\ \sum_{l_2=[k_2-j_2,0]^{+}}^{[m_2-j_2,k_2]^{-}}\ \sum_{l_3=[k_3-j_3,0]^{+}}^{[m_3-j_3,k_3]^{-}}\cdots\sum_{l_\tau=[k_\tau-j_\tau,0]^{+}}^{[m_\tau-j_\tau,k_\tau]^{-}}$$

This proves the first part of the theorem. Next, let:

$$\Re=\{0,1,2,\ \cdot\ \cdot\ ,m_1\}\times\{0,1,2,\ \cdot\ \cdot\ ,m_2\}\times\ \cdot\ \cdot\{0,1,2,\ \cdot\ \cdot\ ,m_\tau\}$$

For each $\bar{K}=\{k_1,k_2,..\,k_\tau\}\in\Re$, let us consider the roots of the equation:

$$z=V_{\bar{K}}(z)=w\prod_{i=1}^{\tau}\lambda_{1i}^{k_i}\lambda_{2i}^{m_i-k_i}. \qquad (3.82)$$

Let $h(z) = z$ and $g_{\bar{K}}(z) = -V_{\bar{K}}(z)$. Since $\forall i \in \{1, 2, \ldots, \tau\}$ $|\lambda_{1i}| \le |\lambda_{2i}| \le 1$ and $|w| < 1$, then for each $\bar{K} \in \Re$:

$$|g_{\bar{K}}(z)| = \left| w \prod_{i=1}^{\tau} \lambda_{1i}^{k_i} \lambda_{2i}^{m_i - k_i} \right| \le \left| \prod_{i=1}^{\tau} \lambda_{2i}^{m_i - k_i} \right|$$

Since $\prod_{i=1}^{\tau} \lambda_{2i}^{m_i - k_i}$ is a valid generating function (GF), then for a small $\varepsilon > 0$ and on $|z| = 1 + \varepsilon$ we have $|g_{\bar{K}}(z)| \le 1 + \varepsilon \sum_{i=1}^{\tau} \frac{(m_i - k_i)(1 - \beta_i)}{2 - \alpha_i - \beta_i} \bar{f}_i$.

On $|z| = 1 + \varepsilon$ we also have $|h(z)| = 1 + \varepsilon$ and therefore if the system is stable, i.e. ($\rho = \sum_{i=1}^{\tau} \frac{m_i(1 - \beta_i)}{2 - \alpha_i - \beta_i} \bar{f}_i < 1$), then for each $\bar{K} \in \Re$ $|h(z)| > |g_{\bar{K}}(z)|$ on $|z| = 1 + \varepsilon$. From Rouche's theorem $h(z)$ and $g_{\bar{K}}(z) + h(z)$ have the same number of zeros inside $|z| = 1 + \varepsilon$ and therefore (3.82) has also 1 root inside $|z| = 1 + \varepsilon$. In addition, since $f_i(0) = 0$ implies that $\lambda_{1i}|_{z=0} = 0$ then for any $\bar{K} \in \Re - \{\bar{0}\}$, the unique root of (3.82) is $z^* = 0$ which also appears in the numerator of (3.81) since $\tilde{C}_{1i}|_{z=0}$ and $\tilde{D}_{1i}|_{z=0}$ are also zero. For $\bar{K} = \bar{0}$, the corresponding term in (3.81) is given by:

$$\tilde{P}(z, w) = \sum_{\iota=0}^{\infty} \sum_{\bar{J}=\bar{0}}^{\bar{M}} \frac{\left[\prod_{i=1}^{\tau} \tilde{D}_{2i}^{j_i} \tilde{C}_{2i}^{m_i - j_i}\right] z^{\iota} p_0(\iota, \bar{J}) z}{z - w \prod_{i=1}^{\tau} \lambda_{2i}^{m_i}} + (z-1) P(w) \frac{\left[\prod_{i=1}^{\tau} \tilde{C}_{2i}^{m_i}\right] w \prod_{i=1}^{\tau} \lambda_{2i}^{m_i}}{z - w \prod_{i=1}^{\tau} \lambda_{2i}^{m_i}} \quad (3.83)$$

Next, let $H(z) = \prod_{i=1}^{\tau} \lambda_{2i}^{m_i}$ and denote by z^* the unique root of the equation, $z = wH(z)$, inside the unit disk. Then the numerator of (3.83) must also be zero at z^*, which implies that:

$$(z^* - 1) P(w) + \sum_{\iota=\bar{0}}^{\infty} \sum_{\bar{J}=\bar{0}}^{\bar{M}} z^{*\iota} \prod_{i=1}^{\tau} \left(\frac{\hat{D}_{2i}}{\hat{C}_{2i}}\right)^{j_i} p_0(\iota, \bar{J}) = 0$$

or equivalently:

$$P(w) = \frac{Q_0(z^*, r_1(z^*), r_2(z^*), \cdots r_\tau(z^*))}{1 - z^*} \quad (3.84)$$

where $r_i(z) = \frac{\bar{D}_{2i}}{\bar{C}_{2i}} = \frac{\lambda_{2i} - \beta_i}{(1 - \beta_i) f_i(z)}$. □

Taking the inverse w transform of (3.84) allows the computation of the unknowns $p_k(0)$'s. Alternatively from Lagrange's theorem (Appendix A2), with $a=0$, $g(z)=H(z)$ and:

$$\psi(z) = \frac{Q_0(z^*, r_1(z), r_2(z), \cdots r_\tau(z))}{1-z}$$

we have:

$$P(w) = p_0(0) + \sum_{k=1}^{\infty} \frac{w^k}{k!} \frac{d^{k-1}}{dz^{k-1}} \left(H(z)^k \left[\frac{Q_0(z^*, r_1(z), r_2(z), \cdots r_\tau(z))}{(1-z)^2} + \frac{Q'_0(z^*, r_1(z), r_2(z), \cdots r_\tau(z))}{1-z} \right] \right)_{z=0}$$

This implies that $\forall (k \geq 1)$:

$$p_k(0) = \frac{1}{k!} \frac{d^{k-1}}{dz^{k-1}} \left(H(z)^k \left[\frac{Q_0(z^*, r_1(z), r_2(z), \cdots r_\tau(z))}{(1-z)^2} + \frac{Q'_0(z^*, r_1(z), r_2(z), \cdots r_\tau(z))}{1-z} \right] \right)_{z=0}$$

In the special case, where the system is initially empty, with all sources being in the Off state (i.e. $Q_0(z, \hat{y}) = 1$), (3.84) reduces to $P(w) = \frac{1}{1-z^*}$. In addition, for a large number of traffic types (τ) the root z^* cannot be easily found. Hence, as a simple *approximation*, we might expand $H(z)$ in a Taylor series around z=1 and keep the first three terms. Then the equation $z = wH(z)$ is reduced to the following quadratic equation (in z):

$$wbz^2 + (aw - 2bw - 1) z + w(1 - a + b) = 0 \tag{3.85}$$

where $a = H'(1)$ and $b = \frac{H''(1)}{2}$, with:

$$H'(1) = \sum_{i=1}^{\tau} m_i \frac{1 - \beta_i}{2 - \alpha_i - \beta_i} \bar{f}_i \tag{3.86}$$

and:

$$H''(1) = \sum_{i=1}^{\tau}\left(m_i(m_i-1)\frac{(1-\beta_i)^2}{(2-\alpha_i-\beta_i)^2}[\bar{f}_i]^2\right.$$

$$\left.+m_i\left[\frac{2(1-\alpha_i)(1-\beta_i)(\alpha_i+\beta_i-1)}{(2-\alpha_i-\beta_i)^3}[\bar{f}_i]^2+\frac{1-\beta_i}{2-\alpha_i-\beta_i}f_i''(1)\right]\right)$$

$$+2\sum_{i=1}^{\tau-1}\sum_{j=i+1}^{\tau}\frac{m_i m_j(1-\beta_i)(1-\beta_j)}{(2-\alpha_i-\beta_i)(2-\alpha_j-\beta_j)}\bar{f}_i\bar{f}_j \quad (3.87)$$

The proof to the last equation can be found in Appendix A.8. Next, the solution to (3.85) which lies inside the unit circle is given by:

$$z^* = \frac{(1+2bw-aw)-\sqrt{(1-aw)^2+4bw(1-w)}}{2bw} \quad (3.88)$$

Hence:

$$P(w) = \frac{1}{1-z^*} = \frac{1}{2}\left[a+\frac{1-a}{1-w}+\frac{\sqrt{(1-aw)^2+4bw(1-w)}}{1-w}\right] \quad (3.89)$$

Taking the inverse w transform of the above equation, it can be shown (Appendix A.9) that:

$$p_k(0) = \frac{1}{2}[1+H'(1)(\delta(k)-1)]$$

$$+\frac{1}{2}\sum_{i=0}^{k}\sum_{j=0}^{[i]\dagger}\begin{bmatrix}1/2\\ i-j\end{bmatrix}\begin{bmatrix}i-j\\ j\end{bmatrix}[H'^2(1)-2H''(1)]^j[2H''(1)-2H'(1)]^{i-2j} \quad (3.90)$$

where $\delta(k)$ is the Kronecker delta function; while:

$$[x]\dagger = \begin{cases}\dfrac{x}{2} & \text{if } x \text{ is even}\\ \dfrac{x-1}{2} & \text{if } x \text{ is odd}\end{cases}$$

Note that the Binomial coefficients in (3.90) are given by the general formula (2.22).

3.3.4.2 Transient Mean of the Queue Length Distribution

Let $\bar{N}_k = \left.\frac{dP_k(z)}{dz}\right|_{z=1}$ denote the average queue length at the end of the k^{th} slot. Then from (3.72):

$$\bar{N}_k = \left.\frac{d\tilde{B}(k)}{dz}\right|_{z=1} - k + \frac{d}{dz}Q_0(z, \tilde{\Phi}_1(k), \tilde{\Phi}_2(k), \ldots, \tilde{\Phi}_\tau(k))\Big|_{z=1} + \sum_{l=1}^{k} p_{k-l}(0) \qquad (3.91)$$

Next let:

$$\chi(z) = Q_0(z, \tilde{\Phi}_1(k), \tilde{\Phi}_2(k), \ldots, \tilde{\Phi}_\tau(k)) = \sum_{\iota=0}^{\infty} \sum_{\bar{J}=\bar{0}}^{\bar{M}} z^{\iota} \left[\prod_{i=1}^{\tau} \tilde{\Phi}_i(k)^{j_i}\right] p_0(\iota, \bar{J})$$

Then:

$$\chi'(1) = \sum_{\iota=0}^{\infty} \sum_{\bar{J}=\bar{0}}^{\bar{M}} \left[\iota + \sum_{i=1}^{\tau} j_i \tilde{\Phi}'_i(k)\big|_{z=1}\right] p_0(\iota, \bar{J}) = \bar{N}_0 + \sum_{i=1}^{\tau} \bar{A}_{i0} \cdot \tilde{\Phi}'_i(k)\big|_{z=1} \qquad (3.92)$$

where $\bar{A}_{i0}$ is the initial average number of active sources of type i. From the above and after substituting:

$$\left.\frac{d\tilde{B}(k)}{dz}\right|_{z=1} = \sum_{i=1}^{\tau} \frac{m_i(1-\beta_i)(1-\alpha_i-\beta_i)}{(2-\alpha_i-\beta_i)^2} \bar{f}_i (1 - (\alpha_i + \beta_i - 1)^k) + k\rho$$

$$\tilde{\Phi}'_i(k)\big|_{z=1} = \frac{(\alpha_i + \beta_i - 1)}{2 - \alpha_i - \beta_i} \bar{f}_i (1 - (\alpha_i + \beta_i - 1)^k)$$

into ((3.91)-(3.92)) we finally get:

$$\bar{N}_k = \bar{N}_0 + k(\rho - 1) + \sum_{i=1}^{\tau} \frac{\alpha_i + \beta_i - 1}{2 - \alpha_i - \beta_i} \bar{f}_i (1 - (\alpha_i + \beta_i - 1)^k) \left[\bar{A}_{i0} - \frac{m_i(1-\beta_i)}{2 - \alpha_i - \beta_i}\right] + \sum_{l=0}^{k-1} p_l(0) \qquad (3.93)$$

We also note that with more complicated algebra, higher moments for the transient queue length distribution can also be obtained by successive differentiation of (3.72).

3.3.4.3 Steady-State PGF of the Queue Occupancy Distribution

The steady-state PGF, $P(z) = \lim_{k \to \infty} P_k(z)$ of the queue length distribution is readily obtained by applying Abel's theorem to (3.72), giving:

$$P(z) = (1-\rho)(z-1)\sum_{\bar{K}=\bar{0}}^{\bar{M}} \frac{\prod_{i=1}^{\tau} \begin{bmatrix} m_i \\ k_i \end{bmatrix} (\tilde{C}_{1i}\lambda_{1i})^{k_i} (\tilde{C}_{2i}\lambda_{2i})^{m_i-k_i}}{z - \prod_{i=1}^{\tau} \lambda_{1i}^{k_i}\lambda_{2i}^{m_i-k_i}} \qquad (3.94)$$

where $\rho = 1 - p_{\infty}(0)$ is the load of the system, which once more, can be obtained from the normalization condition, $P(1) = 1$ as follows:

Since $\tilde{C}_{1i}|_{z=1} = \tilde{D}_{1i}|_{z=1} = 0$, then, except for the case $\bar{K} = \bar{0}$ all the terms under the multidimensional summation in (3.94) become zero when evaluated at z=1. Hence it is convenient to rewrite (3.94) as follows:

$$P(z) = (1-\rho)(z-1)\left[F(z) + \frac{G(z)}{z - H(z)}\right] \qquad (3.95)$$

where:

$$F(z) = \sum_{\substack{\bar{K}=\bar{0} \\ \bar{K}\neq\bar{0}}}^{\bar{M}} \frac{\prod_{i=1}^{\tau} \begin{bmatrix} m_i \\ k_i \end{bmatrix} (\tilde{C}_{1i}\lambda_{1i})^{k_i} (\tilde{C}_{2i}\lambda_{2i})^{m_i-k_i}}{z - \prod_{i=1}^{\tau} \lambda_{1i}^{k_i}\lambda_{2i}^{m_i-k_i}}$$

$$G(z) = \prod_{i=1}^{\tau} (\tilde{C}_{2i}\lambda_{2i})^{m_i} \quad \text{and} \quad H(z) = \prod_{i=1}^{\tau} \lambda_{2i}^{m_i}$$

From (3.95):

$$P(z)[z - H(z)] = (1-\rho)(z-1)\left[F(z)(z - H(z)) + G(z)\right] \qquad (3.96)$$

Next differentiating both sides of the above equation with respect to z, substituting z=1 in the resulting equation yields:

$$\rho = H'(1) = \sum_{i=1}^{\tau} m_i \frac{1-\beta_i}{2-\alpha_i-\beta_i} \bar{f}_i \qquad (3.97)$$

in accordance with the result previously derived in section 3.3.3.

The expression of the steady-state PGF of the buffer occupancy distribution given in (3.94) might look a bit complicated, yet it is another significant result mainly for two reasons:

- First, expression (3.94) is, by far, more explicit than the corresponding result obtained with the matrix geometric/spectral decomposition approaches since the latter is often given in a matrix form which involves the Kronecker products of the individual probability generating matrices of the individual sources, (see for example [41], [75],[76]).

- Second, the observation we made earlier concerning the fact that, except for the case $\bar{K} = \bar{0}$ all the terms under the multidimensional summation in (3.94) become zero when evaluated at z=1, allows a straight forward derivation of any moment of the queue length distribution.

3.3.4.4 Steady-State Mean of the Queue Length Distribution

Let $\bar{N}$ denote the steady-state mean buffer length. Then by differentiating (3.96) twice with respect to z and substituting z=1 in the resulting expression we get:

$$\bar{N} = P'(1) = \frac{H''(1)}{2(1-H'(1))} + G'(1) \qquad (3.98)$$

where:

$$G'(1) = \sum_{i=1}^{\tau} \frac{m_i(1-\beta_i)(3-2\alpha_i-2\beta_i)}{(2-\alpha_i-\beta_i)^2} \bar{f}_i \qquad (3.99)$$

with $H'(1)$ and $H''(1)$ as given in (3.86) and (3.87), respectively.

The above result for the steady-state mean of the buffer length is a generalization of Bruneel results [51] obtained for the single type of traffic case. It should be noted that higher moments of the buffer occupancy distribution can also be obtained by successive differentiation of (3.96). The resulting expressions get however complicated. A derivation for the variance of the queue length can be found in Appendix *A*10.

Under the special case where the number of each type of traffic source, m_i, is restricted to one, with each source generating one cell per active slot $(f_i(z) = z)$, Viterbi [75] derived an explicit expression for the steady-state mean of the buffer length, using a matrix analytical approach. By substituting $m_i = 1$, $\bar{f}_i = 1$ and $\bar{f}''_i(1) = 0$ in (3.98) we have found that our result reduces to hers. However while the derivation of (3.98) was done in a straight-forward and classical fashion, once we have derived $P(z)$ in (3.94), the derivation of the corresponding formula in [75] is quite lengthy and requires clever manipulations of Kronecker products and eigenvalues ([41],pp. 354,360).

3.3.5 The Infinite Source Model

Let us consider the limiting case where $m_i \to \infty$ and $\beta_i \to 1$ such that $m_i(1-\beta_1) \to \Lambda_i$, for $i = 1, 2 \cdot \cdot \tau$. This corresponds to modeling each type of traffic by an infinite source model. Further, without any loss of generality, let us assume that the buffer is initially empty, with all sources being in the *Off* state. Let $\tilde{A}_k(\vec{y})$ denote the PGF of the number of active users under this model. Then from (3.70) we obtain:

$$\tilde{A}_k(\vec{y}) = \lim_{\substack{(m_i \to \infty, \beta_i \to 1) \\ i=1,2,..\tau}} A_k(\vec{y}) = \lim_{\substack{m_i \to \infty \\ i=1,2,..\tau}} \left\{ \prod_{i=1}^{\tau} \left[1 - \frac{\Lambda_i}{m_i}(1-y_i) \sum_{j=0}^{k-1} \alpha_i^j \right]^{m_i} \right\}$$

or equivalently:

$$\tilde{A}_k(\tilde{y}) = e^{\sum_{i=1}^{\tau} \frac{\Lambda_i(1-\alpha_i^k)}{1-\alpha_i}(y_i-1)} = \prod_{i=1}^{\tau} e^{\frac{\Lambda_i(1-\alpha_i^k)}{1-\alpha_i}(y_i-1)} \qquad (3.100)$$

The above equation shows that the number of active users for each type of traffic (i) at any particular slot, k, follows a Poisson process with a slot-dependent rate, $\frac{\Lambda_i(1-\alpha_i^k)}{1-\alpha_i}$. Similarly we can derive the steady-state PGF of the queue length, $\tilde{P}(z)$, under the above asymptotic limit by following the same approach outlined in section 3.2.10 for the single type of traffic case. Hence, with $\rho = \sum_{i=1}^{\tau} \frac{\Lambda_i f_i}{1-\alpha_i}$, we obtain:

$$\tilde{P}(z) = (1-\rho)(z-1)\sum_{k=1}^{\infty} \frac{e^{\sum_{i=1}^{\tau} \frac{\Lambda_i(f_i(z)-1)}{1-\alpha f_i(z)}\left[k-\alpha f_i(z)\frac{1-[\alpha f_i(z)]^k}{1-\alpha f_i(z)}\right]}}{z^k}$$

which can be approximated to yield:

$$\tilde{P}(z) \cong (1-\rho)(z-1)\frac{\hat{H}(z)}{z-\hat{H}(z)}$$

with:

$$\hat{H}(z) = \prod_{i=1}^{\tau} e^{\frac{\Lambda_i(f_i(z)-1)}{1-\alpha f_i(z)}}$$

3.4 Steady-State Distribution of the Packet Delay

So far, we have focused on the derivation of some transient and steady-state performance measures related to the buffer occupancy distribution. In this section we focus on the steady-state packet delay, which is also an important performance measure for the ATM multiplexer, as a real time traffic such as voice or video has very stringent delay and delay jitter requirements. Here we define the packet delay as the period (expressed in integer number of slots) between the end of the slot during which a packet enters the queue and the end of the slot when it departs from the system. Accordingly let $D(z)$ be the PGF of the packet delay in number of slots at the steady-state.

In [67], Steyaert et al, have derived a general relationship between the packet delay and the buffer contents in a discrete-time multiserver queue, assuming generally correlated (not necessarily Markovian) arrivals. More specifically, they derived explicit expressions for the distribution, PGF, mean and variance of the packet delay, in terms of the distribution, PGF, mean and variance of the buffer occupancy. One significant aspect of their result is that the derivations are independent of the exact nature of the arrival process. Therefore, using the results derived in [67], $D(z)$ can be derived from the steady-state PGF of the buffer occupancy distribution, $P(z)$, through the relationship:

$$D(z) = \frac{1}{\rho}[P(z) - (1-\rho)] \qquad (3.101)$$

The average packet delay is therefore $\bar{d} = \frac{\bar{N}}{\rho}$ while the variance of the delay becomes $\sigma_d^2 = \frac{\sigma_N^2}{\rho} - \frac{(1-\rho)}{\rho^2}\bar{N}^2$.

3.5 Idle and Busy Period Analysis

In this section, we focus on the discrete idle and busy periods of the ATM multiplexer with multiple types of traffic case. We feel that this study can give further insights into the behavior of the ATM multiplexer, especially in regard to the smoothing function performed by the multiplexer. We first start with the idle period, since it is easier to analyze. By definition, an idle period starts at the departure instant of the last packet from the buffer (which leaves the system empty) and ends at the end of the first subsequent slot during which at least one arrival occurs [2]. Let the random variable I^* denote the length of an arbitrary idle period, expressed in number of slots, and denote by $I(z)$ the corresponding PGF.

3.5.1 The Idle Period

For the idle period to last for k consecutive slots there must have been no arrivals during each of the first (k-1) of these slots and at least one arrival must have occurred in the k^{th} of these slots. Recall form our model assumption that when the ATM buffer is empty, all the sources must be in the OFF state. Therefore, since an idle period is initiated by an empty system, then all the sources must be in the *OFF* state and must remain so for the first (k-1) slots. At the last slot (k), at least one source must turn to the *ON* state. Because of the independence assumption among all the sources it follows that:

$$Pr[I^* = k] = \left[\prod_{i=1}^{\tau} \beta_i^{m_i}\right]^{k-1} \left[1 - \prod_{i=1}^{\tau} \beta_i^{m_i}\right] \qquad (k \geq 1) \qquad (3.102)$$

and therefore:

$$I(z) = \frac{z\left[1 - \prod_{i=1}^{\tau} \beta_i^{m_i}\right]}{1 - z\prod_{i=1}^{\tau} \beta_i^{m_i}} \qquad (3.103)$$

In other words, the idle periods of the ATM multiplexer are geometrically distributed with parameter $\prod_{i=1}^{\tau} \beta_i^{m_i}$, mean:

$$\overline{I^*} = \frac{1}{1 - \prod_{i=1}^{\tau} \beta_i^{m_i}} \qquad (3.104)$$

and variance:

$$\sigma^2_{I^*} = \frac{\prod_{i=1}^{\tau} \beta_i^{m_i}}{\left(1 - \prod_{i=1}^{\tau} \beta_i^{m_i}\right)^2} \qquad (3.105)$$

3.5.2 The Busy Period

In this section, we study the busy period of the ATM multiplexer. Our main goal is to derive an expression for the PGF of the busy period in terms of the system parameters. As done before, we first illustrate our solution technique by considering the case of the GI/D/1 queue and then show how the approach is directly applicable to our system.

3.5.2.1 Busy Period Analysis of the GI/D/1 Queue

Generally speaking, the analysis of discrete busy periods is far more complicated than that of the idle periods. Further it is well known that the lengths and the positions of the idle and busy periods on the time axis are not affected by the queueing discipline, as long as it is work conserving. Based on this observation, Bruneel and Kim [2] have presented the busy period analysis of the GI/G/1 model, assuming a Last-in-First-Out (LIFO) discipline. When applied to the GI/D/1 model, their result yields a functional equation for the PGF of the busy period and an explicit expression for its mean. Closed-form solutions for the PGF of the busy period can only be obtained under some special cases, corresponding to specific arrival distributions.

This section presents a simple approach for the derivation of the busy period distribution of the GI/D/1 model. Once again, we denote by $Q(z)$ the steady-state PGF of the buffer content and by $V(z)$ the PGF of the number of packet arrivals during a slot. In addition we use the superscript $^{(i)}$ to denote the i^{th}derivative operator.

Without any loss of generality, let us assume that the system is initially empty ($Q_0(z) = 1$). We explain our approach through a tree diagram, as shown in figure 3.3, where, for the purpose of illustration, we assume that the number of arrivals during a slot can take on the values zero, one or two packets.

From figure 3.3, we can see that the event of having an empty buffer at the end of the k^{th}slot can be expressed as the sum of k mutually exclusive events. For instance, in reference to figure 3.3, if we look to slot number 1, the probability of having an empty buffer, $Q_1(0)$, is equal to the probability that the buffer was initially empty and there was no arrival. In other words:

For k=1:

$$Q_1(0) = Q_0(0)V(0)$$

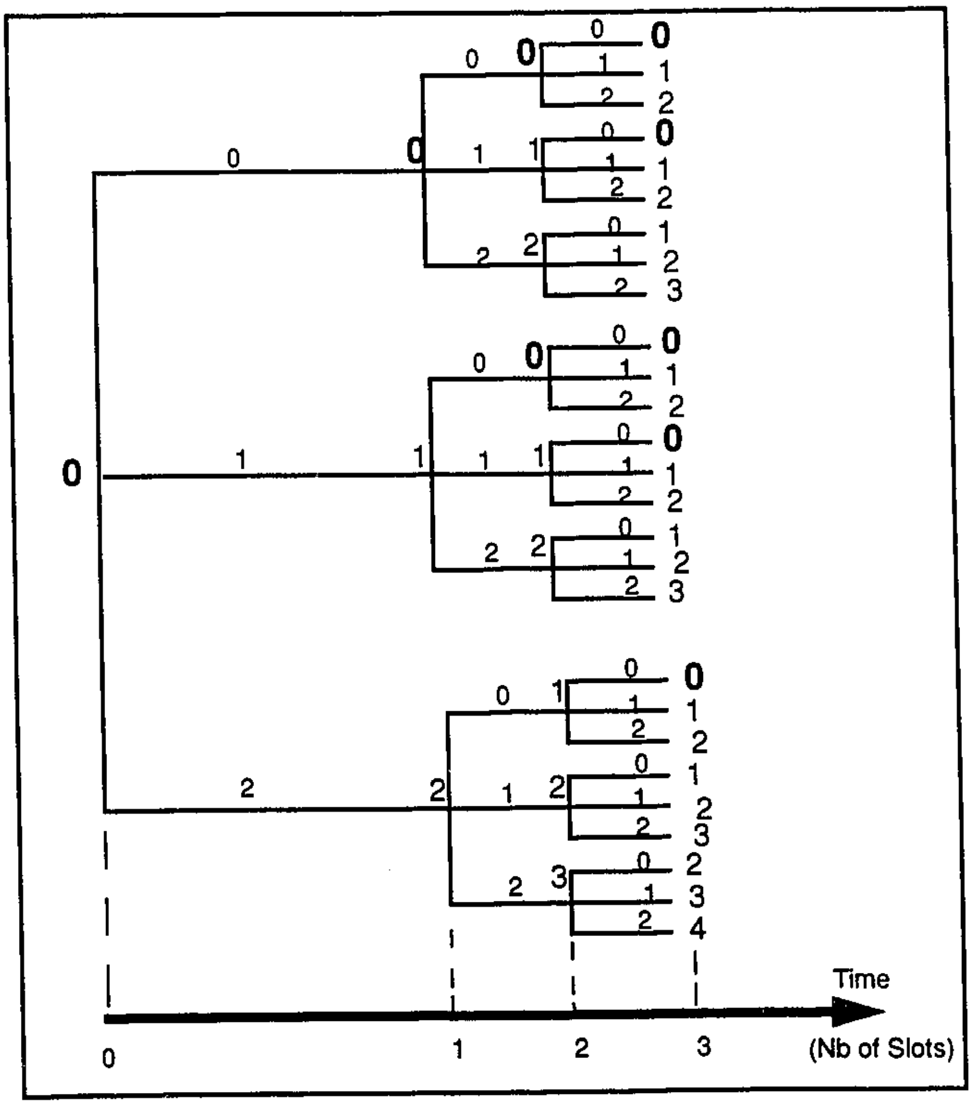

FIGURE.3.3 An Illustrating Example (Numbers above branches represent number of arrivals (we assume a maximum of 2 per-slot) and number at the end leaves represent buffer length at the end of the corresponding slot)

Similarly, from figure 3.3, and for k=2, the probability of having an empty buffer at the end of the second slot is equal to the probability that the buffer was empty at the end of the first slot and there was no arrivals plus the probability that the buffer was initially empty and there were one then zero arrivals, since these events are mutually exclusive. In mathematical terms, this translates to:

$$Q_2(0) = Q_1(0)V(0) + Q_0(0)V^{(1)}(0)V(0)$$

$$= Q_1(0)V(0) + \frac{V^{(1)}(0)^2}{2!}Q_0(0)$$

Similarly, for k=3, it is easy to verify that:

$$Q_3(0) = Q_2(0)V(0) + Q_1(0)V^{(1)}(0)V(0) + Q_0(0)\left[(V^{(1)}(0))^2V(0) + \frac{V^{(2)}(0)}{2!}(V(0))^2\right]$$

$$= V(0)Q_2(0) + \frac{V^{(1)}(0)^2}{2!}Q_1(0) + \frac{V^{(2)}(0)^3}{3!}Q_0(0)$$

and in general, by induction, we can write:

$$p_k(0) = \sum_{j=1}^{k} \xi(j)p_{k-j}(0) \tag{3.106}$$

where:

$$\xi(j) = \begin{cases} V(0) & (j=1) \\ \sum_{i=1}^{j-1} \frac{[V(0)]^i}{i!}[V^{(i)}(0)]^{j-i} & (j \geq 2) \end{cases} \tag{3.107}$$

In (3.106), we have expressed the probability of having an empty buffer at the end of the kthslot as the sum of k mutually exclusive events. Further, in the RHS of (3.106), $p_{k-j}(0)$, is interpreted as the probability that the system was empty for the last time at the end of the $(k-j)^{th}$ slot and therefore the function $\xi(j)$ in (3.107) is the probability that the system is busy for $(j-1)$ slots. With simple algebra, we can prove by recurrence that (3.107) can be further simplified to yield:

$$\xi(j) = pr[System\ is\ Busy\ for\ (j-1)\ slots] = \frac{1}{j!}\frac{d^{(j-1)}}{dz^{j-1}}[V(z)]^j\bigg|_{z=0} \qquad (j \geq 1)$$

and therefore:

$$pr[System\ \text{is busy for' } j \text{ slots}] = \frac{1}{(j+1)!}\frac{d^{(j)}}{dz^j}[V(z)]^{j+1}\bigg|_{z=0} \qquad (j \geq 0) \quad (3.108)$$

Note that the above expression allows a busy period to consist of zero slots. In general, we define the busy period of a system as the time between two consecutive idle periods. Therefore since a busy period is initiated by an arrival, then it must consist of at least one slot. Under this definition, let the random variable B^* denote the length of an arbitrary busy period in number of slots. Let $B^*(z)$ be the corresponding PGF. Then the distribution of the busy period is given by:

$$pr[B^* = j] = \frac{1}{[1-V(0)]}\frac{1}{(j+1)!}\frac{d^{(j)}}{dz^j}[V(z)]^{j+1}\bigg|_{z=0} \qquad (j \geq 1) \quad (3.109)$$

and the corresponding PGF is therefore:

$$B^*(z) = \sum_{j=1}^{\infty} pr[B^* = j]\, z^j$$

$$= \frac{1}{[1-V(0)]}\sum_{j=1}^{\infty}\frac{1}{(j+1)!}\frac{d^{(j)}}{dz^j}[V(z)]^{j+1}\bigg|_{z=0} z^j$$

$$= \frac{1}{[1-V(0)]}\cdot\frac{1}{z}\left\{\sum_{k=1}^{\infty}\frac{z^k}{k!}\frac{d^{(k-1)}}{dz^{k-1}}[V(z)]^k\bigg|_{z=0} - zV(0)\right\}$$

From Lagrange's theorem (Appendix A.2), we can write:

$$\sum_{k=1}^{\infty}\frac{z^k}{k!}\frac{d^{(k-1)}}{dz^{k-1}}[V(z)]^k\bigg|_{z=0} = \sigma^*$$

where σ^* is the unique solution of the equation $\sigma = zV(\sigma)$, inside the unit circle. Hence:

$$B^*(z) = \frac{\sigma^* - zV(0)}{z[1-V(0)]} \qquad (3.110)$$

The above expression for the PGF of the busy period of the GI/D/1 queue has a simpler form than that of the corresponding result in [2], which expresses $B^*(z)$ in terms of the functional equation:

$$B^*(z) = \frac{V(z\{(1-V(0))B^*(z)+V(0)\})-V(0)}{1-V(0)}$$

The mean length of the busy period is readily obtained by differentiating (3.110) with respect to *z*, substituting *z*=1 in the resulting expression and by taking into account the fact that:

$$\left.\frac{d}{dz}\sigma^*\right|_{z=1} = \frac{1}{1-V'(1)}$$

Hence:

$$\overline{B^*} = \left.\frac{dB^*}{dz}\right|_{z=1} = \frac{\left[\left.\frac{d}{dz}\sigma^*\right|_{z=1} - V(0)\right] - (1-V(0))}{(1-V(0))} = \frac{V'(1)}{(1-V'(1))(1-V(0))} \quad (3.111)$$

in accordance with the corresponding result derived in [2].

3.5.2.2 Busy Period Analysis of the ATM Multiplexer

In section 3.2.5.4, it was found that, when the ATM multiplexer is initially empty, the expression of transient probability of an empty buffer, $p_k(0)$, is the same as that of the GI/D/1 queue with $V(z) = H(z) = \prod_{i=1}^{\tau} \lambda_{2i}^{m_i}$. Hence ((3.106)-(3.107)) also hold for the correlated arrivals case (this has also been verified through symbolic computation using the symbolic Maple computational system [77]) and hence:

$$pr[B^*=j] = \frac{1}{\left[1-\prod_{i=1}^{\tau}\beta_i^{m_i}\right]} \frac{1}{(j+1)!} \left.\frac{d^{(j)}}{dz^j}[H(z)]^{j+1}\right|_{z=0} \qquad (j \geq 1) \quad (3.112)$$

and:

$$B^*(z) = \frac{\sigma^* - z\prod_{i=1}^{\tau}\beta_i^{m_i}}{z\left[1 - \prod_{i=1}^{\tau}\beta_i^{m_i}\right]} \qquad (3.113)$$

where σ^* is the unique solution of the equation $\sigma = zH(\sigma)$ inside the unit circle.

In the single type of traffic case, we have dealt with a similar equation, namely $z = wH(z)$, in full details through the application of the Lagrange's theorem, as explained in section 3.2.5.5. The same analysis can also be used, in some cases, to explicitly derive the distribution of the busy period. Further, from (3.113), the mean busy period of the ATM multiplexer is:

$$\overline{B^*} = \frac{H'(1)}{(1-H'(1))(1-H(0))} = \frac{\rho}{(1-\rho)\left(1-\prod_{i=1}^{\tau}\beta_i^{m_i}\right)} \qquad (3.114)$$

where $\rho = \sum_{i=1}^{\tau} m_i \frac{1-\beta_i}{2-\alpha_i-\beta_i}\bar{f}_i$ is the load of the system at steady-state.

Finally we note that our definition of the busy and idle periods implies that a slot will belong to an idle period if and only if the multiplexer is empty at the beginning of this slot, otherwise it belongs to a busy period. Hence, from (3.104) and (3.114), the fraction of slots belonging to an idle period is given by:

$$\frac{\overline{I^*}}{\overline{I^*} + \overline{B^*}} = 1 - \rho \qquad (3.115)$$

which, as expected, equals to the steady state probability of an empty buffer $p_\infty(0)$.

3.6 Numerical Results

In this section we illustrate our solution technique by some numerical examples where we also attempt to draw some conclusions on the general attributes of the transient behavior of the ATM multiplexer. First we consider the case where the system is initially empty, with all sources being in the *OFF* state. In figure 3.4 we plot the transient probabilities of an empty buffer as function of time, with the number of sources as a parameter. We assumed that $f(z) = z$ and kept α (which, as defined before, is the probability that a source is active, given that it was active during the previous slot) fixed at 0.75. In addition, the steady-state load is also kept constant, at $\rho = 0.7826$. As may be seen, for the same steady-state load, different probabilities are obtained for different values of m. Also note that the transient probabilities of an empty buffer approach the steady state value of $1-\rho$ as time increases. In figures 3.5 and 3.6 we plot the corresponding transient mean and variance of the queue length. In particular we observe that for the same offered load, an increase in the number of sources leads to a rise in the transient mean and variance of the queue length. Note also the exponential behavior of the mean-time curve and the nearly-linear profile in the variance-time curve for large values of m. In particular we note that the exponential rise in the mean-time curve, depicted in figure 3.5, is typical in many other queueing systems [78]. More specifically the transient mean curve of the ATM multiplexer, under zero initial conditions, can be approximated by an expression in the form:

$$\overline{N}_k = \overline{N}_\infty [1 - e^{-k/\tau}]$$

where τ is the characteristic time constant (known also as the "relaxation time") of the system, which depends on the parameters of the offered traffic and which is often independent of the initial conditions [78].

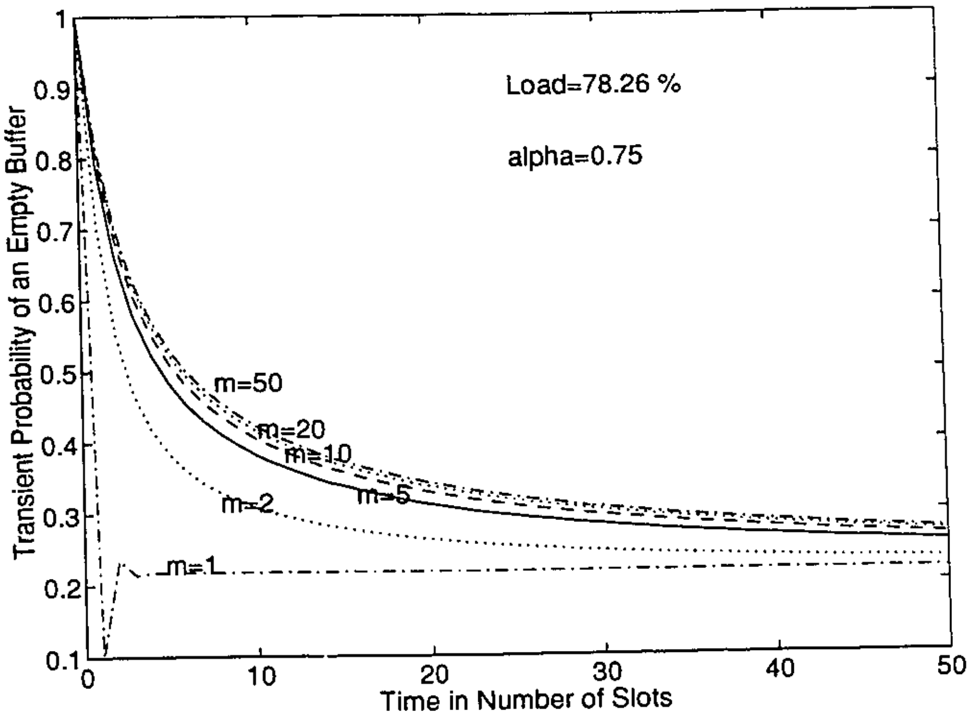

FIGURE.3.4 Transient Probability of an Empty Buffer for a Fixed Offered Load and Different Number of Sources (m)

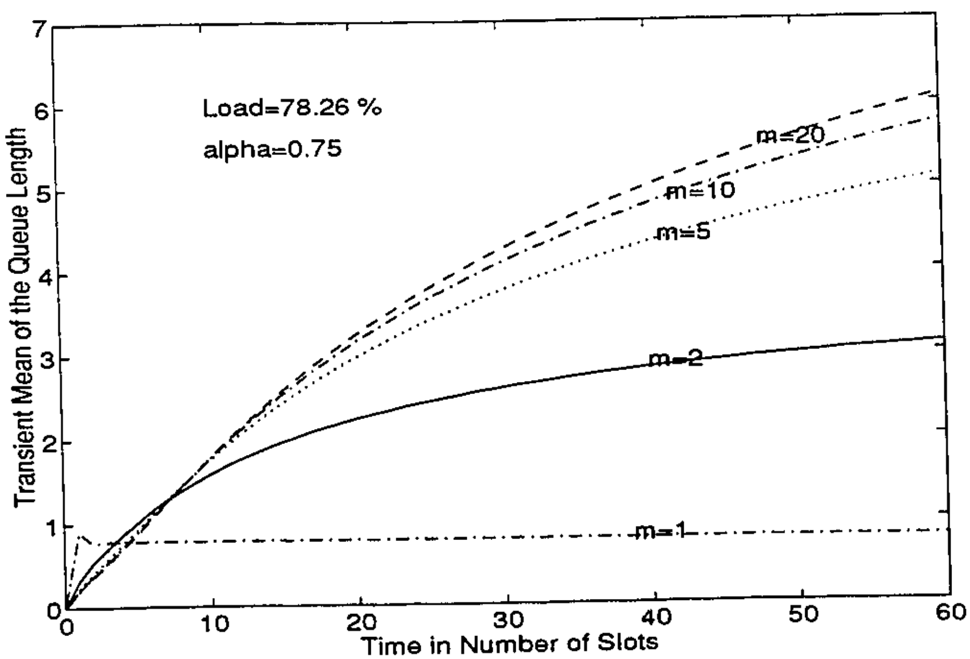

FIGURE.3.5 Transient Mean of the Queue Length, for Different Values of Number of Sources (m)

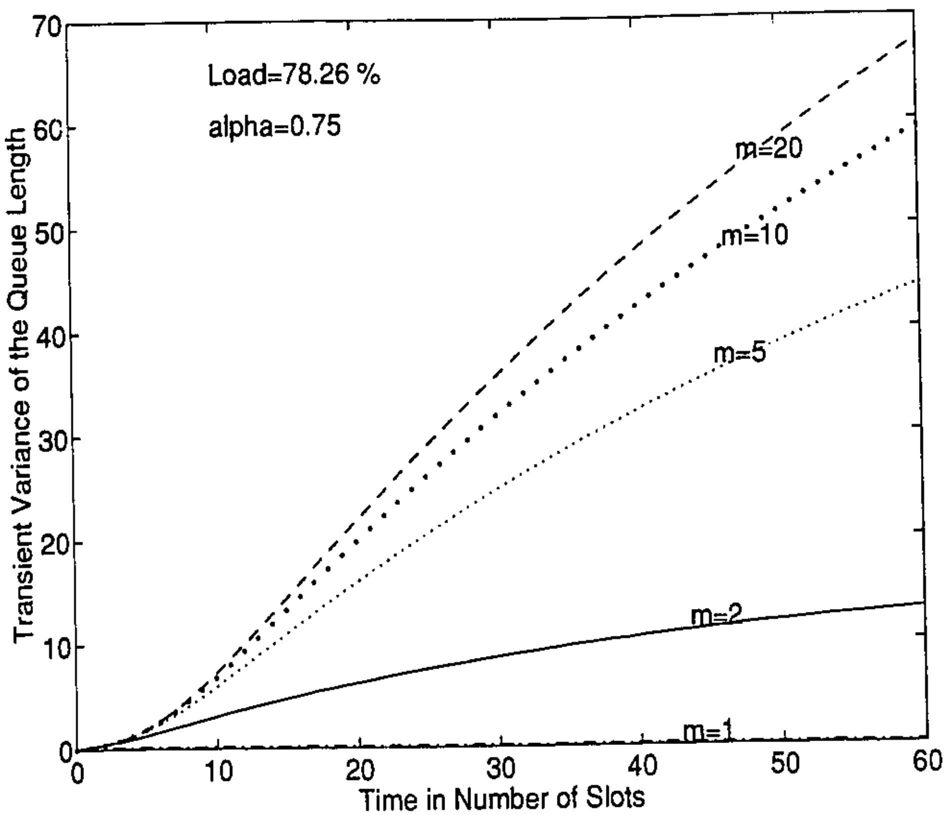

FIGURE.3.6 Transient Variance of the Queue Length for Different Values of m

A very useful measure to *estimate* the transient packet loss ratio, due to a finite buffer size (*n*), is the probability, $q_k(n) = pr[i_k > n]$, that the transient buffer occupancy exceeds the proposed buffer size, which is an upper-bound for the transient probability of overflow. In our case, this probability can be computed from the transient PGF $P_k(z)$, as given in (3.18). Figure 3.7 shows the transient probabilities of buffer overflow for fixed source's statistics and with $f(z) = z$. In this case these probabilities were easily computed by noting that since $\frac{1-P_k(z)}{1-z}$ is found to be a polynomial function in *z*, then $q_k(n) = pr[i_k > n]$ corresponds to the coefficient of z^n in this polynomial. As expected, the transient probabilities of overflow increase as time evolves and this reflects the fact that when the system starts from zero initial conditions, the queue waiting room builds up progressively.

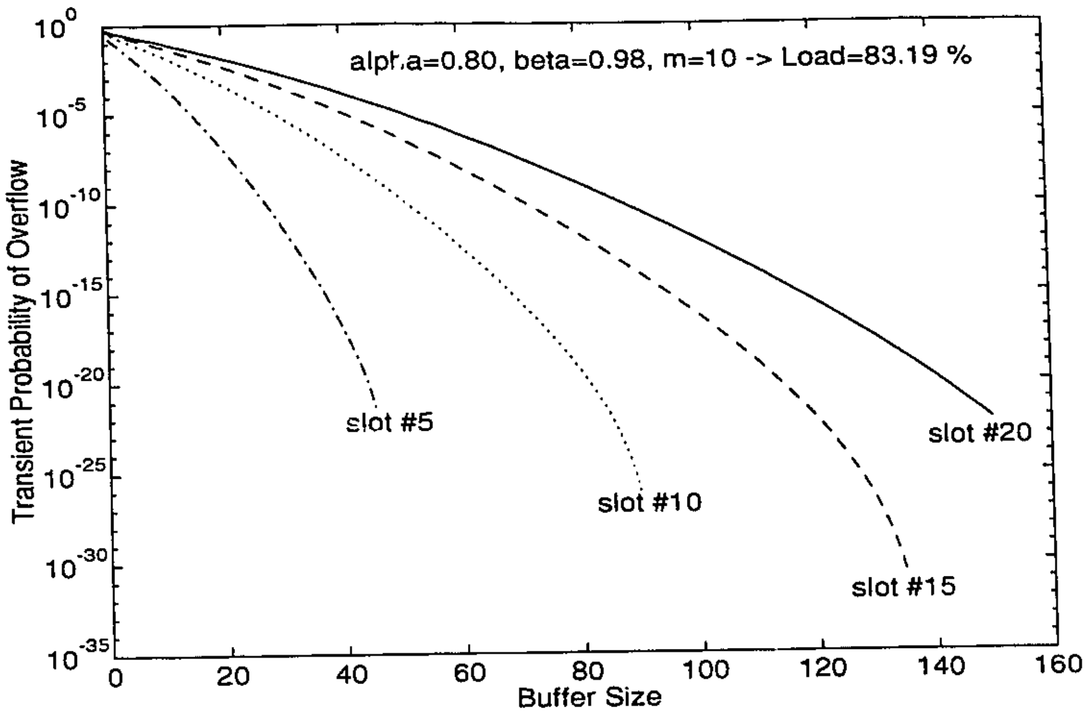

FIGURE.3.7 Transient Probability of Overflow Versus Buffer Size

Next we investigate the transient behavior of the ATM multiplexer under a non-zero initial condition (IC). Without any loss of generality, let us assume a deterministic initial state whereby, at time zero, there are $i_0 = 10$ packets in the buffer, with all sources being active (i.e. $a_0 = m$). Figures 3.8, 3.9 and 3.10 show the transient probabilities of an empty buffer, the transient mean and the transient variance of the queue length for different values of *m*.

On the basis of these plots, we recognize the strong dependency of the multiplexer transient behavior on the initial state of the system, though the steady state is independent of the initial condition. For the same offered load, the manner in which the $p_k(0)$, $\overline{N}_k$ or σ_k^2 curves approach the steady state value does depend on the initial state of the system. Further for deterministic ICs, one can use the previous plots to predict the future behavior of the system. For instance one may use the transient mean curve to determine how many slots it will take before $\overline{N}_k$

becomes very close to $\bar{N}$ (let us say within -2%). Further, as a result of the overshoot displayed in the transient mean curve in figure 3.9, we conclude that steady-state mean does not always reflect the true behavior of the ATM multiplexer.

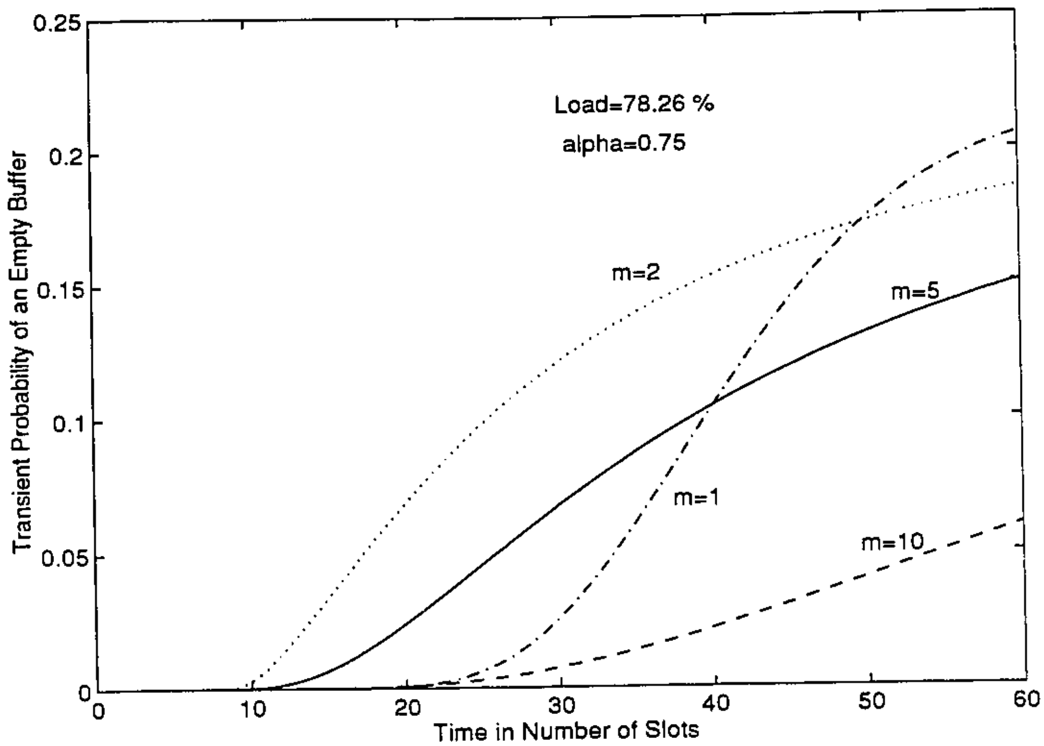

FIGURE.3.8 Transient Probability of an Empty Buffer for a Fixed Offered Load and Different Number of Sources (m)

Next we show, through one simple example, how equilibrium solutions can sometimes be invalid descriptors of the system behavior and how transient solutions become necessary. With the same deterministic ICs, specified by $i_0 = 10$ and $a_0 = m$, we plot the probability of buffer overflow versus buffer size for different number of slots, as shown in figure 3.11. In particular, and as shown in figure 3.12, for small buffer sizes, the transient probability of overflow exceeds the corresponding steady state result. This is a further confirmation that ATM system design and congestion control algorithms which are based on steady state results could sometimes fail as a result of the underestimation of the transient dynamics.

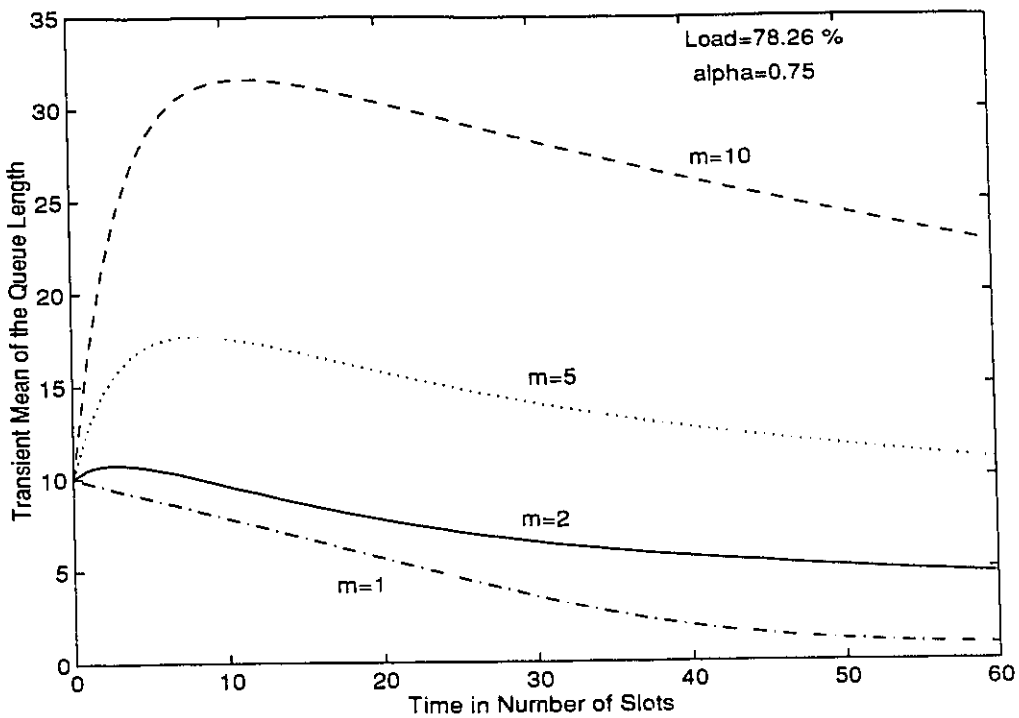

FIGURE.3.9 Transient Mean of the Queue Length, for Different Values of Number of Sources (m)

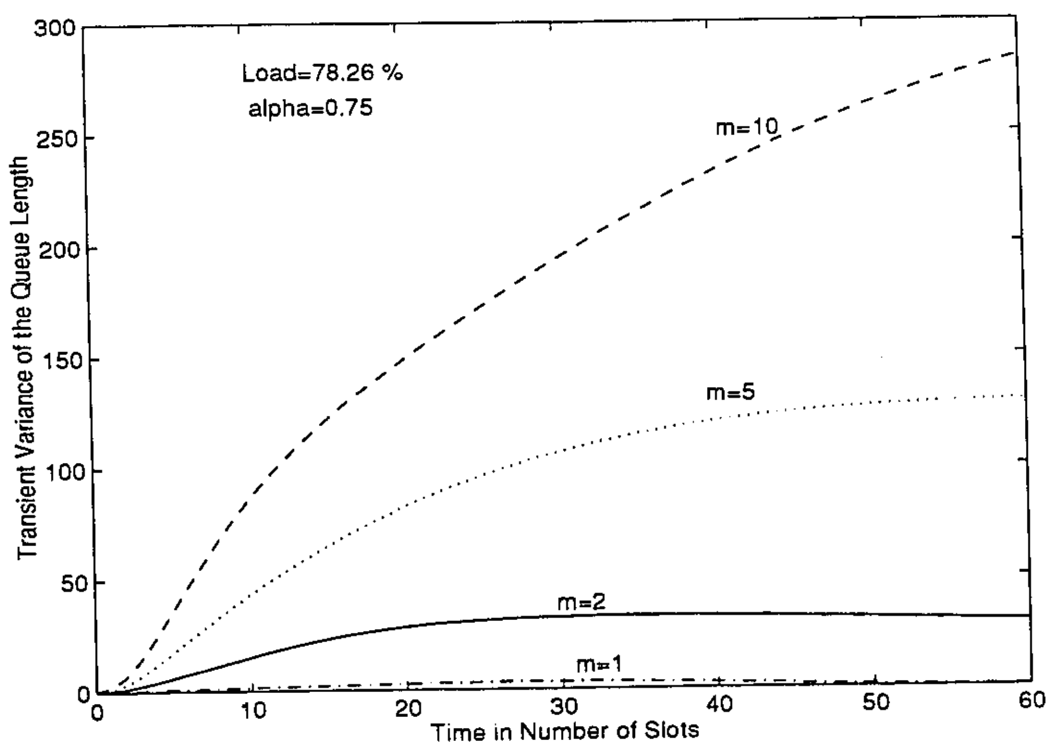

FIGURE.3.10 Transient Variance of the Queue Length for Different Values of m

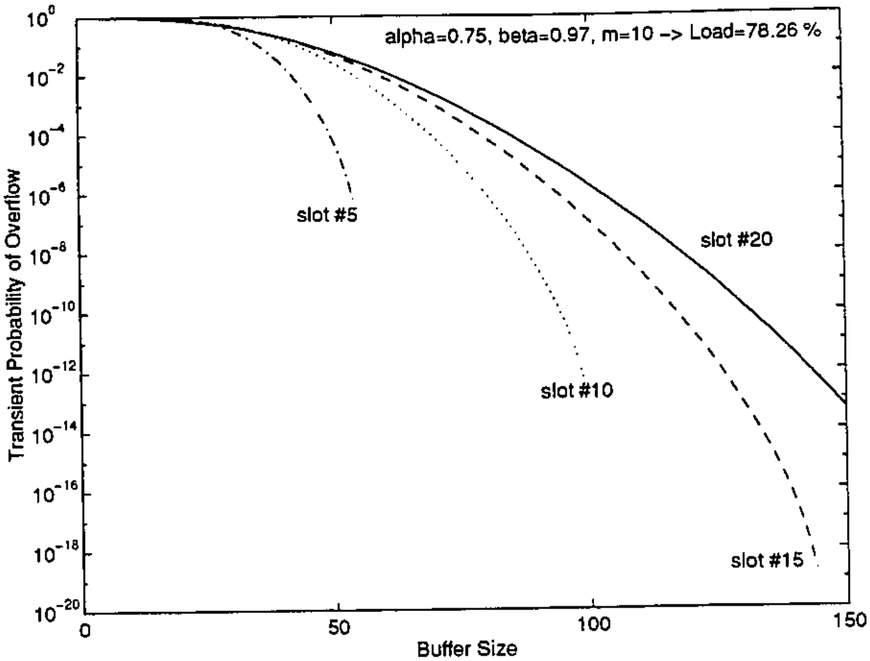

FIGURE.3.11 Transient Probabilities of Overflow Versus Buffer Size

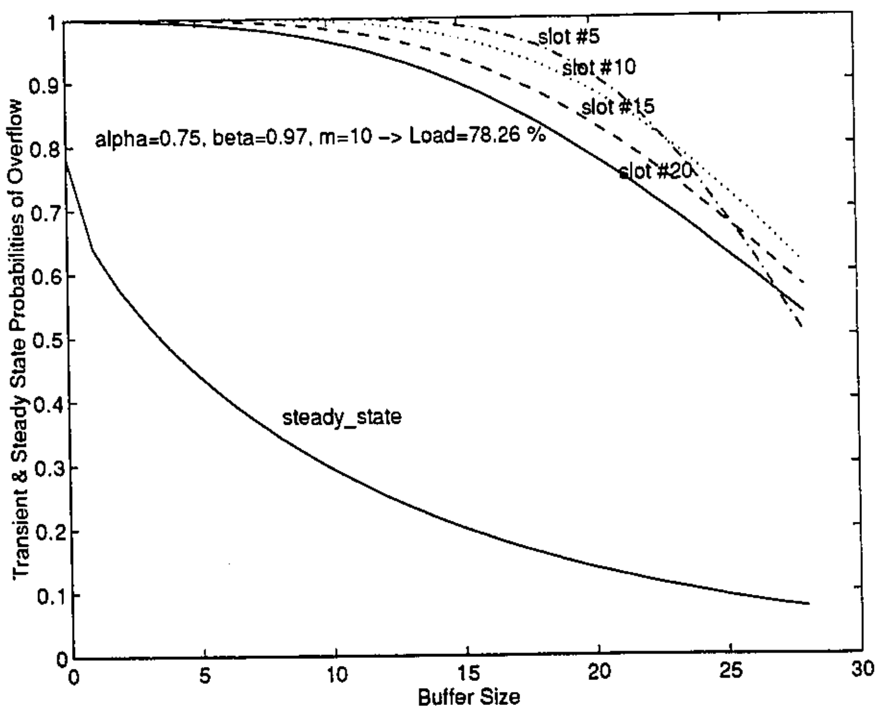

FIGURE.3.12 Transient Versus Steady State Probabilities of Overflow For Small Buffer Size

Figures 3.13 and 3.14 show some results for the mean and variance of the packet delay. Here we assumed that during an active slot we have a batch arrival process whose size is geometrically distributed with a PGF $f(z) = \frac{(1-\nu)z}{1-\nu z}$ where $\nu = \frac{1}{2}$. Observe that under light loading the mean delay is heavily influenced by the mean batch size ($\bar{f} = 2$), while under heavy loading, queuing delay dominates and, for a fixed load, it increases with the number of sources.

We also note the very sharp increase in the variance of the packet delay under heavy loading. In this case the variance-load curve is also tailored by the choice of the parameter ν which controls the first three moments of the batch size distribution.

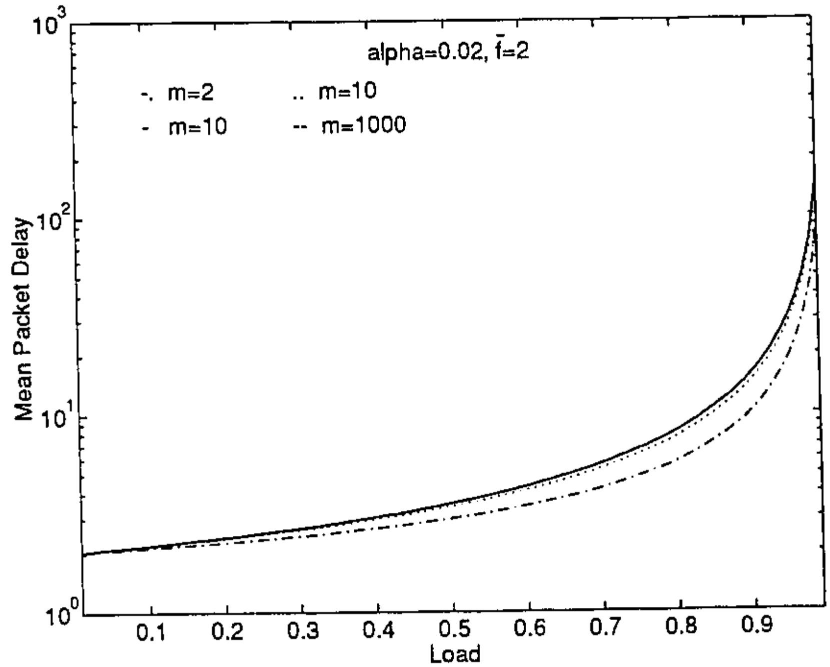

FIGURE.3.13 Mean Packet Delay for Different Numbers of Sources (m)

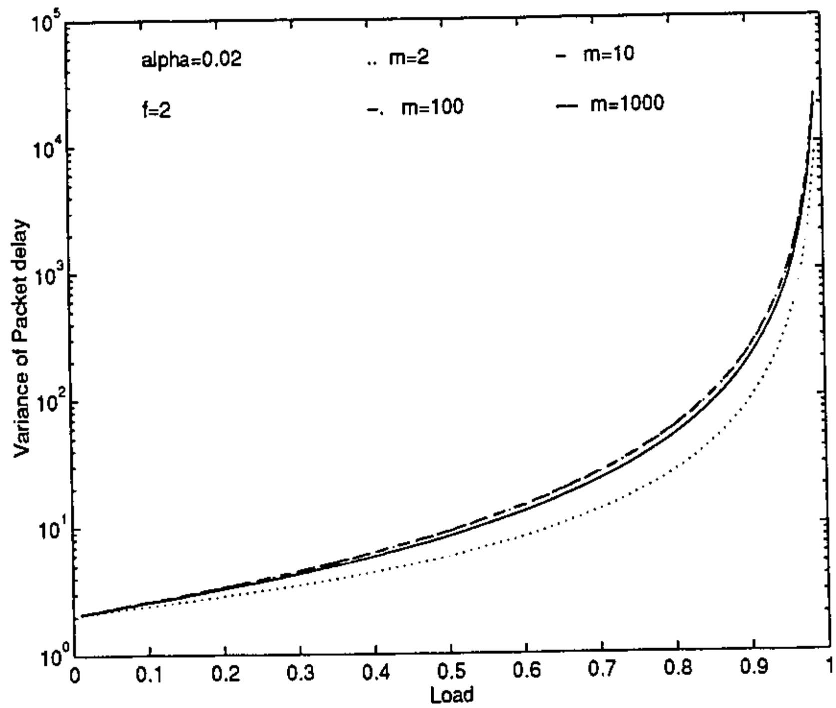

FIGURE.3.14 Variance of the Packet Delay for Different Values of *m*

CHAPTER IV

Transient and Steady-State Analysis of a Multi-Server ATM Multiplexer

4.1 Introduction

In this chapter, we extend the analysis of the queue length behavior of the single-server ATM multiplexer to the general multiserver case. More specifically, we assume that the number of output links (channels), used for the removal of packets from the buffer (i.e. the number of servers in the ATM multiplexer), is equal to the integer, $c > 0$.We also assume that the servers are not subject to interruptions. This means that if at the beginning of a slot, c packets are present at the front of the queue, they will leave the buffer at the end of this slot.

The remaining assumptions for the arrival process are the same as those previously described in section (3.2.1), except that, in this case, we allow a source to generate no packets during an active slot. By allowing some "silence" between cells, generated during a burst, we introduce more variability in the activity of each source. Filipiak [79] has suggested this type of source behavior as being a very versatile model in the ATM environment. However, the possibility of having zero arrivals during an active slot introduces more unknowns in the analysis of the queue length behavior.

Once again, the time axis is slotted in such a way that one slot is dedicated for the transmission of one packet via each of the c output channels of the multiplexer. We further assume that the average number of packet arrivals during an arbitrary slot is strictly less than c so that the system is stable and a steady-state exists.

As we will show shortly, ATM multiserver queues are more complex to analyze than the single server ones and some of the algebra presented in later stages of this chapter might look rather involved. Through careful choice of notation, we hope to have succeeded, with a certain degree, in presenting the material in a relatively simple and unified manner.

We start our analysis by considering the homogeneous case where we assume that all the sources feeding the multiplexer are of the same type.

4.2 The Single Type of Traffic Case

In this section, we consider a multi-server ATM multiplexer whose packet arrival process consists of the superposition of the traffic generated by m mutually independent and identical binary markov sources of the type described in section 3.2.1. Once again, we define the system by the pair (i_k, a_k) where i_k is the queue content at the end of the kth slot and a_k is the number of active sources during slot k. We focus our analysis on the transient as well as on the steady-state system occupancy distribution.

4.2.1 The Imbedded Markov Chain Analysis

The goal of this section is to derive the functional equation relating the joint PGF of the system between two consecutive slots. In this case, the evolution of the queue length is determined by the relationship:

$$i_{k+1} \doteq (i_k - c)^+ + b_{k+1} \tag{4.1}$$

where the notation $(x)^+$ denotes $max(x, 0)$. Next let:

$$Q_k(z, y) = E[z^{i_k} y^{a_k}] = \sum_{i=0}^{\infty} \sum_{j=0}^{m} z^i y^j p_k(i, j)$$

denote the joint PGF of the pair (i_k, a_k), where $p_k(i, j) = Pr(i_k = i, a_k = j)$. Then:

$$Q_{k+1}(z,y) = E\left[z^{(i_k-c)^+ + b_{k+1}} \; y^{a_{k+1}}\right]$$

Using (4.1) and averaging over the distribution of the $f_{j,k}$'s, the c_j's and the d_j's, gives:

$$Q_{k+1}(z,y) = E\left[z^{(i_k-c)^+ + \sum_{j=1}^{a_{k+1}} f_{j,k+1}} \cdot y^{a_{k+1}}\right]$$

$$= E\left[E\left[z^{(i_k-c)^+ + \sum_{j=1}^{a_{k+1}} f_{j,k+1}} \cdot y^{a_{k+1}} \middle| i_k, a_{k+1}\right]\right]$$

$$= E\left[z^{(i_k-c)^+} (y \cdot f(z))^{a_{k+1}}\right]$$

$$= E\left[E\left[z^{(i_k-c)^+} (y \cdot f(z))^{\sum_{j=1}^{a_k} c_j + \sum_{j=1}^{m-a_k} d_j} \middle| i_k, a_k\right]\right]$$

$$= E\left[z^{(i_k-c)^+} (d(y \cdot f(z)))^m \cdot \left[\frac{c(y \cdot f(z))}{d(y \cdot f(z))}\right]^{a_k}\right] \tag{4.2}$$

or equivalently:

$$Q_{k+1}(z,y) = [d(y \cdot f(z))]^m E\left[z^{(i_k-c)^+} \cdot Y^{a_k}\right] \tag{4.3}$$

where:

$$Y = \frac{c(y \cdot f(z))}{d(y \cdot f(z))} = \frac{1-\alpha+\alpha y f(z)}{\beta+(1-\beta) y f(z)}$$

The $(x)^+$ operator in the expectation term in (4.3) can be removed by noting that:

$$E\left[z^{(i_k-c)^+} \cdot Y^{a_k}\right] = \sum_{i=0}^{\infty} \sum_{j=0}^{m} z^{(i-c)^+} Y^j p_k(i,j)$$

$$= \sum_{i=0}^{c-1} \sum_{j=0}^{m} Y^j p_k(i,j) + \sum_{i=c}^{\infty} \sum_{j=0}^{m} z^{i-c} \cdot Y^j p_k(i,j)$$

$$= \sum_{i=0}^{c-1} \sum_{j=0}^{m} Y^j p_k(i,j) + \frac{1}{z^c}\left[\sum_{i=0}^{\infty} \sum_{j=0}^{m} z^i Y^j p_k(i,j) - \sum_{i=0}^{c-1} \sum_{j=0}^{m} z^i Y^j p_k(i,j)\right]$$

Hence:

$$Q_{k+1}(z,y) = [d(y \cdot f(z))]^m \left[\frac{Q_k(z,Y) - \sum_{i=0}^{c-1} \sum_{j=0}^{m} z^i Y^j p_k(i,j)}{z^c} + \sum_{i=0}^{c-1} \sum_{j=0}^{m} Y^j p_k(i,j) \right] \quad (4.4)$$

The above result is a generalization of Bruneel [51] functional equation, obtained for the single server (c=1) condition and under the special case where zero packet arrivals occur only during "passive" slots. In addition, in the sequel, we propose a new approach which enables us to derive explicit closed form expressions for the transient/steady-state joint and marginal PGFs of the multiserver system, as well as for some transient and steady-state performance measures. Our analysis also assumes an arbitrary, but a priori known, initial condition, $Q_0(z,y)$.

4.2.2 The Solution Method

By expanding $Q_{k+1}(z,y)$ in (4.4) for the first few values of k, we can prove by recurrence the following important result:

4.2.2.1 Theorem 4.1:

The joint PGF of the queueing system under consideration, as given by the functional equation (4.4), can be written as follows:

$$Q_k(z,y) = \frac{B(k)}{z^{kc}} Q_0(z, \Phi(k)) + \sum_{l=1}^{k} \sum_{i=0}^{c-1} \sum_{j=0}^{m} \frac{B(l)}{z^{lc}} (z^c - z^i) [\Phi(l)]^j p_{k-l}(i,j) \quad (4.5)$$

where $B(k) = [X(k)]^m$ and $X(k)$, $\Phi(k)$ are as defined in (3.7) and (3.4), respectively.

PROOF

Throughout this proof we make use of the fact that if $B^*(k) = B(k)|_{y=Y}$ then $B^*(k) = \frac{B(k+1)}{B(1)}$. This follows directly from the result of Appendix A5. Hence for k=0, the functional equation (4.4) yields:

$$Q_1(z,y) = B(1)\cdot\left\{\frac{Q_0(z,\Phi(1)) - \sum_{i=0}^{c-1}\sum_{j=0}^{m} z^i\Phi(1)^j p_0(i,j)}{z^c} + \sum_{i=0}^{c-1}\sum_{j=0}^{m}\Phi(1)^j p_0(i,j)\right\}$$

$$= \frac{B(1)}{z^c}Q_0(z,\Phi(1)) + \sum_{i=0}^{c-1}\sum_{j=0}^{m}\frac{B(1)}{z^c}(z^c - z^i)\Phi(1)^j p_0(i,j) \tag{4.6}$$

For k=1:

$$Q_2(z,y) = B(1)\cdot\left\{\frac{Q_1(z,\Phi(1)) - \sum_{i=0}^{c-1}\sum_{j=0}^{m} z^i\Phi(1)^j p_1(i,j)}{z^c} + \sum_{i=0}^{c-1}\sum_{j=0}^{m}\Phi(1)^j p_1(i,j)\right\}$$

Substituting (4.6) in the above gives:

$$Q_2(z,y) = \frac{B(1)}{z^c}\left\{\frac{B^*(1)}{z^c}Q_0(z,\Phi(2)) + \sum_{i=0}^{c-1}\sum_{j=0}^{m}\frac{B^*(1)}{z^c}(z^c - z^i)\Phi(2)^j p_0(i,j)\right\}$$

$$-\frac{B(1)}{z^c}\sum_{i=0}^{c-1}\sum_{j=0}^{m} z^i\Phi(1)^j p_1(i,j) + B(1)\sum_{i=0}^{c-1}\sum_{j=0}^{m}\Phi(1)^j p_1(i,j)$$

$$= \frac{B(2)}{z^{2c}}Q_0(z,\Phi(2)) + \sum_{l=1}^{2}\sum_{i=0}^{c-1}\sum_{j=0}^{m}\frac{B(l)}{z^{lc}}(z^c - z^i)\,[\Phi(l)]^j p_{2-l}(i,j) \tag{4.7}$$

and therefore (4.5) is verified for k=1,2 and obviously for k=0. Next let us suppose that (4.5) is true for the order (k), i.e.:

$$Q_k(z,y) = \frac{B(k)}{z^{kc}}Q_0(z,\Phi(k)) + \sum_{l=1}^{k}\sum_{i=0}^{c-1}\sum_{j=0}^{m}\frac{B(l)}{z^{lc}}(z^c - z^i)\,[\Phi(l)]^j p_{k-l}(i,j) \tag{4.8}$$

Let us prove that is also true for the order (k+1), i.e.:

$$Q_{k+1}(z,y) = \frac{B(k+1)}{z^{(k+1)c}}Q_0(z,\Phi(k+1)) + \sum_{l=1}^{k+1}\sum_{i=0}^{c-1}\sum_{j=0}^{m}\frac{B(l)}{z^{lc}}(z^c - z^i)\,[\Phi(l)]^j p_{k+1-l}(i,j) \tag{4.9}$$

By substituting (4.8) into (4.4) we get:

$$Q_{k+1}(z,y) = B(1)\Bigg\{$$

$$\frac{\frac{B^*(k)}{z^{kc}}Q_0(z,\Phi(k+1)) + \sum_{l=1}^{k}\sum_{i=0}^{c-1}\sum_{j=0}^{m}\frac{B^*(l)}{z^{lc}}(z^c - z^i)\,[\Phi(l+1)]^j p_{k-l}(i,j) - \sum_{i=0}^{c-1}\sum_{j=0}^{m} z^i(\Phi(1))^j p_k(i,j)}{z^c}$$

$$+\sum_{i=0}^{c-1}\sum_{j=0}^{m}[\Phi(1)]^j p_k(i,j)\Bigg\}$$

$$= \frac{B(k+1)}{z^{(k+1)c}}Q_0(z,\Phi(k+1)) + \sum_{l=1}^{k}\sum_{i=0}^{c-1}\sum_{j=0}^{m}\frac{B(l+1)}{z^{(l+1)c}}(z^c - z^i)\,[\Phi(l+1)]^j p_{k-l}(i,j)$$

$$-\frac{B(1)}{z^c}\sum_{i=0}^{c-1}\sum_{j=0}^{m} z^i(\Phi(1))^j p_k(i,j) + B(1)\sum_{i=0}^{c-1}\sum_{j=0}^{m}[\Phi(1)]^j p_k(i,j)$$

$$= \frac{B(k+1)}{z^{(k+1)c}}Q_0(z,\Phi(k+1)) + \sum_{l=1}^{k+1}\sum_{i=0}^{c-1}\sum_{j=0}^{m}\frac{B(l)}{z^{lc}}(z^c - z^i)\,[\Phi(l)]^j p_{k+1-l}(i,j)$$

Hence (4.9) is proved and this completes the proof of the theorem.□

In this case, the load of the system is:

$$\rho = m\frac{1-\beta}{2-\alpha-\beta}\bar{f}$$

and hence for stability we require that $\rho < c$ or equivalently $m\frac{1-\beta}{2-\alpha-\beta}\bar{f} < c$.

We next focus on the transient/steady-state behavior of the buffer occupancy.

4.2.2.2 Transient/Steady-State Analysis of the Buffer Occupancy Distribution

Let $P_k(z) = Q_k(z,1)$ denote the marginal PGF of the queue length at the end of the k^{th} slot. Then from (4.5):

$$P_k(z) = \frac{\tilde{B}(k)}{z^{kc}}Q_0(z,\tilde{\Phi}(k)) + \sum_{l=1}^{k}\sum_{i=0}^{c-1}\sum_{j=0}^{m}\frac{\tilde{B}(l)}{z^{lc}}(z^c - z^i)\,[\tilde{\Phi}(l)]^j p_{k-l}(i,j) \quad (4.10)$$

where:

$$\tilde{B}(k) = B(k)|_{y=1} = [\tilde{X}(k)]^m = (\tilde{C}_1\lambda_1^k + \tilde{C}_2\lambda_2^k)^m \tag{4.11}$$

$$\tilde{\Phi}(k) = \Phi(k)|_{y=1} = \frac{\tilde{U}(k)}{\tilde{X}(k)} = \frac{\tilde{D}_1\lambda_1^k + \tilde{D}_2\lambda_2^k}{\tilde{C}_1\lambda_1^k + \tilde{C}_2\lambda_2^k} \tag{4.12}$$

with $\tilde{C}_i = C_i|_{y=1}$ and $\tilde{D}_i = D_i|_{y=1}$; $\forall i \in \{1, 2\}$, while λ_i, C_i and D_i are as given in (3.8).

Next let $\mathfrak{I} = \{0, 1, \ldots, c-1\}$ and $\mathfrak{R} = \{0, 1, 2, \ldots, m\}$. Then from (4.10) we see that the transient probabilities $p_k(i, j)$'s, $(i, j) \in \aleph = \mathfrak{I} \times \mathfrak{R}$ are the only terms which remains to be evaluated in order to fully characterize the transient PGF of the queue size. The following theorem provides a means to compute them.

4.2.2.3 Theorem 4.2:

Let $P(z, w)$ and $P_{ij}(w)$ be the one-dimensional transforms, defined by:

$$P(z, w) = \sum_{k=0}^{\infty} P_k(z)\, w^k \qquad (|w| < 1) \tag{4.13}$$

and:

$$P_{ij}(w) = \sum_{k=0}^{\infty} p_k(i, j)\, w^k \qquad (|w| < 1) \tag{4.14}$$

Then:

$$P(z, w) = \sum_{i=0}^{\infty}\sum_{j=0}^{m}\sum_{\kappa=0}^{m} \frac{\sum_{l=[\kappa-j,0]^+}^{[m-j,\kappa]^-} \begin{bmatrix} m-j \\ l \end{bmatrix} \begin{bmatrix} j \\ \kappa-l \end{bmatrix} z^i p_0(i, j)\, \tilde{D}_1^{\kappa-l}\tilde{D}_2^{j-\kappa+l}\tilde{C}_1^{l}\tilde{C}_2^{m-j-l} z^c}{z^c - w\lambda_1^{\kappa}\lambda_2^{m-\kappa}}$$

$$+ \sum_{i=0}^{c-1}\sum_{j=0}^{m}\sum_{\kappa=0}^{m} \frac{\sum_{l=[\kappa-j,0]^+}^{[m-j,\kappa]^-} \begin{bmatrix} m-j \\ l \end{bmatrix} \begin{bmatrix} j \\ \kappa-l \end{bmatrix} \tilde{D}_1^{\kappa-l}\tilde{D}_2^{j-\kappa+l}\tilde{C}_1^{l}\tilde{C}_2^{m-j-l} (z^c - z^i)\, P_{ij}(w)\, w\lambda_1^{\kappa}\lambda_2^{m-\kappa}}{z^c - w\lambda_1^{\kappa}\lambda_2^{m-\kappa}} \tag{4.15}$$

where $p_0(i, j) = Pr(i_0 = i, j_0 = j)$.

In addition, for each $\kappa \in \Re$, let $V_\kappa(z) = w\lambda_1^\kappa \lambda_2^{m-\kappa}$ and denote by $z_{p\kappa}$, ($p \in \hat{\Im} = \{1, 2, \ldots, c\}$), the c roots of the equation $z^c = V_\kappa(z)$ inside the unit circle. Then for each $(p, \kappa) \in \hat{\aleph} = \hat{\Im} \times \Re$, $P_{ij}(w)$, $(i, j) \in \aleph = \Im \times \Re$ satisfies:

$$\sum_{i=0}^{\infty}\sum_{j=0}^{m}\sum_{l=[\kappa-j,0]^+}^{[m-j,\kappa]^-}\begin{bmatrix} m-j \\ l \end{bmatrix}\begin{bmatrix} j \\ \kappa-l \end{bmatrix} z_{p\kappa}^i p_0(i,j)\frac{(\hat{\lambda}_2-\beta)^{2l+j}}{(\alpha-1)^l[(1-\beta)f(z_{p\kappa})]^{l+j}}$$

$$+\sum_{i=0}^{c-1}\sum_{j=0}^{m}\sum_{l=[\kappa-j,0]^+}^{[m-j,\kappa]^-}\begin{bmatrix} m-j \\ l \end{bmatrix}\begin{bmatrix} j \\ \kappa-l \end{bmatrix}(z_{p\kappa}^c - z_{p\kappa}^i)P_{ij}(w)\frac{(\hat{\lambda}_2-\beta)^{2l+j}}{(\alpha-1)^l[(1-\beta)f(z_{p\kappa})]^{l+j}} = 0 \qquad (4.16)$$

where $\hat{\lambda} = \lambda_2|_{z = z_{p\kappa}}$.

PROOF

From (4.10), the w transform, $P(z, w) = \sum_{k=0}^{\infty} P_k(z) w^k$ where ($|w| < 1$) is given by:

$$P(z,w) = \sum_{k=0}^{\infty}\tilde{B}(k)Q_0(z,\tilde{\Phi}(k))\left[\frac{w}{z^c}\right]^k + \sum_{k=0}^{\infty}\sum_{l=1}^{k}\sum_{i=0}^{c-1}\sum_{j=0}^{m}\frac{\tilde{B}(l)}{z^{lc}}(z^c - z^i)[\tilde{\Phi}(l)]^j p_{k-l}(i,j)w^k \qquad (4.17)$$

We first look to the first term in (4.17) namely, $I = \sum_{k=0}^{\infty}\tilde{B}(k)Q_0(z,\tilde{\Phi}(k))\left[\frac{w}{z^c}\right]^k$.

Since:

$$Q_0(z,\tilde{\Phi}(k)) = \sum_{i=0}^{\infty}\sum_{j=0}^{m} z^i[\tilde{\Phi}(k)]^j p_0(i,j)$$

then by substituting for $\tilde{B}(k)$ and $\tilde{\Phi}(k)$ as in ((4.11)-(4.12)) and by applying the Binomial theorem we get:

$$I = \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{m} z^i \tilde{U}(k)^j \tilde{X}(k)^{m-j} p_0(i,j)\left[\frac{w}{z^c}\right]^k$$

$$= \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{m}\sum_{r=0}^{m-j}\sum_{s=0}^{j}\begin{bmatrix} m-j \\ r \end{bmatrix}\begin{bmatrix} j \\ s \end{bmatrix} z^i p_0(i,j)(\tilde{D}_1\lambda_1^k)^s(\tilde{D}_2\lambda_2^k)^{j-s}(\tilde{C}_1\lambda_1^k)^r(\tilde{C}_2\lambda_2^k)^{m-j-r}\left[\frac{w}{z^c}\right]^k$$

Interchanging the order of summations gives:

$$I = \sum_{i=0}^{\infty}\sum_{j=0}^{m}\sum_{r=0}^{m-j}\sum_{s=0}^{j}\begin{bmatrix}m-j\\r\end{bmatrix}\begin{bmatrix}j\\s\end{bmatrix} z^i p_0(i,j)\tilde{D}_1^{s}\tilde{D}_2^{j-s}\tilde{C}_1^{r}\tilde{C}_2^{m-j-r}\sum_{k=0}^{\infty}\left[\frac{\lambda_1^{r+s}\lambda_2^{m-(r+s)}w}{z^c}\right]^k$$

Finally the last term in the above expression can be further simplified to yield:

$$I = \sum_{i=0}^{\infty}\sum_{j=0}^{m}\sum_{r=0}^{m-j}\sum_{s=0}^{j}\begin{bmatrix}m-j\\r\end{bmatrix}\begin{bmatrix}j\\s\end{bmatrix}\frac{z^i p_0(i,j)\tilde{D}_1^{s}\tilde{D}_2^{j-s}\tilde{C}_1^{r}\tilde{C}_2^{m-j-r} z^c}{z^c - w\lambda_1^{r+s}\lambda_2^{m-(r+s)}}$$

Next we consider the second term in (4.17) which can be expanded as follows:

$$II = \sum_{k=0}^{\infty}\sum_{l=1}^{k}\sum_{i=0}^{c-1}\sum_{j=0}^{m}\frac{\tilde{B}(l)}{z^{lc}}(z^c - z^i)\,[\tilde{\Phi}(l)]^j p_{k-l}(i,j)\,w^k$$

$$= \sum_{i=0}^{c-1}\sum_{j=0}^{m}(z^c - z^i)\sum_{k=0}^{\infty}\left[\sum_{l=0}^{k}\frac{\tilde{B}(l)}{z^{lc}}[\tilde{\Phi}(l)]^j p_{k-l}(i,j) - p_k(i,j)\right]w^k$$

$$= \sum_{i=0}^{c-1}\sum_{j=0}^{m}(z^c - z^i)\left[P_{ij}(w)\cdot\sum_{k=0}^{\infty}\frac{\tilde{B}(k)}{z^{kc}}[\tilde{\Phi}(k)]^j w^k - P_{ij}(w)\right]$$

$$= \sum_{i=0}^{c-1}\sum_{j=0}^{m}(z^c - z^i)P_{ij}(w)\cdot\sum_{k=1}^{\infty}\frac{\tilde{B}(k)}{z^{kc}}[\tilde{\Phi}(k)]^j w^k$$

Once again, substituting for $\tilde{B}(k)$ and $\tilde{\Phi}(k)$ as in ((4.11)-(4.12)) and using the Binomial theorem gives:

$$II = \sum_{i=0}^{c-1}\sum_{j=0}^{m}(z^c - z^i)P_{ij}(w)\cdot\sum_{k=1}^{\infty}\tilde{X}(k)^{m-j}\tilde{U}(k)^j\left[\frac{w}{z^c}\right]^k$$

$$= \sum_{i=0}^{c-1}\sum_{j=0}^{m}(z^c - z^i)P_{ij}(w)\cdot\sum_{k=1}^{\infty}\sum_{r=0}^{m-j}\sum_{s=0}^{j}\begin{bmatrix}m-j\\r\end{bmatrix}\begin{bmatrix}j\\s\end{bmatrix}(\tilde{D}_1\lambda_1^k)^s(\tilde{D}_2\lambda_2^k)^{j-s}(\tilde{C}_1\lambda_1^k)^r(\tilde{C}_2\lambda_2^k)^{m-j-r}\left[\frac{w}{z^c}\right]^k$$

Next by interchanging the order of summations, we get:

$$II = \sum_{i=0}^{c-1}\sum_{j=0}^{m}(z^c - z^i)P_{ij}(w)\sum_{r=0}^{m-j}\sum_{s=0}^{j}\begin{bmatrix}m-j\\r\end{bmatrix}\begin{bmatrix}j\\s\end{bmatrix}\tilde{D}_1^{s}\tilde{D}_2^{j-s}\tilde{C}_1^{r}\tilde{C}_2^{m-j-r}\sum_{k=1}^{\infty}\left[\frac{\lambda_1^{r+s}\lambda_2^{m-(r+s)}w}{z^c}\right]^k$$

The last term can be further simplified to yield:

$$II = \sum_{i=0}^{c-1}\sum_{j=0}^{m}\sum_{r=0}^{m-j}\sum_{s=0}^{j}\begin{bmatrix} m-j \\ r \end{bmatrix}\begin{bmatrix} j \\ s \end{bmatrix}\frac{\tilde{D}_1^s\tilde{D}_2^{j-s}\tilde{C}_1^r\tilde{C}_2^{m-j-r}\,(z^c - z^i)\,P_{ij}(w)\,w\lambda_1^{r+s}\lambda_2^{m-(r+s)}}{z^c - w\lambda_1^{r+s}\lambda_2^{m-(r+s)}}$$

Therefore:

$$P(z,w) = \sum_{i=0}^{\infty}\sum_{j=0}^{m}\sum_{r=0}^{m-j}\sum_{s=0}^{j}\begin{bmatrix} m-j \\ r \end{bmatrix}\begin{bmatrix} j \\ s \end{bmatrix}\frac{z^i p_0(i,j)\,\tilde{D}_1^s\tilde{D}_2^{j-s}\tilde{C}_1^r\tilde{C}_2^{m-j-r}\,z^c}{z^c - w\lambda_1^{r+s}\lambda_2^{m-(r+s)}}$$

$$+\sum_{i=0}^{c-1}\sum_{j=0}^{m}\sum_{r=0}^{m-j}\sum_{s=0}^{j}\begin{bmatrix} m-j \\ r \end{bmatrix}\begin{bmatrix} j \\ s \end{bmatrix}\frac{\tilde{D}_1^s\tilde{D}_2^{j-s}\tilde{C}_1^r\tilde{C}_2^{m-j-r}\,(z^c - z^i)\,P_{ij}(w)\,w\lambda_1^{r+s}\lambda_2^{m-(r+s)}}{z^c - w\lambda_1^{r+s}\lambda_2^{m-(r+s)}}$$

or equivalently, with the change of variables, $\kappa = r+s$, we get:

$$P(z,w) = \sum_{i=0}^{\infty}\sum_{j=0}^{m}\sum_{\kappa=0}^{m}\frac{\sum_{l=[\kappa-j,0]^+}^{[m-j,\kappa]^-}\begin{bmatrix} m-j \\ l \end{bmatrix}\begin{bmatrix} j \\ \kappa-l \end{bmatrix}z^i p_0(i,j)\,\tilde{D}_1^{\kappa-l}\tilde{D}_2^{j-\kappa+l}\tilde{C}_1^{l}\tilde{C}_2^{m-j-l}\,z^c}{z^c - w\lambda_1^{\kappa}\lambda_2^{m-\kappa}}$$

$$+\sum_{i=0}^{c-1}\sum_{j=0}^{m}\sum_{\kappa=0}^{m}\frac{\sum_{l=[\kappa-j,0]^+}^{[m-j,\kappa]^-}\begin{bmatrix} m-j \\ l \end{bmatrix}\begin{bmatrix} j \\ \kappa-l \end{bmatrix}\tilde{D}_1^{\kappa-l}\tilde{D}_2^{j-\kappa+l}\tilde{C}_1^{l}\tilde{C}_2^{m-j-l}\,(z^c - z^i)\,P_{ij}(w)\,w\lambda_1^{\kappa}\lambda_2^{m-\kappa}}{z^c - w\lambda_1^{\kappa}\lambda_2^{m-\kappa}} \tag{4.18}$$

The only remaining unknowns in (4.18) are the $c(m+1)$ terms, $P_{ij}(w)$'s, which can be determined by invoking the analytical property of $P(z,w)$, inside the poly-disc ($|w| < 1$, $|z| \le 1$), as follows:

For each $\kappa \in \Re$, let us consider the roots of the equation:

$$z^c = V_\kappa(z) = w\lambda_1^{\kappa}\lambda_2^{m-\kappa}. \tag{4.19}$$

We first apply Rouché's theorem (appendix A1) for the evaluation of the number of roots and then use the Lagrange's theorem (Appendix A.2) in order to compute these roots explicitly.

Let $h(z) = z^c$ and $g_\kappa(z) = -V_\kappa(z)$. Since $|\lambda_1| \le |\lambda_2| \le 1$ and $|w| < 1$, then for each $\kappa \in \Re$:

$$|g_\kappa(z)| = |w\lambda_1^{\kappa}\lambda_2^{m-\kappa}| \le |\lambda_2^{m-\kappa}|$$

Further $\lambda_2^{m-\kappa}$ is a valid generating function (GF), and for a small $\varepsilon > 0$ and on $|z| = 1+\varepsilon$, a GF *G(z)* satisfies [68]:

$$|G(z)| \le 1 + \varepsilon G'(1) + \theta(\varepsilon).$$

Using this bound, we get $|g_\kappa(z)| \le 1 + \varepsilon \frac{(m-\kappa)(1-\beta)}{2-\alpha-\beta}\bar{f} + \theta(\varepsilon)$. On $|z| = 1+\varepsilon$ we also have $|h(z)| = (1+\varepsilon)^c = 1 + c\varepsilon + \theta(\varepsilon)$ and therefore if the system is stable, i.e. $(\rho = \frac{m(1-\beta)}{2-\alpha-\beta}\bar{f} < c)$, then for each $\kappa \in \Re$, $|h(z)| > |g_\kappa(z)|$ on $|z| = 1+\varepsilon$.

From Rouché's theorem $h(z)$ and $h(z) + g_\kappa(z)$ have the same number of zeros inside $|z| = 1+\varepsilon$. Evidently $h(z)$ has *c* zeros inside $|z| = 1+\varepsilon$ and therefore (4.19) has also *c* roots inside $|z| = 1+\varepsilon$. These roots can be expressed explicitly in terms of Lagrange's expansion by noting that for each $\kappa \in \Re$,(4.19) implies that $z = \nu_p [V_k(z)]^{1/c}$ where $\nu_p = e^{j2\pi p/c}$ $(p \in \hat{\Im})$ are the c roots of unity and $j = \sqrt{-1}$. Therefore, from Lagrange's theorem, and for each $\kappa \in \Re$,the c roots of (4.19) inside $|z| = 1$ are given by:

$$z_{p\kappa} = \sum_{n=1}^{\infty} \frac{(\nu_p w^{1/c})^n}{n!} \frac{d^{n-1}}{dz^{n-1}} [V_\kappa(z)]^{n/c}\Big|_{z=0} \tag{4.20}$$

where $(p, \kappa) \in \hat{\aleph} = \hat{\Im} \times \Re$.

Now for each $\kappa \in \Re$ we know the values of the c zeros of the denominator of $P(z, w)$. In addition since $P(z, w)$ is analytical inside $\langle |z| \le 1\,;\ |w| < 1 \rangle$ then for each $\kappa \in \Re$, the numerator of $P(z, w)$ must also be zero at $z = z_{p\kappa}$, for each $p \in \hat{\Im}$. Hence $\forall (p, \kappa) \in \hat{\aleph} = \hat{\Im} \times \Re$:

$$\sum_{i=0}^{\infty}\sum_{j=0}^{m}\sum_{l=[\kappa-j,0]^+}^{[m-j,\kappa]^-} \begin{bmatrix} m-j \\ l \end{bmatrix} \begin{bmatrix} j \\ \kappa-l \end{bmatrix} z_{p\kappa}^i p_0(i,j) \left[\frac{\hat{C}_1\hat{D}_2}{\hat{C}_2\hat{D}_1}\right]^l \left[\frac{\hat{D}_2}{\hat{C}_2}\right]^j$$

$$+ \sum_{i=0}^{c-1}\sum_{j=0}^{m}\sum_{l=[\kappa-j,0]^+}^{[m-j,\kappa]^-} \begin{bmatrix} m-j \\ l \end{bmatrix} \begin{bmatrix} j \\ \kappa-l \end{bmatrix} (z_{p\kappa}^c - z_{p\kappa}^i) P_{ij}(w) \left[\frac{\hat{C}_1\hat{D}_2}{\hat{C}_2\hat{D}_1}\right]^l \left[\frac{\hat{D}_2}{\hat{C}_2}\right]^j = 0$$

where $\hat{C}_{1,2} = \tilde{C}_{1,2}\big|_{z=z_{pk}}$ and $\hat{D}_{1,2} = \tilde{D}_{1,2}\big|_{z=z_{pk}}$.

Next, it is easy to verify that $\frac{\tilde{C}_1}{\tilde{D}_1} = \frac{\lambda_2 - \beta}{\alpha - 1}$, $\frac{\tilde{D}_2}{\tilde{C}_2} = \frac{\lambda_2 - \beta}{(1-\beta)f(z)}$ and hence the above equation can be conveniently rewritten as:

$$\sum_{i=0}^{\infty}\sum_{j=0}^{m}\sum_{l=[\kappa-j,0]^+}^{[m-j,\kappa]^-}\begin{bmatrix} m-j \\ l \end{bmatrix}\begin{bmatrix} j \\ \kappa-l \end{bmatrix} z_{p\kappa}^{i} p_0(i,j)\frac{(\hat{\lambda}_2-\beta)^{2l+j}}{(\alpha-1)^l[(1-\beta)f(z_{p\kappa})]^{l+j}}$$

$$+\sum_{i=0}^{c-1}\sum_{j=0}^{m}\sum_{l=[\kappa-j,0]^+}^{[m-j,\kappa]^-}\begin{bmatrix} m-j \\ l \end{bmatrix}\begin{bmatrix} j \\ \kappa-l \end{bmatrix}(z_{p\kappa}^{c}-z_{p\kappa}^{i})P_{ij}(w)\frac{(\hat{\lambda}_2-\beta)^{2l+j}}{(\alpha-1)^l[(1-\beta)f(z_{p\kappa})]^{l+j}} = 0 \tag{4.21}$$

where $\hat{\lambda} = \lambda_2|_{z=z_{p\kappa}}$.□ .

The above defines a set of $c(m+1)$ equations for the determination of the w transforms, $P_{ij}(w)$, $(i,j) \in \aleph = \Im \times \Re$. Taking the inverse w transform enables us to compute the $c(m+1)$ unknown boundary terms $p_k(i,j)$'s.

4.2.3 Transient Mean of the Queue Length Distribution

Let $\bar{N}_k = \left.\frac{dP_k(z)}{dz}\right|_{z=1}$ denote the average queue length of the ATM multiplexer at the end of the k^{th} slot. Differentiating (4.10) with respect to z and substituting z=1 in the resulting expression yields:

$$\bar{N}_k = \left.\frac{d\tilde{B}(k)}{dz}\right|_{z=1} - kc + \frac{d}{dz}Q_0(z,\tilde{\Phi}(k))\Big|_{z=1} + \sum_{l=1}^{k}\sum_{i=0}^{c-1}\sum_{j=0}^{m}(c-i)p_{k-l}(i,j) \tag{4.22}$$

or equivalently:

$$\bar{N}_k = \frac{m(1-\beta)(1-\alpha-\beta)}{(2-\alpha-\beta)^2}\bar{f}(1-(\alpha+\beta-1)^k) - k(c-\rho) + \bar{N}_0$$
$$+\frac{\alpha+\beta-1}{2-\alpha-\beta}\bar{f}(1-(\alpha+\beta-1)^k)\bar{A}_0 + \sum_{l=0}^{k-1}\sum_{i=0}^{c-1}\sum_{j=0}^{m}(c-i)p_l(i,j) \tag{4.23}$$

Using the results of the previous section, we can compute the transient probabilities $p_l(i,j)$'s and hence, from the above equation, we can evaluate the average queue length at the end of any particular slot.

4.2.4 Transient Variance of the Queue Length Distribution

By differentiating (4.10) twice with respect to z and evaluating the resulting expression at z=1, we get:

$$\left.\frac{d^2P_k(z)}{dz^2}\right|_{z=1} = \left.\frac{d^2\tilde{B}(k)}{dz^2}\right|_{z=1} + kc(kc+1) - 2kc\left.\frac{d\tilde{B}(k)}{dz}\right|_{z=1} + 2\left[\left.\frac{d\tilde{B}(k)}{dz}\right|_{z=1} - kc\right]\left.\frac{dQ_0(z,\tilde{\Phi}(k))}{dz}\right|_{z=1}$$

$$+\sum_{l=1}^{k}\sum_{i=0}^{c-1}\sum_{j=0}^{m}\left[2\left\{\left.\frac{d}{dz}\tilde{B}(l)\right|_{z=1} - lc\right\}[c-i] + \{c(c-1) - i(i-1)\} + 2(c-i)j\left.\frac{d}{dz}\tilde{\Phi}(l)\right|_{z=}\right]p_{k-l}(i,j)$$

$$+\frac{d^2}{dz^2}Q_0(z,\tilde{\Phi}(k))\Big|_{z=1} \tag{4.24}$$

The unknowns $\left.\frac{d^2\tilde{B}(k)}{dz^2}\right|_{z=1}$, $\left.\frac{d\tilde{B}(k)}{dz}\right|_{z=1}$, $\left.\frac{dQ_0(z,\tilde{\Phi}(k))}{dz}\right|_{z=1}$ and $\frac{d^2}{dz^2}Q_0(z,\tilde{\Phi}(k))\Big|_{z=1}$ can be readily obtained, using the results of section 3.2.7 and hence the variance of the queue length distribution at the end of the k^{th} slot can be computed from the general formula:

$$\sigma^2_{N_k} = \left.\frac{d^2P_k(z)}{dz^2}\right|_{z=1} + \bar{N}_k(1-\bar{N}_k)$$

4.2.5 Steady-State PGF of the Buffer Occupancy Distribution

The steady-state PGF, $P(z) = \lim_{k\to\infty} P_k(z)$, of the queue length distribution can be found by applying Abel's theorem to (4.15). Hence we can write:

$P(z) = \lim_{w\to 1^-}(1-w)P(z,w)$ or, equivalently:

$$P(z) = \lim_{w\to 1^-}(1-w)\sum_{i=0}^{\infty}\sum_{j=0}^{m}\sum_{\kappa=0}^{m}\frac{\sum_{l=[\kappa-j,0]^+}^{[m-j,\kappa]^-}\begin{bmatrix}m-j\\ l\end{bmatrix}\begin{bmatrix}j\\ \kappa-l\end{bmatrix}z^i p_0(i,j)\tilde{D}_1^{\kappa-l}\tilde{D}_2^{j-\kappa+l}\tilde{C}_1^{l}\tilde{C}_2^{m-j-l}z^c}{z^c - w\lambda_1^{\kappa}\lambda_2^{m-\kappa}}$$

$$+\lim_{w\to 1^-}(1-w)\sum_{i=0}^{c-1}\sum_{j=0}^{m}\sum_{k=0}^{m}\frac{\sum_{l=[\kappa-j,0]^+}^{[m-j,\kappa]^-}\begin{bmatrix}m-j\\ l\end{bmatrix}\begin{bmatrix}j\\ \kappa-l\end{bmatrix}\tilde{D}_1^{\kappa-l}\tilde{D}_2^{j-\kappa+l}\tilde{C}_1^{l}\tilde{C}_2^{m-j-l}(z^c-z^i)P_{ij}(w)w\lambda_1^{\kappa}\lambda_2^{m-\kappa}}{z^c - w\lambda_1^{\kappa}\lambda_2^{m-\kappa}}$$

Since the first limit converges to zero and since from Abel's theorem $\lim_{w \to 1^-} (1-w) P_{ij}(w) = \lim_{k \to \infty} p_k(i,j)$ then the last equation reduces to:

$$P(z) = \sum_{i=0}^{c-1} \sum_{j=0}^{m} (z^c - z^i)\, p(i,j) \sum_{\kappa=0}^{m} \frac{\sum_{l=[\kappa-j,0]^+}^{[m-j,\kappa]^-} \begin{bmatrix} m-j \\ l \end{bmatrix} \begin{bmatrix} j \\ \kappa-l \end{bmatrix} \tilde{D}_1^{\kappa-l} \tilde{D}_2^{j-\kappa+l} \tilde{C}_1^{l} \tilde{C}_2^{m-j-l} \lambda_1^{\kappa} \lambda_2^{m-\kappa}}{z^c - \lambda_1^{\kappa} \lambda_2^{m-\kappa}} \tag{4.25}$$

where $p(i,j) = \lim_{k \to \infty} p_k(i,j) = \lim_{w \to 1^-} (1-w) P_{ij}(w)$.

As expected, and because of the Markovian property of the model, the steady-state solution is independent of the initial conditions $p_0(i,j)$'s (which are imbedded in the first term of (4.10)). The only unknowns in the steady-state PGF are the $c(m+1)$ boundary terms $p(i,j)$'s, $(i,j) \in \aleph = \Im \times \Re$. These could be determined from the application of Abel's theorem to $P_{ij}(w)$, or they could be simply determined by invoking the analytical property of the steady-state PGF, $P(z)$, inside the unit disk as follows:

First by applying Rouché's theorem, and using similar arguments to those presented in the proof of theorem 4.2, it is easy to prove that for each $\kappa \in \Re$, the equation $z^c = V_\kappa(z) = \lambda_1^{\kappa} \lambda_2^{m-\kappa}$ has c distinct roots inside the unit circle. These will be denoted by $z_{\kappa p}$, $(p, \kappa) \in \hat{\aleph} = \hat{\Im} \times \Re$. For $\kappa = 0$, one of these roots is $z_{0c} = 1$,which also appears in the numerator of $P(z)$, and the remaining roots are $z_{01}, z_{02}, \ldots, z_{0c-1}$. The roots $z_{\kappa p}$'s can be computed using standard numerical solution techniques, such as the Newton-Raphson method, by noting that for each $\kappa \in \Re$ the equation $z^c = V_\kappa(z)$ can be replaced by an equivalent set of c equations, each having a unique root inside the unit circle [2]. Alternatively, from Lagrange's theorem, the roots $z_{\kappa p}$($(p,\kappa) \in \tilde{\aleph} = \hat{\aleph} - \{c, 0\}$) are given by:

$$z_{\kappa p} = \sum_{n=1}^{\infty} \frac{\nu_p^n}{n!} \frac{d^{n-1}}{dz^{n-1}} [V_\kappa(z)]^{n/c} \Big|_{z=0} \tag{4.26}$$

where $\nu_p = e^{j2\pi p/c} (p \in \hat{\Im})$ are the c roots of unity and $j = \sqrt{-1}$.

Since $P(z)$ is analytic inside the unit disk, then for each $k \in \Re$ the numerator of $P(z)$ must also be zero at $z = z_{kp}$. Hence:$\forall (p, \kappa) \in \tilde{\aleph} = \hat{\aleph} - \{c, 0\}$

$$\sum_{i=0}^{c-1} \sum_{j=0}^{m} \sum_{l=[\kappa-j,0]^+}^{[m-j,\kappa]^-} \begin{bmatrix} m-j \\ l \end{bmatrix} \begin{bmatrix} j \\ \kappa-l \end{bmatrix} (z_{p\kappa}^c - z_{p\kappa}^i)\, p(i,j) \frac{(\hat{\lambda}_2 - \beta)^{2l+j}}{(\alpha-1)^l [(1-\beta) f(z_{p\kappa})]^{l+j}} = 0 \qquad (4.27)$$

where $\hat{\lambda} = \lambda_2|_{z = z_{p\kappa}}$.

The above defines a set of $c(m+1) - 1$ linear equations for the determination of the unknowns $p(i,j)$'s. The remaining equation is provided by the normalization condition, $P(1) = 1$ as follows:

First we note that since $\tilde{C}_1|_{z=1} = \tilde{D}_1|_{z=1} = 0$, then, except for the term corresponding to $(\kappa = 0)$, all the terms under the third summation in (4.25) become zero when evaluated at z=1. Therefore it is convenient to rewrite (4.25) as follows:

$$P(z) = \sum_{i=0}^{c-1} \sum_{j=0}^{m} (z^c - z^i)\, p(i,j) \left[F_j(z) + \frac{G_j(z)}{z^c - H(z)} \right] \qquad (4.28)$$

where:

$$F_j(z) = \sum_{k=1}^{m} \frac{\sum_{l=[\kappa-j,0]^+}^{[m-j,\kappa]^-} \begin{bmatrix} m-j \\ l \end{bmatrix} \begin{bmatrix} j \\ k-l \end{bmatrix} \tilde{D}_1^{k-l} \tilde{D}_2^{j-k+l} \tilde{C}_1^{l} \tilde{C}_2^{m-j-l} \lambda_1^k \lambda_2^{m-k}}{z^c - \lambda_1^k \lambda_2^{m-k}}$$

$$G_j(z) = (\tilde{C}_2 \lambda_2)^m \left[\frac{\tilde{D}_2}{\tilde{C}_2} \right]^j \qquad \text{and} \qquad H(z) = \lambda_2^m$$

From (4.28):

$$P(z)\,[z^c - H(z)] = \sum_{i=0}^{c-1} \sum_{j=0}^{m} (z^c - z^i)\, p(i,j)\, [G_j(z) + (z^c - H(z)) F_j(z)] \qquad (4.29)$$

Next differentiating both sides of the above equation with respect to z, substituting z=1 in the resulting equation and noting that $F_j(1) = 0$, $G_j(1) = 1$ yields:

$$c - H'(1) = \sum_{i=0}^{c-1} \sum_{j=0}^{m} (c-i)\, p(i,j)$$

or equivalently:

$$c - m\frac{1-\beta}{2-\alpha-\beta}\bar{f} = \sum_{i=0}^{c-1}\sum_{j=0}^{m}(c-i)\,p(i,j) \tag{4.30}$$

This is the remaining equation for the determination of all the *c(m+1)* boundary terms.

Note that (4.30) implies that:

$$c - \rho = c\left[\sum_{i=0}^{\infty}\sum_{j=0}^{m}p(i,j) - \sum_{i=c}^{\infty}\sum_{j=0}^{m}p(i,j)\right] - \sum_{i=0}^{c-1}\sum_{j=0}^{m}ip(i,j)$$

or equivalently:

$$\rho = \sum_{i=0}^{c-1}ip(i) + \sum_{i=c}^{\infty}cp(i) \tag{4.31}$$

This last expression is intuitively clear owing to the conservation principle and the infinite waiting room assumption. In fact the quantity in the right hand side of (4.31) represents the steady-state mean of the number of packets that leave the buffer at the end of a slot, which is also equal to the steady-state mean of the number of packet arrivals during a slot.

4.2.6 Steady-State Mean and Variance of the Buffer length

Let $\bar{N}$ denote the steady-state mean buffer length. Then by differentiating (4.29) twice with respect to z and substituting z=1 in the resulting expression we get:

$$\bar{N} = \frac{H''(1) - c(c-1)}{2(c-H'(1))} + \frac{1}{2[c-H'(1)]}\sum_{i=0}^{c-1}\sum_{j=0}^{m}[c(c-1) - i(i-1) + 2(c-i)G'_j(1)]\,p(i,j) \tag{4.32}$$

where $H'(1) = \rho$ and $H''(1)$ is as given in (3.53b).

Surprisingly, the expression of the average queue length, as given above, depends only on the term of $P(z)$ in (4.25), which corresponds to the case $\kappa = 0$. Further, for the special case, c=1, the average buffer length is given by the formula:

$$\bar{N} = \frac{H''(1)}{2(1-H'(1))} + \frac{1}{[1-H'(1)]}\sum_{j=0}^{m} G'_j(1)\,p(0,j)$$

which is more involved than the expression previously derived in (3.52), since in this chapter we are allowing zero packet generation during an active slot.

Next, we focus on the variance of the buffer occupancy distribution. From (4.29), the second moment of the queue length is given by:

$$P''(1) = \frac{H'''(1) - c(c-1)(c-2)}{3[c-H'(1)]} + \frac{H''(1) - c(c-1)}{c-H'(1)}P'(1)$$

$$+ \frac{1}{3[c-H'(1)]} \times \{\sum_{i=0}^{c-1}\sum_{j=0}^{m} \{c(c-1)(c-2) - i(i-1)(i-2) + 3[c(c-1) - i(i-1)]G'_j(1)$$

$$+ 3(c-i)[G''_j(1) + 2(c-H'(1))F'_j(1)]\}\,p(i,j)\} \tag{4.33}$$

where:

$$G''_j(1) = G''(1) + 2jG'(1)[\tilde{D}'_2(1) - \tilde{C}'_2(1)]$$

$$+ j\{\tilde{D}''_2(1) - \tilde{C}''_2(1) - 2\tilde{C}'_2(1)[\tilde{D}'_2(1) - \tilde{C}'_2(1)] + (j-1)[\tilde{D}'_2(1) - \tilde{C}'_2(1)]^2\} \tag{4.34}$$

$$F'_j(1) = \frac{m(1-\beta)(1-\alpha-\beta)^2}{(2-\alpha-\beta)^3}\bar{f} - j\frac{(\alpha+\beta-1)^2}{(2-\alpha-\beta)^2}\bar{f} \tag{4.35}$$

$$D''_2(1) = \frac{2(1-\alpha)(1-\alpha-\beta)(\alpha^2-2\alpha+\alpha\beta-3\beta+3)}{(2-\alpha-\beta)^4}\bar{f}^2 - \frac{(1-\alpha)(1-\alpha-\beta)f''(1)}{(2-\alpha-\beta)^2} \tag{4.36}$$

and $H'''(1)$, $G'(1)$ and $G''(1)$ are as given in (3.55a), (3.53a) and (3.56), respectively.

The variance of the queue length:

$$\sigma^2_N = P''(1) + P'(1)[1-P'(1)] \tag{4.37}$$

can be then easily computed using ((4.32)-(4.36)).

Finally note that equations ((4.32)-(4.37)) provides us with explicit closed-from analytical expressions for the mean and variance of the queue length distribution.

These expressions are exact and can be easily evaluated, once the boundary terms are computed.

4.3 The Multiple Type of traffic case

In this section, we extend the multiserver analysis of section 4.2 to the more general case where the multiplexer is fed with τ types of binary Markov sources. The model's assumptions for the arrival process are the same as those outlined in sections 3.3. Further throughout this section, it assumed that the equilibrium condition:

$$\rho = \sum_{i=1}^{\tau} m_i \frac{1-\beta_i}{2-\alpha_i-\beta_i} \bar{f}_i < c \tag{4.38}$$

is fulfilled and hence a steady-state exists.

4.3.1 The Imbedded Markov Chain Analysis

Once again, the queueing model under consideration, here, can be formulated as a discrete-time multidimensional Markov chain. The state of the system is defined by $(i_k, a^1_k, a^2_k, \ldots, a^{\tau}_k)$, where i_k is the queue length at the end of slot k. The evolution of the queue length is determined by the equation:

$$i_{k+1} = (i_k - c)^+ + b_{k+1} \tag{4.39}$$

Let:

$$Q_k(z, y_1, y_2, \ldots, y_\tau) = E\left[z^{i_k} \cdot \prod_{i=1}^{\tau} y_i^{a_k^i}\right] = \sum_{\iota=0}^{\infty} \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} \cdots \sum_{j_\tau=0}^{m_\tau} z^{\iota} \left[\prod_{i=1}^{\tau} y_i^{j_i}\right] p_k(\iota, j_1, j_2, \ldots, j_\tau) \tag{4.40}$$

denote the joint PGF of $i_k, a^1_k, a^2_k, \ldots a^{\tau}_k$. Then:

$$Q_{k+1}(z, y_1, y_2, \ldots, y_\tau) = E\left[z^{i_{k+1}} \cdot \prod_{i=1}^{\tau} y_i^{a_{k+1}^i}\right] = E\left[z^{(i_k - c)^+} \cdot z^{\sum_{i=1}^{\tau} \sum_{j=1}^{a^i_{k+1}} f^{(i)}_{j,k+1}} \cdot \prod_{i=1}^{\tau} y_i^{a_{k+1}^i}\right]$$

From the above, using ((3.59)-(3.60)) and (4.39) and by averaging over the distribution of the $f_{j,k}^{(i)}$'s, the c^i_j and the d^i_j, we obtain:

$$Q_{k+1}(z, y_1, y_2, \ldots y_\tau) = E\left[z^{(i_k - c)^+} \cdot z^{\sum_{i=1}^{\tau} \sum_{j=1}^{a^i_{k+1}} f_{j,k+1}^{(i)}} \cdot \prod_{i=1}^{\tau} y_i^{a^i_{k+1}} \right]$$

$$= E\left[E\left[z^{(i_k - c)^+} \cdot z^{\sum_{i=1}^{\tau} \sum_{j=1}^{a^i_{k+1}} f_{j,k+1}^{(i)}} \cdot \prod_{i=1}^{\tau} y_i^{a^i_{k+1}} \middle| i_k,\, a^1_{k+1},\, a^2_{k+1} \ldots a^\tau_{k+1} \right] \right]$$

$$= E\left[z^{(i_k - c)^+} \prod_{i=1}^{\tau} (y_i f_i(z))^{a^i_{k+1}} \right]$$

$$= E\left[E\left[z^{(i_k - c)^+} \prod_{i=1}^{\tau} (y_i f_i(z))^{\sum_{j=1}^{a^i_k} c^i_j + \sum_{j=1}^{m_i - a^i_k} d^i_j} \middle| i_k,\, a^1_k,\, a^2_k \ldots a^\tau_k \right] \right]$$

$$= E\left[z^{(i_k - c)^+} \cdot \prod_{i=1}^{\tau} \left[d^i (y_i f_i(z)) \right]^{m_i} \left[\frac{c^i (y_i f_i(z))}{d^i (y_i f_i(z))} \right]^{a^i_k} \right] \tag{4.41}$$

or equivalently:

$$Q_{k+1}(z, y_1, y_2, \ldots y_\tau) = \prod_{i=1}^{\tau} \left[d^i (y_i f_i(z)) \right]^{m_i} \cdot E\left[z^{(i_k - c)^+} \cdot \prod_{i=1}^{\tau} Y_i^{a^i_k} \right] \tag{4.42}$$

where:

$$Y_i = \frac{c^i (y_i f_i(z))}{d^i (y_i f_i(z))} = \frac{1 - \alpha_i + \alpha_i y_i f_i(z)}{\beta_i + (1 - \beta_i) y_i f_i(z)}$$

In the sequel, the notation $\sum_{\bar{J} = \bar{0}}^{\bar{M}}$ will be used to refer to the multidimensional summation given by $\sum_{j_1 = 0}^{m_1} \sum_{j_2 = 0}^{m_2} \cdots \sum_{j_\tau = 0}^{m_\tau}$. Further, $p_k(\iota, j_1, j_2, \ldots, j_\tau)$ and $Q_k(z, y_1, y_2, \ldots y_\tau)$ will be simply referred by $p_k(\iota, \bar{J})$ and $Q_k(z, \bar{y})$ respectively. Then from (4.42) we can remove the $(x)^+$ operator by noting that since:

$$E\left[z^{(i_k-c)^+}\cdot\prod_{i=1}^{\tau}Y_i^{a_k^i}\right] = \sum_{\iota=0}^{\infty}\sum_{\bar{j}=\bar{0}}^{\bar{M}} z^{(i-c)^+}\cdot\left[\prod_{i=1}^{\tau}Y_i^{j_i}\right]p_k(\iota,\bar{J})$$

$$= \sum_{\iota=0}^{c-1}\sum_{\bar{j}=\bar{0}}^{\bar{M}}\left[\prod_{i=1}^{\tau}Y_i^{j_i}\right]p_k(\iota,\bar{J}) + \sum_{\iota=c}^{\infty}\sum_{\bar{j}=\bar{0}}^{\bar{M}} z^{(i-c)}\cdot\left[\prod_{i=1}^{\tau}Y_i^{j_i}\right]p_k(\iota,\bar{J})$$

$$= \sum_{\iota=0}^{c-1}\sum_{\bar{j}=\bar{0}}^{\bar{M}}\left[\prod_{i=1}^{\tau}Y_i^{j_i}\right]p_k(\iota,\bar{J}) + \frac{1}{z^c}\left[\sum_{\iota=0}^{\infty}\sum_{\bar{j}=\bar{0}}^{\bar{M}} z^i\cdot\left[\prod_{i=1}^{\tau}Y_i^{j_i}\right]p_k(\iota,\bar{J}) - \sum_{\iota=0}^{c-1}\sum_{\bar{j}=\bar{0}}^{\bar{M}} z^i\cdot\left[\prod_{i=1}^{\tau}Y_i^{j_i}\right]p_k(\iota,\bar{J})\right]$$

then:

$$Q_{k+1}(z,\bar{y}) = \prod_{i=1}^{\tau}\left[d^i(y_if_i(z))\right]^{m_i}\left[\frac{Q_k(z,Y_1,Y_2,\ldots,Y_\tau) - \sum_{\iota=0}^{c-1}\sum_{\bar{j}=\bar{0}}^{\bar{M}} z^i\cdot\left[\prod_{i=1}^{\tau}Y_i^{j_i}\right]p_k(\iota,\bar{J})}{z^c} + \sum_{\iota=0}^{c-1}\sum_{\bar{j}=\bar{0}}^{\bar{M}}\left[\prod_{i=1}^{\tau}Y_i^{j_i}\right]p_k(\iota,\bar{J})\right] \tag{4.43}$$

The above is the functional equation relating the joint PGF of the general multiserver system, with multiple types of traffic, between two consecutive slots.

Expanding the above equation for the first few values of k, and using the same recurrence approach outlined in theorem 4.1, we can prove the following result:

4.3.2 Theorem 4.3:

The joint PGF of the system, as described by the functional equation (4.43), is given by:

$$Q_k(z,\bar{y}) = \frac{B(k)}{z^{kc}}Q_0(z,\Phi_1(k),\Phi_2(k),\ldots\Phi_\tau(k)) + \sum_{l=1}^{k}\sum_{\iota=0}^{c-1}\sum_{\bar{j}=\bar{0}}^{\bar{M}}\frac{B(l)}{z^{lc}}(z^c-z^\iota)\left[\prod_{i=1}^{\tau}\Phi_i(l)^{j_i}\right]p_{k-l}(\iota,\bar{J}) \tag{4.44}$$

where:

$$\Phi_i(k) = \frac{U_i(k)}{X_i(k)} = \frac{D_{1i}\lambda_{1i}^k + D_{2i}\lambda_{2i}^k}{C_{1i}\lambda_{1i}^k + C_{2i}\lambda_{2i}^k} \tag{4.45a}$$

with $\lambda_{1i,2i}$, $C_{1i,2i}$ and $D_{1i,2i}$ as specified in (3.68a-c) and:

$$B(k) = \prod_{i=1}^{\tau}[X_i(k)]^{m_i} \tag{4.45b}$$

PROOF

Substituting k=0 then k=1 in the functional equation (4.43) and, once again, using the fact that if $B^*(k) = B(k)|_{y_1=Y_1,\, y_2=Y_2 \ldots y_\tau=Y_\tau}$ then $B^*(k) = \frac{B(k+1)}{B(1)}$ (which is an immediate result of having: $X_i^*(k) = X_i(k)|_{y_i=Y_i} = \frac{X_i(k+1)}{X_i(1)}$), we get:

$$Q_1(z,\vec{y}) = B(1)\left[\frac{Q_0(z,Y_1,Y_2,\ldots,Y_\tau) - \sum_{\iota=0}^{c-1}\sum_{\vec{j}=\vec{0}}^{\vec{M}} z^{\iota}\cdot\left[\prod_{i=1}^{\tau} Y_i^{j_i}\right]p_0(\iota,\vec{j})}{z^c} + \sum_{\iota=0}^{c-1}\sum_{\vec{j}=\vec{0}}^{\vec{M}}\left[\prod_{i=1}^{\tau} Y_i^{j_i}\right]p_0(\iota,\vec{j})\right]$$

$$= \frac{B(1)}{z^c}Q_0(z,\Phi_1(1),\Phi_2(1),\ldots\Phi_\tau(1)) + \sum_{\iota=0}^{c-1}\sum_{\vec{j}=\vec{0}}^{\vec{M}}\frac{B(1)}{z^c}(z^c - z^{\iota})\left[\prod_{i=1}^{\tau}[\Phi_i(1)]^{j_i}\right]p_0(\iota,\vec{j})$$

$$Q_2(z,\vec{y}) = B(1)\left[\frac{Q_1(z,Y_1,Y_2,\ldots,Y_\tau) - \sum_{\iota=0}^{c-1}\sum_{\vec{j}=\vec{0}}^{\vec{M}} z^{\iota}\cdot\left[\prod_{i=1}^{\tau} Y_i^{j_i}\right]p_1(\iota,\vec{j})}{z^c} + \sum_{\iota=0}^{c-1}\sum_{\vec{j}=\vec{0}}^{\vec{M}}\left[\prod_{i=1}^{\tau} Y_i^{j_i}\right]p_1(\iota,\vec{j})\right]$$

$$= \frac{B(1)}{z^c}\left[\frac{B^*(1)}{z^c}Q_0(z,\Phi_1(2),\Phi_2(2),\ldots\Phi_\tau(2)) + \sum_{\iota=0}^{c-1}\sum_{\vec{j}=\vec{0}}^{\vec{M}}\frac{B^*(1)}{z^c}(z^c - z^{\iota})\left[\prod_{i=1}^{\tau}[\Phi_i(2)]^{j_i}\right]p_0(\iota,\vec{j})\right]$$

$$-\frac{B(1)}{z^c}\sum_{\iota=0}^{c-1}\sum_{\vec{j}=\vec{0}}^{\vec{M}} z^{\iota}\left[\prod_{i=1}^{\tau}[\Phi_i(1)]^{j_i}\right]p_1(\iota,\vec{j}) + B(1)\sum_{\iota=0}^{c-1}\sum_{\vec{j}=\vec{0}}^{\vec{M}}\left[\prod_{i=1}^{\tau}[\Phi_i(1)]^{j_i}\right]p_1(\iota,\vec{j})$$

$$= \frac{B(2)}{z^{2c}}Q_0(z,\Phi_1(2),\Phi_2(2),\ldots\Phi_\tau(2)) + \sum_{l=1}^{2}\sum_{\iota=0}^{c-1}\sum_{\vec{j}=\vec{0}}^{\vec{M}}\frac{B(l)}{z^{lc}}(z^c - z^{\iota})\left[\prod_{i=1}^{\tau}[\Phi_i(l)]^{j_i}\right]p_{2-l}(\iota,\vec{j})$$

Therefore (4.44) is true for k=1,2 and also for k=0. Next let us suppose that (4.44) is true for the order k, i.e.:

$$Q_k(z,\vec{y}) = \frac{B(k)}{z^{kc}}Q_0(z,\Phi_1(k),\Phi_2(k),\ldots\Phi_\tau(k)) + \sum_{l=1}^{k}\sum_{\iota=0}^{c-1}\sum_{\vec{j}=\vec{0}}^{\vec{M}}\frac{B(l)}{z^{lc}}(z^c - z^{\iota})\left[\prod_{i=1}^{\tau}[\Phi_i(l)]^{j_i}\right]p_{k-l}(\iota,\vec{j}) \quad (4.46)$$

Let us prove that it is also true for the order (k+1), i.e.:

$$Q_{k+1}(z,\vec{y}) = \frac{B(k+1)}{z^{(k+1)c}} Q_0(z, \Phi_1(k+1), \Phi_2(k+1), \ldots \Phi_\tau(k+1))$$

$$+ \sum_{l=1}^{k+1} \sum_{\iota=0}^{c-1} \sum_{\vec{j}=\vec{0}}^{\vec{M}} \frac{B(l)}{z^{lc}} (z^c - z^\iota) \left[\prod_{i=1}^{\tau} [\Phi_i(l)]^{j_i} \right] p_{k+1-l}(\iota, \vec{j})$$

By substituting (4.46) into the functional equation (4.43) and using the fact that $B^*(k) = \frac{B(k+1)}{B(1)}$ we obtain:

$$Q_{k+1}(z,\vec{y}) = \frac{B(1)}{z^c} \{ \frac{B^*(k)}{z^{kc}} Q_0(z, \Phi_1(k+1), \Phi_2(k+1), \ldots \Phi_\tau(k+1))$$

$$+ \sum_{l=1}^{k} \sum_{\iota=0}^{c-1} \sum_{\vec{j}=\vec{0}}^{\vec{M}} \frac{B^*(l)}{z^{lc}} (z^c - z^\iota) \left[\prod_{i=1}^{\tau} [\Phi_i(l+1)]^{j_i} \right] p_{k-l}(\iota, \vec{j})$$

$$- \sum_{\iota=0}^{c-1} \sum_{\vec{j}=\vec{0}}^{\vec{M}} z^\iota \left[\prod_{i=1}^{\tau} [\Phi_i(1)]^{j_i} \right] p_k(\iota, \vec{j}) \}$$

$$+ B(1) \sum_{\iota=0}^{c-1} \sum_{\vec{j}=\vec{0}}^{\vec{M}} \left[\prod_{i=1}^{\tau} [\Phi_i(1)]^{j_i} \right] p_k(\iota, \vec{j})$$

$$= \frac{B(k+1)}{z^{(k+1)c}} Q_0(z, \Phi_1(k+1), \Phi_2(k+1), \ldots \Phi_\tau(k+1))$$

$$+ \sum_{l=1}^{k+1} \sum_{\iota=0}^{c-1} \sum_{\vec{j}=\vec{0}}^{\vec{M}} \frac{B(l)}{z^{lc}} (z^c - z^\iota) \left[\prod_{i=1}^{\tau} [\Phi_i(l)]^{j_i} \right] p_{k+1-l}(\iota, \vec{j})$$

which completes the proof of the theorem.□

4.3.3 Transient/Steady-State Analysis of the Buffer Occupancy Distribution

Let $P_k(z) = Q_k(z, 1, 1, \ldots, 1)$ denote the marginal PGF of the buffer occupancy distribution at the end of the k^{th} slot. Then from (4.44) we have:

$$P_k(z) = \frac{\tilde{B}(k)}{z^{kc}} Q_0(z, \tilde{\Phi}_1(k), \tilde{\Phi}_2(k), \ldots \tilde{\Phi}_\tau(k)) + \sum_{l=1}^{k} \sum_{\iota=0}^{c-1} \sum_{\vec{j}=\vec{0}}^{\vec{M}} \frac{\tilde{B}(l)}{z^{lc}} (z^c - z^\iota) \left[\prod_{i=1}^{\tau} \tilde{\Phi}_i(l)^{j_i} \right] p_{k-l}(\iota, \vec{j}) \quad (4.47)$$

where:

$$\tilde{B}(k) = B(k)|_{y_1 = y_2 = \ldots = y_\tau = 1} = \prod_{i=1}^{\tau} [\tilde{X}_i(k)]^{m_i} = \prod_{i=1}^{\tau} (\tilde{C}_{1i}\lambda_{1i}^k + \tilde{C}_{2i}\lambda_{2i}^k)^{m_i} \quad (4.47a)$$

$$\tilde{\Phi}_i(k) = \Phi_i(k)|_{y_i = 1} = \frac{\tilde{X}_i(k)}{\tilde{U}_i(k)} = \frac{\tilde{D}_{1i}\lambda_{1i}^k + \tilde{D}_{2i}\lambda_{2i}^k}{\tilde{C}_{1i}\lambda_{1i}^k + \tilde{C}_{2i}\lambda_{2i}^k} \quad (4.47b)$$

with $\tilde{C}_{ri} = C_{ri}|_{y_i = 1}$ and $\tilde{D}_{ri} = D_{ri}|_{y_i = 1}$ $\forall (r \in \{1, 2\})$, while λ_{ri}, C_{ri} and D_{ri} are as given in (3.68).

Next let $\Re = \{0, 1, 2, \ldots, m_1\} \times \{0, 1, 2, \ldots, m_2\} \times \cdot \cdot \cdot \{0, 1, 2, \ldots, m_\tau\}$ and let $\Im = \{0, 1, \ldots, c-1\}$ Then, from (4.47), the $c\prod_{i=1}^{\tau}(m_i + 1)$ transient probabilities $p_k(\iota, \bar{J})$'s, $(\iota, \bar{J}) \in \aleph = \Im \times \Re$ are the only terms which remains to be evaluated in order to fully characterize the transient PGF of the queue size. The following theorem provides a means to compute them.

4.3.3.1 Theorem 4.4

Let $P(z, w)$ and $P_{\iota\bar{J}}(w)$ be the one-dimensional transforms, defined by:

$$P(z, w) = \sum_{k=0}^{\infty} P_k(z) w^k \qquad (|w| < 1) \qquad (4.48)$$

and:

$$P_{\iota\bar{J}}(w) = \sum_{k=0}^{\infty} p_k(\iota, \bar{J}) w^k \qquad (|w| < 1) \qquad (4.49)$$

then:

$$P(z, w) = \sum_{\iota=0}^{\infty} \sum_{\bar{J}=\bar{0}}^{\bar{M}} \sum_{\bar{K}=\bar{0}}^{\bar{M}} \frac{\sum_{\bar{l}=[\bar{K}-\bar{J},\bar{0}]^+}^{[\bar{M}-\bar{J},\bar{K}]^-} \left[\prod_{i=1}^{\tau} \begin{bmatrix} j_i \\ k_i - l_i \end{bmatrix} \begin{bmatrix} m_i - j_i \\ l_i \end{bmatrix} \tilde{D}_{1i}^{k_i - l_i} \tilde{D}_{2i}^{j_i - k_i + l_i} \tilde{C}_{1i}^{l_i} \tilde{C}_{2i}^{m_i - j_i - l_i}\right] z^{\iota} p_0(\iota, \bar{J}) z^c}{z^c - w\prod_{i=1}^{\tau} \lambda_{1i}^{k_i} \lambda_{2i}^{m_i - k_i}}$$

$$+ \sum_{\iota=0}^{c-1} \sum_{\bar{J}=\bar{0}}^{\bar{M}} \sum_{\bar{K}=\bar{0}}^{\bar{M}} \frac{\sum_{=[\bar{K}-\bar{J},\bar{0}]^+}^{[\bar{M}-\bar{J},\bar{K}]^-} \left[\prod_{i=1}^{\tau} \begin{bmatrix} j_i \\ k_i - l_i \end{bmatrix} \begin{bmatrix} m_i - j_i \\ l_i \end{bmatrix} \tilde{D}_{1i}^{k_i - l_i} \tilde{D}_{2i}^{j_i - k_i + l_i} \tilde{C}_{1i}^{l_i} \tilde{C}_{2i}^{m_i - j_i - l_i}\right] (z^c - z^{\iota}) P_{\iota\bar{J}}(w)\, w \prod_{i=1}^{\tau} \lambda_{1i}^{k_i} \lambda_{2i}^{m_i - k_i}}{z^c - w\prod_{i=1}^{\tau} \lambda_{1i}^{k_i} \lambda_{2i}^{m_i - k_i}} \quad (4.50)$$

where $p_0(\iota, \bar{J})$ is the initial joint distribution of the system, $\sum_{\bar{K}=\bar{0}}^{\bar{M}}$ and $\sum_{\bar{l}=[\bar{K}-\bar{J},\bar{0}]^+}^{[\bar{M}-\bar{J},\bar{K}]^-}$ refer to the multidimensional summations $\sum_{k_1=0}^{m_1}\sum_{k_2=0}^{m_2}\cdot\sum_{k_\tau=0}^{m_\tau}$ and

$$\sum_{[k_1-j_1,0]^+}^{[m_1-j_1,k_1]^-}\ \sum_{[k_2-j_2,0]^+}^{[m_2-j_2,k_2]^-}\ \sum_{[k_3-j_3,0]^+}^{[m_3-j_3,k_3]^-}\ \cdot\cdot\ \sum_{[k_\tau-j_\tau,0]^+}^{[m_\tau-j_\tau,k_\tau]^-}$$

respectively.

In addition, for each $\bar{K} = \{k_1, k_2 .. k_\tau\} \in \Re$, let $V_{\bar{K}}(z) = w\prod_{i=1}^{\tau}\lambda_{1i}^{k_i}\lambda_{2i}^{m_i-k_i}$ and denote by $z_{p\bar{K}}$, $(p \in \hat{\Im} = \{1, 2, \ldots .c\})$, the c roots of the equation $z^c = V_{\bar{K}}(z)$ inside the unit circle. Then for each $(p, \bar{K}) \in \hat{\aleph} = \hat{\Im}\times\Re$, $P_{\iota\bar{J}}(w)$ satisfies:

$$\sum_{\iota=0}^{\infty}\sum_{\bar{J}=\bar{0}}^{\bar{M}}\sum_{\bar{l}=[\bar{K}-\bar{J},\bar{0}]^+}^{[\bar{M}-\bar{J},\bar{K}]^-}\left[\prod_{i=1}^{\tau}\begin{bmatrix}m_i-j_i\\ l_i\end{bmatrix}\begin{bmatrix}j_i\\ k_i-l_i\end{bmatrix}\frac{(\hat{\lambda}_{2i}-\beta_i)^{2l_i+j_i}}{(\alpha_i-1)^{l_i}[(1-\beta_i)f_i(z_{p\bar{K}})]^{l_i+j_i}}\right]z_{p\bar{K}}^{\iota}p_0(\iota,\bar{J})$$

$$+\sum_{\iota=0}^{c-1}\sum_{\bar{J}=\bar{0}}^{\bar{M}}\sum_{\bar{l}=[\bar{K}-\bar{J},\bar{0}]^+}^{[\bar{M}-\bar{J},\bar{K}]^-}\left[\prod_{i=1}^{\tau}\begin{bmatrix}m_i-j_i\\ l_i\end{bmatrix}\begin{bmatrix}j_i\\ k_i-l_i\end{bmatrix}\frac{(\hat{\lambda}_{2i}-\beta_i)^{2l_i+j_i}}{(\alpha_i-1)^{l_i}[(1-\beta_i)f_i(z_{p\bar{K}})]^{l_i+j_i}}\right](z_{p\bar{K}}^{c}-z_{p\bar{K}}^{\iota})P_{\iota\bar{J}}(w) = 0 \tag{4.51}$$

where $\hat{\lambda}_{2i} = \lambda_{2i}\big|_{z=z_{p\bar{K}}}$.

PROOF

Define the following w transform, $P(z,w) = \sum_{k=0}^{\infty}P_k(z)w^k$ where ($|w|<1$). Then from (4.47):

$$P(z,w) = \sum_{k=0}^{\infty}\tilde{B}(k)Q_0(z,\tilde{\Phi}_1(k),\tilde{\Phi}_2(k),\ldots\tilde{\Phi}_\tau(k))\left[\frac{w}{z^c}\right]^k$$

$$+\sum_{k=0}^{\infty}\sum_{l=1}^{k}\sum_{\iota=0}^{c-1}\sum_{\bar{J}=\bar{0}}^{\bar{M}}\frac{\tilde{B}(l)}{z^{lc}}(z^c-z^{\iota})\left[\prod_{i=1}^{\tau}[\tilde{\Phi}_i(l)]^{j_i}\right]p_{k-l}(\iota,\bar{J})w^k \tag{4.52}$$

We first look to the first term, $I = \sum_{k=0}^{\infty}\tilde{B}(k)Q_0(z,\tilde{\Phi}_1(k),\tilde{\Phi}_2(k),\ldots\tilde{\Phi}_\tau(k))\left[\frac{w}{z^c}\right]^k$. Since:

$$Q_0(z,\tilde{\Phi}_1(k),\tilde{\Phi}_2(k),\ldots\tilde{\Phi}_\tau(k)) = \sum_{\iota=0}^{\infty}\sum_{\bar{J}=\bar{0}}^{\bar{M}}z^{\iota}\left[\prod_{i=1}^{\tau}[\tilde{\Phi}_i(k)]^{j_i}\right]p_0(\iota,\bar{J})$$

then by substituting for $\tilde{B}(k)$ and $\tilde{\Phi}_i(k)$ as in (4.47a,b) and by applying the Binomial theorem, we get:

$$I = \sum_{k=0}^{\infty} \sum_{\iota=0}^{\infty} \sum_{\bar{J}=\bar{0}}^{\bar{M}} z^{\iota} \left[\prod_{i=1}^{\tau} U_i(k)^{j_i} X_i(k)^{m_i - j_i} \right] p_0(\iota, \bar{J}) \left[\frac{w}{z^c} \right]^k$$

$$= \sum_{k=0}^{\infty} \sum_{\iota=0}^{\infty} \sum_{\bar{J}=\bar{0}}^{\bar{M}} z^{\iota} p_0(\iota, \bar{J}) \left[\prod_{i=1}^{\tau} \sum_{r_i=0}^{m_i - j_i} \sum_{s_i=0}^{j_i} \begin{bmatrix} j_i \\ s_i \end{bmatrix} \begin{bmatrix} m_i - j_i \\ r_i \end{bmatrix} (\tilde{D}_{1i}\lambda_{1i}^k)^{s_i} (\tilde{D}_{2i}\lambda_{2i}^k)^{j_i - s_i} (\tilde{C}_{1i}\lambda_{1i}^k)^{r_i} (\tilde{C}_{2i}\lambda_{2i}^k)^{m_i - j_i - r_i} \right] \left[\frac{w}{z^c} \right]^k$$

Interchanging the order of summations gives:

$$I = \sum_{\iota=0}^{\infty} \sum_{\bar{J}=\bar{0}}^{\bar{M}} \sum_{\bar{R}=\bar{0}}^{\bar{M}-\bar{J}} \sum_{\bar{S}=\bar{0}}^{\bar{J}} z^{\iota} p_0(\iota, \bar{J}) \left[\prod_{i=1}^{\tau} \begin{bmatrix} j_i \\ s_i \end{bmatrix} \begin{bmatrix} m_i - j_i \\ r_i \end{bmatrix} \tilde{D}_{1i}^{s_i} \tilde{D}_{2i}^{j_i - s_i} \tilde{C}_{1i}^{r_i} \tilde{C}_{2i}^{m_i - j_i - r_i} \right] \sum_{k=0}^{\infty} \left[\frac{w \prod_{i=1}^{\tau} \lambda_{1i}^{(r_i + s_i)} \lambda_{2i}^{m_i - ((r_i + s_i))}}{z^c} \right]^k \tag{4.53}$$

where we used the notations $\sum_{\bar{S}=\bar{0}}^{\bar{J}}$ and $\sum_{\bar{R}=\bar{0}}^{\bar{M}-\bar{J}}$ to refer to the multidimensional summations, $\sum_{s_1=0}^{j_1} \sum_{s_2=0}^{j_2} \cdot \cdot \sum_{s_\tau=0}^{j_\tau}$ and $\sum_{r_1=0}^{m_1-j_1} \sum_{r_2=0}^{m_2-j_2} \cdot \cdot \sum_{r_\tau=0}^{m_\tau-j_\tau}$, respectively.

Finally the last term in (4.53) can be further simplified to yield:

$$I = \sum_{\iota=0}^{\infty} \sum_{\bar{J}=\bar{0}}^{\bar{M}} \sum_{\bar{R}=\bar{0}}^{\bar{M}-\bar{J}} \sum_{\bar{S}=\bar{0}}^{\bar{J}} \frac{\left[\prod_{i=1}^{\tau} \begin{bmatrix} j_i \\ s_i \end{bmatrix} \begin{bmatrix} m_i - j_i \\ r_i \end{bmatrix} \tilde{D}_{1i}^{s_i} \tilde{D}_{2i}^{j_i - s_i} \tilde{C}_{1i}^{r_i} \tilde{C}_{2i}^{m_i - j_i - r_i} \right] z^{\iota} p_0(\iota, \bar{J}) z^c}{z^c - w \prod_{i=1}^{\tau} \lambda_{1i}^{(r_i + s_i)} \lambda_{2i}^{m_i - ((r_i + s_i))}}$$

Next we consider the second term in (4.52) which can be expanded as follows:

$$II = \sum_{k=0}^{\infty} \sum_{l=1}^{k} \sum_{\iota=0}^{c-1} \sum_{\bar{J}=\bar{0}}^{\bar{M}} \frac{\tilde{B}(l)}{z^{lc}} (z^c - z^{\iota}) \left[\prod_{i=1}^{\tau} [\tilde{\Phi}_i(l)]^{j_i} \right] p_{k-l}(\iota, \bar{J}) w^k$$

$$= \sum_{\iota=0}^{c-1} \sum_{\bar{J}=\bar{0}}^{\bar{M}} (z^c - z^{\iota}) \sum_{k=0}^{\infty} \left[\sum_{l=0}^{k} \frac{\tilde{B}(l)}{z^{lc}} \left[\prod_{i=1}^{\tau} [\tilde{\Phi}_i(l)]^{j_i} \right] p_{k-l}(\iota, \bar{J}) - p_k(\iota, \bar{J}) \right] w^k$$

Recall that $P_{\iota\bar{J}}(w) = \sum_{k=0}^{\infty} p_k(\iota, \bar{J})\, w^k$. Hence:

$$II = \sum_{\iota=0}^{c-1} \sum_{\bar{J}=\bar{0}}^{\bar{M}} (z^c - z^\iota)\left[P_{\iota\bar{J}}(w) \cdot \sum_{k=0}^{\infty} \frac{\tilde{B}(k)}{z^{kc}}\left[\prod_{i=1}^{\tau} [\tilde{\Phi}_i(k)]^{j_i}\right] w^k - P_{\iota\bar{J}}(w)\right]$$

$$= \sum_{\iota=0}^{c-1} \sum_{\bar{J}=\bar{0}}^{\bar{M}} (z^c - z^\iota) P_{\iota\bar{J}}(w) \cdot \sum_{k=1}^{\infty} \frac{\tilde{B}(k)}{z^{kc}}\left[\prod_{i=1}^{\tau} [\tilde{\Phi}_i(k)]^{j_i}\right] w^k$$

Substituting for $\tilde{\Phi}_i(k)$, $\tilde{B}(k)$ as in (4.47a,b), applying the Binomial theorem, and interchanging the order of summations gives:

$$II = \sum_{\iota=0}^{c-1} \sum_{\bar{J}=\bar{0}}^{\bar{M}} (z^c - z^\iota) P_{\iota\bar{J}}(w) \cdot \sum_{k=1}^{\infty} \frac{\tilde{B}(k)}{z^{kc}}\left[\prod_{i=1}^{\tau} U_i(k)^{j_i} X_i(k)^{m_i - j_i}\right]\left[\frac{w}{z^c}\right]^k$$

$$= \sum_{\iota=0}^{c-1} \sum_{\bar{J}=\bar{0}}^{\bar{M}} \sum_{\bar{R}=\bar{0}}^{\bar{M}-\bar{J}} \sum_{\bar{S}=\bar{0}}^{\bar{J}} (z^c - z^\iota) P_{\iota\bar{J}}(w)\left[\prod_{i=1}^{\tau} \begin{bmatrix} j_i \\ s_i \end{bmatrix}\begin{bmatrix} m_i - j_i \\ r_i \end{bmatrix} \tilde{D}_{1i}^{s_i} \tilde{D}_{2i}^{j_i - s_i} \tilde{C}_{1i}^{r_i} \tilde{C}_{2i}^{m_i - j_i - r_i}\right] \sum_{k=1}^{\infty}\left[\frac{w \prod_{i=1}^{\tau} \lambda_{1i}^{(r_i+s_i)} \lambda_{2i}^{m_i - ((r_i+s_i))}}{z^c}\right]^k$$

From the above, the last infinite sum can be further simplified to give:

$$II = \sum_{\iota=0}^{c-1} \sum_{\bar{J}=\bar{0}}^{\bar{M}} \sum_{\bar{R}=\bar{0}}^{\bar{M}-\bar{J}} \sum_{\bar{S}=\bar{0}}^{\bar{J}} (z^c - z^\iota) P_{\iota\bar{J}}(w) \frac{\left[\prod_{i=1}^{\tau} \begin{bmatrix} j_i \\ s_i \end{bmatrix}\begin{bmatrix} m_i - j_i \\ r_i \end{bmatrix} \tilde{D}_{1i}^{s_i} \tilde{D}_{2i}^{j_i - s_i} \tilde{C}_{1i}^{r_i} \tilde{C}_{2i}^{m_i - j_i - r_i}\right] w \prod_{i=1}^{\tau} \lambda_{1i}^{(r_i+s_i)} \lambda_{2i}^{m_i - ((r_i+s_i))}}{z^c - w \prod_{i=1}^{\tau} \lambda_{1i}^{(r_i+s_i)} \lambda_{2i}^{m_i - ((r_i+s_i))}}$$

Combining I and II yields:

$$P(z,w) = \sum_{\iota=0}^{\infty} \sum_{\bar{J}=\bar{0}}^{\bar{M}} \sum_{\bar{R}=\bar{0}}^{\bar{M}-\bar{J}} \sum_{\bar{S}=\bar{0}}^{\bar{J}} \frac{\left[\prod_{i=1}^{\tau} \begin{bmatrix} j_i \\ s_i \end{bmatrix}\begin{bmatrix} m_i - j_i \\ r_i \end{bmatrix} \tilde{D}_{1i}^{s_i} \tilde{D}_{2i}^{j_i - s_i} \tilde{C}_{1i}^{r_i} \tilde{C}_{2i}^{m_i - j_i - r_i}\right] z^\iota p_0(\iota, \bar{J})\, z^c}{z^c - w \prod_{i=1}^{\tau} \lambda_{1i}^{(r_i+s_i)} \lambda_{2i}^{m_i - ((r_i+s_i))}}$$

$$+ \sum_{\iota=0}^{c-1} \sum_{\bar{J}=\bar{0}}^{\bar{M}} \sum_{\bar{R}=\bar{0}}^{\bar{M}-\bar{J}} \sum_{\bar{S}=\bar{0}}^{\bar{J}} (z^c - z^\iota) P_{\iota\bar{J}}(w) \frac{\left[\prod_{i=1}^{\tau} \begin{bmatrix} j_i \\ s_i \end{bmatrix}\begin{bmatrix} m_i - j_i \\ r_i \end{bmatrix} \tilde{D}_{1i}^{s_i} \tilde{D}_{2i}^{j_i - s_i} \tilde{C}_{1i}^{r_i} \tilde{C}_{2i}^{m_i - j_i - r_i}\right] w \prod_{i=1}^{\tau} \lambda_{1i}^{(r_i+s_i)} \lambda_{2i}^{m_i - ((r_i+s_i))}}{z^c - w \prod_{i=1}^{\tau} \lambda_{1i}^{(r_i+s_i)} \lambda_{2i}^{m_i - ((r_i+s_i))}}$$

or equivalently, with the change of variables: $\bar{K} = \bar{R} + \bar{S}$:

$$P(z,w)=\sum_{\iota=0}^{\infty}\sum_{\bar{J}=\bar{0}}^{\bar{M}}\sum_{\bar{K}=\bar{0}}^{\bar{M}}\frac{\sum_{\bar{l}=[\bar{K}-\bar{J},\bar{0}]^{+}}^{[\bar{M}-\bar{J},\bar{K}]^{-}}\left[\prod_{i=1}^{\tau}\begin{bmatrix}j_i\\k_i-l_i\end{bmatrix}\begin{bmatrix}m_i-j_i\\l_i\end{bmatrix}\bar{D}_{1i}^{k_i-l_i}\bar{D}_{2i}^{j_i-k_i+l_i}\bar{C}_{1i}^{l_i}\bar{C}_{2i}^{m_i-j_i-l_i}\right]z^{\iota}p_0(\iota,\bar{J})\,z^{c}}{z^{c}-w\prod_{i=1}^{\tau}\lambda_{1i}^{k_i}\lambda_{2i}^{m_i-k_i}}$$

$$+\sum_{\iota=0}^{c-1}\sum_{\bar{J}=\bar{0}}^{\bar{M}}\sum_{\bar{K}=\bar{0}}^{\bar{M}}\frac{\sum_{=[\bar{K}-\bar{J},\bar{0}]^{+}}^{[\bar{M}-\bar{J},\bar{K}]^{-}}\left[\prod_{i=1}^{\tau}\begin{bmatrix}j_i\\k_i-l_i\end{bmatrix}\begin{bmatrix}m_i-j_i\\l_i\end{bmatrix}\bar{D}_{1i}^{k_i-l_i}\bar{D}_{2i}^{j_i-k_i+l_i}\bar{C}_{1i}^{l_i}\bar{C}_{2i}^{m_i-j_i-l_i}\right](z^{c}-z^{\iota})P_{\iota\bar{J}}(w)\,w\prod_{i=1}^{\tau}\lambda_{1i}^{k_i}\lambda_{2i}^{m_i-k_i}}{z^{c}-w\prod_{i=1}^{\tau}\lambda_{1i}^{k_i}\lambda_{2i}^{m_i-k_i}} \tag{4.54}$$

which completes the first part of the theorem.

Next, for each $\bar{K}=\{k_1,k_2,..\,k_\tau\}\in\Re$, let us consider the roots of the equation

$$z^{c}=V_{\bar{K}}(z)=w\prod_{i=1}^{\tau}\lambda_{1i}^{k_i}\lambda_{2i}^{m_i-k_i}. \tag{4.55}$$

Let $h(z)=z^{c}$ and $g_{\bar{K}}(z)=-V_{\bar{K}}(z)$. Since $\forall i\in\{1,2,\ldots,\tau\}$ $|\lambda_{1i}|\le|\lambda_{2i}|\le 1$ and $|w|<1$, then for each $\bar{K}\in\Re$:

$$|g_{\bar{K}}(z)|=\left|w\prod_{i=1}^{\tau}\lambda_{1i}^{k_i}\lambda_{2i}^{m_i-k_i}\right|\le\left|\prod_{i=1}^{\tau}\lambda_{2i}^{m_i-k_i}\right|$$

Since $\prod_{i=1}^{\tau}\lambda_{2i}^{m_i-k_i}$ is a valid generating function (GF), then for a small $\varepsilon>0$ and on $|z|=1+\varepsilon$ we have $|g_{\bar{K}}(z)|\le 1+\varepsilon\sum_{i=1}^{\tau}\frac{(m_i-k_i)(1-\beta_i)}{2-\alpha_i-\beta_i}\bar{f}_i+\theta(\varepsilon)$.
On $|z|=1+\varepsilon$ we also have $|h(z)|=(1+\varepsilon)^{c}=1+c\varepsilon+\theta(\varepsilon)$ and therefore if the system is stable, i.e. ($\rho=\sum_{i=1}^{\tau}\frac{m_i(1-\beta_i)}{2-\alpha_i-\beta_i}\bar{f}_i<c$), then for each $\bar{K}\in\Re$, $|h(z)|>|g_{\bar{K}}(z)|$ on $|z|=1+\varepsilon$. From Rouché's theorem $h(z)$ and $h(z)+g_{\bar{K}}(z)$ have the same number of zeros inside $|z|=1+\varepsilon$ and therefore (4.55) has also c roots inside $|z|=1+\varepsilon$. Once again, these roots can be expressed explicitly in terms of Lagrange's expansion by noting that for each $\bar{K}\in\Re$, (4.55) implies that $z=\nu_p[V_{\bar{K}}(z)]^{1/c}$ where $\nu_p=e^{j2\pi p/c}$, ($p\in\hat{\Im}$), are the c roots of unity. Therefore, from Lagrange's theorem, and for each $\bar{K}\in\Re$,the c roots of (4.55) inside $|z|=1$ are given by:

$$z_{p\bar{K}} = \sum_{n=1}^{\infty} \frac{(\nu_p w^{1/c})^n}{n!} \frac{d^{n-1}}{dz^{n-1}} [V_{\bar{K}}(z)]^{n/c} \Big|_{z=0} \tag{4.56}$$

where $(p, \bar{K}) \in \hat{\aleph} = \hat{\Im} \times \Re$. Now for each $\bar{K} \in \Re$ we know the values of the c zeros of the denominator of $P(z, w)$. In addition since $P(z, w)$ is analytical inside $\langle |z| \le 1 \,;\, |w| < 1 \rangle$, then for each $\bar{K} \in \Re$, the numerator of $P(z, w)$ must also be zero at $z = z_{p\bar{K}}$, for each $p \in \hat{\Im}$. Hence $\forall (p, \bar{K}) \in \hat{\aleph} = \hat{\Im} \times \Re$:

$$\sum_{\iota=0}^{\infty} \sum_{\bar{J}=\bar{0}}^{\bar{M}} \sum_{\bar{l}=[\bar{K}-\bar{J},\bar{0}]^+}^{[\bar{M}-\bar{J},\bar{K}]^-} \left[\prod_{i=1}^{\tau} \begin{bmatrix} m_i - j_i \\ l_i \end{bmatrix} \begin{bmatrix} j_i \\ k_i - l_i \end{bmatrix} \left[\frac{\hat{C}_{1i}\hat{D}_{2i}}{\hat{C}_{2i}\hat{D}_{1i}} \right]^{l_i} \left[\frac{\hat{D}_{2i}}{\hat{C}_{2i}} \right]^{j_i} \right] z_{p\bar{K}}^{\iota} p_0(\iota, \bar{J})$$

$$+ \sum_{\iota=0}^{c-1} \sum_{\bar{J}=\bar{0}}^{\bar{M}} \sum_{\bar{l}=[\bar{K}-\bar{J},\bar{0}]^+}^{[\bar{M}-\bar{J},\bar{K}]^-} \left[\prod_{i=1}^{\tau} \begin{bmatrix} m_i - j_i \\ l_i \end{bmatrix} \begin{bmatrix} j_i \\ k_i - l_i \end{bmatrix} \left[\frac{\hat{C}_{1i}\hat{D}_{2i}}{\hat{C}_{2i}\hat{D}_{1i}} \right]^{l_i} \left[\frac{\hat{D}_{2i}}{\hat{C}_{2i}} \right]^{j_i} \right] (z_{p\bar{K}}^{c} - z_{p\bar{K}}^{\iota}) P_{\iota\bar{J}}(w) = 0$$

where $\hat{C}_{1i,2i} = \tilde{C}_{1i,2i}\big|_{z=z_{p\bar{K}}}$ and $\hat{D}_{1i,2i} = \tilde{D}_{1i,2i}\big|_{z=z_{p\bar{K}}}$.

The last expression can be further simplified to yield:

$$\sum_{\iota=0}^{\infty} \sum_{\bar{J}=\bar{0}}^{\bar{M}} \sum_{\bar{l}=[\bar{K}-\bar{J},\bar{0}]^+}^{[\bar{M}-\bar{J},\bar{K}]^-} \left[\prod_{i=1}^{\tau} \begin{bmatrix} m_i - j_i \\ l_i \end{bmatrix} \begin{bmatrix} j_i \\ k_i - l_i \end{bmatrix} \frac{(\hat{\lambda}_{2i} - \beta_i)^{2l_i + j_i}}{(\alpha_i - 1)^{l_i} [(1-\beta_i) f_i(z_{p\bar{K}})]^{l_i + j_i}} \right] z_{p\bar{K}}^{\iota} p_0(\iota, \bar{J})$$

$$+ \sum_{\iota=0}^{c-1} \sum_{\bar{J}=\bar{0}}^{\bar{M}} \sum_{\bar{l}=[\bar{K}-\bar{J},\bar{0}]^+}^{[\bar{M}-\bar{J},\bar{K}]^-} \left[\prod_{i=1}^{\tau} \begin{bmatrix} m_i - j_i \\ l_i \end{bmatrix} \begin{bmatrix} j_i \\ k_i - l_i \end{bmatrix} \frac{(\hat{\lambda}_{2i} - \beta_i)^{2l_i + j_i}}{(\alpha_i - 1)^{l_i} [(1-\beta_i) f_i(z_{p\bar{K}})]^{l_i + j_i}} \right] (z_{p\bar{K}}^{c} - z_{p\bar{K}}^{\iota}) P_{\iota\bar{J}}(w) = 0 \tag{4.57}$$

where $\hat{\lambda}_{2i} = \lambda_{2i}\big|_{z=z_{p\bar{K}}}$. □ .

The above defines a set of $c \prod_{i=1}^{\tau} (m_i + 1)$ equations for the determination of the *w* transforms, $P_{\iota\bar{J}}(w)$, $(\iota, \bar{J}) \in \aleph = \Im \times \Re$. Taking the inverse *w* transform enables us to compute the unknown boundary terms $p_k(\iota, \bar{J})$'s.

4.3.3.2 Transient Mean of the Queue Length Distribution

Let $\bar{N}_k = \left.\frac{dP_k(z)}{dz}\right|_{z=1}$ denote the average queue length at the end of the k^{th} slot. Then from(4.47):

$$\bar{N}_k = \left.\frac{d\bar{B}(k)}{dz}\right|_{z=1} - kc + \frac{d}{dz}Q_0(z,\bar{\Phi}_1(k),\bar{\Phi}_2(k),\ldots,\bar{\Phi}_\tau(k))\Big|_{z=1} + \sum_{l=1}^{k}\sum_{\iota=0}^{c-1}\sum_{\bar{J}=\bar{0}}^{\bar{M}}(c-\iota)\,p_{k-l}(\iota,\bar{J}) \tag{4.58}$$

Using the results of section 3.3.4.2, we finally get:

$$\bar{N}_k = \bar{N}_0 + k(\rho - c) + \sum_{i=1}^{\tau}\frac{\alpha_i+\beta_i-1}{2-\alpha_i-\beta_i}\bar{f}_i\left(1-(\alpha_i+\beta_i-1)^k\right)\left[\bar{A}_{i0} - \frac{m_i(1-\beta_i)}{2-\alpha_i-\beta_i}\right]$$
$$+\sum_{l=0}^{k-1}\sum_{\iota=0}^{c-1}\sum_{\bar{J}=\bar{0}}^{\bar{M}}(c-\iota)\,p_l(\iota,\bar{J}) \tag{4.59}$$

where $\bar{A}_{i0}$ is the initial average number of active sources of type i.

4.3.3.3 Steady-State PGF of the Queue Occupancy Distribution

The steady-state PGF $P(z) = \lim_{k\to\infty}P_k(z)$ of the queue length distribution is readily obtained by applying Abel's theorem to (4.50), giving:

$$P(z) = \sum_{\iota=0}^{c-1}\sum_{\bar{J}=\bar{0}}^{\bar{M}}(z^c - z^\iota)P(\iota,\bar{J})\sum_{\bar{K}=\bar{0}}^{\bar{M}}\frac{\displaystyle\sum_{\bar{l}=[\bar{K}-\bar{J},\bar{0}]^+}^{[\bar{M}-\bar{J},\bar{K}]^-}\left[\prod_{i=1}^{\tau}\begin{bmatrix}j_i\\k_i-l_i\end{bmatrix}\begin{bmatrix}m_i-j_i\\l_i\end{bmatrix}\bar{D}_{1i}^{k_i-l_i}\bar{D}_{2i}^{j_i-k_i+l_i}\bar{C}_{1i}^{l_i}\bar{C}_{2i}^{m_i-j_i-l_i}\right]\prod_{i=1}^{\tau}\lambda_{1i}^{k_i}\lambda_{2i}^{m_i-k_i}}{z^c - \prod_{i=1}^{\tau}\lambda_{1i}^{k_i}\lambda_{2i}^{m_i-k_i}} \tag{4.60}$$

where $p(\iota,\bar{J}) = p_\infty(\iota,\bar{J}) = \lim_{w\to 1-}(1-w)P_{\iota\bar{J}}(w)$.

Alternatively, when there is no need for transient results, the $c\prod_{i=1}^{\tau}(m_i+1)$ unknowns $P(\iota,\bar{J})$'s, $(\iota,\bar{J}) \in \aleph = \Im\times\Re$, can be determined by invoking the analytical property of $P(z)$ inside the unit disk. First, it is easy to verify that for each $\bar{K}\in\Re$, the equation $z^c = V_{\bar{K}}(z) = \prod_{i=1}^{\tau}\lambda_{1i}^{k_i}\lambda_{2i}^{m_i-k_i}$ has c distinct roots inside the unit circle. These will be denoted by $z_{\bar{K}p}$, $(p,\bar{K}) \in \hat{\aleph} = \hat{\Im}\times\Re$. For $\bar{K}=\bar{0}$, one of these roots is $z_{\bar{0}c} = 1$,which also appears in the numerator of $P(z)$, and the remaining roots are $z_{\bar{0}1}, z_{\bar{0}2}, \ldots z_{\bar{0}c-1}$. The roots $z_{\bar{K}p}$'s, ($(p,\bar{K}) \in \tilde{\aleph} = \hat{\aleph} - \{c,\bar{0}\}$) can be expressed in terms of the Lagrange's expansion:

$$z_{\bar{K}p} = \sum_{n=1}^{\infty} \frac{v_p^n}{n!} \frac{d^{n-1}}{dz^{n-1}} [V_{\bar{K}}(z)]^{n/c}\Big|_{z=0} \tag{4.61}$$

where $v_p = e^{j2\pi p/c}$ $(p \in \hat{\Im})$ are the c roots of unity.

Since $P(z)$ is analytical inside the unit disk, then for each $\bar{K} \in \Re$ the numerator of $P(z)$ must also be zero at $z = z_{\bar{K}p}$. Hence: $\forall (p, \bar{K}) \in \tilde{\aleph} = \hat{\aleph} - \{c, \bar{0}\}$:

$$\sum_{\iota=0}^{c-1} \sum_{\bar{J}=\bar{0}}^{\bar{M}} \sum_{i=\lfloor \bar{K}-\bar{J},\bar{0}\rfloor^+}^{\lfloor \bar{M}-\bar{J},\bar{K}\rfloor^-} \left[\prod_{i=1}^{\tau} \begin{bmatrix} m_i - j_i \\ l_i \end{bmatrix} \begin{bmatrix} j_i \\ k_i - l_i \end{bmatrix} \frac{(\hat{\lambda}_{2i} - \beta_i)^{2l_i + j_i}}{(\alpha_i - 1)^{l_i} [(1-\beta_i) f_i(z_{p\bar{K}})]^{l_i + j_i}} \right] (z^c_{\bar{K}p} - z^{\iota}_{\bar{K}p}) p(\iota, \bar{J}) = 0 \tag{4.62}$$

where $\hat{\lambda}_{2i} = \lambda_{2i}\big|_{z = z_{p\bar{K}}}$.

The remaining equation required for the determination of the unknowns $P(\iota, \bar{J})$'s is provided, once again, by the normalization condition, $P(1) = 1$. Since $\tilde{C}_{1i}\big|_{z=1} = \tilde{D}_{1i}\big|_{z=1} = 0$, then, except for the case $\bar{K} = \bar{0}$ all the terms under the third summation in (4.60) become zero when evaluated at z=1. Hence it is convenient to rewrite (4.60) as follows:

$$P(z) = \sum_{\iota=0}^{c-1} \sum_{\bar{J}=\bar{0}}^{\bar{M}} (z^c - z^{\iota}) p(\iota, \bar{J}) \left[F_{\bar{J}}(z) + \frac{G_{\bar{J}}(z)}{z^c - H(z)} \right] \tag{4.63}$$

where:

$$F_{\bar{J}}(z) = \sum_{\substack{\bar{K}=\bar{0} \\ \bar{K} \neq \bar{0}}}^{\bar{M}} \frac{\displaystyle\sum_{i=\lfloor \bar{K}-\bar{J},\bar{0}\rfloor^+}^{\lfloor \bar{M}-\bar{J},\bar{K}\rfloor^-} \left[\prod_{i=1}^{\tau} \begin{bmatrix} j_i \\ k_i - l_i \end{bmatrix} \begin{bmatrix} m_i - j_i \\ l_i \end{bmatrix} \tilde{D}_{1i}^{k_i - l_i} \tilde{D}_{2i}^{j_i - k_i + l_i} \tilde{C}_{1i}^{l_i} \tilde{C}_{2i}^{m_i - j_i - l_i} \right] \prod_{i=1}^{\tau} \lambda_{1i}^{k_i} \lambda_{2i}^{m_i - k_i}}{z^c - \prod_{i=1}^{\tau} \lambda_{1i}^{k_i} \lambda_{2i}^{m_i - k_i}}$$

$$G_{\bar{J}}(z) = \prod_{i=1}^{\tau} (\tilde{C}_{2i} \lambda_{2i})^{m_i} \left[\frac{\tilde{D}_{2i}}{\tilde{C}_{2i}} \right]^{j_i} \qquad \text{and} \qquad H(z) = \prod_{i=1}^{\tau} \lambda_{2i}^{m_i}$$

From (4.63):

$$P(z)[z^c - H(z)] = \sum_{\iota=0}^{c-1} \sum_{\bar{J}=\bar{0}}^{\bar{M}} (z^c - z^{\iota}) p(\iota, \bar{J}) \left[G_{\bar{J}}(z) + (z^c - H(z)) F_{\bar{J}}(z) \right] \tag{4.64}$$

Next differentiating both sides of the above equation with respect to z, substituting z=1 in the resulting equation yields:

$$c - H'(1) = \sum_{\iota=0}^{c-1} \sum_{\bar{J}=\bar{0}}^{\bar{M}} (c-\iota)\, p(\iota, \bar{J})$$

or equivalently:

$$c - \rho = \sum_{\iota=0}^{c-1} (c-\iota)\, p(\iota) \tag{4.65}$$

This is the remaining equation for the determination of all boundary terms in (4.60).

4.3.3.4 Steady-State Mean of the Queue Length Distribution

Let $\bar{N}$ denote the steady-state mean buffer length. Then by differentiating (4.64) twice with respect to z and substituting z=1 in the resulting expression we get:

$$\bar{N} = \frac{H''(1) - c(c-1)}{2(c - H'(1))} + \frac{1}{2[c - H'(1)]} \sum_{\iota=0}^{c-1} \sum_{\bar{J}=\bar{0}}^{\bar{M}} [c(c-1) - \iota(\iota-1) + 2(c-\iota) G'_{\bar{J}}(1)]\, p(\iota, \bar{J}) \tag{4.66}$$

where:

$$G'_{\bar{J}}(1) = \sum_{i=1}^{\tau} \frac{m_i (1-\beta_i)(3 - 2\alpha_i - 2\beta_i)}{(2-\alpha_i-\beta_i)^2} \hat{f}_i + j_i \frac{\alpha_i + \beta_i - 1}{2 - \alpha_i - \beta_i} \hat{f}_i \tag{4.67a}$$

with $H'(1)$ and $H''(1)$ as defined in (3.86) and (3.87), respectively.

Once more, the expression of the average queue length, as given in (4.66), depends only on the term of $P(z)$ (which corresponds $\bar{K} = \bar{0}$). This expression is fairly general since it is applicable for multiserver queues, with multiple types of traffic sources, and where some silence is permitted within each source's active period.

4.4 Steady-State Distribution of the Packet Delay

Let $D(z)$ be the PGF of the packet delay in number of slots at the steady-state. Using the results derived in [67], $D(z)$ can be derived from the steady-state PGF of the buffer occupancy distribution, $P(z)$, through the relationship:

$$D(z^c) = \frac{1}{\rho c}\sum_{l=0}^{c-1}\frac{(1-z^{-c})(1-z^c)}{\left(1-e^{-j\frac{2\pi l}{c}}z^{-1}\right)\left(1-e^{j\frac{2\pi l}{c}}z\right)}P\left(e^{j\frac{2\pi l}{c}}z\right)-\frac{1}{\rho}\sum_{i=0}^{c-1}(c-i)p(i) \qquad (4.68)$$

where $j = \sqrt{-1}$. In [67] it was shown that the mean packet delay is $\bar{d} = \frac{\bar{N}}{\rho}$, in accordance with Little's formula [80], while the expression for the variance of the packet delay, as given by equation (18) in [67] is incorrect and should be modified as follows:

First note that, with $a = e^{\frac{j2\pi}{c}}$, the property $\sum_{k=0}^{c-1} ka^{lk} = c\frac{1-a^{l(c-1)}}{(1-a^l)^2}$ which follows (17) in [67] is incorrect and should read:

$$\sum_{k=0}^{c-1} ka^{lk} = \begin{cases} \dfrac{c}{a^l-1} & (l \neq 0) \\ \dfrac{1}{2}c(c-1) & (l=0) \end{cases} \qquad (4.69)$$

Hence it follows that the variance of the delay becomes:

$$\sigma_d^2 = \frac{\sigma_N^2}{\rho c}+\frac{(c-1)(c+1)}{6c\rho}-\left(1-\frac{\rho}{c}\right)[\bar{d}]^2-\frac{1}{\rho c}\sum_{l=1}^{c-1}\frac{P\left(e^{j\frac{2\pi l}{c}}\right)}{\left(1-\cos\left(\frac{2\pi l}{c}\right)\right)} \qquad (4.70)$$

4.5 Numerical Results

In this section, we present some numerical results in order to illustrate our solution technique. We consider the case where each *active* user independently generates either one packet with probability *p*, or no packet at all. This is a generalization of Bruneel [51] model since zero packet arrivals are now allowed to occur during an *active* slot. In this case, $f(z) = (1-p)+pz$ and we keep $\alpha = 0.75$, c is fixed to 2 and the parameter p is fixed at $\frac{3}{4}$. We therefore consider the case of a single type of traffic and we want to examine the queueing behavior of the multi-server queue for different traffic intensities and different number of sources, *m*. Figure 4.1, shows the average buffer length, as a function of ρ, for *m*=5 and *m*=10. In each of these two cases, we had to solve for a system of 12 and 22 linear equations, respectively, in order to determine the boundary constants $p(i,j)$'s.

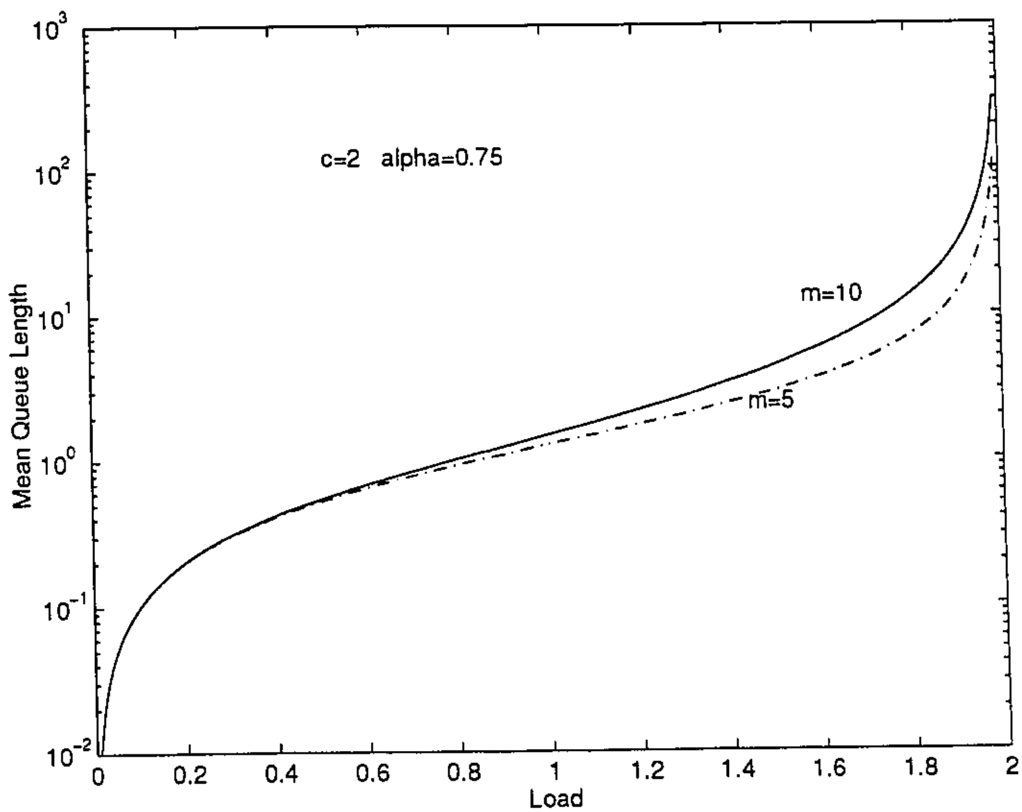

FIGURE.4.1 Steady-State Mean Queue Length as a Function of Load, ρ

Note also that for the same offered load, the average queue length increases with *m* and this becomes more noticeable at heavy loads. figure 4.2 shows the variance of the queue length for different applied loads. In particular, we note the very sharp increase in the variance at high traffic intensities. Finally, in figure 4.3, the average packet delay is plotted as function of ρ.

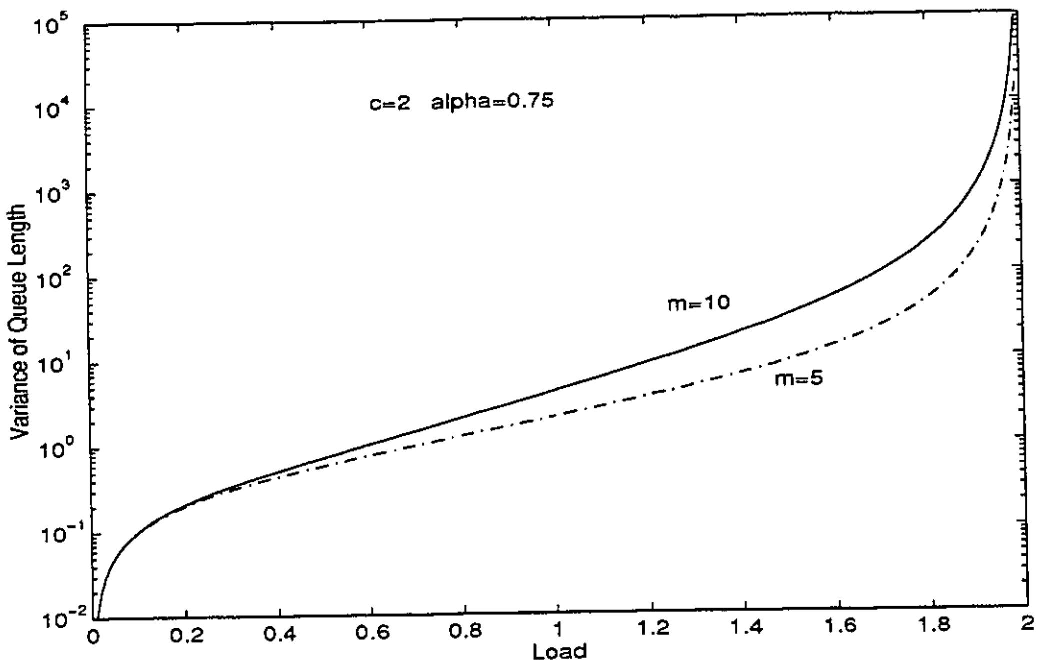

FIGURE.4.2 Steady-State Variance of the Queue Length as a Function of Load

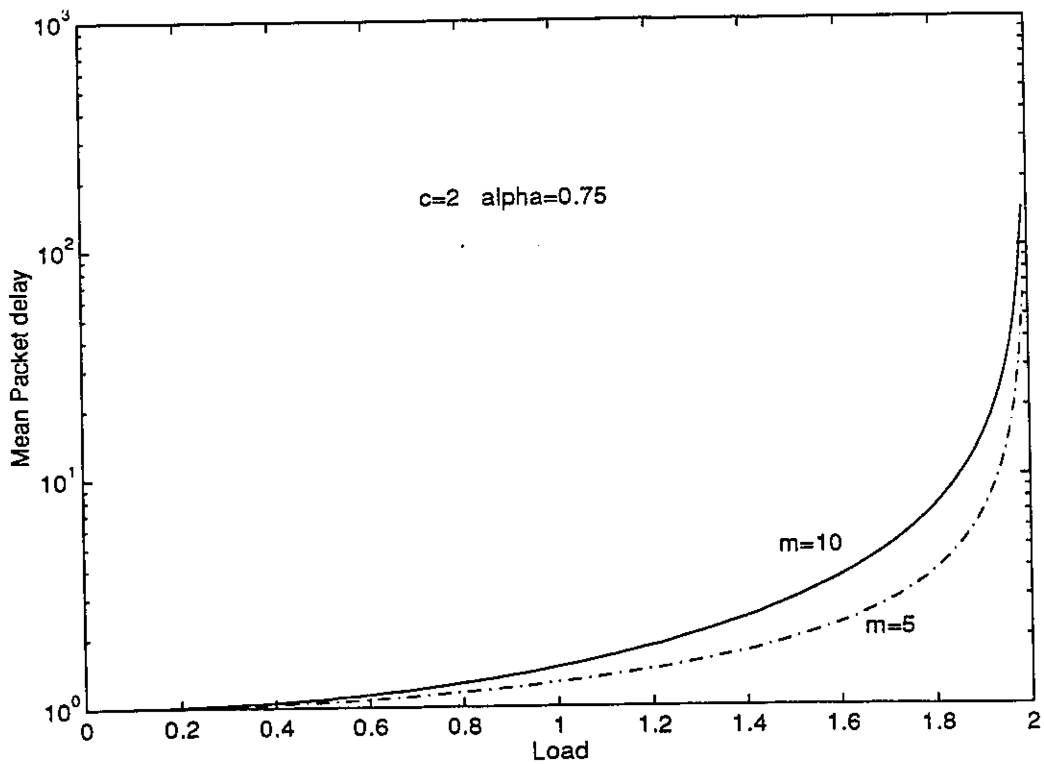

FIGURE.4.3 Mean Packet Delay as a Function of Load, ρ.

CHAPTER V

Queuing Analysis of ATM Tandem Networks

5.1 Preliminary

If most analytical studies related to ATM systems have focused on an isolated component in the network, such as a switch or a buffer, it is mainly because of the difficulty which surrounds the performance analysis at the network level. However because of the single path routing in ATM networks, the performance analysis of an ATM virtual connection, which typically consists of a number of queues in tandem, is of great importance since it might help to understand how ATM cell streams change as they pass through a number of switching nodes.

In chapter 1, we discussed some of the main issues which make it difficult to establish an exact performance evaluation method for a network-wide ATM system. In the lack of exact methods, we also highlighted the frequent use of the decomposition technique as a tool for the approximate analysis of ATM networks.

The main idea behind the decomposition technique consists of decomposing a queueing network into weakly coupled subsystems (ex. individual queues or subnetworks) and then analyzing each subsystem in isolation. To do so requires some approximations and fitting methods for the modeling of the input (arrival) and output (departure) to each subsystem. For example in [81], a four node ATM tandem queueing network is considered as shown in figure 5.1. The output process of each node is approximated by a renewal (GI-stream) process and it is fed to the next node. In addition interfering traffic enters each node and leaves immediately. The interfering traffic in [81] consists of an M-stream which is modeled by a Bernoulli process with batch arrivals and a B-stream which is modeled as a number of N discrete-time interrupted Poisson processes (IPP's). By convolving the delay distribution at each switching node, an approximation for the end-to-end delay distribution is provided.

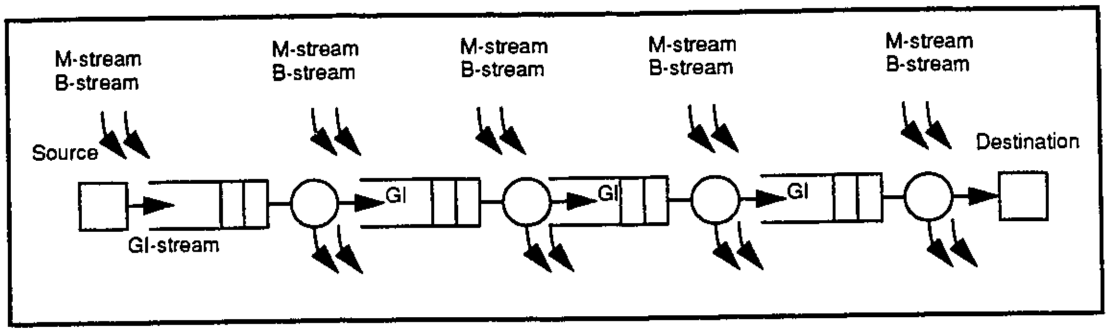

FIGURE.5.1The Four Nodes Tandem Queuing Model Considered in [81]

One of the limitations of this approach lies in the characterization of the output process of each isolated switching node. In fact the renewal approximation for the nodal departure process is hard to justify, as correlation is inherent in the output process of each node. This correlation has significant effect on the queuing behavior of the downstream nodes. This problem has been investigated by Lau and Li [82] where the "output/input" distortion due to multiplexing and splitting of traffic streams has been studied through extensive simulation studies. To take into account this distortion in the departure streams, Ren et al [83] proposed an approximate analytical technique which is based on the observation that when an isolated node is fed with a number of Markov modulated ON/OFF traffic streams, the departure streams from that node can also be approximated by ON/OFF streams with modified parameters. This makes the aggregate arrival process to each node, in a tandem ATM network, consist of ON/OFF streams. Using the performance results of a single ATM multiplexer with Markov modulated ON/OFF streams (ex. the fluid approximation approach [36]), it becomes plausible to analyze each node of the network in isolation and hence develop end to end performance analysis. The main limitation of this approach lies in the need to estimate the new parameters of the departure stream at each node. Also, the estimation has to be accurate since the departure stream at each node becomes the input to the next downstream node. In addition, the approach neglects the effect of the cross-correlation among the different traffic streams in the network and, from a modeling point of view, the approach is also limited because it requires different models for the output of every buffering element in the network. Another limitation of the decomposition method is that, in most cases, one cannot guarantee that there are feasible solutions for the estimation problem of fitting parameters to the output process at each node.

In this chapter, we consider a generic tandem queuing network with correlated arrivals and joining interference, which models a portion of a virtual circuit at the inlet of an ATM network. We focus our main analysis on a two-node tandem network, which we model as a discrete-time queueing system. The same example has been treated by Morrison [84] under the special case where the input to the first queue is geometrically distributed while the external input to the second queue is a Bernoulli process. Very recently Boxma and Resing [85] treated the case where the number of external arrivals to the two queues is modulated by a two-state Markov chain. An expression for the joint generating function of the queue lengths distribution when the modulating Markov chain is in state j, (j=1,2) is derived. To our knowledge, this derivation is the only contribution, we are aware of, in the *exact* queuing analysis of tandem queues with correlated arrivals. However, the two-state Markov chain model for the external arrivals in [85] is too restrictive to be used for the modeling of ATM traffic since it can be viewed as a surrogate model which does not have a concrete relationship with the actual traffic generated by the multimedia sources.Therefore, it is hoped that by generalizing the external arrivals process, this work will be another step towards the exact analysis of more realistic tandem configurations which arise in an ATM environment. We also note that the delay results of tandem queues with joining interference will provide an upper-bound for the corresponding results, with crossing interference. The main contribution of this chapter is that it presents an *exact* analysis of a two-node ATM tandem network, where the external arrival process to each queue is modeled by the traffic generated by the superposition of Markov binary sources, which as mentioned before, are extensively used in the modeling of Broadband traffic. This work can also be viewed as an extension of our previous approach, presented in chapters 3 and 4, in the context of a single multiplexer.

The main advantage of the approach is being exact, that no approximations are made regarding the nature of the departure process from the nodes. Hence the performance measures derived in this chapter faithfully incorporate the "output/input" distortion caused by multiplexing. We also note that our choice for the generating function approach to analyze the tandem queueing network is motivated by the fact that other solution techniques such as global balance equations [86] which are often used in continuous-time systems are not adequate in the discrete domain. This is due to the fact a very large number of state transitions (if not infinite) is required in the discrete case, as multiple arrivals are allowed within a time slot [87]. We have organized this chapter as follows:

In the next section, we give a description of the general model along with the assumptions governing its operation. In section 5.3, we illustrate our solution technique by considering a two-node tandem network. We model the network as a discrete-time queueing system and derive a functional equation relating the joint generating function of this system between two consecutive slots. The functional equation is then put into a suitable form, which enables the derivation of the steady-state joint generating function of the contents of the queues and the number of active sources. From this, any moment of the queue length at each node can be extracted. In addition we derive explicit expressions for the average packet delay at each node as well as for the total average delay in the network. In section 5.4, we discuss the extension of the analysis to the general N-nodes tandem network. Finally, in section 5.5, we illustrate our solution technique by some numerical results.

5.2 General Model Description and Notations

In this chapter, we consider a tandem queuing network, consisting of N nodes, as depicted in figure 5.2.

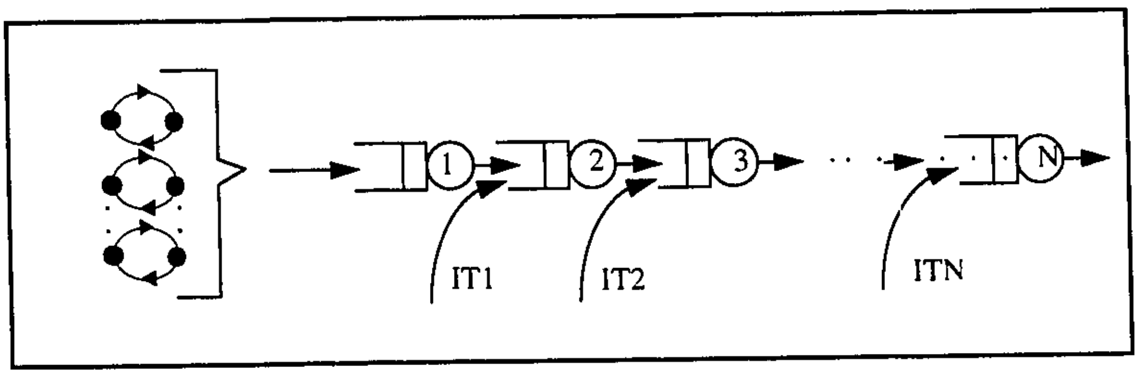

FIGURE.5.2 The General Model for the ATM Tandem Network

Here node (1) is an access node, which is loaded with the traffic generated by the superposition of m_1 independent and homogeneous Markov binary sources. The remaining nodes are also loaded with an intermediate traffic (IT) which consists of the superposition of the traffic streams arriving from other nodes in the network. We further assume joining (as opposed to crossing) interference, whereby the interfering traffic joins the main packet stream until the last queue. This assumption is appropriate at the inlet of the ATM network where joining interference (multiplexing) dominates crossing interference (switching) which often prevails inside the network [88]. In addition, we model the intermediate traffic to node *(i)*, as the traffic generated by the superposition of m_i independent and identical binary

Markov sources. Throughout this chapter, the ATM tandem queueing network that we are considering is assumed to operate under the following conditions:

- All channel time axises are segmented into slots of equal length, each corresponding to the transmission time of one cell. In addition all the nodes are synchronized, so that a node can transmit cells only at the slots' boundaries. Hence a packet which arrives during a slot cannot be served before the beginning of the next slot. We further assume that the cells received by a node from external sources or from other nodes are buffered in an infinite capacity queue and then transmitted on a FCFS basis. However a node can transmit up to one cell per slot.

- It is assumed that the mean combined input rate from all the sources is always less than 1 so that the system is stable and a steady-state exists.

- Each node *(i)* is fed with m_i mutually independent and identical binary Markov sources, each alternating between an *On* and an *Off* state. We assume that during an "active" slot each source, feeding node *i*, generates a strictly positive number of packets with a probability generating function $f_i(z_i)$, while during a "passive" slot no packets are generated. State transitions are assumed to occur at the slots' boundaries and the lengths of the *On* and *Off* periods are, once again, assumed to be geometrically distributed with means $\frac{1}{1-\alpha_i}$ and $\frac{1}{1-\beta_i}$, respectively. At this stage, it should be noted that the extension of this work to the multiple type of traffic case (at each node) is straightforward.

Let $a_{i,k}$ be the number of active sources that feed node *i* during slot k. Then:

$$a_{i,k+1} = \sum_{j=1}^{a_{i,k}} c_j^i + \sum_{j=1}^{m_i - a_{i,k}} d_j^i \tag{5.1}$$

where c_j^i and d_j^i are two sets of i.i.d Bernoulli variables with corresponding PGFs:

$$c_i(z_i) = (1-\alpha_i) + \alpha_i z_i \tag{5.2a}$$

$$d_i(z_i) = \beta_i + (1-\beta_i) z_i \tag{5.2b}$$

The number of packets, $b_{i,k}$, generated by the m_i sources during slot *k* and destined for node *(i)* is given by:

$$b_{i,k} = \sum_{j=1}^{a_{i,k}} f^i_{j,k} \tag{5.2c}$$

where $f^i_{j,k}$ is the number of packets generated by the j^{th} active source during slot *k* and which are destined to node *i*. All the $f^i_{j,k}$'s are assumed to be i.i.d with PGF $f_i(z_i)$.

5.3 A Two-Nodes Tandem Network

The queuing model shown in figure 5.3 can be formulated as a discrete-time multidimensional Markov chain. The state of the system is defined by $(i_k, j_k, a_{1,k}, a_{2,k})$ where i_k and j_k are the queue lengths of nodes 1 and 2 at the end of the k^{th} slot, while $a_{1,k}$ and $a_{2,k}$ are the number of active sources during slot *k* which feed nodes 1 and 2, respectively.

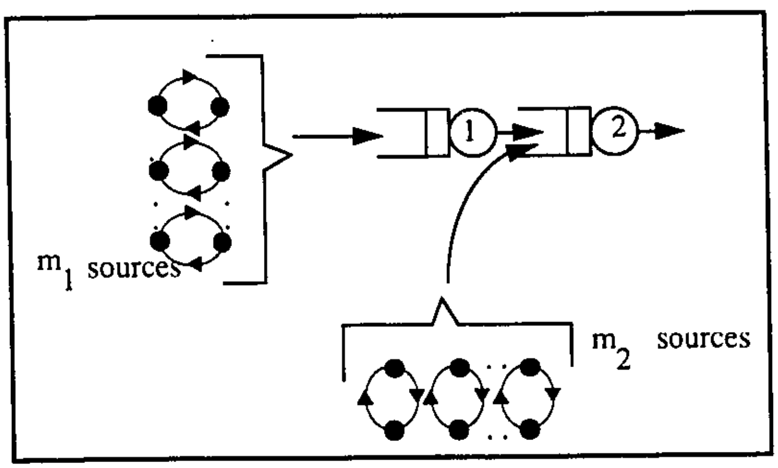

FIGURE.5.3 A Two-Node Tandem Network

In this section, we shall first determine the joint PGF of the queue lengths at the end of the kth slot and then put it into a suitable form, which will be used in conjunction with Abel's theorem to derive the steady-state PGF of the queue lengths.

The evolution of the queue length at each node is described by the following equations:

$$i_{k+1} = i_k - U(i_k) + b_{1,k+1}$$

$$j_{k+1} = j_k - U(j_k) + b_{2,k+1} + U(i_k) \tag{5.3}$$

where $U(x)$ is a binary-valued random variable which takes the value 1 if $x > 0$ and 0 otherwise.

Next let us define the joint generating function of the system at the end of the k^{th} slot as follows:

$$Q_k(z_1, z_2, y_1, y_2) = E[z_1^{i_k} z_2^{j_k} y_1^{a_{1,k}} y_2^{a_{2,k}}] = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{a_1=0}^{m_1} \sum_{a_2=0}^{m_2} z_1^i z_2^j y_1^{a_1} y_2^{a_2} p_k(i, j, a_1, a_2)$$

where $p_k(i, j, a_1, a_2) = pr[i_k = i, j_k = j, a_{1,k} = a_1, a_{2,k} = a_2]$ is the joint distribution of the system at the end of the kth slot. Then from (5.3) it follows that:

$$Q_{k+1}(z_1, z_2, y_1, y_2) = E[z_1^{i_k - U(i_k) + b_{1,k+1}} z_2^{j_k - U(j_k) + b_{2,k+1} + U(i_k)} y_1^{a_{1,k+1}} y_2^{a_{2,k+1}}] \tag{5.4}$$

From the above, using (5.1) and (5.2) and averaging over the distributions of the $f_{j,k+1}^i$'s, the c_j^i's and the d_j^i's yields:

$$Q_{k+1}(z_1, z_2, y_1, y_2) = E\left[z_1^{i_k - U(i_k)} z_2^{j_k - U(j_k) + U(i_k)} \prod_{i=1}^{2} z_i^{\sum_{j=1}^{a_{i,k+1}} f_{j,k+1}} \cdot y_i^{a_{i,k+1}}\right]$$

$$= E\left[E\left[z_1^{i_k - U(i_k)} z_2^{j_k - U(j_k) + U(i_k)} \prod_{i=1}^{2} z_i^{\sum_{j=1}^{a_{i,k+1}} f_{j,k+1}} \cdot y_i^{a_{i,k+1}} \middle| i_k, j_k, a_{1,k+1}, a_{2,k+1}\right]\right]$$

$$= E\left[z_1^{i_k - U(i_k)} z_2^{j_k - U(j_k) + U(i_k)} \prod_{i=1}^{2} (y_i f_i(z_i))^{a_{i,k+1}}\right]$$

$$= E\left[E\left[z_1^{i_k - U(i_k)} z_2^{j_k - U(j_k) + U(i_k)} \prod_{i=1}^{2} (y_i f_i(z_i))^{\sum_{j=1}^{a_{i,k}} c_j' + \sum_{j=1}^{m_i - a_{i,k}} d_j'} \middle| i_k, j_k, a_{1,k}, a_{2,k}\right]\right]$$

$$= E\left[z_1^{i_k - U(i_k)} z_2^{j_k - U(j_k) + U(i_k)} \prod_{i=1}^{2} (d_i(y_i f_i(z_i)))^{m_i} \cdot \left[\frac{c_i(y_i f_i(z_i))}{d_i(y_i f_i(z_i))}\right]^{a_{i,k}}\right]$$

or equivalently:

$$Q_{k+1}(z_1, z_2, y_1, y_2) = \prod_{i=1}^{2} [d_i(y_i f_i(z_i))]^{m_i} \cdot E[z_1^{i_k - U(i_k)} z_2^{j_k - U(j_k) + U(i_k)} Y_1^{a_{1,k}} Y_2^{a_{2,k}}] \qquad (5.5a)$$

where:

$$Y_i = \frac{c_i(y_i f_i(z_i))}{d_i(y_i f_i(z_i))} = \frac{1 - \alpha_i + \alpha_i y_i f_i(z_i)}{\beta_i + (1 - \beta_i) y_i f_i(z_i)} \qquad (5.5b)$$

Next in order to remove the *U(x)*'s from (5.5a), and as illustrated in figure 5.4, we consider the four cases:

$$\begin{cases} i_k > 0, j_k > 0 & \text{Case } (I) \\ i_k > 0, j_k = 0 & \text{Case } (II) \\ i_k = 0, j_k > 0 & \text{Case } (III) \\ i_k = j_k = 0 & \text{Case } (IV) \end{cases}$$

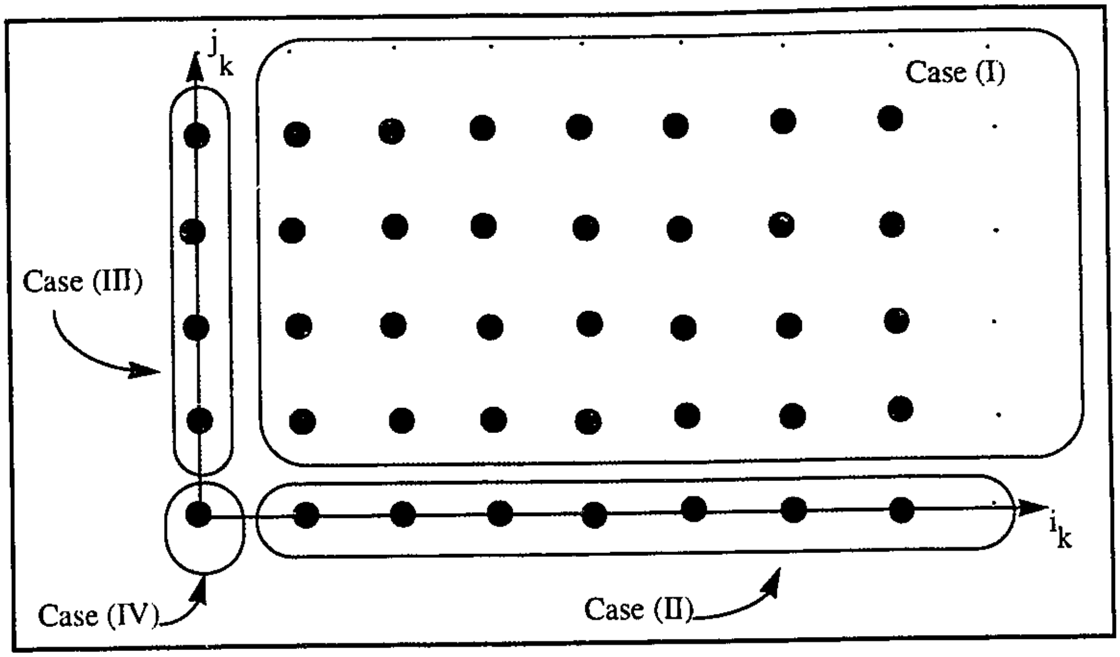

FIGURE.5.4 Partitioning the State Space of the Queue Lengths into 4 Non-Overlapping Regions

In addition, from the model's assumption, we note that if the random variable $i_k(j_k)$ is zero then the random variable $a_{1,k}$ $(a_{2,k})$ has also to be zero. As before, this follows from the observation that if one buffer is empty at the end of a slot, then the sources which feed it should have been empty during that slot. Therefore, from the above two observations, we can write:

$$E\left[z_1^{i_k - U(i_k)} z_2^{j_k - U(j_k) + U(i_k)} \gamma_1^{a_{1,k}} \gamma_2^{a_{2,k}}\right] = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{a_1=0}^{m_1} \sum_{a_2=0}^{m_2} z_1^{i-U(i)} z_2^{j-U(j)+U(i)} \gamma_1^{a_1} \gamma_2^{a_2} p_k(i,j,a_1,a_2)$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{a_1=0}^{m_1} \sum_{a_2=0}^{m_2} z_1^{i-1} z_2^{j} \gamma_1^{a_1} \gamma_2^{a_2} p_k(i,j,a_1,a_2) + \sum_{i=1}^{\infty} \sum_{a_1=0}^{m_1} z_1^{i-1} z_2^{1} \gamma_1^{a_1} p_k(i,0,a_1,0) + \sum_{j=1}^{\infty} \sum_{a_2=0}^{m_2} z_2^{j-1} \gamma_2^{a_2} p_k(0,j,0,a_2) + p_k(0,0,0,0) \quad (5.6)$$

With reference to figure 5.4, the first, second and third term in the above expression can be written as follows:

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{a_1=0}^{m_1}\sum_{a_2=0}^{m_2} z_1^{i-1} z_2^{j} Y_1^{a_1} Y_2^{a_2} p_k(i,j,a_1,a_2) = z_1^{-1}\Bigg[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{a_1=0}^{m_1}\sum_{a_2=0}^{m_2} z_1^{i} z_2^{j} Y_1^{a_1} Y_2^{a_2} p_k(i,j,a_1,a_2)$$

$$-\sum_{i=0}^{\infty}\sum_{a_1=0}^{m_1} z_1^{i} Y_1^{a_1} p_k(i,0,a_1,0) - \sum_{j=0}^{\infty}\sum_{a_2=0}^{m_2} z_2^{j} Y_2^{a_2} p_k(0,j,0,a_2) + p_k(0,0,0,0)\Bigg] \qquad (5.7a)$$

$$\sum_{i=1}^{\infty}\sum_{a_1=0}^{m_1} z_1^{i-1} z_2^{1} Y_1^{a_1} p_k(i,0,a_1,0) = z_2 \cdot z_1^{-1}\Bigg[\sum_{i=0}^{\infty}\sum_{a_1=0}^{m_1} z_1^{i} Y_1^{a_1} p_k(i,0,a_1,0) - p_k(0,0,0,0)\Bigg] \qquad (5.7b)$$

and:

$$\sum_{j=1}^{\infty}\sum_{a_2=0}^{m_2} z_2^{j-1} Y_2^{a_2} p_k(0,j,0,a_2) = z_2^{-1}\Bigg[\sum_{j=0}^{\infty}\sum_{a_2=0}^{m_2} z_2^{j} Y_2^{a_2} p_k(0,j,0,a_2) - p_k(0,0,0,0)\Bigg] \qquad (5.7c)$$

respectively.

Finally, from (5.5a) and after substituting (5.7) into (5.6) we get the following expression relating the joint PGF of the system between two consecutive slots:

$$Q_{k+1}(z_1, z_2, y_1, y_2) = \prod_{i=1}^{2} [d_i(y_i f_i(z_i))]^{m_i} \cdot \{$$

$$z_1^{-1}[Q_k(z_1, z_2, Y_1, Y_2) - Q_k(z_1, 0, Y_1, 0) - Q_k(0, z_2, 0, Y_2) + Q_k(0,0,0,0)]$$

$$+ z_2 \cdot z_1^{-1}[Q_k(z_1, 0, Y_1, 0) - Q_k(0,0,0,0)]$$

$$+ z_2^{-1}[Q_k(0, z_2, 0, Y_2) - Q_k(0,0,0,0)]$$

$$+ Q_k(0,0,0,0) \quad \} \qquad (5.8)$$

In the above, the first term corresponds to case (I), the second to case (II), the third to case (III) and finally, the last term corresponds to case (IV).

As before, we note that taking the limit as $k \to \infty$ on both sides of (5.8) does not help in carrying the analysis further, namely because of the presence of the Y_1 and Y_2 terms on the RHS of (5.8).

Therefore, to handle the functional equation (5.8), we apply a similar approach to the one presented in chapters 3 and 4 in order to put (5.8) into a suitable form, which enables us to carry the analysis further. In addition, since we are interested in the steady-state solution, then without any loss of generality we can assume that the system is initially empty, with all sources being in the *Off* state, i.e. $Q_0(z_1, z_2, y_1, y_2) = 1$. Note that because of the Markovian property of the system, the steady state solution is independent of the initial condition.

5.3.1 The Solution Method

With zero initial conditions, and by expanding $Q_{k+1}(z_1, z_2, y_1, y_2)$ in (5.8) for the first few values of *k*, we can prove by recurrence the following major result which enables us to express $Q_k(z_1, z_2, y_1, y_2)$ in a more suitable form so that the corresponding steady-state result will be readily obtained.

5.3.1.1 Theorem 5.1:

The joint generating function of the system, as described by the functional equation (5.8), can be written as follows:

$$\begin{aligned} Q_k(z_1, z_2, y_1, y_2) = {} & \frac{B(k)}{z_1^{k-1}} + (z_2 - 1)\sum_{j=1}^{k-1}\frac{B(j)}{z_1^j}\; Q_{k-j}(z_1, 0, \Phi_1(j), 0) \\ & + \frac{(z_1 - z_2)}{z_2}\sum_{j=1}^{k-1}\frac{B(j)}{z_1^j}\; Q_{k-j}(0, z_2, 0, \Phi_2(j)) \\ & + \frac{(z_2 - 1)(z_1 - z_2)}{z_2}\sum_{j=1}^{k-1}\frac{B(j)}{z_1^j}\; Q_{k-j}(0, 0, 0, 0) \end{aligned} \qquad (5.9a)$$

or equivalently:

$$Q_k(z_1, z_2, y_1, y_2) = \frac{B(k)}{z_1^k} + (z_2 - 1)\sum_{j=1}^{k} \frac{B(j)}{z_1^j}\; Q_{k-j}(z_1, 0, \Phi_1(j), 0)$$

$$+ \frac{(z_1 - z_2)}{z_2}\sum_{j=1}^{k} \frac{B(j)}{z_1^j}\; Q_{k-j}(0, z_2, 0, \Phi_2(j))$$

$$+ \frac{(z_2 - 1)(z_1 - z_2)}{z_2}\sum_{j=1}^{k} \frac{B(j)}{z_1^j}\; Q_{k-j}(0, 0, 0, 0) \tag{5.9b}$$

where:

$$\Phi_i(k) = \frac{U_i(k)}{X_i(k)} = \frac{D_{1i}\lambda_{1i}^k + D_{2i}\lambda_{2i}^k}{C_{1i}\lambda_{1i}^k + C_{2i}\lambda_{2i}^k} \tag{5.10}$$

with:

$$\lambda_{1i,2i} = \frac{\beta_i + \alpha_i f_i(z_i) \mp \sqrt{(\beta_i + \alpha_i f_i(z_i))^2 + 4(1 - \alpha_i - \beta_i) f_i(z_i)}}{2} \tag{5.11a}$$

$$C_{1i,2i} = \frac{1}{2} \mp \frac{2(y_i - y_i\beta_i - \alpha_i) f_i(z_i) + (\beta_i + \alpha_i f_i(z_i))}{2\sqrt{(\beta_i + \alpha_i f_i(z_i))^2 + 4(1 - \alpha_i - \beta_i) f_i(z_i)}} \tag{5.11b}$$

$$D_{1i,2i} = \frac{y_i}{2} \mp \frac{2(1 - \alpha_i + \alpha_i y_i f_i(z_i)) - (\beta_i + \alpha_i f_i(z_i)) y_i}{2\sqrt{(\beta_i + \alpha_i f_i(z_i))^2 + 4(1 - \alpha_i - \beta_i) f_i(z_i)}} \tag{5.11c}$$

$$B(k) = \prod_{i=1}^{2} [X_i(k)]^{m_i} \tag{5.12}$$

PROOF

Throughout this proof we use the fact that $B^*(k) = B(k)|_{y_1 = Y_1,\, y_2 = Y_2} = \frac{B(k+1)}{B(1)}$. In addition we can easily verify that (5.9a) and (5.9b) are equivalent. Hence by substituting k=0 then k=1 in the functional equation (5.8) we get:

$$Q_1(z_1, z_2, y_1, y_2) = B(1)$$

$$Q_2(z_1,z_2,y_1,y_2) = B(1)\ \Big[z_1^{-1}(B^*(1) - Q_1(z_1,0,\Phi_1(1),0) - Q_1(0,z_2,0,\Phi_2(1)) + Q_1(0,0,0,0))$$

$$+ z_2 z_1^{-1}(Q_1(z_1,0,\Phi_1(1),0) - Q_1(0,0,0,0))$$

$$+ z_2^{-1}(Q_1(0,z_2,0,\Phi_2(1)) - Q_1(0,0,0,0)) + Q_1(0,0,0,0)\Big]$$

$$= \frac{B(2)}{z_1} + (z_2-1)\sum_{j=1}^{1}\frac{B(j)}{z_1^j}Q_{2-j}(z_1,0,\Phi_1(j),0) + \frac{(z_1-z_2)}{z_2}\sum_{j=1}^{1}\frac{B(j)}{z_1^j}Q_{2-j}(0,z_2,0,\Phi_2(1))$$

$$+ \frac{(z_2-1)(z_1-z_2)}{z_2}\sum_{j=1}^{1}\frac{B(j)}{z_1^j}Q_{2-j}(0,0,0,0)$$

which shows that (5.9) is true for k=1,2 and evidently for k=0. Next let us suppose that (5.9a) is true for the order k, i.e.:

$$Q_k(z_1,z_2,y_1,y_2) = \frac{B(k)}{z_1^{k-1}} + (z_2-1)\sum_{j=1}^{k-1}\frac{B(j)}{z_1^j}\ Q_{k-j}(z_1,0,\Phi_1(j),0)$$
$$+ \frac{(z_1-z_2)}{z_2}\sum_{j=1}^{k-1}\frac{B(j)}{z_1^j}\ Q_{k-j}(0,z_2,0,\Phi_2(j))$$
$$+ \frac{(z_2-1)(z_1-z_2)}{z_2}\sum_{j=1}^{k-1}\frac{B(j)}{z_1^j}\ Q_{k-j}(0,0,0,0) \tag{5.13}$$

let us prove that it is also true for the order (k+1), i.e.:

$$Q_{k+1}(z_1,z_2,y_1,y_2) = \frac{B(k+1)}{z_1^k} + (z_2-1)\sum_{j=1}^{k}\frac{B(j)}{z_1^j}\ Q_{k+1-j}(z_1,0,\Phi_1(j),0)$$
$$+ \frac{(z_1-z_2)}{z_2}\sum_{j=1}^{k}\frac{B(j)}{z_1^j}\ Q_{k+1-j}(0,z_2,0,\Phi_2(j))$$
$$+ \frac{(z_2-1)(z_1-z_2)}{z_2}\sum_{j=1}^{k}\frac{B(j)}{z_1^j}\ Q_{k+1-j}(0,0,0,0) \tag{5.14}$$

Substituting (5.13) into the functional equation (5.8) yields:

$$Q_{k+1}(z_1,z_2,y_1,y_2) = B(1)\cdot\left[z_1^{-1}\left(\frac{B^*(k)}{z_1^{k-1}} + (z_2-1)\sum_{j=1}^{k-1}\frac{B^*(j)}{z_1^j}\ Q_{k-j}(z_1,0,\Phi_1(j+1),0)\right.\right.$$

$$+\frac{(z_1-z_2)}{z_2}\sum_{j=1}^{k-1}\frac{B^*(j)}{z_1^j}\ Q_{k-j}(0,z_2,0,\Phi_2(j+1)) + \frac{(z_2-1)(z_1-z_2)}{z_2}\sum_{j=1}^{k-1}\frac{B^*(j)}{z_1^j}Q_{k-j}(0,0,0,0)$$

$$\left. - Q_k(z_1,0,\Phi_1(1),0) - Q_k(0,z_2,0,\Phi_2(1)) + Q_k(0,0,0,0)\ \right)$$

$$+ z_2\cdot z_1^{-1}(Q_k(z_1,0,\Phi_1(1),0) - Q_k(0,0,0,0)) + z_2^{-1}(Q_k(0,z_2,0,\Phi_2(1)) - Q_k(0,0,0,0))$$

$$\left. + (Q_k(0,0,0,0))\ \right]$$

Since $B(1)\cdot B^*(k) = B(k+1)$ then the above reduces to:

$$Q_{k+1}(z_1,z_2,y_1,y_2) = \frac{B(k+1)}{z_1^k} + (z_2-1)\sum_{j=1}^{k-1}\frac{B(j+1)}{z_1^{j+1}}\ Q_{k-j}(z_1,0,\Phi_1(j+1),0)$$

$$+\frac{(z_1-z_2)}{z_2}\sum_{j=1}^{k-1}\frac{B(j+1)}{z_1^{j+1}}\ Q_{k-j}(0,z_2,0,\Phi_2(j+1))$$

$$+\frac{(z_2-1)(z_1-z_2)}{z_2}\sum_{j=1}^{k-1}\frac{B(j+1)}{z_1^{j+1}}Q_{k-j}(0,0,0,0)$$

$$+ (z_2-1)\frac{B(1)}{z_1}Q_k(z_1,0,\Phi_1(1),0) + \frac{(z_1-z_2)}{z_2}\frac{B(1)}{z_1}Q_k(0,z_2,0,\Phi_2(1))$$

$$+\frac{(z_2-1)(z_1-z_2)}{z_2}\frac{B(1)}{z_1}Q_k(0,0,0,0)$$

or equivalently:

$$Q_{k+1}(z_1,z_2,y_1,y_2) = \frac{B(k+1)}{z_1^k} + (z_2-1)\sum_{j=1}^{k}\frac{B(j)}{z_1^j}\ Q_{k+1-j}(z_1,0,\Phi_1(j),0)$$

$$+\frac{(z_1-z_2)}{z_2}\sum_{j=1}^{k}\frac{B(j)}{z_1^j}\ Q_{k+1-j}(0,z_2,0,\Phi_2(j))$$

$$+\frac{(z_2-1)(z_1-z_2)}{z_2}\sum_{j=1}^{k}\frac{B(j)}{z_1^j}\ Q_{k+1-j}(0,0,0,0)$$

which completes the proof of the theorem □ .

5.3.1.2 Steady State Analysis of the System:

Let $Q(z_1, z_2, y_1, y_2) = \lim_{k \to \infty} Q_k(z_1, z_2, y_1, y_2)$ denote the steady state joint generating function of the system under stability conditions and let us define $Q(z_1, z_2, y_1, y_2, w)$ by the following w transform:

$$Q(z_1, z_2, y_1, y_2, w) = \sum_{k=0}^{\infty} Q_k(z_1, z_2, y_1, y_2)\, w^k \qquad ((|w| < 1) \qquad (5.15)$$

Substituting (5.9b) into the above equation gives:

$$\begin{aligned} Q(z_1, z_2, y_1, y_2, w) = {} & \sum_{k=0}^{\infty} B(k)\left[\frac{w}{z_1}\right]^k + (z_2 - 1)\sum_{k=0}^{\infty}\sum_{j=1}^{k}\frac{B(j)}{z_1^j}\; Q_{k-j}(z_1, 0, \Phi_1(j), 0)\, w^k \\ & + \frac{(z_1 - z_2)}{z_2}\sum_{k=0}^{\infty}\sum_{j=1}^{k}\frac{B(j)}{z_1^j}\; Q_{k-j}(0, z_2, 0, \Phi_2(j))\, w^k \\ & + \frac{(z_2 - 1)(z_1 - z_2)}{z_2}\sum_{k=0}^{\infty}\sum_{j=1}^{k}\frac{B(j)}{z_1^j}\; Q_{k-j}(0, 0, 0, 0)\, w^k \end{aligned} \qquad (5.16)$$

We first focus on the first term, $I = \sum_{k=0}^{\infty} B(k)\left[\frac{w}{z_1}\right]^k$, in the last expression. Substituting for $B(k)$ as in (5.12) and applying the Binomial theorem yields:

$$I = \sum_{k=0}^{\infty}\prod_{i=1}^{2}\left[\sum_{j_i=0}^{m_i}\begin{bmatrix} m_i \\ j_i \end{bmatrix} C_{1i}^{j_i} C_{2i}^{m_i - j_i}\left[\lambda_{1i}^{j_i}\lambda_{2i}^{m_i - j_i}\right]^k\right]\left[\frac{w}{z_1}\right]^k$$

Interchanging the order of summations gives:

$$I = \sum_{j_1=0}^{m_1}\sum_{j_2=0}^{m_2}\left[\prod_{i=1}^{2}\begin{bmatrix} m_i \\ j_i \end{bmatrix} C_{1i}^{j_i} C_{2i}^{m_i - j_i}\right]\sum_{k=0}^{\infty}\left[\frac{w\prod_{i=1}^{2}\lambda_{1i}^{j_i}\lambda_{2i}^{m_i - j_i}}{z_1}\right]^k$$

or equivalently:

$$I = \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} \frac{\left[\prod_{i=1}^{2} \begin{bmatrix} m_i \\ j_i \end{bmatrix} C_{1i}^{j_i} C_{2i}^{m_i - j_i}\right] z_1}{z_1 - w \prod_{i=1}^{2} \lambda_{1i}^{j_i} \lambda_{2i}^{m_i - j_i}} \tag{5.17}$$

Next we consider the second term, $II = (z_2 - 1) \sum_{k=0}^{\infty} \sum_{j=1}^{k} \frac{B(j)}{z_1^j} Q_{k-j}(z_1, 0, \Phi_1(j), 0) w^k$, in (5.16). Once again, substituting for $B(j)$ as in (5.12) yields:

$$II = (z_2 - 1) \sum_{k=0}^{\infty} \sum_{j=1}^{k} \frac{\prod_{i=1}^{2} [C_{1i}\lambda_{1i}^{j} + C_{2i}\lambda_{2i}^{j}]^{m_i}}{z_1^j} Q_{k-j}(z_1, 0, \Phi_1(j), 0) w^k$$

or equivalently:

$$II = (z_2 - 1) \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} \frac{\prod_{i=1}^{2} [C_{1i}\lambda_{1i}^{j} + C_{2i}\lambda_{2i}^{j}]^{m_i}}{z_1^j} Q_{k-j}(z_1, 0, \Phi_1(j), 0) w^k$$

By making the change of variables, $k - j = l$, the above expression becomes:

$$II = (z_2 - 1) \sum_{j=1}^{\infty} \sum_{l=0}^{\infty} \frac{\prod_{i=1}^{2} [C_{1i}\lambda_{1i}^{j} + C_{2i}\lambda_{2i}^{j}]^{m_i}}{z_1^j} Q_l(z_1, 0, \Phi_1(j), 0) w^{j+l}$$

Next, since $Q_l(z_1, 0, \Phi_1(j), 0) = \sum_{k=0}^{\infty} \sum_{r_1=0}^{m_1} z_1^k [\Phi_1(j)]^{r_1} p_l(k, 0, r_1, 0)$ then:

$$II = (z_2 - 1) \sum_{j=1}^{\infty} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \sum_{r_1=0}^{m_1} \frac{\prod_{i=1}^{2} [C_{1i}\lambda_{1i}^{j} + C_{2i}\lambda_{2i}^{j}]^{m_i}}{z_1^j} z_1^k [\Phi_1(j)]^{r_1} p_l(k, 0, r_1, 0) w^{j+l}$$

Substituting for $\Phi_1(j)$,as in (5.10), in the above expression gives:

$$II = (z_2 - 1) \sum_{j=1}^{\infty} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \sum_{r_1=0}^{m_1} [C_{11}\lambda_{11}^{j} + C_{21}\lambda_{21}^{j}]^{m_1 - r_1} [C_{12}\lambda_{12}^{j} + C_{22}\lambda_{22}^{j}]^{m_2} [D_{11}\lambda_{11}^{j} + D_{21}\lambda_{21}^{j}]^{r_1} z_1^k p_l(k, 0, r_1, 0) \frac{w^{j+l}}{z_1^j}$$

By applying the Binomial theorem to the above and after rearranging the terms we get:

$$II = (z_2 - 1) \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \sum_{r_1=0}^{m_1} \sum_{j_1=0}^{m_1-r_1} \sum_{j_2=0}^{m_2} \sum_{j_3=0}^{r_1}$$

$$\begin{bmatrix} m_1 - r_1 \\ j_1 \end{bmatrix} \begin{bmatrix} m_2 \\ j_2 \end{bmatrix} \begin{bmatrix} r_1 \\ j_3 \end{bmatrix} C_{11}^{j_1} C_{21}^{m_1 - r_1 - j_1} C_{12}^{j_2} C_{22}^{m_2 - j_2} D_{11}^{j_3} D_{21}^{r_1 - j_3} z_1^k p_l(k, 0, r_1, 0) w^l \sum_{j=1}^{\infty} \left[\frac{w \lambda_{11}^{j_1 + j_3} \lambda_{21}^{m_1 - j_1 - j_3} \lambda_{12}^{j_2} \lambda_{22}^{m_2 - j_2}}{z_1} \right]$$

or equivalently:

$$II = (z_2 - 1) \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \sum_{r_1=0}^{m_1} \sum_{j_1=0}^{m_1-r_1} \sum_{j_2=0}^{m_2} \sum_{j_3=0}^{r_1} \begin{bmatrix} m_1 - r_1 \\ j_1 \end{bmatrix} \begin{bmatrix} m_2 \\ j_2 \end{bmatrix} \begin{bmatrix} r_1 \\ j_3 \end{bmatrix} \frac{C_{11}^{j_1} C_{21}^{m_1 - r_1 - j_1} C_{12}^{j_2} C_{22}^{m_2 - j_2} D_{11}^{j_3} D_{21}^{r_1 - j_3} w \lambda_{11}^{j_1 + j_3} \lambda_{21}^{m_1 - j_1 - j_3} \lambda_{12}^{j_2} \lambda_{22}^{m_2 - j_2} z_1^k p_l(k, 0, r_1, 0) w^l}{z_1 - w \lambda_{11}^{j_1 + j_3} \lambda_{21}^{m_1 - j_1 - j_3} \lambda_{12}^{j_2} \lambda_{22}^{m_2 - j_2}} \quad (5.17a)$$

The third term, $III = \frac{(z_1 - z_2)}{z_2} \sum_{k=0}^{\infty} \sum_{j=1}^{k} \frac{B(j)}{z_1^j} Q_{k-j}(0, z_2, 0, \Phi_2(j)) w^k$, can also be expanded in a similar fashion, as follows:

$$III = \frac{(z_1 - z_2)}{z_2} \sum_{k=0}^{\infty} \sum_{j=1}^{k} \frac{\prod_{i=1}^{2} [C_{1i} \lambda_{1i}^j + C_{2i} \lambda_{2i}^j]^{m_i}}{z_1^j} Q_{k-j}(0, z_2, 0, \Phi_2(j)) w^k$$

$$= \frac{(z_1 - z_2)}{z_2} \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} \frac{\prod_{i=1}^{2} [C_{1i} \lambda_{1i}^j + C_{2i} \lambda_{2i}^j]^{m_i}}{z_1^j} Q_{k-j}(0, z_2, 0, \Phi_2(j)) w^k$$

$$= \frac{(z_1 - z_2)}{z_2} \sum_{j=1}^{\infty} \sum_{l=0}^{\infty} \frac{\prod_{i=1}^{2} [C_{1i} \lambda_{1i}^j + C_{2i} \lambda_{2i}^j]^{m_i}}{z_1^j} Q_l(0, z_2, 0, \Phi_2(j)) w^{j+l}$$

Substituting $Q_l(0, z_2, 0, \Phi_2(j)) = \sum_{k=0}^{\infty} \sum_{r_1=0}^{m_2} z_2^k [\Phi_2(j)]^{r_1} p_l(0, k, 0, r_1)$ in the above, expanding $\Phi_2(j)$ as in (5.10) and applying the Binomial theorem yields:

$$III = \frac{(z_1 - z_2)}{z_2} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \sum_{r_1=0}^{m_2} \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2-r_1} \sum_{j_3=0}^{r_1}$$

$$\begin{bmatrix} m_1 \\ j_1 \end{bmatrix} \begin{bmatrix} m_2 - r_1 \\ j_2 \end{bmatrix} \begin{bmatrix} r_1 \\ j_3 \end{bmatrix} C_{11}^{j_1} C_{21}^{m_1 - j_1} C_{12}^{j_2} C_{22}^{m_2 - r_1 - j_2} D_{12}^{j_3} D_{22}^{r_1 - j_3} z_1^k p_l(k, 0, r_1, 0) w^l \sum_{j=1}^{\infty} \left[\frac{w \lambda_{12}^{j_2 + j_3} \lambda_{22}^{m_2 - j_2 - j_3} \lambda_{11}^{j_1} \lambda_{22}^{m_1 - j_1}}{z_1} \right]^j$$

or equivalently:

$$III = \frac{(z_1 - z_2)}{z_2} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \sum_{r_1=0}^{m_2} \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2 - r_1} \sum_{j_3=0}^{r_1}$$

$$\begin{bmatrix} m_2 - r_1 \\ j_2 \end{bmatrix} \begin{bmatrix} m_1 \\ j_1 \end{bmatrix} \begin{bmatrix} r_1 \\ j_3 \end{bmatrix} \frac{C_{11}^{j_1} C_{21}^{m_1 - j_1} C_{12}^{j_2} C_{22}^{m_2 - r_1 - j_2} D_{12}^{j_3} D_{22}^{r_1 - j_3} w \lambda_{11}^{j_1} \lambda_{21}^{m_1 - j_1} \lambda_{12}^{j_2 + j_3} \lambda_{22}^{m_2 - j_2 - j_3} z_2^k P_l (0, k, 0, r_1) w^l}{z_1 - w \lambda_{11}^{j_1} \lambda_{21}^{m_1 - j_1} \lambda_{12}^{j_2 + j_3} \lambda_{22}^{m_2 - j_2 - j_3}} \tag{5.18}$$

Finally the forth term $IV = \frac{(z_2 - 1)(z_1 - z_2)}{z_2} \sum_{k=0}^{\infty} \sum_{j=1}^{k} \frac{B(j)}{z_1^j} Q_{k-j}(0,0,0,0) w^k$ can be expressed as follows:

$$IV = \frac{(z_2 - 1)(z_1 - z_2)}{z_2} \sum_{k=0}^{\infty} \sum_{j=1}^{k} \frac{\prod_{i=1}^{2} [C_{1i} \lambda_{1i}^j + C_{2i} \lambda_{2i}^j]^{m_i}}{z_1^j} Q_{k-j}(0,0,0,0) w^k$$

$$= \frac{(z_2 - 1)(z_1 - z_2)}{z_2} \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} \frac{\prod_{i=1}^{2} [C_{1i} \lambda_{1i}^j + C_{2i} \lambda_{2i}^j]^{m_i}}{z_1^j} Q_{k-j}(0,0,0,0) w^k$$

$$= \frac{(z_2 - 1)(z_1 - z_2)}{z_2} \sum_{j=1}^{\infty} \sum_{l=0}^{\infty} \frac{\prod_{i=1}^{2} [C_{1i} \lambda_{1i}^j + C_{2i} \lambda_{2i}^j]^{m_i}}{z_1^j} Q_l(0,0,0,0) w^{l+j}$$

$$= \frac{(z_2 - 1)(z_1 - z_2)}{z_2} \sum_{l=0}^{\infty} Q_l(0,0,0,0) w^l \sum_{j=1}^{\infty} \prod_{i=1}^{2} \left[\sum_{j_i=0}^{m_i} \begin{bmatrix} m_i \\ j_i \end{bmatrix} C_{1i}^{j_i} C_{2i}^{m_i - j_i} \left[\lambda_{1i}^{j_i} \lambda_{2i}^{m_i - j_i} \right]^j \right] \left[\frac{w}{z_1} \right]^j$$

$$= \frac{(z_2 - 1)(z_1 - z_2)}{z_2} \sum_{l=0}^{\infty} Q_l(0,0,0,0) w^l \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} \left[\prod_{i=1}^{2} \begin{bmatrix} m_i \\ j_i \end{bmatrix} C_{1i}^{j_i} C_{2i}^{m_i - j_i} \right] \sum_{j=1}^{\infty} \left[\frac{w \prod_{i=1}^{2} \lambda_{1i}^{j_i} \lambda_{2i}^{m_i - j_i}}{z_1} \right]^j$$

or equivalently:

$$IV = \frac{(z_2 - 1)(z_1 - z_2)}{z_2} \sum_{l=0}^{\infty} Q_l(0,0,0,0) w^l \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} \frac{\left[\prod_{i=1}^{2} \begin{bmatrix} m_i \\ j_i \end{bmatrix} C_{1i}^{j_i} C_{2i}^{m_i - j_i} \right] \left[\prod_{i=1}^{2} \lambda_{1i}^{j_i} \lambda_{2i}^{m_i - j_i} \right] w}{z_1 - w \prod_{i=1}^{2} \lambda_{1i}^{j_i} \lambda_{2i}^{m_i - j_i}} \tag{5.19}$$

Combining the previous intermediate results yields the following expression for the w transform of the joint PGFs of the system at the end of the k^{th} slot:

$$Q(z_1,z_2,y_1,y_2,w) = \sum_{j_1=0}^{m_1}\sum_{j_2=0}^{m_2} \frac{\left[\prod_{i=1}^{2}\begin{bmatrix} m_i \\ j_i \end{bmatrix} C_{1i}^{j_i} C_{2i}^{m_i-j_i}\right] z_1}{z_1 - w\prod_{i=1}^{2}\lambda_{1i}^{j_i}\lambda_{2i}^{m_i-j_i}}$$

$$+ (z_2-1)\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r_1=0}^{m_1}\sum_{j_1=0}^{m_1-r_1}\sum_{j_2=0}^{m_2}\sum_{j_3=0}^{r_1}\begin{bmatrix} m_1-r_1 \\ j_1 \end{bmatrix}\begin{bmatrix} m_2 \\ j_2 \end{bmatrix}\begin{bmatrix} r_1 \\ j_3 \end{bmatrix}\frac{C_{11}^{j_1}C_{21}^{m_1-r_1-j_1}C_{12}^{j_2}C_{22}^{m_2-j_2}D_{11}^{j_3}D_{21}^{r_1-j_3}w\lambda_{11}^{j_1+j_3}\lambda_{21}^{m_1-j_1-j_3}\lambda_{12}^{j_2}\lambda_{22}^{m_2-j_2}z_1^k p_l(k,0,r_1,0)\,w^l}{z_1 - w\lambda_{11}^{j_1+j_3}\lambda_{21}^{m_1-j_1-j_3}\lambda_{12}^{j_2}\lambda_{22}^{m_2-j_2}}$$

$$+ \frac{(z_1-z_2)}{z_2}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r_1=0}^{m_2}\sum_{j_1=0}^{m_1}\sum_{j_2=0}^{m_2-r_1}\sum_{j_3=0}^{r_1}\begin{bmatrix} m_2-r_1 \\ j_2 \end{bmatrix}\begin{bmatrix} m_1 \\ j_1 \end{bmatrix}\begin{bmatrix} r_1 \\ j_3 \end{bmatrix}\frac{C_{11}^{j_1}C_{21}^{m_1-j_1}C_{12}^{j_2}C_{22}^{m_2-r_1-j_2}D_{12}^{j_3}D_{22}^{r_1-j_3}w\lambda_{11}^{j_1}\lambda_{21}^{m_1-j_1}\lambda_{12}^{j_2+j_3}\lambda_{22}^{m_2-j_2-j_3}z_2^k p_l(0,k,0,r_1)\,w^l}{z_1 - w\lambda_{11}^{j_1}\lambda_{21}^{m_1-j_1}\lambda_{12}^{j_2+j_3}\lambda_{22}^{m_2-j_2-j_3}}$$

$$+ \frac{(z_2-1)(z_1-z_2)}{z_2}\sum_{l=0}^{\infty}Q_l(0,0,0,0)\,w^l\sum_{j_1=0}^{m_1}\sum_{j_2=0}^{m_2}\frac{\left[\prod_{i=1}^{2}\begin{bmatrix} m_i \\ j_i \end{bmatrix} C_{1i}^{j_i} C_{2i}^{m_i-j_i}\right]\left[\prod_{i=1}^{2}\lambda_{1i}^{j_i}\lambda_{2i}^{m_i-j_i}\right] w}{z_1 - w\prod_{i=1}^{2}\lambda_{1i}^{j_i}\lambda_{2i}^{m_i-j_i}} \qquad (5.20)$$

The steady state joint generating function of the system can be found by applying Abel's theorem to (5.20) giving:

$$Q(z_1,z_2,y_1,y_2) = \lim_{w\to 1^-}(1-w)\,Q(z_1,z_2,y_1,y_2,w)$$

$$= (z_2-1)\sum_{k=0}^{\infty}\sum_{r_1=0}^{m_1}\sum_{j_1=0}^{m_1-r_1}\sum_{j_2=0}^{m_2}\sum_{j_3=0}^{r_1}\begin{bmatrix} m_1-r_1 \\ j_1 \end{bmatrix}\begin{bmatrix} m_2 \\ j_2 \end{bmatrix}\begin{bmatrix} r_1 \\ j_3 \end{bmatrix}\frac{C_{11}^{j_1}C_{21}^{m_1-r_1-j_1}C_{12}^{j_2}C_{22}^{m_2-j_2}D_{11}^{j_3}D_{21}^{r_1-j_3}\lambda_{11}^{j_1+j_3}\lambda_{21}^{m_1-j_1-j_3}\lambda_{12}^{j_2}\lambda_{22}^{m_2-j_2}z_1^k p(k,0,r_1,0)}{z_1 - \lambda_{11}^{j_1+j_3}\lambda_{21}^{m_1-j_1-j_3}\lambda_{12}^{j_2}\lambda_{22}^{m_2-j_2}}$$

$$+ \frac{(z_1-z_2)}{z_2}\sum_{k=0}^{\infty}\sum_{r_1=0}^{m_2}\sum_{j_1=0}^{m_1}\sum_{j_2=0}^{m_2-r_1}\sum_{j_3=0}^{r_1}\begin{bmatrix} m_2-r_1 \\ j_2 \end{bmatrix}\begin{bmatrix} m_1 \\ j_1 \end{bmatrix}\begin{bmatrix} r_1 \\ j_3 \end{bmatrix}\frac{C_{11}^{j_1}C_{21}^{m_1-j_1}C_{12}^{j_2}C_{22}^{m_2-r_1-j_2}D_{12}^{j_3}D_{22}^{r_1-j_3}\lambda_{11}^{j_1}\lambda_{21}^{m_1-j_1}\lambda_{12}^{j_2+j_3}\lambda_{22}^{m_2-j_2-j_3}z_2^k p(0,k,0,r_1)}{z_1 - \lambda_{11}^{j_1}\lambda_{21}^{m_1-j_1}\lambda_{12}^{j_2+j_3}\lambda_{22}^{m_2-j_2-j_3}}$$

$$+ \frac{(z_2-1)(z_1-z_2)}{z_2}Q(0,0,0,0)\sum_{j_1=0}^{m_1}\sum_{j_2=0}^{m_2}\frac{\left[\prod_{i=1}^{2}\begin{bmatrix} m_i \\ j_i \end{bmatrix} C_{1i}^{j_i} C_{2i}^{m_i-j_i}\right]\left[\prod_{i=1}^{2}\lambda_{1i}^{j_i}\lambda_{2i}^{m_i-j_i}\right]}{z_1 - \prod_{i=1}^{2}\lambda_{1i}^{j_i}\lambda_{2i}^{m_i-j_i}} \qquad (5.21)$$

where we also used the fact that:

$$p(k,0,r_1,0) = \lim_{l \to \infty} p_l(k,0,r_1,0) = \lim_{w \to 1^-} (1-w) \sum_{l=0}^{\infty} p_l(k,0,r_1,0)\, w^l$$

$$p(0,k,0,r_1) = \lim_{l \to \infty} p_l(0,k,0,r_1) = \lim_{w \to 1^-} (1-w) \sum_{l=0}^{\infty} p_l(0,k,0,r_1)\, w^l$$

$$Q(0,0,0,0) = \lim_{l \to \infty} Q_l(0,0,0,0) = \lim_{w \to 1^-} (1-w) \sum_{l=0}^{\infty} Q_l(0,0,0,0)\, w^l$$

Next, and as previously done in the single multiplexer case, it is convenient to combine the j_1 and j_3 terms which appear in the first part of (5.21) and also combine the j_2 and j_3 terms which appear in the second part of the same equation, in order to rewrite (5.21) in the following form:

$$Q(z_1,z_2,y_1,y_2) = (z_2-1) \sum_{k=0}^{\infty} \sum_{r_1=0}^{m_1} \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} \frac{\sum_{l=[j_1-r_1,0]^+}^{[m_1-r_1,j_1]^-} \begin{bmatrix} m_1-r_1 \\ l \end{bmatrix} \begin{bmatrix} m_2 \\ j_2 \end{bmatrix} \begin{bmatrix} r_1 \\ j_1-l \end{bmatrix} C_{11}^{l} C_{21}^{m_1-r_1-l} C_{12}^{j_2} C_{22}^{m_2-j_2} D_{11}^{j_1-l} D_{21}^{r_1-j_1+l} \left[\prod_{i=1}^{2} \lambda_{1i}^{j_i} \lambda_{2i}^{m_i-j_i} \right] z_1^k p(k,0,r_1,0)}{z_1 - \prod_{i=1}^{2} \lambda_{1i}^{j_i} \lambda_{2i}^{m_i-j_i}}$$

$$+ \frac{(z_1-z_2)}{z_2} \sum_{k=0}^{\infty} \sum_{r_1=0}^{m_2} \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} \frac{\sum_{l=[j_2-r_1,0]^+}^{[m_2-r_1,j_2]^-} \begin{bmatrix} m_2-r_1 \\ l \end{bmatrix} \begin{bmatrix} m_1 \\ j_1 \end{bmatrix} \begin{bmatrix} r_1 \\ j_2-l \end{bmatrix} C_{11}^{j_1} C_{21}^{m_1-j_1} C_{12}^{l} C_{22}^{m_2-r_1-l} D_{12}^{j_2-l} D_{22}^{r_1-j_2+l} \left[\prod_{i=1}^{2} \lambda_{1i}^{j_i} \lambda_{2i}^{m_i-j_i} \right] z_2^k p(0,k,0,r_1)}{z_1 - \prod_{i=1}^{2} \lambda_{1i}^{j_i} \lambda_{2i}^{m_i-j_i}}$$

$$+ \frac{(z_2-1)(z_1-z_2)}{z_2} Q(0,0,0,0) \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} \frac{\left[\prod_{i=1}^{2} \begin{bmatrix} m_i \\ j_i \end{bmatrix} C_{1i}^{j_i} C_{2i}^{m_i-j_i} \right] \left[\prod_{i=1}^{2} \lambda_{1i}^{j_i} \lambda_{2i}^{m_i-j_i} \right]}{z_1 - \prod_{i=1}^{2} \lambda_{1i}^{j_i} \lambda_{2i}^{m_i-j_i}} \qquad (5.22)$$

The above equation fully describes the steady state joint generating function of the system where the only unknowns correspond to the two boundary functions $Q(z_1,0,y_1,0)$, $Q(0,z_2,0,y_2)$ and the boundary term $Q(0,0,0,0)$.

Next we determine the marginal generating functions of the queue lengths at each node as follows:

By substituting $z_2 = y_2 = y_1 = 1$ in (5.22), the marginal generating function of the queue length at node 1 is readily obtained, giving:

$$V_1(z_1) = (z_1-1)(1-\rho_1)\sum_{j_1=0}^{m_1}\begin{bmatrix}m_1\\j_1\end{bmatrix}\frac{(\bar{C}_{11}\lambda_{11})^{j_1}(\bar{C}_{21}\lambda_{21})^{m_1-j_1}}{z_1-\lambda_{11}^{j_1}\lambda_{21}^{m_1-j_1}} \tag{5.23a}$$

where $\bar{C}_{i1} = C_{i1}|_{y_1=1}$ for i=1,2 and

$$\rho_1 = 1-Q(0,1,0,1) = \frac{m_1(1-\beta_1)}{2-\alpha_1-\beta_1}\bar{f}_1 \tag{5.23b}$$

with $\bar{f}_1 = \left.\frac{d}{dz_1}f(z_1)\right|_{z_1=1}$ is easily found from the normalization condition, $V_1(1) = 1$.

As expected, (5.23a) corresponds to the same result as the one previously derived in chapter 3, for the single multiplexer case. From (5.52), the average queue length at node 1, denoted by $\bar{N}_1$, is:

$$\bar{N}_1 = \frac{H''(1)}{2(1-H'(1))}+G'(1) \tag{5.24}$$

where:

$$H'(1) = \rho_1 \quad ; \quad G'(1) = \frac{m_1(1-\beta_1)(3-2\alpha_1-2\beta_1)}{(2-\alpha_1-\beta_1)^2}\bar{f}_1 \tag{5.25}$$

$$H''(1)=m_1(m_1-1)\left[\frac{(1-\beta_1)}{2-\alpha_1-\beta_1}\right]^2[\bar{f}_1]^2+m_1\left[\frac{2(1-\alpha_1)(1-\beta_1)(\alpha_1+\beta_1-1)}{(2-\alpha_1-\beta_1)^3}[\bar{f}_1]^2+\frac{1-\beta_1}{2-\alpha_1-\beta_1}f''_1(1)\right] \tag{5.26}$$

Similarly, from (5.22), the marginal generating function of the queue length at node 2, is given by:

$$\begin{aligned}V_2(z_2) = Q(1,z_2,1,1) &= (z_2-1)(1-\rho_2)\sum_{j_2=0}^{m_2}\begin{bmatrix}m_2\\j_2\end{bmatrix}\frac{(\bar{C}_{12}\lambda_{12})^{j_2}(\bar{C}_{22}\lambda_{22})^{m_2-j_2}}{1-\lambda_{12}^{j_2}\lambda_{22}^{m_2-j_2}}\\
&+\frac{(1-z_2)}{z_2}\sum_{k=0}^{\infty}\sum_{r_1=0}^{m_2}\sum_{j_2=0}^{m_2}\frac{\sum_{l=\{j_2-r_1,0\}^+}^{\{m_2-r_1,j_2\}^-}\begin{bmatrix}m_2-r_1\\l\end{bmatrix}\begin{bmatrix}r_1\\j_2-l\end{bmatrix}\bar{C}_{12}^{l}\bar{C}_{22}^{m_2-r_1-l}\bar{D}_{12}^{j_2-l}\bar{D}_{22}^{r_1-j_2+l}\lambda_{12}^{j_2}\lambda_{22}^{m_2-j_2}z_2^k p(0,k,0,r_1)}{1-\lambda_{12}^{j_2}\lambda_{22}^{m_2-j_2}}\\
&+\frac{(z_2-1)(1-z_2)}{z_2}Q(0,0,0,0)\sum_{j_2=0}^{m_2}\begin{bmatrix}m_2\\j_2\end{bmatrix}\frac{(\bar{C}_{12}\lambda_{12})^{j_2}(\bar{C}_{22}\lambda_{22})^{m_2-j_2}}{1-\lambda_{12}^{j_2}\lambda_{22}^{m_2-j_2}}\end{aligned} \tag{5.27}$$

where $\tilde{C}_{i2} = C_{i2}|_{y_2=1}$ and $\tilde{D}_{i2} = D_{i2}|_{y_2=1}$ for i=1,2. The boundary constant, $\rho_2 = 1 - Q(1,0,1,0)$, can be determined from (5.27) by using the normalization condition, $V_2(1) = 1$, as follows: Since $\tilde{C}_{12}|_{z_2=1} = \tilde{D}_{12}|_{z_2=1} = 0$, then it is convenient to rewrite (5.27) as:

$$V_2(z_2) = (z_2-1)(1-\rho_2)\left\{F_1(z_2) + \frac{(\tilde{C}_{22}\lambda_{22})^{m_2}}{1-\lambda_{22}^{m_2}}\right\} + \frac{(1-z_2)}{z_2}\left\{F_2(z_2) + \frac{(\tilde{C}_{22}\lambda_{22})^{m_2}}{1-\lambda_{22}^{m_2}} Q\left(0, z_2, 0, \frac{\tilde{D}_{22}}{\tilde{C}_{22}}\right)\right\}$$

$$+ \frac{(z_2-1)(1-z_2)}{z_2} Q(0,0,0,0)\left\{F_1(z_2) + \frac{(\tilde{C}_{22}\lambda_{22})^{m_2}}{1-\lambda_{22}^{m_2}}\right\} \tag{5.28}$$

where:

$$F_1(z_2) = \sum_{j_2=1}^{m_2} \begin{bmatrix} m_2 \\ j_2 \end{bmatrix} \frac{(\tilde{C}_{12}\lambda_{12})^{j_2}(\tilde{C}_{22}\lambda_{22})^{m_2-j_2}}{1-\lambda_{12}^{j_2}\lambda_{22}^{m_2-j_2}}$$

$$F_2(z_2) = \sum_{k=0}^{\infty}\sum_{r_1=0}^{m_2}\sum_{j_2=1}^{m_2} \frac{\displaystyle\sum_{l=[j_2-r_1,0]^+}^{[m_2-r_1,j_2]^-} \begin{bmatrix} m_2-r_1 \\ l \end{bmatrix}\begin{bmatrix} r_1 \\ j_2-l \end{bmatrix} \tilde{C}_{12}^{l}\tilde{C}_{22}^{m_2-r_1-l}\tilde{D}_{12}^{j_2-l}\tilde{D}_{22}^{r_1-j_2+l}\lambda_{12}^{j_2}\lambda_{22}^{m_2-j_2} z_2^k p(0,k,0,r_1)}{1-\lambda_{12}^{j_2}\lambda_{22}^{m_2-j_2}}$$

are both zero at $z_2 = 1$. From (5.28) we have:

$$V_2(z_2)(1-\lambda_{22}^{m_2}) = (z_2-1)(1-\rho_2)\{F_1(z_2)(1-\lambda_{22}^{m_2}) + (\tilde{C}_{22}\lambda_{22})^{m_2}\}$$

$$+ \frac{(1-z_2)}{z_2}\{F_2(z_2)(1-\lambda_{22}^{m_2}) + (\tilde{C}_{22}\lambda_{22})^{m_2} Q(0, z_2, 0, r_2(z_2))\}$$

$$+ \frac{(z_2-1)(1-z_2)}{z_2} Q(0,0,0,0)\{F_1(z_2)(1-\lambda_{22}^{m_2}) + (\tilde{C}_{22}\lambda_{22})^{m_2}\} \tag{5.29}$$

where $r_2(z_2) = \dfrac{\tilde{D}_{22}}{\tilde{C}_{22}} = \dfrac{\lambda_{22}-\beta_2}{(1-\beta_2)f_2(z_2)}$. Differentiating both sides of the above equation with respect to z_2 and substituting $z_2 = 1$ in the resulting expression yields:

$$\rho_2 = \sum_{i=1}^{2} \frac{m_i(1-\beta_i)\bar{f}_i}{2-\alpha_i-\beta_i} \tag{5.30}$$

where $\bar{f}_i = \frac{d}{dz_i} f(z_i) \Big|_{z_i = 1}$. The last expression is an expected result, as the average departure rate from node 1 equals to the average arrival rate, ρ_1, under steady state conditions. Next, in order to evaluate the boundary function $Q_k(z_1, 0, y_1, 0)$, we proceed as follows: First, taking the limit of the functional equation (5.8) as $z_2 \to 0, y_2 \to 0$ yields:

$$Q_{k+1}(z_1, 0, y_1, 0) = \beta_2^{m_2} [\beta_1 + (1-\beta_1) y_1 f_1(z_1)]^{m_1} \left[\frac{d}{dz_2} Q_k(0, z_2, 0, Y_2) \Big|_{z_2 = 0} + Q_k(0, 0, 0, 0) \right]$$

Then by substituting $z_1 = y_1 = 1$ in the above equation we get:

$$Q_{k+1}(1, 0, 1, 0) = \beta_2^{m_2} \left[\frac{d}{dz_2} Q_k(0, z_2, 0, Y_2) \Big|_{z_2 = 0} + Q_k(0, 0, 0, 0) \right] \tag{5.31}$$

and therefore, from the above two equations, we can write:

$$Q_k(z_1; 0, y_1, 0) = Q_k(1, 0, 1, 0) \cdot [\beta_1 + (1-\beta_1) y_1 f_1(z_1)]^{m_1} \tag{5.32}$$

Taking the limit as $k \to \infty$ in the last expression gives:

$$Q(z_1, 0, y_1, 0) = Q_\infty(z_1, 0, y_1, 0) = (1-\rho_2) [\beta_1 + (1-\beta_1) y_1 f_1(z_1)]^{m_1} \tag{5.33}$$

We also note that (5.33) could have also been derived by noting that if node 2 is empty at the end of the k^{th} slot, then node 1 has also to be empty at the end of the previous slot. This also implies that all the sources feeding node 1 were in the *Off* state during this previous slot. Hence each source will remain passive with probability β_1 or go active with probability $(1-\beta_1)$, in which case it generates packets according to the generating function $f_1(z_1)$.

The boundary constant $Q(0,0,0,0)$ is readily obtained by substituting $z_1 = y_1 = 0$ in (5.33), giving:

$$Q(0, 0, 0, 0) = (1-\rho_2) \beta_1^{m_1} \tag{5.34}$$

Next, from Rouché's theorem, it can be shown (Appendix A11) that the equation:

$$z_1 - \prod_{i=1}^{2} \lambda_{1i}^{j_i} \lambda_{2i}^{m_i - j_i} = 0$$

has a unique root $z_1{*}$ in $(|z_1| < 1)$, given z_2, $(|z_2| < 1)$. Further, for $j_1 \neq 0$, $z_1 = 0$ is the root, which cancels out with the numerator since $C_{11}|_{z_1=0} = \lambda_{11}|_{z_1=0} = 0$. Hence and as before, consider the remaining term in (5.22), which corresponds to $j_1 = j_2 = 0$, namely:

$$\hat{Q}(z_1, z_2, y_1, y_2) = (z_2 - 1) Q(z_1, 0, r_1(z_1), 0) \frac{\prod_{i=1}^{2} (C_{2i}\lambda_{2i})^{m_i}}{z_1 - \prod_{i=1}^{2} \lambda_{2i}^{m_i}}$$

$$+ \frac{(z_1 - z_2)}{z_2} Q(0, z_2, 0, r_2(z_2)) \frac{\prod_{i=1}^{2} (C_{2i}\lambda_{2i})^{m_i}}{z_1 - \prod_{i=1}^{2} \lambda_{2i}^{m_i}} + \frac{(z_2 - 1)(z_1 - z_2)}{z_2} Q(0,0,0,0) \frac{\prod_{i=1}^{2} (C_{2i}\lambda_{2i})^{m_i}}{z_1 - \prod_{i=1}^{2} \lambda_{2i}^{m_i}} \quad (5.35)$$

where:

$$r_i(z_i) = \frac{D_{2i}}{C_{2i}} = \frac{\lambda_{2i} - \beta_i}{(1 - \beta_i) f_i(z_i)}$$

Once again, it can be shown that the equation:

$$z_1 - \prod_{i=1}^{2} \lambda_{2i}^{m_i} = 0 \quad (5.36)$$

has a unique root $z_1{*}$ in $(|z_1| < 1)$,given z_2,. Further from Lagrange's theorem:

$$z_1{*} = \sum_{k=1}^{\infty} \frac{1}{k!} \frac{d^{k-1}}{dz_1^{k-1}} \{ \prod_{i=1}^{2} \lambda_{2i}^{km_i} \} \Big|_{z_1 = 0} \quad (5.37)$$

Since $Q(z_1, z_2, y_1, y_2)$ is analytical in $|z_1| < 1$, given z_2, $(|z_2| < 1)$ then the numerator of (5.35) should also be zero at $z_1{*}$ and it follows that:

$$z_2 (z_2 - 1) Q(z_1{*}, 0, r_1(z_1^{*}), 0) + (z_1{*} - z_2) Q(0, z_2, 0, r_2(z_2)) + (z_2 - 1)(z_1{*} - z_2) Q(0,0,0,0) = 0 \quad (5.38a)$$

A similar argument appears in the analysis of the M/G/1 queue with priority batch arrivals [68]. Using (5.33) we can rewrite (5.38a) as follows:

$$Q(0, z_2, 0, r_2(z_2)) = (1-z_2) Q(0,0,0,0) + \frac{z_2(z_2-1)}{z_2 - z_1^*}(1-\rho_2)\left[\beta_1 + (1-\beta_1) r_1(z_1^*) \cdot f_1(z_1^*)\right]^{m_1} \qquad (5.38b)$$

Although we do not have the complete expression for the boundary function, $Q(0, z_2, 0, y_2)$, the above equation contains sufficient information, which can be combined to our previous results, to derive moments of the queue length behavior at node 2. In fact, from the above, the average queue length, $\bar{N}_2$, at node 2 can be obtained as follows:

First by differentiating (5.29) twice with respect to z_2 and substituting $z_2 = 1$ in the resulting expression, we get:

$$\bar{N}_2 = \frac{(\tilde{C}'_{22}(1) + \lambda'_{22}(1))}{\lambda'_{22}(1)}(\rho_2 - \rho_1) + \frac{1}{m_2 \lambda'_{22}(1)} \left\{ \frac{d}{dz_2} Q(0, z_2, 0, r_2(z_2))\Big|_{z_2=1} - (1-\rho_1) + Q(0,0,0,0) \right\}$$
$$- \frac{(m_2-1)\lambda'_{22}(1)}{2} - \frac{\lambda''_{22}(1)}{2\lambda'_{22}(1)} \qquad (5.39)$$

where:

$$\tilde{C}'_{22}(1) = \frac{d\tilde{C}_{22}}{dz_2}\Big|_{z_2=1} = \frac{(1-\beta_2)(1-\alpha_2-\beta_2)}{(2-\alpha_2-\beta_2)^2}\bar{f}_2 \;;\quad \lambda'_{22}(1) = \frac{d\lambda_{22}}{dz_2}\Big|_{z_2=1} = \frac{1-\beta_2}{2-\alpha_2-\beta_2}\bar{f}_2$$

$$\lambda''_{22}(1) = \frac{d^2\lambda_{22}}{dz_2^2}\Big|_{z_2=1} = \frac{2(1-\alpha_2)(1-\beta_2)(1-\alpha_2-\beta_2)}{(\alpha_2+\beta_2-2)^3}[\bar{f}_2^2] + \frac{1-\beta_2}{2-\alpha_2-\beta_2} f''_2(1)$$

The only remaining unknown in (5.39) is the term:

$$\frac{d}{dz_2} Q(0, z_2, 0, r_2(z_2))\Big|_{z_2=1}$$

which can be computed from (5.38b) in following way:

First, from (5.36), we can write:

$$z_1^* = H_1(z_1^*) \cdot H_2(z_2) \qquad (5.40)$$

where $H_1(z_1) = \lambda_{21}^{m_1}$ and $H_2(z_2) = \lambda_{22}^{m_2}$. Since $\lim_{z_2 \to 1} z_1{}^*=1$, then by differentiating both sides of (5.40) with respect to z_2, and substituting $z_2 = 1$ in the resulting expression, we get:

$$\left.\frac{dz_1{}^*}{dz_2}\right|_{z_2=1} = \left.\frac{dH_1(z_1{}^*)}{dz_2}\right|_{z_2=1} + H'_2(1) = H'_1(1)\left.\frac{dz_1{}^*}{dz_2}\right|_{z_2=1} + H'_2(1)$$

or equivalently:

$$\left.\frac{dz_1{}^*}{dz_2}\right|_{z_2=1} = \frac{H'_2(1)}{1-H'_1(1)} \tag{5.41}$$

where $H'_i(1) = \dfrac{m_i(1-\beta_i)\bar{f}_i}{2-\alpha_i-\beta_i}$. Similarly from (5.40) we can show (Appendix A12) that:

$$\left.\frac{d^2z_1{}^*}{dz_2^2}\right|_{z_2=1} = \frac{H''_2(1)}{1-H'_1(1)} + \frac{2[H'_2(1)]^2H'_1(1)}{[1-H'_1(1)]^2} + \frac{H''_1(1)[H'_2(1)]^2}{[1-H'_1(1)]^3} \tag{5.42}$$

where:

$$H''_i(1) = m_i(m_i-1)\left[\frac{(1-\beta_i)\bar{f}_i}{2-\alpha_i-\beta_i}\right]^2 + m_i\left[\frac{2(1-\alpha_i)(1-\beta_i)(1-\alpha_i-\beta_i)}{(\alpha_i+\beta_i-2)^3}[\bar{f}_i^2] + \frac{1-\beta_i}{2-\alpha_i-\beta_i}f''_i(1)\right]$$

Finally from (5.38a) it can be shown (Appendix A13) that:

$$\left.\frac{dQ(0,z_2,0,r_2(z_2))}{dz_2}\right|_{z_2=1} = \frac{1-\rho_2}{1-\left.\frac{dz_1{}^*}{dz_2}\right|_{z_2=1}} + \rho_1(1-\rho_2)\frac{\left.\frac{dz_1{}^*}{dz_2}\right|_{z_2=1}}{1-\left.\frac{dz_1{}^*}{dz_2}\right|_{z_2=1}} + \frac{\left.\frac{d^2z_1{}^*}{dz_2^2}\right|_{z_2=1}}{2\left[1-\left.\frac{dz_1{}^*}{dz_2}\right|_{z_2=1}\right]}(1-\rho_1) - Q(0,0,0,0) \tag{5.43}$$

and therefore by substituting (5.43) into (5.39), we obtain the following expression for the steady state mean of the buffer length at node 2:

$$\bar{N}_2 = \frac{(\bar{C}'_{22}(1)+\lambda'_{22}(1))}{\lambda'_{22}(1)}(\rho_2-\rho_1) - \frac{(m_2-1)\lambda'_{22}(1)}{2} - \frac{\lambda''_{22}(1)}{2\lambda'_{22}(1)}$$
$$+\frac{1}{m_2\lambda'_{22}(1)}\left\{\frac{1-\rho_2}{1-\left.\frac{dz_1{}^*}{dz_2}\right|_{z_2=1}} + \rho_1(1-\rho_2)\frac{\left.\frac{dz_1{}^*}{dz_2}\right|_{z_2=1}}{1-\left.\frac{dz_1{}^*}{dz_2}\right|_{z_2=1}} + \frac{\left.\frac{d^2z_1{}^*}{dz_2^2}\right|_{z_2=1}}{2\left[1-\left.\frac{dz_1{}^*}{dz_2}\right|_{z_2=1}\right]}(1-\rho_1) - (1-\rho_1)\right\} \tag{5.44}$$

Using ((5.41)-(5.42)) we can further simplify the above expression to obtain:

$$\bar{N}_2 = m_2(\tilde{C}'_{22}(1) + \lambda'_{22}(1)) - \frac{(m_2-1)\lambda'_{22}(1)}{2} - \frac{\lambda''_{22}(1)}{2\lambda'_{22}(1)}$$

$$+ \frac{1}{m_2\lambda'_{22}(1)}\{\rho_1 H'_2(1) + (1-\rho_1)\frac{H''_2(1)}{2(1-\rho_2)} + \rho_1\frac{[H'_2(1)]^2}{1-\rho_2} + \frac{H''_1(1)[H'_2(1)]^2}{2(1-\rho_2)(1-\rho_1)}\} \quad (5.45)$$

From the above analysis, we can use Little's formula [80] to get the average time delay at node *i*, denoted by $\bar{T}_i$, as follows:

$$\bar{T}_i = \frac{\bar{N}_i}{\rho_i} \quad (5.46)$$

where ρ_i is the total arrival rate at node *i* as defined in ((5.23b), (5.30)).
Similarly the total average delay in the system, denoted by $\bar{T}$ can be obtained by applying Little's law to the whole system and is given by:

$$\bar{T} = \frac{\sum_{i=1}^{2} \bar{N}_i}{\rho_2} \quad (5.47)$$

where ρ_2, as defined in (5.30), is the total arrival rate to the network which is also the sum of the arrival rates to the two nodes from their corresponding sources.Note that in the above we have considered queueing delay only and hence did not incorporate the propagation delay. This, however, might be assumed to be constant [83].
Finally we note that higher moments of the queue length at node 2, can also be obtained by successive differentiation of (5.29).

5.4 The General N-Node Tandem Network

In this section we show how to extend the analysis of section 5.3 to tandem networks with an arbitrary number of nodes. Further, let n_i and a_i denote the queue length of node *i* and the number of active sources which feed it, respectively. For an N-node tandem network, the functional equation relating the joint PGF of the system between two consecutive slots can be written by inspection (see [89]). For example, a three-node network will be described by the following dynamic equation:

$$Q_{k+1}(z_1, z_2, z_3, Y_1, Y_2, Y_3) = \prod_{i=1}^{3} [d_i(y_i f_i(z_i))]^{m_i} \cdot \{$$

$$z_1^{-1} [Q_k(z_1, z_2, z_3, Y_1, Y_2, Y_3) - Q_k(z_1, z_2, 0, Y_1, Y_2, 0) - Q_k(z_1, 0, z_3, Y_1, 0, Y_3)$$

$$- Q_k(0, z_2, z_3, 0, Y_2, Y_3) + Q_k(z_1, 0, 0, Y_1, 0, 0) + Q_k(0, 0, z_3, 0, 0, Y_3)$$

$$+ Q_k(0, z_2, 0, 0, Y_2, 0) - Q_k(0, 0, 0, 0, 0, 0)]$$

$$+ z_3 \cdot z_1^{-1} [Q_k(z_1, z_2, 0, Y_1, Y_2, 0) - Q_k(z_1, 0, 0, Y_1, 0, 0) - Q_k(0, z_2, 0, 0, Y_2, 0)$$

$$+ Q_k(0, 0, 0, 0, 0, 0)]$$

$$+ z_2 \cdot (z_1 z_3)^{-1} [Q_k(z_1, 0, z_3, Y_1, 0, Y_3) - Q_k(z_1, 0, 0, Y_1, 0, 0) - Q_k(0, 0, z_3, 0, 0, Y_3)$$

$$+ Q_k(0, 0, 0, 0, 0, 0)]$$

$$+ z_2^{-1} [Q_k(0, z_2, z_3, 0, Y_2, Y_3) - Q_k(0, 0, z_3, 0, 0, Y_3) - Q_k(0, z_2, 0, 0, Y_2, 0)$$

$$+ Q_k(0, 0, 0, 0, 0, 0)]$$

$$+ z_2 \cdot z_1^{-1} [Q_k(z_1, 0, 0, Y_1, 0, 0) - Q_k(0, 0, 0, 0, 0, 0)]$$

$$+ z_3 \cdot z_2^{-1} [Q_k(0, z_2, 0, 0, Y_2, 0) - Q_k(0, 0, 0, 0, 0, 0)]$$

$$+ z_3^{-1} [Q_k(0, 0, z_3, 0, 0, Y_3) - Q_k(0, 0, 0, 0, 0, 0)]$$

$$+ Q_k(0, 0, 0, 0) \quad \} \tag{5.48}$$

or, equivalently:

$$Q_k(z_1, z_2, z_3, Y_1, Y_2, Y_3) = \frac{B(k)}{z_1^k} + (z_3 - 1) \sum_{j=1}^{k} \frac{B(j)}{z_1^j} Q_{k-j}(z_1, z_2, 0, \Phi_1(j), \Phi_2(j), 0)$$

$$+ \frac{(z_2 - z_3)}{z_3} \sum_{j=1}^{k} \frac{B(j)}{z_1^j} Q_{k-j}(z_1, 0, z_3, \Phi_1(j), 0, \Phi_3(j))$$

$$+ \frac{(z_1 - z_2)}{z_2} \sum_{j=1}^{k} \frac{B(j)}{z_1^j} Q_{k-j}(0, z_2, z_3, 0, \Phi_2(j), \Phi_3(j))$$

$$+ \frac{(z_3 - z_2)(1 - z_3)}{z_3} \sum_{j=1}^{k} \frac{B(j)}{z_1^j} Q_{k-j}(z_1, 0, 0, \Phi_1(j), 0, 0)$$

$$+ \frac{(z_3 - z_2)(z_2 - z_1)}{z_2 z_3} \sum_{j=1}^{k} \frac{B(j)}{z_1^j} Q_{k-j}(0, 0, z_3, 0, 0, \Phi_3(j))$$

$$+ \frac{(z_2 - z_1)(1 - z_3)}{z_2} \sum_{j=1}^{k} \frac{B(j)}{z_1^j} Q_{k-j}(0, z_2, 0, 0, \Phi_2(j), 0)$$

$$+ \frac{(z_1 - z_2)(z_3 - z_2)(1 - z_3)}{z_2 z_3} \sum_{j=1}^{k} \frac{B(j)}{z_1^j} Q_{k-j}(0, 0, 0, 0, 0, 0)$$

where $B(k) = \prod_{i=1}^{3} [X_i(k)]^{m_i}$ and $X_i(k)$ and $\Phi_i(k)$ are as defined in (5.10-511).

From the above, the steady-state PGF of the buffer occupancy distribution at node 3 is given by:

$$V_3(z_3) = (z_3 - 1)(1 - \rho_3) \sum_{j_3=0}^{m_3} \begin{bmatrix} m_3 \\ j_3 \end{bmatrix} \frac{(\tilde{C}_{13}\lambda_{13})^{j_3} (\tilde{C}_{23}\lambda_{23})^{m_3 - j_3}}{1 - \lambda_{13}^{j_3} \lambda_{23}^{m_3 - j_3}}$$

$$+ \frac{(1 - z_3)}{z_3} \sum_{k=0}^{\infty} \sum_{r_1=0}^{m_3} \sum_{j_3=0}^{m_3} \frac{\sum_{l=[j_3 - r_1, 0]^+}^{[m_3 - r_1, j_3]^-} \begin{bmatrix} m_3 - r_1 \\ l \end{bmatrix} \begin{bmatrix} r_1 \\ j_3 - l \end{bmatrix} \tilde{C}_{13}^{l} \tilde{C}_{23}^{m_2 - r_1 - l} \bar{D}_{13}^{j_3 - l} \bar{D}_{23}^{r_1 - j_3 + l} \lambda_{13}^{j_3} \lambda_{23}^{m_3 - j_3} z_3^k p(0, k, 0, r_1)}{1 - \lambda_{13}^{j_3} \lambda_{23}^{m_3 - j_3}}$$

$$+ \frac{(z_3 - 1)(1 - z_3)}{z_3} Q(1, 0, 0, 1, 0, 0) \sum_{j_3=0}^{m_3} \begin{bmatrix} m_3 \\ j_3 \end{bmatrix} \frac{(\tilde{C}_{13}\lambda_{13})^{j_3} (\tilde{C}_{23}\lambda_{23})^{m_3 - j_3}}{1 - \lambda_{13}^{j_3} \lambda_{23}^{m_3 - j_3}} \qquad (5.49)$$

and the corresponding steady-state mean is:

$$\bar{N}_3 = \frac{(\tilde{C}'_{23}(1) + \lambda'_{23}(1))}{\lambda'_{23}(1)}(\rho_3 - \rho_2) + \frac{1}{m_3\lambda'_{23}(1)}\left\{\frac{d}{dz_3}Q(1,0,z_3,1,0,r_3(z_3))\Big|_{z_3=1}\cdot(1-\rho_2) + Q(1,0,0,1,0,0)\right\} - \frac{(m_3-1)\lambda'_{23}(1)}{2} - \frac{\lambda''_{23}(1)}{2\lambda'_{23}(1)} \quad (5.50)$$

where $r_3(z_3) = \dfrac{\lambda_{23} - \beta_3}{(1-\beta_3)f_3(z_3)}$ and $p(0,k,0,r_1) = pr(n_2{=}0, n_3{=}k, a_3{=}r_1)$.

We have also studied a four-node network and found that the steady-state PGF of the buffer occupancy distribution at node 4 is given by:

$$V_4(z_4) = (z_4 - 1)(1-\rho_4)\sum_{j_4=0}^{m_4}\begin{bmatrix} m_4 \\ j_4 \end{bmatrix}\frac{(\tilde{C}_{14}\lambda_{14})^{j_4}(\tilde{C}_{24}\lambda_{24})^{m_4-j_4}}{1-\lambda_{14}^{j_4}\lambda_{24}^{m_4-j_4}}$$

$$+\frac{(1-z_4)}{z_4}\sum_{k=0}^{\infty}\sum_{r_1=0}^{m_4}\sum_{j_4=0}^{m_4}\frac{\sum_{l=[j_4-r_1,0]^+}^{[m_4-r_1,j_4]^-}\begin{bmatrix} m_4-r_1 \\ l \end{bmatrix}\begin{bmatrix} r_1 \\ j_4-l \end{bmatrix}\tilde{C}_{14}^{l}\tilde{C}_{24}^{m_4-r_1-l}\tilde{D}_{14}^{j_4-l}\tilde{D}_{24}^{r_1-j_4+l}\lambda_{14}^{j_4}\lambda_{24}^{m_4-j_4}z_4^k p(0,k,0,r_1)}{1-\lambda_{14}^{j_4}\lambda_{24}^{m_4-j_4}}$$

$$+\frac{(z_4-1)(1-z_4)}{z_4}Q(1,1,0,0,1,1,0,0)\sum_{j_4=0}^{m_4}\begin{bmatrix} m_4 \\ j_4 \end{bmatrix}\frac{(\tilde{C}_{14}\lambda_{14})^{j_4}(\tilde{C}_{24}\lambda_{24})^{m_4-j_4}}{1-\lambda_{14}^{j_4}\lambda_{24}^{m_4-j_4}} \quad (5.51)$$

The corresponding steady-state mean is:

$$\bar{N}_4 = \frac{(\tilde{C}'_{24}(1) + \lambda'_{24}(1))}{\lambda'_{24}(1)}(\rho_4 - \rho_3) + \frac{1}{m_4\lambda'_{24}(1)}\left\{\frac{d}{dz_4}Q(1,1,0,z_4,1,1,0,r_4(z_4))\Big|_{z_4=1}\cdot(1-\rho_3) + Q(1,1,0,0,1,1,0,0)\right\}$$

$$-\frac{(m_4-1)\lambda'_{24}(1)}{2} - \frac{\lambda''_{24}(1)}{2\lambda'_{24}(1)} \quad (5.52)$$

where $r_4(z_4) = \dfrac{\lambda_{24} - \beta_4}{(1-\beta_4)f_4(z_4)}$ and $p(0,k,0,r_1) = pr(n_3{=}0, n_4{=}k, a_4{=}r_1)$.

From equations (5.27, 5.49, 5.51) and (5.39, 5.50, 5.52) we conjecture that for the N-node tandem network, depicted in figure 5.2, the steady state PGF of the buffer occupancy distribution at node i $(1 < i \leq N)$ is given by the general expression:

$$V_i(z_i) = (z_i - 1)(1 - \rho_i) \sum_{j_i = 0}^{m_i} \begin{bmatrix} m_i \\ j_i \end{bmatrix} \frac{(\bar{C}_{1i}\lambda_{1i})^{j_i} (\bar{C}_{2i}\lambda_{2i})^{m_i - j_i}}{1 - \lambda_{1i}^{j_i}\lambda_{2i}^{m_i - j_i}}$$

$$+ \frac{(1 - z_i)}{z_i} \sum_{k=0}^{\infty} \sum_{r_1 = 0}^{m_i} \sum_{j_i = 0}^{m_i} \frac{\sum_{l = [j_i - r_1, 0]^+}^{[m_i - r_1, j_i]^-} \begin{bmatrix} m_i - r_1 \\ l \end{bmatrix} \begin{bmatrix} r_1 \\ j_i - l \end{bmatrix} \bar{C}_{1i}^{l} \bar{C}_{2i}^{m_i - r_1 - l} \bar{D}_{1i}^{j_i - l} \bar{D}_{2i}^{r_i - j_i + l} \lambda_{1i}^{j_i} \lambda_{2i}^{m_i - j_i} z_i^k p(0, k, 0, r_1)}{1 - \lambda_{1i}^{j_i}\lambda_{2i}^{m_i - j_i}}$$

$$+ \frac{(z_i - 1)(1 - z_i)}{z_i} Q(1, 1, \ldots, 0, 0, 1, 1, \ldots, 0, 0) \sum_{j_i = 0}^{m_i} \begin{bmatrix} m_i \\ j_i \end{bmatrix} \frac{(\bar{C}_{1i}\lambda_{1i})^{j_i} (\bar{C}_{2i}\lambda_{2i})^{m_i - j_i}}{1 - \lambda_{1i}^{j_i}\lambda_{2i}^{m_i - j_i}} \qquad (5.53)$$

while the corresponding steady-state mean is:

$$\bar{N}_i = \frac{(\bar{C}'_{2i}(1) + \lambda'_{2i}(1))}{\lambda'_{2i}(1)} (\rho_i - \rho_{i-1}) + \frac{1}{m_i \lambda'_{2i}(1)} \left\{ \frac{d}{dz_i} Q(1, 1, \ldots, 0, z_i, 1, 1, \ldots, 0, r_i(z_i)) \Big|_{z_i = 1} \cdot (1 - \rho_{i-1}) + Q(1, 1, \ldots, 0, 0, 1, 1, \ldots, 0, 0) \right\}$$
$$- \frac{(m_i - 1)\lambda'_{2i}(1)}{2} - \frac{\lambda''_{2i}(1)}{2\lambda'_{2i}(1)} \qquad . \qquad (5.54)$$

where $\rho_i = \sum_{j=1}^{i} \frac{m_j(1 - \beta_j)}{2 - \alpha_j - \beta_j} \bar{f}_j$ and $r_i(z_i) = \frac{\bar{D}_{2i}}{\bar{C}_{2i}} = \frac{\lambda_{2i} - \beta_i}{(1 - \beta_i) f_i(z_i)}$.

Note that in (5.53), $p(0, k, 0, r_1) = pr(n_{i-1} = 0, n_i = k, a_i = r_1)$ where n_{i-1}, n_i denote the steady-state queue lengths at nodes (i-1) and i, respectively, while a_i is the steady-state number of active sources feeding node i. It is also interesting to note from (5.53-5.54) that the steady -state queue length behavior at node i is influenced by two factors, namely the mean combined input rate from all the sources feeding the downstream nodes (as demonstrated by the ρ_i term) and the joint behavior of nodes i and i-1, when the latter is empty. The unknowns in (5.53-5.54) can in principle be found by applying similar approach to that of section 5.3 (i.e. through direct substitution into the functional equation of the system and by applying Rouché's theorem) and this is left for future research.

5.4.1 Numerical Results

In this section, we present some numerical results related to the average queue sizes and the average time delay at nodes 1 and 2, as well as to the total delay in the network. We have assumed that all the sources feeding the network are identical. In addition when a source is active it generates, at each slot, a geometrically distributed number of packets with a PGF $f_i(z_i) = \frac{(1-\nu)z_i}{1-\nu z_i}$, where $\nu = \frac{4}{5}$. This corresponds to an average batch size of $\bar{f} = \bar{f}_i = 5$ packet arrivals per active slot. We have also kept $\alpha = 0.02$ and assumed that $m_1 = m_2 = m = 2$. In Figure 5.5, we plot the mean queue length at nodes 1 and 2 as a function of $\rho = \rho_1$

In figure 5.6, the average delay at each node as well as the total average delay in the network are also plotted as a function of ρ. As expected, we can see that, compared to node 1, the average queue length and the mean delay, at node 2, saturate much faster.

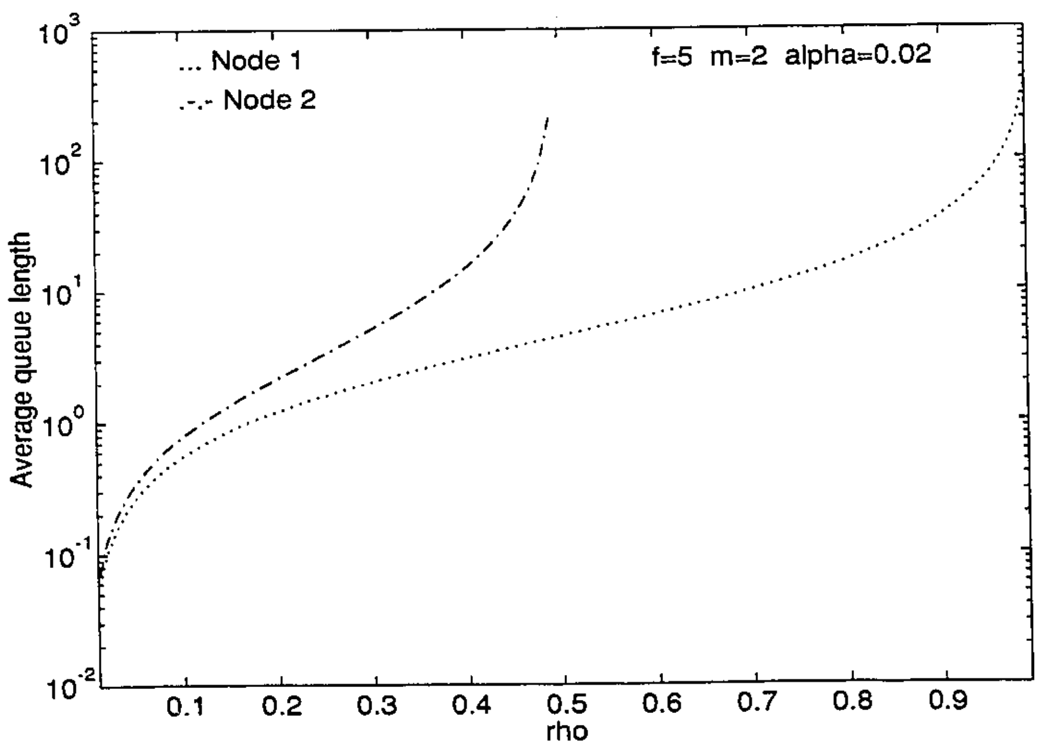

FIGURE.5.5 Average Queue Length at Nodes 1 and 2 as a Function of $\rho = \rho_1$

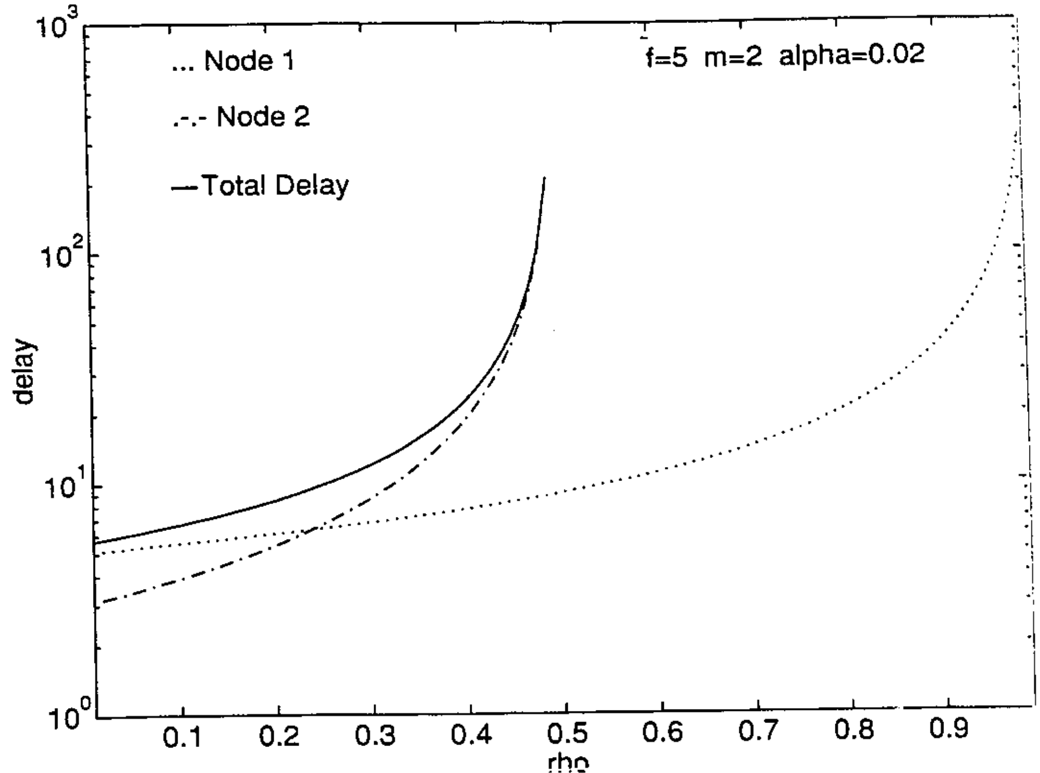

FIGURE.5.6 Average Packet Delay as a Function of $\rho = \rho_1$

CHAPTER VI

Conclusion and Suggestions for Future Research

6.1 Conclusion

In this dissertation, we have proposed a new theoretical approach for the transient analysis of ATM multiplexers with a correlated arrival process, consisting of the traffic generated by the superposition of independent binary Markov sources. We demonstrated the effectiveness of the approach in deriving closed-form expressions for some transient performance measures as well as the corresponding steady-state results. We also presented some results related to the steady-state packet delay as well as the busy and idle periods. We then showed how the proposed approach can as well be applied to analyze tandem queues with correlated arrivals.The solution technique developed here can be applied to a variety of design and performance analysis problems which arise in an ATM environment.

The main thrust of our work was an attempt to tighten the bridge that exists today between the classical transform analysis of discrete-time queues with uncorrelated arrivals and the relatively new analysis which takes into account the correlation in the arrival process. While the former analysis is relatively easy to understand, the latter is often complex and requires some knowledge of the matrix geometric and spectral decomposition approaches. In addition most of the results previously derived using matrix geometric approaches are given in general matrix forms which are often not very handy.

In this thesis, it has been shown that, with some slight modifications, we can extend the classical analysis of discrete time queues with uncorrelated arrivals in order to handle the correlation in the arrival process. We proposed a novel approach which enables us to handle the functional equation relating the joint PGF of the ATM system between two consecutive slots. First, we expressed this PGF in terms of the unknown boundary terms and then derived a sufficient number of linear equations to determine these unknowns. We then applied Abel's theorem to extract the steady-state joint PGF of the system. We have illustrated this technique for the case of a single multiplexer, as well as for the case of two multiplexers in tandem.

Compared to matrix geometric and spectral decomposition approaches, we feel that this work offers a smoother transition from the classical theory of queues which appears in Takács [66] and Kleinrock [90] and which most communication researchers are familiar with. Further, compared to other methods, our approach has another advantage which lies in the fact that it enables the derivation of explicit closed-form expressions for many performance measures, with a remarkable ease.

6.2 Main Contributions

The *main* contributions of this thesis can be briefly summarized as follows:

In chapter 3, we derived closed-form expressions for the transient PGFs of the queue length and the number of active sources for the single-server case. Through transform techniques, we were able to extract explicit expressions for the transient probabilities of an empty buffer with arbitrary deterministic initial conditions. From these, time dependent performance measures for the buffer length behavior, such as transient mean, variance and buffer overflow probabilities were derived. We then presented a fully explicit expression for the steady-state PGF of the buffer content. This is also the solution of the non-linear functional equation (3.3) considered by Bruneel [51]. We also carried the steady-state analysis of the busy and idle periods of the system.

In chapter 4, we extended the above analysis to the multi-server case where we also added more variability in the activity of each source. Our numerical results have shed light on many interesting aspects in the transient behavior of the ATM multiplexer. For instance, under zero initial conditions, we noted the exponential rise in the mean-time curve and the nearly linear profile in the variance-time curve for large number of sources. We also observed the strong dependency of the multiplexer transient behavior on the initial state of the system. We have shown that equilibrium solutions can sometimes be invalid descriptors of the ATM multiplexer behavior. For instance, we observed that the transient probabilities of overflow may exceed the corresponding steady-state results and highlighted the importance of this observation in the ATM design problem, where it is required to estimate the right buffer size needed to meet a specified QoS for fixed source statistics.

In chapter 5, we have shown how to extend the queuing analysis of the single multiplexer to handle tandem queues. We focused our analysis on a two-nodes tandem network. We derived a functional equation relating the joint PGF of the system between two consecutive slots. From this we obtained the steady-state joint PGF of the system in terms of the boundary functions. Further, we have derived expressions for the average queue length and the average delay at each node, as well as the total average delay in the network. We finally discussed the extension of the approach to the analysis of tandem networks, with more than two nodes.

6.3 Suggestions for Future Research

This work can be explored in many directions and possible extensions of the results obtained so far include tne following:

- Extending the analysis of the multiserver queue to the finite buffer case and investigating the effect of a limited buffer size on the performance of the system.

- Since the inversion of the steady-state PGFs of the queue lengths is not trivial, either numerical techniques or approximate formulas need to be introduced in order to derive estimates for the buffer occupancy probabilities. In particular, tight upper-bounds for the tail distribution of the buffer contents could be derived. As mentioned in [71], the distribution of the buffer content of a wide range of infinite capacity queueing systems, including the *G/G/c* queue has a geometric tail behavior which is dominated by the smallest pole, outside the unit circle, of the PGF of the queue length. This tail distribution provides a good approximation to the original queue length distribution under heavy traffic.

- Applying the proposed approach to deal with similar types of functional equations that arise in other contexts in the performance analysis of ATM switching systems (see for example [91]).

- Extending the two-node tandem network analysis to handle the case where packets leaving the first node are allowed to leave the system with a pre-assigned probability.

- Generalizing the steady-state analysis of the two-node tandem network, considered here, by deriving some transient results under different initial conditions.

- Investigating the end-to-end cell queuing delay characterization for the two queues in tandem.

Finally, as any research, there will be no end to this work and many issues that we did not cover here will need to be investigated. Perhaps, among the most important issue is the need to exploit our explicit formulas for the steady-state PGFs of the buffer content in order to derive formulas for the tail probabilities of both the buffer occupancy and the packet delay. Also the significance of our transient analysis in the context of ATM design and congestion control needs further investigation.These will be the subject of our future research.

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Appendix A1

Rouchés Theorem [92]

If $f(z)$ and $g(z)$ are analytic inside and on a closed contour *C*, and if on *C* we have $f(z) \neq 0$ and $|f(z)| > |g(z)|$, then $f(z)$ and $f(z) + g(z)$ have the same number of zeros within *C*.

Appendix A2

Lagrange's Theorem ([92],[93])

If $\psi(z)$ and $g(z)$ are functions of *z*, analytic on and inside a contour *C* surrounding a point *a*, and if *w* is such that

$$|wg(z)| < |z-a| \tag{A.1}$$

is satisfied at all points *z* on the perimeter *C*, then the equation

$$z = a + wg(z) \tag{A.2}$$

regarded as an equation in *z*, has exactly one root in the interior of *C*. Further any function $\psi(z)$ of *z* analytic on and inside *C* can be expanded as a power series in *w* by the formula:

$$\psi(z) = \psi(a) + \sum_{k=1}^{\infty} \frac{w^k}{k!} \left[\frac{d^{k-1}[\psi'(z) g(z)^k]}{dz^{k-1}} \right]_{z=a} \tag{A.3}$$

For the special case, *a=0,* and provided that $g(0) \neq 0$, the following identities also hold:

$$\frac{\psi(z)}{1 - wg'(z)} = \sum_{k=0}^{\infty} \frac{w^k}{k!} \left[\frac{d^k[\psi(z) g(z)^k]}{dz^k} \right]_{z=0} \tag{A.4}$$

$$\frac{\psi'(z) g(z)}{1 - wg'(z)} = \sum_{k=1}^{\infty} \frac{w^{k-1}}{(k-1)!} \left[\frac{d^{k-1}[\psi'(z) g(z)^k]}{dz^{k-1}} \right]_{z=0} \tag{A.5}$$

Appendix A3

Proof of Equation (2.14)

Recall that the Leibniz's rule for the kthderivative of a product $(A \cdot B)$, denoted by $(A \cdot B)^{(k)}$, states that:

$$(A \cdot B)^{(k)} = \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix} A^{(k-i)} B^{(i)} \tag{A.6}$$

where $\begin{bmatrix} k \\ i \end{bmatrix} = \frac{k!}{(k-i)!i!}$. Next let: $A = H(z)^k$ and let $B = (1-z)^{-2}$. Then:

$$(A \cdot B)^{(k-1)}\Big|_{z=0} = \sum_{i=0}^{k-1} \begin{bmatrix} k-1 \\ i \end{bmatrix} A^{(k-1-i)} B^{(i)}\Bigg|_{z=0} \tag{A.7}$$

In addition since $B^{(i)}|_{z=0} = (i+1)!$, then:

$$(A \cdot B)^{(k-1)}\Big|_{z=0} = (k-1)! \sum_{i=0}^{k-1} \frac{(i+1)}{(k-1-i)!} A^{(k-1-i)}\Bigg|_{z=0} \tag{A.8}$$

and therefore, using (2.13), we can write: $\forall (k \geq 1)$:

$$p_k(0) = \frac{1}{k} \sum_{i=0}^{k-1} \frac{(i+1)}{(k-1-i)!} A^{(k-1-i)}\Bigg|_{z=0} = \frac{1}{k} \sum_{i=0}^{k-1} \frac{(k-i)}{i!} \frac{d^i}{dz^i} [H(z)^k]\Big|_{z=0} \tag{A.9}$$

which completes the proof □ .

Appendix A4

Abel's Theorem [94]

If $\lim_{n \to \infty} a_n = a$ and $\sum_{n=0}^{\infty} a_n w^n$ converges for $|w| < 1$, then:

$$\lim_{w \to 1^-} \left[(1-w) \sum_{n=0}^{\infty} a_n w^n \right] = a \tag{A.10}$$

Appendix A5

To prove that if $X^*(k) = X(k)|_{y=Y}$ then:

$$X^*(k) = \frac{X(k+1)}{X(1)} \tag{A.11}$$

we proceed by recurrence, as follows. For k=0, $X^*(0) = 1 = \frac{X(1)}{X(1)}$, hence (A.11) is satisfied. For k=1, (A.11) is also satisfied since:

$$X^*(1) = \beta + (1-\beta)f(z)\left(\frac{(1-\alpha)+\alpha y f(z)}{\beta+(1-\beta)yf(z)}\right)$$

$$= \frac{\beta(\beta+(1-\beta)yf(z))+(1-\beta)f(z)((1-\alpha)+\alpha y f(z))}{\beta+(1-\beta)yf(z)}$$

$$= \frac{((\beta+\alpha f(z))(\beta+(1-\beta)yf(z))+(1-\alpha-\beta)f(z))}{\beta+(1-\beta)yf(z)}$$

$$= \frac{(\beta+\alpha f(z))X(1)+(1-\alpha-\beta)f(z)X(0)}{\beta+(1-\beta)yf(z)} = \frac{X(2)}{X(1)}$$

Next let us suppose that (A.11) is true for the order k, let us prove that is also true for the order (k+1), that is: $X^*(k+1) = \frac{X(k+2)}{X(1)}$. Using (3.5b), we have:

$$X^*(k+1) = (\beta+\alpha f(z))X^*(k) + (1-\alpha-\beta)f(z)X^*(k-1)$$

$$= (\beta+\alpha f(z))\frac{X(k+1)}{X(1)} + (1-\alpha-\beta)f(z)\frac{X(k)}{X(1)}$$

$$= \frac{X(k+2)}{X(1)}$$

which completes the proof.□ .

Appendix A6

To prove that if $U^*(k) = U(k)|_{y=Y}$ then:

$$U^*(k) = \frac{U(k+1)}{X(1)} \qquad \text{(A.12)}$$

we proceed by recurrence, as follows. For k=0, $U^*(0) = Y = \frac{1-\alpha+\alpha y f(z)}{\beta+(1-\beta)yf(z)} = \frac{U(1)}{X(1)}$, hence (A.12) is satisfied. For k=1, (A.12) is also satisfied since:

$$U^*(1) = 1-\alpha+\alpha\frac{1-\alpha+\alpha yf(z)}{\beta+(1-\beta)yf(z)}f(z)$$

$$= \frac{(1-\alpha)(\beta+(1-\beta)yf(z))+\alpha f(z)((1-\alpha)+\alpha yf(z))}{\beta+(1-\beta)yf(z)}$$

$$= \frac{(\beta+\alpha f(z))(1-\alpha+\alpha yf(z))+(1-\alpha-\beta)f(z)y}{\beta+(1-\beta)yf(z)}$$

$$= \frac{(\beta+\alpha f(z))U(1)+(1-\alpha-\beta)f(z)U(0)}{\beta+(1-\beta)yf(z)} = \frac{U(2)}{X(1)}$$

Next let us suppose that (A.12) is true for the order k, let us prove that is also true for the order (k+1), that is: $U^*(k+1) = \frac{U(k+2)}{X(1)}$. Using (3.5a), we have:

$$U^*(k+1) = (\beta+\alpha f(z))U^*(k)+(1-\alpha-\beta)f(z)U^*(k-1)$$

$$= (\beta+\alpha f(z))\frac{U(k+1)}{X(1)}+(1-\alpha-\beta)f(z)\frac{U(k)}{X(1)}$$

$$= \frac{U(k+2)}{X(1)}$$

which completes the proof.□ .

Appendix A7

In order to proof that:

$$\frac{d^k}{dx^k}(x^N) = \begin{bmatrix}N\\k\end{bmatrix}k!x^{N-k} \tag{A.13}$$

we proceed by recurrence as follows:

For k=1, (A.13) is obviously true. Next let us suppose that (A.13) is true for the order k and let us prove that it is also true for the order (k+1); i.e.

$$\frac{d^{(k+1)}}{dx^{k+1}}(x^N) = \begin{bmatrix}N\\k+1\end{bmatrix}(k+1)!x^{N-k-1} \tag{A.14}$$

$$\frac{d^{(k+1)}}{dx^{k+1}}(x^N) = \frac{d}{dx}\left(\frac{d^k}{dx^k}(x^N)\right) = \frac{d}{dx}\left(\begin{bmatrix} N \\ k \end{bmatrix} k! x^{N-k}\right) = \begin{bmatrix} N \\ k \end{bmatrix} k!\,(N-k)\,x^{N-k-1}$$

or equivalently:

$$\frac{d^{(k+1)}}{dx^{k+1}}(x^N) = \begin{bmatrix} N \\ k+1 \end{bmatrix} \frac{k+1}{N-k} k!\,(N-k)\,x^{N-k-1} = \begin{bmatrix} N \\ k+1 \end{bmatrix} (k+1)!\,x^{N-k-1} \qquad \text{(A.15)}$$

which completes the proof.□ .

Appendix A8

The PGF $H(z) = \prod_{i=1}^{\tau} \lambda_2^{m_i}$ can be viewed as the PGF of the sum:

$$Y = \sum_{i=1}^{\tau} X_i$$

of τ independent random variables, each with a PGF $H_i(z) = \lambda_2^{m_i}$. Therefore we can write:

$$\begin{aligned} H''(1) &= E[Y^2] - E[Y] = E\left[\left(\sum_{i=1}^{\tau} X_i\right)^2\right] - \sum_{i=1}^{\tau} E[X_i] \\ &= \sum_{i=1}^{\tau} E[X_i^2] + 2\sum_{i=1}^{\tau-1} \sum_{j=i+1}^{\tau} E[X_i] E[X_j] - \sum_{i=1}^{\tau} E[X_i] \\ &= \sum_{i=1}^{\tau} H''_i(1) + 2\sum_{i=1}^{\tau-1} \sum_{j=i+1}^{\tau} H'_i(1) H'_j(1) \end{aligned} \qquad \text{(A.16)}$$

This last expression leads to the result obtained in (3.87).

Appendix A9

The main complexity behind the inversion of the transform:

$$P(w) = \frac{1}{2}\left[a + \frac{1-a}{1-w} + \frac{\sqrt{(1-aw)^2 + 4bw(1-w)}}{1-w}\right] \qquad \text{(A.17)}$$

resides in inverting the radical term:

$$\Phi(w) = \sqrt{(1-aw)^2 + 4bw(1-w)} = \sqrt{(a^2-4b)w^2 + (4b-2a)w + 1}$$

We first note that since the transform $P(w)$, as defined in (3.75), is convergent within $(|w| < 1)$, then we can apply the binomial theorem to express $\Phi(w)$ in terms of the infinite series expansion:

$$\Phi(w) = \sum_{k=0}^{\infty} \begin{bmatrix} 1/2 \\ k \end{bmatrix} ((a^2-4b)w^2 + (4b-2a)w)^k$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \begin{bmatrix} 1/2 \\ k \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} (a^2-4b)^j (4b-2a)^{k-j} w^{j+k}$$

Next we rewrite the above equation as follows:

$$\Phi(w) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \varphi(j,k) w^{j+k} = \sum_{k=0}^{\infty} \phi(k) w^k \qquad \text{(A.18)}$$

with:

$$\varphi(j,k) = \begin{bmatrix} 1/2 \\ k \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} (a^2-4b)^j (4b-2a)^{k-j}$$

By equating the coefficients of equal powers of w in (A.18) it is easy to verify that:

$$\phi(k) = \sum_{j=0}^{[k]\dagger} \varphi(j, k-j) = \sum_{j=0}^{[k]\dagger} \begin{bmatrix} 1/2 \\ k-j \end{bmatrix} \begin{bmatrix} k-j \\ j \end{bmatrix} (a^2-4b)^j (4b-2a)^{k-2j}$$

and therefore:

$$p_k(0) = W^{-1}[Q(w)] = \frac{1}{2}\left[a\delta(k) + (1-a) + \sum_{i=0}^{k} \phi(i)\right]$$

Appendix A10

Steady-State Variance of the Queue Length Distribution for the Multiple Type of Traffic Case

By differentiating (3.96) three times with respect to z and substituting z=1 in the resulting expression we get:

$$P''(1) = \frac{H''(1)}{1-H'(1)}P'(1) + \frac{H'''(1)}{3[1-H'(1)]} + 2[1-H'(1)]F'(1) + G''(1) \qquad \text{(A.19)}$$

where $H'(1), H''(1)$ and $P'(1)$ are as given in (3.86), (3.87) and (3.98), respectively. To find $H'''(1)$ we follow the same procedure outlined in Appendix A8. Hence we can write:

$$H'''(1) = E[Y^3] - 3H''(1) - H'(1) \qquad \text{(A.20)}$$

The only unknown, $E[Y^3]$, in the above equation can be computed through the application of the multinomial theorem, as follows:

$$E[Y^3] = E\left[\left(\sum_{i=1}^{\tau} X_i\right)^3\right] = \sum_{n_1+n_2+\dots+n_\tau=3} \begin{bmatrix} 3 \\ n_1, n_2, \cdot \quad \cdot n_\tau \end{bmatrix} \prod_{i=1}^{\tau} E[X_i^{n_i}] \qquad \forall (n_i \in \{0,1,2,3\})$$

where:

$$\begin{bmatrix} 3 \\ n_1, n_2, \cdot \quad \cdot n_\tau \end{bmatrix} = \frac{3!}{n_1! n_2! n_3! \dots n_\tau!}$$

and:

$$E[X_i^{n_i}] = \begin{cases} 1 & n_i = 0 \\ H_i'(1) & n_i = 1 \\ H_i''(1) + H_i'(1) & n_i = 2 \\ H_i'''(1) + 3H_i''(1) + H_i'(1) & n_i = 3 \end{cases}$$

Note that $H_i'''(1)$ in the above can be computed from (3.55a) by replacing m, α, β and f by m_i, α_i, β_i and f_i respectively. Next it is easy to verify that $F'(1) = 0$ for all $\tau > 1$ and that $G''(1)$ is given by:

$$G''(1) = \sum_{i=1}^{\tau} G''_i(1) + 2\sum_{i=1}^{\tau-1} \sum_{j=i+1}^{\tau} G_i'(1) G_j'(1) \tag{A.21}$$

where $G'_i(1)$ and $G''_i(1)$ are directly computed from equations (3.53a) and (3.56) by replacing m, α, β and f by m_i, α_i, β_i and f_i, respectively. From the above and using (3.98) the variance of the queue length, σ^2_N, can be obtained from the general formula:

$$\sigma^2_N = P''(1) + P'(1)[1 - P'(1)] \tag{A.22}$$

Appendix A11

Proposition

For each $(j_1, j_2) \in \{0, 1, 2, \cdot \cdot, m_1\} \times \{0, 1, 2, \cdot \cdot, m_2\}$, the equation:

$$z_1 - \prod_{i=1}^{2} \lambda_{1i}^{j_i} \lambda_{2i}^{m_i - j_i} = 0 \tag{A.23}$$

has a unique root z_1* in $(|z_1| < 1)$, given z_2, $(|z_2| < 1)$.

PROOF

Let $h(z_1) = z_1$ and let $g(z_1) = -\prod_{i=1}^{2} \lambda_{1i}^{j_i} \lambda_{2i}^{m_i - j_i}$. Since $\forall i \in \{1, 2\}$, $|\lambda_{1i}| \le |\lambda_{2i}| \le 1$ inside the poly-disc $(|z_1| \le 1, |z_2| \le 1)$ then:

$$|g(z_1)| \le |\lambda_{21}^{m_1 - j_1}|$$

Recall that $\lambda_{21}^{m_1 - j_1}$ is a valid generating function and hence for a small $\varepsilon > 0$ and on $|z_1| = 1 + \varepsilon$ we have $|g(z_1)| \le 1 + \varepsilon(m_1 - j_1) \frac{(1 - \beta_1)}{2 - \alpha_1 - \beta_1} f_1 + \theta(\varepsilon)$. On $|z_1| = 1 + \varepsilon$, $|h(z_1)| = 1 + \varepsilon$ and therefore if node 1 is stable $(\rho_1 < 1)$ then for each

$j_1 \in \{0, 1, 2, \cdot, m_1\}$ $|h(z_1)| > |g(z_1)|$ on $|z_1| = 1+\varepsilon$. From Rouché's theorem it follows that $h(z_1)$ and $g(z_1) + g(z_1)$ have the same number of zeros inside $|z_1| = 1+\varepsilon$, given z_2. Therefore (A.23) has also one root $z_1{*}$ in $(|z_1| < 1)$,given z_2.

Appendix A12

Given:

$$z_1{*} = H_1(z_1{*}) \cdot H_2(z_2) \quad \text{(A.24)}$$

Then by differentiating both sides of the above equation with respect to z_2 we get:

$$\frac{dz_1{*}}{dz_2} = \frac{dH_1(z_1{*})}{dz_2} H_2(z_2) + H_1(z_1{*}) \frac{dH_2(z_2)}{dz_2}$$

Once again, differentiation both sides of the above equation with respect to z_2 and taking the limit as $z_2 \to 1$ gives:

$$\left.\frac{d^2 z_1{*}}{dz_2^2}\right|_{z_2=1} = \left.\frac{d^2 H_1(z_1{*})}{dz_2^2}\right|_{z_2=1} + 2\left.\frac{dH_1(z_1{*})}{dz_2}\right|_{z_2=1} H'_2(1) + H''_2(1)$$

It is easy to verify that:

$$\left.\frac{dH_1(z_1{*})}{dz_2}\right|_{z_2=1} = \left.\frac{dz_1{*}}{dz_2}\right|_{z_2=1} H'_1(1)$$

$$\left.\frac{d^2 H_1(z_1{*})}{dz_2^2}\right|_{z_2=1} = H''_1(1)\left(\left.\frac{dz_1{*}}{dz_2}\right|_{z_2=1}\right)^2 + \left.\frac{d^2 z_1{*}}{dz_2^2}\right|_{z_2=1} H'_1(1)$$

and therefore:

$$\left.\frac{d^2 z_1{*}}{dz_2^2}\right|_{z_2=1} = H''_1(1)\left(\left.\frac{dz_1{*}}{dz_2}\right|_{z_2=1}\right)^2 + \left.\frac{d^2 z_1{*}}{dz_2^2}\right|_{z_2=1} H'_1(1) + 2\left.\frac{dz_1{*}}{dz_2}\right|_{z_2=1} H'_1(1) H'_2(1) + H''_2(1)$$

or equivalently:

$$\left.\frac{d^2 z_1*}{dz_2^2}\right|_{z_2=1} = \frac{H''_1(1)}{1-H'_1(1)}\left(\left.\frac{dz_1*}{dz_2}\right|_{z_2=1}\right)^2 + 2\frac{H'_1(1)H'_2(1)}{1-H'_1(1)}\left.\frac{dz_1*}{dz_2}\right|_{z_2=1} + \frac{H''_2(1)}{1-H'_1(1)} \qquad \text{(A.25)}$$

Substituting for $\left.\frac{dz_1*}{dz_2}\right|_{z_2=1}$ as in (5.41) in the above equation, yields the expression given in (5.42).

Appendix A13

Recall from (5.38a) that:

$$z_2(z_2-1)Q(z_1*,0,r_1(z_1^*),0) + (z_1*-z_2)Q(0,z_2,0,r_2(z_2)) + (z_2-1)(z_1*-z_2)Q(0,0,0,0) = 0$$

Differentiating both side of the above equation with respect to z_2 gives:

$$(2z_2-1)Q(z_1*,0,r_1(z_1^*),0) + z_2(z_2-1)\frac{dQ(z_1*,0,r_1(z_1^*),0)}{dz_2} + \left(\frac{dz_1*}{dz_2}-1\right)Q(0,z_2,0,r_2(z_2))$$

$$+ (z_1*-z_2)\frac{dQ(0,z_2,0,r_2(z_2))}{dz_2} + \left(1+(z_2-1)\frac{dz_1*}{dz_2}+z_1*-2z_2\right)Q(0,0,0,0) = 0 \qquad \text{(A.26)}$$

Once again we differentiate (A.26) with respect to z_2 and take the limit as $z_2 \to 1$ to get:

$$2Q(1,0,1,0) + 2\left.\frac{dQ(z_1*,0,r_1(z_1^*),0)}{dz_2}\right|_{z_2=1} + \left.\frac{d^2 z_1*}{dz_2^2}\right|_{z_2=1} Q(0,1,0,1)$$

$$+2\left(\left.\frac{dz_1*}{dz_2}\right|_{z_2=1}-1\right)\left.\frac{dQ(0,z_2,0,r_2(z_2))}{dz_2}\right|_{z_2=1} + 2\left(\left.\frac{dz_1*}{dz_2}\right|_{z_2=1}-1\right)Q(0,0,0,0) = 0$$

or equivalently:

$$\left.\frac{dQ(0,z_2,0,r_2(z_2))}{dz_2}\right|_{z_2=1} = \frac{1-\rho_2}{1-\left.\frac{dz_1*}{dz_2}\right|_{z_2=1}} + \frac{\left.\frac{dQ(z_1*,0,r_1(z_1^*),0)}{dz_2}\right|_{z_2=1}}{1-\left.\frac{dz_1*}{dz_2}\right|_{z_2=1}} + \frac{\left.\frac{d^2 z_1*}{dz_2^2}\right|_{z_2=1}}{2\left[1-\left.\frac{dz_1*}{dz_2}\right|_{z_2=1}\right]}(1-\rho_1) - Q(0,0,0,0) \qquad \text{(A.27)}$$

Next we note that:

$$\left.\frac{dQ(z_1^*, 0, r_1(z_1^*), 0)}{dz_2}\right|_{z_2=1} = \left.\frac{dz_1^*}{dz_2}\right|_{z_2=1} \cdot \left.\frac{dQ(z_1, 0, r_1(z_1), 0)}{dz_1}\right|_{z_1=1}$$

Using (5.33), the above reduces to:

$$\left.\frac{dQ(z_1^*, 0, r_1(z_1^*), 0)}{dz_2}\right|_{z_2=1} = \left.\frac{dz_1^*}{dz_2}\right|_{z_2=1} \cdot \rho_1(1-\rho_2)$$

Finally substituting this last equation into (A.27) leads to the expression given in (5.43).