Estimation of Relative Intensity and Relative Finality of fBm

Ron Addie, 21 May, 2019

This notebook is based on the one called fBmShiftTransformation10b.nb, which was in turn based on animate/animate3.nb. Section 1 includes basic definitions, Section 2 defines a function which estimates exit probability, for a process with a given down-crossing, by simulation. Section 3 then estimates the last exit density, which is carried out in detail for the cases H=0.75, 0.5, 0.85, and 0.35. Relative finality is found by fitting to the simulations, and this is then used to estimate the last exit density.

1. Definitions

```
κ = 2; H = 0.8;
T = Table[j / 2^κ, {j, 1, κ 2^κ}]; n = Length[T];
Num[κ_] := κ 2^κ; nd = NormalDistribution[];
var[t_] := t^(2 H);
Clear[ρ];
ρ[tlist_, i_, j_, h_] := (1 / 2) (tlist[[i]] tlist[[j]])^(1-2 h)
   (tlist[[i]]^(2 h) + tlist[[j]]^(2 h) - Abs[tlist[[i]] - tlist[[j]]]^(2 h));
ρ[ti_, tj_, h_] := (1 / 2) (ti tj)^(1-2 h) (ti^(2 h) + tj^(2 h) - Abs[ti - tj]^(2 h));
ρ[{10}, 1, 1, 0.5]
```

```
Out[27]= 10.
```

Elementary Paths

Elementary paths have a “locality”, which is where their natural “kink” occurs. In the two parameter function below, for example, the locality of phi is at t. In the 3 parameter version (with H as a parameter), v is the locality.

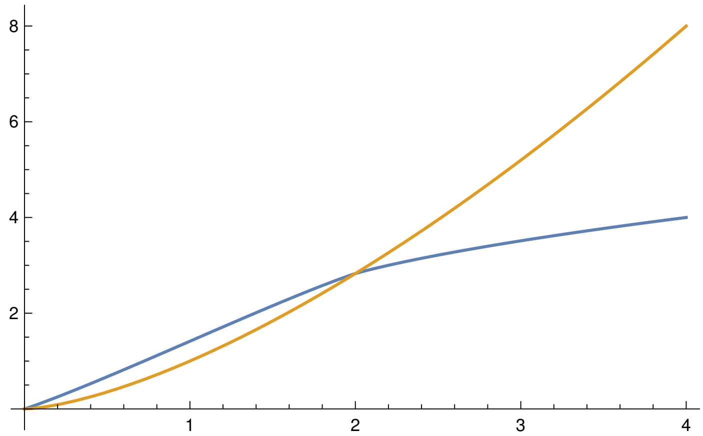
```
In[28]:= H = 0.85;
Var[w_, h_] := Abs[w]^(2 h);
Gamm[u_, v_, h_] := (Var[u, h] + Var[v, h] - Var[Abs[v - u], h]) / (2.0);
Var[w_] := Var[w, H]
phi[u_, v_, h_] := (v / Gamm[v, v, h]) Gamm[v, u, h];
phiv[u_, v_, h_] := (Var[u, h] + Var[v, h] - Var[Abs[v - u], h]) / 2;
(* This rescaling of phi takes the value V[v] at v *)
phi[s_, t_] := phi[s, t, H];
```

```
In[30]:= Evaluate[phi[tt, tt] /. {tt → uu}]
```

```
Out[30]= 1. uu
```

In[31]:=
```
H = 0.75; Plot[{phiv[s, 2, H], Var[s, H]}, {s, 0, 4}]
```

Out[31]=



2. Generation of fBm (simulation)

2.1 Basic Definitions

In[32]:=
```
chol[times_, h_] :=
  Module[{P}, P = Table[ρ[times, i, j, h], {i, Length[times]}, {j, Length[times]}];
   CholeskyDecomposition[N[P, 48]]];
fBm[times_, h_, Z_, U_] := Module[{Y},
   Y = Z.Transpose[Inverse[U]] ;
   Function[s, Evaluate[
     Sum[Evaluate[Y[[m]]] phi[s, Evaluate[times[[m]]], h], {m, Length[times]}]]]
  ];
fBm[times_, h_, U_] :=
  Module[{Z = RandomVariate[nd, Length[times]]}, fBm[times, h, Z, U]];
fBm[t1_, x1_, times_, h_, Z_, U_] := fBm[Join[{t1}, times], h, Join[{x1}, Z], U];
fBm[times_, h_] := fBm[times, h, chol[times, h]];
(* See (10) on page 3 of animatecg.pdf *)
```

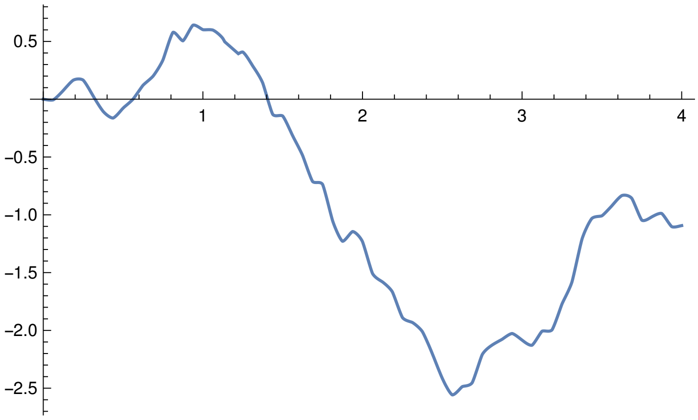
In[33]:=
```
A = 4; K = 4; T = Table[j / 2^K, {j, A (2^K), 1, -1}]
```

Out[33]=

$$\left\{4, \frac{63}{16}, \frac{31}{8}, \frac{61}{16}, \frac{15}{4}, \frac{59}{16}, \frac{29}{8}, \frac{57}{16}, \frac{7}{2}, \frac{55}{16}, \frac{27}{8}, \frac{53}{16}, \frac{13}{4}, \frac{51}{16}, \frac{25}{8}, \frac{49}{16}, 3, \frac{47}{16}, \frac{23}{8}, \frac{45}{16}, \frac{11}{4}, \frac{43}{16}, \frac{21}{8}, \frac{41}{16}, \frac{5}{2}, \frac{39}{16}, \frac{19}{8}, \frac{37}{16}, \frac{9}{4}, \frac{35}{16}, \frac{17}{8}, \frac{33}{16}, 2, \frac{31}{16}, \frac{15}{8}, \frac{29}{16}, \frac{7}{4}, \frac{27}{16}, \frac{13}{8}, \frac{25}{16}, \frac{3}{2}, \frac{23}{16}, \frac{11}{8}, \frac{21}{16}, \frac{5}{4}, \frac{19}{16}, \frac{9}{8}, \frac{17}{16}, 1, \frac{15}{16}, \frac{7}{8}, \frac{13}{16}, \frac{3}{4}, \frac{11}{16}, \frac{5}{8}, \frac{9}{16}, \frac{1}{2}, \frac{7}{16}, \frac{3}{8}, \frac{5}{16}, \frac{1}{4}, \frac{3}{16}, \frac{1}{8}, \frac{1}{16}\right\}$$

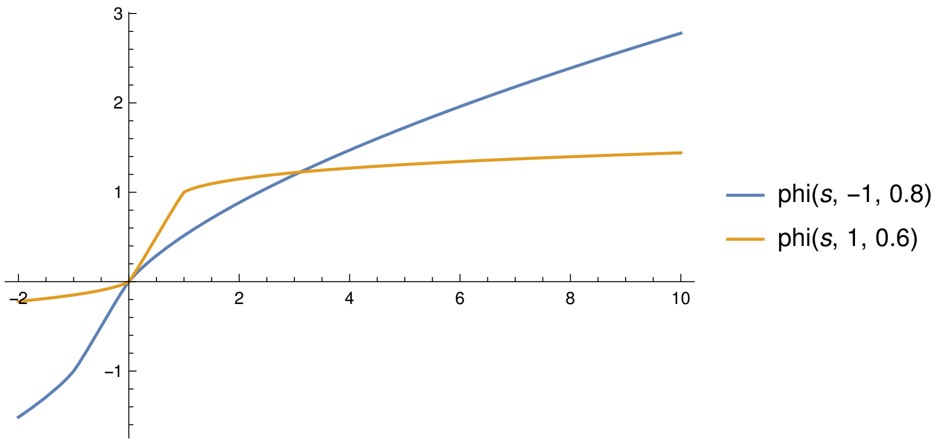
In[34]:=
```
(* T={1,2,4,6,8,10,15,20,30}; *)
f = fBm[T, 0.75];
Plot[f[s], {s, 0, A}]
```

Out[34]=



In[35]:=
```
Plot[{phi[s, -1, 0.8], phi[s, 1, 0.6]}, {s, -2, 10}, PlotLegends -> "Expressions"]
```

Out[35]=




```
Inverse[U][[1]][[1]]
```

0.623899

3. Simulation of Last Exit Density

This section is based on the definitions in Section 2, and in particular those in Section 2.1. The idea is to estimate the density by multipying the density of hitting the boundary by the probability that the path does not return to the boundary later, with the latter estimated by simulation, by using the method developed in Section 2.

3.1 Relative Finality Estimated by Simulation

In[36]:=
```
ExitDensity[x_, h_, n_, bdry_, λ_, μ_, locale_] :=
  Module[{ndens = PDF[NormalDistribution[0, x^h], bdry[x]],
    ψ, T0, T, p, U, Z, exits, itsanex}, exits = 0;
   T0 = Table[s, {s, x + 1 / λ, x + μ / λ, 1 / λ}];
   T = Join[{x}, T0];
U = chol[T, H];
For[k = 1, k ≤ n, k = k + 1, Z  = RandomVariate[nd, Length[T0]];
    Z  = RandomVariate[nd, Length[T0]];
    Z1 = Join[{bdry[x] / (x Inverse[U][[1]][[1]])}, Z];
ψ = fBm[T, h, Z1, U];
    itsanex  = True;
    For[j = locale + 2, j ≤ Length[T],
     j = j + 1, If[ψ[T[[j]]] > bdry[T[[j]]], itsanex = False;
      Break]];
    If[itsanex, exits = exits + 1]];
   p = exits / n;
   {p, N[p ndens]}];
```

In[37]:=
```
λ = 5; μ = 20; ExitDensity[12, 0.5, 100, bdry, λ, μ, λ]
```

Out[37]= $\left\{\frac{29}{100}, 0.0000827848\right\}$

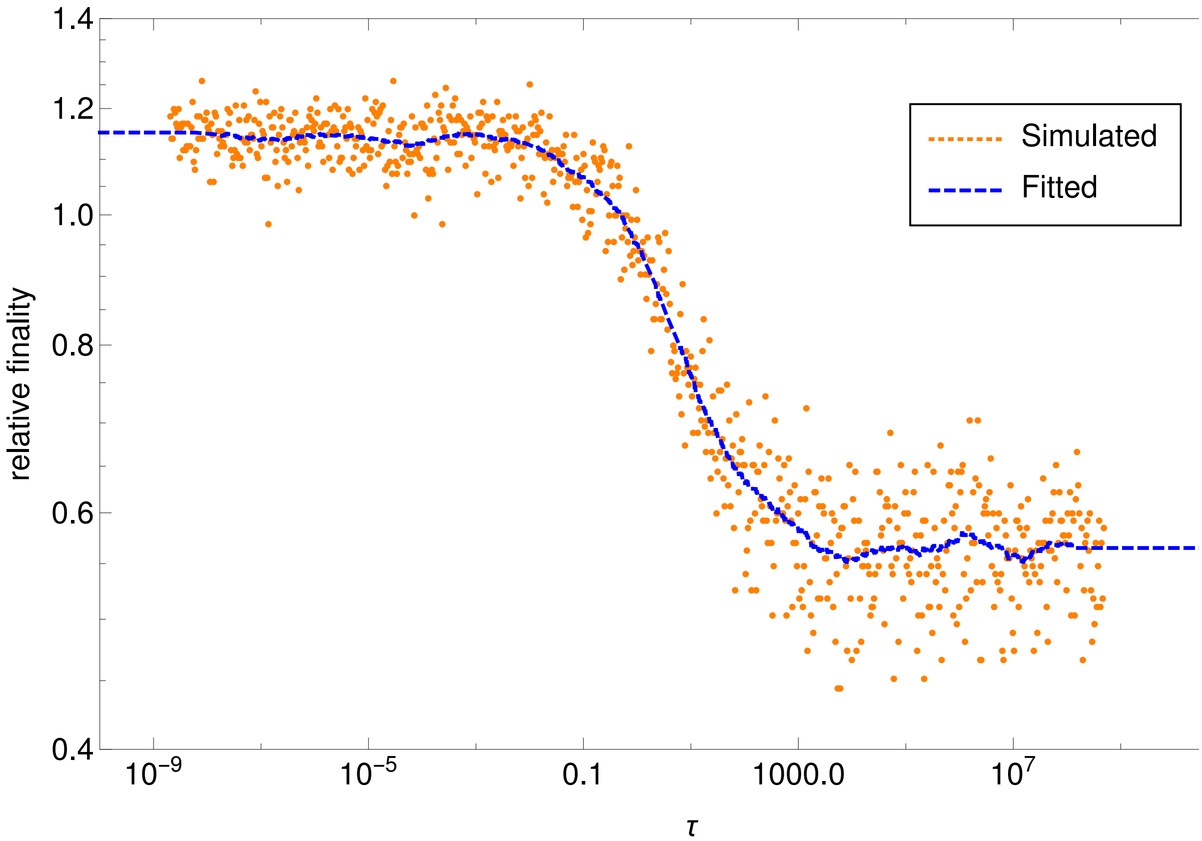
3.2 Case 1, H = 0.75

In[38]:=
```
λ = 4;
μ = 40;
x0 = 0;
bdry[x_] := x - x0;
R = 40;
H = 0.75;
low = -20;
high = 20;
inc = 0.05;
xx = Table[Exp[w], {w, low, high, inc}];
<< ~/usq/pg/PhD/Hardy/PHD/mypapers/stflow/exitpair.mx;

exitpair = Table[ExitDensity[xx[[k]], H, 200, bdry, λ, μ, λ], {k, 1, Length[xx]}];
DumpSave["~/usq/pg/PhD/Hardy/PHD/mypapers/stflow/exitpair.mx", exitpair];
```

In[40]:=
```
exitprob  = Table[exitpair[[k]][[1]], {k, 1, Length[exitpair]}];
exitprobtable  = Table[{xx[[k]], exitpair[[k]][[1]]}, {k, 1, Length[exitpair]}];
exitprobtableloglog  = Table[{Log[xx[[k]]], Log[exitpair[[k]][[1]]]},
   {k, 1, Length[exitpair]}]; exitprobtablelog  =
  Table[{Log[xx[[k]]], exitpair[[k]][[1]]}, {k, 1, Length[exitpair]}];
exitp  = Table[exitpair[[k]][[2]], {k, 1, Length[exitpair]}];
```

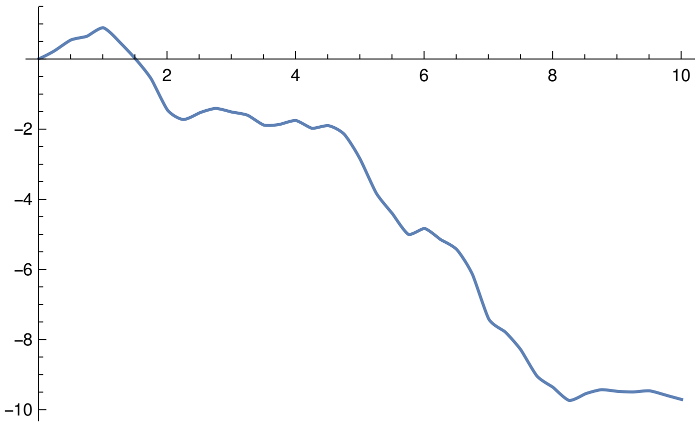
```
In[41]:= exitfit =
          Predict[Table[exitprobtablelog[[kk]][[1]] → exitprobtablelog[[kk]][[2]],
            {kk, 1, Length[exitprobtablelog]}]];
         Clear[xfunc];
         xfunc[x_?NumericQ] := exitfit[x]
         H = 0.75;
         exitpwfit = Table[xfunc[Log[xx[[k]]]] (2 - 2 H) PDF[
             NormalDistribution[0, xx[[k]]^H], bdry[xx[[k]]]], {k, 1, Length[exitpair]}];
         exitpfunctemp[x_, h_] := xfunc[Log[x]] (2 - 2 h)
            PDF[NormalDistribution[0, x^h], bdry[x]];
         B = NIntegrate[exitpfunctemp[x, H], {x, 0, 1 000 000}, PrecisionGoal → 2];
         B

Out[46]= 0.675672

In[47]:= relfinality[x_, h_] := B^-1 xfunc[Evaluate[Log[x]]];
         exitpfunc[x_, h_] :=
          relfinality[x, h] (2 - 2 h) PDF[NormalDistribution[0, x^h], bdry[x]];
         exitprobtableB = Table[{xx[[k]], B^-1 exitpair[[k]][[1]]},
           {k, 1, Length[exitpair]}];
         Show[ListLogLogPlot[exitprobtableB, PlotStyle → Orange,
           PlotRange → {{10^-10, 10^12}, {0.4, 1.4}}], LogLogPlot[{, B^-1 xfunc[Log[x]]},
           {x, 10^-10, 10^12}, PlotStyle → {{Orange, Thick, Dotted}, {Blue, Thick, Dashed}},
           PlotLegends → {Placed[LineLegend[{"Simulated", "Fitted"},
               LabelStyle → {{Directive[16], Dotted}, {Directive[16]}}, LegendFunction →
                (Framed[#] &)], {0.8, 0.8}]}, PlotRange → {{10^-10, 10^12}, {0.4, 1.6}}],
          Frame → True, FrameLabel → {"τ", "relative finality"}, LabelStyle → Directive[16]]
```

Out[48]=



In[49]:=
```
ψ[2]
```
Out[49]=
```
ψ[2]
```

```
Plot[ψ[s], {s, T[[1]], T[[Length[T]]]}]
```



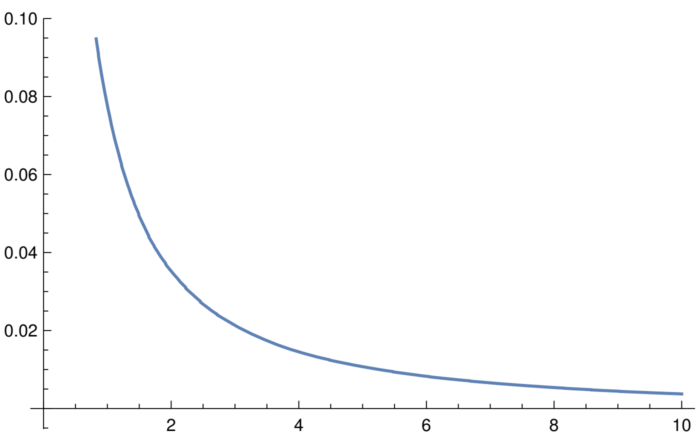
In[50]:=
```
interp[s_, tab_, xx_] := Module[{left = Select[Range[1, Length[xx]],
      (# == 1 ∧ s < xx[[1]]) ∨ (# == Length[xx] ∧ s ≥ xx[[Length[xx]]]) ∨
        (xx[[#]] < s ∧ ((# ≥ Length[xx]) ∨ (xx[[# + 1]] > s))) &][[1]],
    leftvalue, rightvalue}, leftvalue = xx[[left]];
  rightvalue = If[left < Length[xx], xx[[left + 1]], xx[[left]]];
  If[left < Length[xx], ((rightvalue - s) tab[[left]] +
      (s - leftvalue) tab[[left + 1]]) / (rightvalue - leftvalue), tab[[left]]]]
```

In[51]:=
```
lastxdensity[x_, x0_, h_, K_] :=
 Which[h == 0.5, (2^(1 + 1/2 (x0 - 1)) Gamma[1 + 1/2 (x0 - 1)])^-1 x^(1/2 (x0 - 1)) Exp[-1/2 x], x0 == 0,
  (Sqrt[π/2]/(1 - h))^-1 x^-h Exp[-1/2 x^(2 - 2 h)], h ≠ 0.5, K x^-h Exp[-1/2 x^(2 - 2 h) - ((1 - h)/(2 h - 1)) x0 x^(1 - 2 h)]]
```

In[52]:=
```
A = Sum[exitp[[k]] (xx[[k + 1]] - xx[[k]]), {k, 1, Length[xx] - 1}]
```
Out[52]=
```
1.37888
```

In[53]:=
```
Plot[exitpfunctemp[x, H], {x, 0, 10}]
```

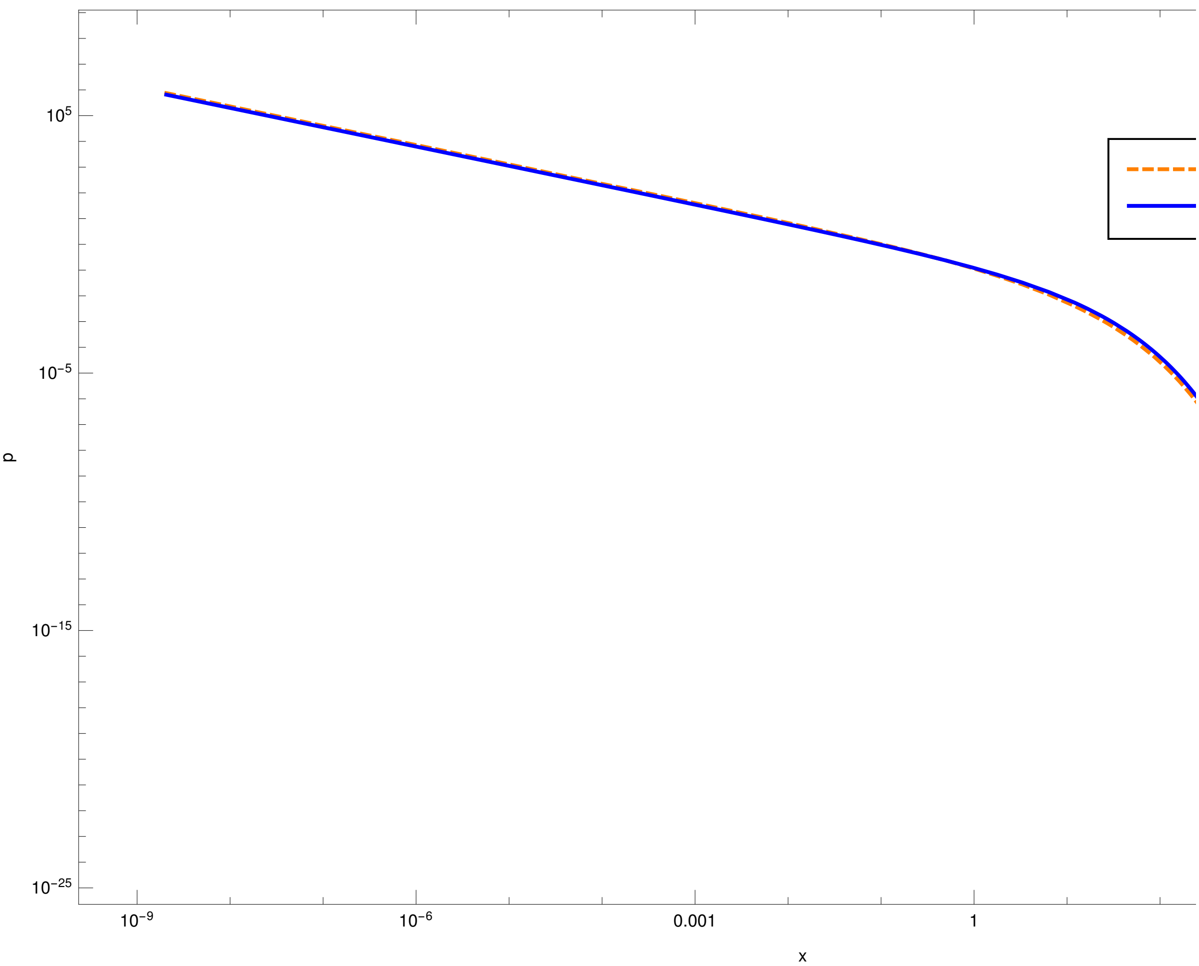
Out[53]=



In[54]:=
```
B
```
Out[54]= 0.675672

In[55]:=
```
H = 0.75; LogLogPlot[{exitpfunc[s, H], lastxdensity[s, x0, H, 1]},
  {s, xx[[1]], 100 000}, PlotStyle → {{Orange, Dashed, Thick}, {Blue, Thick}},
  ImageSize → {800, 600}, PlotLegends →
   {Placed[LineLegend[Style → {{Blue, Directive[16]}, {Orange, Directive[16]}},
      {"Proposition 5.5", "(43)"}, LegendFunction → (Framed[#] &)], {0.8, 0.8}]},
  Frame → True, FrameLabel → {"x", "p"}]
```

Out[55]=



3.3 Case 2, H = 0.5

In[56]:=
```
λ = 4; μ = 40; x0 = 0; bdry[x_] := x - x0; R = 40; H = 0.5; low = -5;
high = 5; inc = 0.02; xx2 = Table[Exp[w], {w, low, high, inc}];
<< ~/usq/pg/PhD/Hardy/PHD/mypapers/stflow/exitpair2.mx;
```

In[57]:=
```
exitpair2 =
  Table[ExitDensity[xx2[[k]], H, 200, bdry, λ, μ, λ], {k, 1, Length[xx2]}];
DumpSave["~/usq/pg/PhD/Hardy/PHD/mypapers/stflow/exitpair2.mx", exitpair2];
```

In[73]:=
```
exitprobtable2 =
   Table[{xx2[[k]], exitpair2[[k]][[1]]}, {k, 1, Length[exitpair2]}];
exitprobtablelog2 = Table[{Log[xx2[[k]]], exitpair2[[k]][[1]]},
   {k, 1, Length[exitpair2]}];
exitprobtableloglog2 = Table[{Log[xx2[[k]]], Log[exitpair2[[k]][[1]]]},
   {k, 1, Length[exitpair2]}];
exitprob2 = Table[exitpair2[[k]][[1]], {k, 1, Length[exitpair2]}];
exitp2 = Table[exitpair2[[k]][[2]], {k, 1, Length[exitpair2]}];
exitfit2 =
   Predict[Table[exitprobtablelog2[[k]][[1]] -> exitprobtablelog2[[k]][[2]],
      {k, 1, Length[exitprobtablelog2]}]];
```

In[76]:=
```
Clear[xfunc2]
xfunc2[x_?NumericQ] := exitfit2[x];
exitpwfit2 =
   Table[Exp[xfunc2[Log[xx2[[k]]]]] PDF[NormalDistribution[0, xx2[[k]]^H],
       bdry[xx2[[k]]]], {k, 1, Length[exitpair2]}];
A2 = Sum[exitp2[[k]] (xx2[[k + 1]] - xx2[[k]]), {k, 1, Length[xx2] - 1}]
```

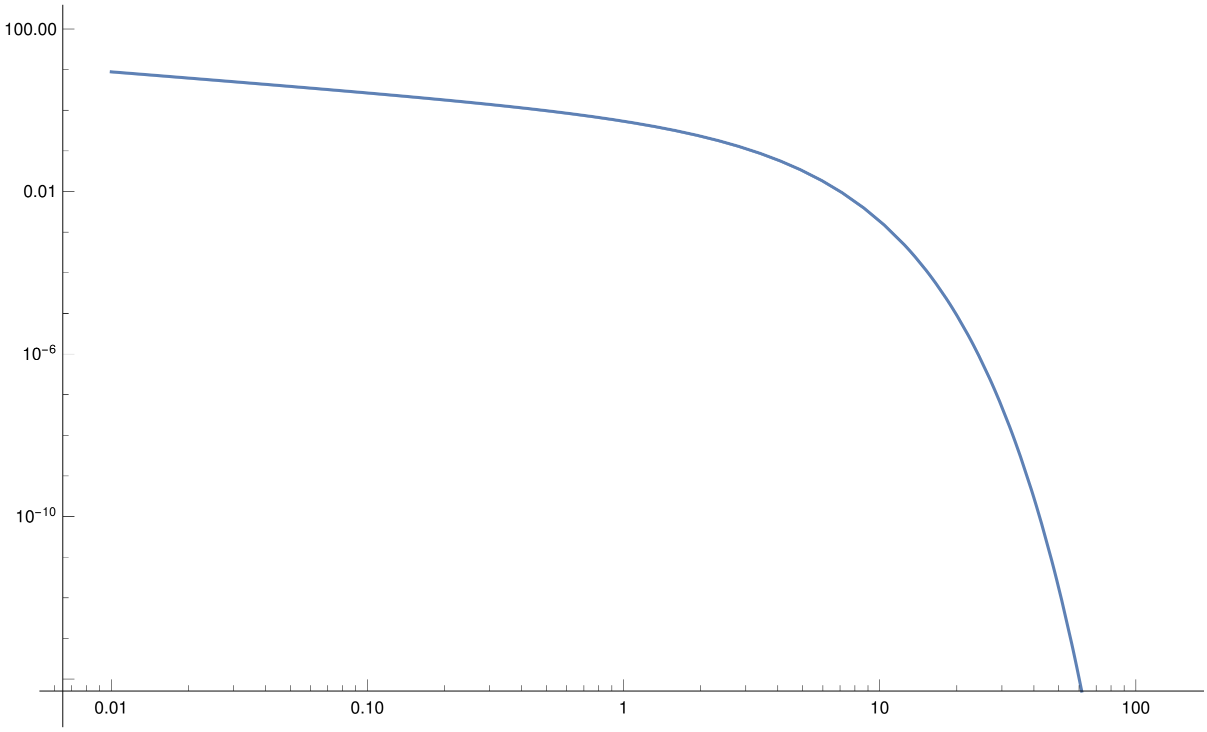
Out[79]= 0.751057

In[80]:=
```
H = 0.5;
exitpfunc2temp[x_, h_] :=
  Exp[xfunc2[Log[x]]] (2 - 2 h) PDF[NormalDistribution[0, x^h], bdry[x]];
B2 = NIntegrate[exitpfunc2temp[x, H], {x, 0, 1 000 000}, PrecisionGoal → 2];
relfinality2[x_, h_] := B2^-1 Exp[xfunc2[Log[x]]];
exitpfunc2[x_, h_] :=
  relfinality2[x, h] (2 - 2 h) PDF[NormalDistribution[0, x^h], bdry[x]];
B2
```

Out[82]= 2.21929

In[87]:=
```
LogLogPlot[exitpfunc2temp[x, H], {x, 0.01, 100}]
```

Out[87]=

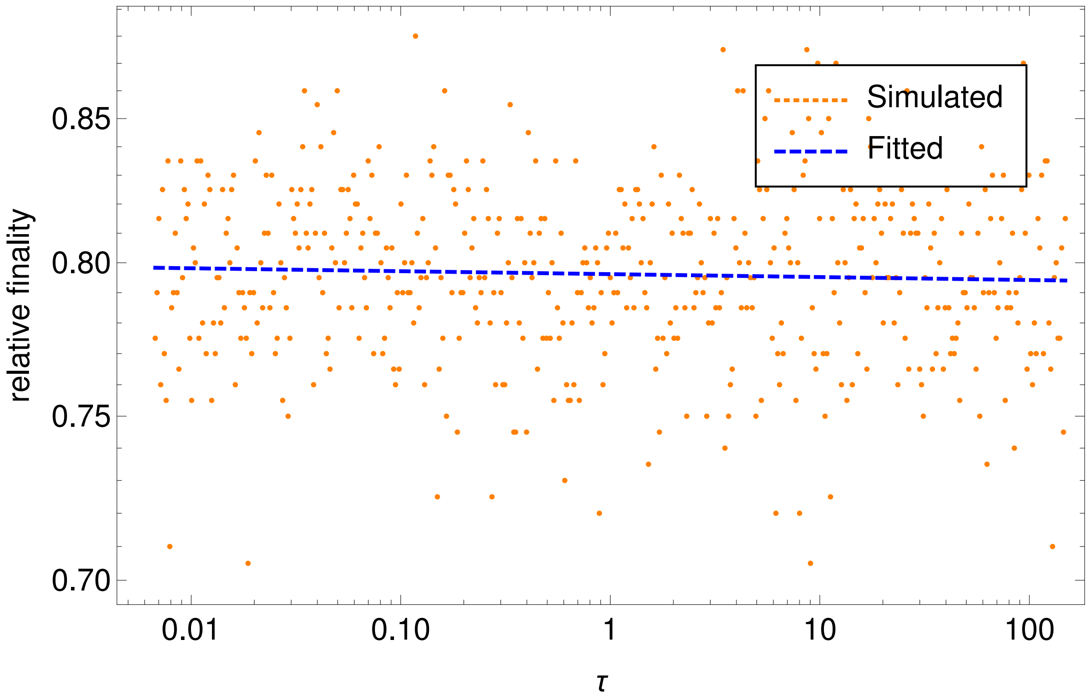


In[84]:=
```
B2
```

Out[84]= 2.21929

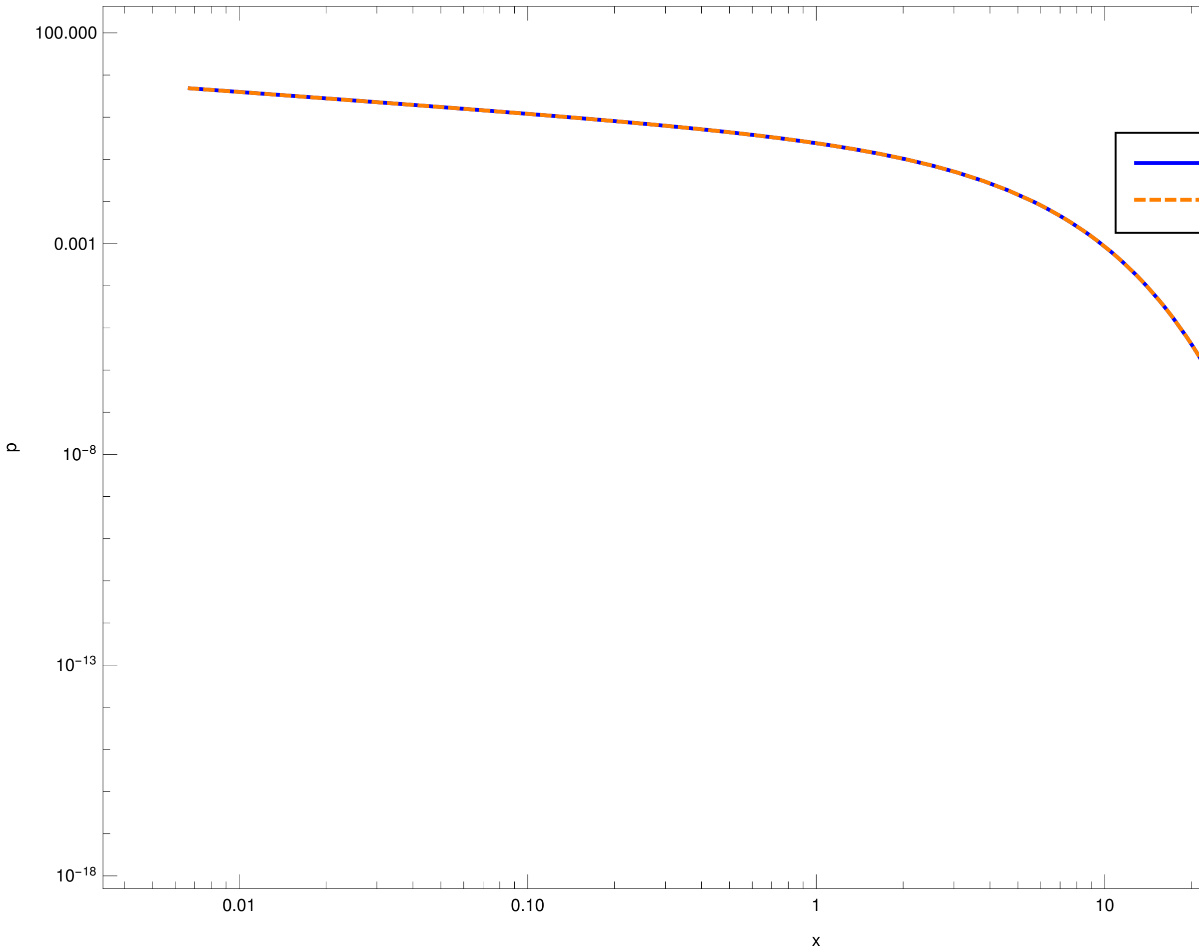
In[85]:=
```
Show[ListLogLogPlot[exitprobtable2, PlotStyle → Orange],
  LogLogPlot[{, xfunc2[Log[x]]}, {x, xx2[[1]], xx2[[Length[xx2]]]},
   PlotStyle → {{Orange, Thick, Dotted}, {Blue, Thick, Dashed}},
   PlotLegends → {Placed[LineLegend[{"Simulated", "Fitted"},
       LabelStyle → {{Directive[16], Dotted}, {Directive[16]}},
       LegendFunction → (Framed[#] &)], {0.8, 0.8}]}],
  Frame → True, FrameLabel → {"τ", "relative finality"},
  LabelStyle → Directive[16]]
```

Out[85]=



In[86]:=
```
H = 0.5; LogLogPlot[{exitpfunc2[s, H], lastxdensity[s, x0, H, 1]},
  {s, xx2[[1]], xx2[[Length[xx2]]]},
  PlotStyle → {{Blue, Thick}, {Orange, Dashed, Thick}},
  ImageSize → {800, 600}, PlotLegends →
   {Placed[LineLegend[Style → {{Orange, Directive[16]}, {Blue, Directive[16]}},
      {"Proposition 5.5", "(43)"}, LegendFunction → (Framed[#] &)], {0.8, 0.8}]},
  Frame → True, FrameLabel → {"x", "p"}]
```

Out[86]=



3.4 Case 3, H = 0.85

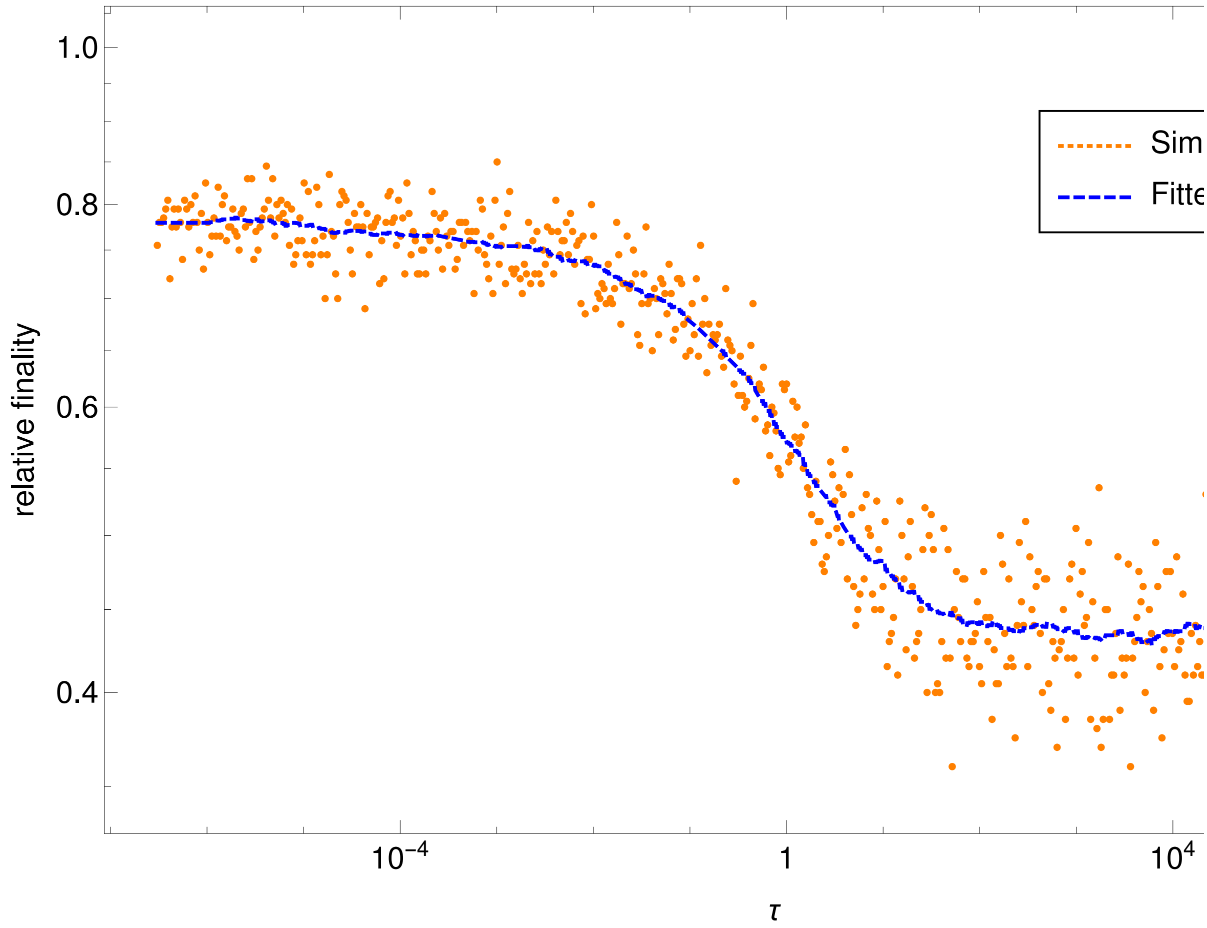
In[88]:=
```
λ = 12;
μ = 40;
x0 = 0;
bdry[x_] := x - x0;
R = 400;
H = 0.85;
low = -15;
high = 15;
inc = 0.05;
xx3 = Table[Exp[w], {w, low, high, inc}];
exitpair3 =
 Table[ExitDensity[xx3[[k]], H, 200, bdry, λ, μ, λ], {k, 1, Length[xx3]}];
exitprob3 = Table[exitpair3[[k]][[1]], {k, 1, Length[exitpair3]}];
exitp3 = Table[exitpair3[[k]][[2]], {k, 1, Length[exitpair3]}];
<< ~/usq/pg/PhD/Hardy/PHD/mypapers/stflow/exitpair3.mx;
```

In[91]:=
```
exitprobtable3 =
  Table[{xx3[[k]], exitpair3[[k]][[1]]}, {k, 1, Length[exitpair3]}];
DumpSave["~/usq/pg/PhD/Hardy/PHD/mypapers/stflow/exitpair3.mx", exitpair3];
```

In[93]:=
```
exitprobtablelog3 =
  Table[{Log[xx3[[k]]], exitpair3[[k]][[1]]}, {k, 1, Length[exitpair3]}];
exitprobtableloglog3 = Table[{Log[xx3[[k]]], Log[exitpair3[[k]][[1]]]},
   {k, 1, Length[exitpair3]}];
exitfit3 = Predict[Table[exitprobtablelog3[[k]][[1]] ->
     exitprobtablelog3[[k]][[2]], {k, 1, Length[exitprobtablelog3]}]];
Clear[xfunc3]; xfunc3[x_?NumericQ] := exitfit3[x];
exitpwfit3 = Table[Exp[xfunc3[Log[xx3[[k]]]]] lastxdensity[xx3[[k]], x0, H, 1],
   {k, 1, Length[exitpair3]}];
exitprobtablelog3 = Table[{Log[xx3[[k]]], Log[exitpair3[[k]][[1]]]},
  {k, 1, Length[exitpair3]}];
exitprobtable3 = Table[{xx3[[k]], exitpair3[[k]][[1]]},
  {k, 1, Length[exitpair3]}];
```

In[99]:=
```
Show[ListLogLogPlot[exitprobtable3, PlotStyle → Orange],
 LogLogPlot[{, xfunc3[Log[x]]}, {x, xx3[[1]], xx3[[Length[xx3]]]},
  PlotStyle → {{Orange, Thick, Dotted}, {Blue, Thick, Dashed}},
  PlotLegends → {Placed[LineLegend[{"Simulated", "Fitted"},
      LabelStyle → {{Directive[16], Dotted}, {Directive[16]}},
      LegendFunction → (Framed[#] &)], {0.8, 0.8}]}],
 Frame → True, FrameLabel → {"τ", "relative finality"},
 LabelStyle → Directive[16]]
```

Out[99]=

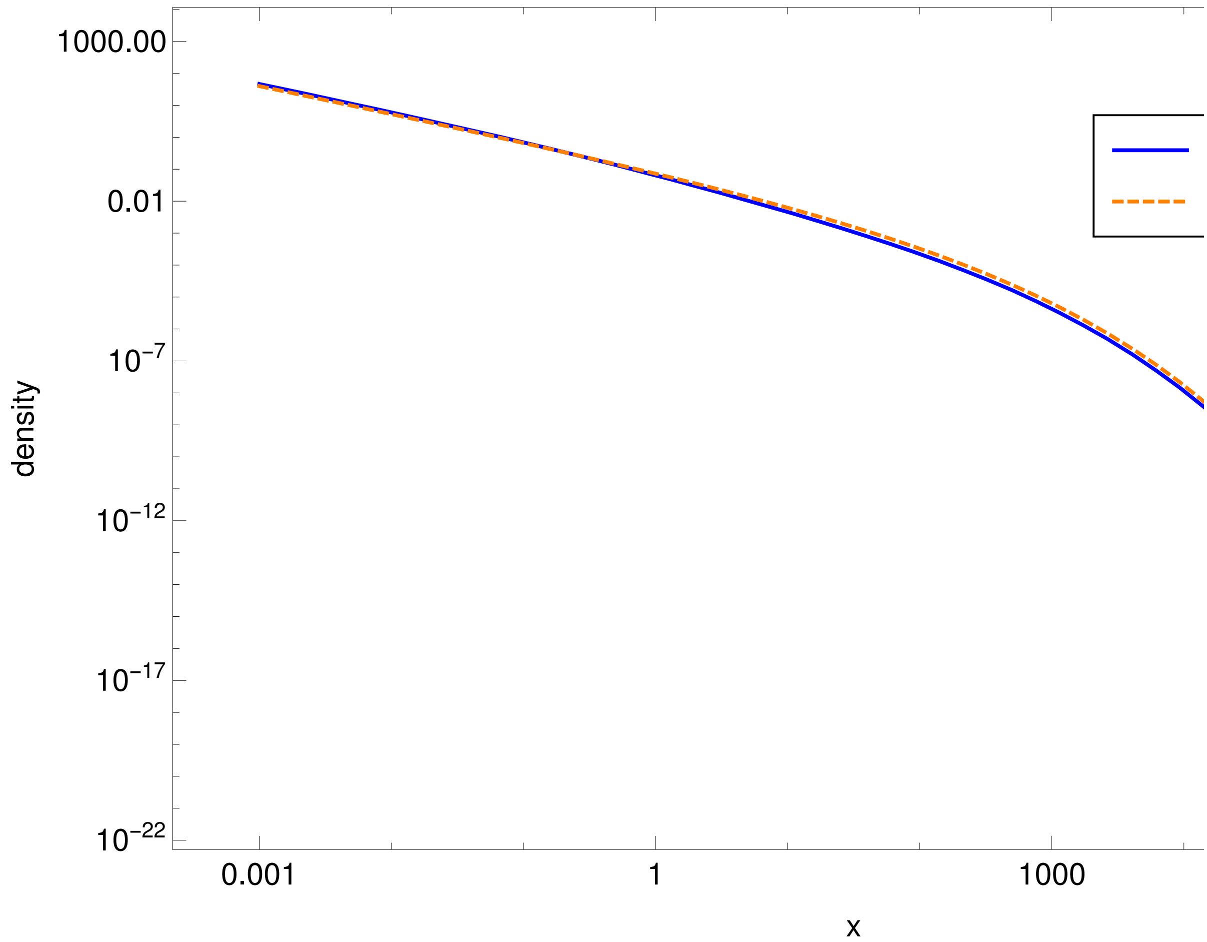


In[100]:=
```
H = 0.85;
A3 = Sum[exitp3[[k]] (xx3[[k + 1]] - xx3[[k]]),
   {k, 1, Min[Length[exitp3], (Length[xx3] - 1)]}];
exitpfunc3temp[x_, h_] := xfunc3[Log[x]] lastxdensity[x, x0, H, 1];
B3 = NIntegrate[exitpfunc3temp[x, H], {x, 0, 1 000 000}, PrecisionGoal → 2];
relfinality3[x_, h_] := B3^-1 xfunc3[Log[x]];
exitpfunc3[x_, h_] := relfinality3[x, h] lastxdensity[x, x0, h, 1];
B3
```

Out[103]= 0.647303

In[104]:=
```
H = 0.85; LogLogPlot[{exitpfunc3[s, H], lastxdensity[s, x0, H, 1]},
  {s, 0.001, 10^6}, PlotStyle → {{Blue, Thick}, {Orange, Dashed, Thick}},
  ImageSize → {800, 600}, PlotLegends →
   {Placed[LineLegend[Style → {{Orange, Directive[16]}, {Blue, Directive[16]}},
      {"Proposition 5.5", "(43)"}, LegendFunction → (Framed[#] &)], {0.8, 0.8}]},
  Frame → True, FrameLabel → {"x", "density"}, LabelStyle → Directive[16]]
```

Out[104]=



3.5 Case 4, H = 0.35

In[105]:=
```
λ = 4; μ = 40; x0 = 0; bdry[x_] := x - x0; low = -15; high = 10;
inc = 0.05; xx4 = Table[Exp[w], {w, low, high, inc}]; R = 40;
H = 0.35; << ~/usq/pg/PhD/Hardy/PHD/mypapers/stflow/exitpair4.mx;
```

In[116]:=
```
exitpair4 =
  Table[ExitDensity[xx4[[k]], H, 200, bdry, λ, μ, λ], {k, 1, Length[xx4]}];
Save["~/usq/pg/PhD/Hardy/PHD/mypapers/stflow/exitpair4.mx", exitpair4];
```

In[118]:=
```
exitprob4 = Table[exitpair4[[k]][[1]], {k, 1, Length[exitpair4]}];
exitp4 = Table[exitpair4[[k]][[2]], {k, 1, Length[exitpair4]}];
exitprobtablelog4 =
 Table[{Log[xx4[[k]]], exitpair4[[k]][[1]]}, {k, 1, Length[exitpair4]}];
exitprobtableloglog4 =
 Table[{Log[xx4[[k]]], Log[exitpair4[[k]][[1]]]}, {k, 1, Length[exitpair4]}];
exitprobtable4 = Table[{xx4[[k]], exitpair4[[k]][[1]]},
  {k, 1, Length[exitpair4]}];
```

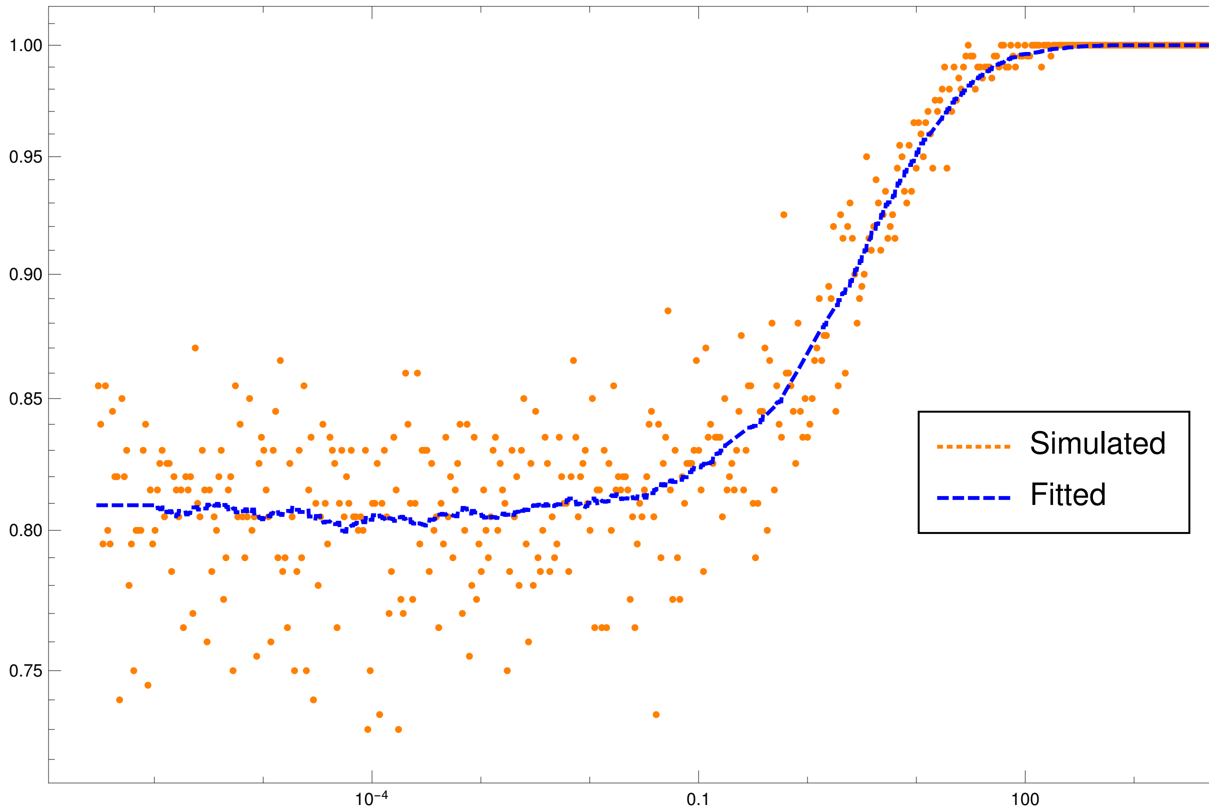
In[120]:=
```
H = 0.35;
exitfit4 = Predict[Table[exitprobtablelog4[[k]][[1]] ->
    exitprobtablelog4[[k]][[2]], {k, 1, Length[exitprobtablelog4]}]];
Clear[
 xfunc4];
xfunc4[x_?NumericQ] := exitfit4[x];
exitpwfit4 = Table[xfunc4[Log[xx4[[k]]]] PDF[
    NormalDistribution[0, xx4[[k]]^H], bdry[xx4[[k]]]], {k, 1, Length[exitpair4]}];
```

In[122]:=
```
H = 0.35;
A4 = Sum[exitp4[[k]] (xx4[[k + 1]] - xx4[[k]]), {k, 1, Length[xx4] - 1}];
exitpfunc4temp[x_, h_] :=
 xfunc4[Log[x]] (2 - 2 h) PDF[NormalDistribution[0, x^h], bdry[x]];
B4 = NIntegrate[exitpfunc4temp[x, H], {x, 0, 1000}, PrecisionGoal → 2];
relfinality4[x_, h_] := B4^-1 xfunc4[Log[x]];
exitpfunc4[x_, h_] :=
 relfinality4[x, h] (2 - 2 h) PDF[NormalDistribution[0, x^h], bdry[x]];
N[
 B4]
```

Out[124]= 0.852936

In[125]:=
```
Show[ListLogLogPlot[exitprobtable4, PlotStyle → Orange],
 LogLogPlot[{, xfunc4[Log[x]]}, {x, xx4[[1]], xx4[[Length[xx4]]]},
  PlotStyle → {{Orange, Thick, Dotted}, {Blue, Thick, Dashed}},
  PlotLegends → {Placed[LineLegend[{"Simulated", "Fitted"},
      LabelStyle → {{Directive[16], Dotted}, {Directive[16]}},
      LegendFunction → (Framed[#] &)], {0.8, 0.4}]}], Frame → True]
```

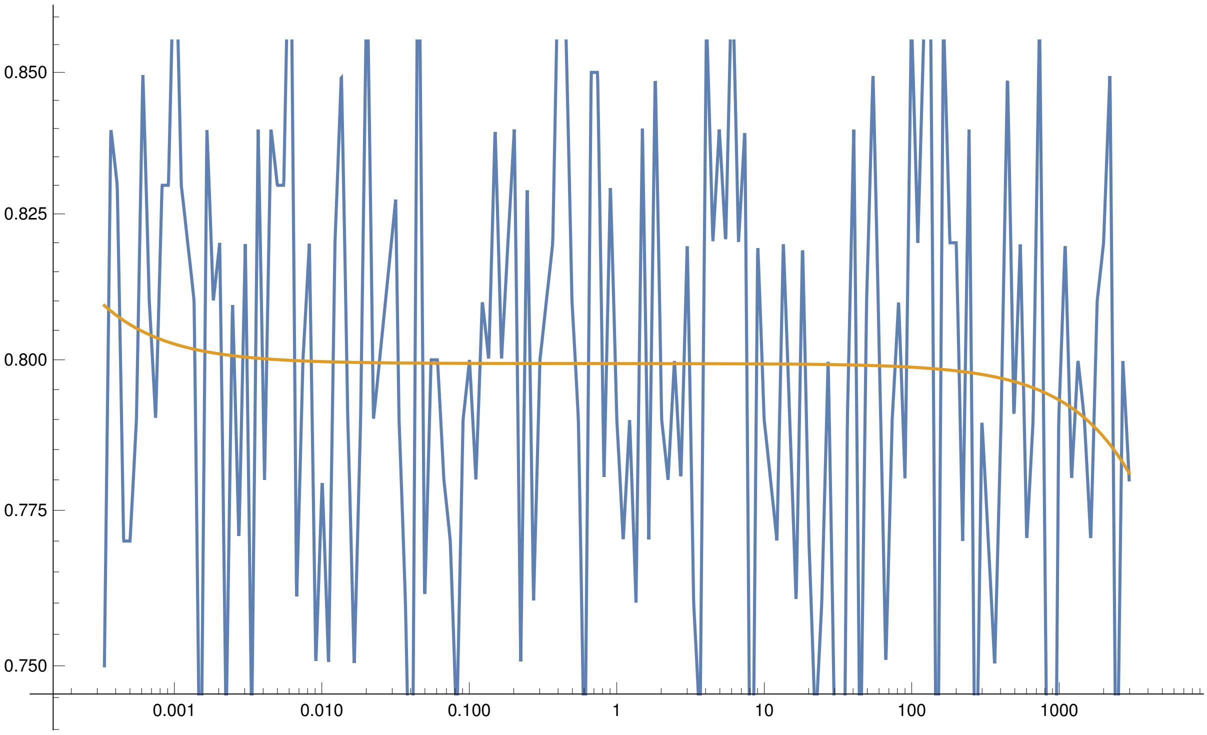
Out[125]=



```
LogLogPlot[{interp[x, exitprob4, xx4], xfunc4[x]},
 {x, xx4[[1]], xx4[[Length[xx4]]]}]
```

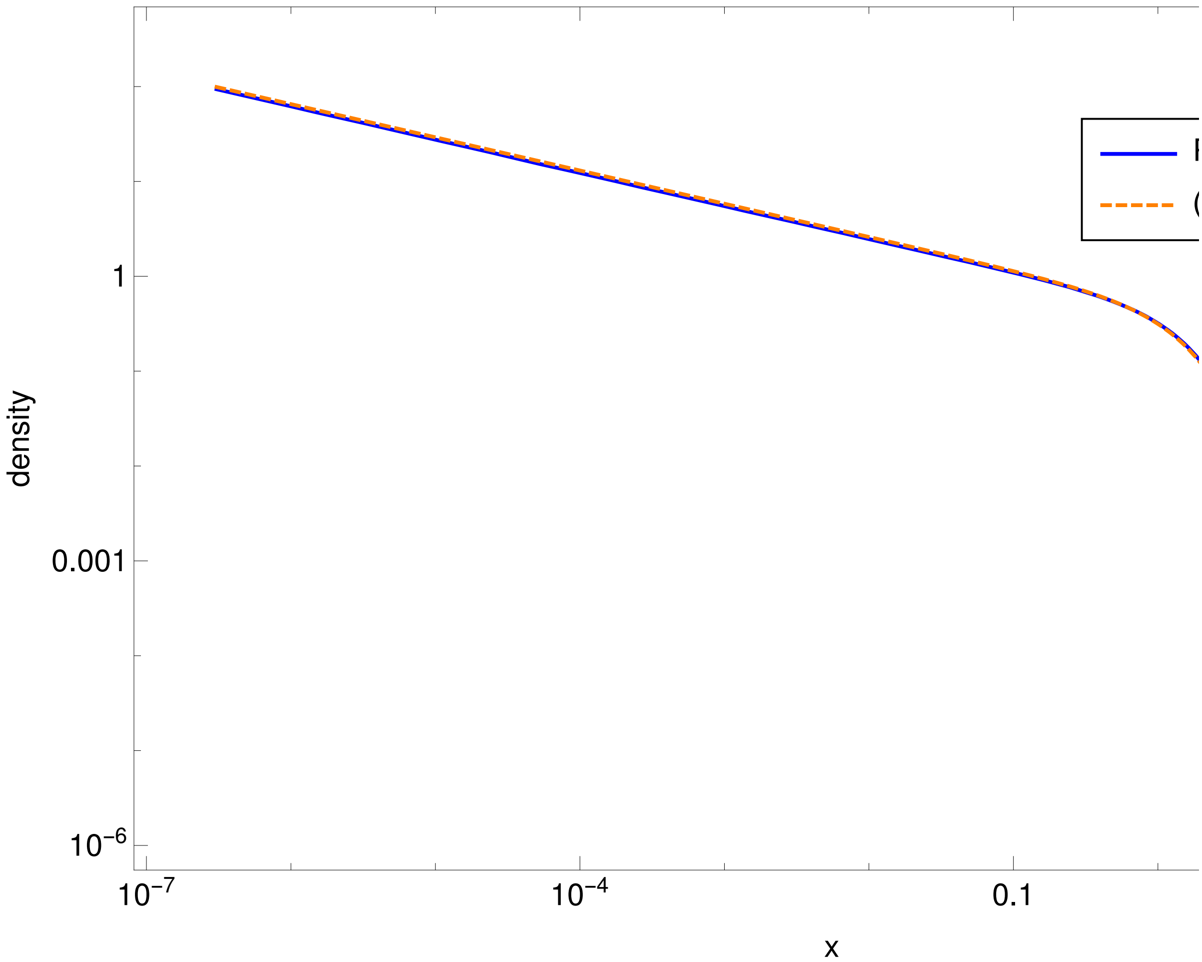
Part::partw : Part 1 of {} does not exist. ≫

Part::pkspec1 : The expression {}⟦1⟧ cannot be used as a part specification. ≫



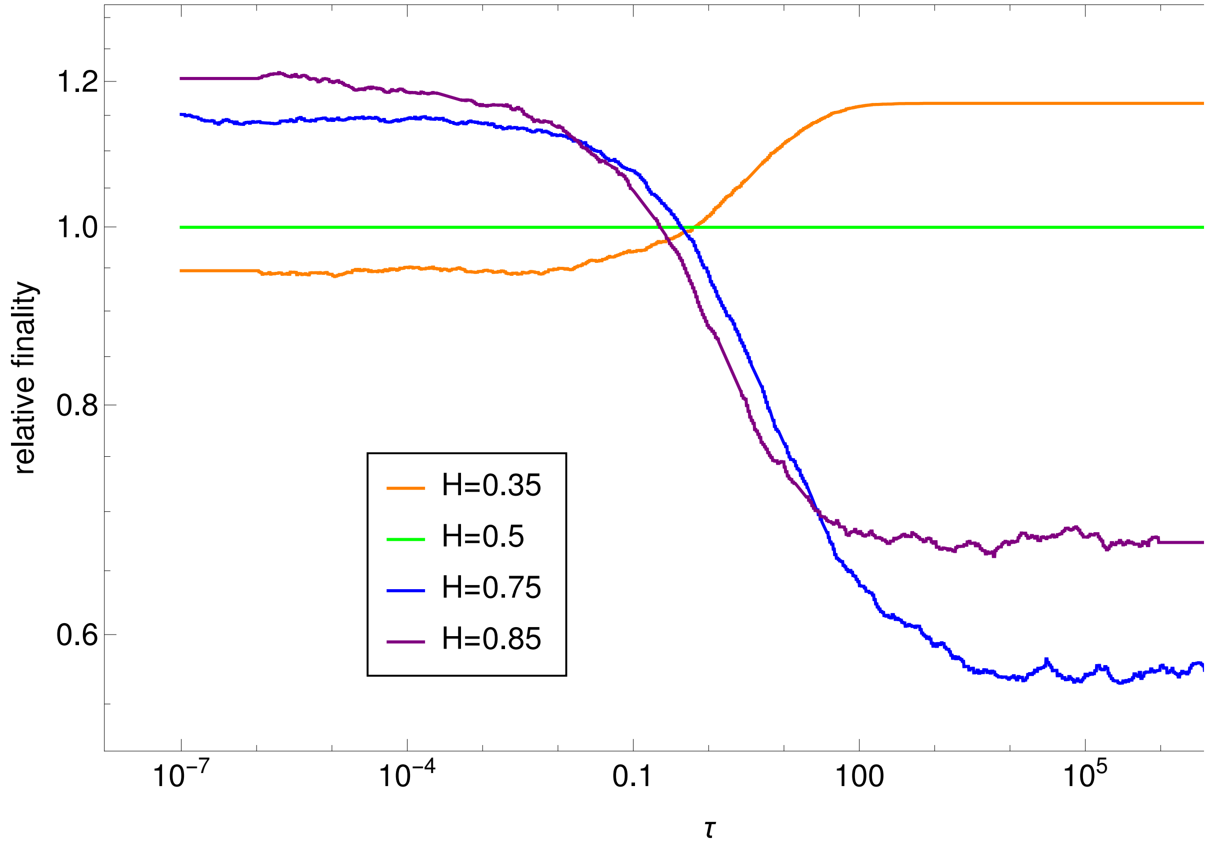
In[126]:=
```
LogLogPlot[{exitpfunc4[s, H], lastxdensity[s, x0, H, 1]},
  {s, xx4[[1]], 100}, ImageSize → {800, 600},
  PlotStyle → {{Blue, Thick}, {Orange, Dashed, Thick}}, PlotLegends →
   {Placed[LineLegend[Style → {{Orange, Directive[16]}, {Blue, Directive[16]}},
      {"Proposition 5.5", "(43)"}, LegendFunction → (Framed[#] &)], {0.8, 0.8}]},
  Frame → True, FrameLabel → {"x", "density"}, LabelStyle → Directive[16]]
```

Out[126]=



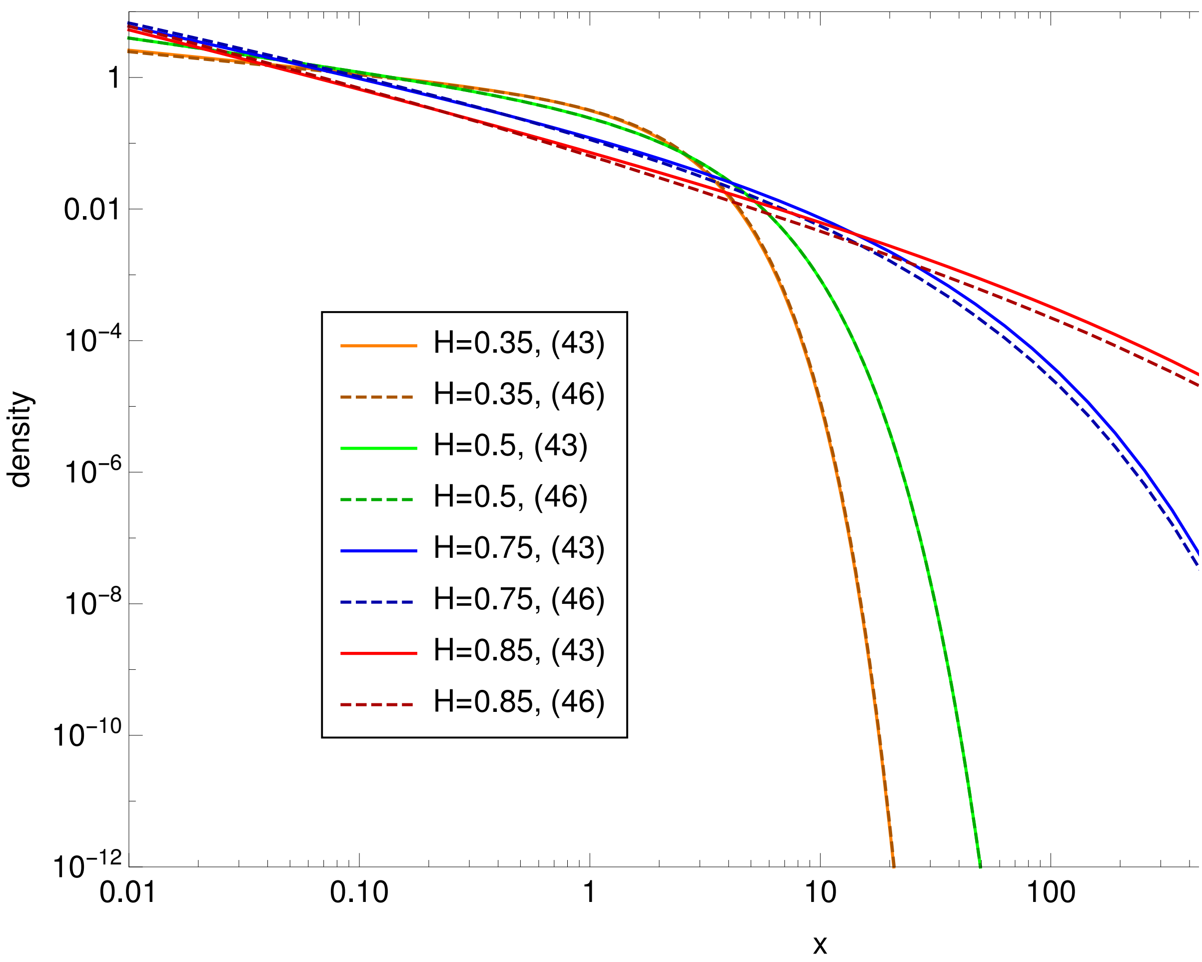
3.7 Relative Finality Curves

```
LogLogPlot[{relfinality4[x, 0.35], relfinality2[x, 0.5],
  relfinality[x, 0.75], relfinality3[x, 0.85]}, {x, 0.0000001, 10 000 000},
 PlotStyle → {Orange, Green, Blue, Purple, Magenta},
 PlotLegends → {Placed[LineLegend[{"H=0.35", "H=0.5", "H=0.75", "H=0.85"},
     LabelStyle → {Directive[16]}, LegendFunction → (Framed[#] &)], {0.3, 0.25}]},
 Frame → True, FrameLabel → {"τ", "relative finality"}, LabelStyle → Directive[16]]
```



3.8 Last Exit Densities

```
LogLogPlot[{If[s > 100, 0, lastxdensity[s, x0, 0.35, 1]],
  If[s > 100, 0, exitpfunc4[s, 0.35]], If[s > 100, 0, lastxdensity[s, x0, 0.5, 1]],
  If[s > 100, 0, exitpfunc2[s, 0.5]], lastxdensity[s, x0, 0.75, 1],
  exitpfunc[s, 0.75], lastxdensity[s, x0, 0.85, 1], exitpfunc3[s, 0.85]},
 {s, 0.01, 10 000}, ImageSize → {800, 600}, PlotStyle →
  {{Orange}, {Darker[Orange], Dashed}, {Green}, {Darker [Green], Dashed},
   {Blue}, {Darker[Blue] , Dashed}, {Red}, {Darker[Red] , Dashed}},
 PlotLegends → {Placed[LineLegend[{"H=0.35,  (43)", "H=0.35,  (46)", "H=0.5,  (43)",
      "H=0.5,  (46)", "H=0.75,  (43)", "H=0.75,  (46)", "H=0.85,  (43)",
      "H=0.85,  (46)"}, LegendFunction → (Framed[#] &)], {0.25, 0.4}]},
 Frame → True, FrameLabel → {"x", "density"}, LabelStyle → Directive[16],
 PlotRange → {{0.01, 10 000}, {10^-12, 10}}]
```



⋮

3.9 Check on last exit density

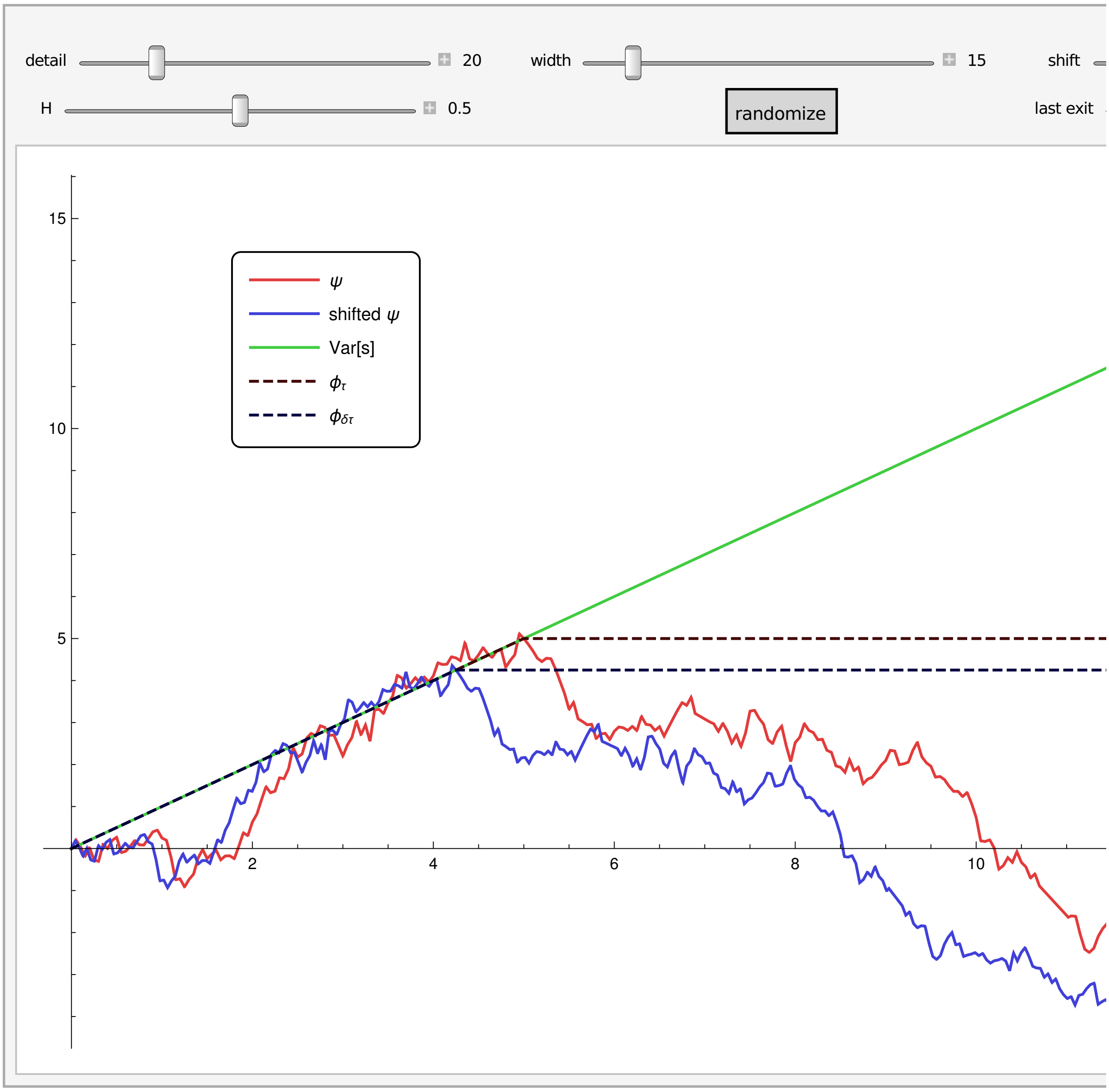
```
Integrate[lastxdensity[x, x0, H, 1], {x, 0, ∞}]
```

```
1.
```

4. Example for Ramer Paper

The following animation has been used to create an illustration in ramer.pdf.

```
Clear[λ, κ, s, H, δ, τ]; Manipulate[BlockRandom[SeedRandom[rand];
  T0 = Join[Table[s, {s, Ceiling[λ τ + 1 / 2] / λ, κ, 1 / λ}],
    Table[s, {s, Ceiling[λ τ - 1] / λ, 1 / λ, -1 / λ}]];
  T1 = Join[{τ}, T0];
  T2 = δ T1;
Z = RandomVariate[nd, Length[T0]];
  U = chol[T1, H];
  V = chol[T2, H];
  Z1 = Join[{Var[τ, H] / (τ Inverse[U][[1]][[1]])}, Z];
  Z2 = Join[{Var[δ τ, H] / (δ τ Inverse[V][[1]][[1]])}, Z];
f1 = fBm[T1, H, Z1, U]; f2 = fBm[T2, H, Z2, V];
Plot[{f1[s], f2[s], Var[s, H],
    Var[τ, H] phi[s, τ, H] / τ, Var[δ τ, H] phi[s, δ τ, H] / (δ τ)}, {s, 0, κ},
   PlotLegends → Placed[LineLegend[{"ψ", "shifted ψ", "Var[s]", "ϕτ", "ϕδτ"},
      LegendFunction → (Framed[#, RoundingRadius → 5] &), LegendMargins → 5],
     {0.2, 0.8}], ImageSize → 800, PlotStyle → {RGBColor[0.89, 0.23, 0.23],
     RGBColor[0.25, 0.25, 0.85], RGBColor[0.25, 0.8, 0.25],
     {Dashed, RGBColor[0.25, 0, 0]}, {Dashed, RGBColor[0, 0, 0.25]}}]],
 Grid[{{Control[{{λ, 20, "detail"}, 1.1, 100, 1.14, Appearance → "Labeled"}],
    Control[{{κ, 15, "width"}, 5, 100, 1, Appearance → "Labeled"}],
    Control[{{δ, 0.85, "shift"}, 0.25, 4, 0.1, Appearance → "Labeled"}]},
   {Control[{{H, 0.5, "H"}, 0.01, 0.99, 0.01, Appearance → "Labeled"}], Control[
     {{rand, 0, ""}, Button["randomize", rand = RandomInteger[2^64 - 1]] &}],
    Control[{{τ, 5, "last exit"}, 1, 10, 0.85, Appearance → "Labeled"}]}}],
 SaveDefinitions → True
]
```


detail
20
width
15
shift
H
0.5
randomize
last exit
15
10
5
2
4
6
8
10
ψ
shifted ψ
Var[s]
ϕτ
ϕδτ