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THE EFFECT OF RHEOLOGY WITH GAS BUBBLES ON LINEAR ELASTIC WAVES IN FLUID-SATURATED GRANULAR MEDIA

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Elastic waves in fluid-saturated granular media depend on the grain rheology, which can be complicated by the presence of gas bubbles. We investigated the effect of the bubble dynamics and their role in rheological scheme, on the linear Frenkel-Biot waves of P1 type. For the wave with the bubbles the scheme consists of three segments representing the solid continuum, fluid continuum and bubbles surrounded by the fluid. We derived the Nikolaevskiy-type equation describing the velocity of the solid matrix in the moving reference system. The equation is linearized to yield the decay rate λ as a function of the wave number k . We compared the $\lambda(k)$ -dependence for the cases with and without the bubbles, using typical values of the input mechanical parameters. For both the cases, the $\lambda(k)$ curve lies entirely below zero, which implies a global decay of the wave. We found that the increase of the radius of the bubbles leads to a faster decay, while the increase in the number of the bubbles leads to slower decay of the wave.

Key words: Frenkel-Biots waves, bubbles, rheology, porous media.

1. Introduction

The fundamentals of the theory of wave propagation in porous elastic solids can be found in [1, 2] or, for a more recent review, [3]. In [1, 2] Biot generalised the first principles of linear elasticity and today, most studies in acoustics, geophysical and geological mechanics rely on his theory. Biot also deduced [4, 5] the dynamical equations for the wave propagation in poroelastic media. The elastic modulus of the porous matrix with first-order nonlinearity was described in [6]. According to the Frenkel-Biots theory, there are two types of longitudinal waves propagating in a saturated porous medium. The first type is fast and weakly damped (P1-wave), whereas the other type is slow and strongly damped (P2-wave). The slow P-wave has first been observed in a laboratory by Plona [7]. Yang *et al.* [8] showed that the dispersion of velocity and attenuation of the fast P1-wave are both affected by the viscoelasticity of the medium, but has almost no effect on the slow P2-wave. In addition, they proved that the dominant frequency of the fast P1-wave shifts linearly toward lower frequencies due to the conditions of low permeability and low porosity; this plays a significant role in exploration for gas and oil.

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Nikolaevskiy [9] derived a model describing the propagation of nonlinear longitudinal waves in a viscoelastic medium taking into account complicated rheology of grains

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = \sum_{p=1}^5 A_{p+1} \frac{\partial^{p+1} v}{\partial x^{p+1}} \tag{1.1}$$

where v is the velocity of the solid matrix and the coefficients A_{p+1} are constants linked to mechanical parameters of the system. The terms in Eq.(1.1) account for the effects of non-linearity, dissipation and dispersion. Based on [9] Nikolaevskiy extended [10] the evolution Eq.(1.1) to include the global dissipation

$$\frac{\partial v}{\partial t} + Nv \frac{\partial v}{\partial x} + \zeta v = \sum_{p=1}^5 A_{p+1} \frac{\partial^{p+1} v}{\partial x^{p+1}} \tag{1.2}$$

where ζ and N are constants. Experimental evidence indicated that the existence of gas bubbles in the saturated porous medium changes the characteristics of this medium [11]–[13], acoustic properties and the velocity of the wave. As Nikolaevskiy pointed out [10], in rocks saturated with fluids, the P1-wave is the only observable wave. However, the presence of gas, even in small proportion can affect the wave type [14], so that P2-wave may be visible.

Dunin *et al.* studied [11] the effect of gas bubbles on the P1- and P2-waves. They found that the dispersion curve of the P2-wave consists of two branches: a low-frequency branch and a high-frequency branch. In this work, a simple stress-strain relation was used, $\sigma = Ee$ (in standard notations). Nikolaevskiy used a much more complicated stress-strain relation that involves higher-order time derivatives of the stress σ and strain e . This relation is the result of the rheological scheme shown in Fig.1. Eventually, it leads to the higher-order partial differential Eq.(1.1). However, the original rheological scheme [9] does not include gas bubbles. Nikolaevskiy and Strunin [14] pointed out the place in this scheme that the bubbles should take, see Fig.1. In the present work, we aim to include the bubble into the rheological scheme and, based on this, derive the Nikolaevskiy-type Eq.(1.1), where the coefficients A_p will depend on the bubble-related parameters.

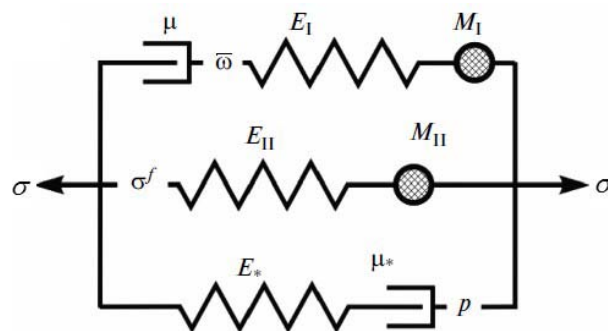


Fig.1. Rheological scheme for the grain. The branch ϖ corresponds to the bubble [14].

2. Basic equations of one-dimensional dynamics

2.1. Conservation of mass and momentum

For a one-dimensional case the momentum and mass balance equations are

$$\begin{aligned}
\frac{\partial}{\partial t}(I-m)\rho^{(s)}v + \frac{\partial}{\partial x}(I-m)\rho^{(s)}vw &= \frac{\partial}{\partial x}\sigma^{(ef)} - (I-m)\frac{\partial p}{\partial x} - I, \\
\frac{\partial}{\partial t}m\rho^{(f)}u + \frac{\partial}{\partial x}m\rho^{(f)}uu &= -m\frac{\partial p}{\partial x} + I, \\
\frac{\partial}{\partial t}(I-m)\rho^{(s)} + \frac{\partial}{\partial x}(I-m)\rho^{(s)}v &= 0, \\
\frac{\partial}{\partial t}m\rho^{(f)} + \frac{\partial}{\partial x}m\rho^{(f)}u &= 0
\end{aligned} \tag{2.1}$$

where the subscripts s and f label the solid and gas-liquid mixture, respectively, ρ , v , and u are the corresponding densities and mass velocities, m is the porosity, $\sigma^{(ef)}$ is the effective Terzaghi stress, p is the pore pressure, and I is the interfacial viscous force approximated by

$$I = \delta m(v - u), \quad \delta = \frac{\mu^{(f)}m}{k}$$

where $\mu^{(f)}$ is the gas-liquid mixture viscosity and k is the intrinsic permeability.

2.2. Dynamics of bubbles

The equation of the dynamics of a bubble [11] has the form

$$R\frac{\partial^2}{\partial t^2}R + \frac{3}{2}\left(\frac{\partial}{\partial t}R\right)^2 + \frac{4\mu}{\rho^{(L)}}\left(\frac{I}{R} + \frac{m}{4k}R\right)\frac{\partial}{\partial t}R = (p_g - p)/\rho^{(L)} \tag{2.2}$$

where R is the bubble radius, p is the pressure in the liquid, $p_g = (R_0/R)^\chi$ is the gas pressure inside the bubble (here $\chi = 3\zeta$, ζ is the adiabatic exponent), $\rho^{(L)}$ is the density of the liquid without the bubbles, and μ is the viscosity of the liquid without the bubbles. The density equations for the solid and liquid without gas are

$$\rho^{(s)} = \rho_0^{(s)}(I - \beta^{(s)}\sigma) = \rho_0^{(s)}\left[I + \beta^{(s)}p - \frac{\beta^{(s)}\sigma^{(ef)}}{I-m}\right], \tag{2.3}$$

$$\approx \rho_0^{(s)}\left[I + \beta^{(s)}p - \beta^{(s)}\sigma^{(ef)}\right],$$

$$\rho^{(L)} = \rho_0^{(L)}(I + \beta^{(L)}p). \tag{2.4}$$

The mean density of the gas-liquid mixture is

$$\rho^{(f)} = (I - \phi)\rho^{(L)} + \phi\rho^{(g)} \tag{2.5}$$

where $\phi = (4\pi/3)R^3n_0$.

Here σ is the true stress, ϕ is the volume gas content and n_0 is the number density of the bubbles per unit volume. In Eq.(2.5) we can neglect the density of the gas $\rho^{(g)}$ due to the low gas content. The change in ϕ is due to the change in the bubble radius R . Then Eq.(2.5) becomes

$$\rho^{(f)} = \rho_0^L \left(I + \beta^{(L)} p \right) \left(I - \frac{4\pi}{3} R_0^3 n_0 \right). \quad (2.6)$$

Similarly to [15], we also assume that the pore pressure p is equal to the pressure in the liquid far from the bubble.

2.3. Stress-strain relation

In this section we derive the stress-strain relation for the viscoelastic medium based on the rheological Maxwell-Voigt model, which includes the gas bubble. The model includes two friction elements with viscosities μ_1 and μ_2 , three elastic springs with the elastic moduli E_1 , E_2 , and E_3 , and three oscillating masses M_1 , M_2 , and M_3 . The total stress is denoted σ . We also denote the displacements of the elements of the model by e with respective subscripts as shown in Fig.2. Now we write the second Newton's law for the elements and the kinematic relations

$$\begin{aligned} M_1 \frac{d^2 e_1}{dt^2} + M_2 \frac{d^2 e_2}{dt^2} &= \sigma - E_1 e_1 - E_2 e_2, \\ e &= e_2 = e_1 + e_3 + e_4 + e_5, \\ M_3 \frac{d^2 e_3}{dt^2} &= E_1 e_1 - E_3 e_3, \\ E_3 e_3 &= \mu_2 \frac{de_4}{dt} = \mu_1 \frac{de_5}{dt}. \end{aligned} \quad (2.7)$$

Equations (2.7) generate the following relation between the stress and strain

$$\begin{aligned} \left[E_1 E_3 \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \right] \sigma + (E_1 + E_3) \frac{d\sigma}{dt} + M_3 \frac{d^3 \sigma}{dt^3} &= \left[E_1 E_2 E_3 \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \right] e + \\ + \left[((E_1 + E_2) E_3 + E_1 E_2) \right] \frac{de}{dt} + \left[E_1 E_3 M_2 \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \right] \frac{d^2 e}{dt^2} + \left[((E_1 + E_2) M_3 + \right. & \\ \left. + (E_3 + E_1) M_2) + E_3 M_1 \right] \frac{d^3 e}{dt^3} + \left[(M_1 + M_2) M_3 \right] \frac{d^5 e}{dt^5}. \end{aligned} \quad (2.8)$$

Generalizing Eq.(2.8) using a similar approach to [9], we get

$$\sigma^{(ef)} + \eta \sum_{q=1,3} b_q \frac{D^q \sigma^{(ef)}}{Dt^q} = E_2 e + \beta^{(s)} k_b p + \eta \sum_{q=1,2,3,5} a_q \frac{D^q e}{Dt^q}, \quad (2.9)$$

where $\sigma^{(ef)}$, is the effective stress, k_b is the bulk elastic module of the porous matrix,

$$\eta = \left(E_1 E_3 \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \right)^{-1}$$

and the coefficients a_q and b_q are expressed as

$$\begin{aligned} a_1 &= [(E_2 + E_1) E_3 + E_1 E_2], & a_2 &= M_2, \\ a_3 &= [(E_2 + E_1) M_3 + (E_3 + E_1) M_2 + E_3 M_1], & a_5 &= [(M_2 + M_1) M_3], \\ b_1 &= (E_3 + E_1), & b_3 &= M_3. \end{aligned}$$

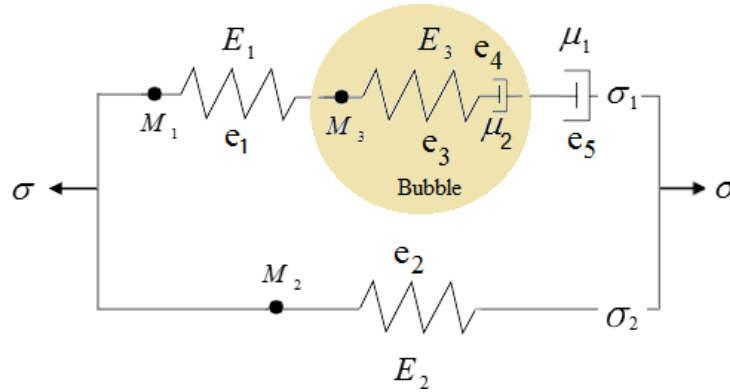


Fig.2. Rheological scheme including a gas bubble.

Finally, we add the closing relation between the deformation e and the velocity v of the solid

$$\frac{De}{Dt} \equiv \frac{\partial e}{\partial t} + v \frac{\partial e}{\partial x} = \frac{\partial v}{\partial x}. \tag{2.10}$$

3. Waves in saturated media including gas bubbles

Following the approach of Nikolaevskiy [10], we consider the P1-wave in porous media under full saturation. Accordingly, we assume that the mass velocities v and u have the same sign

$$v = u + O(\varepsilon v) \tag{3.1}$$

where ε is the small parameter. The Darcy force has the order as shown

$$I = \varepsilon^\gamma \delta m (v - u) = \varepsilon^\gamma \delta m v, \quad \delta = m \mu / kO(I). \tag{3.2}$$

Describing a weakly non-linear wave, we use the running coordinate system with simultaneous scale change

$$\varepsilon = \varepsilon^\alpha (x - ct), \quad \tau = \frac{1}{2} \varepsilon^\beta t, \tag{3.3}$$

$$\frac{\partial}{\partial x} = \varepsilon^\alpha \frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial t} = \varepsilon^\alpha \left(\frac{1}{2} \varepsilon^{\beta-\alpha} \frac{\partial}{\partial \tau} - c \frac{\partial}{\partial \xi} \right).$$

Thus, the constitutive law (2.9) transforms into the following form

$$\begin{aligned} \sigma^{(ef)} + \eta \sum_{q=1,3} b_q \varepsilon^{q\alpha} \left(\frac{1}{2} \varepsilon^{\beta-\alpha} \frac{\partial}{\partial \tau} + (v-c) \frac{\partial}{\partial \xi} \right)^q \sigma^{(ef)} = E_2 e + \beta^{(s)} k_b p + \\ + \eta \sum_{q=1,2,3,5} a_q \varepsilon^{q\alpha} \left(\frac{1}{2} \varepsilon^{\beta-\alpha} \frac{\partial}{\partial \tau} + (v-c) \frac{\partial}{\partial \xi} \right)^q e. \end{aligned} \quad (3.4)$$

Now, we seek the unknown functions as power series

$$\begin{aligned} v &= \varepsilon v_1 + \varepsilon^2 v_2 + \dots, \\ u &= \varepsilon u_1 + \varepsilon^2 u_2 + \dots, \\ \sigma^{(ef)} &= \sigma_0^{(ef)} + \varepsilon \sigma_1^{(ef)} + \varepsilon^2 \sigma_2^{(ef)} + \dots, \\ p &= p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \dots, \\ m &= m_0 + \varepsilon m_1 + \varepsilon^2 m_2 + \dots, \\ e &= e_0 + \varepsilon e_1 + \varepsilon^2 e_2 + \dots, \\ \phi &= \phi_0 + \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \dots, \\ R &= R_0 \left(1 + \varepsilon R_1 + \varepsilon^2 R_2 + \dots \right). \end{aligned} \quad (3.5)$$

3.1. The first approximation

By assuming $\beta = \alpha + 1$, $\alpha = 1$ and using Eqs (3.5), we collect the linear terms $\sim \varepsilon$ in system (2.1)

$$\begin{aligned} -(1-m_0) \rho_0^{(s)} c \frac{\partial v_1}{\partial \xi} &= \frac{\partial \sigma_1^{(ef)}}{\partial \xi} - (1-m_0) \frac{\partial p_1}{\partial \xi}, \\ -m_0 \rho_0^{(f)} c \frac{\partial u_1}{\partial \xi} &= -m_0 \frac{\partial p_1}{\partial \xi}, \\ \rho_0^{(s)} c \frac{\partial m_1}{\partial \xi} - (1-m_0) c \frac{\partial \rho_1^{(s)}}{\partial \xi} + (1-m_0) \rho_0^{(s)} \frac{\partial v_1}{\partial \xi} &= -\frac{1}{2} (1-m_0) \frac{\partial \rho_0^{(s)}}{\partial \tau}, \\ -m_0 c \frac{\partial \rho_1^{(f)}}{\partial \xi} - \rho_0^{(f)} c \frac{\partial m_1}{\partial \xi} + m_0 \rho_0^{(f)} \frac{\partial u_1}{\partial \xi} &= -\frac{1}{2} m_0 \frac{\partial \rho_0^{(f)}}{\partial \tau}. \end{aligned} \quad (3.6)$$

The system (3.6) gives the integrals

$$\begin{aligned}
 (1 - m_0)\rho_0^{(s)}cv_I &= -\sigma_I^{(ef)} + (1 - m_0)p_I, \\
 m_0\rho_0^{(f)}cu_I &= m_0p_I, \\
 (1 - m_0)\rho_0^{(s)}v_I &= ((1 - m_0)\rho_I^{(s)} - \rho_0^{(s)}m_I)c, \\
 m_0\rho_0^{(f)}u_I &= (\rho_0^{(f)}m_I + m_0\rho_I^{(f)})c.
 \end{aligned}
 \tag{3.7}$$

According to Eqs (2.3) and (2.6) the terms $\sim \varepsilon$ in the density series are

$$\rho_I^{(s)} = \rho_0^{(s)} \left(\beta^{(s)}p_I - \frac{\beta^{(s)}\sigma_I^{(ef)}}{(1 - m_0)} \right),
 \tag{3.8}$$

$$\rho_I^{(f)} = \rho_0^{(L)} (\beta^{(L)}\kappa_1p_I - 4\pi n_0\kappa_2R_0^3R_I),$$

and also

$$\rho_0^{(f)} = \kappa_1\kappa_2\rho_0^{(L)}
 \tag{3.9}$$

where $\kappa_1 = 1 - \frac{4\pi}{3}R_0^3n_0$, $\kappa_2 = 1 + \beta^{(L)}p$.

Inserting Eqs (3.8) and (3.9) into the last two Eqs in (3.7) (mass equations) we get

$$(1 - m_0)v_I = \left[(1 - m_0)\beta^{(s)}p_I - \beta^{(s)}\sigma_I^{(ef)} - m_I \right] c,
 \tag{3.10}$$

$$m_0u_I = \left[m_I + \frac{m_0\beta^{(L)}p_I}{\kappa_2} - \frac{4\pi n_0m_0R_0^3R_I}{\kappa_1} \right].
 \tag{3.11}$$

The combination of Eqs (3.10) and (3.11) gives

$$(1 - m_0)v_I + m_0u_I = \left[\frac{(\beta + (1 - m_0)\beta^{(s)}\beta^{(L)}p_0)p_I}{\kappa_2} - \beta^{(s)}\sigma_I^{(ef)} - \frac{4\pi n_0m_0R_0^3R_I}{\kappa_1} \right] c.
 \tag{3.12}$$

The condition (3.1) means $v_I = u_I$, therefore Eq.(3.12) becomes

$$v_I = \left[\frac{(\beta + (1 - m_0)\beta^{(s)}\beta^{(L)}p_0)p_I}{\kappa_2} - \beta^{(s)}\sigma_I^{(ef)} - \frac{4\pi n_0m_0R_0^3R_I}{\kappa_1} \right] c.
 \tag{3.13}$$

Due to the conditions $v_I = u_I$, $\rho_0 = (1 - m_0)\rho_0^{(s)} + m_0\rho_0^{(L)}$ and using Eq.(3.9), the first two of the momentum Eqs (3.7) give

$$\rho_0 c v_l = -\sigma_l^{(ef)} + A p_l, \quad (3.14)$$

where $A = (I - m_0) + \frac{m_0}{\kappa_1 \kappa_2}$. Now, the linear terms $\sim \varepsilon$ in relations (2.10) and (3.4) give

$$\frac{I}{2} \frac{\partial e_0}{\partial \tau} - c \frac{\partial e_l}{\partial \xi} + v_l \frac{\partial e_0}{\partial \xi} = \frac{\partial v_l}{\partial \xi}, \quad (3.15)$$

$$\sigma_l^{(ef)} - E_2 e_l - \beta^{(s)} k_b p_l = T, \quad (3.16)$$

where

$$T \equiv \eta \left[\sum_{q=1,2,3,5} a_q (-c)^q \varepsilon^{q-1} \frac{\partial^q e_0}{\partial \xi^q} + \sum_{q=1,3} b_q c^q \varepsilon^{q-1} \frac{\partial^q \sigma_0^{(ef)}}{\partial \xi^q} \right].$$

The linear terms $\sim \varepsilon$ in the bubble Eq.(2.2) give

$$-\frac{\mu c}{\rho_0^{(L)} \kappa_2} \left[\frac{4}{R_0} + \frac{m_0 R_0}{k} \right] \frac{\partial R_0}{\partial \xi} = -\frac{I}{\rho_0^{(L)} \kappa_2} (p_0 \chi R_l + p_l). \quad (3.17)$$

Equations (3.15), (3.16) and (3.17) lead to the integrals

$$e_l = -\frac{v_l}{c}, \quad \sigma_l^{(ef)} = E_2 e_l + \beta^{(s)} k_b p_l, \quad p_l = -p_0 \chi R_l. \quad (3.18)$$

The effective stress $\sigma_l^{(ef)}$ in Eq.(3.18) can be rewritten as

$$\sigma_l^{(ef)} = -\left[\frac{E_2 v_l}{c} + p_0 \chi \beta^{(s)} k_b R_l \right]. \quad (3.19)$$

Substituting (3.19) and the value of p_l from Eq.(3.18) into (3.13), leads to

$$(I - \beta^{(s)} E_2) v_l + (B - \beta^{(s)} \beta^{(s)} k_b) p_0 \chi R_l c = 0, \quad (3.20)$$

where

$$B = \frac{(\beta + (I - m_0) \beta^{(s)} \beta^{(L)} p_0)}{\kappa_2} + \frac{4\pi n_0 m_0 R_0^3}{\kappa_1 p_0 \chi}.$$

Now, from Eq.(3.14) and using the value of p_l from Eq.(3.18), we obtain the effective stress as

$$\sigma_l^{(ef)} = -(p_0 c v_l + A) p_0 \chi R_l. \quad (3.21)$$

The combination of Eqs (3.19) and (3.21) results in

$$(E_2 - p_0 c^2) v_l - (A - \beta^{(s)} k_b) p_0 \chi R_l c = 0. \quad (3.22)$$

Equations (3.20) and (3.22) must coincide, therefore

$$\begin{vmatrix} (1-\beta^{(s)}E_2) & (B-\beta^{(s)}\beta^{(s)}k_b)p_0\chi \\ (E_2-p_0c^2) & (A-\beta^{(s)}k_b)p_0\chi \end{vmatrix} = 0. \tag{3.23}$$

Equation (3.23) gives the velocity of the wave

$$c^2 = \frac{(A-\beta^{(s)}k_b)Z + E_2}{\rho_0}, \tag{3.24}$$

where

$$Z = \frac{(1-\beta^{(s)}E_2)}{B-\beta^{(s)}\beta^{(s)}k_b}.$$

Thus, all the variables are expressed through any one selected variable, for example, the velocity v_l

$$\begin{aligned} e_l &= -\frac{v_l}{c}, & \sigma_l^{(ef)} &= -(E_2 - \beta^{(s)}k_b Z)\frac{v_l}{c}, & p_l &= Z\frac{v_l}{c}, \\ R_l &= -\frac{Z}{p_0\chi}\frac{v_l}{c}, & m_l &= \left[((1-m_0) - \beta^{(s)}k_b)\beta^{(s)}Z + \beta^{(s)}E_2 - (1-m_0) \right]\frac{v_l}{c}, \\ \rho_l^{(f)} &= \rho_0^{(L)}Z\left(\beta^{(L)}\kappa_l + \frac{4\pi n_0\kappa_2 R_0^3}{p_0\chi}\right)\frac{v_l}{c}, & \rho_l^{(s)} &= \rho_0^{(s)}\beta^{(s)}\left(Z(1-\beta^{(s)}k_b) + E_2\right)\frac{v_l}{c}. \end{aligned} \tag{3.25}$$

3.1. The second approximation

Collecting the quadratic terms $\sim \varepsilon^2$ in Eq.(3.4), we get

$$\sigma_2^{(ef)} - E_2 e_2 - \beta^{(s)}k_b p_2 = T, \tag{3.26}$$

where

$$T \equiv \eta \left[\sum_{q=1,2,3,5} a_q (-c)^q \varepsilon^{q-1} \frac{\partial^q e_l}{\partial \xi^q} + \sum_{q=1,3} b_q c^q \varepsilon^{q-1} \frac{\partial^q \sigma_l^{(ef)}}{\partial \xi^q} \right].$$

Note that here we keep (as Nikolaevskiy did in [10]) higher powers of ε to represent small corrections to the leading terms. These corrections will eventually translate into small corrections in the Nikolaevskiy equation derived further in this paper; they will be the object of our study. Thus

$$\frac{\partial \sigma_2^{(ef)}}{\partial \xi} - E_2 \frac{\partial e_2}{\partial \xi} - \beta^{(s)}k_b \frac{\partial p_2}{\partial \xi} = \frac{\partial T}{\partial \xi}. \tag{3.27}$$

From Eq.(2.10) in the order $\sim \varepsilon^2$, we get

$$\frac{\partial}{\partial \xi}(ce_2 + v_2) = F, \quad (3.28)$$

$$F = -\frac{I}{2} \left(\frac{1}{2} \frac{\partial v_l}{\partial \tau} + \frac{\partial v_l v_l}{\partial \xi} \right).$$

Therefore

$$\frac{\partial e_2}{\partial \xi} = \frac{F}{c} - \frac{1}{c} \frac{\partial v_2}{\partial \xi}. \quad (3.29)$$

Substituting Eq.(3.29) into Eq.(3.27), we obtain

$$\frac{\partial}{\partial \xi} \left(c\sigma_2^{(ef)} + E_2 v_2 - c\beta^{(s)} k_b p_2 \right) = E_2 F + c \frac{\partial T}{\partial \xi}. \quad (3.30)$$

From the momentum Eq.(2.1) for the solid and liquid, we get

$$(1 - m_0) \rho_0^{(s)} c \frac{\partial v_2}{\partial \xi} + \frac{\partial \sigma_2^{(ef)}}{\partial \xi} - (1 - m_0) \frac{\partial p_2}{\partial \xi} = \Sigma_1, \quad (3.31)$$

where

$$\Sigma_1 = (1 - m_0) \rho_0^{(s)} \left(\frac{1}{2} \frac{\partial v_l}{\partial \tau} + \frac{\partial v_l v_l}{\partial \xi} \right) - (1 - m_0) \rho_l^{(s)} c \frac{\partial v_l}{\partial \xi} + m_l \rho_0^{(s)} c \frac{\partial v_l}{\partial \xi} - m_l \frac{\partial p_l}{\partial \xi} + \varepsilon^{\gamma-1} \delta m_0 v_l$$

and

$$m_0 \rho_0^{(f)} c \frac{\partial u_2}{\partial \xi} - m_0 \frac{\partial p_2}{\partial \xi} = \Sigma_2, \quad (3.32)$$

where

$$\Sigma_2 = m_0 \rho_0^{(f)} \left(\frac{1}{2} \frac{\partial u_l}{\partial \tau} + \frac{\partial u_l u_l}{\partial \xi} \right) - m_0 \rho_l^{(f)} c \frac{\partial u_l}{\partial \xi} - m_l \rho_0^{(f)} c \frac{\partial u_l}{\partial \xi} + m_l \frac{\partial p_l}{\partial \xi} - \varepsilon^{\gamma-1} \delta m_0 u_l.$$

Due to condition (3.1), the combination of Eq.(3.31) with (3.32) give

$$\rho_0 c \frac{\partial v_2}{\partial \xi} + \frac{\partial \sigma_2^{(ef)}}{\partial \xi} - \frac{\partial p_2}{\partial \xi} = \Sigma, \quad (3.33)$$

where $\Sigma = \Sigma_1 + \Sigma_2$, so that

$$\begin{aligned} \Sigma = & \rho_0 \left(\frac{1}{2} \frac{\partial v_l}{\partial \tau} + \frac{\partial v_l v_l}{\partial \xi} \right) - c \left((1 - m_0) \rho_0^{(s)} \beta^{(s)} + m_0 \rho_0^{(L)} \beta^{(L)} \kappa_l \right) \frac{\partial p_l v_l}{\partial \xi} + c \rho_0^{(s)} \beta^{(s)} \frac{\partial \sigma_l^{(ef)} v_l}{\partial \xi} + \\ & + c \rho_0^{(L)} 4\pi n_0 m_0 \kappa_2 R_0^3 \frac{\partial R_l v_l}{\partial \xi} + c \left(\rho_0^{(s)} - \kappa_l \kappa_2 \rho_0^{(L)} \right) \frac{\partial m_l v_l}{\partial \xi}. \end{aligned}$$

Equations (3.30) and (3.33) result in

$$\frac{\partial}{\partial \xi} \left[\left(E_2 - p_0 c^2 \right) v_2 + \left(1 - \beta^{(s)} k_b \right) c p_2 \right] = E_2 F - c \Sigma + c \frac{\partial T}{\partial \xi}. \quad (3.34)$$

From the bubble Eq.(2.2), in the order $\sim \varepsilon^2$,

$$-\frac{\mu c}{\rho_0^{(L)} \kappa_2} \left[4 + \frac{m_0 R_0^2}{k} \right] \frac{\partial R_l}{\partial \xi} = \frac{I}{\rho_0^{(L)} \kappa_2} \left[\frac{\beta^{(L)} p_0 \chi}{\kappa_2} p_l R_l + \frac{\beta^{(L)}}{\kappa_2} p_l^2 + \right. \\ \left. + \frac{p_0 \chi (\chi + I)}{2} R_l^2 - p_0 \chi R_2 - p_2 \right]. \tag{3.35}$$

We re-write Eq.(3.35) as

$$p_2 = \Gamma - p_0 \chi R_2, \tag{3.36}$$

where
$$\Gamma = \mu c \left(4 + \frac{m_0 R_0^2}{k} \right) \frac{\partial R_l}{\partial \xi} + \frac{\beta^{(L)} p_l}{\kappa_2} (p_0 \chi R_l + p_l) + \frac{p_0 \chi (\chi + I)}{2} R_l^2.$$

Now we substitute the value of p_2 from Eq.(3.36) into Eq.(3.34) to get

$$\frac{\partial}{\partial \xi} \left[(E_2 - p_0 c^2) v_2 - (I - \beta^{(s)} k_b) p_0 \chi R_2 c \right] = E_2 F - c \Sigma + \\ + c \frac{\partial T}{\partial \xi} - c (I - \beta^{(s)} k_b) \frac{\partial \Gamma}{\partial \xi}. \tag{3.37}$$

In the second order, the mass balances (2.1) for the solid and liquid-gas mixture have the form

$$\frac{\partial}{\partial \xi} \left((I - m_0) v_2 - \left[(I - m_0) \beta^{(s)} p_2 - \beta^{(s)} \sigma_2^{(ef)} - m_2 \right] c \right) = \Lambda^{(s)} / \rho_0^{(s)}, \tag{3.38}$$

$$\frac{\partial}{\partial \xi} \left(m_0 u_2 - \left[m_2 + \frac{m_0 \beta^{(L)} p_2}{\kappa_2} - \frac{4 \pi n_0 m_0 R_0^3 (R_2 + R_l^2)}{\kappa_l} \right. \right. \\ \left. \left. - \frac{4 \pi n_0 m_0 R_0^3 p_0 \chi \beta^{(L)} R_l^2}{\kappa_l \kappa_2} \right] c \right) = \Lambda^{(L)} / \rho_0^{(L)}, \tag{3.39}$$

where

$$\Lambda^{(s)} = \rho_0^{(s)} \frac{I}{2} \frac{\partial}{\partial \tau} \left[(m_l - (I - m_0) \beta^{(s)} p_l + \beta^{(s)} \sigma_2^{(ef)}) \right] + \\ + \rho_0^{(s)} \frac{\partial}{\partial \xi} \left[m_l v_l - \left((I - m_0) p_l + \sigma_l^{(ef)} \right) \beta^{(s)} v_l - c \beta^{(s)} m_l \left(p_l - \frac{\sigma_l^{(ef)}}{(I - m_0)} \right) \right], \tag{3.40}$$

$$\Lambda^{(L)} = -\rho_0^{(L)} \frac{I}{2} \frac{\partial}{\partial \tau} \left[\kappa_l (m_l \kappa_2 + m_0 \beta^{(L)} p_l) - 4 \pi n_0 \kappa_2 R_0^3 R_l \right] + \\ + \rho_0^{(L)} \frac{\partial}{\partial \xi} \left[(\beta^{(L)} \kappa_l p_l - 4 \pi n_0 \kappa_2 R_0^3 R_l) (c m_l - m_0 u_l) \right] - \kappa_l \kappa_2 \rho_0^{(L)} \frac{\partial m_l u_l}{\partial \xi}. \tag{3.41}$$

The combination of Eqs (3.38) and (3.39) gives

$$\frac{\partial}{\partial \xi} \left(v_2 - \left[\frac{(\beta + (1 - m_0)\beta^{(s)}\beta^{(L)} p_0) p_2}{\kappa_2} - \beta^{(s)} \sigma_2^{(ef)} - \frac{4\pi n_0 m_0 R_0^3 (R_2 + R_1^2)}{\kappa_1} + \frac{4\pi n_0 m_0 R_0^3 p_0 \lambda \beta^{(L)} R_1^2}{\kappa_1 \kappa_2} \right] c \right) = \Lambda, \quad (3.42)$$

where $\Lambda = \Lambda^{(s)}/\rho_0^{(s)} + \Lambda^{(L)}/\rho_0^{(L)}$.

From Eq.(3.30) we have

$$\frac{\partial \sigma_2^{(ef)}}{\partial \xi} = \frac{\partial}{\partial \xi} \left(T + \beta^{(s)} k_b \Gamma - \beta^{(s)} k_b p_0 \lambda R_2 - \frac{E_2}{c} v_2 \right) + \frac{1}{c} E_2 F. \quad (3.43)$$

Now we insert Eq.(3.43) and the value of p_2 represented by Eq.(3.36) into Eq.(3.42)

$$\begin{aligned} & \frac{\partial}{\partial \xi} \left[(1 - E_2 \beta^{(s)}) v_2 - \left(\omega_1 p_0 \lambda - \frac{4\pi n_0 m_0 R_0^3}{\kappa_1} \right) R_2 c \right] = \\ & = \Lambda - \beta^{(s)} E_2 F - c \beta^{(s)} \frac{\partial T}{\partial \xi} - c \omega_1 \frac{\partial \Gamma}{\partial \xi} + c \omega_2 \frac{\partial R_1^2}{\partial \xi} \end{aligned} \quad (3.44)$$

where

$$\begin{aligned} \omega_1 &= k_b \beta^{(s)} \beta^{(s)} + \frac{(\beta + (1 - m_0)\beta^{(s)}\beta^{(L)} p_0)}{\kappa_2}, \\ \omega_2 &= \frac{4\pi n_0 m_0 R_0^3 \beta^{(L)} p_0 \lambda}{\kappa_1 \kappa_2} - \frac{4\pi n_0 m_0 R_0^3}{\kappa_1}. \end{aligned}$$

The determinant of the left-hand side of the system of Eqs (3.37) and (3.44) coincides with the determinant of Eq.(3.23), which equals zero. A non-zero solution for v_2 exists only if the following compatibility condition takes place

$$\begin{vmatrix} (E_2 - p_0 c^2) & \frac{\partial}{\partial \xi} \left(E_2 F - c \Sigma + c \frac{\partial T}{\partial \xi} - c (1 - \beta^{(s)} k_b) \frac{\partial \Gamma}{\partial \xi} \right) \\ (1 - \beta^{(s)} E_2) & \frac{\partial}{\partial \xi} \left(\Lambda - \beta^{(s)} E_2 F - c \beta^{(s)} \frac{\partial T}{\partial \xi} - c \omega_1 \frac{\partial \Gamma}{\partial \xi} + c \omega_2 \frac{\partial R_1^2}{\partial \xi} \right) \end{vmatrix} = 0. \quad (3.45)$$

(see Appendix). This is the evolution equation with respect to $v \cong v_1$

$$cM \frac{\partial \Gamma}{\partial \xi} - cN \frac{\partial T}{\partial \xi} + c\omega_2 \psi \frac{\partial R_1^2}{\partial \xi} + \Lambda \psi + c \Sigma (1 - E_2 \beta^{(s)}) - E_2 FN = 0, \quad (3.46)$$

where $\psi = (E_2 - p_0 c^2)$, $M = (1 - \beta^{(s)} k_b)(1 - E_2 \beta^{(s)}) - \omega_1 \psi$, $N = (1 - \beta^{(s)} \rho_0 c^2)$.

Now, we re-write Eq.(3.46) in terms of v and re-group

$$\begin{aligned} & \frac{I}{2} \left[Y_1 + \psi \left((I - \kappa_1 \kappa_2) \hat{m}_1 - E_2 \beta^{(s)} - \left(Y_2 + \frac{4\pi n_0 R_0^3}{p_0 \chi} \right) Z \right) \right] \frac{\partial v}{\partial \tau} + \\ & - \left[N \eta c^2 \left(a_1 - b_1 (E_2 - k_b \beta^{(s)} Z) \right) + \frac{M Z \mu c^2}{p_0 \chi} \left(4 + \frac{m_0 R_0^2}{k} \right) \right] \frac{\partial^2 v}{\partial \xi^2} + \varepsilon N \eta c^3 a_2 \frac{\partial^3 v}{\partial \xi^3} + \\ & - \varepsilon^2 N \eta c^4 \left[a_3 - b_3 (E_2 - k_b \beta^{(s)} Z) \right] \frac{\partial^4 v}{\partial \xi^4} - \varepsilon^4 N \eta a_5 c^6 \frac{\partial^6 v}{\partial \xi^6} - [\zeta_1 + \zeta_2] \frac{\partial v v}{\partial \xi} = 0, \end{aligned} \quad (3.47)$$

where

$$\hat{m}_1 = \left((I - m_0) - \beta^{(s)} k_b \right) \beta^{(s)} Z + \beta^{(s)} E_2 - (I - m_0), \quad Y_1 = \left(E_2 N + c^2 \rho_0 (I - E_2 \beta^{(s)}) \right),$$

$$Y_2 = m_0 \left(\kappa_1 \beta^{(L)} - \beta^{(s)} \right) + \beta^{(s)} \left(I - \beta^{(s)} k_b \right),$$

$$\begin{aligned} \zeta_1 = \psi \left(\hat{m}_1 - (I - m_0) \beta^{(s)} Z + \beta^{(s)} \left(E_2 - k_b \beta^{(s)} Z \right) (I - \hat{m}_1) - \beta^{(s)} Z \hat{m}_1 + \kappa_1 \beta^{(L)} Z \hat{m}_1 - \kappa_1 \kappa_2 + \right. \\ \left. + 4\pi n_0 \kappa_2 R_0^3 \frac{Z}{p_0 \chi} (\hat{m}_1 - m_0) - m_0 \kappa_1 \beta^{(L)} Z + \omega_2 \frac{Z^2}{(p_0 \chi)^2} \right), \end{aligned}$$

$$\begin{aligned} \zeta_2 = c^2 (I - E_2 \beta^{(s)}) \left(\rho_0 - \rho_0 \kappa_1 \beta Z - \rho_0^{(s)} \beta^{(s)} (E_2 - k_b \beta^{(s)} Z) - m_0 \kappa_2 \rho^{(L)} \frac{Z}{p_0 \chi} + \right. \\ \left. + \hat{m}_1 \left(\rho^{(s)} - \kappa_1 \kappa_2 \rho^{(L)} \right) \right) + \frac{M(\chi + I)}{2 p_0 \chi} Z^2 + E_2 N. \end{aligned}$$

In short, the evolution Eq.(3.47) can be written as

$$C_1 \frac{\partial v}{\partial \tau} - C_2 \frac{\partial^2 v}{\partial \xi^2} + \varepsilon C_3 \frac{\partial^3 v}{\partial \xi^3} - \varepsilon^2 C_4 \frac{\partial^4 v}{\partial \xi^4} - \varepsilon^4 C_6 \frac{\partial^6 v}{\partial \xi^6} - \zeta \frac{\partial v v}{\partial \xi} = 0, \quad (3.48)$$

where

$$C_1 = \frac{I}{2} \left[Y_1 + \psi \left((I - \kappa_1 \kappa_2) \hat{m}_1 - E_2 \beta^{(s)} - \left(Y_2 + \frac{4\pi n_0 R_0^3}{p_0 \chi} \right) Z \right) \right],$$

$$C_2 = \left[N \eta c^2 \left(a_1 - b_1 (E_2 - k_b \beta^{(s)} Z) \right) + \frac{M Z \mu c^2}{p_0 \chi} \left(4 + \frac{m_0 R_0^2}{k} \right) \right], \quad C_3 = N \eta c^3 a_2,$$

$$C_4 = N \eta c^4 \left(a_3 - b_3 (E_2 - k_b \beta^{(s)} Z) \right), \quad C_6 = N \eta a_5 c^6, \quad \zeta = \zeta_1 + \zeta_2.$$

4. Linearized model

In this section, we consider the linearized version of the model (3.48). Our particular interest is its dissipative part responsible for decay of the wave.

4.1. Evaluation of the parameters and the wave velocity

From [15]–[21], the values of the parameters are: densities, $\rho^{(L)} = 1000 \text{ kg/m}^3$ for water, $\rho^{(g)} = 2 \text{ kg/m}^3$ for gas, $\rho^{(s)} = 2500 \text{ kg/m}^3$ for solid, porosity $m_0 = 0.25$; bulk modulus $k_b = 1.7 \times 10^9 \text{ Pa}$; compressibility $\beta^{(L)} = 2 \times 10^{-9} \text{ Pa}^{-1}$ for water, $\beta^{(g)} = 2.4 \times 10^{-6} \text{ Pa}^{-1}$ for gas, $\beta^{(s)} = 2 \times 10^{-10} \text{ Pa}^{-1}$ for solid; steady pressure $p_0 = 10^7 \text{ Pa}$; bubble radius; $R_0 = 5 \times 10^{-5} \text{ m}$ volume gas content $\phi_0 = 10^{-3}$; viscosities $\mu_1 = 10^{-3} \text{ Pa} \cdot \text{s}$ for water, $\mu_2 = 2 \times 10^{-5} \text{ Pa} \cdot \text{s}$ for gas; adiabatic exponent $\zeta = 1.4$, and permeability $k = 2 \times 10^{-11} \text{ m}^2$. Using the data from [14, 16, 21, 22], the values of the parameters of the rheological scheme in Fig.2 are

$$M_1 = \rho^{(L)} L_s^2 = 10^{-2} \text{ kg/m}, \quad M_2 = \rho^{(s)} L_s^2 = 0.02 \text{ kg/m}, \quad M_3 = \rho^{(g)} L_s^2 = 2 \times 10^{-6} \text{ kg/m}$$

and

$$(a) \quad E_1 = 1/\beta^{(L)} = 4 \times 10^5 \text{ Pa}, \quad E_2 = c^2 \rho_0 = 2 \times 10^7 \text{ Pa}, \quad E_3 = 3\chi p_0 = 4 \times 10^7 \text{ Pa},$$

where we used, just for the purpose of calculation of E_i and M_i , the typical velocity $c \sim 100 \text{ m/s}$ and the linear size of the oscillator $L_s = 0.3 \text{ cm}$ from [16, 23]. We will also explore the values of E_i obtained by a different method, namely by using the formula $c^2 \rho$ for all three phases, with ρ being the density of the liquid, solid and gas, respectively,

$$(b) \quad E_1 = c^2 \rho^{(L)} = 1000 \times 10^4 \text{ Pa}, \quad E_2 = c^2 \rho^{(s)} = 2500 \times 10^4 \text{ Pa}, \quad E_3 = c^2 \rho^{(g)} = 2 \times 10^4 \text{ Pa}.$$

Now we apply the formula for the wave velocity (3.24) to show that it gives reasonable order of magnitude. For the wave with the bubbles (3.24) gives $c \approx 582 \text{ m/s}$ for both variants (a) and (b). For the wave without the bubbles ($n_0 = 0$, $R_0 = 0$, and $M_3 = 0$) $c \approx 740 \text{ m/s}$. This illustrates, in line with the previous studies that the bubbles may result in a considerable change of the wave velocity. Furthermore, we will also explore the following values of E_i obtained experimentally.

$$(c) \quad E_1 = 10^6 \text{ Pa}, \quad E_2 = 10^9 \text{ Pa}, \quad E_3 = 10^6 \text{ Pa}.$$

This results in the wave velocity with the bubbles $c \approx 855 \text{ m/s}$, and the wave velocity without the bubbles $c \approx 950 \text{ m/s}$.

4.2. Dissipation relation

Analysing the linearized model, we are interested in the influence of the bubbles on the decay rate of the wave. This effect is controlled by even derivatives. Therefore, we truncate the linearized Eq.(3.48) to the form

$$\frac{\partial v}{\partial \tau} = \frac{C_2}{C_1} \frac{\partial^2 v}{\partial \xi^2} + \varepsilon^2 \frac{C_4}{C_1} \frac{\partial^4 v}{\partial \xi^4} + \varepsilon^4 \frac{C_6}{C_1} \frac{\partial^6 v}{\partial \xi^6}. \quad (4.1)$$

For the Fourier modes $v \sim \exp(\lambda t + ikx)$, we get the dissipation relation

$$\lambda(k) = -\frac{C_2}{C_1} k^2 + \varepsilon^2 \frac{C_4}{C_1} k^4 - \varepsilon^4 \frac{C_6}{C_1} k^6, \quad (4.2)$$

where λ is the decay rate and k is the wave number.

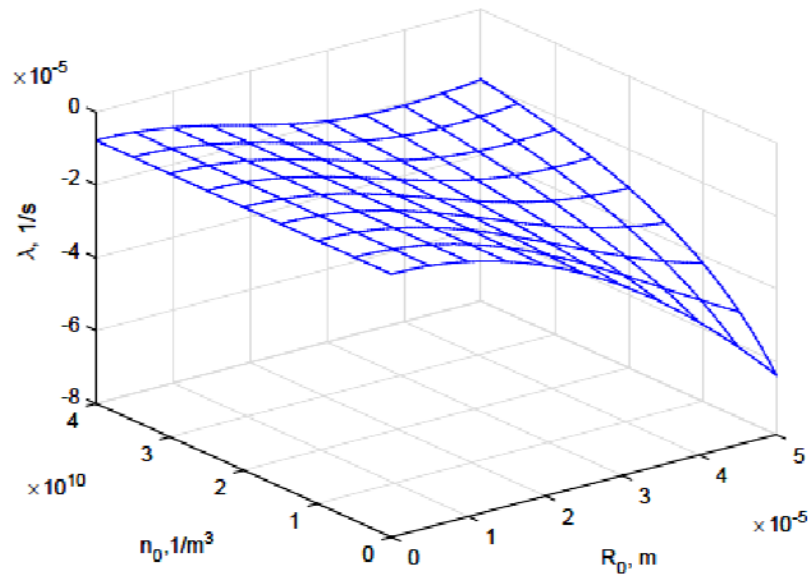


Fig.3. The decay rate for variant (a), $k_* = 0.251 / m$.

The plot in Fig.3 shows the decay rate at fixed $k_* = 0.251 / m$ [9] against R_0 and n_0 . See that the increase in R_0 significantly affects the decay rate and makes it in absolute value larger as the bubbles affect the system through the pressure $p_l = -p_0 \chi R_l$. As for n_0 , one should disregard the region of small n_0 in Fig.3 where the equations of continuum mechanics are not valid. This is because the assumption that each bubble is embedded in its own fluid particle (see Eq.(2.2)) becomes inapplicable due to the large size of the particle.

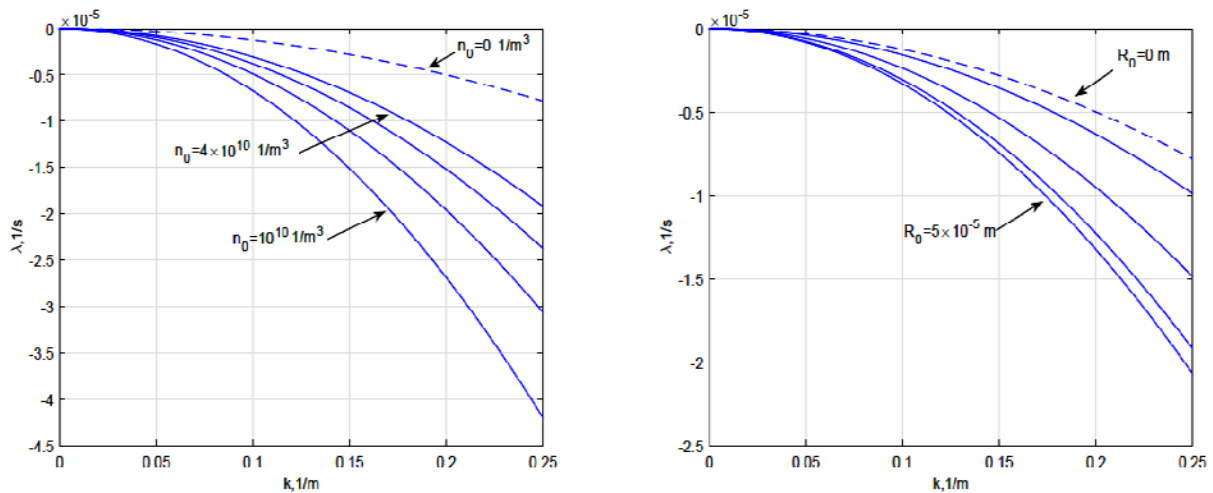


Fig.4. The attenuation curves by formula (4.2) for variant (a). Left: n_0 varies, $R_0 = 5 \times 10^{-5}$; right: R_0 varies, $n_0 = 4 \times 10^{10}$.

Figure 4 compares the attenuation curves of the wave with and without the bubbles. The dashed line describes the case without the bubbles that is $n_0 = 0$, and $R_0 = 0$ and the other lines correspond to the wave with the bubbles. The figure on the left is for varying n_0 and fixed R_0 . The figure on the right is for varying R_0 and fixed n_0 . See that the curves lie entirely below zero, which means that the wave decays and the decay rate depends on the number and radius of the bubbles. This result agrees with the conception emphasized in [24, 25] about the essentially passive nature of the freely propagating elastic wave. Similar results are obtained for variants (b) and (c) as shown in Figs 5–8.

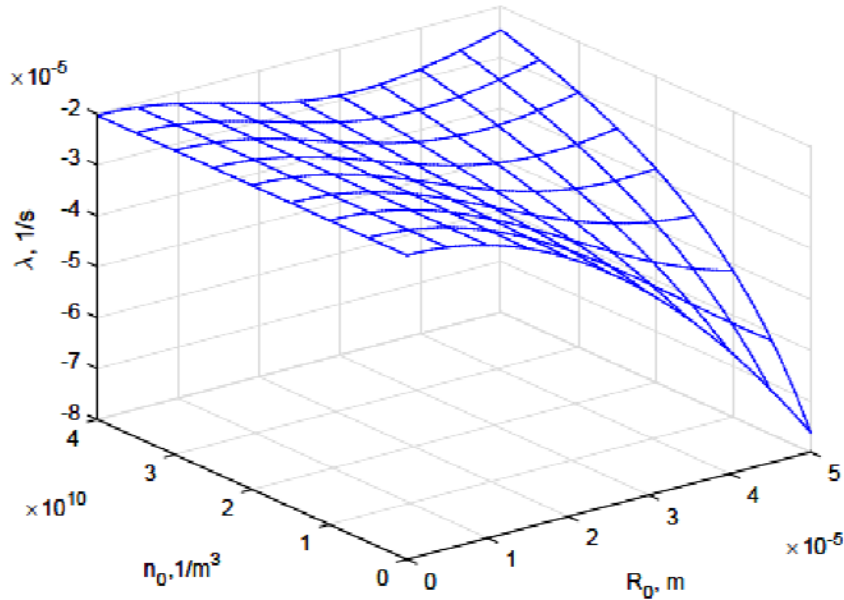


Fig.5. The decay rate for variant (b), $k_* = 0.251 / m$.

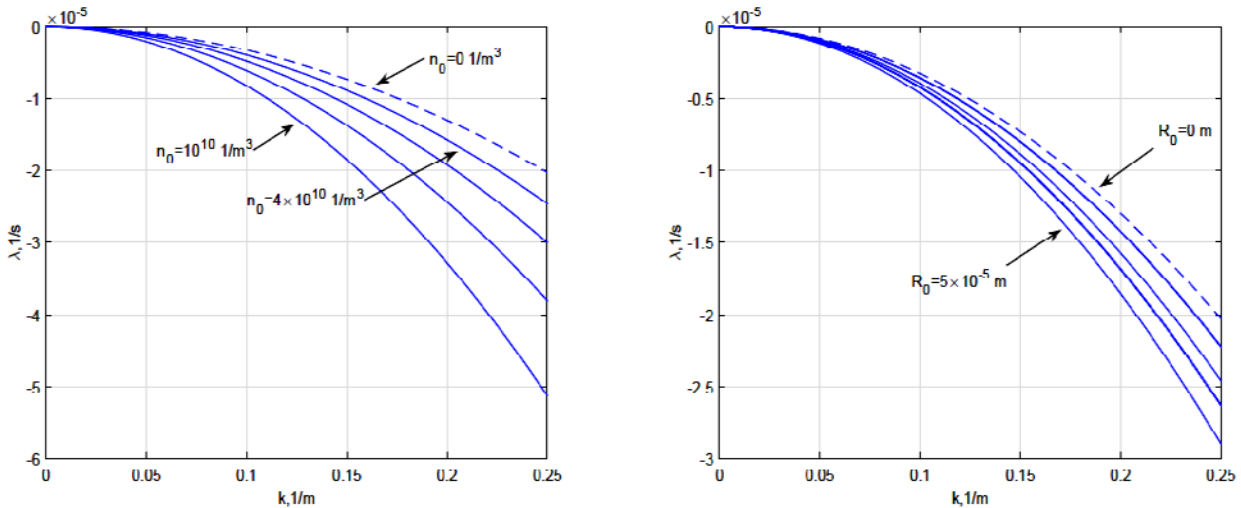


Fig.6. The attenuation curves by formula (4.2) for variant (b). Left: n_0 varies, $R_0 = 5 \times 10^{-5}$; right: R_0 varies, $n_0 = 4 \times 10^{10}$.

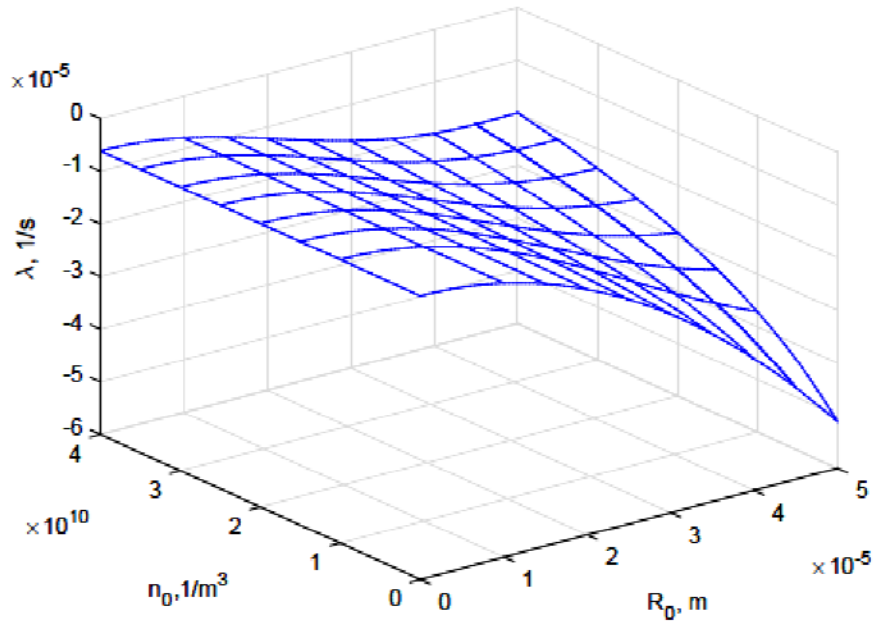


Fig.7. The decay rate for variant (c), $k_* = 0.251 / m$.

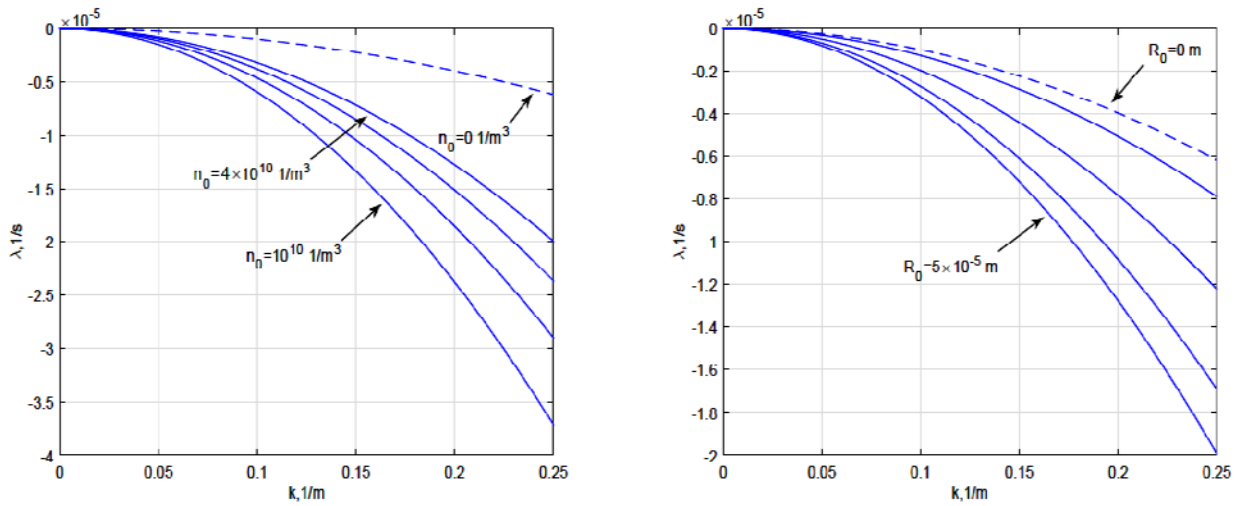


Fig.8. The attenuation curves by formula (4.2) for variant (c). Left: n_0 varies, $R_0 = 5 \times 10^{-5}$; right: R_0 varies, $n_0 = 4 \times 10^{10}$.

For a different $k_* = 0.521 / m$ [26], the results are similar, see Fig.9.

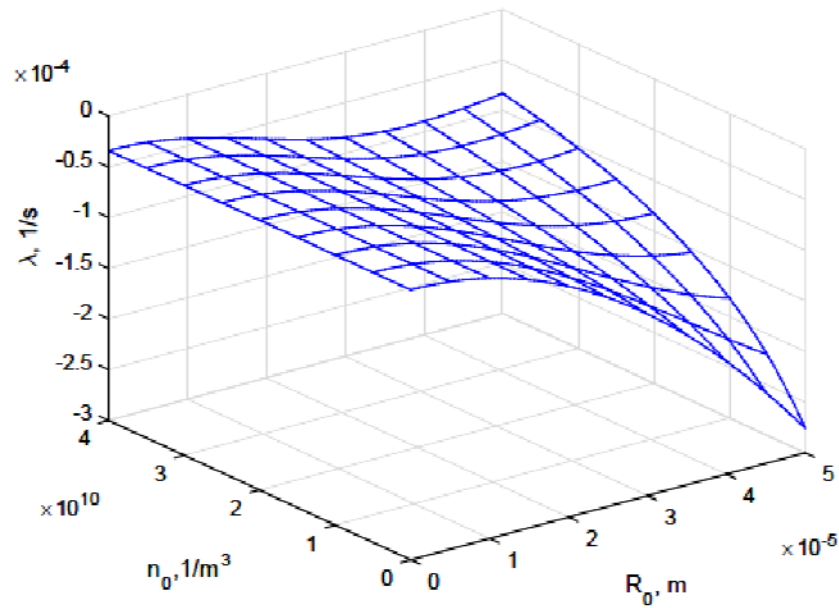


Fig.9. The decay rate for variant (a), $k_* = 0.521 / m$.

5. Conclusions

We studied the effect of the rheology with gas bubbles and bubble dynamics on the elastic wave in a fluid-saturated medium. The P1 Frenkel-Biot wave is analysed, which corresponds to the fully saturated medium. Using three-segment rheology, we derived the Nikolaevskiy-type equation for the velocity of the solid matrix in the wave. The linearized version of the equation is compared in terms of the decay rate $\lambda(k)$ of the Fourier modes. For both the cases with and without the bubbles, the $\lambda(k)$ -curve lies entirely below zero. We discovered that $\lambda(k)$ increases with the increase of the radius of the bubbles but decreases with the increase of the number of the bubbles.

Nomenclature

- k – permeability, [m^2]
- k_b – bulk modulus, [Pa]
- m – porosity
- p – pressure, [Pa]
- R – bubble radius, [m]
- $\beta^{(s)}$ – compressibility for solid, [Pa^{-1}]
- $\beta^{(L)}$ – compressibility for water, [Pa^{-1}]
- $\beta^{(g)}$ – compressibility for gas, [Pa^{-1}]
- ζ – adiabatic exponent
- μ – viscosity, [Pa.s]
- $\rho^{(s)}$ – density for solid, [kg/m^3]
- $\rho^{(L)}$ – density for water, [kg/m^3]
- $\rho^{(g)}$ – density for gas, [kg/m^3]
- ϕ – volume gas content

Appendix

Equation (3.45) can be illustrated with this simple example. We will use the same notations v_I and c as in the main text just to resemble the particle velocity and wave velocity

$$v_I + cv_I = 0,$$

$$2v_I + 4v_I = 0.$$

A non-zero solution of the system exists only if $c = 2$ (the eigenvalue of the problem). Here v_I is the analogy of the first approximation from our main text. The second approximation, v_2 , satisfies the system

$$v_2 + cv_2 = f[v_I],$$

$$2v_2 + 4v_2 = g[v_I],$$

which is solvable only if the operators satisfy the equation $g[v_I] = 2f[v_I]$. This is the analogy to the Nikolaevskiy-type equation that we aim to derive.

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