Local Stability of a Plate with a Circular Inclusion Under Tensile Stress

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ABSTRACT: The paper deals with the problem of the local buckling caused by uniaxial stretching of an infinite plate with a circular inclusion from a different material. The effect of elastic modulus of the inclusion on the value of the critical load is investigated. In order to find the first critical load a variational principle is applied. The comparison of numerical results that were obtained in the Maple 18 and the results obtained by the finite element method in the ANSYS 13.1. The influence is analysed the ratio of the elastic properties of the inclusion and plate on the value of the critical load and the form of the loss of stability.

1 INTRODUCTION

A wide range of monographs [1-5] is focused on the problems of stability of thin plates with holes and cracks. It is noted that the compressive stresses can occur not only when the plates are compressed, which is obvious, but near the boundary of the holes and under tension. At a certain value of the tensile external forces, these stresses can cause a local loss of stability of plates, which significantly affects their bearing capacity. The loss of the plane shape of the deformation of plates with different types of cuts under uniaxial tension was investigated in [2-3].

It is known that compressive stresses are also observed in plates with inclusions. The simplest example is the problem of the possible loss of stability of a flat form of equilibrium under uniaxial tension by stresses of a plate with a circular inclusion made of another material.

In the work the problem is studied of the local buckling caused by uniaxial stretching of the infinite plate with a circular inclusion from the different material under uniaxial tension. The influence of the ratio of the elastic properties of the inclusion and plate on the value of the critical load and the form of the loss of stability is analyzed.

2 PROBLEM STATEMENT

Suppose that: E_1, ν_1 – are Young's modulus and Poison's ratio of the plate, and E_2, ν_2 – parameters of the inclusion. R - radius of the inclusion, and x, y Cartesian coordinates. The plate is presented in Fig.1.

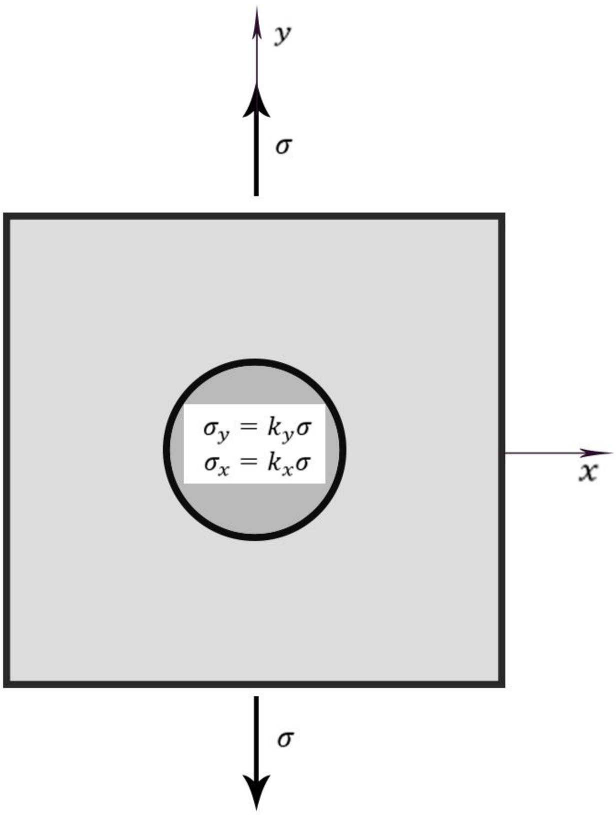

Figure 1. Plate with a Circular Inclusion Under Tensile Stress.

The stresses act along the y axis. It is known [4] that in the case of elliptical inclusion, the inclusion's stress field is homogeneous ($\tau_{xy} = 0$) and symmetric with respect to the extension axis. Let us denote the stress field inside the inclusion as $\sigma_{xx} = k_x\sigma$ and $\sigma_{yy} = k_y\sigma$, where k_x and k_y are the coefficients which were defined in [2]:

$$k_x = \frac{E_2[(3\nu_2 - 1)E_1 + (1 - 3\nu_1)E_2]}{(E_1 + 2E_2)^2 - [\nu_2 E_1 + (1-\nu_1)E_2]^2},$$
$$k_y = \frac{E_2[(3 - \nu_2)E_1 + (5 + \nu_1)E_2]}{(E_1 + 2E_2)^2 - [\nu_2 E_1 + (1-\nu_1)E_2]^2}. \qquad (1)$$

In polar coordinates, the dimensionless stresses in the plate are given by formulas:

$$\sigma^0_{(1,\rho\rho)} = \frac{\sigma_{\rho\rho}}{\sigma} = \frac{1}{2}\left[1 - \frac{(1 - k_y - k_x)}{\rho^2} + \left(1 - 4\frac{(1 - k_y + k_x)}{\rho^2} + 3\frac{(1 - k_y + k_x)}{\rho^4}\right)cos(2\varphi)\right],$$
$$\sigma^0_{(1,\varphi\varphi)} = \frac{\sigma_{\varphi\varphi}}{\sigma} = \frac{1}{2}\left[1 + \frac{(1 - k_y - k_x)}{\rho^2} - \left(1 + 3\frac{(1 - k_y + k_x)}{\rho^4}\right)cos(2\varphi)\right], \qquad (2)$$
$$\tau^0_{(1,\rho\varphi)} = \frac{\tau_{\rho\varphi}}{\sigma} = -\frac{1}{2}\left(1 + 2\frac{1 - k_y + k_x}{\rho^2} - 3\frac{1 - k_y + k_x}{\rho^4}\right)sin(2\varphi).$$

and for the inclusion:

$$\sigma^0_{(2,\rho\rho)} = \frac{\sigma_{\rho\rho}}{\sigma} = \frac{1}{2}[k_y + k_x + (k_y - k_x)cos(2\varphi)],$$
$$\sigma^0_{(2,\varphi\varphi)} = \frac{\sigma_{\varphi\varphi}}{\sigma} = \frac{1}{2}[k_y + k_x - (k_y - k_x)cos(2\varphi)], \qquad (3)$$
$$\tau^0_{(2,\rho\varphi)} = \frac{\tau_{\rho\varphi}}{\sigma} = -\frac{1}{2}(k_y - k_x)\sin(2\varphi).$$

It follows from the formulas (2,3) that in general the stresses σ_{yy}/σ are positive. The zones of negative stresses can appear in limited cases: for a “rigid” inclusion or its absence. Negative stresses do not exist when the inclusion and plate are of the same material.

The stresses σ_{xx}/σ can also be negative (Fig. 2). For a more rigid inclusion (Fig.3), decrement of the square of the area of negative stresses occurs with an increment of its elastic properties. The absolute value of the negative stresses is also reduced.

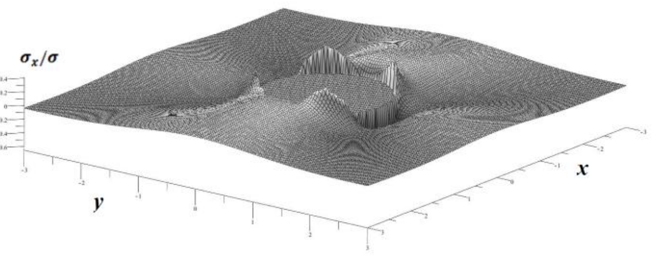

Figure 2. Stress distribution for the material of inclusion is 10 times softer than the material of the plate. (Maple results) $\sigma_{xx}^{max}/\sigma=0.42$, $\sigma_{xx}^{min}/\sigma=-0.75$

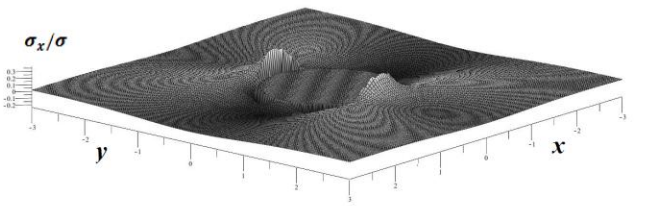

Figure 3. Stress distribution for the material of inclusion is 10 times more rigid than the material of the plate. (Maple results) $\sigma_{xx}^{max}/\sigma=0.42$, $\sigma_{xx}^{min}/\sigma=-0.24$

In the case when the inclusion is softer than the plate, the region of negative stress appears again and the local loss of stability may appear. Where the inclusion is more rigid than the plate, the region of negative stresses is displaced by 90 degrees compared to the soften insent.

The analytical calculations agree well with the results of the problem is solved with the use of the finite element method in the ANSYS 13.1 (Figure 4-5).

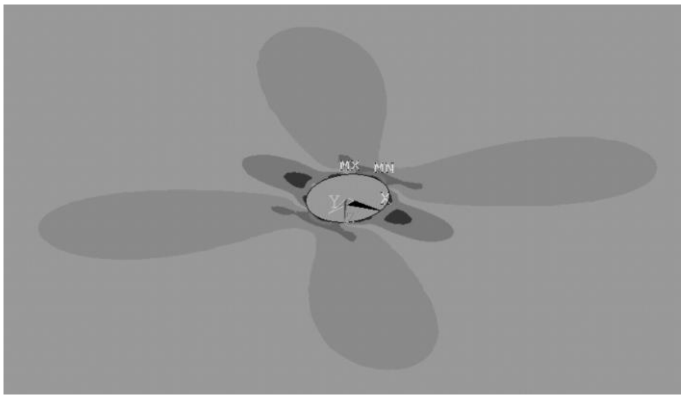

Figure 4. Stress distribution σ_{xx}/σ for the material of inclusion is 10 times softer than the material of the plate. (ANSYS results.)

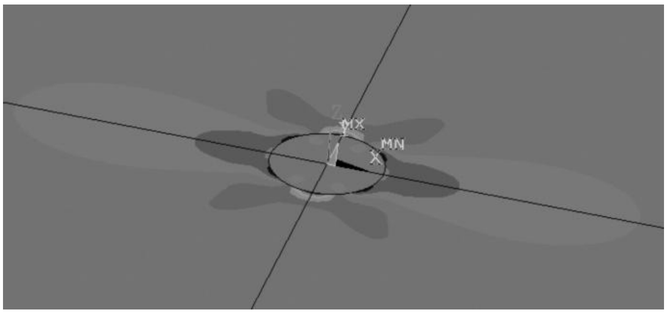

Figure 5. Stress distribution σ_{xx}/σ for the material of inclusion is 10 times more rigid than the material of the plate. (ANSYS results.)

3 NUMERICAL SOLUTION

The first critical load that causes the loss of stability can be evaluated with the use of the energy method customized by Timoshenko and Ritz [6,7].

Let us present the deflection functions in series, where representations for deflection of the plate w_1 and the inclusion w_2, take the following forms [8]:

$$w_1(\rho,\varphi) = R\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} A_{(k,2l-2)}\frac{cos[2(l-1)\varphi]}{\rho^k}, \tag{4}$$

$$w_2(\rho,\varphi) = R\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} B_{(k,2l-2)}cos[2(l-1)\varphi]\rho^{k+1}. \tag{5}$$

These expressions satisfy the boundary condition of the symmetry of deformations, and the deflection decrement at infinity.

The continuity of the deflection function at the boundaries gives the following conditions for a rigidly fixed inclusion:

$$\begin{cases} w_1(\rho,\varphi) = w_2(\rho,\varphi) \\ \dfrac{\partial w_1(\rho,\varphi)}{\partial\rho} = \dfrac{\partial w_2(\rho,\varphi)}{\partial\rho} \end{cases} \tag{6}$$

With the use of these two relations (6) we can express the two first members of the second series $B_{(1,0)}$, $B_{(2,2)}$ through the others members.

The full energy of the system is a sum the energy of the plate and the energy of the inclusion. The same applies to the work of the median plane, so the total energy is represented as:

$$\Delta V = U_2 + U_1 + \lambda(W_2 + W_1) \tag{7}$$

The second integrals (9-11) for U_1, W_1 and U_2, W_2 show different limits by $\{\rho_1, \rho_2\}$: from 1 to infinity in the case of the plate, and from 0 to 1 for the inclusion (where 1 corresponds to the dimensionless radius, $\rho = R/r = 1$).

$$U_i = \frac{D_i}{2}[U_{i1} - 2(1-\nu_1)U_{i2}], \quad where \quad D_i = \frac{E_i h^3}{12(1-\nu_1^2)} \tag{8}$$

$$U_{i1} = \frac{1}{R^2}\int_0^{2\pi}\int_{\rho_1}^{\rho_2}\left(\frac{\partial^2 w_i}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial w_i}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 w_i}{\partial\varphi^2}\right)^2 \rho\, d\rho\, d\varphi \tag{9}$$

$$\begin{aligned} U_{i2} = \frac{1}{R^2}\int_0^{2\pi}\int_{\rho_1}^{\rho_2} & \frac{\partial^2 w_i}{\partial\rho^2}\frac{\partial w_i}{\partial\rho} + \frac{1}{\rho}\frac{\partial^2 w_i}{\partial\rho^2}\frac{\partial^2 w_i}{\partial\varphi^2} - \\ & -\frac{1}{\rho}\left(\frac{\partial^2 w_i}{\partial\rho\partial\varphi}\right)^2 + \frac{2}{\rho^2}\frac{\partial^2 w_i}{\partial\rho\,\partial\varphi}\frac{\partial w_i}{\partial\varphi} - \frac{1}{\rho^3}\left(\frac{\partial w_i}{\partial\varphi}\right)^2 d\rho d\varphi \end{aligned} \tag{10}$$

$$\begin{aligned} W_i = \frac{h}{2}\int_0^{2\pi}\int_{\rho_1}^{\rho_2} & \left[\sigma^0_{(i,\rho\rho)}\left(\frac{\partial w_i}{\partial\rho}\right)^2 + \sigma^0_{(i,\varphi\varphi)}\frac{1}{\rho}\left(\frac{\partial w_i}{\partial\varphi}\right)^2 + \right. \\ & \left. + 2\tau^0_{(i,\rho\varphi)}\frac{\partial w_i}{\partial\rho}\frac{\partial w_i}{\partial\varphi}\right] d\rho d\varphi \end{aligned} \tag{11}$$

Substituting the following relations (8-11) into the formula (7) we obtain the full functional of the potential energy as a combination of these series of displacements.

According to the principle of possible displacements, the minimum potential energy can be found from the equality of partial derivatives of the increment of the potential energy from the generalized coordinates.

Collecting the coefficients $A_{(k,2l-2)}$, $B_{(k,2l-2)}$ from the partial derivatives of U and W into the corresponding matrixes, we come to the classical eigenvalue problem:

$$U + \lambda W = 0 \tag{12}$$

The first critical load p^*, that caused the buckling of the plate is equal to the minimum positive eigenvalue λ that can be found from the system (12).

4 THE RESULTS OF CALCULATIONS

The forms of stability loss are constructed and the corresponding critical loads are determined. A good agreement is achieved of the first critical loads obtained by the finite element method ANSYS 13 and the results of the analytical approach.

Calculations show that the loss of stability of a plate with a circular rigidly fixed inclusion happens at lower loads when the modulus of elasticity of the inclusion is either much smaller than the plate (i.e. the inclusion is very "soft") or, conversely, much larger (i.e., inclusion very "tough").

Figures 6 and 7 show the forms of loss of stability of a plate with an inclusion stretched along the Y-axis.

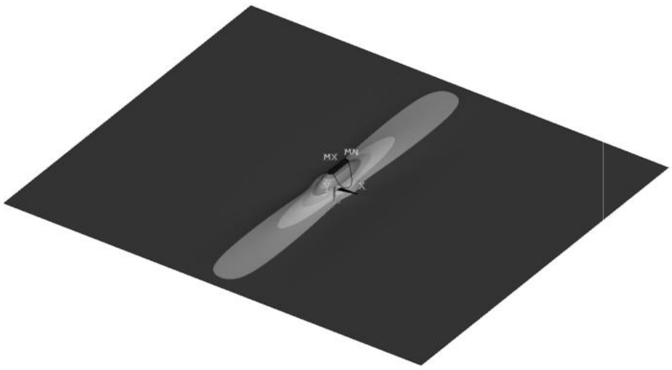

Figure 6. the inclusion is 10 times softer than the plate.

In Fig. 6 the inclusion is 10 times softer than the plate (E_2/E_1=1/10). When the inclusion becomes "softer" (the Young's modulus of the inclusion $E_2 \rightarrow 0$), the λ is closer to the minimum eigenvalue corresponding to the problem of the loss of stability of a plate with an aperture.

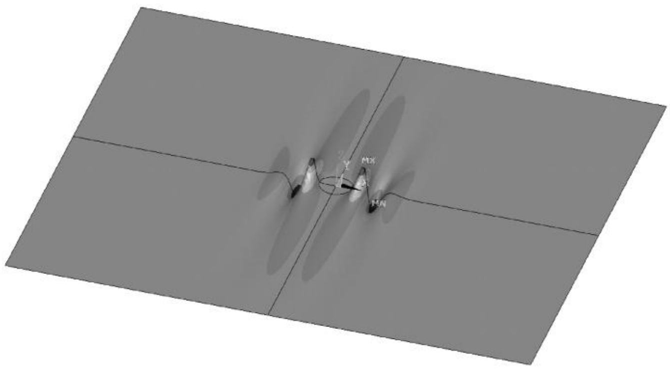

Figure 7. The inclusion is 10 times stiffer than the plate.

In Fig. 7, the inclusion is 10 times stiffer than the plate. In this case, the regions of the compressive stress zone are located along *x* axis.
Figure 8 shows the dependence of the critical load on the ratio between the modulus of the inclusion and plate (σ_0 is the critical load corresponding to the plate with a hole of radius R). Even the smaller stresses can cause the loss of stability for the more rigid inclusion (the right part of the graph in Figure 8). It should be noted that for the more rigid inclusion, we need more series members for the convergence of the solution. We can see that in the vicinity of the point E_2/E_1=1, significant stresses are needed to cause loss of stability. When modulus of elasticity of the inclusion and plate are equal to one another, (ie, a homogeneous isotropic plate), there is no loss of stability.

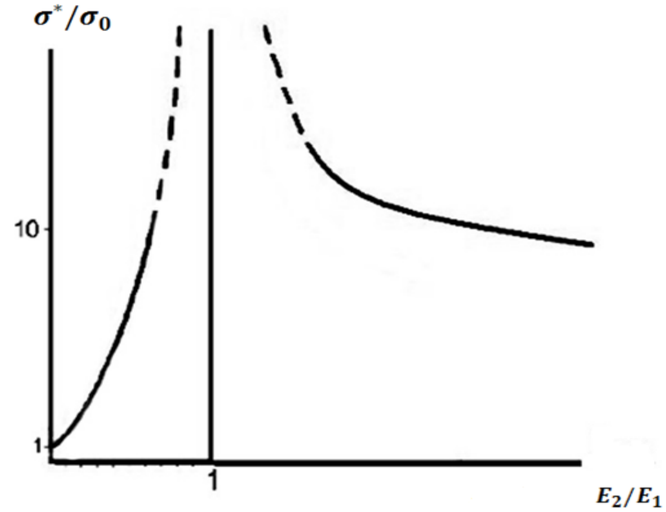

Figure 8. Dependence of the critical load on the ratio of the inclusion module to the plate module.

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