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δ-equality of intuitionistic fuzzy sets: a new proximity measure and applications in medical diagnosis

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Abstract Intuitionistic fuzzy set is capable of handling uncertainty with counterpart falsities which exist in nature. Proximity measure is a convenient way to demonstrate impractical significance of values of memberships in the intuitionistic fuzzy set. However, the related works of Pappis (Fuzzy Sets Syst 39(1):111–115, 1991), Hong and Hwang (Fuzzy Sets Syst 66(3):383–386, 1994), Virant (2000) and Cai (IEEE Trans Fuzzy Syst 9(5):738–750, 2001) did not model the measure in the context of the intuitionistic fuzzy set but in the Zadeh's fuzzy set instead. In this paper, we examine this problem and propose new notions of δ-equalities for the intuitionistic fuzzy set and δ-equalities for intuitionistic fuzzy relations. Two fuzzy sets are said to be δ-equal if they are equal to an extent of δ. The applications of δ-equalities are important to fuzzy statistics and fuzzy reasoning. Several characteristics of δ-equalities that were not discussed in the previous works are also investigated. We apply the δ-equalities to the application of medical diagnosis to investigate a patient's diseases from symptoms. The idea is using δ-equalities for intuitionistic fuzzy relations to find groups of intuitionistic fuzzified set with certain equality or similar degrees then combining them. Numerical examples are given to illustrate validity of the proposed algorithm. Further, we conduct experiments on real medical datasets to check the efficiency and applicability on real-world problems. The results obtained are also better in comparison with 10 existing diagnosis methods namely De et al. (Fuzzy Sets Syst 117:209–213, 2001), Samuel and Balamurugan (Appl Math Sci 6(35):1741–1746, 2012), Szmidt and Kacprzyk (2004), Zhang et al. (Procedia Eng 29:4336–4342, 2012), Hung and Yang (Pattern Recogn Lett 25:1603–1611, 2004), Wang and Xin (Pattern Recogn Lett 26:2063–2069, 2005), Vlachos and Sergiadis (Pattern Recogn Lett 28(2):197–206, 2007), Zhang and Jiang (Inf Sci 178(6):4184–4191, 2008), Maheshwari and Srivastava (J Appl Anal Comput 6(3):772–789, 2016) and Support Vector Machine (SVM).

Keywords δ-equalities · Algebraic operations · Intuitionistic fuzzy set · Intuitionistic fuzzy relations · Medical diagnosis

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1 Introduction

Fuzzy set was proposed by Zadeh in 1965 to handle uncertainty and ambiguity [67, 68]. A fuzzy set is defined by a membership degree function with range in the unit interval [0,1]. It defines a multilevel extension to the classical set such that a proposition can get any value in the unit interval instead of 'True' or 'False'. Based on the fuzzy set, several additional and hybrid concepts such as theinterval-valued fuzzy set [69], the type-2 fuzzy set [69], the intuitionistic fuzzy set [2] were developed. Fuzzy sets play a

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tremendous role in signal processing [25], control theory [14], reasoning [7], decision making [23], medical diagnosis [31], geo-demographic analysis [33, 37, 41, 42, 65], dental segmentation [47, 48, 59], compression [43], recommender systems [34, 36, 38] and other fields [8, 10, 35, 39, 40, 46, 49, 50, 56–58].

In Zadeh's fuzzy set, the degree of membership is a single value between 0 and 1. Nonetheless, this may not always be valid in real-life applications due to the existence of hesitation margin or degree. To deal with this issue, the *intuitionistic fuzzy set* (IFS) [2] extended the fuzzy set by incorporating the degree of non-membership. In the other words, IFS is branded and characterized by the degrees of membership and non-membership with the condition that their sum does not exceed 1. It has been observed that IFS can better designate fuzziness. In the practical point of view, IFS gained much attention from the research community which have been successfully tested in the fields of modeling imprecision [12], decision making [5], pattern recognition [62], computational intelligence [6] and medical diagnosis [27, 44, 45, 51, 52, 54, 55]. The strength of these approaches evolves from those cases where conflicting information concerning membership taints the ability to determine the actual fuzzy membership of objects.

Proximity measure was firstly discussed by Pappis [26] to demonstrate the impractical significance of values of membership. Let A and B be two fuzzy sets on a universe U, and $\mu_A(x)$ and $\mu_B(x)$ representing their membership functions, respectively. A and B are said to be approximately equal if $\sup_x |\mu_A(x) - \mu_B(x)| \leq \varepsilon$, where ε is a small nonnegative number and called the proximity measure. Pappis believed that the max-min compositional rule of inference is preserved with approximately equal fuzzy sets. Another approach considered by Hong and Hwang [17], as a generalization of the work of Pappis [26], was mainly based on the same philosophy of the max-min compositional rule of inference that is preserved with respect to approximately equal fuzzy sets and approximately equal fuzzy relation respectively. Cai [4] argued that both the Pappis et al. approaches were limited to a fixed value of ε, i.e. they assumed that ε is constant and disregarded what "small nonnegative number" means. However in reality, different values of ε may make different senses and the role of context is indeed important. We also note that the notion ε-equality was introduced by Dubois and Prade [15]. Two fuzzy sets A and B are said to be ε-equality if $S(A, B) \geq \varepsilon$, where $S(A, B)$ is a similarity measure between A and B. Evidently, there is an inherent relationship between proximity measure and ε-equality, i.e. $S(A, B)$ can be interpreted in terms of $\sup_x |\mu_A(x) - \mu_B(x)|$. Cai introduced *$\delta$-equalities of fuzzy set* to overcome this problem in which two fuzzy sets are said to be δ-equal if they equal to a degree of δ. In the other words, two fuzzy sets A and B are said to be δ-equality if $\sup_x |\mu_A(x) - \mu_B(x)| \leq 1 - \delta$. As Cai explained in his paper, the advantage of using $1 - \delta$ rather than ε is that interpretation of δ can comply with common sense. That is, the greater the value of δ is, the 'more equal' the two fuzzy sets are; and they become 'strictly equal' when $\delta = 1$. The applications of δ-equalities have important roles to fuzzy statistics and fuzzy reasoning. Virant [61] tested δ-equalities of fuzzy sets in synthesis of real-time fuzzy systems while Cai [4] used them for validating the robustness of fuzzy reasoning accompanied with several reliability examples through δ-equalities. Nonetheless, there is no such notion in the context of the IFS set.

In this paper, we propose a new notion of *δ-equalities for the universe of IFS set*. The notions of δ-equalities for intuitionistic fuzzy relations and intuitionistic fuzzy norms are also proposed herein. The **aim of those proposals** in comparison with the work of Cai [4] is to extend the existing definitions in a new context of IFS which was shown to better model real-life applications than the fuzzy set [2] and to examine several characteristics and theorems of δ-equalities that were not (or partly) discussed in the previous works. The mentioned proposals are significant to understand the behavior of δ-equalities in IFS which is helpful to select appropriate setting for applications.

The significance and practical implication of the proposed approach is not limited to the theoretical aspects but also the establishment in practice. In this regards, we apply the δ-equalities to the application of medical diagnosis, which is always one of the leading research interest areas, to investigate a patient's diseases from his symptoms. The Sanchez's approach [32] using the theory of fuzzy sets was long recognized as the traditional method. De et al. [9] extended the Sanchez's method with the theory of intuitionistic fuzzy sets (IFSs). Samuel and Balamurugan [31], Szmidt and Kacprzyk [53], Zhang et al. [71], Hung and Yang [19], Wang and Xin [63], Vlachos and Sergiadis [62], Zhang and Jiang [70], Wei and Ye [64] and Hung [18], Junjun et al. [20], Maheshwari and Srivastava [24] continued to work on the IFS theory to improve the method of De et al. [9], i.e., by using new score functions, new distance functions, or new measures instead of the score function in the method of De et al. [9]. In this paper, the proposed algorithm combines the δ-equalities with the extended Sanchez's approach for intuitionistic fuzzy sets. The idea is using δ-equalities for intuitionistic fuzzy relations to find groups of intuitionistic fuzzified set with certain equality or similar degrees then combining them. Numerical examples and experimental validation on real-world datasets are given to illustrate the activities of the proposed algorithm.

The rest of the paper is organized as follows. Section 2 provides some fundamental concepts of the IFS set. Section 3 proposes the δ-equalities for IFS accompanied with theoretical investigation with set theoretic operations

of IFS such as the union, intersection, complement, product, addition and some other operations. Section 4 extends the δ-equalities to intuitionistic fuzzy relations and intuitionistic fuzzy norms. Section 5 presents an application of δ-equalities to the medical diagnosis problem including a new algorithm and numerical examples. Section 6 shows the experimental results on real-world datasets. Finally, conclusions and further studies of this research are given in Section 7.

2 Preliminary

Definition 1 [67] Fuzzy Set

Let U be a space of points and let $u \in U$. A fuzzy set S in U is characterized by a membership function μ_S with a range in [0,1]. A fuzzy set can be represented as

$$S = \{(u, \mu_S(u)) : u \in U\}.$$

Definition 2 [2] Intuitionistic Fuzzy Set

Let U be a space of points and let $u \in U$. An intuitionistic fuzzy set S in U is characterized by a membership function μ_S and a non-membership function ν_S with a range in [0,1] such that $0 \leq \mu_S + \nu_S \leq 1$. Intuitionistic fuzzy set can be represented as a triplet in the following way

$$S = \{\langle u, \mu_S(u), \nu_S(u)\rangle : u \in U\}.$$

We now give some set theoretic operations of intuitionistic sets.

Definition 3 [2] Inclusion relation between two intuitionistic fuzzy sets

Let A and B be two intuitionistic fuzzy sets in a universe of discourse U. Then the inclusion relation $\subseteq$ between A and B is defined by

$$A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x), \nu_A(x) \geq \nu_B(x), \forall x \in U.$$

Especially,

$$A = B \Leftrightarrow \mu_A(x) = \mu_B(x), \nu_A(x) = \nu_B(x), \forall x \in U.$$

Definition 4 [2] Complement of Intuitionistic Set

The complement of an intuitionistic fuzzy set S is denoted by S^c and is given as

$$\mu_{S^c}(u) = \nu_S(u), \ \ \nu_{S^c}(u) = \mu_S(u), \quad \forall u \in U.$$

Definition 5 [2] Union of intuitionistic fuzzy sets

Let A and B be two intuitionistic fuzzy sets in a universe of discourse U. Then the union of A and B is denoted by $A \cup B$, which is defined by

$$A \cup B = \{\langle u, \mu_A(u) \vee \mu_B(u), \nu_A(u) \wedge \nu_B(u)\rangle : u \in U\},$$

where $\vee$ denote the max-operator, and $\wedge$ denote the min-operator.

Definition 6 [2] Intersection of intuitionistic fuzzy sets

Let A and B be two intuitionistic fuzzy sets in a universe of discourse U. Then the intersection of A and B is denoted as $A \cap B$, which is defined by

$$A \cap B = \{\langle u, \mu_A(u) \wedge \mu_B(u), \nu_A(u) \vee \nu_B(u)\rangle : u \in U\},$$

where $\vee$ denote the max-operator, and $\wedge$ denote the min-operator.

Definition 7 [2] Addition of two intuitionistic fuzzy sets

Let A and B be two intuitionistic fuzzy sets in a universe of discourse U. Then the addition of A and B is denoted as $A + B$, which is defined by

$$\begin{aligned} A + B = \{&\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x).\mu_B(x), \\ &\nu_A(x).\nu_B(x)\rangle : x \in U\}. \end{aligned}$$

Definition 8 Difference of two intuitionistic fuzzy sets

Let A and B be two intuitionistic fuzzy sets in a universe of discourse U. Then the difference of A and B is denoted as $A - B$, which is defined by

$$A - B = A + B^c.$$

Definition 9 [2] Product of two intuitionistic fuzzy sets

Let A and B be two intuitionistic fuzzy sets in a universe of discourse U. Then the product of A and B is denoted as $A \cdot B$(or AB), which is defined by

$$\begin{aligned} A \cdot B = \{&\langle x, \mu_A(x).\mu_B(x), \nu_A(x) \\ &+\nu_B(x) - \nu_A(x).\nu_B(x)\rangle : x \in U\}. \end{aligned}$$

Definition 10 [13] The set L^* defined by

$$L^* = \{x = (x_1, x_2) | x_1, x_2 \in [0, 1], x_1 + x_2 \leq 1\},$$
$$0_{L^*} = (0, 1), 1_{L^*} = (1, 0),$$

and the order relation $\leq_{L^*}$ on L^* defined by

$$x = (x_1, x_2), y = (y_1, y_2) \in L^*, x \leq_{L^*} y \Leftrightarrow x_1 \leq y_1, x_2 \geq y_2,$$

The first and second projection mappings pr_1 and pr_2 on L^* are defined as

$$pr_1(x) = x_1, pr_1(x) = x_2, \ \forall x \in L^*.$$

Definition 11 [13] An intuitionistic fuzzy triangular norm $\mathcal{T}$ is a function $\mathcal{T} : L^{*2} \to L^*$ defined by

$$\mathcal{T}(x, y) = (pr_1\mathcal{T}(x, y)), pr_2\mathcal{T}(x, y)), \forall x, y \in L^*,$$

and $\mathcal{T}$ has to satisfy the following conditions:

1. $\mathcal{T}(1_{L^*}, x) = x, \forall x \in L^*$;
2. $\mathcal{T}(x, y) \leq_{L^*} \mathcal{T}(w, z)$, whenever $x \leq_{L^*} w$ and $y \leq_{L^*} z$, $\forall x, y, w, z \in L^*$;
3. $\mathcal{T}(x, y) = \mathcal{T}(y, x), \forall x, y \in L^*$;
4. $\mathcal{T}(\mathcal{T}(x, y), z) = \mathcal{T}(x, \mathcal{T}(y, z)), \forall x, y, w, z \in L^*$.

Example 1 Some intuitionistic fuzzy triangular norms are given below.

- $\mathcal{T}_1(x, y) = (x_1 \wedge y_1, x_2 \vee y_2), \forall x, y \in L^*$.
- $\mathcal{T}_2(x, y) = (x_1y_1, x_2 + y_2 - x_2y_2), \forall x, y \in L^*$.
- $\mathcal{T}_3(x, y) = (\max(0, x_1 + y_1 - 1), \min(1, x_2 + y_2))$, $\forall x, y \in L^*$.
- $\mathcal{T}_4(x, y) = (\max(0, \lambda(x_1 + y_1) - \lambda + (1 - \lambda)x_1y_1), \min(1, x_2 + y_2)), \forall x, y \in L^*, 0 < \lambda < 1$.
- $\mathcal{T}_5(x, y) = (\max(0, x_1 + y_1 - 1), x_2 + y_2 - x_2y_2)$, $\forall x, y \in L^*$.
- $\mathcal{T}_6(x, y) = (\max(0, \lambda(x_1 + y_1) - \lambda + (1 - \lambda)x_1y_1), x_2 + y_2 - x_2y_2), \forall x, y \in L^*, 0 < \lambda < 1$.

Definition 12 [13]
An intuitionistic fuzzy triangular co-norm $\mathcal{S}$ is a function $\mathcal{S} : L^{*2} \to L^*$ defined by

$$\mathcal{S}(x, y) = (pr_1\mathcal{S}(x, y), pr_2\mathcal{S}(x, y)), \forall x, y \in L^*,$$

where $\mathcal{S}$ has to satisfy the following conditions:

1. $\mathcal{S}(0_{L^*}, x) = x, \forall x \in L^*$;
2. $\mathcal{S}(x, y) \leq_{L^*}, \mathcal{S}(w, z)$ whenever $x \leq_{L^*} w$ and $y \leq_{L^*} z$, $\forall x, y, w, z \in L^*$;
3. $\mathcal{S}(x, y) = \mathcal{S}(y, x), \forall x, y \in L^*$;
4. $\mathcal{S}(\mathcal{S}(x, y), z) = \mathcal{S}(x, \mathcal{S}(y, z)), \forall x, y, z \in L^*$.

Example 2 Some intuitionistic fuzzy triangular co-norms are presented below.

- $\mathcal{S}_1(x, y) = (x_1 \vee y_1, x_2 \wedge y_2), \forall x, y \in L^*$.
- $\mathcal{S}_2(x, y) = (x_1 + y_1 - x_1y_1, x_2y_2), \forall x, y \in L^*$.
- $\mathcal{S}_3(x, y) = (\min(1, x_1 + y_1), \max(0, x_2 + y_2 - 1))$, $\forall x, y \in L^*$.
- $\mathcal{S}_4(x, y) = (\min(1, x_1 + y_1), \max(0, \lambda(x_2 + y_2) - \lambda + (1 - \lambda)x_2y_2)))$, $\forall x, y \in L^*, 0 < \lambda < 1$.
- $\mathcal{S}_5(x, y) = (x_1 + y_1 - x_1y_1, \max(0, x_2 + y_2 - 1))$, $\forall x, y \in L^*$.
- $\mathcal{S}_6(x, y) = (x_1 + y_1 - x_1y_1, \max(0, \lambda(x_2 + y_2) - \lambda + (1 - \lambda)x_2y_2)), \forall x, y \in L^*$, $0 < \lambda < 1$.

Definition 13 General extension intersection of two intuitionistic fuzzy sets

Let A and B be intuitionistic fuzzy sets define on U and $\mathcal{T}$ be an intuitionistic fuzzy triangular norm, then the general extension intersection of A and B is denoted as $A \cap_{\mathcal{T}} B$, which is defined by

$$A \cap_{\mathcal{T}} B = \{\langle x, pr_1\mathcal{T}(\mu_A(x), \mu_B(x)), pr_2\mathcal{T}(\nu_A(x), \nu_B(x))\rangle : x \in U\}.$$

For example: $A \cap_{\mathcal{T}_1} B = A \cap B$ and $A \cap_{\mathcal{T}_2} B = AB$.

Definition 14 General extension union of two intuitionistic fuzzy sets

Let A and B be intuitionistic fuzzy sets define on U and $\mathcal{S}$ be an intuitionistic fuzzy triangular co-norm, then the general extension union of A and B is denoted as $A \cup_{\mathcal{S}} B$, which is defined by

$$A \cup_{\mathcal{S}} B = \{\langle x, pr_1\mathcal{S}(\mu_A(x), \mu_B(x)), pr_2\mathcal{S}(\nu_A(x), \nu_B(x))\rangle : x \in U\}.$$

For example: $A \cup_{\mathcal{S}_1} B = A \cup B$ and $A \cup_{\mathcal{S}_2} B = A + B$.

Definition 15 [4] Let U be a universe of discourse. Let A and B be two fuzzy sets on U, and $\mu_A(x)$ and $\mu_B(x)$ their membership functions, respectively. Then A and B are said to be δ-equal denoted by $A = (\delta)B$, if

$$\sup_{x \in U} |\mu_A(x) - \mu_B(x)| \leq 1 - \delta, \quad 0 \leq \delta \leq 1.$$

In this way, we say A and B construct δ-equality.

Lemma 1 [4] *Let*

$$\delta_1 * \delta_2 = \max(0, \delta_1 + \delta_2 - 1), \ 0 \leq \delta_1, \delta_2 \leq 1, \tag{1}$$

Then

1. $0 * \delta_1 = 0$*; for all* $\delta_1 \in [0, 1]$,
2. $1 * \delta_1 = \delta_1$*; for all* $\delta_1 \in [0, 1]$,
3. $0 \leq \delta_1 * \delta_2 \leq 1$*; for all* $\delta_1, \delta_2 \in [0, 1]$,
4. $\delta_1 \leq \delta_1' \Rightarrow \delta_1 * \delta_2 \leq \delta_1' * \delta_2$*; for all* $\delta_1, \delta_1', \delta_2 \in [0, 1]$,
5. $\delta_1 * \delta_2 = \delta_2 * \delta_1$*; for all* $\delta_1, \delta_2 \in [0, 1]$
6. $(\delta_1 * \delta_2) * \delta_3 = \delta_1 * (\delta_2 * \delta_3)$*; for all* $\delta_1, \delta_2, \delta_3 \in [0, 1]$.

Lemma 2 *Let f, g be bounded, real valued function on a set U. Then*

$$\sup_U (f + g) \leq \sup_U f + \sup_U g,$$
$$\inf_U (f + g) \geq \inf_U f + \inf_U g.$$

Proof Since $f(u) \leq \sup_U f$ and $g(u) \leq \sup_U g$ for every $u \in U$, we have

$$f(u) + g(u) \leq \sup_U f + \sup_U g.$$

Thus,

$$\sup_U (f + g) \leq \sup_U f + \sup_U g.$$

Now, since $f(u) \geq \inf_U f$ and $g(u) \geq \inf_U g$ for every $u \in U$, we have

$$f(u) + g(u) \geq \inf_U f + \inf_U g.$$

Thus,

$$\inf_U (f + g) \geq \inf_U f + \inf_U g.$$

□

Lemma 3 *Let f, g be bounded, real valued function on a set U. Then*

$$\left|\sup_U f - \sup_U g\right| \leq \sup_U |f - g|,$$

$$\left|\inf_U f - \inf_U g\right| \leq \sup_U |f - g|.$$

Proof Since $f = f - g + g$, $f - g \leq |f - g|$ and from Lemma 2, we have

$$\sup_U f \leq \sup_U (f - g) + \sup_U g \leq \sup_U |f - g| + \sup_U g,$$

Then

$$\sup_U f - \sup_U g \leq \sup_U |f - g|.$$

Exchanging f and g in this inequality, we have

$$\sup_U g - \sup_U f \leq \sup_U |f - g|.$$

Therefore, we obtain

$$\left|\sup_U f - \sup_U g\right| \leq \sup_U |f - g|.$$

Replacing f by $-f$ and g by $-g$ in this inequality and using the inequality $\sup(-f) = -\inf f$, we obtain

$$\left|\inf_U f - \inf_U g\right| \leq \sup_U |f - g|.$$

□

Definition 16 [3] An intuitionistic fuzzy relation (IFR) R between X and $Y (R \in IFR(X \times Y))$ is defined as an intuitionistic fuzzy set on $X \times Y$, that is, R is given by

$$R = \{\langle (x, y), \mu_R(x, y), \nu_R(x, y)\rangle : (x, y) \in X \times Y\},$$

where $\mu_R, \nu_R : X \times Y \to [0, 1]$ satisfy the condition $\mu_R(x, y) + \nu_R(x, y) \leq 1$ for every $(x, y) \in X \times Y$. For each $(x, y) \in X \times Y$, $\mu_R(x, y)$ and $\nu_R(x, y)$ express the degree of membership of (x, y) to relation R and the degree of non-membership of (x, y) to relation R, respectively.

3 δ-equalities of intuitionistic fuzzy sets

In what follows, we define the new concept of δ-equalities for the intuitionistic fuzzy set.

Definition 17 Let U be a universe of discourse. Let A and B be two intuitionistic fuzzy sets on U, and $\mu_A(u)$, $\nu_A(u)$ and $\mu_B(u)$, $\nu_B(u)$ be their membership functions and non-membership functions respectively. Then A and B are said to be δ-equal if and only if

$$\sup_{u \in U} |\mu_A(u) - \mu_B(u)| \leq 1 - \delta, \tag{2}$$

$$\sup_{u \in U} |\nu_A(u) - \nu_B(u)| \leq 1 - \delta, \tag{3}$$

for all $u \in U$ and $0 \leq \delta \leq 1$. This can be denote it as $A = (\delta)B$.

From Definition 17, it is clear that $(1 - \delta)$ is the maximum difference or proximity measure between A and B, and δ is the degree of equality between them. It is customary to be noted that δ-equality of intuitionistic fuzzy sets construct the class of intuitionistic fuzzy relations. Considering the set (IFSs) of all intuitionistic fuzzy sets on U, based on this δ-equality, we can know the sets belong to IFSs which are most similar. This recognition is very important for the classification of information.

Remark 1 Some remarks for δ-equalities of intuitionistic fuzzy sets are below.

1. Because the new concept δ-equalities states about the equal degree of intuitionistic fuzzy sets, and the left side of (2) and (3) describes about the different level of two intuitionistic fuzzy sets. Then, the right side of (2) and (3) is defined by $1 - \delta$.
2. The two conditions (2) and (3) occur simultaneously. The natures of the two concepts δ-equalities of intuitionistic fuzzy sets and the order relation on L^* are different.
3. In the fact, the standard value of δ is depending on the each material model. Usually, the selected standard value of δ equal to the maximum value of δ in the material model.

4. An illustration is given as follows. Let $U = \{x, y, z\}$, $A, B \in \text{IFS}(U)$ and

$$A = \{\langle x, 0.25, 0.4\rangle, \langle y, 0.3, 0.41\rangle, \langle z, 0.18, 0.5\rangle\},$$
$$B = \{\langle x, 0.29, 0.32\rangle, \langle y, 0.33, 0.38\rangle, \langle z, 0.2, 0.54\rangle\}.$$

Thus, $\sup_{u\in U} |\mu_A(u) - \mu_B(u)| = \sup_{u\in U} vz\{0.04, 0.03, 0.02\}$ $x = 0.04 = 1 - 0.96$,

$$\sup_{u\in U} |\nu_A(u) - \nu_B(u)| = \sup_{u\in U}\{0.08, 0.03, 0.04\} = 0.08 = 1 - 0.92$$

Therefore $\sup_{u\in U} |\mu_A(u) - \mu_B(u)| \le 1 - 0.92$

$$\sup_{u\in U} |\nu_A(u) - \nu_B(u)| \le 1 - 0.92$$

We choose $\delta = 0.92$ and say that A and B have the same δ-equality (0.92).

5. Intuitionistic fuzzy set as generalized fuzzy set is quite interesting and useful in many application areas [16], such as in the fields of decision making [5], and medical diagnosis [52]. The new concept δ-equalities of intuitionistic fuzzy sets is a direct extension of the old concept δ-equalities of fuzzy sets [4]. We propose this extension to study more deeply about intuitionistic fuzzy theory in practical applications as the medical diagnostic problem.

We consider Example 3 to see more clearly the meaning of the concept δ-equalities of intuitionistic fuzzy sets.

Example 3 Assume that there are 3 medical experts A, B and C who diagnose for 3 patients x, y and z about s symptoms contracted. A, B and C are also denoted for the diagnostic results of 3 experts which, respectively, are expressed as form of the intuitionistic fuzzy sets as following

$$A = \{\langle x, 0.25, 0.4\rangle, \langle y, 0.3, 0.41\rangle, \langle z, 0.18, 0.5\rangle\},$$
$$B = \{\langle x, 0.29, 0.32\rangle, \langle y, 0.33, 0.38\rangle, \langle z, 0.2, 0.54\rangle\},$$
$$C = \{\langle x, 0.2, 0.1\rangle, \langle y, 0.32, 0.36\rangle, \langle z, 0.17, 0.14\rangle\}.$$

In order to assess the equal level between the results, we can use δ-equalities measure and we obtain $A = (0.92)B$, $A = (0.7)C$ and $B = (0.6)C$. Then, we say that between 3 diagnostic results, A and B have the largest equal level. In other words, A and B are closest together.

Moreover, the proposed notions overcome the limitation in the work of Pappis [26] in which the max-min compositional rule of inference is preserved with approximately equal fuzzy sets as well as the approach considered by Hong and Hwang [17] which was mainly based on the same philosophy of the max-min compositional rule of inference that is preserved with respect to approximately equal fuzzy sets and approximately equal fuzzy relation respectively. It also generalizes the work of Cai [4] regarding the δ-equalities for fuzzy sets. The applications of δ-equalities have important roles to fuzzy statistics and fuzzy reasoning. The aim of those proposals in comparison with the work of Cai [4] is to extend the existing definitions in a new context of IFS which has been shown to be better at modeling real-life applications than the fuzzy set [2] and to examine several characteristics and theorems of δ-equalities that were not (or partly) discussed in the previous works. The mentioned proposals are significant to understand the behavior of δ-equalities in IFS which is helpful to select appropriate setting for applications.

Now, we examine some characteristics of the δ-equalities for intuitionistic fuzzy sets in Definition 17.

Proposition 1 *For two intuitionistic fuzzy sets A and B, defined on U. The following assertions hold.*

1. $A = (0)B$,
2. $A = (1)B$ *if and only if* $A = B$,
3. $A = (\delta)B$ *if and only if* $B = (\delta)A$,
4. $A = (\delta_1)B$ *and if* $\delta_1 \ge \delta_2$, *then* $A = (\delta_2)B$,
5. *If* $A = (\delta_\alpha)B$ *for all* $\alpha \in J$, *where* J *is an index set, then* $A = \left(\sup_{\alpha\in J}\delta_\alpha\right) B$,
6. *For all* A, B, *there exist a unique* δ *such that* $A = (\delta)B$ *and if* $A = (\delta')B$, *then* $\delta' \le \delta$.

Proof Properties 1–4 can be proved easily. We only prove 5 and 6.

(5). Suppose that $A = (\delta_\alpha)B$, we have

$$\sup_{u\in U} |\mu_A(u) - \mu_B(u)| \le 1 - \delta_\alpha, \text{ or } \delta_\alpha \le 1 - \sup_{u\in U} |\mu_A(u) - \mu_B(u)| \text{ for all } \alpha \in J.$$

Thus, $\sup \delta_\alpha \le 1 - \sup_{u\in U} |\mu_A(u) - \mu_B(u)|$.

Similarly,

$$\sup_{u\in U} |\nu_A(u) - \nu_B(u)| \le 1 - \delta_\alpha, \text{ or } \delta_\alpha \le 1 - \sup_{u\in U} |\nu_A(u) - \nu_B(u)| \text{ for all } \alpha \in J.$$

Therefore, $\sup \delta_\alpha \le 1 - \sup_{u\in U} |\nu_A(u) - \nu_B(u)|$. Thus $A = \left(\sup_{\alpha\in J}\delta_\alpha\right) B$.

(6). Let $\delta = \min\left(1 - \sup_{u\in U} |\mu_A(u) - \mu_B(u)|, 1 - \sup_{u\in U} |\nu_A(u) - \nu_B(u)|\right)$ then $\sup_{u\in U} |\mu_A(u) - \mu_B(u)| \le 1 - \delta$ and $\sup_{u\in U} |\nu_A(u) - \mu_B(u)| \le \delta$.

This implies $A = (\delta)B$ and if $A = (\delta')B$, then $\delta' \le \delta$. Now suppose that there exist δ_1 and δ_2 such that they

simultaneously satisfy the required properties, then $\delta_1 \leq \delta_2$ and $\delta_2 \leq \delta_1$ which implies $\delta_1 = \delta_2$. Hence δ is unique. □

Proposition 2 *Let A, B and C be intuitionistic fuzzy sets define on U. If $A = (\delta_1)B$ and $B = (\delta_2)C$, then $A = (\delta)C$ where*

$$\delta = \delta_1 * \delta_2. \tag{4}$$

Proof Since $A = (\delta_1)B$, we have

$$\sup_{u\in U} |\mu_A(u) - \mu_B(u)| \leq 1 - \delta_1,$$
$$\sup_{u\in U} |\nu_A(u) - \nu_B(u)| \leq 1 - \delta_1.$$

Also $B = (\delta_2)C$, we have

$$\sup_{u\in U} |\mu_B(u) - \mu_C(u)| \leq 1 - \delta_2,$$
$$\sup_{u\in U} |\nu_B(u) - \nu_C(u)| \leq 1 - \delta_2.$$

Now

$$\sup_{u\in U} |\mu_A(u) - \mu_C(u)| \leq \sup_{u\in U} |\mu_A(u) - \mu_B(u)| + \sup_{u\in U} |\mu_B(u) - \mu_C(u)| \leq 1 - \delta_1 + 1 - \delta_2 \leq 1 - (\delta_1 + \delta_2 - 1).$$

Further, $\sup_{u\in U} |\mu_A(u) - \mu_C(u)| \leq 1$; so $\sup_{u\in U} |\mu_A(u) - \mu_C(u)| \leq 1 - \max(0, \delta_1 + \delta_2 - 1) = 1 - \delta_1 * \delta_2$.

Finally,

$$\sup_{u\in U} |\nu_A(u) - \nu_C(u)| \leq \sup_{u\in U} |\nu_A(u) - \nu_B(u)| + \sup_{u\in U} |\nu_B(u) - \nu_C(u)| \leq 1 - \delta_1 + 1 - \delta_2 \leq 1 - (\delta_1 + \delta_2 - 1).$$

We note that $\sup_{u\in U} |\nu_A(u) - \nu_C(u)| \leq 1$, therefore,

$$\sup_{u\in U} |\nu_A(u) - \nu_C(u)| \leq 1 - \max(0, \delta_1 + \delta_2 - 1) = 1 - \delta_1 * \delta_2.$$

Thus, $A = (\delta)C$ where $\delta = \delta_1 * \delta_2$. □

Now, the δ-equalities are applied to set theoretic operations of intuitionistic fuzzy set such as union, intersection, complement as following.

Proposition 3 *Let A and B be intuitionistic fuzzy sets define on U. Let A^c be the complement of A and B^c be the complement of B. Further let $A = (\delta)B$. Then*

$$A^c = (\delta)B^c. \tag{5}$$

Proof This is because

$$\sup_{u\in U} |\mu_{A^c}(u) - \mu_{B^c}(u)| = \sup_{u\in U} |\nu_A(u) - \nu_B(u)| \leq 1 - \delta,$$

And next

$$\sup_{u\in U} |\nu_{A^c}(u) - \nu_{B^c}(u)| = \sup_{u\in U} |\mu_A(u) - \mu_B(u)| \leq 1 - \delta.$$

This shows that $A^c = (\delta)B^c$. □

Proposition 4 *Let A_1, A_2, B_1 and B_2 be intuitionistic fuzzy sets define on U. Let $A_1 = (\delta_1)B_1$, $A_2 = (\delta_2)B_2$. Then*

$$A_1 \cap A_2 = (\min(\delta_1, \delta_2))B_1 \cap B_2. \tag{6}$$

Proof From Lemma 3, we have

$$\begin{aligned}
&\sup_{u\in U} \left|\mu_{A_1\cap A_2}(u) - \mu_{B_1\cap B_2}(u)\right| \\
&= \sup_{u\in U} \left|\min(\mu_{A_1}(u), \mu_{A_2}(u)) - \min(\mu_{B_1}(u), \mu_{B_2}(u))\right| \\
&\leq \sup_{u\in U} \max\left(\sup_{u\in U}\left|\mu_{A_1}(u), \mu_{B_1}(u)\right|, \sup_{u\in U}\left|\mu_{A_2}(u), \mu_{B_2}(u)\right|\right) \\
&\leq \sup_{u\in U} \max(1 - \delta_1, 1 - \delta_2) \leq 1 - \min(\delta_1, \delta_2).
\end{aligned}$$

Thus

$$\sup_{u\in U} \left|\mu_{A_1\cap A_2}(u) - \mu_{B_1\cap B_2}(u)\right| \leq 1 - \min(\delta_1, \delta_2).$$

Next,

$$\begin{aligned}
&\sup_{u\in U} \left|\nu_{A_1\cap A_2}(u) - \nu_{B_1\cap B_2}(u)\right| \\
&= \left|\max(\nu_{A_1}(u), \nu_{A_2}(u)) - \max(\nu_{B_1}(u), \nu_{B_2}(u))\right| \\
&\leq \sup_{u\in U} \max\left(\sup_{u\in U}\left|\nu_{A_1}(u), \nu_{B_1}(u)\right|, \left|\nu_{A_2}(u) - \nu_{B_2}(u)\right|\right) \\
&\leq \sup_{u\in U} \max(1 - \delta_1, 1 - \delta_2) \leq 1 - \min(\delta_1, \delta_2).
\end{aligned}$$

Therefore,

$$\sup_{u\in U} \left|\nu_{A_1\cap A_2}(u) - \nu_{B_1\cap B_2}(u)\right| \leq 1 - \min(\delta_1, \delta_2).$$

This implies that $A_1 \cap A_2 = (\min(\delta_1, \delta_2))B_1 \cap B_2$. □

Remark 2 Proposition 4 has the important meaning in supporting aggregate information. We can consider Example 4 as follows.

Example 4 A patient p was diagnosed with liver disease by two hospitals h_1 and h_2 through 3 indexes x_1, x_2, x_3. Let two sets A_1 and A_2 which are proposed by h_1 and h_2, respectively, be standard levels of indexes x_1, x_2, x_3. Let two sets B_1 and B_2 be test results of patient p by h_1 and h_2, respectively through 3 indexes x_1, x_2, x_3 and $B_1 = (\delta_1)A_1$, $B_2 = (\delta_2)A_2$. Then, $A_1 \cap A_2$ is aggregated standard level of indexes x_1, x_2, x_3 and $B_1 \cap B_2$ is the aggregated test result of patient p. From Proposition 4, we have $B_1 \cap B_2 =$

$(\min(\delta_1, \delta_2))A_1 \cap A_2$, thus $\delta = \min(\delta_1, \delta_2)$ is the final diagnosis which describes the severity of the disease of patient p.

Proposition 5 *Let A_α, B_α be intuitionistic fuzzy sets define on U, for all $\alpha \in J$, where J is an index set. Let $A_\alpha = (\delta_\alpha)B_\alpha$, for all $\alpha \in J$. Let $\bigcap_{\alpha\in J} A_\alpha$ represents the intersection of $\{A_\alpha : \alpha \in J\}$ and $\bigcap_{\alpha\in J} B_\alpha$ represents the intersection of $\{B_\alpha : \alpha \in J\}$, and $\mu_{\bigcap_{\alpha\in J} A_\alpha}(u) = \inf_{\alpha\in J}\mu_{A_\alpha}(u)$, $\nu_{\bigcap_{\alpha\in J} A_\alpha}(u) = \sup_{\alpha\in J}\nu_{A_\alpha}(u)$ and $\mu_{\bigcap_{\alpha\in J} B_\alpha}(u) = \inf_{\alpha\in J}\mu_{B_\alpha}(u)$, $\nu_{\bigcap_{\alpha\in J} B_\alpha}(u) = \sup_{\alpha\in J}\nu_{B_\alpha}(u)$ their membership functions and non-membership functions respectively. Then*

$$\bigcap_{\alpha\in J} A_\alpha = \left(\inf_{\alpha\in J}\delta_\alpha\right)\bigcap_{\alpha\in J} B_\alpha. \tag{7}$$

Proof This is because

$$\begin{aligned}
&\sup_{u\in U}\left|\mu_{\bigcap_{\alpha\in J} A_\alpha}(u) - \mu_{\bigcap_{\alpha\in J} B_\alpha}(u)\right| = \sup_{u\in U}\left|\inf_{\alpha\in J}\mu_{A_\alpha}(u) - \inf_{\alpha\in J}\mu_{B_\alpha}(u)\right| \\
&\le \sup_{u\in U}\sup_{\alpha\in J}\left|\mu_{A_\alpha}(u) - \mu_{B_\alpha}(u)\right| \\
&\le \sup_{u\in U}\sup_{\alpha\in J}(1-\delta_\alpha) = 1 - \inf_{\alpha\in J}\delta_\alpha.
\end{aligned}$$

This implies

$$\sup_{u\in U}\left|\mu_{\bigcap_{\alpha\in J} A_\alpha}(u) - \mu_{\bigcap_{\alpha\in J} B_\alpha}(u)\right| \le 1 - \inf_{\alpha\in J}\delta_\alpha.$$

Finally, we have

$$\begin{aligned}
&\sup_{u\in U}\left|\nu_{\bigcap_{\alpha\in J} A_\alpha}(u) - \nu_{\bigcap_{\alpha\in J} B_\alpha}(u)\right| = \sup_{u\in U}\left|\sup_{\alpha\in J}\nu_{A_\alpha}(u) - \sup_{\alpha\in J}\nu_{B_\alpha}(u)\right| \\
&\le \sup_{u\in U}\sup_{\alpha\in J}\left|\nu_{A_\alpha}(u) - \nu_{B_\alpha}(u)\right| \\
&\le \sup_{u\in U}\sup_{\alpha\in J}(1-\delta_\alpha) = 1 - \inf_{\alpha\in J}\delta_\alpha.
\end{aligned}$$

Therefore,

$$\sup_{u\in U}\left|\nu_{\bigcap_{\alpha\in J} A_\alpha}(u) - \nu_{\bigcap_{\alpha\in J} B_\alpha}(u)\right| \le 1 - \inf_{\alpha\in J}\delta_\alpha.$$

Thus $\bigcap_{\alpha\in J} A_\alpha = \left(\inf_{\alpha\in J}\delta_\alpha\right)\bigcap_{\alpha\in J} B_\alpha$. □

Proposition 6 *Let A_α, B_α be intuitionistic fuzzy sets define on U for all $\alpha \in J$, where J is an index set. Let $A_\alpha = (\delta_\alpha)B_\alpha$, for all $\alpha \in J$. Let $\bigcup_{\alpha\in J} A_\alpha$ represents the union of $\{A_\alpha : \alpha \in J\}$ and $\bigcup_{\alpha\in J} B_\alpha$ represents the union of $\{B_\alpha : \alpha \in J\}$, and $\mu_{\bigcup_{\alpha\in J} A_\alpha}(u) = \sup_{\alpha\in J}\mu_{A_\alpha}(u)$, $\nu_{\bigcup_{\alpha\in J} A_\alpha}(u) = \inf_{\alpha\in J}\nu_{A_\alpha}(u)$ and $\mu_{\bigcup_{\alpha\in J} B_\alpha}(u) = \sup_{\alpha\in J}\mu_{B_\alpha}(u)$, $\nu_{\bigcup_{\alpha\in J} B_\alpha}(u) = \inf_{\alpha\in J}\nu_{B_\alpha}(u)$ their membership functions, and non-membership functions, respectively. Then*

$$\bigcup_{\alpha\in J} A_\alpha = \left(\inf_{\alpha\in J}\delta_\alpha\right)\bigcup_{\alpha\in J} B_\alpha. \tag{8}$$

Proof From Propositions 4 and 5, we have $A_\alpha^c = (\delta_\alpha)B_\alpha^c$ and $\bigcap_{\alpha\in J} A_\alpha^c = \left(\inf_{\alpha\in J}\delta_\alpha\right)\bigcap_{\alpha\in J} B_\alpha^c$. Thus

$$\left(\bigcap_{\alpha\in J} A_\alpha^c\right)^c = \left(\inf_{\alpha\in J}\delta_\alpha\right)\left(\bigcap_{\alpha\in J} B_\alpha^c\right)^c \text{ or } \bigcup_{\alpha\in J} A_\alpha = \left(\inf_{\alpha\in J}\delta_\alpha\right)\bigcup_{\alpha\in J} B_\alpha.$$

□

Remark 3 Proposition 6 is an extension from Proposition 4 by considering α initial intuitionistic fuzzy sets instead of considering two initial intuitionistic fuzzy sets as in Proposition 4.

Corollary 1 *Let $A_{\alpha\beta}$, $B_{\alpha\beta}$ be intuitionistic fuzzy sets define on U, for all $\alpha \in J_1$ and $\beta \in J_2$ where J_1 and J_2 are index sets. Let $A_{\alpha\beta} = (\delta_{\alpha\beta})B_{\alpha\beta}$; $\alpha \in J_1$ and $\beta \in J_2$. Then*

$$\bigcup_{\alpha\in J_1}\bigcap_{\beta\in J_2} A_{\alpha\beta} = \left(\inf_{\alpha\in J_1}\inf_{\beta\in J_2}\delta_\alpha\right)\bigcup_{\alpha\in J_1}\bigcap_{\beta\in J_2} B_{\alpha\beta}, \tag{9}$$

$$\bigcap_{\alpha\in J_1}\bigcup_{\beta\in J_2} A_{\alpha\beta} = \left(\inf_{\alpha\in J_1}\inf_{\beta\in J_2}\delta_\alpha\right)\bigcap_{\alpha\in J_1}\bigcup_{\beta\in J_2} B_{\alpha\beta}. \tag{10}$$

Proof This follows from Propositions 4 and 6. □

Corollary 2 *Let A_k, B_k be intuitionistic fuzzy sets define on U, for all $k = 1, 2, 3 \ldots$. Let $A_k = (\delta_k)B_k$, $k = 1, 2, 3, \cdots$ and let*

$$\lim_{n\to\infty}\sup A_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k,\ \lim_{n\to\infty}\inf A_n = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k, \tag{11}$$

$$\lim_{n\to\infty}\sup \mathrm{B}_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} B_k,\ \lim_{n\to\infty}\inf B_n = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} B_k. \tag{12}$$

Then

$$\lim_{n\to\infty}\sup A_n = \left(\inf_{n\ge 1}\delta_n\right)\lim_{n\to\infty}\sup B_n, \tag{13}$$

$$\lim_{n\to\infty}\inf A_n = \left(\inf_{n\ge 1}\delta_n\right)\lim_{n\to\infty}\inf B_n. \tag{14}$$

Proof From Corollary 1, we have

$$\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k = \left(\inf_{n\geq 1}\inf_{k\geq n}\delta_\alpha\right)\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} B_k,$$

$$\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k = \left(\inf_{n\geq 1}\inf_{k\geq n}\delta_\alpha\right)\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} B_k,$$

which implies that

$$\lim_{n\to\infty}\sup A_n = \left(\inf_{n\geq 1}\delta_n\right)\lim_{n\to\infty}\sup B_n,$$

$$\lim_{n\to\infty}\inf A_n = \left(\inf_{n\geq 1}\delta_n\right)\lim_{n\to\infty}\inf B_n.$$

□

Proposition 7 *Let A_1, A_2, B_1 and B_2 be intuitionistic fuzzy sets define on U. Let $A_1 = (\delta_1)B_1$ and $A_2 = (\delta_2)B_2$. Then*

$$A_1A_2 = (\delta_1 * \delta_2)\, B_1B_2. \tag{15}$$

Proof Since, we have

$$\begin{aligned}
&\sup_{u\in U}\left|\mu_{A_1A_2}(u)-\mu_{B_1B_2}(u)\right| = \sup_{u\in U}\left|\mu_{A_1}(u)\mu_{A_2}(u) - \mu_{B_1}(u)\mu_{B_2}(u)\right| \\
&= \sup_{u\in U}\left|\mu_{A_1}(u)\mu_{A_2}(u) - \mu_{A_2}(u)\mu_{B_1}(u) + \mu_{A_2}(u)\mu_{B_1}(u)\right. \\
&\left.-\mu_{B_1}(u)\mu_{B_2}(u)\right| \\
&\leq \sup_{u\in U}\left[\mu_{A_2}(u)\left|\mu_{A_1}(u) - \mu_{B_1}(u)\right| + \mu_{B_1}(u)\left|\mu_{A_2}(u) - \mu_{B_2}(u)\right|\right] \\
&\leq \sup_{u\in U}(1-\delta_1+1-\delta_2) = 1-(\delta_1+\delta_2-1).
\end{aligned}$$

Further, we have $\sup_{u\in U}\left|\mu_{A_1A_2}(u) - \mu_{B_1B_2}(u)\right| \leq 1$; so

$$\sup_{u\in U}\left|\mu_{A_1A_2}(u) - \mu_{B_1B_2}(u)\right| \leq 1-\delta_1*\delta_2.$$

Finally, we show that

$$\begin{aligned}
&\sup_{u\in U}\left|\nu_{A_1A_2}(u) - \nu_{B_1B_2}(u)\right| = \sup_{u\in U}\left|(\nu_{A_1}(u) + \nu_{A_2}(u)\right. \\
&\left.-\nu_{A_1}(u)\nu_{A_2}(u)) - (\nu_{B_1}(u) + \nu_{B_2}(u) - \nu_{B_1}(u)\nu_{B_2}(u))\right| \\
&= \sup_{u\in U}\left|(1-\nu_{B_2}(u))(\nu_{A_1}(u) - \nu_{B_1}(u))\right. \\
&\left.+(1-\nu_{A_1}(u))(\nu_{A_2}(u) - \nu_{B_2}(u))\right| \\
&\leq 1-\delta_1+1-\delta_2 = 1-(\delta_1+\delta_2-1).
\end{aligned}$$

Since $\sup_{u\in U}\left|\nu_{A_1A_2}(u) - \nu_{B_1B_2}(u)\right| \leq 1$; so

$$\sup_{u\in U}\left|\nu_{A_1A_2}(u) - \nu_{B_1B_2}(u)\right| \leq 1-\delta_1*\delta_2.$$

Thus $A_1A_2 = (\delta_1*\delta_2)B_1B_2$. □

Corollary 3 *Let A_j and B_j be intuitionistic fuzzy sets define on U, for all $j = 1, 2, 3, \cdots, n$. Let $A_j = (\delta_j)B_j$, where $j = 1, 2, 3, \cdots, n$. Then*

$$A_1\cdots A_n = (\delta_1*\cdots*\delta_n)B_1\cdots B_n. \tag{16}$$

Proof The proof is followed from Proposition 7. □

Proposition 8 *Let A_1, A_2, B_1 and B_2 be intuitionistic fuzzy sets define on U. Let $A_1 = (\delta_1)B_1$ and $A_2 = (\delta_2)B_2$. Then*

$$A_1 + A_2 = (\delta_1*\delta_2)\, B_1 + B_2. \tag{17}$$

Proof Since, we have

$$\begin{aligned}
&\sup_{u\in U}\left|\mu_{A_1+A_2}(u) - \mu_{B_1+B_2}(u)\right| = \sup_{u\in U}\left|\left(\mu_{A_1}(u) + \mu_{A_2}(u)\right.\right. \\
&\left.\left.-\mu_{A_1}(u)\,\mu_{A_2}(u)\right) - \left(\mu_{B_1}(u) + \mu_{B_2}(u) - \mu_{B_1}(u)\,\mu_{B_2}(u)\right)\right| \\
&= \sup_{u\in U}\left|\left(1-\mu_{B_2}(u)\right)\left(\mu_{A_1}(u) - \mu_{B_1}(u)\right)\right. \\
&\left.+\left(1-\mu_{A_1}(u)\right)\left(\mu_{A_2}(u) - \mu_{B_2}(u)\right)\right| \\
&\leq 1-\delta_1+1-\delta_2 = 1-(\delta_1+\delta_2-1).
\end{aligned}$$

Further, we note that $\sup_{u\in U}\left|\mu_{A_1+A_2}(u) - \mu_{B_1+B_2}(u)\right| \leq 1$, so

$$\sup_{u\in U}\left|\mu_{A_1+A_2}(u) - \mu_{B_1+B_2}(u)\right| \leq 1-(\delta_1*\delta_2).$$

Next,

$$\begin{aligned}
&\sup_{u\in U}\left|\nu_{A_1+A_2}(u) - \nu_{B_1+B_2}(u)\right| = \sup_{u\in U}\left|\nu_{A_1}(u)\,\nu_{A_2}(u)\right. \\
&\left.-\nu_{B_1}(u)\,\nu_{B_2}(u)\right| \\
&= \sup_{u\in U}\left|\nu_{A_1}(u)\,\nu_{A_2}(u) - \nu_{A_2}(u)\,\nu_{B_1}(u) + \nu_{A_2}(u)\,\nu_{B_1}(u)\right. \\
&\left.-\nu_{B_1}(u)\,\nu_{B_2}(u)\right| \\
&\leq \sup_{u\in U}\left[\nu_{A_2}(u)\left|\nu_{A_1}(u) - \nu_{B_1}(u)\right| + \nu_{B_1}(u)\left|\nu_{A_2}(u) - \nu_{B_2}(u)\right|\right] \\
&\leq \sup_{u\in U}(1-\delta_1+1-\delta_2) = 1-(\delta_1+\delta_2-1).
\end{aligned}$$

Further, we have, $\sup_{u\in U}\left|\nu_{A_1+A_2}(u) - \nu_{B_1+B_2}(u)\right| \leq 1$, it follows that

$$\sup_{u\in U}\left|\nu_{A_1+A_2}(u) - \nu_{B_1+B_2}(u)\right| \leq 1-(\delta_1*\delta_2).$$

Thus $A_1 + A_2 = (\delta_1*\delta_2)\, B_1 + B_2$. □

Corollary 4 *Let A_j and B_j be intuitionistic fuzzy sets define on U, for all $j = 1, 2, 3, \cdots, n$. Suppose $A_j = (\delta_j)B_j$, where $j = 1, 2, 3, \cdots, n$. Then*

$$A_1 + \cdots + A_n = (\delta_1*\cdots*\delta_n)\, B_1 + \cdots + B_n. \tag{18}$$

Proof This is followed from Proposition 8. □

Proposition 9 *Let A_1, A_2, B_1 and B_2 be intuitionistic fuzzy sets define on U. Suppose $A_1 = (\delta_1)B_1$, $A_2 = (\delta_2)B_2$, and*

$$\mathcal{T}_3(x, y) = (\max(0, x_1 + y_1 - 1), \min(1, x_2 + y_2)), \forall x, y \in L^*.$$

Then

$$A_1 \cap_{\mathcal{T}_3} A_2 = (\delta_1 * \delta_2) B_1 \cap_{\mathcal{T}_3} B_2. \tag{19}$$

Proof Since

$$\begin{aligned}
&\sup_{u\in U}\left|\mu_{A_1\cap_{\mathcal{T}_3}A_2}(u) - \mu_{B_1\cap_{\mathcal{T}_3}B_2}(u)\right| \\
&= \sup_{u\in U}\left|\max\left(0, \mu_{A_1}(u) + \mu_{A_2}(u) - 1\right)\right. \\
&\left. - \max\left(0, \mu_{B_1}(u) + \mu_{B_2}(u) - 1\right)\right| \\
&\le \sup_{u\in U}\left|\mu_{A_1}(u) + \mu_{A_2}(u) - \mu_{B_1}(u) - \mu_{B_2}(u)\right| \\
&\le \sup_{u\in U}\left(\left|\mu_{A_1}(u) - \mu_{B_1}(u)\right| + \left|\mu_{A_2}(u) - \mu_{B_2}(u)\right|\right) \\
&\le 1 - \delta_1 + 1 - \delta_2 = 1 - (\delta_1 + \delta_2 - 1),
\end{aligned}$$

Since we have $\sup_{u\in U}\left|\mu_{A_1\cap_{\mathcal{T}_3}A_2}(u) - \mu_{B_1\cap_{\mathcal{T}_3}B_2}(u)\right| \le 1$. Therefore

$$\sup_{u\in U}\left|\mu_{A_1\cap_{\mathcal{T}_3}A_2}(u) - \mu_{B_1\cap_{\mathcal{T}_3}B_2}(u)\right| \le 1 - (\delta_1 * \delta_2).$$

Further,

$$\begin{aligned}
&\sup_{u\in U}\left|\nu_{A_1\cap_{\mathcal{T}_3}A_2}(u) - \nu_{B_1\cap_{\mathcal{T}_3}B_2}(u)\right| = \sup_{u\in U}\left|\min\left(1, \nu_{A_1}(u)\right.\right. \\
&\left.\left. + \nu_{A_2}(u)\right) - \min\left(1, \nu_{B_1}(u) + \nu_{B_2}(u)\right)\right| \\
&\le \sup_{u\in U}\left|\nu_{A_1}(u) + \nu_{A_2}(u) - \nu_{B_1}(u) - \nu_{B_2}(u)\right| \\
&\le \sup_{u\in U}\left(\left|\nu_{A_1}(u) + \nu_{B_1}(u)\right| + \left|\nu_{A_2}(u) - \nu_{B_2}(u)\right|\right) \\
&\le 1 - \delta_1 + 1 - \delta_2 = 1 - (\delta_1 + \delta_2 - 1),
\end{aligned}$$

and we have $\sup_{u\in U}\left|\nu_{A_1\cap_{\mathcal{T}_3}A_2}(u) - \nu_{B_1\cap_{\mathcal{T}_3}B_2}(u)\right| \le 1$, so

$$\sup_{u\in U}\left|\nu_{A_1\cap_{\mathcal{T}_3}A_2}(u) - \nu_{B_1\cap_{\mathcal{T}_3}B_2}(u)\right| \le 1 - (\delta_1 * \delta_2).$$

Thus $A_1 \cap_{\mathcal{T}_3} A_2 = (\delta_1 * \delta_2) B_1 \cap_{\mathcal{T}_3} B_2$. □

Proposition 10 *Let A_1, A_2, B_1 and B_2 be intuitionistic fuzzy sets define on U. Suppose $A_1 = (\delta_1)B_1$, $A_2 = (\delta_2)B_2$ and*

$$\mathcal{S}_3(x, y) = (\min(1, x_1 + y_1), \max(0, x_2 + y_2 - 1)), \forall x, y \in L^*.$$

Then

$$A_1 \cup_{\mathcal{S}_3} A_2 = (\delta_1 * \delta_2) B_1 \cup_{\mathcal{S}_3} B_2. \tag{20}$$

Proof Since

$$\begin{aligned}
&\sup_{u\in U}\left|\mu_{A_1\cup_{\mathcal{S}_3}A_2}(u) - \mu_{B_1\cup_{\mathcal{S}_3}B_2}(u)\right| \\
&= \sup_{u\in U}\left|\min\left(1, \mu_{A_1}(u) + \mu_{A_2}(u)\right) - \min\left(1, \mu_{B_1}(u) + \mu_{B_2}(u)\right)\right| \\
&\le \sup_{u\in U}\left|\mu_{A_1}(u) + \mu_{A_2}(u) - \mu_{B_1}(u) - \mu_{B_2}(u)\right| \\
&\le \sup_{u\in U}\left(\left|\mu_{A_1}(u) - \mu_{B_1}(u)\right| + \left|\mu_{A_2}(u) - \mu_{B_2}(u)\right|\right) \\
&\le 1 - \delta_1 + 1 - \delta_2 = 1 - (\delta_1 + \delta_2 - 1),
\end{aligned}$$

Since we have $\sup_{u\in U}\left|\mu_{A_1\cup_{\mathcal{S}_3}A_2}(u) - \mu_{B_1\cup_{\mathcal{S}_3}B_2}(u)\right| \le 1$. Therefore

$$\sup_{u\in U}\left|\mu_{A_1\cup_{\mathcal{S}_3}A_2}(u) - \mu_{B_1\cup_{\mathcal{S}_3}B_2}(u)\right| \le 1 - (\delta_1 * \delta_2).$$

Next,

$$\begin{aligned}
&\sup_{u\in U}\left|\nu_{A_1\cup_{\mathcal{S}_3}A_2}(u) - \nu_{B_1\cup_{\mathcal{S}_3}B_2}(u)\right| \\
&= \sup_{u\in U}\left|\max\left(0, \nu_{A_1}(u) + \nu_{A_2}(u) - 1\right)\right. \\
&\left. - \max\left(0, \nu_{B_1}(u) + \nu_{B_2}(u) - 1\right)\right| \\
&\le \sup_{u\in U}\left|\nu_{A_1}(u) + \nu_{A_2}(u) - \nu_{B_1}(u) - \nu_{B_2}(u)\right| \\
&\le \sup_{u\in U}\left(\left|\nu_{A_1}(u) + \nu_{B_1}(u)\right| + \left|\nu_{A_2}(u) - \nu_{B_2}(u)\right|\right) \\
&\le 1 - \delta_1 + 1 - \delta_2 = 1 - (\delta_1 + \delta_2 - 1),
\end{aligned}$$

We also have $\sup_{u\in U}\left|\nu_{A_1\cup_{\mathcal{S}_3}A_2}(u) - \nu_{B_1\cup_{\mathcal{S}_3}B_2}(u)\right| \le 1$, so

$$\sup_{u\in U}\left|\nu_{A_1\cup_{\mathcal{S}_3}A_2}(u) - \nu_{B_1\cup_{\mathcal{S}_3}B_2}(u)\right| \le 1 - (\delta_1 * \delta_2).$$

Hence, $A_1 \cup_{\mathcal{S}_3} A_2 = (\delta_1 * \delta_2) B_1 \cup_{\mathcal{S}_3} B_2$. □

Proposition 11 *Let A_1, A_2, B_1 and B_2 be intuitionistic fuzzy sets define on U. Suppose $A_1 = (\delta_1)B_1$ and $A_2 = (\delta_2)B_2$. Let*

$$\mathcal{T}_4(x, y) = (\max(0, x_1 + y_1 - 1), x_2 + y_2 - x_2 y_2), \forall x, y \in L^*.$$

Then

$$A_1 \cap_{\mathcal{T}_4} A_2 = (\delta_1 * \delta_2) B_1 \cap_{\mathcal{T}} B_2. \tag{21}$$

Proof This is followed from Propositions 7 and 9. □

Proposition 12 *Let A_1, A_2, B_1 and B_2 be intuitionistic fuzzy sets define on U. Suppose $A_1 = (\delta_1)B_1$, $A_2 = (\delta_2)B_2$ and*

$$\mathcal{S}_4(x, y) = (x_1 + y_1 - x_1 y_1, \max(0, x_2 + y_2 - 1)), \forall x, y \in L^*.$$

Then

$$A_1 \cup_{\mathcal{S}_4} A_2 = (\delta_1 * \delta_2)\, B_1 \cup_{\mathcal{S}_4} B_2. \tag{22}$$

Proof This is followed from Propositions 8 and 10. □

Proposition 13 *Let A_1, A_2, B_1 and B_2 be intuitionistic fuzzy sets define on U. Suppose $A_1 = (\delta_1)B_1$ and $A_2 = (\delta_2)B_2$. Let*

$$\mathcal{T}_5(x, y) = (\max(0, \lambda(x_1 + y_1) - \lambda + (1-\lambda)x_1 y_1),\ \min(1, x_2 + y_2)), \forall x, y \in L^*, 0 < \lambda < 1.$$

Then

$$A_1 \cap_{\mathcal{T}_5} A_2 = (\delta_1 * \delta_2)\, B_1 \cap_{\mathcal{T}_5} B_2. \tag{23}$$

Proof Since for all $0 < \lambda < 1$, we have

$$\begin{aligned}
&\sup_{u\in U}\left|\mu_{A_1\cap_{\mathcal{T}_5}A_2}(u) - \mu_{B_1\cap_{\mathcal{T}_5}B_2}(u)\right| \\
&= \sup_{u\in U}\left|\begin{array}{l}\max\left(0, \lambda\left(\mu_{A_1}(u) + \mu_{A_2}(u)\right) - \lambda + (1-\lambda)\mu_{A_1}(u)\mu_{A_2}(u)\right) \\ -\max\left(0, \lambda\left(\mu_{B_1}(u) + \mu_{B_2}(u)\right) - \lambda + (1-\lambda)\mu_{B_1}(u)\mu_{B_2}(u)\right)\end{array}\right| \\
&\le \sup_{u\in U}\left|\begin{array}{l}\lambda(\mu_{A_1}(u) + \mu_{A_2}(u)) - \lambda + (1-\lambda)\mu_{A_1}(u)\mu_{A_2}(u) \\ -\lambda(\mu_{B_1}(u) + \mu_{B_2}(u)) + \lambda - (1-\lambda)\mu_{B_1}(u)\mu_{B_2}(u)\end{array}\right| \\
&\le \sup_{u\in U}\left|\lambda(\mu_{A_1}(u) - \mu_{B_1}(u)) + \lambda(\mu_{A_2}(u) - \mu_{B_2}(u)) + (1-\lambda)(\mu_{A_1}(u)\mu_{A_2}(u) - \mu_{B_1}(u)\mu_{B_2}(u))\right| \\
&\le \sup_{u\in U}\left|\begin{array}{l}\lambda(\mu_{A_1}(u) - \mu_{B_1}(u)) + \lambda(\mu_{A_2}(u) - \mu_{B_2}(u)) + \\ (1-\lambda)(\mu_{A_1}(u)\mu_{A_2}(u) - \mu_{A_2}(u)\mu_{B_1}(u) + \mu_{A_2}(u)\mu_{B_1}(u) - \mu_{B_1}(u)\mu_{B_2}(u)\end{array}\right| \\
&\le \sup_{u\in U}\left[\begin{array}{l}\lambda\left|\mu_{A_1}(u) - \mu_{B_1}(u)\right| + \lambda\left|\mu_{A_2}(u) - \mu_{B_2}(u)\right| \\ +(1-\lambda)\left|\mu_{A_1}(u) - \mu_{B_1}(u)\right| + (1-\lambda)\left|\mu_{A_2}(u) - \mu_{B_2}(u)\right|\end{array}\right] \\
&\le \sup_{u\in U}(1 - \delta_1 + 1 - \delta_2) = 1 - (\delta_1 + \delta_2 - 1),
\end{aligned}$$

Since we have $\sup_{u\in U}\left|\mu_{A_1\cap_{\mathcal{T}_5}A_2}(u) - \mu_{B_1\cap_{\mathcal{T}_5}B_2}(u)\right| \le 1$. Therefore

$$\sup_{u\in U}\left|\mu_{A_1\cap_{\mathcal{T}_5}A_2}(u) - \mu_{B_1\cap_{\mathcal{T}_5}B_2}(u)\right| \le 1 - (\delta_1 * \delta_2).$$

Further,

$$\begin{aligned}
&\sup_{u\in U}\left|\nu_{A_1\cap_{\mathcal{T}_5}A_2}(u) - \nu_{B_1\cap_{\mathcal{T}_5}B_2}(u)\right| \\
&= \sup_{u\in U}\left|\min\left(1, \nu_{A_1}(u) + \nu_{A_2}(u)\right) - \min\left(1, \nu_{B_1}(u) + \nu_{B_2}(u)\right)\right| \\
&\le \sup_{u\in U}\left|\nu_{A_1}(u) + \nu_{A_2}(u) - \nu_{B_1}(u) - \nu_{B_2}(u)\right| \\
&\le \sup_{u\in U}\left(\left|\nu_{A_1}(u) - \nu_{B_1}(u)\right| + \left|\nu_{A_2}(u) - \nu_{B_2}(u)\right|\right) \\
&\le (1 - \delta_1 + 1 - \delta_2) = 1 - (\delta_1 + \delta_2 - 1),
\end{aligned}$$

and we have $\sup_{u\in U}\left|\nu_{A_1\cap_{\mathcal{T}_5}A_2}(u) - \nu_{B_1\cap_{\mathcal{T}_5}B_2}(u)\right| \le 1$, so

$$\sup_{u\in U}\left|\nu_{A_1\cap_{\mathcal{T}_5}A_2}(u) - \nu_{B_1\cap_{\mathcal{T}_5}B_2}(u)\right| \le 1 - (\delta_1 * \delta_2).$$

Thus $A_1 \cap_{\mathcal{T}_5} A_2 = (\delta_1 * \delta_2)\, B_1 \cap_{\mathcal{T}_5} B_2$. □

Proposition 14 *Let A_1, A_2, B_1 and B_2 be intuitionistic fuzzy sets define on U. Suppose $A_1 = (\delta_1)B_1$ and $A_2 = (\delta_2)B_2$. Let*

$$\mathcal{T}_6(x, y) = (\max(0, \lambda(x_1 + y_1) - \lambda + (1-\lambda)x_1 y_1),\ x_2 + y_2 - x_2 y_2), \forall x, y \in L^*, 0 < \lambda < 1.$$

Then

$$A_1 \cap_{\mathcal{T}_6} A_2 = (\delta_1 * \delta_2)\, B_1 \cap_{\mathcal{T}_6} B_2. \tag{24}$$

Proof This is followed from above propositions. □

Proposition 15 *Let A_1, A_2, B_1 and B_2 be intuitionistic fuzzy sets define on U. Suppose $A_1 = (\delta_1)B_1$ and $A_2 = (\delta_2)B_2$. Let*

$$\mathcal{S}_5(x, y) = (\min(1, x_1 + y_1), \max(0, \lambda(x_2 + y_2) - \lambda + (1-\lambda)x_2 y_2)), \forall x, y \in L^*, 0 < \lambda < 1.$$

Then

$$A_1 \cup_{\mathcal{S}_5} A_2 = (\delta_1 * \delta_2)\, B_1 \cup_{\mathcal{S}_5} B_2. \tag{25}$$

Proof This is followed from above propositions. □

Proposition 16 *Let A_1, A_2, B_1 and B_2 be intuitionistic fuzzy sets define on U. Suppose $A_1 = (\delta_1)B_1$ and $A_2 = (\delta_2)B_2$. Let*

$$\mathcal{S}_6(x, y) = (x_1 + y_1 - x_1 y_1, \max(0, \lambda(x_2 + y_2) - \lambda + (1-\lambda)x_2 y_2)), \forall x, y \in L^*, 0 < \lambda < 1.$$

Then

$$A_1 \cup_{\mathcal{S}_6} A_2 = (\delta_1 * \delta_2) B_1 \cup_{\mathcal{S}_6} B_2. \tag{26}$$

Proof This is followed from above propositions. □

Remark 4 From the hypothetical part of Proposition 4, we replace the operation $\cap$ by $\cdot, \cap_{\mathcal{T}_3}, \cap_{\mathcal{T}_4}, \cap_{\mathcal{T}_5}, \cap_{\mathcal{T}_6}, +, \cup_{\mathcal{S}_3}, \cup_{\mathcal{S}_4}, \cup_{\mathcal{S}_5}$ and $\cup_{\mathcal{S}_6}$ as in the hypothetical part of Proposition 7 and from Propositions 8 to 16, then we obtain $\delta = \delta_1 * \delta_2$ instead of $\delta = \min(\delta_1, \delta_2)$ as in conclusion part of Proposition 4.

Definition 18 Let $B \in IFS(U)$ and $B^\delta = \{A \in IFS(U) | A = (\delta)B\}$ then B^δ is called δ-equal ball of the set B

Proposition 17 *Let $B \in IFS(U)$ and $A_i, A_j \in B^\delta, i \neq j$. Then*

$$A_i = (\delta * \delta)A_j. \tag{27}$$

Proof This is followed from Proposition 2. □

Remark 5 Proposition 17 shows that two any sets are in δ-equal ball of the set B, i.e., they have the same δ-equality degree with the set B, then they have the max δ-equal degree equal to $\delta * \delta$.

4 δ-equalities for intuitionistic fuzzy relations

Proposition 18 *Let X, Y and Z be initial universes, and Σ be the collection of all intuitionistic fuzzy sets defined on $X \times Y$ and Π denote the collection of all intuitionistic fuzzy sets defined on $Y \times Z$ respectively. Let $R, R' \in \Sigma$ and $S, S' \in \Pi$, i.e., and S' are intuitionistic fuzzy relations on $X \times Y$ and $Y \times Z$ respectively. Further, let $R \circ S$ and $R' \circ S'$ be their composition, $\mu_{R\circ S}(x, y)$, $\nu_{R\circ S}(x, y)$, and $\mu_{R'\circ S'}(x, y)$, $\nu_{R'\circ S'}(x, y)$ be their membership and non-membership functions respectively, where $\mu_{R\circ S}(x, z) = \sup_{y\in Y} \min(\mu_R(x, y), \mu_S(y, z))$, $\nu_{R\circ S}(x, z) = \inf_{y\in Y} \max(\nu_R(x, y), \nu_S(y, z))$, and $\mu_{R'\circ S'}(x, z) = \sup_{y\in Y} \min(\nu_{R'}(x, y), \mu_{S'}(y, z))$, $\nu_{R'\circ S'}(x, z) = \inf_{y\in Y} \max(\nu_{R'}(x, y), \nu_{S'}(y, z))$, $\forall x \in X$, $z \in Z$. Suppose $R = (\delta_1)R'$ and $S = (\delta_2)S'$. Then*

$$R \circ S = (\min(\delta_1, \delta_2))R' \circ S'. \tag{28}$$

Proof Since we have

$$\begin{aligned}
&|\mu_{R\circ S}(x, z) - \mu_{R'\circ S'}(x, z)| \\
&= \left|\sup_{y\in Y} \min(\mu_R(x, y), \mu_S(y, z)) - \sup_{y\in Y} \min(\mu_{R'}(x, y), \mu_{S'}(y, z))\right|, \\
&\leq \sup_{y\in Y} |\min(\mu_R(x, y), \mu_S(y, z)) - \min(\mu_{R'}(x, y), \mu_{S'}(y, z))|, \\
&\leq \sup_{y\in Y} |\max(\mu_R(x, y) - \mu_{R'}(x, y)), (\mu_S(y, z) - \mu_{S'}(y, z))|, \\
&\leq \sup_{y\in Y} |\max(1 - \delta_1, 1 - \delta_2) = 1 - \min(\delta_1, \delta_2).
\end{aligned}$$

This implies that $|\mu_{R\circ S}(x, z) - \mu_{R'\circ S'}(x, z)| \leq 1 - \min(\delta_1, \delta_2)$.

Now, we have

$$\begin{aligned}
|\nu_{R\circ S}(x, z) - \nu_{R'\circ S'}(x, z)| &= \left|\inf_{y\in Y} \max(\nu_R(x, y), \nu_S(y, z)) - \inf_{y\in Y} \max(\nu_{R'}(x, y), \nu_{S'}(y, z))\right|, \\
&\leq \sup_{y\in Y} |\max(\nu_R(x, y), \nu_S(y, z)) - \max(\nu_{R'}(x, y), \nu_{S'}(y, z))|, \\
&\leq \sup_{y\in Y} |\max(\nu_R(x, y) - \nu_{R'}(x, y)), (\nu_S(y, z) - \nu_{S'}(y, z))|, \\
&\leq \sup_{y\in Y} \max(1 - \delta_1, 1 - \delta_2) = 1 - \min(\delta_1, \delta_2).
\end{aligned}$$

Therefore $|\nu_{R\circ S}(x, z) - \nu_{R'\circ S'}(x, z)| \leq 1 - \min(\delta_1, \delta_2)$. Hence $R \circ S = (\min(\delta_1, \delta_2))R' \circ S'$. □

Remark 6 Proposition 18 demonstrates that we can determine the δ-equality degree of the compositions of relations if we know δ-equality degrees between those relations. In some applications like medical diagnosis, the compositions of intuitionistic fuzzy relations are very important. Let Q_i be a intuitionistic fuzzy relation between the set of patients P and the set of symptoms S, and R_i be a intuitionistic fuzzy relation between the set of symptoms S and the set of diagnoses D, then $R_i \circ Q_i$ be a intuitionistic fuzzy relation between the set of patients P and the set of diagnoses D. In the case when there are many medical experts making a diagnosis, we can obtain many corresponding sets Q_i and R_i which are different. From Proposition 18, if we know δ-equality degrees between Q_i and Q_j, R_i and R_j, then δ-equality degree between $R_i \circ Q_i$ and $R_j \circ Q_j$ is determined. Thus, we can compute the δ-equality degree of final diagnosis.

Proposition 19 *Let $U_1, U_2, \cdots U_n$ be universes and A_j, B_j be intuitionistic sets defined on U_j, $j = 1, 2, \cdots, n$. Let $A_j = (\delta_j)B_j$, where $j = 1, 2, \cdots, n$. Let $A = A_1 \times A_2 \times \cdots \times A_n$ and $B = B_1 \times B_2 \times \cdots \times B_n$ and $\mu_A(\mu_1, \mu_2, \cdots, \mu_n)$, $\nu_A(u_1, u_2, \cdots, u_n)$ and*

$\mu_B(u_1, u_2, \cdots, u_n)$, $\nu_B(u_1, u_2, \cdots, u_n)$ be their membership and non-membership functions respectively, where

$$\mu_A(u_1, u_2, \cdots, u_n) = \min(\mu_{A_1}(u_1), \mu_{A_2}(u_2), \cdots, \mu_{A_n}(u_n)),$$
$$\nu_A(u_1, u_2, \cdots, u_n) = \max(\nu_{A_1}(u_1), \nu_{A_2}(u_2), \cdots, \nu_{A_n}(u_n)),$$

and

$$\mu_B(u_1, u_2, \cdots, u_n) = \min(\mu_{B_1}(u_1), \mu_{B_2}(u_2), \cdots, \mu_{B_n}(u_n)),$$
$$\nu_B(u_1, u_2, \cdots, u_n) = \max(\nu_{B_1}(u), \nu_{B_2}(u), \cdots, \nu_{B_n}(u)).$$

Then

$$A = \left(\inf_{1\le j\le n} \delta_j\right) B. \tag{29}$$

Proof This is because

$$\begin{aligned}
&\sup_{u_j\in U_j} |\mu_A(u_1, u_2, \cdots, u_n) - \mu_B(u_1, u_2, \cdots, u_n)| \\
&= \sup_{u_j\in U_j} \big|\min(\mu_{A_1}(u_1), \mu_{A_2}(u_2), \cdots, \mu_{A_n}(u_n)) \\
&- \min(\mu_{B_1}(u_1), \mu_{B_2}(u_2), \cdots, \mu_{B_n}(u_n))\big| \\
&\le \sup_{u_j\in U_j} \sup_{1\le j\le n} \left|\mu_{A_j}(u_j) - \mu_{B_j}(u_j)\right|, \\
&\le \sup_{u_j\in U_j} \sup_{1\le j\le n} (1-\delta_j) = 1 - \inf_{1\le j\le n} \delta_j.
\end{aligned}$$

Now,

$$\begin{aligned}
&\sup_{u_j\in U_j} |\nu_A(u_1, u_2, \cdots, u_n) - \nu_B(u_1, u_2, \cdots, u_n)| \\
&= \sup_{u_j\in U_j} \big|\max(\nu_{A_1}(u_1), \nu_{A_2}(u_2), \cdots, \nu_{A_n}(u_n)) \\
&\quad - \max(\nu_{B_1}(u_1), \nu_{B_2}(u_2), \cdots, \nu_{B_n}(u_n))\big| \\
&\le \sup_{u_j\in U_j} \sup_{1\le j\le n} \left|\nu_{A_j}(u_j) - \nu_{B_j}(u_j)\right|, \\
&\le \sup_{u_j\in U_j} \sup_{1\le j\le n} (1-\delta_j) = 1 - \inf_{1\le j\le n} \delta_j.
\end{aligned}$$

Thus $A = \left(\inf_{1\le j\le n} \delta_j\right) B$. □

Remark 7 Proposition 19 is the result from the combination of δ-equalities of intuitionistic fuzzy sets and Cartesian product.

5 An application of δ-equalities for medical diagnosis

This section presents an application of δ-equalities for medical diagnosis. Medicine is always one of the areas which leads research interests. Medical diagnosis is the process of investigation of diseases from a patient's symptoms [32]. Medical data are often uncertain, ambiguous and difficult to retrieve. A categorized relationship between a symptom and a disease is usually depended on uncertain information which affects the decision making process. The medical diagnosis has successful practical applications in several areas such as telemedicine, space medicine and rescue services. Thus, medical diagnosis has got full attention from both the computer science and computer applicable mathematics research societies. The traditional approach for medical diagnosis is using fuzzy relation to represent the relationships between patients-symptoms, symptoms-diseases and patients-diseases [32]. De et al. [9] extended the Sanchez's method with the theory of intuitionistic fuzzy sets. The extended Sanchez's approaches for type-2 fuzzy sets, neutrosophic sets and other ones were introduced in [1, 21, 29, 66]. The methods listed above have significant differences in the domain of problems and used datasets.

In medical diagnosis, normal level reference value ranges for attributes are given by different experts or different referenced ranges provided by a specific laboratory, for instance, heretofore, normal level reference value range for Alanine Aminotransferase (ALT) index is less than 40 International Unit/ Lit (IU/L) (female: 6-34 IU/L, male: 8-40 IU/L). Lee et al. [22], based on their experiments on population, suggested new normal values of ALT such as 30 IU/L for males and 19 IU/L for females. The normalAlbumin/Globulin (A/G) ratio is pointed out in [0.8, 2.0] [28], but it was shown in [1.2, 1.5] according by another reference [30]. Therefore, if we use the traditional medical diagnosis method of Sanchez [32] and De et al. [9] with multiple medical references then the initial crisp symptoms of patients such as ALT, A/G, etc. will give several different (intuitionistic) fuzzy sets, which result in the problem of choosing inappropriate (intuitionistic) fuzzified results to use in the next step. As such, our idea is to use the concept of δ-equalities to find groups of (intuitionistic) fuzzified set with certain equality or similar degrees then combining them. This is exactly the meaning of δ-equalities which are given in this paper. The new method involves mainly the basic steps:

1. Determining the relation between patients and symptoms.
2. Formulating the relation between symptoms and diagnoses.
3. Determining diagnoses for all patients on the basis of composition of relations.

Let us draw those steps in details. Suppose P, S, D are the set of patients, the set of symptoms, and the set of diagnoses, respectively. Let $\Delta Q = \{Q_1, Q_2, ..., Q_n\} \subset IFR(P \times S)$.

Step 1: Calculating δ_{ij} is maximum delta-equalities degree of Q_i, Q_j:

$$Q_i = (\delta_{ij})\, Q_j, \ \left(i, j = \overline{1, n}, i \ne j\right).$$

$$\delta_{ij} = \min\left(1 - \sup_{(p,s)\in P\times S} \left|\mu_{Q_i}(p,s) - \mu_{Q_j}(p,s)\right|,\right.$$
$$\left. 1 - \sup_{(p,s)\in P\times S} \left|\nu_{Q_i}(p,s) - \nu_{Q_i}(p,s)\right|\right),$$

Step 2: Finding $\delta = \max\{\delta_{ij} : i, j = \overline{1, n}\}$. Suppose exist k pairs $(Q_{i_t}, Q_{j_t}) \subset \Delta Q, t = \overline{1, k}$ satisfy $\delta_{i_t j_{j_t}} = \delta$, then unionizing the set Q_{i_t} and Q_{j_t}. Let Q_t^* be defined by $Q_t^* = Q_{i_t} \cup_{\mathcal{S}} Q_{j_t}$, with

$$\mathcal{S} \in \{\cup; +; \mathcal{S}_3; \mathcal{S}_4; \mathcal{S}_5; \mathcal{S}_6\}.$$

Then $Q_t^* \in IFR(P \times S), t = \overline{1, k}$.

Let $\Delta Q \cap Q_t^{*\delta} = \{Q_i \in \Delta Q | Q_i = (\delta) Q_t^*\}$. Then, calculating $Q_{\hat{t}} = \bigcup_{\Delta Q \cap Q_t^{*\delta}} Q_i$

$$\mu_{Q_{\hat{t}}}(u) = \max_{\Delta Q \cap Q_t^{*\delta}} \mu_{Q_i}(u); \nu_{Q_{\hat{t}}}(u) = \min_{\Delta Q \cap Q_t^{*\delta}} \nu_{Q_i}(u).$$

Then $Q_{\hat{t}} \in IFR(P \times S), t = \overline{1, k}$..

Step 3: We define "intuitionistic medical knowledge" as a intuitionistic fuzzy relation R between the set of symptoms S and the set of diagnoses D which reveals the degree of positive association and negative association between symptoms and the diagnosis. Then $R \in IFR(S \times D)$, clearly, the composition $R \circ Q_{\hat{t}}$ of R and $Q_{\hat{t}}$ describes the state of patients in terms of the diagnosis.

Step 4: The composed relation $R \circ Q_{\hat{t}}$ between P and D is identified as following

$$\mu_{R \circ Q_{\hat{t}}}(p, d) = \sup_{s \in S} \min(\mu_{Q_{\hat{t}}}(p, s), \mu_R(s, d))$$
$$\mu_{R \circ Q_{\hat{t}}}(p, d) = \inf_{s \in S} \max(\nu_{Q_{\hat{t}}}(p, s), \nu_R(s, d)), \forall p \in P, d \in D.$$

Then, the correspondence between patient p and diagnosis d is expressed as a couple containing $\mu_{R \circ Q_{\hat{t}}}(p, d), \nu_{R \circ Q_{\hat{t}}}(p, d)$.

Step 5: For each $p, d \in P \times D$, we calculate $S_{R \circ Q_{\hat{t}}}(p, d)$ as below:

$$S_{R \circ Q_{\hat{t}}}(p, d) = \mu_{R \circ Q_{\hat{t}}}(p, d) - \nu_{R \circ Q_{\hat{t}}}(p, d) \pi_{R \circ Q_{\hat{t}}}(p, d),$$

where $\pi_{R \circ Q_{\hat{t}}}(p, d) = 1 - [\mu_{R \circ Q_{\hat{t}}}(p, d) + \nu_{R \circ Q_{\hat{t}}}(p, d)]$.

It is easily seen that if $\mu_{R \circ Q_{\hat{t}}}(p, d) + \nu_{R \circ Q_{\hat{t}}}(p, d) = 1$, then $S_{R \circ Q_{\hat{t}}}(p, d) = \mu_{R \circ Q_{\hat{t}}}(p, d)$.

Step 6: If $S_{R \circ Q_{\hat{t}}}(p, d) = \max_{t=1,2,\ldots k} S_{R \circ Q_{\hat{t}}}(p, d) \geq y^*$ where y^* is a trained value from a fact data set about disease d, then patient p is said to be suffered from illness d.

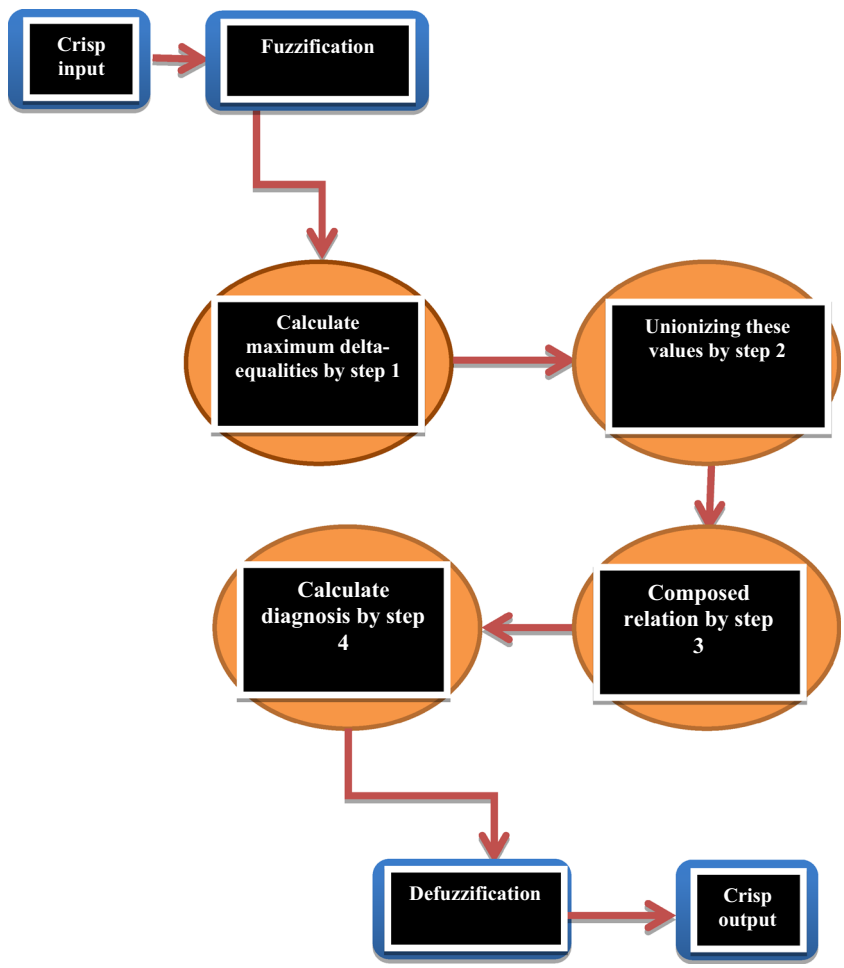

Fig. 1 The proposed model for medical diagnosis

The proposed model is illustrated in Fig. 1. Now, we define the following options:

- In Step 2, Q_t^* can be defined by $Q_t^* = Q_{i_t} \cap_{\mathcal{T}} Q_{j_t}$, with $\mathcal{T} \in \{\cap; \cdot; \mathcal{T}_3; \mathcal{T}_4; \mathcal{T}_5; \mathcal{T}_6\}$.
- In Step 3, calculating $Q_{\hat{t}} = \bigcap_{\widehat{Q} \cap Q_t^{*\delta}} Q_i$

  $\mu_{Q_{\hat{t}}}(u) = \min_{\widehat{Q} \cap Q_t^{*\delta}} \mu_{Q_i}(u), \nu_{Q_{\hat{t}}}(u) = \max_{\widehat{Q} \cap Q_t^{*\delta}} \nu_{Q_i}(u).$
- In Step 6, if $S_{R \circ Q_{\hat{t}}}(p, d) = \min_{t=1,2,\ldots k} S_{R \circ Q_{\hat{t}}}(p, d)$ then the patient p is said to be suffered from illness d.

Now, we present two numerical examples based on the proposed algorithm to illustrate the application of δ-equalities to medical diagnosis.

Example 5 Consider the dataset adapted from [31].

- X contains four patients ($x_1 = Ram, x_2 = Mari, x_3 = Sugu, x_4 = Somu$;).
- Y is the set of five symptoms:

  $(y_1 = Temperature, y_2 = Headache,$
  $y_3 = Stomach\ pain, y_4 = Cough, y_5 = Chest\ pain;)$
- Z includes five diseases:

  $(z_1 = Viral\_Fever, z_2 = Malaria, z_3 = Malaria,$
  $z_4 = Stomach, z_5 = Heart.)$

Case 1 We illustrate results of the Sanchez's approach [32] for medical diagnosis. There is one initial intuitionistic fuzzy data set P which describes the relations from patients to symptoms.

The intuitionistic fuzzy relations (IFRs) from the patients to the symptoms as well as the symptoms to the diseases are given in Tables 1 and 2, respectively. The IFR from the patients to the diseases determined by the fuzzy max-min composition is drawn as in Table 3, in which the first values in each pair are larger than 0.5 implying the possible diseases. In here, note that we take $\nu_P = 1 - \mu_P$ and $\nu_R = 1 - \mu_R$, so that $\nu_{R \circ P} = 1 - \mu_{R \circ P}$. Thus $S_{R \circ P} = \mu_{R \circ P}$.

Case 2 Now, we illustrate the proposed method. There are three initial intuitionistic fuzzy data sets P_1, P_2, P_3, which describe the relationsfrom patients to symptoms (Tables 4, 5 and 6).

- At the step 1, we need to calculate $\delta_{P_1 P_2}, \delta_{P_1 P_3}, \delta_{P_2 P_3}$:

  $\delta_{P_1 P_2} = 0.94, \delta_{P_1 P_3} = 0.7, \delta_{P_2 P_3} = 0.7$
- At the step 2, we see that $\delta_{P_1 P_2} = \max\{\delta_{P_1 P_2}, \delta_{P_1 P_3}, \delta_{P_2 P_3}\}$, then combine P_1 and P_2 (Table 7).
- At the step 3, we use the set R in the case 1 again.
- At the step 4, we calculate $R \circ P^*$ similar with calculating $R \circ P$ in the case 1 (Table 8).
- At the step 5, we calculate $S_{R \circ P^*}$ as in Table 9.
- At the step 6, in Table 9, values are larger than 0.5 implying the possible diseases. It is recognized that the results in Table 9 are identical to those in Table 3.

Example 6 Let consider four patients p_1, p_2, p_3 and p_4. Their symptoms are temperature, headache, stomach pain, cough, and chest pain. Then, the set of patients is $P = \{p_1, p_2, p_3, p_4\}$ and the set of symptoms is $S = \{temperature, headache, stomach\ pain, cough, and\ chest\ pain\}$. The intuitionistic fuzzy relations $Q_1, Q_2, Q_3 \in IFR(P \times S)$ are evaluated by three decision makers groups and are given as in Tables 10, 11 and 12, respectively. The data of Q_1 is real data, and the data of Q_2, Q_2 are hypothetical.

Now, we illustrate the proposed method. First of all, we calculate the maximum δ-equality degree of Q_1 and Q_2:

$$\delta_{12} = \min\left(1 - \sup_{(p,s)\in P\times S} \left|\mu_{Q_1}(p,s) - \mu_{Q_2}(p,s)\right|, \right.$$
$$\left. 1 - \sup_{(p,s)\in P\times S} \left|\nu_{Q_1}(p,s) - \nu_{Q_2}(p,s)\right|\right) = 0.95,$$

The maximum δ-equality degree of Q_1 and Q_3:

$$\delta_{13} = \min\left(1 - \sup_{(p,s)\in P\times S} \left|\mu_{Q_1}(p,s) - \mu_{Q_3}(p,s)\right|, \right.$$
$$\left. 1 - \sup_{(p,s)\in P\times S} \left|\nu_{Q_1}(p,s) - \nu_{Q_3}(p,s)\right|\right) = 0.2,$$

The maximum δ-equality degree of Q_2 and Q_3:

$$\delta_{23} = \min\left(1 - \sup_{(p,s)\in P\times S} \left|\mu_{Q_2}(p,s) - \mu_{Q_3}(p,s)\right|, \right.$$
$$\left. 1 - \sup_{(p,s)\in P\times S} \left|\nu_{Q_2}(p,s) - \nu_{Q_3}(p,s)\right|\right) = 0.17,$$

Because $\delta_{12} \geq \delta_{13} \geq \delta_{23}$, consider the set

$$Q = Q_1 \cup Q_2 = \left\{\left\langle (p,s), \mu_{Q_1}(p,s) \vee \mu_{Q_2}(p,s), \right.\right.$$
$$\left.\left. \nu_{Q_1}(p,s) \wedge \mu_{Q_2}(p,s)\right\rangle : (p,s) \in P \times S\right\},$$

It turns out that $Q = (0.95)Q_1$ (Table 13).

Let the set of diagnoses be $D = \{viral\ \ fever, malaria, typhoid, stomach\ problem\}$. The intuitionistic fuzzy relation $R \in IFR(S \times D)$ is given as in Table 14. The composed relation $R \circ Q$ is given as in Table 15.

From Proposition 18, we have $R \circ Q = (0.95)R \circ Q_1$. For each $p, d \in P \times D$, we calculate $S_{R \circ Q}(p, d)$ as in Table 16. If $S_{R \circ Q}(p, d) \geq 0.5$, then the patient p is said to be suffered

Table 1 IFR from patients to symptoms

P	y_1	y_2	y_3	y_4	y_5
x_1	(0.79,0.21)	(0.57,0.43)	(0.2,0.8)	(0.57,0.43)	(0,1)
x_2	(0,1)	(0.32,0.68)	(0.57,0.43)	(0,1)	(0.02,0.98)
x_3	(0.79,0.21)	(0.79,0.21)	(0,1)	(0.13,0.87)	(0,1)
x_4	(0.57,0.43)	(0.46,0.54)	(0.18,0.82)	(0.68,0.32)	(0.18,0.82)

Table 2 IFR from symptoms to diseases

R	z_1	z_2	z_3	z_4	z_5
y_1	(0.4,0.6)	(0.7,0.3)	(0.18,0.82)	(0,1)	(0.02,0.98)
y_2	(0,1)	(0,1)	(0.13,0.87)	(0.8,0.2)	(0.2,0.8)
y_3	(0.79,0.21)	(0.79,0.21)	(0,1)	(0.13,0.87)	(0,1)
y_4	(0.31,0.69)	(0.7,0.3)	(0.08,0.92)	(0.13,0.87)	(0.2,0.8)
y_5	(0,1)	(0.02,0.98)	(0.1,0.9)	(0.13,0.87)	(0.79,0.21)

Table 3 IFR from patients to diseases

$R \circ P$	z_1	z_2	z_3	z_4	z_5
x_1	(0.4,0.6)	(**0.7**,0.3)	(0.18,0.82)	(**0.57**,0.43)	(0.2,0.8)
x_2	(**0.57**,0.43)	(**0.57**,0.43)	(0.13,0.87)	(0.32,0.68)	(0.2,0.8)
x_3	(0.4,0.6)	(**0.7**,0.3)	(0.18,0.82)	(**0.79**,0.21)	(0.2,0.8)
x_4	(0.4,0.6)	(**0.68**,0.32)	(0.18,0.82)	(0.46,0.54)	(0.2,0.8)

Table 4 IFR from patients to symptoms – P_1

P_1	y_1	y_2	y_3	y_4	y_5
x_1	(0.79,0.21)	(0.57,0.43)	(0.2,0.8)	(0.57,0.43)	(0,1)
x_2	(0,1)	(0.32,0.68)	(0.57,0.43)	(0,1)	(0.02,0.98)
x_3	(0.79,0.21)	(0.79,0.21)	(0,1)	(0.13,0.87)	(0,1)
x_4	(0.57,0.43)	(0.46,0.54)	(0.18,0.82)	(0.68,0.32)	(0.18,0.82)

Table 5 IFR from patients to symptoms – P_2

P_2	y_1	y_2	y_3	y_4	y_5
x_1	(0.79,0.21)	(0.58,0.4)	(0.2,0.78)	(0.57,0.42)	(0,0.98)
x_2	(0,0.97)	(0.32,0.66)	(0.57,0.43)	(0,1)	(0.05,0.93)
x_3	(0.79,0.21)	(0.79,0.2)	(0,0.94)	(0.13,0.87)	(0,0.95)
x_4	(0.57,0.42)	(0.46,0.54)	(0.18,0.8)	(0.68,0.32)	(0.18,0.82)

Table 6 IFR from patients to symptoms – P_3

P_3	y_1	y_2	y_3	y_4	y_5
x_1	(0.66,0.25)	(0.57,0.43)	(0.23,0.7)	(0.57,0.43)	(0,0.7)
x_2	(0,0.8)	(0.32,0.68)	(0.57,0.43)	(0,0.75)	(0.02,0.7)
x_3	(0.79,0.21)	(0.49,0.21)	(0,0.8)	(0.13,0.87)	(0,0.9)
x_4	(0.57,0.43)	(0.5,0.32)	(0.18,0.52)	(0.68,0.32)	(0.18,0.82)

Table 7 IFR from patients to symptoms – $P^* = P_1 \cup P_2$

$P^* = P_1 \cup P_2$	y_1	y_2	y_3	y_4	y_5
x_1	(0.79,0.21)	(0.58,0.4)	(0.2,0.78)	(0.57,0.42)	(0,0.98)
x_2	(0,0.97)	(0.32,0.66)	(0.57,0.43)	(0,1)	(0.05,0.93)
x_3	(0.79,0.21)	(0.79,0.2)	(0,0.94)	(0.13,0.87)	(0,0.95)
x_4	(0.57,0.42)	(0.46,0.54)	(0.18,0.8)	(0.68,0.32)	(0.18,0.82)

Table 8 IFR from patients to diseases – $R \circ P^*$

$R \circ P^*$	z_1	z_2	z_3	z_4	z_5
x_1	(0.4,0.6)	(0.7,0.3)	(0.18,0.82)	(0.58,0.4)	(0.2,0.8)
x_2	(0.57,0.43)	(0.57,0.43)	(0.13,0.87)	(0.32,0.66)	(0.2,0.8)
x_3	(0.4,0.6)	(0.7,0.3)	(0.18,0.82)	(0.79,0.2)	(0.2,0.8)
x_4	(0.4,0.6)	(0.68,0.32)	(0.18,0.82)	(0.46,0.54)	(0.2,0.8)

Table 9 FR from patients to diseases – $S_{R \circ P^*}$

$S_{R \circ P^*}$	z_1	z_2	z_3	z_4	z_5
x_1	0.4	**0.7**	0.18	**0.572**	0.2
x_2	**0.57**	**0.57**	0.13	0.3068	0.2
x_3	0.4	**0.7**	0.18	**0.788**	0.2
x_4	0.4	**0.68**	0.18	0.46	0.2

Table 10 Q_1 is intuitionistic fuzzy relation between the set of patients P and the set of symptoms S with the data from 1st decision makers group

Q_1	Temperature	Headache	Stomach pain	Cough	Chest pain
P_1	(0.8, 0.1)	(0.7,0 .2)	(0.1, 0.6)	(0.7, 0.1)	(0.2, 0.5)
P_2	(0.01, 0.7)	(0.5, 0.3)	(0.65, 0.1)	(0.05, 0.7)	(0.07, 0.6)
P_3	(0.75, 0.05)	(0.8, 0.08)	(0.15, 0.5)	(0.3, 0.6)	(0.1, 0.5)
P_4	(0.6, 0.1)	(0.4, 0.4)	(0.2, 0.3)	(0.6, 0.15)	(0.35, 0.2)

Table 11 Q_2 is intuitionistic fuzzy relation between the set of patients P and the set of symptoms S with the data from 2nd decision makers group

Q_2	Temperature	Headache	Stomach pain	Cough	Chest pain
P_1	(0.81, 0.1)	(0.7, 0.22)	(0.09, 0.6)	(0.67, 0.1)	(0.25, 0.5)
P_2	(0.02, 0.7)	(0.51, 0.28)	(0.66, 0.13)	(0.02, 0.7)	(0.08, 0.55)
P_3	(0.7, 0.05)	(0.8, 0.08)	(0.14, 0.5)	(0.32, 0.61)	(0.06, 0.5)
P_4	(0.6, 0.14)	(0.44, 0.4)	(0.2, 0.3)	(0.57, 0.14)	(0.35, 0.22)

Table 12 Q_3 is intuitionistic fuzzy relation between the set of patients P and the set of symptoms S with the data from 3rd decision makers group

Q_3	Temperature	Headache	Stomach pain	Cough	Chest pain
P_1	(0.12, 0.8)	(0.2, 0.5)	(0.9, 0.05)	(0.2, 0.6)	(0.3, 0.6)
P_2	(0.7, 0.24)	(0.1, 0.25)	(0.35, 0.4)	(0.85, 0.01)	(0.4, 0.4)
P_3	(0.25, 0.15)	(0.2, 0.3)	(0.45, 0.4)	(0.7, 0.15)	(0.2, 0.4)
P_4	(0.4, 0.5)	(0.3, 0.6)	(0.4, 0.1)	(0.2, 0.5)	(0.4, 0.5)

Table 13 $Q = Q_1 \cup Q_2 \in IFR(P \times S)$

Q	Temperature	Headache	Stomach pain	Cough	Chest pain
P_1	(0.81, 0.1)	(0.7, 0.2)	(0.1, 0.6)	(0.7, 0.1)	(0.25, 0.5)
P_2	(0.02, 0.7)	(0.51, 0.28)	(0.66, 0.1)	(0.05, 0.7)	(0.08, 0.55)
P_3	(0.75, 0.05)	(0.8, 0.08)	(0.15, 0.5)	(0.32, 0.6)	(0.1, 0.5)
P_4	(0.6, 0.1)	(0.44, 0.4)	(0.2, 0.3)	(0.6, 0.14)	(0.35, 0.2)

Table 14 R is intuitionistic fuzzy relation between the set of symptoms S and the set of diagnoses D

R	Fever	Malaria	Typhoid	Stomach	Chest problem
Temperature	(0.4, 0.05)	(0.8, 0.1)	(0.3, 0.3)	(0.15, 0.6)	(0.05, 0.7)
Headache	(0.4, 0.3)	(0.1, 0.6)	(0.75, 0.03)	(0.3, 0.05)	(0.01, 0.8)
Stomach pain	(0.1, 0.6)	(0.01, 0.9)	(0.1, 0.7)	(0.8, 0.01)	(0.1, 0.75)
Cough	(0.45, 0.1)	(0.65, 0.05)	(0.2, 0.6)	(0.25, 0.5)	(0.15, 0.7)
Chest pain	(0.05, 0.6)	(0.03, 0.8)	(0.01, 0.85)	(0.1, 0.7)	(0.9, 0.05)

Table 15 $R \circ Q$ is intuitionistic fuzzy relation between the set of symptoms P and the set of diagnoses D

$R \circ Q$	Fever	Malaria	Typhoid	Stomach	Chest problem
p_1	(0.45, 0.1)	(0.8, 0.1)	(0.7, 0.2)	(0.3, 0.2)	(0.25, 0.5)
p_2	(0.4, 0.3)	(0.1, 0.6)	(0.51, 0.28)	(0.66, 0.1)	(0.1, 0.55)
p_3	(0.4, 0.05)	(0.75, 0.1)	(0.75, 0.08)	(0.3, 0.08)	(0.15, 0.5)
p_4	(0.45, 0.1)	(0.6, 0.1)	(0.44, 0.3)	(0.3, 0.3)	(0.35, 0.2)

Table 16 $S_{R \circ Q}(p, d)$ where red values show the most suffered diseases of a patient

$S_{R \circ Q}(p, d)$	Fever	Malaria	Typhoid	Stomach	Chest problem
p_1	0.405	**0.79**	**0.68**	0.2	0.125
p_2	0.31	−0.08	0.4428	**0.636**	−0.0925
p_3	0.3725	**0.735**	**0.7364**	0.2504	−0.025
p_4	0.405	**0.57**	0.362	0.18	0.26

Table 17 $S_{R \circ Q_1}$ where red values show the most suffered diseases of a patient

$S_{R \circ Q}(p, d)$	Fever	Malaria	Typhoid	Stomach	Chest problem
p_1	0.405	**0.79**	**0.68**	0.2	0.05
p_2	0.31	−0.08	0.44	**0.625**	−0.08
p_3	0.3725	**0.735**	**0.7364**	0.2504	−0.025
p_4	0.405	**0.57**	0.31	0.18	0.26

Table 18 The descriptions of experimental datasets

Dataset	No. elements	No. attributes	No. classes
ILPD	583	8	2
LD	345	5	2
PIDD	768	5	2
Diabetes	389	4	2
Heart	270	4	2

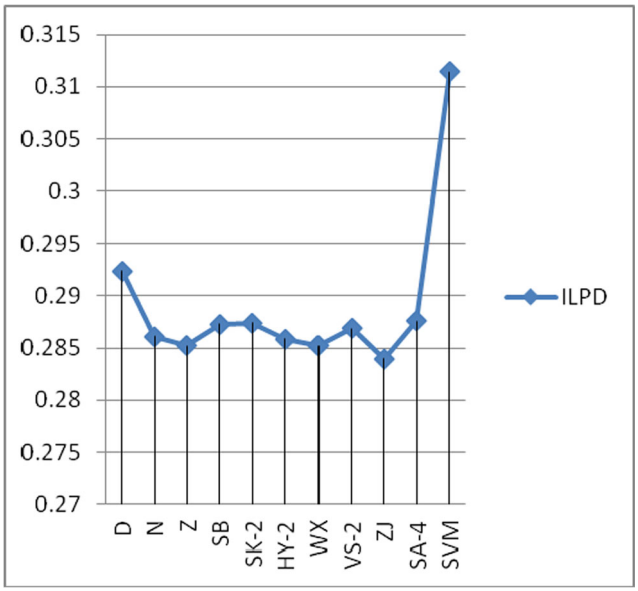

Fig. 2 MAE of algorithms on ILPD

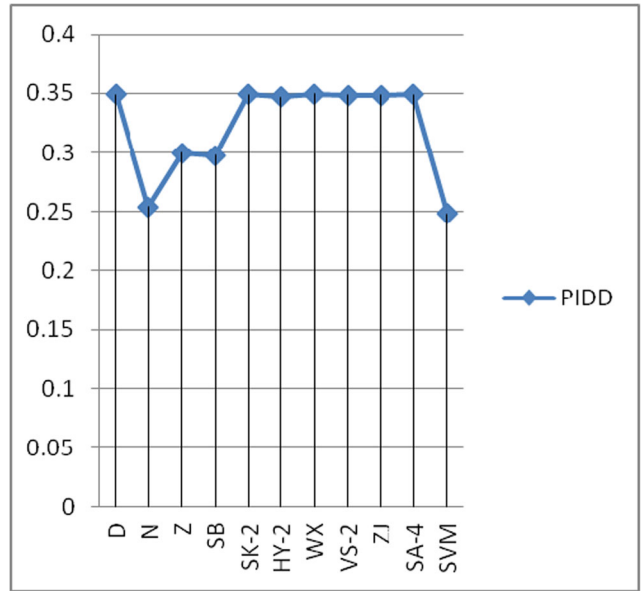

Fig. 4 MAE of algorithms on PIDD

from illness d. It is obvious that if the doctor agrees, then p_1, p_3 and p_4 suffer from Malaria, p_1 and p_3 suffer from Typhoid whereas p_2 faces Stomach problem.

The results of the Sanchez's approach [32] are expressed in Table 17 where p_1, p_3 and p_4 suffer from Malaria, p_1 and p_3 suffer from Typhoid whereas p_2 faces Stomach problem.

6 Experiments on real-world datasets

6.1 Experimental environments

Experimental tools We compare the proposed method (N) against the related diagnosis methods of De et al. [9] (D), Samuel and Balamurugan [31] (SB), Szmidt and Kacprzyk [53] (SK), Zhang et al. [71] (Z), Hung and Yang [19] (HY-2 with the similarity measure), Wang and Xin [63] (WX), Vlachos and Sergiadis [62] (VS-2 with the divergence measure), Zhang and Jiang [70] (ZJ), Maheshwari and Srivastava [24] (SA) and Support Vector Machine (SVM) in the combination of Matlab 2015a programming language and R programming language. Among all, the SK has 2 versions: SK-1 and SK-2 corresponding to the distance measures

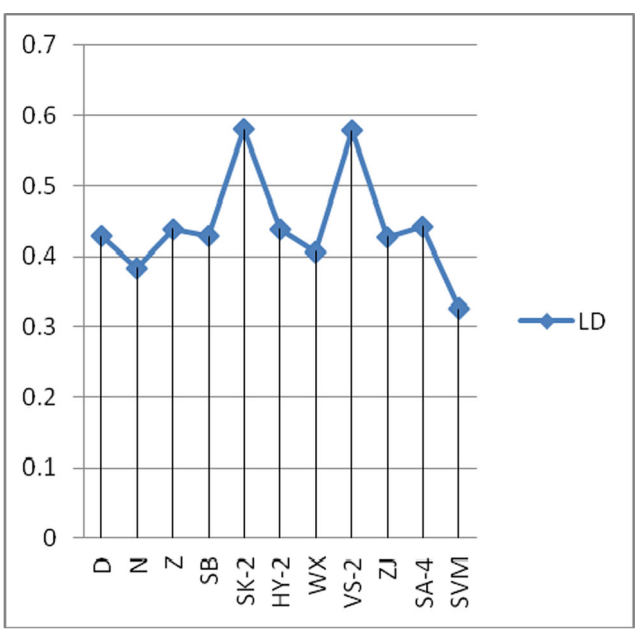

Fig. 3 MAE of algorithms on LD

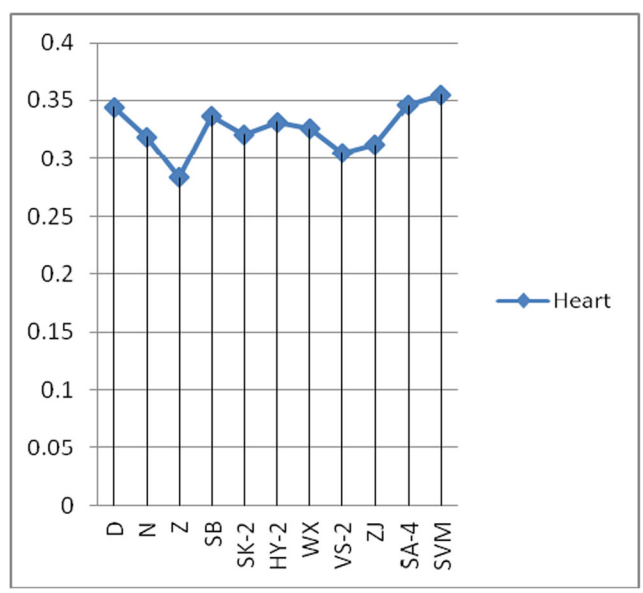

Fig. 5 MAE of algorithms on Heart

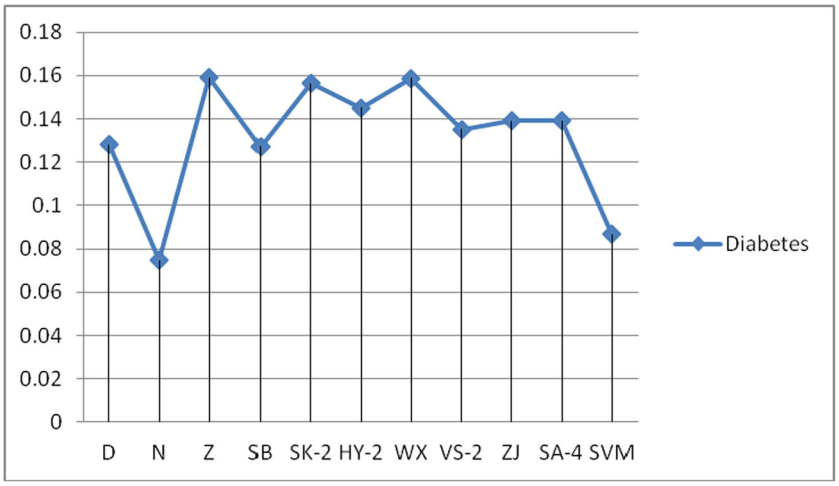

Fig. 6 MAE of algorithms on Diabetes

published in 2000 and 2004 respectively. Analogously, SA has 2 versions namely SA-2 and SA-4 with two cases of parameters: $\alpha = 0.1$ and $\alpha = 0.3$. Please refer to the equivalent articles for their definitions and formulae in details. The source codes and datasets of this section can be found in the Appendix.

Experimental datasets The benchmark datasets Heart, ILPD Indian Liver Patient Dataset, PIDD (Pima Indians Diabetes Data Set), Liver-Disorders (LD) have been taken from UCI Machine Learning Repository [60] while the remaining benchmark dataset Diabetes has been taken from [11]. Table 18 gives an overview of all those datasets.

6.2 Performance comparison

Table 19 presents the average MAE and computational time (Sec.) of the proposed method with all above introduced

Table 19 Mean Absolute Error (MAE) and Computational time (Sec) (NaN means undetermined)

Dataset	MAE													
	D	Z	SB	SK		HY-2	N-	WX	VS-2	ZJ	SA			SVM
				SK-1	SK-2		proposed method				SA-2 $\alpha = 0.1$	SA-4 $\alpha = 0.1$	SA-4 $\alpha = 0.3$	
ILPD	0.2924	0.2852	0.2873	0.2914	0.2874	0.285	0.2861	0.2852	0.2869	0.284	0.2872	0.2964	0.2876	0.3114
LD	0.4281	0.4388	0.4281	0.3452	0.58	0.4388	0.3838	0.407	0.5786	0.4275	NaN	0.4223	0.443	0.3258
PIDD	0.3491	0.2999	0.2977	0.3504	0.3488	0.347	0.2544	0.3497	0.3484	0.3487	0.3466	0.3516	0.3489	0.2482
Diabetes	0.128	0.159	0.1274	0.1628	0.1564	0.145	0.0748	0.1588	0.1352	0.1391	0.141	0.139	0.139	0.0868
Heart	0.3445	0.2836	0.3368	0.304	0.321	0.3313	0.3186	0.3258	0.3052	0.312	0.3136	0.3216	0.3459	0.3547
	Sec													
	D	Z	SB	SK		HY-2	N-	WX	VS-2	ZJ	SA			SVM
				SK-1	SK-2		proposed method				SA-2 $\alpha = 0.1$	SA-4 $\alpha = 0.1$	SA-4 $\alpha = 0.3$	
ILPD	0.67	0.5426	0.6401	0.5976	0.7076	0.5551	0.669	0.7501	0.9401	0.6601	0.8401	1.0801	0.9476	0.11
LD	0.4098	0.3373	0.3848	0.4148	0.4048	0.4198	0.4216	0.4248	0.4498	0.5223	NaN	0.6348	0.6348	0.035
PIDD	0.8709	0.8909	0.7659	0.8034	0.8859	0.8359	0.8754	0.9209	1.1134	1.1584	0.9759	1.3834	1.2784	0.12
Diabetes	0.3924	0.4274	0.4074	0.4499	0.4049	0.4374	0.4846	0.4474	0.5149	0.4874	0.4324	0.6349	0.5724	0.0275
Heart	0.2883	0.3233	0.2683	0.2883	0.3233	0.3433	0.3779	0.3508	0.4008	0.4133	0.3383	0.4483	0.4008	0.0275

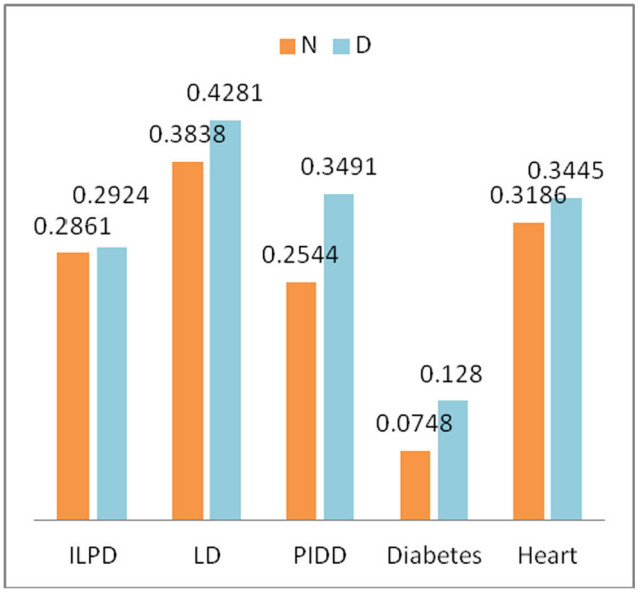

Fig. 7 MAE values of D and N

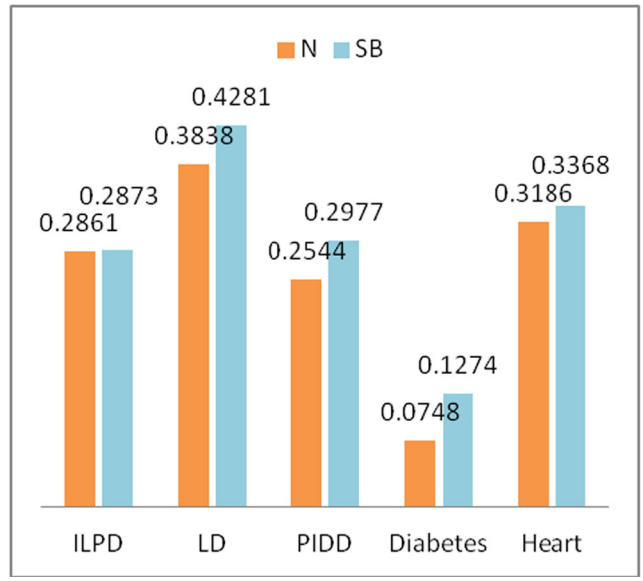

Fig. 9 MAE values of SB and N

methods which denoted by D, Z, SB, SK-1, SK-2, HY-2, WX, VS-2, ZJ, SA-2, SA-4 and SVM on the medical datasets of ILPD, LD, PIDD, Diabetes and Heart. It is clearly seen that the MAE of the proposed method is better than the rest of methods on the Diabetes dataset. Specifically in Table 19, the average MAE values of D, Z, SB, SK-1, SK-2, HY-2, the proposed method N, WX, VS-2, ZJ, SA-2, SA-4 ($\alpha = 0.1$), SA-4 ($\alpha = 0.3$) and SVM are **0.128, 0.159, 0.1274, 0.1628, 0.1564, 0.145, 0.0748, 0.1588, 0.1352, 0.1391, 0.141, 0.139, 0.139** and **0.0868** respectively.

The MAE of the proposed algorithm is clearly better than SVM for the ILPD dataset. Their average values in Table 19 for the data set ILPD are **0.2924, 0.2852, 0.2873, 0.2914, 0.2874, 0.285, 0.2861, 0.2852, 0.2869, 0.284, 0.2872,**

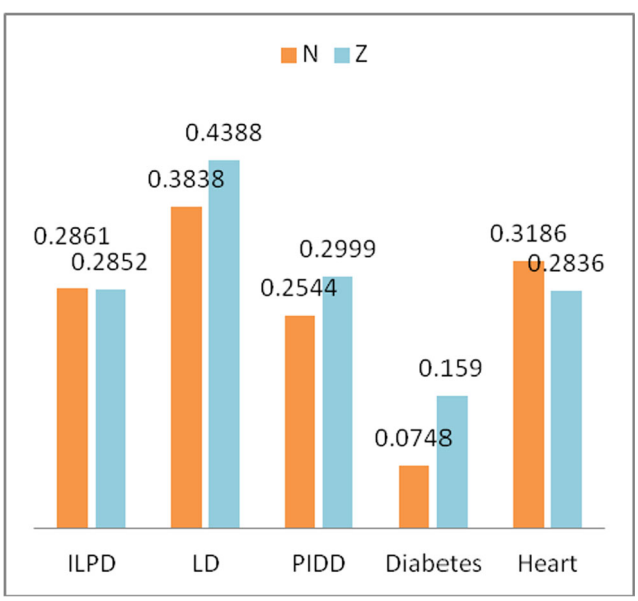

Fig. 8 MAE values of Z and N

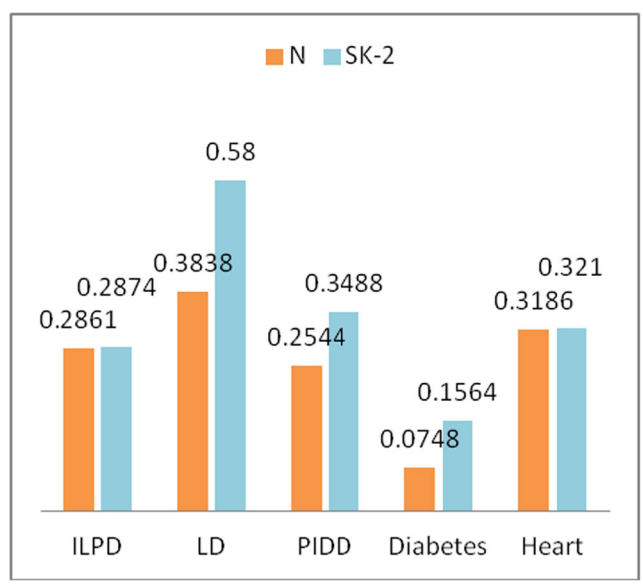

Fig. 10 MAE values of SK-2 and N

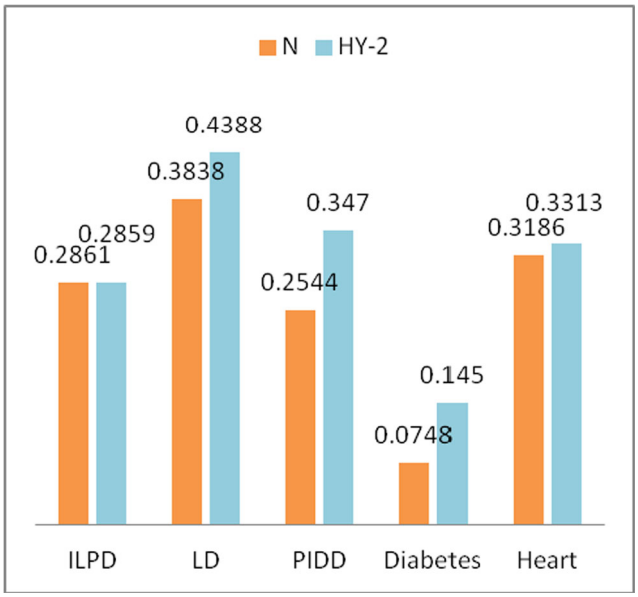

Fig. 11 MAE values of HY-2 and N

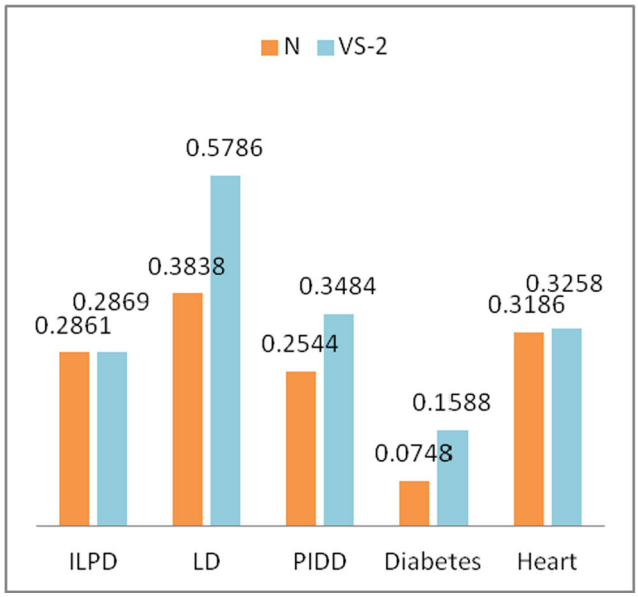

Fig. 13 MAE values of VS-2 and N

0.2964, 0.2876 and **0.3114** respectively. Analogously, the proposed method has better MAE value over D, Z, SB, SK-1, SK-2, HY-2, WX, VS-2, ZJ, SA-4 ($\alpha = 0.1$) and SA-4 ($\alpha = 0.3$) which are **0.3838, 0.4281, 0.4388, 0.4281, 0.3452, 0.58, 0.4388, 0.407, 0.5786, 0.4275, 0.4223, 0.443** respectively whereas the SVM has the MAE average value is **0.3258** and the SA-2 does not give out any value on the LD dataset. Similarly, the proposed method is quite advantageous on PIDD. These values in the Table 19 calculated for PIDD dataset are **0.3491, 0.2999, 0.2977, 0.3504, 0.3488, 0.347, 0.2544, 0.3497, 0.3484, 0.3487, 0.3466, 0.3516, 0.3489** and **0.2482** respectively.

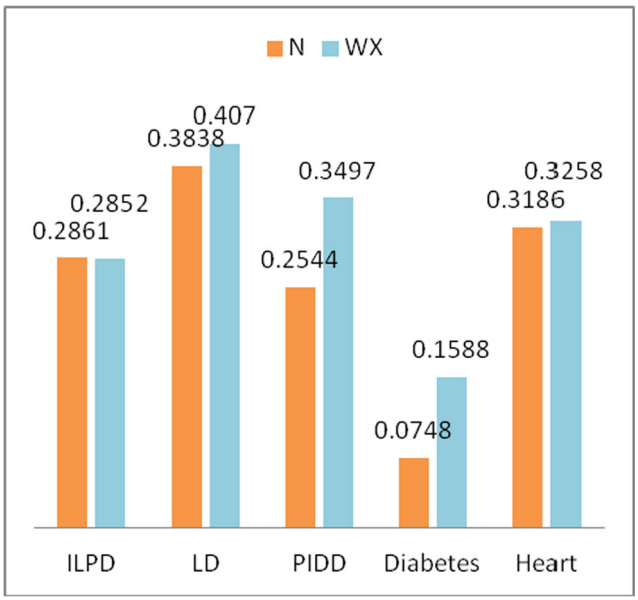

Fig. 12 MAE values of WX and N

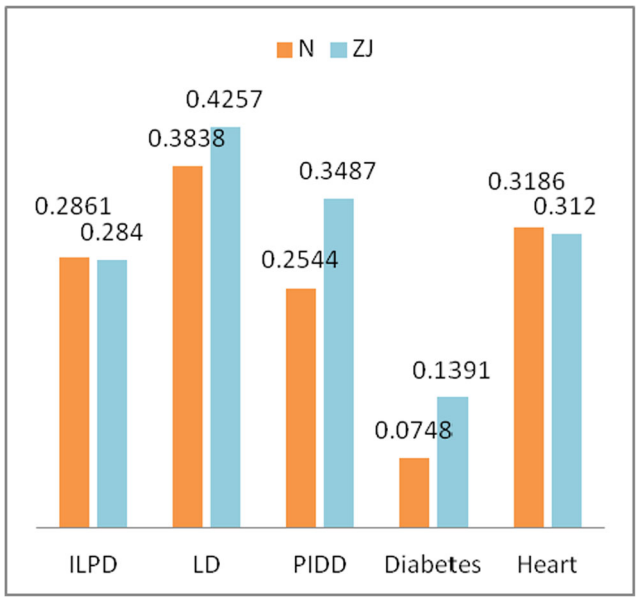

Fig. 14 MAE values of ZJ and N

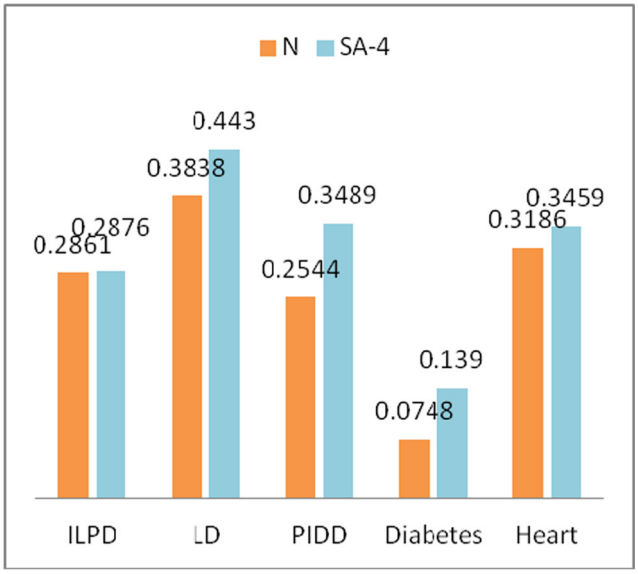

Fig. 15 MAE values of SA-4 ($\alpha = 0.3$) and N

Consider the remaining Heart dataset, MAE of the proposed algorithm is better than those of D, SB, SK-2, HY-2, WX, SA-4 ($\alpha = 0.1$), SA-4 ($\alpha = 0.3$) and SVM but it not good as that of Z method. It provides a good result which is approximate to those of SK-1, VS-2, ZJ and SA-2. The average MAE values computed on the Heart dataset are **0.3445, 0.2836, 0.3368, 0.304, 0.321, 0.3313, 0.3186, 0.3258, 0.3052, 0.312, 0.3136, 0.3216, 0.3459** and **0.3547** respectively. Overall, the average MAE values of the proposed algorithm are better than those of the other algorithms. This fact can be observed in the Figs. 2, 3, 4, 5 and 6. For example in Fig. 6, we can see that the point on the blue graph of Diabetes has MAE value less than **0.08** and is lower than other points on the graph correspond with other methods. In those figures, the best values of SK and SA namely SK-2 and SA-4 ($\alpha = 0.3$) are used to compared with those of the other algorithms. Eventually, the MAE value of proposed method is better than those of other methods on the Diabetes.

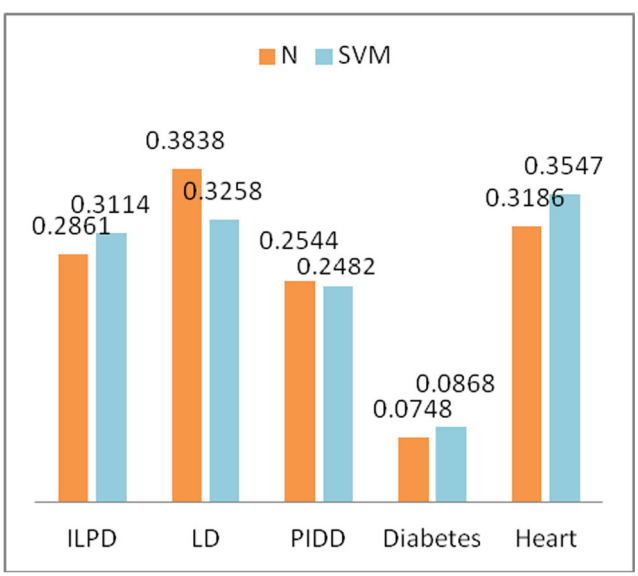

Fig. 16 MAE values of SVM and N

There is no huge difference in the computational time taken by the proposed method and other algorithms. From Table 19, it is clear that the computational time of the algorithms D, Z, SB, SK-1, SK-2, HY-2, Proposed method N, WX, VS-2, ZJ, SA-2, SA-4 ($\alpha = 0.1$), SA-4 ($\alpha = 0.3$) and SVM are **0.67, 0.5426, 0.6401, 0.5976, 0.7076, 0.5551, 0.669, 0.7501, 0.9401, 0.6601, 0.8401, 1.0801, 0.9476** and **0.11** seconds (sec) on the dataset of ILPD. This scenario can also be seen on the datasets of PIDD, Diabetes and Heart (Table 19). On LD, except for the method SA-2 which cannot run, the remaining algorithms have computational time that belong to the interval from **0.035** to **0.6348** sec, where the computational time of the proposed method is **0.4216** sec.

In order to see more clearly about the MAE of 10 algorithms, we draw Figs. 7, 8, 9, 10, 11, 12, 13, 14, 15, 16 and 17. The orange bars show the MAE of the proposed method while the light green ones demonstrate the MAE of other algorithms. In Fig. 7, we can see that the MSE values of **D** and the **proposed method** on the datasets of **ILPD, LD, PIDD, Diabetes** and **Heart**. We clearly see that the MAE of the proposed method on each dataset is better (smaller) than that of **D**.

We can see the same things in Figs. 9, 10, 11, 13, 15, which imply the proposed method is better than **SB, SK-2, HY-2, VS-2** and **SA-4** in accuracy.

In Fig. 8, MAE of the proposed method is better than that of **Z** on the datasets of **LD, PIDD** and **Diabetes**. Although this is not the same on the datasets of **LIPD** and **Heart**, the difference between MAE values of the proposed method and **Z** are less than or equal to **0.035**.

Similarly, in Figs. 12, 14 16, MAE value of the proposed method is better than those of the remaining methods on most datasets.

In Fig. 17, we present the MAE values of all algorithms. Once again, it is easy to see that the orange bars are smaller (or better) than most of the rest of the algorithms for each dataset.

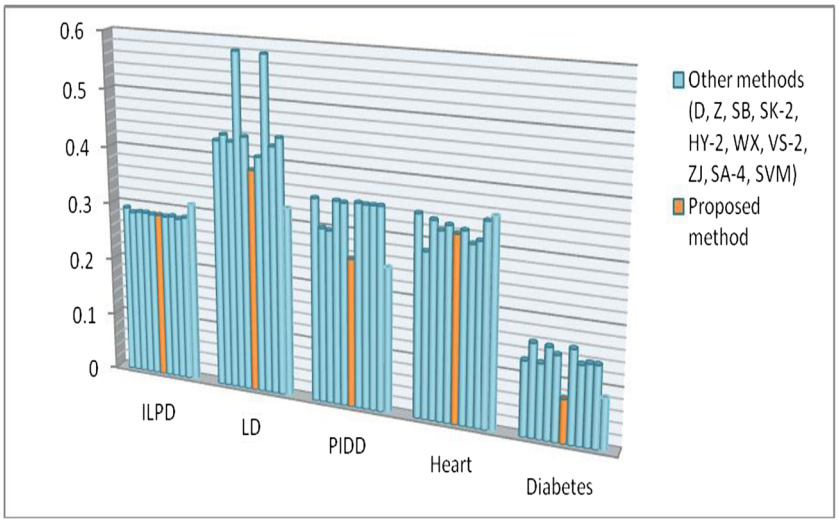

Fig. 17 MAE values of all algorithms

7 Conclusions

This paper concentrated on developing the notions of δ-equalities for the intuitionistic fuzzy set. Two fuzzy sets are said to be δ-equal if they equal to a degree of δ. δ-equalities have been used widely in fuzzy statistics and fuzzy reasoning such as in the applications of real-time fuzzy systems and the validation of robustness of fuzzy reasoning. This research extended the work of Cai [4] regarding the δ-equalities of fuzzy sets in a new context of intuitionistic fuzzy sets, which were shown to be better of modeling real-life applications than the fuzzy sets, and examined several characteristics and theorems of δ-equalities that were not (or partly) discussed in the previous works. The notions of δ-equalities for intuitionistic fuzzy relations and intuitionistic fuzzy norms were also proposed herein. Theoretical investigation of δ-equalities for intuitionistic fuzzy sets with set theoretic operations, such as the union, intersection, complement, product, probabilistic sum, bold sum, bold intersection, bounded difference, symmetrical difference, and convex linear sum of min and max, was mentioned. They are significant to understand the behavior of δ-equalities for intuitionistic fuzzy sets which is helpful to select appropriate settings for applications.

The last part of this paper applied the δ-equalities to the application of medical diagnosis, which investigates a patient's diseases from his symptoms. Medical data are often uncertain, ambiguous and difficult to retrieve. A categorized relationship between a symptom and a disease is usually dependant on uncertain information which affects the decision making process. The traditional approach from Sanchez [32] for medical diagnosis is using fuzzy relation to represent the relationships between patients-symptoms, symptoms-diseases and patients-diseases. However, as in medical diagnosis, normal level reference value ranges for attributes are given by different experts or different referenced ranges provided by a specific laboratory. Therefore, initial crisp symptoms of patients will give several different (intuitionistic) fuzzy sets, which result in the problem of choosing inappropriate (intuitionistic) fuzzified results to use in the next step. As such, our idea is using the concept of δ-equalities to find groups of (intuitionistic) fuzzified set with certain equality or similar degrees then combining them. Two numerical examples on a public dataset from the paper of Samuel and Balamurugan [31] and a real dataset were given to illustrate the application of δ-equalities to medical diagnosis. We ran the proposed algorithm and others on five real datasets to compare accuracy degree and computational time of them. The computing process in the algorithm is equipped with the propositions and theorems that have been mentioned lately.

Further works of this research will investigate new notions of sub δ-equalities such as the weighted δ-equalities. Specifically, let A and B be two intuitionistic fuzzy sets on a universe U. With δ-equalities, we have $A = (\delta)B$. But if U is accompanied with a corresponding set of weights $W = \{w_1, w_2, ..., w_n\}$ then we need to define a new notion called the weighted δ-equalities in order to adapt with the weights. We also study enhanced methods using the weighted δ-equalities for accelerating the diagnosis algorithm in this paper both in accuracy and computational complexity. Lastly, we may use δ-equalities in different decision making applications that incorporate intuitionistic fuzzy information in processing knowledge and information regarding inputs and outputs.

Compliance with Ethical Standards

Disclosure of potential conflicts of interest Conflict of interest: The authors declare that they have no conflict of interest.

Appendix

Source code and datasets of this paper can be found at this link: https://sourceforge.net/projects/ifs-delta-equalities-code1/.

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