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The Correlated Bivariate Noncentral F Distribution and its Application

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Abstract

This paper proposes the *singly* and *doubly* correlated bivariate noncentral F (BNCF) distributions. The probability density function (pdf) and the cumulative distribution function (cdf) of the distributions are derived for arbitrary values of the parameters. The pdf and cdf of the distributions for different arbitrary values of the parameters are computed, and their graphs are plotted by writing and implementing new R codes.

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An application of the correlated BNCF distribution is illustrated in the computations of the power function of the pre-test test for the multivariate simple regression model (MSRM).

Keywords: Correlated bivariate noncentral F distribution; noncentrality parameter; bivariate noncentral chi-square distribution, compounding distribution, pre-test test, and power function.

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1 Introduction

The bivariate central F (BCF) distribution has been studied by many authors, including Krishaniah (1965a), Amos and Bulgren (1972), Schuurmann et al. (1975), Johnson et al. (1995) and El-Bassiouny and Jones (2009). Krishnaiah (1965b) described the use of the BCF distribution in a problem of simultaneous statistical inference. Krishnaiah (1965c) and Krishnaiah and Armitage (1965) later studied the multivariate central F distribution. Hewett and Bulgren (1971) studied the prediction interval for failure times in certain life testing experiments using the multivariate central F distribution.

Many authors have also studied the univariate noncentral F distribution, including Mudholkar et al. (1976), Muirhead (1982), Johnson et al. (1995), and Shao (2005). Johnson et al. (1995) provided the definition of the univariate noncentral F distribution known as the *singly* noncentral F distribution. The authors also described the *doubly* noncentral F distribution with (ν_1, ν_2) degrees of freedom and noncentrality parameters λ_1 and λ_2 as the ratio of two independent noncentral chi-square variables, $\chi'^2_{\nu_1}(\lambda_1)/\nu_1$ and $\chi'^2_{\nu_2}(\lambda_2)/\nu_2$. Tiku (1966) proposed an approximation to the multivariate noncentral F distribution.

In the study of improving the power of a statistical test by pre-testing the uncertain non-sample prior information (NSPI) on the value of a set of parameters (cf. Saleh and Sen, 1983, and Yunus and Khan, 2011a), the cdf of a bivariate noncentral chi-square distribution is used to compute the power function of the test. For large sample studies, the cdf of the bivariate noncentral chi-square (BNCC) distribution is used to compute the power function of the test for testing one subset of regression parameters after pre-testing on another subset of parameters of a multivariate simple regression model (MSRM) (cf. Saleh and Sen, 1983, Yunus and Khan, 2011a). For small sample sizes, the computation of the power function and the size of the test after a pre-test (PT) requires the cdf of a correlated bivariate noncentral F (BNCF) distribution, which has not been reported in the literature because unlike those for the bivariate central F (BCF) distribution, the formulae for the pdf and cdf of the correlated BNCF distribution are more complex; hence, there are no easy computational formulae available. As such, no statistical packages include this distribution.

Yunus and Khan (2011b) derived the bivariate noncentral chi-square (BNCC) distribution by compounding the Poisson distribution with the correlated bivariate central chi-square distribution, aiming to compute the power function of the test after pre-testing. Therefore, using the same method of derivation, we derive the pdf and cdf of the *singly* and *doubly* correlated BNCF distributions in this paper. The *doubly* correlated BNCF is defined by mixing the correlated BNCC distribution with an independent central chi-square distribution. This definition allows for two noncentrality parameters from the two correlated noncentral chi-square variables in the numerator of the noncentral F variables. Additionally, by compounding the BCF and Poisson distributions, we derive the *singly* correlated BNCF distribution. This form of the BNCF distribution has only one noncentrality parameter. We also propose the computational formulae of the pdf and cdf of the correlated BNCF distribution and illustrate their application in the derivation of the power function of the pre-test test (PTT) (for details on the PTT, see Khan and Pratikno, 2013). The R codes are written to compute the values of the pdf and cdf of the correlated BNCF distribution and the power curve of the PTT of the MSRM. In addition to suggesting the computational formulae, we also compute and tabulate the critical values of the distribution for selected values of the parameters and significance levels using the R codes.

The next section derives the expression for the pdf and cdf of the correlated BNCF distribution. The computational method and graphical presentation of the pdf and cdf and the critical values of the correlated BNCF distribution for different values of the noncentrality parameter are presented in Section 3. An application of the BNCF distribution to the power function of the PTT is discussed in Section 4, and concluding remarks are provided in Section 5.

2 The Bivariate Noncentral F Distribution

In this section, the cdfs of the *singly* and *doubly* correlated bivariate noncentral F distributions are obtained using the compounding of distributions technique. The *doubly* correlated BNCF is obtained by compounding the correlated BNCC distribution with an independent central chi-square distribution, thus allowing for two noncentrality parameters and a correlation coefficient parameter. Additionally, by compounding the BCF and Poisson distributions, the *singly* correlated BNCF distribution is derived. This form of the BNCF distribution has only one noncentrality parameter and a correlation coefficient parameter.

2.1 The Singly Correlated Bivariate Noncentral F Distribution

Let a random variable X_i, $i = 1, 2$ follow an F distribution with ν_i degrees of freedom, and let another random variable R follow a Poisson distribution with mean λ. The proposed *singly* correlated BNCF distribution is an extension of the univariate noncentral F distribution introduced by Krishnaiah (1965a) and Johnson et al. (1995) for the bivariate case. The pdf of the *singly* BNCF distribution with noncentrality parameter λ is defined as

$$f(x_1, x_2, \nu_r, \nu_2, \lambda) \quad = \quad \sum_{r=0}^{\infty} \left(\frac{e^{-\lambda/2} \left(\frac{\lambda}{2}\right)^r}{r!} \right) f_1(x_1, x_2, \nu_r, \nu_2), \tag{2.1}$$

where $f_1(x_1, x_2, \nu_r, \nu_2)$ is the pdf of a BCF distribution with ν_r and ν_2 degrees of freedom in which $\nu_r = \nu_1 + 2r$; that is,

$$\begin{aligned} f_1(x_1, x_2, \nu_r, \nu_2) \quad &= \quad \left(\frac{\nu_2^{\nu_2/2} (1-\rho^2)^{(\nu_r+\nu_2)/2}}{\Gamma(\nu_r/2)\Gamma(\nu_2/2)} \right) \sum_{j=0}^{\infty} \left(\frac{\rho^{2j} \Gamma(\nu_r + (\nu_2/2) + 2j)}{j! \Gamma((\nu_r/2) + j)} \right) \nu_r^{\nu_r+2j} \\ &\times \left(\frac{(x_1 x_2)^{(\nu_r/2)+j-1}}{[\nu_2(1-\rho^2) + \nu_r(x_1+x_2)]^{\nu_r+(\nu_2/2)+2j}} \right). \end{aligned}$$

Note that the density function of the *singly* correlated BNCF distribution is obtained by compounding the BCF distribution with the Poisson probabilities.

Therefore, the cdf of the *singly* correlated BNCF distribution is defined as

$$\begin{aligned} P(.) &= P(X_1 < d, X_2 < d, \nu_r, \nu_2, \lambda) \\ &= \sum_{r=0}^{\infty} \left(\frac{e^{-\lambda/2} \left(\frac{\lambda}{2}\right)^r}{r!} \right) P_2(X_1 < d, X_2 < d, \nu_r, \nu_2), \end{aligned} \tag{2.2}$$

where

$$P_2(X_1 < d, X_2 < d, \nu_r, \nu_2) = \left(\frac{(1-\rho^2)^{\nu_r/2}}{\Gamma(\nu_r/2)\Gamma(\nu_2/2)} \right) \sum_{j=0}^{\infty} \left(\frac{\rho^{2j}\Gamma(\nu_r + (\nu_2/2) + 2j)}{j!\Gamma((\nu_r/2)+j)} \right) L_{jr}$$

and L_{jr} is defined as

$$L_{jr} = \int_0^{h_r}\int_0^{h_r} \frac{(x_1x_2)^{(\nu_r/2)+j-1}dx_1dx_2}{(1+x_1+x_2)^{\nu_r+(\nu_2/2)+2j}}$$

with $h_r = \frac{d\nu_r}{\nu_2(1-\rho^2)}$.

For the computation of the value of the cdf of the *singly* correlated BNCF distribution, R codes are used. To make the computations easier, we represent the formulae of the cdf of the *singly* correlated BNCF distribution in equation (2.2) as the sum of infinite series as follows:

$$\begin{aligned}
P(.) &= \sum_{r=0}^{\infty}\left(\frac{e^{-\lambda/2}\left(\frac{\lambda}{2}\right)^r}{r!}\right)\left(\frac{(1-\rho^2)^{\nu_r/2}}{\Gamma(\nu_r/2)\Gamma(\nu_2/2)}\right)\sum_{j=0}^{\infty}\left(\frac{\rho^{2j}\Gamma(\nu_r+(\nu_2/2)+2j)}{j!\Gamma((\nu_r/2)+j)}\right)L_{jr} \\
&= \sum_{r=0}^{\infty}T_r\left[\left(\frac{1\Gamma(\nu_r+(\nu_2/2))}{0!\Gamma((\nu_r/2))}\right)L_{0r}+\left(\frac{\rho^2\Gamma(\nu_r+(\nu_2/2)+2)}{1!\Gamma((\nu_r/2)+1)}\right)L_{1r}+\cdots\cdots\right] \\
&= \sum_{r=0}^{\infty}T_r\left[H_{0r}L_{0r}+H_{1r}L_{1r}+H_{2r}L_{2r}+\ \cdots\cdots\right] \\
&= \sum_{r=0}^{\infty}T_rH_{0r}L_{0r}+T_rH_{1r}L_{1r}+T_rH_{2r}L_{2r}+\cdots\cdots \\
&= \left[T_0H_{00}L_{00}+T_0H_{10}L_{10}+T_0H_{20}L_{20}\ +\cdots\cdots\right] \\
&\qquad +\left[T_1H_{01}L_{01}+T_1H_{11}L_{11}+T_1H_{21}L_{21}\ +\cdots\cdots\right] \\
&\qquad +\left[T_2H_{02}L_{02}+T_2H_{12}L_{12}+T_2H_{22}L_{22}\ +\cdots\cdots\right] \\
&\qquad +\cdots\cdots, \qquad (2.3)
\end{aligned}$$

where

$$T_r=\left(\frac{e^{-\lambda/2}\left(\frac{\lambda}{2}\right)^r}{r!}\right)\left(\frac{(1-\rho^2)^{\nu_r/2}}{\Gamma(\nu_r/2)\Gamma(\nu_2/2)}\right),$$

$$H_{0r}=\frac{1\Gamma(\nu_r+(\nu_2/2))}{0!\Gamma((\nu_r/2))},\ H_{1r}=\frac{\rho^2\Gamma(\nu_r+(\nu_2/2)+2)}{1!\Gamma((\nu_r/2)+1)},\ H_{2r}=\frac{\rho^4\Gamma(\nu_r+(\nu_2/2)+4)}{2!\Gamma((\nu_r/2)+2)},\cdots,$$

and P_0 is defined as

$$\begin{aligned}
P_0 &= \sum_{r=0}^{\infty}T_rH_{0r}L_{0r}=T_0H_{00}L_{00}+T_1H_{01}L_{01}+T_2H_{02}L_{02}+\cdots\cdots,\ \text{for } j=0, \\
&= \left(\frac{e^{-\lambda/2}}{0!}\times\frac{(1-\rho^2)^{\nu_1/2}}{\Gamma(\nu_1/2)\Gamma(\nu_2/2)}\right)\left(\frac{1\Gamma(\nu_1+\nu_2/2)}{0!\Gamma(\nu_1/2)}\right)L_{00}\ + \\
&\quad \left(\frac{e^{-\lambda/2}\left(\frac{\lambda}{2}\right)}{1}\times\frac{(1-\rho^2)^{\nu_1+2/2}}{\Gamma((\nu_1+2)/2)\Gamma(\nu_2/2)}\right)\left(\frac{1\Gamma(\nu_1+2+(\nu_2/2))}{0!\Gamma((\nu_1+2)/2)}\right)L_{01}+\cdots. \qquad (2.4)
\end{aligned}$$

Similarly, we obtain the expressions for $P_1 = \sum_{r=0}^{\infty} T_r H_{1r} L_{1r}$, $P_2 = \sum_{r=0}^{\infty} T_r H_{2r} L_{2r}$ and so on. Finally, we write

$$P(.) = P_0 + P_1 + P_2 + P_3 + \cdots\cdots = \sum_{j=0}^{\infty} P_j.$$

Some properties of the *singly* correlated BNCF distribution are given as follows:

(i) Note that $f_1(x_1, x_2, \nu_r, \nu_2)$ in equation (2.1) is a pdf of a BCF distribution with ν_r and ν_2 degrees of freedom; hence, $\int_0^\infty \int_0^\infty f_1(x_1, x_2, \nu_r, \nu_2) dx_1 dx_2 = \lim_{d\to\infty} P_2(X_1 < d, X_2 < d)$ in equation (2.2) is equal to one. Furthermore, $\sum_{r=0}^{\infty} \frac{e^{-\lambda/2}(\lambda/2)^r}{r!} = 1$. Thus, it can easily be observed that $\int_0^\infty \int_0^\infty f(x_1, x_2, \nu_r, \nu_2, \lambda) dx_1 dx_2 = 1$.

(ii) Because $f_1(x_1, x_2, \nu_r, \nu_2)$ is a pdf of a BCF, $f_1(\cdot) \geq 0$. It is noted that the quantity $\frac{e^{-\lambda/2}(\lambda/2)^r}{r!}$ is always positive. Therefore, $f(\cdot) \geq 0$.

(iii) From equation (2.1), the central case of the bivariate F distribution proposed by Krishnaiah (1965a) is a special case of the *singly* correlated BNCF distribution when the noncentrality parameter, λ, is equal to zero.

2.2 The Doubly Correlated Bivariate Noncentral F Distribution

Let the random variables (X_1, X_2) jointly follow a correlated BNCC distribution with m degrees of freedom, noncentrality parameters θ_1 and θ_2, and a correlation coefficient ρ, and let the random variable Z follow a central chi-square distribution with n degrees of freedom. We propose the cdf of the *doubly* correlated BNCF by compounding the two aforementioned distributions.

The pdf of the correlated BNCC variables X_1 and X_2 proposed by Yunus and Khan

(2011b) is given by

$$
\begin{aligned}
g(x_1, x_2) \quad = \quad & \sum_{j=0}^{\infty} \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \left[\rho^{2j}(1-\rho^2)^{m/2}\Gamma(m/2+j)\right] \\
& \times \left[\frac{(x_1)^{m/2+j+r_1-1} e^{-\frac{(x_1)}{2(1-\rho^2)}}}{[2(1-\rho^2)]^{m/2+j+r_1}\Gamma(m/2+j+r_1)} \times \frac{e^{-\theta_1/2}(\theta_1/2)^{r_1}}{r_1!} \right] \\
& \times \left[\frac{(x_2)^{m/2+j+r_2-1} e^{-\frac{(x_2)}{2(1-\rho^2)}}}{[2(1-\rho^2)]^{m/2+j+r_2}\Gamma(m/2+j+r_2)} \times \frac{e^{-\theta_2/2}(\theta_2/2)^{r_2}}{r_2!} \right],
\end{aligned}
\tag{2.5}
$$

and the pdf of a central chi-square variable Z with n degrees of freedom is given by

$$
f(z) = \frac{z^{(n/2)-1} e^{-z/2}}{2^{n/2}\Gamma(n/2)}, \tag{2.6}
$$

where Z is independent of X_1 and X_2.

Therefore, the random variables (Y_1, Y_2), where

$$
Y_i = \frac{X_i/m}{Z/n}, \quad \text{for} \;\; i = 1, 2, \tag{2.7}
$$

have a joint cdf given by

$$
P(Y_1 \leq a, Y_2 \leq b) = \int_{z=0}^{\infty} f(z) \int_{x_2=0}^{\frac{bmz}{n}} \int_{x_1=0}^{\frac{amz}{n}} g(x_1, x_2) dx_1 dx_2 dz. \tag{2.8}
$$

The distribution function given in equation (2.8) is the cdf of the proposed *doubly* correlated BNCF distribution with m and n degrees of freedom, noncentrality parameters θ_1 and θ_2, and correlation coefficient ρ.

In addition, equation (2.8) can be expressed as the following sum of infinite series

$$\begin{aligned} P(Y_1 \le a, Y_2 \le b) &= (1-\rho^2)^{\frac{m}{2}} \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\sum_{j=0}^{\infty} \frac{(\frac{m}{2})_j}{j!}\rho^{2j} I_2(\tilde{\alpha}_j, \tilde{c}, \beta) \\ &\quad \times \frac{e^{-\theta_1/2}(\theta_1/2)^{r_1}}{r_1!}\frac{e^{-\theta_2/2}(\theta_2/2)^{r_2}}{r_2!}, \end{aligned} \tag{2.9}$$

where

$$I_2(\tilde{\alpha}_j, \tilde{c}, \beta) = \int_0^\infty \frac{e^{-z}z^{\beta-1}}{\Gamma(\beta)}\frac{\gamma(\alpha_1, c_1 z)}{\Gamma(\alpha_1)}\frac{\gamma(\alpha_2, c_2 z)}{\Gamma(\alpha_2)} dz$$

and

$$\beta = \frac{n}{2}, \quad \tilde{c} = \left(\frac{am}{n(1-\rho^2)}, \frac{bm}{n(1-\rho^2)}\right), \quad \tilde{\alpha}_j = \left(\frac{m}{2}+j+r_1, \frac{m}{2}+j+r_2\right).$$

Here, $\gamma(\alpha, x) = \int_0^x e^{-t}t^{\alpha-1}dt$, and $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$.

To ease the computational difficulties of the cdf, we use the following form of I_2 given by Amos and Bulgren (1972),

$$\begin{aligned} I_2(\tilde{\alpha}_j, \tilde{c}, \beta) &= I_u(\alpha_1, \beta) - \frac{(1-u)^\beta}{\alpha_1}\frac{\Gamma(\beta+\alpha_1)}{\Gamma(\beta)\Gamma(\alpha_1)} \\ &\quad \times \sum_{r=0}^{\infty} \frac{(\beta+\alpha_1)_r}{(1+\alpha_1)_r} u^{r+\alpha_1} I_{1-y}(r+\beta+\alpha_1, \alpha_2), \end{aligned} \tag{2.10}$$

with

$$u = c_1/(1+c_1), \quad 1-y = (1+c_1)/(1+c_1+c_2),$$

and

$$\begin{aligned}\int_0^\infty \frac{e^{-z}z^{\beta-1}}{\Gamma(\beta)}\frac{\gamma(\alpha,cz)}{\Gamma(\alpha)}dz &= I_z(\alpha,\beta) \quad \text{and} \\ \int_0^\infty \frac{e^{-z}z^{\beta-1}}{\Gamma(\beta)}\frac{\Gamma(\alpha,cz)}{\Gamma(\alpha)}dz &= I_{1-z}(\beta,\alpha)\end{aligned}$$

are the regularized beta functions, with $\alpha > 0, \beta > 0$, $x = c/(1+c)$, and $1 - x = 1/(1+c)$.

See the Appendix for the pdf of the *doubly* correlated BNCF distribution, which is derived using the transformation of variables method.

Some properties of the *doubly* correlated BNCF distribution are given as follows:

(i) From equation (2.9), we find that $F_1(a,b;r_1,r_2) = (1-\rho^2)^{\frac{m}{2}}\sum_{j=0}^{\infty}\frac{(\frac{m}{2})_j}{j!}\rho^{2j}I_2(\tilde{\alpha}_j,\tilde{c},\beta)$ is the cdf of a BCF distribution (Amos and Bulgren, 1972); thus, $F_1(a,b)$ approaches 1 as both a and b go to infinity. It is clear that both quantities $\sum_{r_1=0}^{\infty}\frac{e^{-\theta_1/2}(\theta_1/2)^{r_1}}{r_1!}$ and $\sum_{r_2=0}^{\infty}\frac{e^{-\theta_2/2}(\theta_2/2)^{r_2}}{r_2!}$ are equal to one. It follows that the cdf of the *doubly* correlated BNCF distribution approaches 1 as both a and b go to infinity.

(ii) When a and b are zero, it is easy to show that the cdf of the *doubly* correlated BNCF is zero.

(iii) Note that $F_1(a,b)$ is an increasing function because it is a cdf of a BCF distribution. It follows that the cdf of the *doubly* correlated BNCF distribution is also an increasing function.

(iv) As b approaches infinity, $\gamma(\alpha_2, c_2z) = \Gamma(\alpha_2)$ through $c_2 = bm/n(1-\rho^2)$, and the second term on the right-hand side of equation (2.10) becomes zero because $1-y$ approaches zero as b goes to infinity. Further simplifications yield the following marginal distribution function

$$\sum_{r_1=0}^{\infty} I_{\frac{my_1}{n+my_1}}(m/2+r_1, n/2)\frac{e^{-\theta_1/2}(\theta_1/2)^{r_1}}{r_1!} = F(y_1; m, n, \theta_1), \tag{2.11}$$

which is the cdf of the noncentral F distribution of Y_1, with noncentrality parameter θ_1 and degrees of freedom m and n. In the same manner, the marginal distribution function for Y_2 can be derived.

(v) The central F distribution can be obtained from the noncentral distribution if the noncentrality parameters, θ_1 and θ_2, are equal to 0. Because r_1 and r_2 are both zero, we rewrite (2.9) as

$$P(Y_1 \leq a, Y_2 \leq b) \quad = \quad (1-\rho^2)^{\frac{m}{2}} \sum_{j=0}^{\infty} \frac{(\frac{m}{2})_j}{j!} \rho^{2j} I_2(\hat{\alpha}_j, \hat{c}, \beta)$$

with

$$\beta = \frac{n}{2}, \quad \hat{c} = \left(\frac{am}{(1-\rho^2)n}, \frac{bm}{(1-\rho^2)n}\right), \quad \hat{\alpha} = \left(\frac{m}{2}+j, \frac{m}{2}+j\right). \tag{2.12}$$

Thus, we arrive at the central correlated bivariate F distribution proposed by Amos and Bulgren (1972) after allowing both noncentrality parameters equal zero in the *doubly* correlated BNCF distribution.

(vi) For $\rho = 0$, which implies $j = 0$, we write equation (2.9) as

$$P(Y_1 \leq a, Y_2 \leq b) \quad = \quad \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty} I_2(\check{\alpha}, \check{c}, \beta)\frac{e^{-\theta_1/2}(\theta_1/2)^{r_1}}{r_1!}\frac{e^{-\theta_2/2}(\theta_2/2)^{r_2}}{r_2!} \tag{2.13}$$

with

$$\beta = \frac{n}{2}, \quad \check{c} = \left(\frac{am}{n}, \frac{bm}{n}\right), \quad \check{\alpha} = \left(\frac{m}{2} + r_1, \frac{m}{2} + r_2\right). \tag{2.14}$$

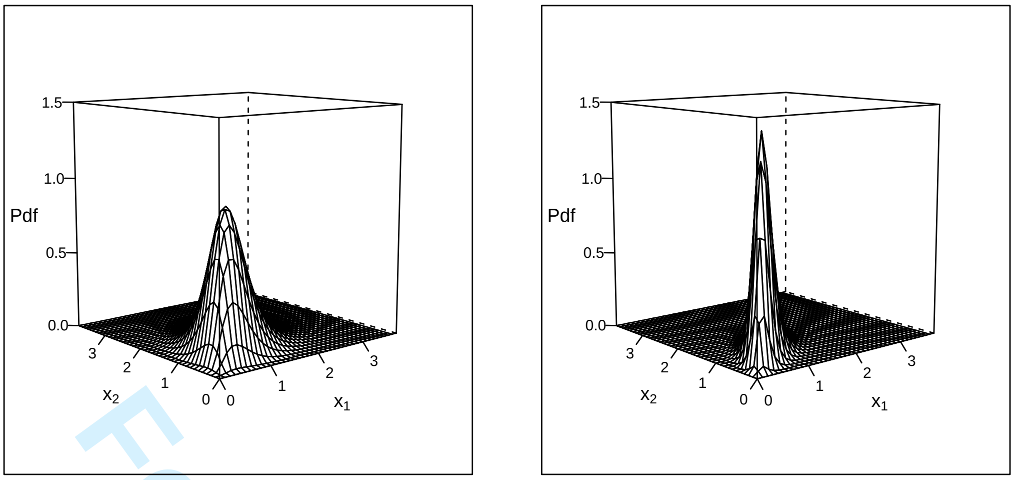
It can be observed that Y_1 and Y_2 are not independent, although the correlation coefficient between Y_1 and Y_2 is zero. In other words, the *doubly* correlated BNCF can have a zero correlation, but the marginal distributions do not support statistical independence.

3 Computation of the pdf and cdf

To compute the values of the pdf and cdf of the BNCF distributions, R codes are written. The R package is also used for the graphical representation of the pdf and cdf. The pdf of the *singly* BNCF distribution is computed using equation (2.1) and plotted in Figure 1. The graph in Figure 1(iii) has a wider spread than that in Figure 1(i) due to the smaller value of ν_1. Comparing Figures 1(i) and 1(iv), the spread of the distribution in Figure 1(iv) decreases due to the increase in the noncentrality parameter. As the value of ρ increases, the spread of the distribution decreases and the pdf shrinks, as shown in Figure 1(ii). For the *doubly* correlated BNCF distribution, the pdf is calculated using equation (5.4) and plotted in Figure 2. The graphs in Figure 2 show properties similar to those shown in Figure 1 but with varying probabilities.

To compute the cdf of the *singly* correlated BNCF distribution in equation (2.3), we choose arbitrary values of the degrees of freedom (ν_1, ν_2), noncentrality parameter (λ), correlation coefficient (ρ) and upper limit (d) of the variable. Figure 3 shows that the cdf of the *singly* correlated BNCF distribution increases as the value of any of the parameters, namely, the degrees of freedom ν_1 (for fixed ν_2), λ, or d, increases.

(i) The pdf for ν_1=10, ν_2=20, ρ =0.5, λ =1 (ii) The pdf for ν_1=10, ν_2=20, ρ =0.9, λ =1



(iii) The pdf for ν_1=5, ν_2=20, ρ =0.5, λ =1 (iv) The pdf for ν_1=10, ν_2=20, ρ =0.5, λ =3

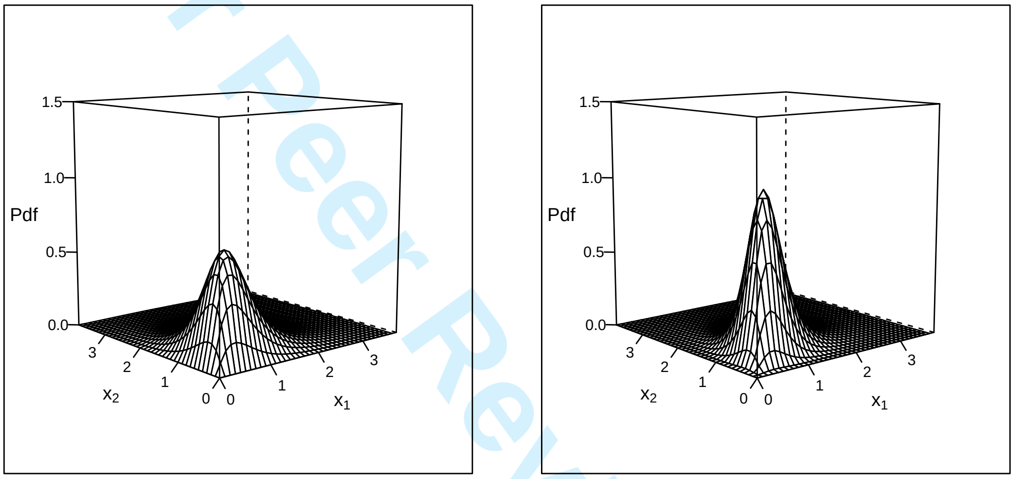
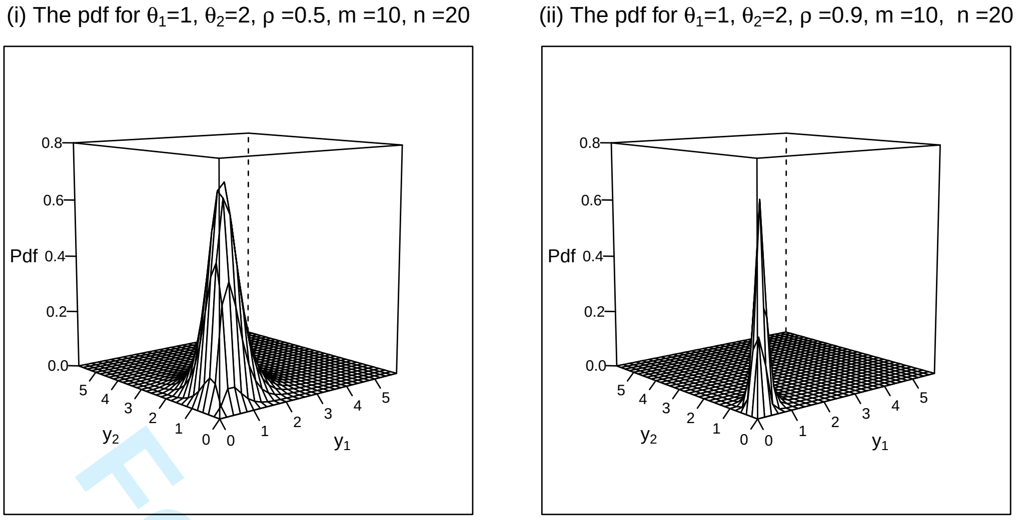


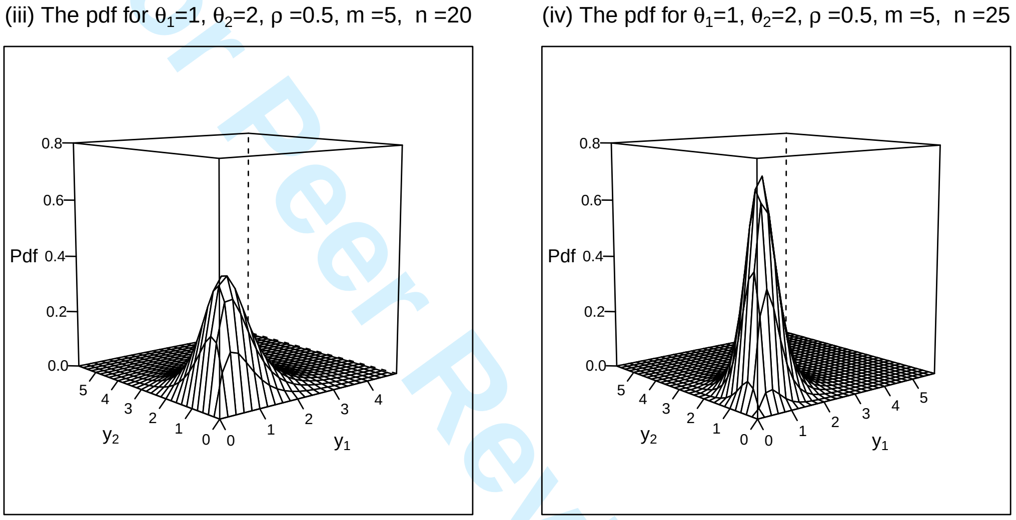
Figure 1: The pdf of the *singly* correlated bivariate noncentral F distribution.

The cdf of the *doubly* correlated BNCF distribution is computed using equation (2.9) for arbitrary degrees of freedom (m, n), noncentrality parameters (θ_1, θ_2), correlation coefficient (ρ), and upper limit $(a = b = d)$. The graphs of the cdf of the *doubly* correlated BNCF distribution are presented in Figure 4. Interestingly, the cdf curve approaches 1 more rapidly for a larger correlation coefficient (see Figure 4(i)), a smaller noncentrality parameter (see Figure 4(ii)) and a greater number of degrees of freedom (m, n) (see Figures 4(iii) and 4(iv)). Figure 4 shows that the shape of the cdf curve is sigmoidal, which depends on the values of



(i) The pdf for θ_1=1, θ_2=2, ρ =0.5, m =10, n =20

(ii) The pdf for θ_1=1, θ_2=2, ρ =0.9, m =10, n =20



(iii) The pdf for θ_1=1, θ_2=2, ρ =0.5, m =5, n =20

(iv) The pdf for θ_1=1, θ_2=2, ρ =0.5, m =5, n =25

Figure 2: The pdf of the *doubly* correlated bivariate noncentral F distribution.

the noncentrality parameters $(\theta_1,\ \theta_2)$, degrees of freedom (m, n), and correlation coefficient (ρ). Tables 1 and 2 provide the values of d for different values of m, n, θ_1, θ_2, and α, where

$$P(Y_1 < d, Y_2 < d) = \int_0^d \int_0^d f(y_1, y_2) dy_1 dy_2 = (1-\alpha),$$

for the case in which $\rho = 0.5$.

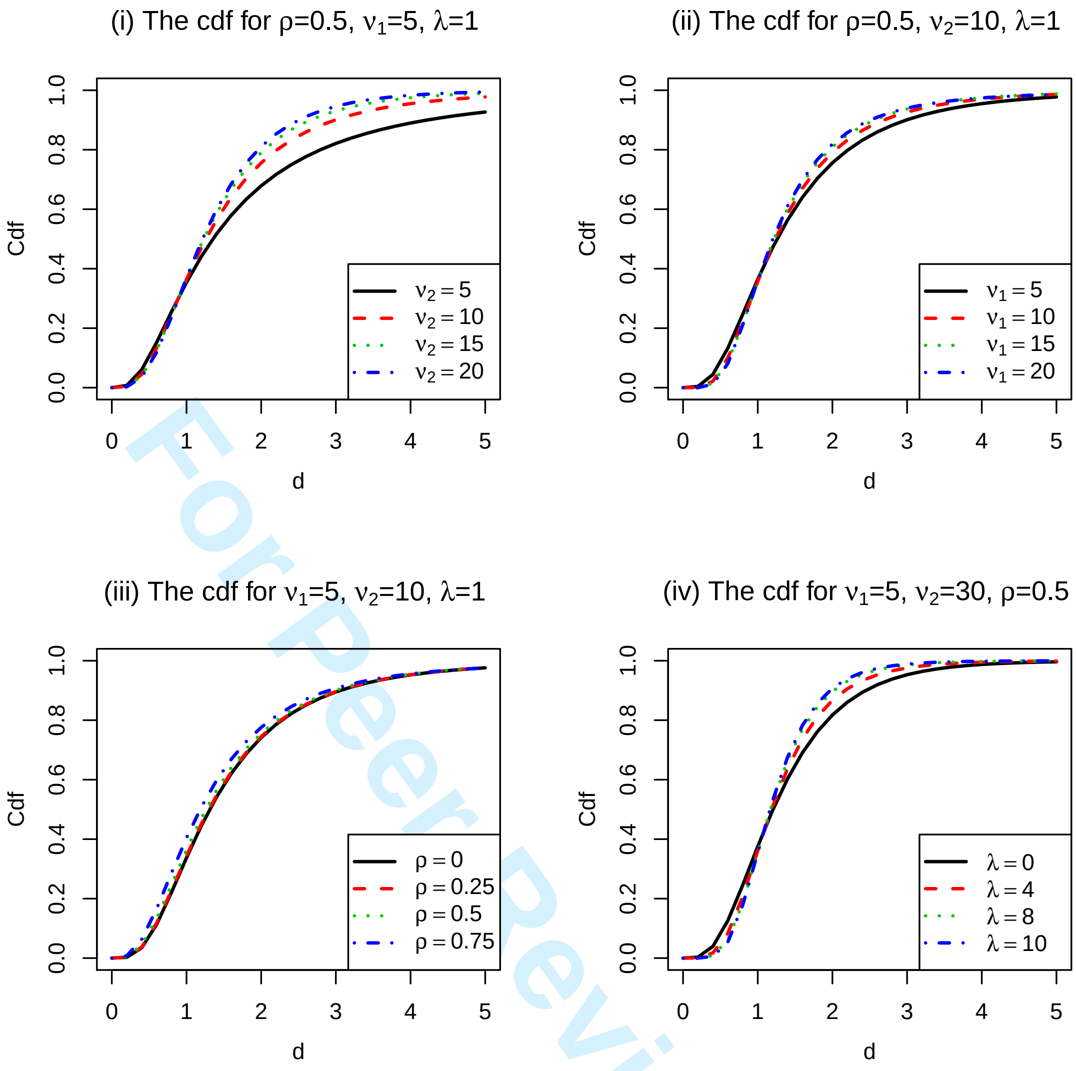


Figure 3: The cdf of the singly correlated BNCF distribution with arbitrary values of ρ, λ, ν_1 and ν_2.

4 Application to the Power Function of the PTT

To test the null hypothesis on the intercept vector $H_0 : \boldsymbol{\beta}_0 = \boldsymbol{\beta}_{00}$ (given known vector) against $H_a : \boldsymbol{\beta}_0 > \boldsymbol{\beta}_{00}$ in the multivariate simple regression model

$$\boldsymbol{y}_i = \boldsymbol{\beta}_0 + \boldsymbol{\beta}_1 x_i + \boldsymbol{e}_i, \tag{4.1}$$

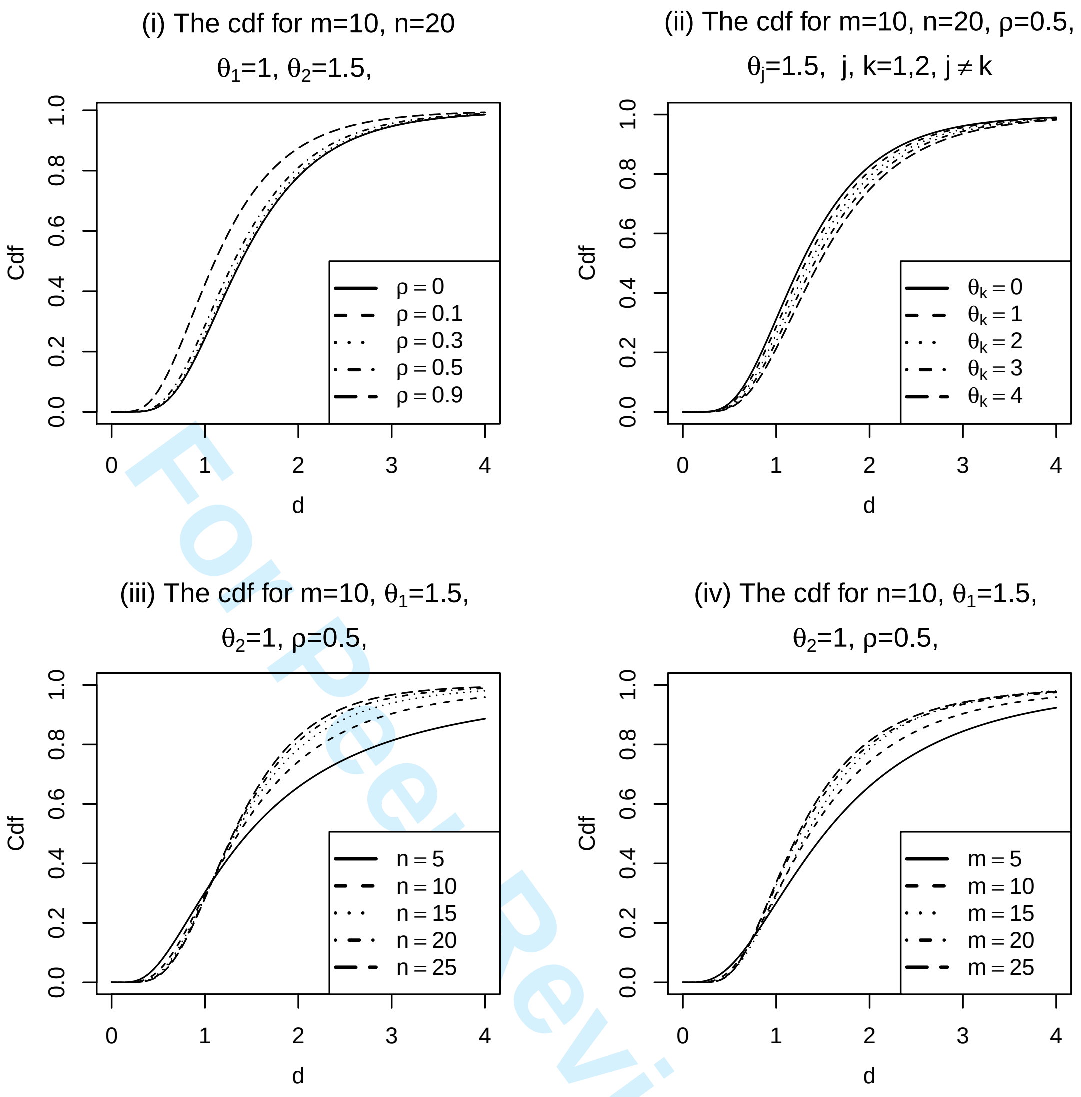


Figure 4: The cdf of the doubly correlated BNCF distribution with arbitrary values of ρ, θ_k, m and n.

(for details, see Khan, 2006) when there is non-sample prior information on the slope vector $\boldsymbol{\beta}_1$, the test statistic follows a correlated bivariate F distribution. The ultimate test on H_0 is called the pre-test test (PTT) because it depends on the outcome of the pre-test on the suspected slope; that is, $H_0^* = \boldsymbol{\beta}_1 = \boldsymbol{\beta}_{10}$ (cf. Khan and Pratikno, 2013). The cdf of the *doubly* correlated BNCF distribution is involved in the formulae for the power function of the PTT. To illustrate the method, we conduct a simulation study by generating random data using the R package.

The explanatory variable (x) is generated from the uniform distribution between 0 and 1. The error vector ($\boldsymbol{e}$) is generated from a $p = 3$ dimensional multivariate normal distribution with $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \sigma^2 \boldsymbol{I}_3$, where I_3 is the identity matrix of order 3. Then, the dependent variable (y_1) is determined by $y_1 = \beta_0^{'} + \beta_1^{'} x + e_1$ for $\beta_0^{'} = 3$ and $\beta_1^{'} = 1.5$. Similarly, y_2 and y_3 are determined by $y_2 = \beta_0^{''} + \beta_1^{''} x + e_2$ for $\beta_0^{''} = 5$ and $\beta_1^{''} = 2.5$ and by $y_3 = \beta_0^{'''} + \beta_1^{'''} x + e_3$ for $\beta_0^{'''} = 6$ and $\beta_1^{'''} = 3$. For each of the three cases, $n = 20$ random variates are generated.

Considering the three cases (i) unspecified $\boldsymbol{\beta}_1$, (ii) specified $\boldsymbol{\beta}_1$ and (iii) uncertain prior information on $\boldsymbol{\beta}_1$, we define the unrestricted, restricted and pre-test test statistics as follows: $T^{UT} = \sum_{i=1}^n (x_i - \bar{x})^2 [(\tilde{\boldsymbol{\beta}}_0 - \boldsymbol{\beta}_{00})' \hat{\Sigma}^{-1} (\tilde{\boldsymbol{\beta}}_0 - \boldsymbol{\beta}_{00})]$, $T^{RT} = \sum_{i=1}^n (x_i - \bar{x})^2 [(\bar{\boldsymbol{y}} - \boldsymbol{\beta}_{10}\bar{x} - \boldsymbol{\beta}_{00})' \hat{\Sigma}^{-1} (\bar{\boldsymbol{y}} - \boldsymbol{\beta}_{10}\bar{x} - \boldsymbol{\beta}_{00})]$ and $T^{PT} = \sum_{i=1}^n (x_i - \bar{x})^2 [(\tilde{\boldsymbol{\beta}}_1 - \boldsymbol{\beta}_{10})' \hat{\Sigma}^{-1} (\tilde{\boldsymbol{\beta}}_1 - \boldsymbol{\beta}_{10})]$, respectively. Here, $\hat{\Sigma}^{-1} = \frac{1}{n-p} \sum_{i=1}^n (\boldsymbol{y}_i - \tilde{\boldsymbol{\beta}}_0 - \tilde{\boldsymbol{\beta}}_1 x_i)(\boldsymbol{y}_i - \tilde{\boldsymbol{\beta}}_0 - \tilde{\boldsymbol{\beta}}_1 x_i)'$, where $\hat{\boldsymbol{\beta}}_0 = \bar{\boldsymbol{y}} - \boldsymbol{\beta}_{10}\bar{x}$, $\tilde{\boldsymbol{\beta}}_0 = \bar{\boldsymbol{y}} - \tilde{\boldsymbol{\beta}}_1 \bar{x}$, $\tilde{\boldsymbol{\beta}}_1 = \sum_{i=1}^n \frac{(\boldsymbol{y}_i - \bar{\boldsymbol{y}})(x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2}$, $\bar{x} = \sum_{i=1}^n x_i/n$, and $\bar{\boldsymbol{y}} = \sum_{i=1}^n \boldsymbol{y}_i/n$.

Under $H_a : \boldsymbol{\beta}_0 > \boldsymbol{\beta}_{00}$, T^{UT} and T^{RT} follow a noncentral F distribution with $(p, n-p)$ degrees of freedom and noncentrality parameters $\Delta_1^2/2$ and $\Delta_2^2/2$, respectively, where $\Delta_1^2 = \sum_{i=1}^n (x_i - \bar{x})^2 [(\boldsymbol{\beta}_0 - \boldsymbol{\beta}_{00})' \Sigma^{-1} (\boldsymbol{\beta}_0 - \boldsymbol{\beta}_{00})]$ and $\Delta_2^2 = \sum_{i=1}^n (x_i - \bar{x})^2 [(\bar{Y} - \boldsymbol{\beta}_{10}\bar{x} - \boldsymbol{\beta}_{00})' \hat{\Sigma}^{-1} (\bar{\boldsymbol{y}} - \boldsymbol{\beta}_{10}\bar{x} - \boldsymbol{\beta}_{00})]$. Under $H_a^\star : \boldsymbol{\beta}_1 > \boldsymbol{\beta}_{10}$, T^{PT} follows a noncentral F distribution with $(p, n-p)$ degrees of freedom and noncentrality parameter $\Delta_3^2/2$, where $\Delta_3^2 = \sum_{i=1}^n (x_i - \bar{x})^2 [(\boldsymbol{\beta}_1 - \boldsymbol{\beta}_{10})' \hat{\Sigma}^{-1} (\boldsymbol{\beta}_1 - \boldsymbol{\beta}_{10})]$.

Let $\{K_n\}$ be a sequence of alternative hypotheses

$$K_n : \boldsymbol{\beta}_0 = \boldsymbol{\beta}_{00} + \boldsymbol{\lambda}_1/\sqrt{n}, \boldsymbol{\beta}_1 = \boldsymbol{\beta}_{10} + \boldsymbol{\lambda}_2/\sqrt{n}, \tag{4.2}$$

where $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ are vectors of fixed real numbers. Under $\{K_n\}$, the power function of the

PTT is given by

$$\begin{aligned}\pi^{PTT}(\boldsymbol{\lambda}) &= P\left(T^{PT} < a, T^{RT} > c\right) + P\left(T^{PT} \geq a, T^{UT} > b\right) \\ &= P\left(T^{PT} < a\right)\ P\left(T^{RT} > c\right) + d_{1r}\left(a, b, \rho\right) \\ &= \left[1 - P\left(T^{PT} > a\right)\right]\ P\left(T^{RT} > c\right) + d_{1r}\left(a, b, \rho\right), \qquad (4.3)\end{aligned}$$

where $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_1/\sqrt{n}, \boldsymbol{\lambda}_2/\sqrt{n})$, $a = F_{\alpha_3,p,n-p} - \phi_2$, $b = F_{\alpha_1,p,n-p} - \phi_1$ and $c = F_{\alpha_2,n,n-1} - [\phi_1 + \phi_2\bar{x}] + \omega\bar{x}$ for $\phi_1 = \boldsymbol{\lambda}_1'\Sigma^{-1}\boldsymbol{\lambda}_1$, $\phi_2 = \boldsymbol{\lambda}_2'\Sigma^{-1}\boldsymbol{\lambda}_2$, $\omega = \boldsymbol{\lambda}_1'\Sigma^{-1}\boldsymbol{\lambda}_2 + \boldsymbol{\lambda}_2'\Sigma^{-1}\boldsymbol{\lambda}_1$, and $d_{1r}(a, b, \rho)$ is a correlated bivariate F probability integral defined as

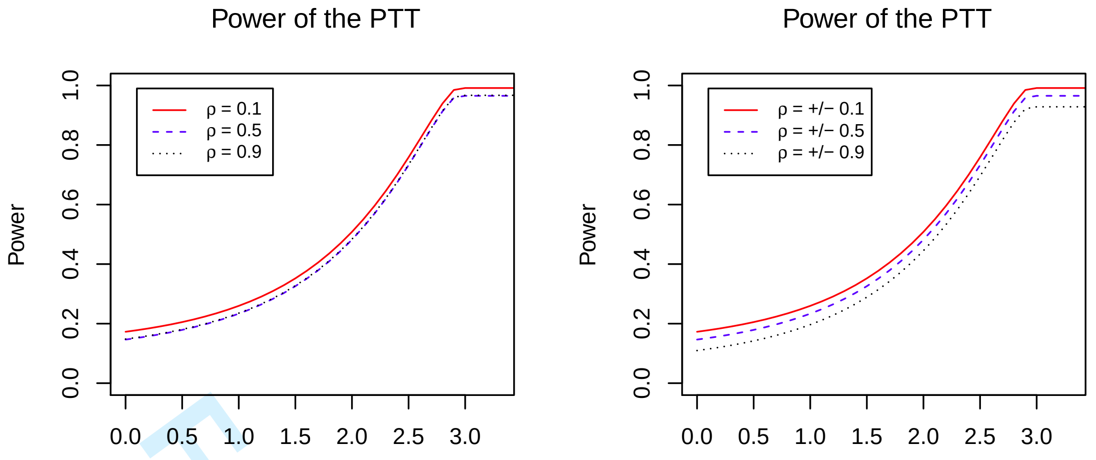
$$d_{1r}(a, b, \rho) = \int_b^\infty \int_a^\infty f(F^{PT}, F^{UT})dF^{PT}dF^{UT} \qquad (4.4)$$

with $\rho = \frac{n^2(n-p-4)}{(2n-p-2)^2(n-p-4)}$. Clearly, the power of the PTT is defined in terms of the powers of the RT and PT as well as the cdf of the *doubly* correlated BNCF distribution.

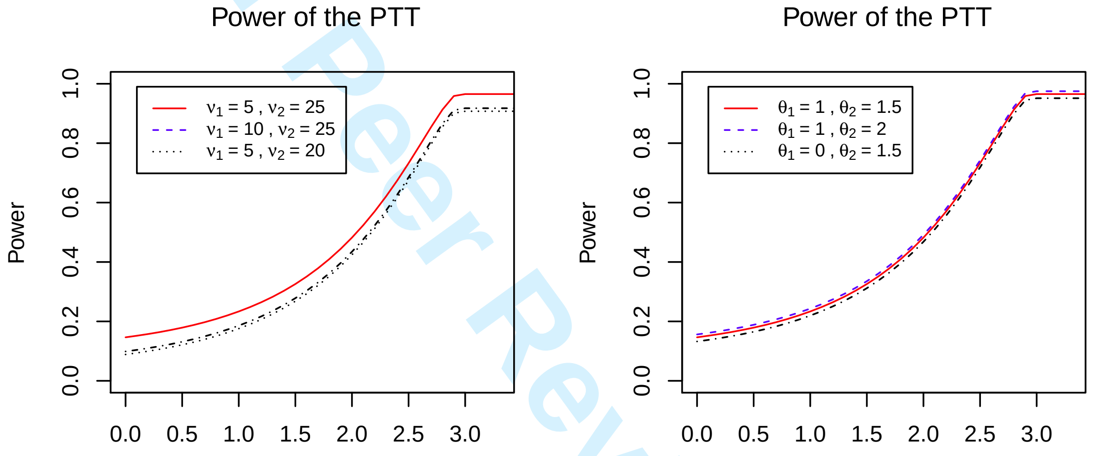
Figure 5 shows the graphs of the power function of the PTT in terms of $d_{1r}(d, d, \rho)$ for selected values of the correlation coefficients (ρ), noncentrality parameters (θ_1, θ_2) and degrees of freedom (m, n). The power of the PTT decreases as the values of ρ increases. The power of the PTT is identical for a fixed value of ρ, regardless of its sign. This figure shows that the power of the PTT increases as the values of the noncentrality parameters increase. The power of the PTT decreases as the value of the first degrees of freedom (m) increases and that of the second degrees of freedom (n) decreases.

5 Concluding Remarks

This paper derives the pdf and cdf of both the *singly* and *doubly* correlated BNCF distributions. The R codes are written to calculate and plot the pdf and cdf of the distributions as



(i) ϕ_1 , ϕ_2 =1 , d=3 , ν_1 =3 , ν_2 =17 , θ_1 =1 , θ_2 =1.5 (ii) ϕ_1 , ϕ_2 =1 , d=3 , ν_1 =5 , ν_2 =25 , θ_1 =1 , θ_2 =1.5



(iii) ϕ_1 , ϕ_2 = 1 , d = 3 , θ_1 = 1 , θ_2 = 1.5 , ρ = 0.5 (iv) ϕ_1 , ϕ_2 = 1 , d= 3 , ν_1 = 5 , ν_2 = 25 , ρ = 0.5

Figure 5: The power of the PTT using the cdf of the *doubly* correlated bivariate noncentral F distribution.

well as the power function of the PTT. Two tables of critical values of the *doubly* correlated BNCF distribution for selected values of the noncentrality parameters and $\rho = 0.5$ at the significance levels 0.01 and 0.05 are presented. As an application of the distribution, the power function of the PTT for the MSRM is calculated and plotted.

The cdf of both the *singly* and *doubly* correlated BNCF distributions depend on the values of the noncentrality parameters, degrees of freedom and correlation coefficient. The cdf curves for both *singly* and *doubly* correlated BNCF distributions are closer to one when there is an increase in the value of the degrees of freedom, correlation coefficient, and the variables for which the cdf is required. However, a smaller value of the noncentrality parameter leads to a larger value of the cdf for the *doubly* correlated BNCF distribution.

The power function of the PTT depends on the number of degrees of freedom, the correlation coefficient and the noncentrality parameters. It decreases as the value of the correlation coefficient ρ, the number of degrees of freedom of the numerator ν_1 or both increase, but it increases as the value of the noncentrality parameter increases.

We find that the central bivariate F distribution proposed by Krishnaiah (1965a) is a special case of the proposed *singly* correlated BNCF distribution, whereas the central bivariate F distribution introduced by Amos and Bulgren (1972) is a special case of the proposed *doubly* correlated BNCF distribution when the noncentrality parameters are zero. We also observe that the two variables of the BNCF distributions are not independent even if the value of ρ is 0. This is another example of a case in which zero correlation between two random variables does not imply the variables' independence.

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Appendix

Using the transformation of variables method for the multivariable case (see, for instance, Wackerly et al., 2008, p.325), we obtain the joint pdf of $\boldsymbol{y} = [y_1, y_2]'$ and z variables as

$$f(\boldsymbol{y}, z) = f(\boldsymbol{x})f(z) \mid J((\boldsymbol{x}, z) \to (\boldsymbol{y}, z)) \mid, \tag{5.1}$$

where $y_1 = \frac{nx_1}{mz}$, $y_2 = \frac{nx_2}{mz}$ and the Jacobian of the transformation $(x_1, x_2, z) \to (y_1, y_2, z)$ is given by

$$det. \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial z} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_2}{\partial z} \\ \frac{\partial z}{\partial y_1} & \frac{\partial z}{\partial y_2} & \frac{\partial z}{\partial z} \end{pmatrix} = det. \begin{pmatrix} \frac{m}{n}z & 0 & \frac{m}{n}y_1 \\ 0 & \frac{m}{n}z & \frac{m}{n}y_2 \\ 0 & 0 & 1 \end{pmatrix} = \left(\frac{m}{n}z\right)^2.$$

Therefore, the joint pdf of $\boldsymbol{y}$ and z is given by

$$\begin{aligned} f(\boldsymbol{y}, z) &= \sum_{j=0}^{\infty}\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\left[\rho^{2j}(1-\rho^2)^{m/2}\Gamma(m/2+j)\right] \\ &\times \left[\frac{(\frac{m}{n}y_1 z)^{m/2+j+r_1-1}e^{-\frac{(\frac{m}{n}y_1 z)}{2(1-\rho^2)}}}{[2(1-\rho^2)]^{m/2+j+r_1}\Gamma(m/2+j+r_1)} \times \frac{e^{-\theta_1/2}(\theta_1/2)^{r_1}}{r_1!}\right] \\ &\times \left[\frac{(\frac{m}{n}y_2 z)^{m/2+j+r_2-1}e^{-\frac{(\frac{m}{n}y_2 z)}{2(1-\rho^2)}}}{[2(1-\rho^2)]^{m/2+j+r_2}\Gamma(m/2+j+r_2)} \times \frac{e^{-\theta_2/2}(\theta_2/2)^{r_2}}{r_2!}\right] \\ &\times \frac{z^{(n/2)-1}e^{-z/2}}{2^{n/2}\Gamma(n/2)} \times \left(\frac{m}{n}z\right)^2 . \end{aligned} \tag{5.2}$$

Thus, the density function of $\boldsymbol{y}$ is obtained as

$$f(\boldsymbol{y}) = f(y_1, y_2) = \int_z f(\boldsymbol{y}, z)dz. \tag{5.3}$$

Therefore, by applying some algebra and calculus, the pdf of the *doubly* correlated BNCF distribution becomes

$$\begin{aligned} f(y_1, y_2) &= \left(\frac{m}{n}\right)^m \left[\frac{(1-\rho^2)^{\frac{m+n}{2}}}{\Gamma(m/2)\Gamma(n/2)}\right]\sum_{j=0}^{\infty}\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\left[\frac{\rho^{2j}}{j!}\left(\frac{m}{n}\right)^{2j}\Gamma(m/2+j)\right] \\ &\times \left[\left(\frac{e^{-\theta_1/2}(\theta_1/2)^{r_1}}{r_1!}\right)\left(\frac{\left(\frac{m}{n}\right)^{r_1}}{\Gamma(m/2+j+r_1)}\right)\left(y_1^{m/2+j+r_1-1}\right)\right] \\ &\times \left[\left(\frac{e^{-\theta_2/2}(\theta_2/2)^{r_2}}{r_2!}\right)\left(\frac{\left(\frac{m}{n}\right)^{r_2}}{\Gamma(m/2+j+r_2)}\right)\left(y_2^{m/2+j+r_2-1}\right)\right] \\ &\times \Gamma(q_{rj})\left[(1-\rho^2)+\frac{m}{n}y_1+\frac{m}{n}y_2\right]^{-(q_{rj})}, \end{aligned} \tag{5.4}$$

where $q_{rj} = m + (n/2) + 2j + r_1 + r_2$.

Table 1: Percentage points for the *doubly* correlated BNCF distribution for $\rho = 0.5$ and $\alpha = 0.05$

m	n	θ_1								
		2			4			10		
		θ_2			θ_2			θ_2		
		2	4	10	2	4	10	2	4	10
2	5	11.9	14.1	22.8	14.5	16.2	23.4	23.6	24.2	27.9
	6	10.4	12.3	19.8	12.6	14.0	20.3	20.4	20.8	24.0
	8	8.8	10.4	16.6	10.6	11.8	17.0	16.9	17.3	19.8
	10	8.4	9.5	15.0	9.6	10.6	15.2	15.1	15.4	17.6
4	5	8.4	9.5	13.6	9.7	10.5	14.0	14.1	14.4	16.3
	6	7.3	8.2	11.1	8.3	9.0	12.0	12.1	12.3	14.0
	8	6.1	6.8	9.8	6.9	7.5	10.0	10.0	10.2	11.4
	10	5.4	6.1	8.8	6.2	6.7	8.9	8.9	9.0	10.1
6	5	7.2	7.9	10.6	8.0	8.6	10.9	10.9	11.1	12.4
	6	6.2	6.8	9.1	6.9	7.3	9.3	9.3	9.5	10.6
	8	5.1	5.6	7.5	5.7	6.0	7.7	7.6	7.8	8.7
	10	4.6	5.0	6.7	5.1	5.4	6.8	6.8	6.9	7.7
8	5	6.6	7.1	9.0	7.2	7.6	9.3	9.3	9.5	10.5
	6	5.6	6.1	7.7	6.2	6.5	8.0	8.0	8.1	8.9
	8	4.7	5.0	6.4	5.1	5.3	6.5	6.5	6.6	7.3
	10	4.1	4.5	5.7	4.5	4.7	5.8	5.8	5.9	6.4
10	5	6.2	6.7	8.1	6.7	7.0	8.3	8.4	8.5	9.3
	6	5.3	5.7	6.9	5.7	6.0	7.1	7.1	7.3	7.9
	8	4.4	4.6	5.7	4.7	4.9	5.8	5.8	5.9	6.4
	10	3.9	4.1	5.1	4.1	4.3	5.2	5.1	5.2	5.7

Table 2: Percentage points for the *doubly* correlated BNCF distribution for $\rho = 0.5$ and $\alpha = 0.01$

m	n	θ_1								
		2			4			10		
		θ_2			θ_2			θ_2		
		2	4	10	2	4	10	2	4	10
2	5	26.0	30.7	49.1	31.4	35.0	50.3	50.5	51.6	59.3
	6	20.9	24.7	39.0	25.1	27.7	39.8	39.8	40.6	46.5
	8	15.9	18.8	29.3	19.0	20.9	29.8	29.7	30.2	34.3
	10	13.6	16.0	24.7	16.1	17.6	25.1	24.9	25.3	28.6
4	5	18.0	20.3	29.0	20.6	22.3	29.7	29.8	30.4	34.4
	6	14.3	16.1	22.9	16.3	17.6	23.4	23.4	23.9	26.8
	8	10.7	12.0	17.0	12.2	13.1	17.4	17.3	17.6	19.7
	10	9.0	10.1	14.3	10.2	11.0	14.5	14.4	14.6	16.3
6	5	15.3	16.8	22.4	17.0	18.1	23.0	23.0	23.5	26.1
	6	12.0	13.2	17.6	13.3	14.2	18.0	17.9	18.3	20.3
	8	8.9	9.7	13.0	9.9	10.5	13.2	13.2	13.5	14.9
	10	7.5	8.2	10.8	8.2	8.7	11.0	10.9	11.1	12.3
8	5	14.0	15.0	19.1	15.2	16.0	19.6	19.6	20.0	21.9
	6	10.9	11.7	14.9	11.8	12.5	15.2	15.2	15.5	17.1
	8	8.1	8.7	11.0	8.7	9.2	11.2	11.1	11.3	12.4
	10	6.7	7.2	9.1	7.2	7.6	9.3	9.2	9.4	10.2
10	5	13.1	14.0	17.1	14.1	14.8	17.5	17.5	17.8	19.5
	6	10.2	10.9	13.3	11.0	11.5	13.6	13.6	13.8	15.1
	8	7.5	8.0	9.8	8.0	8.4	10.0	9.9	10.1	11.0
	10	6.2	6.6	8.1	6.6	6.9	8.2	8.2	8.3	9.0

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The Correlated Bivariate Noncentral F Distribution ~~with~~ and its Application

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Abstract

This paper proposes the ~~correlated~~ *singly* and *doubly* correlated bivariate noncentral F (BNCF) distributions. The probability density function (pdf) and the cumulative distribution function (cdf) of the distributions are derived for ~~some~~ arbitrary values of the parameters. The pdf and cdf of the distributions for different arbitrary values of the parameters are computed, and their graphs are plotted by writing and implementing

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new R codes. An application of the correlated BNCF distribution is illustrated in the computations of the power function of the pre-test test for the multivariate simple regression model (MSRM).

Keywords: Correlated bivariate noncentral F distribution; noncentrality parameter; bivariate noncentral chi-square distribution, compounding distribution, pre-test test, and power function.

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Short heading for running head: The bivariate noncentral F distribution

1 Introduction

The bivariate central F (BCF) distribution has been studied by many authors, including Krishaniah (1965a), Amos and Bulgren (1972), Schuurmann et al. (1975), Johnson et al. (1995) and El-Bassiouny and Jones (2009). Krishnaiah (1965b) described the use of the BCF distribution in a problem of simultaneous statistical inference. Krishnaiah (1965c) ~~,~~ and Krishnaiah and Armitage (1965) later studied the multivariate central F distribution. Hewett and Bulgren (1971) studied the prediction interval for failure times in certain life testing experiments using the multivariate central F distribution.

Many authors have also studied the univariate noncentral F distribution, including Mudholkar et al. (1976), Muirhead (1982), Johnson et al. (1995), and Shao (2005). Johnson et al. (1995) ~~gave~~ provided the definition of the univariate noncentral F distribution known as the *singly* noncentral F distribution. ~~They~~ The authors also described the *doubly* noncentral F distribution with (ν_1, ν_2) degrees of freedom and noncentrality parameters λ_1 and λ_2 as the ratio of two independent noncentral chi-square variables, $\chi'^2_{\nu_1}(\lambda_1)/\nu_1$ and $\chi'^2_{\nu_2}(\lambda_2)/\nu_2$. Tiku (1966) proposed an approximation to the multivariate noncentral F distribution.

In the study of improving the power of a statistical test by pre-testing the uncertain non-sample prior information (NSPI) on the value of a set of parameters (cf. Saleh and Sen, 1983, and Yunus and Khan, 2011a), the cdf of a bivariate noncentral chi-square distribution is used to compute the power function of the test. For large sample studies, the cdf of the bivariate noncentral chi-square (BNCC) distribution is used to compute the power function of the test for testing one subset of regression parameters after pre-testing on ~~the~~ another subset of parameters of a multivariate simple regression model (MSRM) (cf. Saleh and Sen, 1983, Yunus and Khan, 2011a). For small sample sizes, the computation of the power function and the size of the test after a pre-test (PT) requires the cdf of a correlated bivariate noncentral F (BNCF) distribution, which ~~is unavailable~~ has not been reported in the literature ~~. This~~

~~is because unlike~~ because unlike those for the bivariate central F (BCF) distribution, the formulae ~~of~~ for the pdf and cdf of the correlated BNCF distribution are more ~~complicated, and hencethere is~~ complex; hence, there are no easy computational formulae available~~; as suchnone of the~~. As such, no statistical packages include this distribution.

Yunus and Khan (2011b) derived the bivariate noncentral chi-square (BNCC) distribution by compounding the Poisson distribution with the correlated bivariate central chi-square distribution~~with a view to computing~~, aiming to compute the power function of the test after ~~pre-test~~pre-testing. Therefore, using the same method of derivation, we derive the pdf and cdf of the ~~correlated~~ *singly* and *doubly* correlated BNCF distributions in this paper. The *doubly* correlated BNCF is defined by mixing the correlated BNCC distribution with an independent central chi-square distribution. This definition allows for two noncentrality parameters from the two correlated noncentral chi-square variables in the numerator of the noncentral F variables. ~~On the other hand~~Additionally, by compounding the BCF and Poisson distributions, we derive the *singly* correlated BNCF distribution. This form of the BNCF distribution has only one noncentrality parameter. We also propose the computational formulae of the pdf and cdf of the correlated BNCF distribution ~~, and illustrate its~~ and illustrate their application in the derivation of the power function of the pre-test test (PTT) (for details on the PTT, see Khan and Pratikno, 2013). The R codes are written to compute the values of the pdf and cdf of the correlated BNCF distribution and the power curve of the PTT of the MSRM. ~~Along with~~ In addition to suggesting the computational ~~formulas~~ formulae, we also compute and tabulate the critical values of the distribution for selected values of the parameters and significance levels using the R codes.

The next section derives the expression for the pdf and cdf of the correlated BNCF distribution. The computational method and graphical presentation of the pdf and cdf ~~,~~ and the critical values of the correlated BNCF distribution for different values of the noncentrality parameter are ~~given~~ presented in Section 3. An application of the BNCF distribution to the power function of the PTT is ~~included in Section 4. Some~~ discussed in Section 4, and concluding remarks are provided in Section 5.

2 The Bivariate Noncentral F Distribution

In this section, the cdfs of the *singly* and *doubly* correlated bivariate noncentral F distributions are obtained using the compounding of distributions technique. The *doubly* correlated BNCF is obtained by compounding the correlated BNCC distribution with an independent central chi-square distribution, thus allowing for two noncentrality parameters and a correlation coefficient parameter. ~~On the other hand~~Additionally, by compounding the BCF and Poisson distributions, the *singly* correlated BNCF distribution is derived. This form of the BNCF distribution has only one noncentrality parameter and a correlation coefficient parameter.

2.1 The Singly Correlated Bivariate Noncentral F Distribution

Let a random variable X_i, ~~$i = 1, 2,$~~ $i = 1, 2$ follow an F distribution with ν_i degrees of freedom, and let another random variable R ~~follows~~ follow a Poisson distribution with mean λ. The proposed *singly* correlated BNCF distribution is an extension of the univariate noncentral F distribution introduced by Krishnaiah (1965a) and Johnson et al. (1995) for the bivariate case. The pdf of the *singly* BNCF distribution with noncentrality parameter λ is defined as

$$f(x_1, x_2, \nu_r, \nu_2, \lambda) \quad = \quad \sum_{r=0}^{\infty} \left(\frac{e^{-\lambda/2} \left(\frac{\lambda}{2}\right)^r}{r!} \right) f_1(x_1, x_2, \nu_r, \nu_2), \qquad (2.1)$$

where $f_1(x_1, x_2, \nu_r, \nu_2)$ is the pdf of a BCF distribution with ν_r and ν_2 degrees of freedom in which $\nu_r = \nu_1 + 2r$~~,~~ ; that is,

$$\begin{aligned} f_1(x_1, x_2, \nu_r, \nu_2) &= \left(\frac{\nu_2^{\nu_2/2}(1-\rho^2)^{(\nu_r+\nu_2)/2}}{\Gamma(\nu_r/2)\Gamma(\nu_2/2)}\right)\sum_{j=0}^{\infty}\left(\frac{\rho^{2j}\Gamma(\nu_r+(\nu_2/2)+2j)}{j!\Gamma((\nu_r/2)+j)}\right)\nu_r^{\nu_r+2j} \\ &\quad\times\left(\frac{(x_1x_2)^{(\nu_r/2)+j-1}}{[\nu_2(1-\rho^2)+\nu_r(x_1+x_2)]^{\nu_r+(\nu_2/2)+2j}}\right). \end{aligned}$$

Note that the density function of the *singly* correlated BNCF distribution is obtained by compounding the BCF distribution with the Poisson probabilities.

~~Then~~ Therefore, the cdf of the *singly* correlated BNCF distribution is defined as

$$\begin{aligned} P(.) &= P(X_1 < d, X_2 < d, \nu_r, \nu_2, \lambda) \\ &= \sum_{r=0}^{\infty}\left(\frac{e^{-\lambda/2}\left(\frac{\lambda}{2}\right)^r}{r!}\right)P_2(X_1 < d, X_2 < d, \nu_r, \nu_2), \end{aligned} \qquad (2.2)$$

where

$$P_2(X_1 < d, X_2 < d, \nu_r, \nu_2) = \left(\frac{(1-\rho^2)^{\nu_r/2}}{\Gamma(\nu_r/2)\Gamma(\nu_2/2)}\right)\sum_{j=0}^{\infty}\left(\frac{\rho^{2j}\Gamma(\nu_r+(\nu_2/2)+2j)}{j!\Gamma((\nu_r/2)+j)}\right)L_{jr}$$

~~in which~~ and L_{jr} is defined as

$$L_{jr} = \int_0^{h_r}\int_0^{h_r}\frac{(x_1x_2)^{(\nu_r/2)+j-1}dx_1dx_2}{(1+x_1+x_2)^{\nu_r+(\nu_2/2)+2j}}$$

with $h_r = \frac{d\nu_r}{\nu_2(1-\rho^2)}$.

For the computation of the value of the cdf of the *singly* correlated BNCF distribution, R codes are used. To make the computations easier, we represent the formulae of the cdf of the *singly* correlated BNCF distribution in equation (2.2) as the sum of infinite ~~sums of~~

series as follows:

$$
\begin{aligned}
P(.) &= \sum_{r=0}^{\infty}\left(\frac{e^{-\lambda/2}\left(\frac{\lambda}{2}\right)^r}{r!}\right)\left(\frac{(1-\rho^2)^{\nu_r/2}}{\Gamma(\nu_r/2)\Gamma(\nu_2/2)}\right)\sum_{j=0}^{\infty}\left(\frac{\rho^{2j}\Gamma(\nu_r+(\nu_2/2)+2j)}{j!\Gamma((\nu_r/2)+j)}\right)L_{jr}\\
&= \sum_{r=0}^{\infty}T_r\left[\left(\frac{1\Gamma(\nu_r+(\nu_2/2))}{0!\Gamma((\nu_r/2))}\right)L_{0r}+\left(\frac{\rho^2\Gamma(\nu_r+(\nu_2/2)+2)}{1!\Gamma((\nu_r/2)+1)}\right)L_{1r}+\cdots\cdots\right]\\
&= \sum_{r=0}^{\infty}T_r\left[H_{0r}L_{0r}+H_{1r}L_{1r}+H_{2r}L_{2r}+\ \cdots\cdots\right]\\
&= \sum_{r=0}^{\infty}T_rH_{0r}L_{0r}+T_rH_{1r}L_{1r}+T_rH_{2r}L_{2r}+\ \cdots\cdots\\
&= [T_0H_{00}L_{00}+T_0H_{10}L_{10}+T_0H_{20}L_{20}\ +\ \cdots\cdots]\\
&\qquad +[T_1H_{01}L_{01}+T_1H_{11}L_{11}+T_1H_{21}L_{21}\ +\ \cdots\cdots]\\
&\qquad +[T_2H_{02}L_{02}+T_2H_{12}L_{12}+T_2H_{22}L_{22}\ +\ \cdots\cdots]\\
&\qquad +\cdots\cdots, \qquad (2.3)
\end{aligned}
$$

where

$$T_r=\left(\frac{e^{-\lambda/2}\left(\frac{\lambda}{2}\right)^r}{r!}\right)\left(\frac{(1-\rho^2)^{\nu_r/2}}{\Gamma(\nu_r/2)\Gamma(\nu_2/2)}\right),$$

$$H_{0r}=\frac{1\Gamma(\nu_r+(\nu_2/2))}{0!\Gamma((\nu_r/2))},\ H_{1r}=\frac{\rho^2\Gamma(\nu_r+(\nu_2/2)+2)}{1!\Gamma((\nu_r/2)+1)},\ H_{2r}=\frac{\rho^4\Gamma(\nu_r+(\nu_2/2)+4)}{2!\Gamma((\nu_r/2)+2)},\ \cdots,$$

and P_0 is defined as

$$
\begin{aligned}
P_0 &= \sum_{r=0}^{\infty}T_rH_{0r}L_{0r}=T_0H_{00}L_{00}+T_1H_{01}L_{01}+T_2H_{02}L_{02}+\cdots\cdots,\ \text{for } j=0,\\
&= \left(\frac{e^{-\lambda/2}}{0!}\times\frac{(1-\rho^2)^{\nu_1/2}}{\Gamma(\nu_1/2)\Gamma(\nu_2/2)}\right)\left(\frac{1\Gamma(\nu_1+\nu_2/2)}{0!\Gamma(\nu_1/2)}\right)L_{00}\ +\\
&\left(\frac{e^{-\lambda/2}\left(\frac{\lambda}{2}\right)}{1}\times\frac{(1-\rho^2)^{\nu_1+2/2}}{\Gamma((\nu_1+2)/2)\Gamma(\nu_2/2)}\right)\left(\frac{1\Gamma(\nu_1+2+(\nu_2/2))}{0!\Gamma((\nu_1+2)/2)}\right)L_{01}+\cdots.
\end{aligned}
\qquad (2.4)
$$

Similarly, we obtain the expressions for $P_1 = \sum_{r=0}^{\infty} T_r H_{1r} L_{1r}$, $P_2 = \sum_{r=0}^{\infty} T_r H_{2r} L_{2r}$ and so on. Finally, we write

$$P(.) = P_0 + P_1 + P_2 + P_3 + \cdots\cdots = \sum_{j=0}^{\infty} P_j.$$

Some properties of the *singly* correlated BNCF distribution are given as follows:

(i) Note that $f_1(x_1, x_2, \nu_r, \nu_2)$ in equation (2.1) is a pdf of a BCF distribution with ν_r and ν_2 degrees of freedom~~, hence $\int_0^\infty \int_0^\infty f_1(x_1, x_2, \nu_r, \nu_2)dx_1dx_2 = \lim_{d\to\infty} P_2(X_1 < d, X_2 < d)$,~~ ; hence, $\int_0^\infty \int_0^\infty f_1(x_1, x_2, \nu_r, \nu_2)dx_1dx_2 = \lim_{d\to\infty} P_2(X_1 < d, X_2 < d)$ in equation (2.2) ~~,~~ is equal to one. Furthermore, $\sum_{r=0}^{\infty} \frac{e^{-\lambda/2}(\lambda/2)^r}{r!} = 1$. Thus, it ~~is easy to see~~ can easily be observed that $\int_0^\infty \int_0^\infty f(x_1, x_2, \nu_r, \nu_2, \lambda)dx_1dx_2 = 1$.

(ii) ~~Since~~ Because $f_1(x_1, x_2, \nu_r, \nu_2)$ is a pdf of a BCF, ~~thus~~ $f_1(\cdot) \geq 0$. It is noted that the quantity $\frac{e^{-\lambda/2}(\lambda/2)^r}{r!}$ is always positive. Therefore, $f(\cdot) \geq 0$.

(iii) From equation (2.1), the central case of the bivariate F distribution proposed by Krishnaiah (1965a) is a special case of the *singly* correlated BNCF distribution when the noncentrality parameter, λ, is equal to zero.

2.2 The Doubly Correlated Bivariate Noncentral F Distribution

Let the random variables (X_1, X_2) jointly follow a correlated BNCC distribution with m degrees of freedom, noncentrality parameters θ_1 and θ_2, and a correlation coefficient ρ, and let the random variable Z follow a central chi-square distribution with n degrees of freedom. We propose the cdf of the *doubly* correlated BNCF by compounding the two aforementioned distributions.

The pdf of the correlated BNCC variables X_1 and X_2 proposed by Yunus and Khan

(2011b) is given by

$$\begin{aligned} g(x_1, x_2) &= \sum_{j=0}^{\infty}\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\left[\rho^{2j}(1-\rho^2)^{m/2}\Gamma(m/2+j)\right] \\ &\times \left[\frac{(x_1)^{m/2+j+r_1-1}e^{-\frac{(x_1)}{2(1-\rho^2)}}}{[2(1-\rho^2)]^{m/2+j+r_1}\Gamma(m/2+j+r_1)} \times \frac{e^{-\theta_1/2}(\theta_1/2)^{r_1}}{r_1!}\right] \\ &\times \left[\frac{(x_2)^{m/2+j+r_2-1}e^{-\frac{(x_2)}{2(1-\rho^2)}}}{[2(1-\rho^2)]^{m/2+j+r_2}\Gamma(m/2+j+r_2)} \times \frac{e^{-\theta_2/2}(\theta_2/2)^{r_2}}{r_2!}\right], \end{aligned} \tag{2.5}$$

and the pdf of a central chi-square variable Z with n degrees of freedom is given by

$$f(z) = \frac{z^{(n/2)-1}e^{-z/2}}{2^{n/2}\Gamma(n/2)}, \tag{2.6}$$

where Z is independent of X_1 and X_2.

~~Then~~ Therefore, the random variables (Y_1, Y_2), where

$$Y_i = \frac{X_i/m}{Z/n}, \quad \text{for} \quad i = 1, 2, \tag{2.7}$$

~~has~~ have a joint cdf given by

$$P(Y_1 \le a, Y_2 \le b) = \int_{z=0}^{\infty} f(z) \int_{x_2=0}^{\frac{bmz}{n}} \int_{x_1=0}^{\frac{amz}{n}} g(x_1, x_2)dx_1dx_2dz. \tag{2.8}$$

The distribution function given in equation (2.8) is the cdf of the proposed *doubly* correlated BNCF distribution with m and n degrees of freedom, noncentrality parameters θ_1 and θ_2, and correlation coefficient ρ.

~~Also~~In addition, equation (2.8) can be expressed as the following sum of infinite series

$$\begin{aligned} P(Y_1 \leq a, Y_2 \leq b) &= (1-\rho^2)^{\frac{m}{2}} \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \sum_{j=0}^{\infty} \frac{(\frac{m}{2})_j}{j!} \rho^{2j} I_2(\tilde{\alpha}_j, \tilde{c}, \beta) \\ &\quad \times \frac{e^{-\theta_1/2}(\theta_1/2)^{r_1}}{r_1!} \frac{e^{-\theta_2/2}(\theta_2/2)^{r_2}}{r_2!}, \end{aligned} \tag{2.9}$$

where

$$I_2(\tilde{\alpha}_j, \tilde{c}, \beta) = \int_0^{\infty} \frac{e^{-z} z^{\beta-1}}{\Gamma(\beta)} \frac{\gamma(\alpha_1, c_1 z)}{\Gamma(\alpha_1)} \frac{\gamma(\alpha_2, c_2 z)}{\Gamma(\alpha_2)} dz$$

and

$$\beta = \frac{n}{2}, \quad \tilde{c} = \left(\frac{am}{n(1-\rho^2)}, \frac{bm}{n(1-\rho^2)} \right), \quad \tilde{\alpha}_j = \left(\frac{m}{2} + j + r_1, \frac{m}{2} + j + r_2 \right).$$

Here, $\gamma(\alpha, x) = \int_0^x e^{-t} t^{\alpha-1} dt$, and $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt$.

To ease the computational difficulties of the cdf, we use the following form of I_2 given by Amos and Bulgren (1972),

$$\begin{aligned} I_2(\tilde{\alpha}_j, \tilde{c}, \beta) &= I_u(\alpha_1, \beta) - \frac{(1-u)^{\beta}}{\alpha_1} \frac{\Gamma(\beta+\alpha_1)}{\Gamma(\beta)\Gamma(\alpha_1)} \\ &\quad \times \sum_{r=0}^{\infty} \frac{(\beta+\alpha_1)_r}{(1+\alpha_1)_r} u^{r+\alpha_1} I_{1-y}(r+\beta+\alpha_1, \alpha_2), \end{aligned} \tag{2.10}$$

with

$$u = c_1/(1+c_1), \;\; 1-y = (1+c_1)/(1+c_1+c_2),$$

and

$$\begin{aligned}\int_0^\infty \frac{e^{-z}z^{\beta-1}}{\Gamma(\beta)}\frac{\gamma(\alpha,cz)}{\Gamma(\alpha)}dz &= I_z(\alpha,\beta) \quad \text{and} \\ \int_0^\infty \frac{e^{-z}z^{\beta-1}}{\Gamma(\beta)}\frac{\Gamma(\alpha,cz)}{\Gamma(\alpha)}dz &= I_{1-z}(\beta,\alpha)\end{aligned}$$

are the regularized beta functions, with $\alpha > 0, \beta > 0$, $x = c/(1+c)$, and $1-x = 1/(1+c)$.

See the Appendix for the pdf of the *doubly* correlated BNCF distribution, which is derived using ~~transformation of variable~~ the transformation of variables method.

Some properties of the *doubly* correlated BNCF distribution are given as follows:

(i) From equation (2.9), we find that $F_1(a,b;r_1,r_2) = (1-\rho^2)^{\frac{m}{2}} \sum_{j=0}^{\infty} \frac{(\frac{m}{2})_j}{j!}\rho^{2j} I_2(\tilde{\alpha}_j,\tilde{c},\beta)$ is the cdf of a BCF distribution (Amos and Bulgren, 1972)~~, thus~~; thus, $F_1(a,b)$ approaches 1 as both a and b go to infinity. It is ~~obvious~~ clear that both quantities $\sum_{r_1=0}^{\infty} \frac{e^{-\theta_1/2}(\theta_1/2)^{r_1}}{r_1!}$ and $\sum_{r_2=0}^{\infty} \frac{e^{-\theta_2/2}(\theta_2/2)^{r_2}}{r_2!}$ are equal to one. It follows that the cdf of the *doubly* correlated BNCF distribution approaches 1 as both a and b go to infinity.

(ii) When a and b are zero, it is easy to show that the cdf of the *doubly* correlated BNCF is zero.

(iii) Note that $F_1(a,b)$ is an increasing function ~~,~~ because it is a cdf of a BCF ~~distribuiton~~distribution. It follows that the cdf of the *doubly* correlated BNCF distribution is also an increasing function.

(iv) As b approaches infinity, $\gamma(\alpha_2, c_2 z) = \Gamma(\alpha_2)$ through $c_2 = bm/n(1-\rho^2)$, and the second term on the ~~right hand~~ right-hand side of equation (2.10) becomes zero ~~, since~~ because $1-y$ approaches zero as b goes to infinity. Further simplifications yield the following marginal distribution function

$$\sum_{r_1=0}^{\infty} I_{\frac{my_1}{n+my_1}}(m/2+r_1, n/2)\frac{e^{-\theta_1/2}(\theta_1/2)^{r_1}}{r_1!} = F(y_1; m, n, \theta_1), \tag{2.11}$$

which is the cdf of the noncentral F distribution of Y_1, with noncentrality parameter θ_1 ~~,~~ and degrees of freedom m and n. In the same manner, the marginal distribution function for Y_2 can be derived.

(v) The central F distribution can be obtained from the noncentral distribution if the noncentrality parameters, θ_1 and θ_2, are equal to 0. ~~Since~~ Because r_1 and r_2 are both zero, we rewrite (2.9) as

$$P(Y_1 \le a, Y_2 \le b) \quad = \quad (1-\rho^2)^{\frac{m}{2}} \sum_{j=0}^{\infty} \frac{(\frac{m}{2})_j}{j!} \rho^{2j} I_2(\hat{\alpha}_j, \hat{c}, \beta)$$

with

$$\beta = \frac{n}{2}, \quad \hat{c} = \left(\frac{am}{(1-\rho^2)n}, \frac{bm}{(1-\rho^2)n}\right), \quad \hat{\alpha} = \left(\frac{m}{2}+j, \frac{m}{2}+j\right). \tag{2.12}$$

Thus, we arrive at the central correlated bivariate F distribution proposed by Amos and Bulgren (1972) after allowing both noncentrality parameters equal zero in the *doubly* correlated BNCF distribution.

(vi) For $\rho = 0$, which implies $j = 0$, we write equation (2.9) as

$$P(Y_1 \le a, Y_2 \le b) \quad = \quad \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty} I_2(\check{\alpha}, \check{c}, \beta) \frac{e^{-\theta_1/2}(\theta_1/2)^{r_1}}{r_1!} \frac{e^{-\theta_2/2}(\theta_2/2)^{r_2}}{r_2!} \tag{2.13}$$

with

$$\beta = \frac{n}{2}, \quad \check{c} = \left(\frac{am}{n}, \frac{bm}{n}\right), \quad \check{\alpha} = \left(\frac{m}{2} + r_1, \frac{m}{2} + r_2\right). \tag{2.14}$$

~~We see~~ It can be observed that Y_1 and Y_2 are not independent, although the correlation coefficient between Y_1 and Y_2 is zero. In other words, the *doubly* correlated BNCF can have a zero correlation, but the marginal distributions do not support statistical independence.

3 Computation of the pdf and cdf

To compute the values of the pdf and cdf of the BNCF distributions, R codes are written. The R package is also used for the graphical representation of the pdf and cdf. The pdf of the *singly* BNCF distribution ~~was~~ is computed using equation (2.1) and plotted in Figure 1. The graph in Figure 1(iii) has a wider spread than that in Figure 1(i) due to the smaller value of ν_1. Comparing Figures 1(i) and 1(iv), the spread of the distribution in Figure 1(iv) decreases due to the increase in the noncentrality parameter. As the value of ρ increases, the spread of the distribution decreases and the pdf shrinks~~as can be seen~~ , as shown in Figure 1(ii). For the *doubly* correlated BNCF distribution, the pdf is calculated using equation (5.4) ~~,~~ and plotted in Figure 2. The graphs in Figure 2 show ~~similar properties as those of Figure 1 ,~~ properties similar to those shown in Figure 1 but with varying probabilities.

Figure 1: The pdf of the *singly* correlated bivariate noncentral F distribution.

Figure 2: The pdf of the *doubly* correlated bivariate noncentral F distribution.

To compute the cdf of the *singly* correlated BNCF distribution in equation (2.3), we choose ~~some~~ arbitrary values of the degrees of freedom (ν_1, ν_2), noncentrality parameter (λ), correlation coefficient (ρ) and upper limit (d) of the variable. Figure 3 shows that the cdf of the *singly* correlated BNCF distribution increases as the value of any of the parameters, namely, the degrees of freedom ν_1 (for fixed ν_2), λ, or d, increases.

Figure 3: The cdf of the singly correlated BNCF distribution with ~~some~~ arbitrary values of ρ, λ, ν_1 and ν_2.

Figure 4: The cdf of the doubly correlated BNCF distribution with ~~some~~ arbitrary values of ρ, θ_k, m and n.

The cdf of the *doubly* correlated BNCF distribution is computed using equation (2.9) for arbitrary degrees of freedom (m, n), noncentrality parameters (θ_1, θ_2), correlation coefficient (ρ), and upper limit ($a = b = d$). The graphs of the cdf of the *doubly* correlated BNCF distribution are presented in Figure 4. Interestingly, the cdf curve approaches 1 ~~quicker~~ more rapidly for a larger correlation coefficient (see Figure 4(i)), a smaller noncentrality parameter (see Figure 4(ii)) and ~~larger~~ a greater number of degrees of freedom (m, n) (see Figures 4(iii) and 4(iv)). Figure 4 shows that the shape of the cdf curve is ~~sigmoid whose shape~~ sigmoidal, which depends on the values of the noncentrality parameters (θ_1, θ_2), degrees of freedom (m, n), and correlation coefficient (ρ). Tables 1 and 2 provide the values of d for different values of m, n, θ_1, θ_2, and α, where

$$P(Y_1 < d, Y_2 < d) = \int_0^d \int_0^d f(y_1, y_2) dy_1 dy_2 = (1 - \alpha),$$

for the case ~~where~~ in which $\rho = 0.5$.

4 Application to the Power Function of the PTT

~~For testing~~ To test the null hypothesis on the intercept vector ~~,~~ $H_0 : \boldsymbol{\beta}_0 = \boldsymbol{\beta}_{00}$ (given known vector) against $H_a : \boldsymbol{\beta}_0 > \boldsymbol{\beta}_{00}$ in the multivariate simple regression model

$$\boldsymbol{y}_i = \boldsymbol{\beta}_0 + \boldsymbol{\beta}_1 x_i + \boldsymbol{e}_i, \tag{4.1}$$

(for details, see Khan, 2006) when there is non-sample prior information on the slope vector $\boldsymbol{\beta}_1$, the test statistic follows a correlated bivariate F distribution. The ultimate test on H_0 is called the pre-test test (PTT) because it depends on the outcome of the pre-test on the suspected slope~~,~~; that is, $H_0^* = \boldsymbol{\beta}_1 = \boldsymbol{\beta}_{10}$ (cf. Khan and Pratikno, 2013). The cdf of the *doubly* correlated BNCF distribution is involved in the formulae for the power function of the PTT. To illustrate the method, we conduct a simulation study by generating random data using the R package.

The explanatory variable (x) is generated from the uniform distribution between 0 and 1. The error vector $(\boldsymbol{e})$ is generated from a $p = 3$ dimensional multivariate normal distribution with $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \sigma^2 \boldsymbol{I}_3$, where I_3 is the identity matrix of order 3. Then, the dependent variable (y_1) is determined by $y_1 = \beta_0' + \beta_1' x + e_1$ for $\beta_0' = 3$ and $\beta_1' = 1.5$. Similarly, y_2 and y_3 are determined by $y_2 = \beta_0'' + \beta_1'' x + e_2$ for $\beta_0'' = 5$ and $\beta_1'' = 2.5$ ~~, and~~ and by $y_3 = \beta_0''' + \beta_1''' x + e_3$ for $\beta_0''' = 6$ and $\beta_1''' = 3$~~, respectively~~. For each of the three cases, $n = 20$ random variates are generated.

Considering the three cases ~~:~~ (i) unspecified $\boldsymbol{\beta}_1$, (ii) specified $\boldsymbol{\beta}_1$ and (iii) uncertain prior information on $\boldsymbol{\beta}_1$, we define the unrestricted, restricted and pre-test test statistics as follows: $T^{UT} = \sum_{i=1}^n (x_i - \bar{x})^2 [(\tilde{\boldsymbol{\beta}}_0 - \boldsymbol{\beta}_{00})' \hat{\Sigma}^{-1} (\tilde{\boldsymbol{\beta}}_0 - \boldsymbol{\beta}_{00})]$, $T^{RT} = \sum_{i=1}^n (x_i - \bar{x})^2 [(\bar{\boldsymbol{y}} - \boldsymbol{\beta}_{10}\bar{x} - \boldsymbol{\beta}_{00})' \hat{\Sigma}^{-1} (\bar{\boldsymbol{y}} - \boldsymbol{\beta}_{10}\bar{x} - \boldsymbol{\beta}_{00})]$ and $T^{PT} = \sum_{i=1}^n (x_i - \bar{x})^2 [(\tilde{\boldsymbol{\beta}}_1 - \boldsymbol{\beta}_{10})' \hat{\Sigma}^{-1} (\tilde{\boldsymbol{\beta}}_1 - \boldsymbol{\beta}_{10})]$, respectively. Here, $\hat{\Sigma}^{-1} = \frac{1}{n-p} \sum_{i=1}^n (\boldsymbol{y}_i - \tilde{\boldsymbol{\beta}}_0 - \tilde{\boldsymbol{\beta}}_1 x_i)(\boldsymbol{y}_i - \tilde{\boldsymbol{\beta}}_0 - \tilde{\boldsymbol{\beta}}_1 x_i)'$, where $\hat{\boldsymbol{\beta}}_0 = \bar{\boldsymbol{y}} - \boldsymbol{\beta}_{10}\bar{x}$, $\tilde{\boldsymbol{\beta}}_0 = \bar{\boldsymbol{y}} - \tilde{\boldsymbol{\beta}}_1 \bar{x}$, $\tilde{\boldsymbol{\beta}}_1 = \sum_{i=1}^n \frac{(\boldsymbol{y}_i - \bar{\boldsymbol{y}})(x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2}$, $\bar{x} = \sum_{i=1}^n x_i/n$, and $\bar{\boldsymbol{y}} = \sum_{i=1}^n \boldsymbol{y}_i/n$.

Under $H_a : \boldsymbol{\beta}_0 > \boldsymbol{\beta}_{00}$, T^{UT} and T^{RT} follow a noncentral F distribution with $(p, n-p)$ degrees of freedom and noncentrality parameters $\Delta_1^2/2$ and $\Delta_2^2/2$, respectively, where $\Delta_1^2 =$

$\sum_{i=1}^{n}(x_i-\bar{x})^2[(\boldsymbol{\beta}_0-\boldsymbol{\beta}_{00})'\Sigma^{-1}(\boldsymbol{\beta}_0-\boldsymbol{\beta}_{00})]$ and $\Delta_2^2=\sum_{i=1}^{n}(x_i-\bar{x})^2[(\bar{Y}-\boldsymbol{\beta}_{10}\bar{x}-\boldsymbol{\beta}_{00})'\hat{\Sigma}^{-1}(\bar{\boldsymbol{y}}-\boldsymbol{\beta}_{10}\bar{x}-\boldsymbol{\beta}_{00})]$. Under $H_a^\star:\boldsymbol{\beta}_1>\boldsymbol{\beta}_{10}$, T^{PT} follows a noncentral F distribution with $(p,n-p)$ degrees of freedom and noncentrality parameter $\Delta_3^2/2$, where $\Delta_3^2=\sum_{i=1}^{n}(x_i-\bar{x})^2[(\boldsymbol{\beta}_1-\boldsymbol{\beta}_{10})'\hat{\Sigma}^{-1}(\boldsymbol{\beta}_1-\boldsymbol{\beta}_{10})]$.

Let $\{K_n\}$ be a sequence of alternative hypotheses

$$K_n:\boldsymbol{\beta}_0=\boldsymbol{\beta}_{00}+\boldsymbol{\lambda}_1/\sqrt{n},\boldsymbol{\beta}_1=\boldsymbol{\beta}_{10}+\boldsymbol{\lambda}_2/\sqrt{n}, \tag{4.2}$$

where $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ are vectors of fixed real numbers. Under $\{K_n\}$, the power function of the PTT is given by

$$\begin{aligned}\pi^{PTT}(\boldsymbol{\lambda}) &= P\left(T^{PT}<a,T^{RT}>c\right)+P\left(T^{PT}\geq a,T^{UT}>b\right)\\ &= P\left(T^{PT}<a\right)\ P\left(T^{RT}>c\right)+d_{1r}\left(a,b,\rho\right)\\ &= \left[1-P\left(T^{PT}>a\right)\right]\ P\left(T^{RT}>c\right)+d_{1r}\left(a,b,\rho\right), \qquad (4.3)\end{aligned}$$

where $\boldsymbol{\lambda}=(\boldsymbol{\lambda}_1/\sqrt{n},\boldsymbol{\lambda}_2/\sqrt{n})$, $a=F_{\alpha_3,p,n-p}-\phi_2$, $b=F_{\alpha_1,p,n-p}-\phi_1$ and $c=F_{\alpha_2,n,n-1}-[\phi_1+\phi_2\bar{x}]+\omega\bar{x}$ for $\phi_1=\boldsymbol{\lambda}_1'\Sigma^{-1}\boldsymbol{\lambda}_1$, $\phi_2=\boldsymbol{\lambda}_2'\Sigma^{-1}\boldsymbol{\lambda}_2$, $\omega=\boldsymbol{\lambda}_1'\Sigma^{-1}\boldsymbol{\lambda}_2+\boldsymbol{\lambda}_2'\Sigma^{-1}\boldsymbol{\lambda}_1$, and $d_{1r}(a,b,\rho)$ is a correlated bivariate F probability integral defined as

$$d_{1r}(a,b,\rho) = \int_b^\infty\int_a^\infty f(F^{PT},F^{UT})dF^{PT}dF^{UT} \tag{4.4}$$

with $\rho=\frac{n^2(n-p-4)}{(2n-p-2)^2(n-p-4)}$. ~~Obviously~~Clearly, the power of the PTT is defined in terms of the powers of the RT and PT as well as the cdf of the *doubly* correlated BNCF distribution.

Figure 5: The power of the PTT using the cdf of the *doubly* correlated bivariate noncentral F distribution.

Figure 5 shows the graphs of the power function of the PTT in ~~term~~ terms of $d_{1r}(d,d,\rho)$

for ~~some~~ selected values of the correlation coefficients (ρ), noncentrality parameters (θ_1, θ_2) and degrees of freedom (m, n). The power of the PTT decreases as the values of ρ increases. The power of the PTT is identical for a fixed value of ρ, regardless of its sign. This figure shows that the power of the PTT increases as the values of the noncentrality parameters increase. The power of the PTT decreases as the value of the first degrees of freedom (m) increases and that of the second degrees of freedom (n) decreases.

5 Concluding Remarks

~~The~~ This paper derives the pdf and cdf of both the *singly* and *doubly* correlated BNCF distributions. The R codes are written to calculate and plot the pdf and cdf of the ~~distribution~~ distributions as well as the power function of the PTT. Two tables of critical values of the *doubly* correlated BNCF distribution for selected values of the noncentrality parameters and $\rho = 0.5$ at the significance levels 0.01 and 0.05 are presented. As an application of the distribution, the power function of the PTT for the MSRM is calculated and plotted.

The cdf of both the *singly* and *doubly* correlated BNCF distributions depend on the values of the noncentrality ~~parameter~~parameters, degrees of freedom and correlation coefficient. The cdf curves for both *singly* and *doubly* correlated BNCF distributions are closer to one when there is an increase in the value of the degrees of freedom, correlation coefficient, and the ~~value of the~~ variables for which the cdf is required. However, a smaller value of the noncentrality parameter leads to a larger value of the cdf for the *doubly* correlated BNCF distribution.

The power function of the PTT depends on the ~~value of~~ number of degrees of freedom, the correlation coefficient and the noncentrality parameters. It decreases as the value of the correlation coefficient ρ~~or~~ , the number of degrees of freedom of the numerator ν_1 ~~,~~ or both increase, but it increases as the value of the noncentrality parameter increases.

We find that the central bivariate F distribution proposed by Krishnaiah (1965a) is a special case of the proposed *singly* correlated BNCF distribution, ~~while~~ whereas the central bivariate F distribution introduced by Amos and Bulgren (1972) is a special case of the proposed *doubly* ~~BNCF distribution,~~ correlated BNCF distribution when the noncentrality parameters are zero. We also observe that the two variables of the BNCF distributions are not independent even if the value of ρ is 0. This is another example of a case in which zero correlation between two random variables does not imply ~~their~~ the variables' independence.

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Appendix

Using ~~transformation of variable~~ the transformation of variables method for the multivariable case (see, for instance, Wackerly et al., 2008, p.325), we obtain the joint pdf of $\boldsymbol{y} = [y_1, y_2]'$ and z variables as

$$f(\boldsymbol{y}, z) = f(\boldsymbol{x})f(z) \mid J((\boldsymbol{x}, z) \rightarrow (\boldsymbol{y}, z)) \mid, \tag{5.1}$$

where $y_1 = \frac{nx_1}{mz}$, $y_2 = \frac{nx_2}{mz}$ and the Jacobian of the transformation $(x_1, x_2, z) \rightarrow (y_1, y_2, z)$ ~~,~~ is given by

$$det. \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial z} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_2}{\partial z} \\ \frac{\partial z}{\partial y_1} & \frac{\partial z}{\partial y_2} & \frac{\partial z}{\partial z} \end{pmatrix} = det. \begin{pmatrix} \frac{m}{n}z & 0 & \frac{m}{n}y_1 \\ 0 & \frac{m}{n}z & \frac{m}{n}y_2 \\ 0 & 0 & 1 \end{pmatrix} = \left(\frac{m}{n}z\right)^2.$$

Therefore, the joint pdf of $\boldsymbol{y}$ and z is given by

$$\begin{aligned} f(\boldsymbol{y}, z) &= \sum_{j=0}^{\infty}\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty} \left[\rho^{2j}(1-\rho^2)^{m/2}\Gamma(m/2+j)\right] \\ &\times \left[\frac{(\frac{m}{n}y_1 z)^{m/2+j+r_1-1} e^{-\frac{(\frac{m}{n}y_1 z)}{2(1-\rho^2)}}}{[2(1-\rho^2)]^{m/2+j+r_1}\Gamma(m/2+j+r_1)} \times \frac{e^{-\theta_1/2}(\theta_1/2)^{r_1}}{r_1!}\right] \\ &\times \left[\frac{(\frac{m}{n}y_2 z)^{m/2+j+r_2-1} e^{-\frac{(\frac{m}{n}y_2 z)}{2(1-\rho^2)}}}{[2(1-\rho^2)]^{m/2+j+r_2}\Gamma(m/2+j+r_2)} \times \frac{e^{-\theta_2/2}(\theta_2/2)^{r_2}}{r_2!}\right] \\ &\times \frac{z^{(n/2)-1}e^{-z/2}}{2^{n/2}\Gamma(n/2)} \times \left(\frac{m}{n}z\right)^2. \end{aligned} \tag{5.2}$$

Thus, the density function of $\boldsymbol{y}$ is obtained as

$$f(\boldsymbol{y}) = f(y_1, y_2) = \int_z f(\boldsymbol{y}, z)dz. \tag{5.3}$$

~~Then,~~ Therefore, by applying some algebra and calculus, the pdf of the ~~correlated~~ *doubly* correlated BNCF distribution becomes

$$\begin{aligned} f(y_1,y_2) &= \left(\frac{m}{n}\right)^m \left[\frac{(1-\rho^2)^{\frac{m+n}{2}}}{\Gamma(m/2)\Gamma(n/2)}\right] \sum_{j=0}^{\infty}\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty} \left[\frac{\rho^{2j}}{j!}\left(\frac{m}{n}\right)^{2j}\Gamma(m/2+j)\right] \\ &\quad \times \left[\left(\frac{e^{-\theta_1/2}(\theta_1/2)^{r_1}}{r_1!}\right)\left(\frac{\left(\frac{m}{n}\right)^{r_1}}{\Gamma(m/2+j+r_1)}\right)\left(y_1^{m/2+j+r_1-1}\right)\right] \\ &\quad \times \left[\left(\frac{e^{-\theta_2/2}(\theta_2/2)^{r_2}}{r_2!}\right)\left(\frac{\left(\frac{m}{n}\right)^{r_2}}{\Gamma(m/2+j+r_2)}\right)\left(y_2^{m/2+j+r_2-1}\right)\right] \\ &\quad \times \Gamma(q_{rj})\left[(1-\rho^2)+\frac{m}{n}y_1+\frac{m}{n}y_2\right]^{-(q_{rj})}, \end{aligned} \tag{5.4}$$

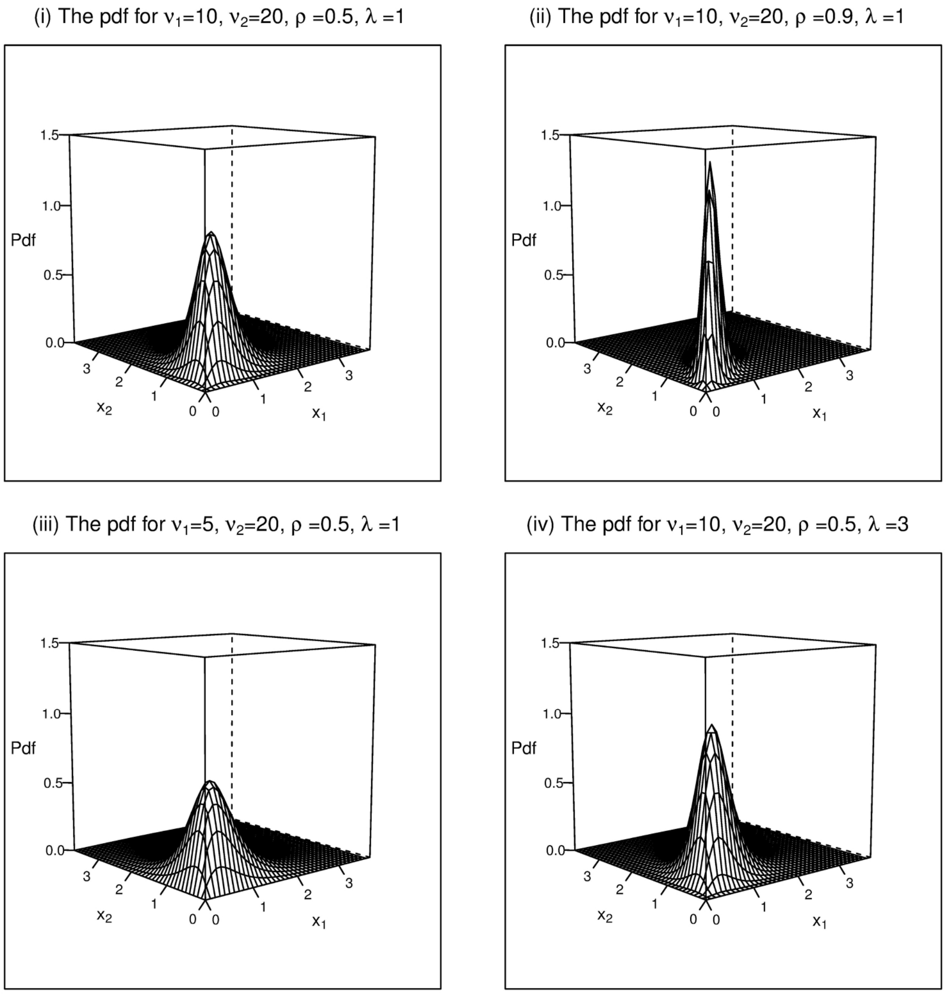
where $q_{rj} = m + (n/2) + 2j + r_1 + r_2$.

Table 1: Percentage points for the *doubly* correlated BNCF distribution for $\rho = 0.5$ and $\alpha = 0.05$

m	n	θ_1								
		2			4			10		
		θ_2			θ_2			θ_2		
		2	4	10	2	4	10	2	4	10
2	5	11.9	14.1	22.8	14.5	16.2	23.4	23.6	24.2	27.9
	6	10.4	12.3	19.8	12.6	14.0	20.3	20.4	20.8	24.0
	8	8.8	10.4	16.6	10.6	11.8	17.0	16.9	17.3	19.8
	10	8.4	9.5	15.0	9.6	10.6	15.2	15.1	15.4	17.6
4	5	8.4	9.5	13.6	9.7	10.5	14.0	14.1	14.4	16.3
	6	7.3	8.2	11.1	8.3	9.0	12.0	12.1	12.3	14.0
	8	6.1	6.8	9.8	6.9	7.5	10.0	10.0	10.2	11.4
	10	5.4	6.1	8.8	6.2	6.7	8.9	8.9	9.0	10.1
6	5	7.2	7.9	10.6	8.0	8.6	10.9	10.9	11.1	12.4
	6	6.2	6.8	9.1	6.9	7.3	9.3	9.3	9.5	10.6
	8	5.1	5.6	7.5	5.7	6.0	7.7	7.6	7.8	8.7
	10	4.6	5.0	6.7	5.1	5.4	6.8	6.8	6.9	7.7
8	5	6.6	7.1	9.0	7.2	7.6	9.3	9.3	9.5	10.5
	6	5.6	6.1	7.7	6.2	6.5	8.0	8.0	8.1	8.9
	8	4.7	5.0	6.4	5.1	5.3	6.5	6.5	6.6	7.3
	10	4.1	4.5	5.7	4.5	4.7	5.8	5.8	5.9	6.4
10	5	6.2	6.7	8.1	6.7	7.0	8.3	8.4	8.5	9.3
	6	5.3	5.7	6.9	5.7	6.0	7.1	7.1	7.3	7.9
	8	4.4	4.6	5.7	4.7	4.9	5.8	5.8	5.9	6.4
	10	3.9	4.1	5.1	4.1	4.3	5.2	5.1	5.2	5.7

Table 2: Percentage points for the *doubly* correlated BNCF distribution for $\rho = 0.5$ and $\alpha = 0.01$

m	n	θ_1								
		2			4			10		
		θ_2			θ_2			θ_2		
		2	4	10	2	4	10	2	4	10
2	5	26.0	30.7	49.1	31.4	35.0	50.3	50.5	51.6	59.3
	6	20.9	24.7	39.0	25.1	27.7	39.8	39.8	40.6	46.5
	8	15.9	18.8	29.3	19.0	20.9	29.8	29.7	30.2	34.3
	10	13.6	16.0	24.7	16.1	17.6	25.1	24.9	25.3	28.6
4	5	18.0	20.3	29.0	20.6	22.3	29.7	29.8	30.4	34.4
	6	14.3	16.1	22.9	16.3	17.6	23.4	23.4	23.9	26.8
	8	10.7	12.0	17.0	12.2	13.1	17.4	17.3	17.6	19.7
	10	9.0	10.1	14.3	10.2	11.0	14.5	14.4	14.6	16.3
6	5	15.3	16.8	22.4	17.0	18.1	23.0	23.0	23.5	26.1
	6	12.0	13.2	17.6	13.3	14.2	18.0	17.9	18.3	20.3
	8	8.9	9.7	13.0	9.9	10.5	13.2	13.2	13.5	14.9
	10	7.5	8.2	10.8	8.2	8.7	11.0	10.9	11.1	12.3
8	5	14.0	15.0	19.1	15.2	16.0	19.6	19.6	20.0	21.9
	6	10.9	11.7	14.9	11.8	12.5	15.2	15.2	15.5	17.1
	8	8.1	8.7	11.0	8.7	9.2	11.2	11.1	11.3	12.4
	10	6.7	7.2	9.1	7.2	7.6	9.3	9.2	9.4	10.2
10	5	13.1	14.0	17.1	14.1	14.8	17.5	17.5	17.8	19.5
	6	10.2	10.9	13.3	11.0	11.5	13.6	13.6	13.8	15.1
	8	7.5	8.0	9.8	8.0	8.4	10.0	9.9	10.1	11.0
	10	6.2	6.6	8.1	6.6	6.9	8.2	8.2	8.3	9.0

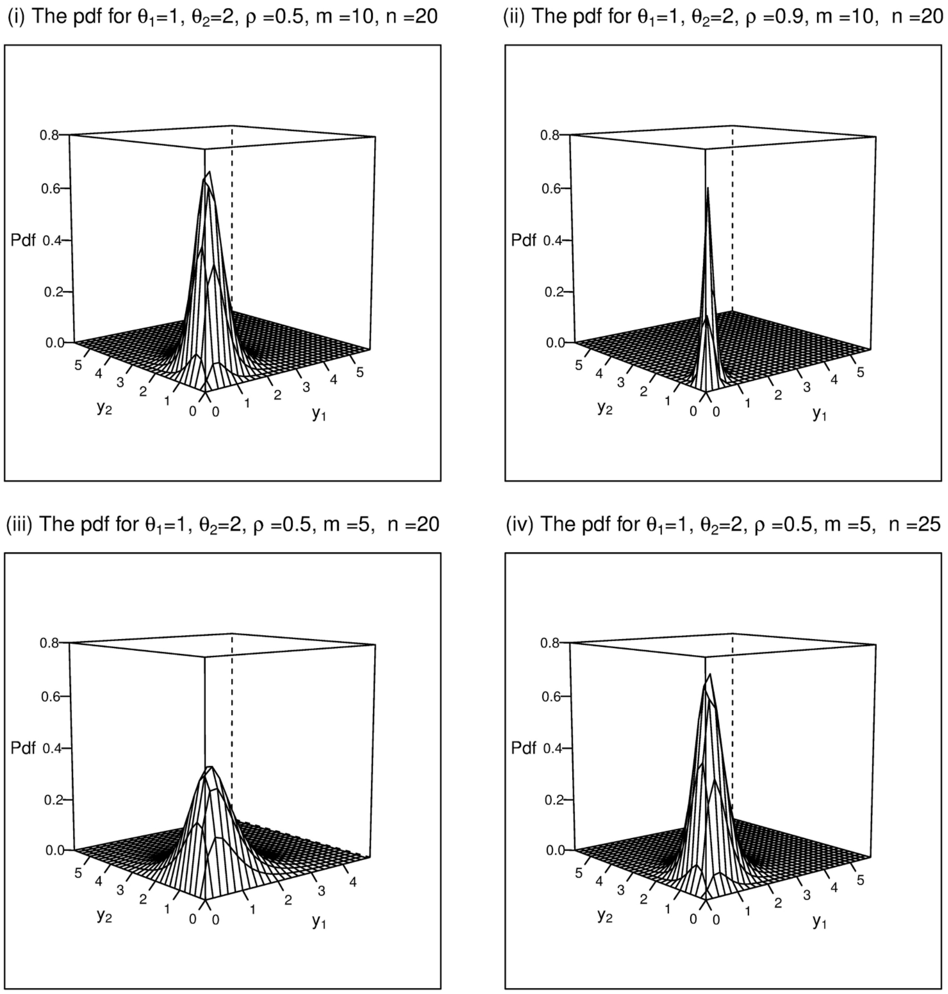

(i) The pdf for ν_1=10, ν_2=20, ρ =0.5, λ =1

(ii) The pdf for ν_1=10, ν_2=20, ρ =0.9, λ =1

(iii) The pdf for ν_1=5, ν_2=20, ρ =0.5, λ =1

(iv) The pdf for ν_1=10, ν_2=20, ρ =0.5, λ =3

The pdf of the singly bivariate noncentral F distribution.
206x209mm (300 x 300 DPI)

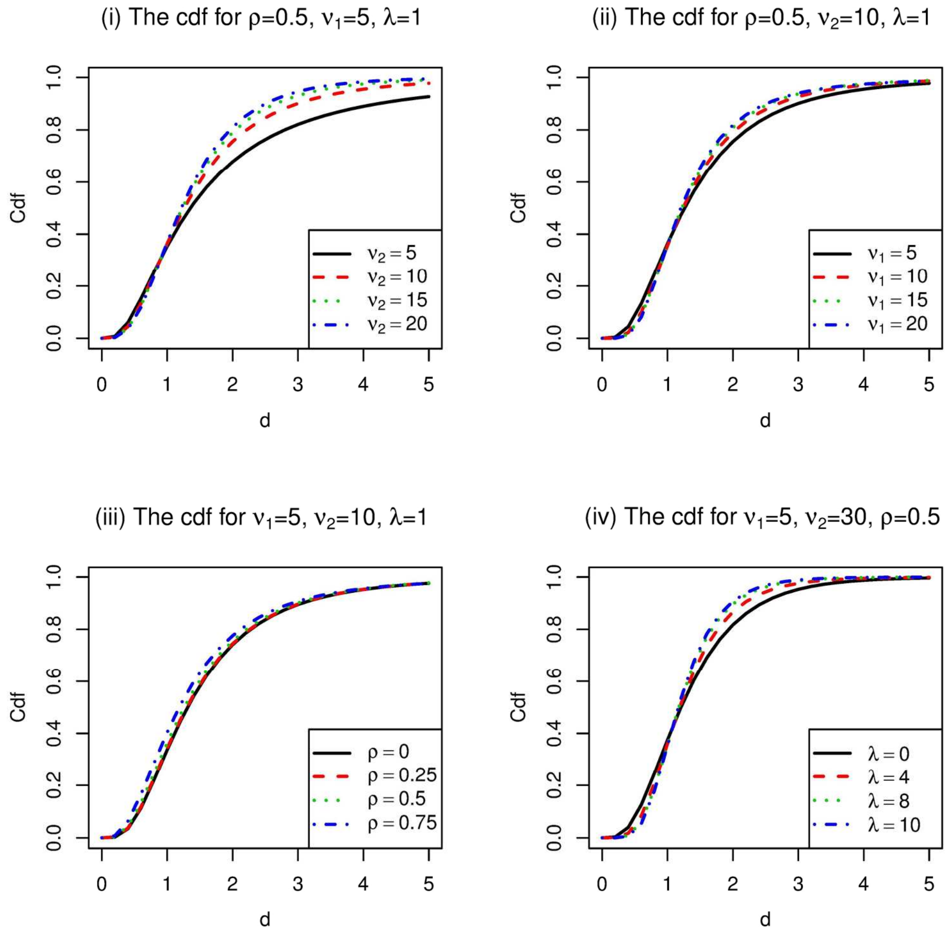

(i) The pdf for θ_1=1, θ_2=2, ρ =0.5, m =10, n =20

(ii) The pdf for θ_1=1, θ_2=2, ρ =0.9, m =10, n =20

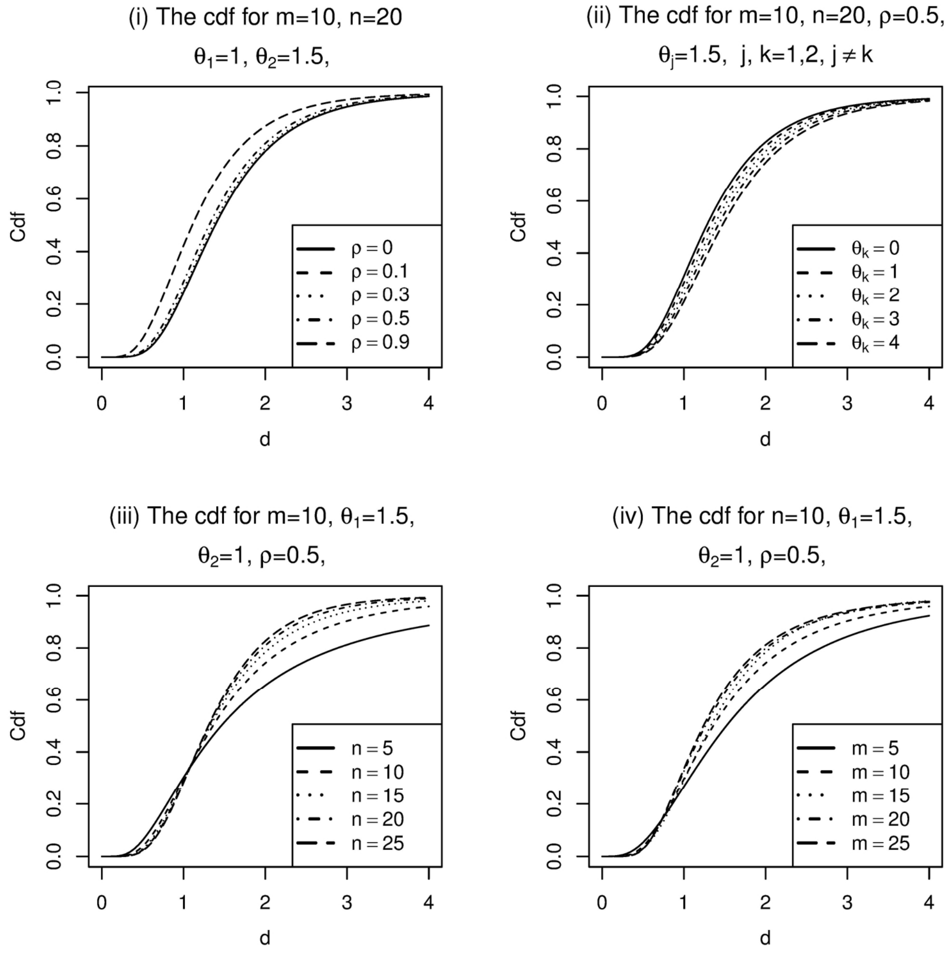
(iii) The pdf for θ_1=1, θ_2=2, ρ =0.5, m =5, n =20

(iv) The pdf for θ_1=1, θ_2=2, ρ =0.5, m =5, n =25

The pdf of the doubly bivariate noncentral F distribution.
206x209mm (300 x 300 DPI)



The cdf of the singly BNCF distribution with some arbitrary values of ρ, λ, v1 and v2.
177x177mm (300 x 300 DPI)


(i) The cdf for m=10, n=20 θ_1=1, θ_2=1.5,

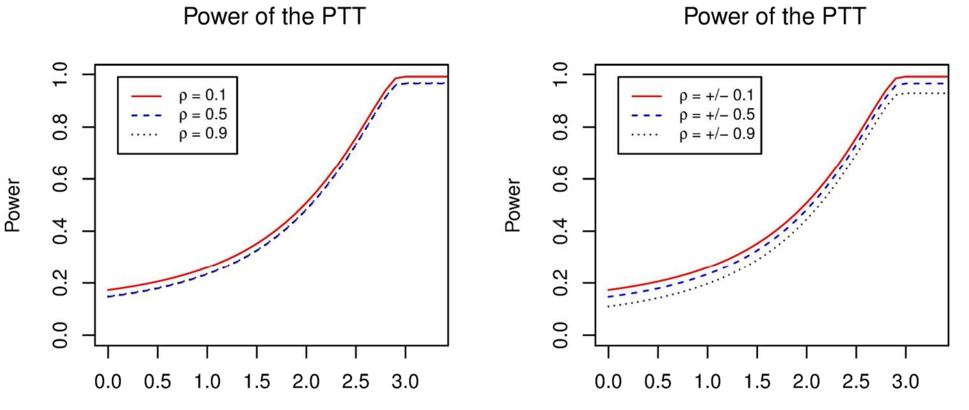
(ii) The cdf for m=10, n=20, ρ=0.5, θ_j=1.5, j, k=1,2, j ≠ k

(iii) The cdf for m=10, θ_1=1.5, θ_2=1, ρ=0.5,

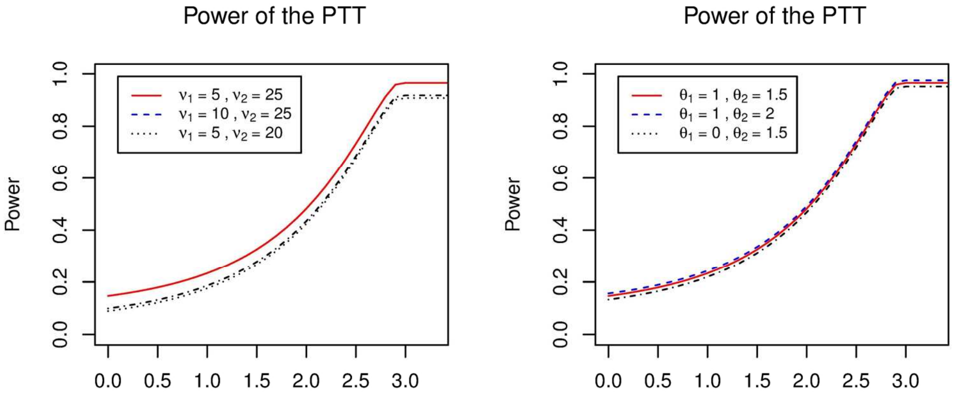
(iv) The cdf for n=10, θ_1=1.5, θ_2=1, ρ=0.5,

The cdf of the doubly BNCF distribution with some arbitrary values of ρ, θk, m and n.

177x177mm (300 x 300 DPI)



(i) ϕ_1 , ϕ_2 =1 , d=3 , ν_1 =3 , ν_2 =17 , θ_1 =1 , θ_2 =1.5 (ii) ϕ_1 , ϕ_2 =1 , d=3 , ν_1 =5 , ν_2 =25 , θ_1 =1 , θ_2 =1.5



(iii) ϕ_1 , ϕ_2 = 1 , d = 3 , θ_1 = 1 , θ_2 = 1.5 , ρ = 0.5 (iv) ϕ_1 , ϕ_2 = 1 , d= 3 , ν_1 = 5 , ν_2 = 25 , ρ = 0.5

The power of the PTT using the cdf of the doubly bivariate noncentral F distribution
177x177mm (300 x 300 DPI)