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Beyond the KdV: Post-explosion development

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Several threads of the last 25 years' developments in nonlinear wave theory that stem from the classical Korteweg–de Vries (KdV) equation are surveyed. The focus is on various generalizations of the KdV equation which include higher-order nonlinearity, large-scale dispersion, and a non-local integral dispersion. We also discuss how relatively simple models can capture strongly nonlinear dynamics and how various modifications of the KdV equation lead to qualitatively new, non-trivial solutions and regimes of evolution observable in the laboratory and in nature. As the main physical example, we choose internal gravity waves in the ocean for which all these models are applicable and have genuine importance. We also briefly outline the authors' view of the future development of the chosen lines of nonlinear wave theory. *© 2015 AIP Publishing LLC.*
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In the aftermath of the revolutionary development in the theory of nonlinear waves in the 1960s–1970s, the intensity of studies in this field does not show signs of decreasing. Here, several lines of studies related to the KdV equation are surveyed. Focusing on generalizations of the KdV equation, we trace the development of main ideas and concepts of nonlinear wave theory yielding qualitatively new solutions such as "fat" and table-top solitons, breathers, and slowly radiating solitons. The reasons underpinning the unmatched universality of the KdV equation as a mathematical model applicable in many physical contexts are quite natural: for long waves the model combines the most typical, small quadratic nonlinearity and weak, small-scale dispersion. The balance between the nonlinearity and dispersion allows the possibility of solitary waves possessing astonishing particle-like properties such as robustness and persistence in collisions not only with each other but also with other perturbations. This particle-like behavior is at the origin of the term soliton introduced to emphasize the affinity of such waves with elementary particles (electrons, protons, etc). As the understanding of nonlinear waves matured, the limitations of the KdV model and necessity to go beyond it became apparent; hence, the trend towards developing more general and rich models which generalize the KdV equation and yield many new and non-trivial results. Here, we briefly discuss their appearance in various mathematical and physical contexts and some of the results which follow, such as qualitatively new types of solitons, their limiting shapes and parameters, and interactions. It is also demonstrated that these features are not mathematical artefacts, which is illustrated by examples mainly related to nonlinear internal gravity waves in the ocean.

I. INTRODUCTION: MODEL EVOLUTION EQUATIONS

The basic equations describing evolution of long nonlinear dispersive waves were first derived in the end of the 19th century by Boussinesq (1872) and Korteweg and de Vries (1895). They were nearly forgotten until the 1960s, when the interest in these equations exploded. They proved central in the revolution in the understanding of nonlinear waves. The revolution was unfolding in several different, albeit related, directions. First, the "wave theory" supported by experiments began to establish itself as an important branch of mathematics and physics in the sense that various physical (as well as chemical, biological, and other) phenomena can be described by similar mathematical models, depending on such general characteristics as dispersion and nonlinearity. This, in turn, has resulted in identifying the key universal "model" equations, such as the KdV equation and many others. It has been realized that these asymptotically derived model evolution equations are universal and represent a powerful tool for studying nature. The surge in interest coincided and was partly caused by the discovery of remarkable mathematical properties of the key evolution equations: many of these nonlinear partial differential equations were found to be exactly solvable, and a new branch of mathematical physics, often referred to as integrable systems, was born. These radically new mathematical techniques complemented by numerical simulations and novel asymptotic approaches have revealed the key role of solitary waves in the wave field evolution. Solitary waves first observed in

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nature by Scott Russell in 1834 proved to be far from the structurally unstable homoclinic solutions such as separatrices in the ODE theory; on the contrary, they proved to be robust solutions representing asymptotic behavior of a wide class of initial conditions. Moreover, the solitary solutions exhibited particle-like properties: in integrable equations solitary waves collide elastically, which gave rise to the term "soliton."

At the same time, new asymptotic methods of solution of nonlinear evolution equations were developed, allowing to examine many physically important cases which cannot be reduced to integrable equations.

The revolutionary phase of the sixties and seventies of the last century was followed by a period of very extensive development characterized by an explosive growth of publications in which the primary ideas were further developed and numerous new model equations, mostly integrable ones, were suggested and analyzed (many of them were written just as examples of integrability, without any asymptotic derivation from physically meaningful equations). Then, starting shortly before 1990, a deeper and more systematic development of the nonlinear wave theory began; both integrable and non-integrable equations found broader physical applications and, due to the progress in numerical modeling, were verified versus primitive physical equations.

The present paper aims to outline some of the latter developments related to the KdV-like systems, mainly those of the last 25 years. Note that nonlinear Schrödinger equation could have been the subject of an equally interesting story, but here we concentrate entirely on the KdV and its "close relatives." Here, we follow several key lines. One is studying broader classes of equations that describe wave fields with different dispersion and nonlinearity as compared to the KdV equation. Second, we trace the enormous extension of the family of fundamental localized solutions which are often qualitatively different from those of KdV. Third, we discuss new applications. We mostly use as examples those taken from the context of internal gravity waves in the ocean; all main types of equations discussed below have established applications in the internal wave context. Moreover, for now the most numerous and diverse observations of solitons in nature apply to the oceanic internal waves.

Thus, what happened beyond the studies of the KdV equation? The universality of the latter is due to the simple physical approximation of generic weak (quadratic) nonlinearity and generic weak dispersion represented by the third-order derivative

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + \varepsilon\left(\alpha u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3}\right) = 0, \tag{1}$$

where c is the speed of long linear waves, α and β are constant coefficients, and $\varepsilon \ll 1$ is a small parameter, characterizing smallness of nonlinear and dispersive terms. By making the Galilean transformation $x' = x - ct$ and $t' = \varepsilon t$, this equation can be reduced to the canonical form

$$\frac{\partial u}{\partial t} + \alpha u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} = 0, \tag{2}$$

where primes are omitted and all terms are of the same order. The balance between the nonlinear and dispersive terms in this equation leads, in particular, to stationary solitary waves—the solitons (we use here the term "stationary" for a translational wave of a permanent form, $u(x - Vt)$; in the coordinate system moving with the wave speed V such wave is indeed stationary as it depends only on spatial coordinate). The remarkable mathematical properties of this equation have been exhaustively studied (e.g., Refs. 1–4).

To capture different wave dynamics, the first natural step is a straightforward extension of the nonlinearity and dispersion by retaining the next-order terms in the asymptotic expansion. A rather general form of this extension, which has been derived by many authors to describe nonlinear waves in different physical contexts (see, e.g., Ref. 5 for surface and internal waves) is

$$\frac{\partial u}{\partial t} + \alpha u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} + \varepsilon\left(\alpha_1 u^2\frac{\partial u}{\partial x} + \gamma_1 u\frac{\partial^3 u}{\partial x^3} + \gamma_2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + \beta_1\frac{\partial^5 u}{\partial x^5}\right) = 0. \tag{3}$$

This equation combines quadratic and cubic nonlinear terms, linear dispersion of the 3rd and 5th order, and also nonlinear dispersion with the coefficients γ_1 and γ_2. In general, the equation is not exactly integrable. However, Fokas and Liu[6] found an asymptotic transformation of solutions of the extended KdV equation (3) for function u to solutions of the KdV equation for a new auxiliary function v

$$u = v + \varepsilon\left[\lambda_1 v^2 + \lambda_2 v_{xx} + \lambda_3 v_x\int_{x_0}^{x} v\,dx + \lambda_4 x(\alpha v v_x + \beta v_{xxx})\right], \tag{4}$$

where the coefficients λ_i are rational functions of coefficients of Eq. (3). Thus, approximate solutions $u(x, t)$ to Eq. (3) can be obtained from the appropriate solutions $v(x, t)$ of the KdV equation (2) that are valid up to $O(\varepsilon^2)$. Inasmuch as the KdV equation is completely integrable, one can say, that Eq. (3) is asymptotically integrable, i.e., it tends to an integrable one when $\varepsilon \to 0$. This approach, although interesting, was not exploited thus far; the terms $O(\varepsilon)$ in Eq. (4) actually give just small corrections to the KdV solutions.

On the other hand, particular cases of (3) can lead to equations that describe much richer and qualitatively different classes of equations as compared to KdV, particularly when there is a degeneracy of either leading order nonlinearity or dispersion terms, and these higher-order terms can be comparable with the lower-order terms. Even in relatively simple cases, the family of solutions, including solitary solutions and of the corresponding equations, can be enormously broadened. The KdV equation possesses a single class of localized solutions—the already mentioned solitons which are asymptotics of all localized initial conditions for which $(\alpha/\beta)\int_{-\infty}^{+\infty}(1 + |x|)u(x)dx > 0$, see Ref. 73. The

KdV solitons have a single free parameter apart from the "phase" specifying its position. The examples considered below are much richer in this respect.

For example, keeping the fifth-order derivative term results in the non-integrable equation known as the Kawahara equation[7]

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} + \beta_1 \frac{\partial^5 u}{\partial x^5} = 0. \tag{5}$$

It possesses solitary waves with oscillating and decaying asymptotics which can form fairly complicated multisolitonic structures. Moreover, the Kawahara equation demonstrates important extensions of the very notion of a solitary wave. It possesses a class of coupled solitons also with more degrees of freedom than the classical KdV-type solitons.[8] It also admits a class of "non-local" steady solitary waves consisting of a central "core" which resembles a classical soliton, and oscillatory, non-damping small amplitude "wings" which extend to both infinities. Such solutions were christened "nanopterons,"[9] which means "dwarf-wing"; the name is not often used, however. These objects can be viewed as solitary waves simultaneously radiating and absorbing a linear wave of small amplitude. These and other "quasi-solitons" which cannot be exactly localized or slowly attenuate due to radiation form a broad class beyond the scope of KdV; even in the case of slow damping they can form a useful intermediate asymptotics in the process.[10] Various aspects of their description are often a mathematical challenge; they attracted a lot of attention in the literature (see, e.g., Ref. 11). Some aspects of radiating solitary waves are discussed below in Sec. III.

Another generalization of KdV is adding a comparable cubic term to obtain the so-called Gardner equation which also exhibits important new classes of solitary waves, table-top solitons, and breathers. It will be considered in more detail in Sec. II.

The form of Eq. (3), even without a small parameter, is not universal. A more general form of a one-dimensional weakly nonlinear and weakly dispersive evolution equation resulting from asymptotic derivation can be written as

$$\frac{\partial u}{\partial t} + C(u)\frac{\partial u}{\partial x} + G\left(\frac{\partial u}{\partial x}\right) = 0, \tag{6}$$

where $C(u)$ is the quadratic polynomial and G is a differential-integral operator obtained by the inverse Fourier transform of the linear dispersion relation $c\,(k)$[1]

$$G = \int_{-\infty}^{\infty} K(x-z)\frac{\partial u}{\partial z}(z,t)dz,$$
$$K(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[c(k)-c_0]e^{-ikx}dk, \tag{7}$$

and $c_0 \equiv c\,(0)$ is the linear long-wave velocity. Evidently, the KdV equation, as well as other evolution equations mentioned above, are just particular cases of Eq. (6). For example, for the KdV equation $C(u) = \alpha u$ and $G = \beta\partial^2/\partial x^2$. A more complicated example of an integral-differential equation for which the very existence of a solitary wave is a non-trivial problem, is considered in Sec. III.

Finally, generalizations are also possible for some strongly nonlinear cases, although the applicability of "one-wave" evolution equations cannot be taken for granted. In these cases, the nonlinearity and dispersion cannot be completely separated, and the term responsible for the long-wave dispersion remains nonlinear. In Sec. IV we show how much insight can be obtained using the understanding provided by weakly nonlinear models even for strongly nonlinear dynamics.

The paper is organized as follows. In Sec. II, we discuss solitary solutions and long wave evolution in the Gardner equation. Sec. III deals with the models containing two dispersions, the short-scale (KdV-type) and long-scale (integral) dispersion, and discusses some peculiar properties of their solutions. Sec. IV describes strongly nonlinear model equations, their limitations and applications. Finally, Section V summarizes the present state of understanding and gives some thoughts regarding the future of development in nonlinear wave theory and applications.

II. GARDNER EQUATION: SOLITONS, KINKS, BREATHERS

A. Solitons

The Gardner equation

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + \alpha_1 u^2 \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = 0 \tag{8}$$

appeared in mathematical physics at the end of 1960s as an important example of completely integrable equations (e.g., Refs. 2 and 3). Later, Lee and Beardsley[5] used this equation as the model describing strongly nonlinear internal waves observed in the ocean. Solitary solutions to the Gardner equation can be very different and depend on the sign of the cubic coefficient α_1. Nonetheless, they can be presented as the one-parameter family[12]

$$u(x,t) = \frac{A}{1 + B\cosh[(x-Vt)/\Delta]},$$
$$A = \frac{6\beta}{\alpha\Delta^2}, \quad B^2 = 1 + \frac{6\alpha_1\beta}{\alpha^2\Delta^2}, \quad V = \frac{\beta}{\Delta^2}, \quad U_0 = \frac{A}{1+B}, \tag{9}$$

where V is the soliton speed, Δ is its characteristic width, and U_0 is the amplitude. The parameter B is in the limits $0 < B < 1$ for $\alpha_1 < 0$ and $B^2 > 1$ for $\alpha_1 > 0$. Soliton shapes are shown in Fig. 1 for $\alpha > 0$ and $\beta > 0$. At fixed signs of coefficients α and β, the soliton characteristics are qualitatively different for different signs of the cubic nonlinear term.

The first important qualitative departure from the KdV solitons is that when $\alpha_1 < 0$, the shape of solitons changes significantly with increase of amplitude U_0 from zero to the limiting value $U_{lim} = -\alpha/\alpha_1$; namely, it goes from the bell-shaped KdV soliton ($B \to 1$), to the table-top pattern ($B \to 0$, $\Delta^2 \to -6\alpha_1\beta/\alpha$) with the infinitely increasing width (see Fig. 1(a)). In this case the sign of the coefficient α controls the soliton polarity which can be either positive or negative so that αu

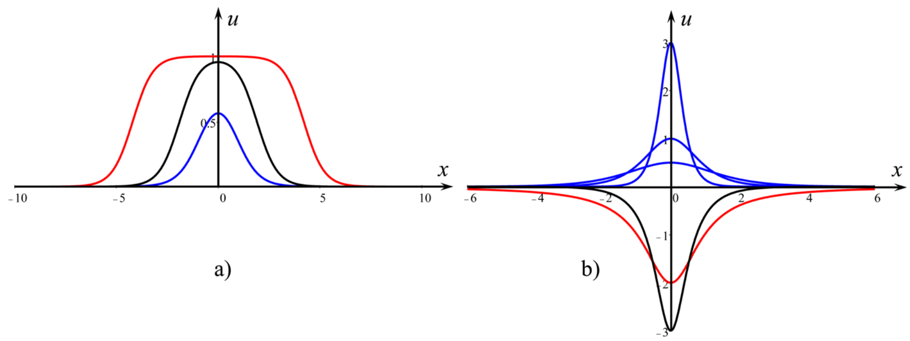

FIG. 1. Soliton shapes in the Gardner equation (5) with $\alpha = \beta = 1$ for different soliton amplitudes. (a) $\alpha_1 = -1$: line 1—KdV-like soliton, 2—"fat" soliton, 3—table-top soliton with kink and anti-kink edges. (b) $\alpha_1 = 1$: lines in the upper half-plane—"positive" solitons, line 4—one of the family of "negative solitons," line 5—the limiting case of "negative" solitons: the algebraic soliton. Reprinted with permission from Grimshaw *et al.*, Physica D **132**, 40 (1999). Copyright 1999 Elsevier.

is always positive. The edges of this soliton are close to step-like transitions (*kinks*, or non-dissipative shock waves)[13]

$$u = \pm \frac{\alpha}{2\alpha_1}\left(1 \mp \tanh \frac{x - Vt}{2\Delta}\right), \quad V = -\frac{\alpha^2}{6\alpha_1},$$
$$\Delta = \sqrt{-\frac{6\alpha_1 \beta}{\alpha^2}}, \tag{10}$$

where the signs correspond to the frontal (descending in space) kink and the rear (ascending) "anti-kink." Note that these patterns are different from the kinks and anti-kinks in the sine-Gordon equation[2,3] where they are the only solitary solutions, whereas here they exist as the limit for a family of solitons.

In the case of $\alpha_1 > 0$ (see Fig. 1(b)), there exist two different families of solitons: "positive" solitons which, with increase in amplitude, vary from the KdV soliton to the soliton of the modified KdV (mKdV) equation[2,3] (Eq. (8) with $\alpha = 0$), with no limitation on its amplitude; and "negative" solitons with the amplitude $U_0 < U_{min} = -2\alpha/\alpha_1$. In the case of $U_0 = U_{min}$, the Gardner soliton (9) reduces to its limiting form having algebraically (rather than exponentially) decaying asymptotics ($u \sim 1/x^2$, when $x \to \infty$). This "algebraic" soliton has zero speed V in (8) (i.e., in the coordinate frame propagating with the linear long-wave velocity.) With the increase of amplitude $|U_0|$ Gardner solitons transform into the mKdV solitons. In the range $0 > U_0 > U_{min}$, there are no stationary solitons at all, instead there are more complicated nonstationary wave patterns called *breathers*[2,3,73] which are discussed below in more detail.

B. Interaction of solitons and kinks. Formation of soliton trains

Two-soliton interaction processes in Gardner equation are also quite peculiar.[14] The interaction of solitons of the same polarity is similar to that in the KdV equation. This result can be interpreted within the framework of the approximate theory of interaction of solitons as particles with repulsing potential.[15] If solitons have different polarities in the case of positive α_1, the result is more interesting, since the solitons can attract each other and form breathers, i.e., localized waves pulsating in the course of propagation. In this case, the breather solution can be found analytically.[16] Depending on their parameters, they can take the form of either bound states of solitary waves of opposite polarity periodically interacting with each other (see Fig. 2(a)), or in the form of envelope solitons whose carrier wave moves with a different velocity than the envelope wave, as in the nonlinear Schrödinger (NLS) equation (see Fig. 2(b)). We emphasize that breathers exist only in the case of a positive α_1 and has no analogues in the KdV equation.

As mentioned, a non-trivial feature of the Gardner soliton close to the flat-top one is that it can be considered as a compound of kink and anti-kink (see above). The kinks behave quasi-independently in the process of interaction of two such solitons. In Ref. 17, this problem was studied using asymptotic theory. As a result, the "double Toda" equation was derived for interacting kinks (the parameters $\alpha = -\alpha_1 = 6$ and $\beta = 1$ were used):

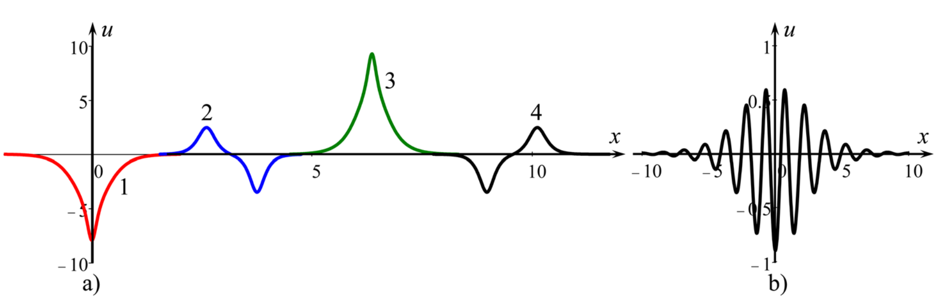

FIG. 2. Gardner breathers. (a) Oscillating pair of solitons within one period of oscillation T: $1 - t = 0$, $2 - t = T/4$, $3 - t = T/2$, $4 - t = 3T/4$. (b) Breather in the form of a wave packet. Reprinted with permission from Grimshaw *et al.*, Chaos **20**, 013102 (2010). Copyright 2010 American Institute of Physics.

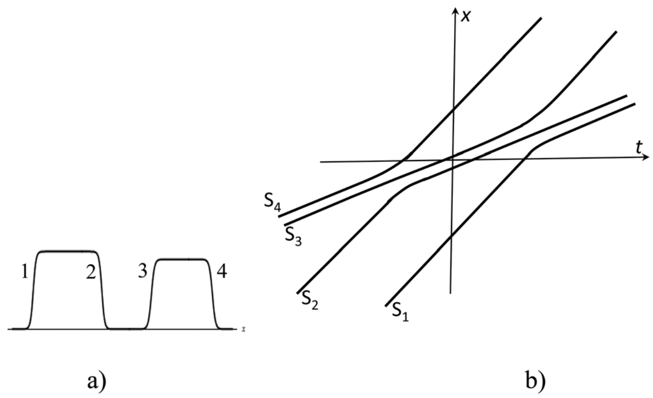

FIG. 3. (a) Two interacting flat-top solitons as a compound of kinks and anti-kinks. (b) Trajectories of kink and anti-kink centers. Reprinted with permission from Gorshkov *et al*., Phys. Rev. E **69**, 016614 (2004). Copyright 2004 American Physical Society.

$$\frac{d^2S_n}{dt^2} = 4\left(e^{S_n - S_{n+2}} - e^{S_{n-2} - S_n}\right), \tag{11}$$

where S_n are coordinates of kinks and anti-kinks; even numbers n correspond to frontal kinks, odd numbers n—to rear anti-kinks. If a kink–anti-kink pair belongs to the same Gardner soliton, then they are linked by the Katz–van Moerbeke equation realizing the degenerated Bäcklund transform for the Toda equation.[2,3,73] The main new element of this process as compared to interaction of solitons considered as a whole object is that, along with their behavior as particles, they reveal the wavelike features. Figure 3 illustrates this by showing trajectories of soliton fronts and rears in two interacting solitons having almost limiting amplitudes: the front of the rear soliton (numbered by 2) first affects the front of the first soliton (numbered by 4) rather than the much closer rear anti-kink (3) of the latter, and vice versa; this testifies to the remote action between the kinks.

When Eq. (11) is applied to chains of kinks, slow modulation of the chain can again be described by the Toda system, and the modulating envelope propagates with the "group" velocity three times greater than the "phase velocity" of the carrier.

As can be expected from the integrability of the Gardner equation, its multisoliton solutions can be found by the inverse scattering method.[14,18] For negative α_1 and small initial amplitudes ($<0.5\ U_{\min}$), the process of the solitons' formation from a long initial pulse is similar to that in the KdV equation. For larger initial amplitudes, a leading table-top soliton is formed followed by smaller KdV-type solitons and dispersive tails.

C. Internal waves in the ocean: Solibores

Along with the KdV equation, the Gardner equation is being actively applied to various physical problems, in particular, to studies of oceanic internal waves.[27,65] The coefficients of the KdV and Gardner equations are determined by the integrals over depth of products of eigenfunctions of the linear boundary value problem. These eigenfunctions, in turn, depend on the vertical profiles of water density $\rho(z)$ through the Brunt–Väisälä (buoyancy) frequency

$$N(z) = \sqrt{-\frac{g}{\rho}\frac{d\rho}{dz}} \tag{12}$$

and shear flow $U(z)$.

In coastal oceanic areas, nonlinear internal waves are typically generated by tides and often have a form of oscillating fronts (undular bores or *solibores*) close to a sequence of solitons. Disintegration of a long initial perturbation into the Gardner solitons can be described using quasi-periodic solutions of the Gardner equation which have been found for both signs of α_1. As in the case of the KdV equation, these solutions are expressed in terms of Jacobi elliptic functions. The periodic, slowly varying solutions are used to analyze the evolution of an initial front within the framework of the "dam break" problem, as disintegration of an initial stepwise perturbation. This approach was earlier suggested by Gurevich and Pitaevsky[22] for the KdV equation based on the Whitham equations[1] for a cnoidal wave with slowly varying parameters. It was then effectively applied to many physical problems, including the Gardner equation.[23] If $\alpha_1 < 0$, a weak initial step develops similarly to that in the KdV equation; for a larger step, the leading wave is the table-top soliton. If $\alpha_1 > 0$, the evolution of a front is qualitatively different. Both these cases are studied by Kamchatnov *et al*.[23] The Gardner-like solibores were observed, in particular, by Henyey and Hoering.[24]

Unlike solitons, it is not easy to observe breathers in the ocean; still there exists some indirect indications of their occurence in the ocean.[25] Meanwhile, direct numerical simulations within the fully nonlinear Euler equations for three-layer water flow suggest the existence of long-lived internal wave breathers.[26]

In the context of internal waves in the ocean, the dispersion coefficient β is always positive, whereas the nonlinearity coefficients α and α_1 can be of either sign. The mapping of all coefficients of the Gardner equation was undertaken in Ref. 27 for the World Ocean based on the available hydrological data.

III. DOUBLE-DISPERSION MODELS. ROTATIONAL KdV: CAN IT SUPPORT SOLITONS?

A. Rotation modified KdV equation

Now we consider another extension of the KdV model which makes the equation non-integrable. As mentioned, the KdV equation combines the effects of weak nonlinearity and weak, "small-scale" dispersion (the dispersion which manifests itself in the short wavelength range). In the linear limit, it corresponds to the dispersion relation in the form $\omega = ck - \beta k^3$ where the last term is small. Another practically interesting class of waves includes the "large-scale" dispersion in addition to the small-scale one. If it is also weak, then the dispersion relation takes the form $\omega = ck - \beta\ k^3 + \gamma/k$. This kind of dispersion is characteristic of waveguides in electrodynamics, acoustics, optics, and, as will be discussed here, waves in rotating fluids.[28] The corresponding generalization of the KdV equation contains an integral dispersive term

$$\frac{\partial u}{\partial t}+c_0\frac{\partial u}{\partial x}+\alpha u\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}=\gamma\int_{-\infty}^{x}u dx'. \qquad (13)$$

After differentiation with respect to x, this equation takes a more convenient form

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+c_0\frac{\partial u}{\partial x}+\alpha u\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}\right)=\gamma u. \qquad (14)$$

As with the Gardner equation, this equation was obtained long before the 1990s for oceanic waves experiencing the influence of Earth rotation[28] and then has attracted much attention of both mathematicians and physicists due to its unusual properties; later it was dubbed the "Ostrovsky equation" (see, e.g., Refs. 29–31); here we call it the rKdV equation. Similar equations were derived in other contexts, in particular, for waves in random media[32] and for waves in rotating plasmas.[33]

Note first that in its general form Eq. (14) is non-integrable; moreover, even its stationary solutions have not been obtained in the analytical form thus far. However, some interesting rigorous results were formulated for this equation from the very beginning. It is easy to see that any finite perturbation described by this equation has zero total "mass": $M\equiv\int u(x,t)dx=0$, where the integration is taken either over the period of a periodic wave or over the entire axis x for localized perturbations. This zero-mass restriction, apparently, has no grounds in basic physics and appears only as a consequence of the approximations adopted in derivation of Eq. (14); there is no such constraint within the set of primitive equations. An interesting issue of how an arbitrary initial perturbation with the non-zero total mass adjusts to suit Eq. (14) has been considered by Grimshaw.[29]

Another important rigorous result is the "anti-soliton theorem" established in Ref. 34 and then reproduced in numerous other papers. The theorem states that when $\beta>0$ (which is the oceanic case) there are no stationary solitary solutions to Eq. (14). However, the existence of such solutions is not prohibited if $\beta<0$; the specific solitary solutions with zero total mass were constructed numerically in Ref. 33. It was shown that they can form stationary bound states in the form of multisolitons and even regular or random chains of solitons (cf. notes about the Kawahara equation (5) above). The phase space of the stationary version of Eq. (14) can be fairly complicated; some stationary solutions were studied in Refs. 35 and 30. The detailed description of all possible stationary solutions is a challenge which, hopefully, will be resolved in forthcoming studies.

B. Reduced rKdV

More successful were analytical studies of the reduced case of Eq. (14) which is valid if the wave is long enough to neglect the short-wave (KdV-type) dispersion

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+c_0\frac{\partial u}{\partial x}+\alpha u\frac{\partial u}{\partial x}\right)=\gamma u. \qquad (15)$$

The stationary solutions of Eq. (15) depending on one variable $\zeta=x-V\,t$, $V=\text{const}$, satisfy a second order ODE and thus can be studied relatively easily. It was shown[28] that there exists a class of smooth "fast" ($V>0$) periodic waves, limited in amplitude by a wave of parabolic shape; evidently, an unlimited parabola is also a solution of the full equation (14). The "slow" stationary waves are non-smooth and limited in space. Later, stationary solutions of Eq. (15) were classified in Ref. 36. Among them there are singular solitary waves with sharp crests dubbed peakons and cuspons. Even more complicated loop soliton solutions were constructed and their interaction was studied (see, e.g., Ref. 37 and references therein), but all such non-smooth solutions are structurally unstable; they disappear as soon as any of neglected physical factors such as dissipation or small-scale dispersion are taken into account;[36] on the contrary, solutions of the full equation are smooth as long as the initial condition is smooth.

Thus, a non-trivial problem concerning the reduced rKdV equation (15) is to find the condition under which the initial perturbation eventually becomes singular (breaks) or, on the contrary, remains smooth at all times in the process of wave propagation. It has been shown in Ref. 38 that if the initial condition $u(x,0)=u_0(x)$ is such that $d^2u_0(x)/dx^2>\gamma/3\alpha$ at some x, then wave breaking eventually occurs and the solution becomes singular. If $d^2u_0(x)/dx^2<\gamma/3\alpha$ for all x, the solution remains smooth at all times. Note that earlier a parameter[39] $\text{Os}=3\alpha K/\gamma$ dubbed as the "Ostrovsky number" was introduced, where K is the maximum curvature in the initial condition; on the basis of this parameter, the authors estimated the possibility of wave breaking in Eq. (15). The physical basis for that is a "competition" between wave steepening due to nonlinearity (as in non-dispersive simple waves) and its deformation due to the long wave dispersion which, unlike the small-scale dispersion in the KdV equation, is not always able to prevent wave breaking. Moreover, in the non-breaking case, Eq. (15) can be reduced to the completely integrable Tzitzeica equation (known also as the Dodd–Bullough–Mikhailov equation) (for details and further references see Ref. 39).

C. Soliton evolution in rKdV

Returning to the full equation (14), we note that the first attempt to classify its stationary translational solutions was undertaken in Ref. 35. It was found that at certain conditions there exist nonlinear periodic waves consisting of parabolic sections mentioned above for Eq. (15) and narrow pulses on the wave crests which are close to the KdV solitons. These solutions were constructed analytically[40] using an asymptotic approach. The direct numerical modeling has shown that such solutions are stable with respect to small perturbations if $\beta\gamma>0$ and unstable in the opposite case. A more detailed numerical investigation of possible stationary solutions of Eq. (14) was undertaken in Refs. 41 and 30. Still, the stability of such solutions within the more general, non-stationary class of solutions remains an open question.

Rather non-trivial are non-stationary solutions of Eq. (14). According to the aforementioned "antisoliton theorem," an initial KdV soliton cannot exist infinitely; it gradually decays due to radiation.[42] As a result its amplitude $A(t)$ attenuates adiabatically as $(t_0 - t)^2$ where $t_0 \propto \sqrt{A(t=0)}$, and at $t \approx t_0$ the soliton is extinct, being completely converted into a radiated wave (*terminal damping*). The situation can, however, radically change if a KdV soliton interacts with a long wave close to the solution of Eq. (15). These two waves exchange energies, and the losses due to radiation from a soliton can be compensated by pumping from a long wave so that a solitary wave can exist in this environment. In the numerical example shown in Fig. 4, the soliton amplitude periodically changes due to the energy exchange with a long wave.

In particular, a stationary wave can be formed in which a soliton is sitting on the long wave crest or trough. As follows from the numerical modeling, the former configuration is stable and the latter is unstable. It is interesting that, as our preliminary studies show, if *only one* soliton is traveling on a periodic wave, the result is the opposite: the soliton equilibrium position on the crest is unstable, whereas its position at the trough is stable. This difference is due to the fact that in the latter case, soliton radiation is a pure loss, whereas in the fully periodic case the soliton radiation can be compensated by that from the solitons ahead of it.

The important outcome of these studies is that, notwithstanding the "anti-soliton theorem," waves close to solitons can exist on a variable background such as a long periodic wave co-propagating with them. Chen and Boyd[41] found some other non-trivial wave shapes; among them are solitary waves of alternative polarity sitting on the crests of a background long wave.

The list of non-trivial features of rKdV equation (14) can be continued. Recently, it was shown that in the long-term evolution, the initial KdV soliton in Eq. (14) with $\beta\gamma > 0$ ends up as a specific envelope soliton which looks similar to that in the NLS equation, but its carrier sinusoid frequency always corresponds to the inflection point of the dispersion curve $\omega(k)$ where the group velocity has a local maximum.[31] The same phenomenon was observed also in the context of waves in periodic lattice structures.[44] Figure 5 demonstrates the evolution of the KdV soliton into the NLS envelope soliton. At the beginning, the KdV soliton experiences a terminal decay as described above, but after a long time it evolves into a stable NLS-like soliton whose phase and group speeds are different. If $\beta\gamma < 0$ in Eq. (14), then one can construct a *stationary* NLS-like soliton[33] whose phase and group speeds are equal $V_p = V_g = 2(\beta\gamma)^{1/2}$; moreover, the phase speed is maximal on such a soliton.

It is noteworthy that the effect of transformation of a KdV soliton into an NLS-like soliton was observed in the laboratory experiment in a rotating tank.[45] Furthermore, a general analysis of the modulation stability of quasi-harmonic waves in the NLS equation following from Eq. (14) has shown that the modulational instability occurs for waves with wavenumbers $k > k_c$, where $k_c = (\gamma/3\beta)^{1/4}$. In the case of KdV equation ($\gamma = 0$), quasi-harmonic waves are always stable with respect to modulational instability for any wavenumbers.[2] There is an apparent paradox in that the rotation effect ($\gamma \neq 0$) leads to modulation instability at large rather than at small wavenumbers. This phenomenon is explained by the suppression of a zero harmonic by rotation. It appears in higher orders and cannot contribute to the

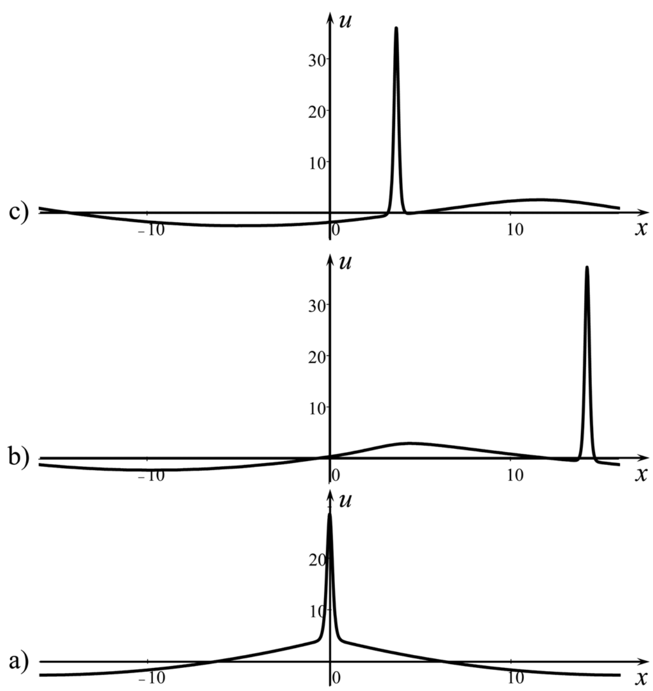

FIG. 4. Adiabatic interaction of a strong KdV soliton with a periodic wave of quasi-parabolic profile satisfying Eq. (14).[43] From bottom to top $t = 0$ (a), $t = 0.4$ (b), $t = 1$ (c). Reprinted with permission from Gilman *et al.*, Dyn. Atmos. Oceans **23**, 403 (1996). Copyright 1996 Elsevier.

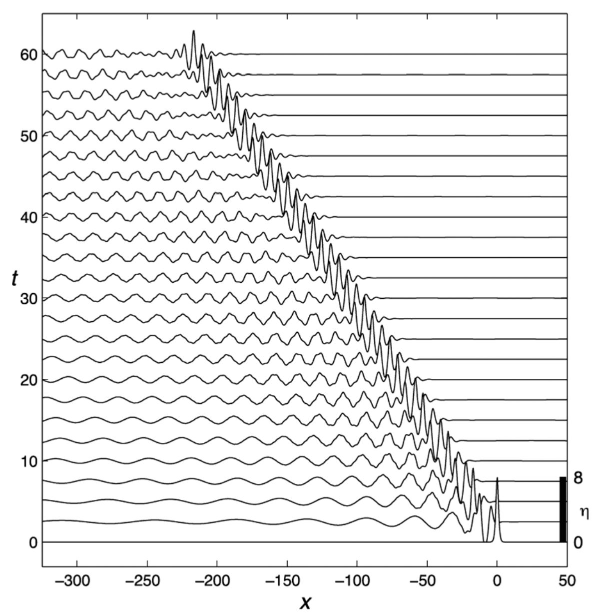

FIG. 5. Numerical solution of Eq. (14) (for $\alpha = \beta = \gamma = 1$) showing the formation of a leading nonlinear wave packet (top) from a soliton (bottom). Reprinted with permission from R. Grimshaw and K. Helfrich, Stud. Appl. Math. **121**, 71 (2008). Copyright 2008 John Wiley & Sons.

nonlinear coefficient of the NLS equation making the equation modulationally unstable for $k > k_c$.

D. Generalizations of rKdV

Similar to the KdV case, Eq. (14) can be supplemented by cubic nonlinearity to obtain the rotation modified Gardner equation

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + c_0\frac{\partial u}{\partial x} + \alpha u\frac{\partial u}{\partial x} + \alpha_1 u^2\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3}\right) = \gamma u. \quad (16)$$

This equation, currently known as the Gardner–Ostrovsky equation, was suggested in Ref. 50 for internal waves in a rotating ocean when the interface between two layers (the pycnocline) is located at the depth close to the half of the fluid depth. The terminal decay of solitons at $\alpha_1 < 0$ due to large-scale dispersion was studied in Ref. 46. It was revealed that solitons of relatively small amplitude (quasi-KdV solitons) decay in accordance with the prediction of adiabatic theory, whereas the decay of large amplitude solitons having a table-top shape can occur non-adiabatically from the very beginning. The direct numerical modeling of Eq. (16) shows that the shape of table-top solitons distorts at the very early stage of evolution. The influence of large-scale dispersion on decay of another kind of solitons (when $\alpha_1 > 0$) and breathers has not been studied thus far.

The particular case of Eq. (16) with $\alpha = 0$ (rotation modified mKdV equation) has been studied more thoroughly. This equation was derived and studied in a different physical context, in particular, for ultrashort optical impulses in nonlinear dielectrics (see, e.g., Ref. 47 and references therein). The large-scale dispersion in this case is caused by the polarization effect.

The reduced version of Eq. (16) with $\alpha = \beta = 0$ can be integrable, but the criterion for integrability is rather intricate. This problem was studied by Johnson and Grimshaw.[48] They thoroughly studied the wave breaking conditions depending on the initial wave steepness and found a link of that equation to the sine-Gordon equation.

E. Oceanic applications

Equations (14) and (15) are often applied in physical oceanography where, as mentioned, soliton-like groups of internal gravity waves are common. In particular (e.g., Refs. 49 and 50) it was shown that rotation decreases the number of solitons formed in a given tidal cycle. Later, Grimshaw *et al.*[51] analyzed non-stationary processes for realistic oceanic parameters. Li and Farmer[39] made specific calculations within the framework of these equations together with the direct numerical simulations of the corresponding hydrodynamic equations to analyze the data of experiments in the South China Sea. In particular, they confirmed the aforementioned result regarding suppressing soliton formation by rotation.

IV. STRONGLY NONLINEAR KdV-TYPE MODELS

A. Stationary waves: History

Can KdV and its modifications be extended to strongly nonlinear waves? A measure of nonlinearity can be, for example, the ratio of maximal fluid velocity in a wave to its propagation velocity in linear approximation; this ratio corresponds to the Mach number in gas dynamics. Classical examples are non-dispersive waves in mechanics of compressible media where the important basic classes of solutions are the simple (Riemann) and shock waves. The theory of strongly nonlinear dispersive waves is not well developed, especially for multi-dimensional waves such as internal waves in the ocean. In many cases, the direct numerical simulation (DNS) of the basic equations is used for each specific problem. Still, a noticeable progress in derivation and application of strongly nonlinear model equations can be reported. We consider this problem in the context of stratified fluid flows, including internal waves.

An additional difficulty in obtaining model equations as compared with the weakly nonlinear waves considered above is that the dependence on vertical coordinate z cannot be separated from the horizontal ones and, consequently, there is no fixed modal structure. The early results were obtained for stationary 2D motions in which all variables depend on z and $\zeta = x - Vt$ where $V = \text{const}$. For these flows, the hydrodynamic equations can be reduced to one equation[52] for the stream function ψ

$$\Psi_{\zeta\zeta} + \Psi_{zz} + V^{-2}N^2(z - \Psi/V)\Psi = 0, \quad (17)$$

where $N(z)$ is the Brunt–Väisälä frequency (see Eq. (12)).

Equation (17) has been widely used in numerical modeling. A remarkable feature of this equation is that it evidently becomes linear if $N(z) = \text{const}$. Therefore, stationary progressive waves (but only stationary!) are described by a linear equation and hence, can have, in particular, a sinusoidal profile of arbitrary amplitude. The majority of other analytical results were obtained for stationary waves in a two-layer fluid with a density jump between the homogeneous layers. The first study using this approach was, apparently, made in Ref. 53. It was found that there exists maximal possible solitary wave amplitude at which it acquires a flat-top shape and tends to two infinitely separated kinks. Qualitatively this pattern is similar to the solutions of the Gardner equation discussed above, but now it is valid for an arbitrarily strong wave. For a two-layer fluid limited by immovable horizontal surfaces from top and bottom, with upper layer thickness h_1 and lower h_2 and densities ρ_1 and $\rho_2 > \rho_1$, the amplitude of the interface displacement $\eta(x, t)$ and propagation velocity of such a limiting soliton are

$$A_{\max} = \frac{h_1 - h_2\sqrt{a}}{2} \approx \frac{h_1 - h_2}{2};$$
$$V_{\max} = \frac{\sqrt{g(1-a)(h_1+h_2)}}{1+\sqrt{a}} \approx \frac{\sqrt{g'(h_1+h_2)}}{2}, \quad (18)$$

where $a = \rho_1/\rho_2$, $g' = g(1-a)a$. The approximate equalities in (18) are valid when density variation is small and a is close to unity; this so-called Boussinesq approximation is well-suited to oceanic conditions where density variations in water are always small. Moreover, in the same case, the free water surface can still be assumed immovable (rigid lid approximation) since surface waves do not significantly affect internal motions (not vice versa, though). In what follows for simplicity we consider this case.

From Eq. (18) follows a significant result: the polarity of a soliton (the sign of A_{max}) is specified entirely by the difference $h_1 - h_2$. As a consequence of that in the oceanic solitons, the displacement is always directed towards the thicker layer. There are no solitary solutions when $h_1 = h_2$; for small $|h_1 - h_2|$ nonlinearity is weak and solitary solutions can be described by the Gardner equation. Figure 6 illustrates shapes of solitary waves at different amplitudes.[54]

B. Non-dispersive waves

Subsequently, effective numerical codes for directly solving the fully nonlinear 2D problems were developed both for stationary and non-stationary solutions. Our goal, however, is to answer the same question as above: is it possible to reduce the 2D problem to one-dimensional equations (or the 3D problem to 2D equations) by separating the z-dependence? As mentioned, the strict answer is no: one cannot separate vertical dependence from horizontal. However, some effective long-wave equations have been obtained for strong nonlinearity, primarily for the two-layer model, where at least one exact result can be obtained. For very long waves, dispersion can be completely neglected (the quasi-hydrostatic approximation). In this case, as expected, the interface wave can propagate as a simple wave with the local velocity[55]

$$c(\eta) = \pm c_0 \left\{ 1 + 3\frac{(h_1 - h_2)(h_1 - h_2 - 2\eta)}{(h_1 + h_2)^2} \left[\sqrt{\frac{(h_1 - \eta)(h_2 + \eta)}{h_1 h_2}} - \frac{h_1 - h_2 - 2\eta}{h_1 - h_2} \right] \right\}, \tag{19}$$

where $c_0 \equiv c(0)$ is the long linear wave velocity. The dependence (19) on local displacement η is shown in Fig. 7 for three depth ratios corresponding to available observational data.

Here, the long wave velocity non-monotonically depends on the local displacement. Thus, unlike the classical dynamics of compressible fluid, the waves of "compression" and "rarefaction" can exist for any sign of displacement gradient. It is interesting that $c(\eta)$ returns to its linear value at $\eta = (h_2 - h_1)/2$, which coincides with the maximal amplitude of the soliton given in (18). As in the 1D dynamics of compressible fluids based on simple waves, the full description of bi-directional non-dispersive internal waves can be developed using Riemann invariants.

Recently, a 2D generalization of the notion of a simple wave for a continuously stratified fluid was found.[57] These solutions have the form $f(\xi, \beta)$, where β is the vertical coordinate of a fixed isoline of density (isopycnal line), and $\xi = x - c(x)t$ is the implicit variable. In such waves all perturbations in a fixed vertical cross section propagate at the same velocity, but vertical structure of the field is different in each cross-section. This solution was validated by comparison with the direct numerical simulation.

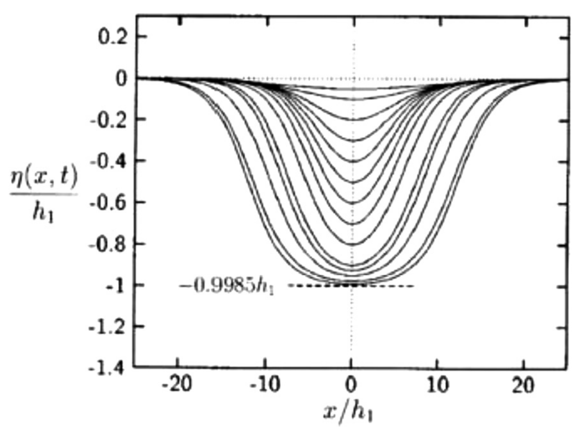

FIG. 6. Displacement profiles in a soliton, $\eta(x)/h_1$ vs x/h_1 for two-layer fluid with $h_2/h_1 = 3$, $\rho_2/\rho_1 = 0.997$ (surface is at +1, and bottom is at −3 on the vertical axis).[54] The lines depict the pycnocline depressions caused by a soliton from 0.05 to 0.99 at the center. The dashed line marks the level of limiting amplitude. Reprinted with permission from W. Evans and M. Ford, Phys. Fluids **8**, 2032 (1996). Copyright 1996 American Institute of Physics.

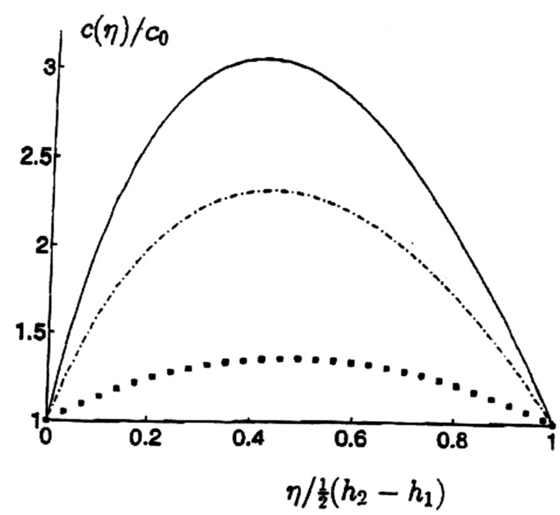

FIG. 7. Simple wave velocity versus local interface displacement for $h_1/h_2 = 3.86$ (dots), 12 (dashed-dotted line), and 20.4 (solid line). Reprinted with permission from L. A. Ostrovsky and J. Grue, Phys. Fluids **15**, 2934 (2003). Copyright 2003 American Institute of Physics.

C. Model equations for dispersive waves

When the dispersion, albeit weak, is crucial (which is always true for solitary waves), it is natural to try to obtain model equations generalizing the weakly nonlinear models such as Boussinesq, KdV, and Benjamin–Ono equations. For these cases, a long-wave approximation can be constructed using the corresponding expansion of dispersive terms while keeping the nonlinearity strong. Actually, this basic approach was first suggested by Whitham[58] for surface waves based on the expansion of the Lagrangian; the later work by Green and Naghdi[59] includes a sloping bottom. For internal waves in a two-layer case, Miyata[60] suggested Boussinesq-type long-wave equations for strongly nonlinear, weakly dispersive waves in a two-layer fluid, and constructed a stationary solitary solution of these equations. A comprehensive analysis of this problem for a two-layer fluid was performed by Choi and Camassa.[61] Thus, we call the corresponding equations the MCC system. They have the form

$$\eta_t = [(h_2 - h_1)u_2], \quad (u_1 - u_2)_t + u_1 u_{2x} - u_2 u_{1x} + g'\eta_x = D, \tag{20}$$

where

$$D = \frac{1}{3(h_1+\eta)}\left\{(h_1+\eta)^3\left(u_{1xt} + u_1 u_{1xx} - u_{1x}^2\right)\right\} - \frac{1}{3(h_2-\eta)}\left\{(h_2-\eta)^3\left(u_{2xt} + u_2 u_{2xx} - u_{2x}^2\right)\right\}. \tag{20a}$$

The term D is responsible for the weak nonlinear dispersion (non-quasistatic approximation); it is obtained by expansion of the higher-order terms in hydrodynamic equations. Even earlier, Choi and Camassa[62] obtained similar equations for the case of an infinitely deep lower layer, in the approximation analogous to that used for the Benjamin–Ono equation.

Within the framework of Eqs. (20), the description of stationary waves can be reduced to a second-order ODE. In particular, the velocity V of a soliton of amplitude A is $\sqrt{g'(h_1 - A)(h_2 + A)/(h_1 + h_2)}$.

Equations (20) were generalized to include a bilinear shear current profile.[63] For stationary waves, an interesting development was suggested by Voronovich:[64] stationary long-wave solitary solutions were found for a two-layer fluid in which each layer is exponentially stratified so that the buoyancy frequency N is constant in each layer (the "2.5-layer model"). According to Eq. (17), the solution in each layer is still linear which allows a relatively simple description. Note that in this case the solution may include an internal vortex core.

The above models, however, have a common inconsistency. They assume a weak dispersion (long-wave approximation), whereas the nonlinearity can be arbitrarily strong. On the other hand, a soliton can exist in the case of essential balance between the nonlinearity and dispersion so that the corresponding approximations should be verified in each case. Fortunately, the direct numerical modeling shows that they work well in a rather broad range of wave parameters.[56]

The next step was to write a one-directional evolution equation generalizing the KdV and BO equations in which the nonlinear long-wave velocity for the two-layer model is taken in the exact form (19), whereas the dispersion parameter corresponds to a local instantaneous displacement at each point of the wave profile.[56] Thus, the strongly nonlinear analog of the KdV equation (the "β-model") has the form

$$\frac{\partial \eta}{\partial t} + c(\eta)\frac{\partial \eta}{\partial x} + \frac{\partial}{\partial x}\left[\beta(\eta)\frac{\partial^2 \eta}{\partial x^2}\right] = 0, \tag{21}$$

where $\beta = c(\eta)(h_1 - \eta)(h_2 + \eta)/6$ is the dispersion parameter which locally corresponds to the KdV dispersion at each point of the wave. A modification of the "β-model" (21), the so-called "E-model," as well as a similar generalization of the Benjamin–Ono (BO) equation, were also suggested in Ref. 56.

D. Oceanic observations

From the 1990s experimental observations of strongly nonlinear internal solitons became ubiquitous,[65] although some single observations were reported even earlier. In certain cases, weakly nonlinear models such as the KdV and Gardner equations describe them well. In general, however, strongly nonlinear models are necessary. A characteristic example is the Coastal Ocean Probing Experiment (COPE) off the coast of northern Oregon performed in 1995 (Fig. 8), which shows a long sequence of tide-generated solitary impulses.[66] As seen from Fig. 8, the depression of the sharp pycnocline (often approximated by a density jump in theory) reaches a depth 5–6 times its initial position (from 5 to 30 m in Fig. 8). Although even stronger solitons were observed in the ocean, this ratio is, perhaps, the "world record of nonlinearity."

In Ref. 66, the Gardner equation was used beyond its formal range of applicability as a fit for the shape of an individual soliton. In Ref. 17, the kink interaction model (11) was applied to the same data to predict evolution of a group of strong solitons. A more consistent and detailed theoretical analysis of these and other data was performed in Ref. 56 using the MCC equations and the β- and E-models. Remarkably, the dependencies of soliton velocity on its amplitude provided by these models are close to those obtained from the direct numerical simulation. For the soliton width,

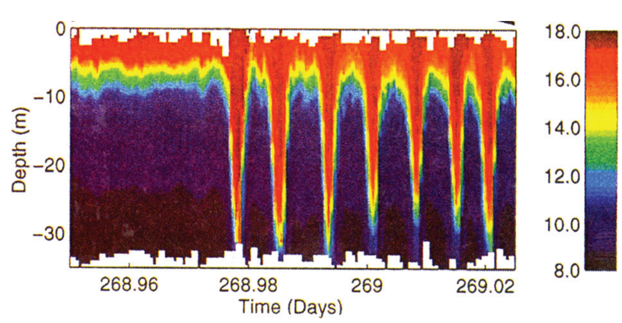

FIG. 8. Color contours of temperature variation in °C in a strong internal wave measured off the NW coast of the USA. Reprinted with permission from T. Stanton and L. Ostrovsky, Geophys. Res. Lett. **25**, 2695 (1998). Copyright 1998 American Geophysical Union.

these results are close for moderate ratios $q = h_2/h_1$; for the case of $q > 10$–12, significant discrepancies occurred; however, even in these cases the agreement with the DNS and experimental results is much better than with those using the KdV solitons.

Numerous other observations of strong internal solitons and their groups have been reported in different areas of the ocean.[65,67]

V. CONCLUSIONS AND PERSPECTIVES

From a close distance, a panorama of any scientific field might look chaotic, but a quarter of a century span allows us to choose and follow up upon a few coherent threads in the recent phase of our chosen corner of nonlinear wave theory. Here, we will briefly summarize the key points and ideas described above and try to make a guess about their possible development in the future.

It has been demonstrated how the account of higher-order nonlinearity and/or dispersion allows one to capture qualitatively new wave dynamics as compared to the classical KdV model. The point to be emphasized first is that for a qualitative change to occur in a weakly nonlinear model, a mere degeneracy of coefficients of the quadratic nonlinearity and/or leading order dispersion is quite often sufficient to obtain these new dynamics. The Gardner equation provides an excellent example of such an extension: its solitary solutions vary from bell-shaped to "fat" and table-top solitons; kinks are also solutions; solitons of any polarity and breathers can co-exist when the cubic term coefficient is positive. Integrability of the Gardner equation made it possible to analytically obtain the full picture of interactions between solitons of various types and between solitons and non-localized waves. Further development of asymptotic methods made it possible to describe multisoliton and multikink ensembles in both integrable and non-integrable systems, including the "hierarchy" of such ensembles allowing one to build a high-order ensemble as an envelope over the previous one.

Extensions of the KdV equation with higher-order dispersion and integral dispersion considerably enrich the family of solitary waves and possible scenarios of wave dynamics. They provide examples of solitons with non-monotonic structures, coupled solitons, multisoliton fronts and groups, as well as stationary random sequences of coupled solitons. When steady solitary waves cannot exist on a constant background due to radiation losses, they can propagate long enough in the form of gradually decaying "radiating solitons" and serve as intermediate asymptotics, possibly evolving into wave packets—the envelope solitons mentioned in Sec. III. Solitons, which asymptotically decay in the absence of a background can, exist indefinitely if they exchange energy with a variable background. The richness of the evolution scenarios provided by the evolution equations with integral dispersion seems to be limitless, but to advance in their understanding new methods of analysis have to be developed. We definitely expect this thread to continue well into the future.

The generalizations of KdV mentioned above still correspond to weakly nonlinear and weakly dispersive waves in physical applications. The theory of "genuinely" strongly nonlinear waves in dispersive media is an extremely difficult and not very well developed part of the wave theory. Direct numerical simulations of the evolution of strongly nonlinear waves are necessarily computationally expensive and not always easy for interpretation, which makes it practically impossible to sweep a multidimensional parameter space. Remarkably, relatively straightforward modifications of weakly nonlinear equations including KdV have proved able to capture strongly nonlinear patterns in good agreement with the DNS and experiments.

We want to emphasize that all, sometimes bizarre, properties of the soliton zoo described above are not exotic: such waves do exist in many real physical environments as exemplified by internal waves in the oceans. It is essential that, in the latter case, the 1D (2D) evolution equations describe 2D (3D) waves with an appropriate depth distribution. For weak nonlinearity, these waves are multimodal, and the results are different for different vertical modes. For strongly nonlinear waves, which cannot be presented by a few fixed modes, manageable results have so far been either for non-dispersive (simple) waves or for a two-layer model of stratification. Nonetheless, in all cases considered above, not only a qualitative but also a quantitative (albeit approximate) agreement with observation was obtained in a number of cases.

Let us now briefly speculate about what can be expected in short- and long-term perspectives in the corner of the nonlinear wave theory visited in this paper, and beyond that.

A shortage of room forced us to leave aside many promising threads related to our main topics. Among them is the development of a statistical description of ensembles of essentially non-sinusoidal waves whose deterministic evolution we have discussed above. One of the developing directions is "soliton turbulence" for integrable models where substantial progress has been made in kinetic description of soliton ensembles ("soliton gas").[19] In contrast to hydrodynamic turbulence where there are clear sources and sinks of energies providing universal turbulence spectra (such as in the Kolmogorov turbulence), in the integrable models the random soliton ensembles can produce various scenarios of very complex dynamics, but retain dependence on the initial distribution of their parameters. Studies of evolution of the wave field momenta during elementary interactions of solitons carried out within the KdV and mKdV equations[20,21] will certainly be extended for other models, including non-integrable ones.

Another topic to be mentioned is wave dynamics in coupled evolution equations, e.g., KdV and rKdV equations.[68] Recent findings show that two wave trains each described by KdV and modulationally stable when considered in isolation become unstable when coupling is taken into account.[69] The results derived from these model equations containing only quadratic nonlinearity should be treated with a certain caution, because within the framework of more accurate or primitive equations the outcome may be different due to the influence of cubic nonlinearity. In a somewhat related problem of the Zakharov equation for deep water, it has been shown[71] that the modulation instability of a quasi-monochromatic wavetrain is affected by the presence of a second wave which leads to modification of the instability domain. Coupled equations describing vector solitons in plasmas and chains of particles[70] yield a new type of solitary waves—helical solitons. Study of their properties, unusual features of interaction, the role in the energy transport in biomolecules represents intriguing issues which are likely to attract attention in the nearest future.

We were unable to survey significant progress in studies of two-dimensional generalizations of the KdV equation, the Kadomtsev–Petviashvili, Zakharov–Kuznetsov, and other equations which possess multidimensional fully localized solitons, the lumps.[2,3,73] Returning to the area of our main physical example, the oceanic internal waves, we only very superficially mentioned a rich body of field observations and totally passed over laboratory modeling of internal solitons. We expect all these threads we barely mentioned here and some others which were not mentioned to flourish and bring important new results yet in the next decade. We also anticipate progress in derivation and numerical justification of strongly nonlinear evolutional models.

Much more difficult is to forecast, even roughly, the subsequent development in the upcoming decades. What shall a reader see in the Chaos issue dedicated to the 50th aniversary of the journal in 2041? All we dare to predict is a series of "Grand Unifications" (borrowing the terminology from quantum field theory).

The first of them is a much closer intertwining of analytical models, computations, and physical experiments. We believe that, in spite of the increasing prominence of the latter two, model equations will retain their key role as the first step in identifying and qualitative understanding of new phenomena, as well as selection of the most promising future directions of research. We will see increasing numerical efforts applied to both the model evolution equations and primitive equations.

Second is the overlapping of different effects and models in nonlinear wave theory. We have already seen that the long-wave soliton can be transformed to an envelope soliton due to radiation in a rotational system. An opposite process can be represented by super-short (femtosecond) laser impulses which can have a length of only a few carrier wave periods[47] and their description as "envelope waves" becomes insufficient.

Third is the merging of various approaches originated in different areas of nonlinear wave theory. In particular, lack of space forced us to abandon touching a very important class of "autowaves" existing, in particular, in biological media such as nerve fibers and in some chemical reactions (e.g., Ref. 72). They are typically considered separately from quasi-conservative waves like KdV solitons and even from oscillating waves in active media such as laser impulses. We will see development and adaptation of relevant asymptotic methods for description of such processes; in fact, such a development has already started.[74]

Fourth, the progress of experimental techniques with application of new methods and equipment can provide surprising discoveries in nonlinear waves (a good recent example is the development of the method of direct observation of the dispersion relation of surface water waves in a laboratory[45]).

And it is most easy to predict that there will be many unpredictable events in both theory and experiment. We are looking forward to seeing these events and hope that young generation of scientists will bring them forward.

ACKNOWLEDGMENTS

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