Evaluations of Infinite Utility Streams: Pareto-Efficient and Egalitarian Axiomatics*

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Abstract: This investigation focuses on the aggregation of infinite utility streams by social welfare functions. We analyze the possibility of combining Pareto-efficiency and Hammond Equity principles when the feasible utilities for each generation are $[0, 1]$ and the natural numbers. In the latter case, the Hammond Equity ethics can be combined with non-trivial specifications of the Pareto postulate, even through anonymous social welfare functions. As a consequence, any evaluation of infinite utility streams that verifies a mild specification of the Paretian axiom must exert some interference on the affairs of particular generations.

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1 Introduction

The problem of evaluating a stream emerges from some economic problems that have the common characteristic "lacking a natural termination date", such as the optimization of economic growth with streams (of consumption, for example) that extend over an infinite future, or the analysis of infinitely repeated games. The resolution of distributional conflicts among a countably infinite number of generations or periods is subject to intense debate and research. We contribute to qualify the approach to this aggregation problem on the basis of evaluations of the streams using social welfare functions (e.g., Addler (2011) provides a comprehensive, philosophically grounded argument for the use of social welfare functions as a framework for governmental policy analysis).

In research pertaining to intertemporal welfare aggregation, the fundamental Diamond (1965) approach suggests the use of social welfare relations that are continuous with respect to suitable topologies. A classical result of this approach reveals that strongly Paretian welfare relations, continuous in the sup norm, cannot treat all generations equally. We adopt a different position that we call the Basu-Mitra approach, concerned with the possible existence of social welfare functions (SWFs) that evaluate the infinite streams by real numbers. No topological consideration is made in this case. This line of inquiry is inspired by Basu and Mitra (2003), whose main result implies that it is possible to dispense of the continuity axiom in Diamond's impossibility theorem (see also Crespo et al. (2009)).

The resolution of the conflict among infinite generations depends on the specific form of the Pareto criterion that accounts for efficiency as well as the equity-related postulates requested. Another factor that influences the response is the domain of utilities that each generation can possess and, in particular, whether that domain is discrete or not. We call this domain the *feasible utilities* or sometimes *feasible social states*. Although many of the analyses rely on common domains, such as bounded intervals of numbers, the use of discrete sets of feasible utilities is justified by the recognition that human perception is not endlessly fine. It is a natural setting if the utilities have a well-defined smallest unit (as happens with measures of monetary amounts), or if we focus on payoffs of infinitely repeated *finite* games.

Following Basu and Mitra (2003), much has been established about SWFs that verify the Anonymity axiom, as we summarize in our concluding section 4. In particular, the structure of the set of utility streams creates the incompatibility of that egalitarian property with Weak Pareto (see Alcantud (2012a)) or Dominance for example. Dubey and Mitra (2011) explore this issue in depth and prove that the domains of feasible states for which there are SWFs that verify the Weak Pareto and Anonymity axioms are precisely the domains that do not contain a set of the order type of the set of positive and negative integers.

A parallel, similarly exhaustive analysis for other focal equity axioms has not been performed. The Hammond Equity axiom deserves special attention, because it is preponderant in literature on inequality-averse distributions of allocations to both finite and infinite societies (e.g. Hammond (1976); Asheim and Tungodden (2004b); Mariotti and Veneziani (2009)). With this study, we aim to specify what can be achieved by using Paretian numerical assignements that reflect this distributional principle. We also attend to the role of the set of utility streams in tracing what can be done in that respect, though we do not endorse any concrete structure for such a set. We are concerned with two salient domain restrictions, namely $[0, 1]$ and $\mathbb{N}$. Similar to the analysis of anonymous SWFs, we find a generalized incompatibility in the case of $[0, 1]$ that vanishes when we turn to the case of $\mathbb{N}$, except for the strongest version of the Pareto principle. Then we show the applicability of our analysis by exploring its implications with regard to the interfering properties of efficient SWFs. Non-interference is an ethical principle that has been introduced by Mariotti and Veneziani (2009) for the study of fair allocations to finite societies. The generic idea of liberal non-interference is that any individual has the right to make society remain passive in all circumstances that change her or his welfare, provided that the welfare of no other individual is affected. As Mariotti and Veneziani explain, this idea does not embody any egalitarian consideration. However, when coupled with Anonymity and the Pareto axiom, non-interference entails Hammond Equity, even in its restricted form where 'change' means only 'damage'. As a consequence, a surprising new characterization of the leximin criterion arises, which can be exported to the infinite setting (e.g., the case of infinitely lived societies where allocations are made to their generations) by Lombardi and Veneziani (2009, 2012). Mariotti and Veneziani (2011) and Alcantud (2012b) provide further implications of Paretian axioms and non-interference for social welfare criteria. Here we take advantage of our results to prove that if a mild specification of the Paretian axiom is assumed, then any evaluation of the infinite utility streams must interfere with the interest of some generation, in the sense that both under adverse and favorable changes in its endowments (the other endowments remaining unchanged) the social comparison changes against its interests. This reasoning brings to light the tension between responsiveness to the interest of individual generations and liberal respect to the interests of individual generations.

This article is organized as follows: We introduce our setting and present our axioms in section 2, along with some simple relationships among the requirements that we employ. In section 3, we investigate whether the Hammond Equity postulate is compatible with efficient social welfare functions, considering the cases of both $[0,1]$ and $\mathbb{N}$. As an application, we deduce some interfering properties of SWFs that verify the very mild Restricted Weak Dominance axiom. Our conclusions and related results are summarized in section 4.

2 Notation and definitions

Let $\mathbf{X}$ denote a subset of $\mathbb{R}^{\mathbb{N}}$ that represents a domain of utility sequences or infinite-horizon utility streams. We adopt the usual notation for such utility streams: $\mathbf{x} = (x_1, ..., x_n, ...) \in \mathbf{X}$. By $(y)_{con}$, we mean the constant sequence $(y, y, ...)$, $(x, (y)_{con})$ holds for $(x, y, y, y, ...)$, and $(x_1, ..., x_k, (y)_{con}) = (x_1, ..., x_k, y, y, ...)$ denotes an eventually constant sequence. We write $\mathbf{x} \geqslant \mathbf{y}$ if $x_i \geqslant y_i$ for each $i = 1, 2, ...$, and $\mathbf{x} \gg \mathbf{y}$ if $x_i > y_i$ for each $i = 1, 2, ...$ Also, $\mathbf{x} > \mathbf{y}$ means $\mathbf{x} \geqslant \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$. We use the notation $\mathbb{N}^* = \mathbb{N} \cup \{0\}$. Furthermore, l_∞ denotes the set of bounded real-valued infinite sequences.

A *social welfare function* (SWF) is a function $\mathbf{W} : \mathbf{X} \longrightarrow \mathbb{R}$.

2.1 The axioms

We present some axioms on SWFs, beginning with two consequentialist equity axioms. They establish preference for more egalitarian allocations of utilities among generations in different senses.

Axiom 1 accounts for an equity principle: When there is a conflict between two generations, every other generation being as well off, the stream where the least favoured generation is better off must be weakly preferred.

Axiom 1 (*Hammond Equity* [HE]). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are such that $x_j > y_j > y_k > x_k$ for some $j, k \in \mathbb{N}$, and $x_t = y_t$ when $j \neq t \neq k$, then $\mathbf{W}(\mathbf{y}) \geqslant \mathbf{W}(\mathbf{x})$.

Axiom 1 captures a formulation of inequality aversion in line with related proposals like the (weak) Pigou-Dalton transfer principle (Sakai (2006); Hara et al. (2008)) or the strong forms of the transfer principle in Fleurbaey and Michel (2001).

The related Axiom 2 was introduced by Asheim and Tungodden (2004a).

Axiom 2 (*Hammond Equity for the Future* [HEF]). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are such that $\mathbf{x} = (x_1, (x)_{con})$ and $\mathbf{y} = (y_1, (y)_{con})$ $(x_1 > y_1 > y > x)$, then $\mathbf{W}(\mathbf{y}) \geqslant \mathbf{W}(\mathbf{x})$.

This axiom places an ethical restriction on the ranking of streams where the level of utility is constant from the second period on and the present generation is better off than the future: If the sacrifice by the present generation conveys a higher utility for all future generations, then such trade is weakly preferred. Asheim and Tungodden (2004a) and Asheim et al. (2007), Section 4.3, explain that it is a very weak equity condition –such that with certain consistency requirements on the social preferences, the "condition HEF is much weaker and more compelling than the standard 'Hammond Equity' condition"– that can be endorsed from both egalitarian and utilitarian points of view.

In the preceding axioms, when $\mathbf{W}(\mathbf{y}) > \mathbf{W}(\mathbf{x})$ replaces $\mathbf{W}(\mathbf{y}) \geqslant \mathbf{W}(\mathbf{x})$, we refer to HE^+ and HEF^+, respectively. Property HE^+ is used by d'Aspremont and Gevers (1977), who adopt the term *extremist equity*.

In addition, we intend to account for some kind of efficiency. In this sense the strongest axiom we deal with is the following:

Axiom 3 (*Strong Pareto* [SP]). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $\mathbf{x} > \mathbf{y}$, then $\mathbf{W}(\mathbf{x}) > \mathbf{W}(\mathbf{y})$.

The next efficiency axioms are implied by Strong Pareto:

Axiom 4 (*Monotonicity* [MON]). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $\mathbf{x} > \mathbf{y}$ then $\mathbf{W}(\mathbf{x}) \geqslant \mathbf{W}(\mathbf{y})$.

Axiom 5 (*Weak Pareto* [WP]). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $\mathbf{x} \gg \mathbf{y}$, then $\mathbf{W}(\mathbf{x}) > \mathbf{W}(\mathbf{y})$.

Monotonicity is regarded as a necessary condition for efficiency. A relaxed version of WP is the *Restricted Weak Pareto* axiom (RWP), that is, the property that states that if $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are eventually constant sequences and

$\mathbf{x} \gg \mathbf{y}$, then $\mathbf{W}(\mathbf{x}) > \mathbf{W}(\mathbf{y})$. Other, weaker versions of the Strong Pareto include the following:

Axiom 6 (*Weak Monotonicity* [WM]). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $\mathbf{x} \gg \mathbf{y}$, then $\mathbf{W}(\mathbf{x}) \geqslant \mathbf{W}(\mathbf{y})$.

Axiom 7 (*Weak Dominance* [WD]). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, and there is $j \in \mathbb{N}$ such that $x_j > y_j$, and $x_i = y_i$ for all $i \neq j$, then $\mathbf{W}(\mathbf{x}) > \mathbf{W}(\mathbf{y})$.

A relaxed version of WD is *Restricted Weak Dominance* (RWD), which is the property that states that if $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are eventually constant sequences and there is $j \in \mathbb{N}$ such that $x_j > y_j$, and $x_i = y_i$ for all $i \neq j$, then $\mathbf{W}(\mathbf{x}) > \mathbf{W}(\mathbf{y})$. The conjunction of WP and MON is a reasonable efficiency property for a social evaluation.

Note that SP $\Rightarrow$ MON $\Rightarrow$ WM, SP $\Rightarrow$ WP $\Rightarrow$ WM, and SP $\Rightarrow$ WD. In prior literature, the combination of WP and WD has been referred to as Partial Pareto, whereas the conjunction of WM and WD has been referred to as Dominance.

Remark 1 *Suppose that $\boldsymbol{W}$ is a SWF on either $\boldsymbol{X} = l_\infty$ or $\boldsymbol{X} = Y^{\mathbb{N}}$ with $Y \subseteq \mathbb{R}$ order-dense.*

If $\boldsymbol{W}$ verifies WD and HE, then $\boldsymbol{W}$ must verify HE$^+$ too. To prove this assertion, take $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$, such that $x_j > y_j > y_k > x_k$ for some $j, k \in \mathbb{N}$, and $x_t = y_t$ when $j \neq t \neq k$. We generate $\boldsymbol{z} \in \boldsymbol{X}$ such that $z_t = x_t$ when $t \neq k$, and $y_k > z_k > x_k$. Then WD implies $\boldsymbol{W}(\boldsymbol{z}) > \boldsymbol{W}(\boldsymbol{x})$, and the conclusion follows because $\boldsymbol{W}(\boldsymbol{y}) \geqslant \boldsymbol{W}(\boldsymbol{z})$ under HE.

Similarly, if $\boldsymbol{W}$ is Weakly Paretian and it verifies HEF, then $\boldsymbol{W}$ must verify HEF$^+$ too. To prove it, take $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$ such that $\boldsymbol{x} = (x_1, (x)_{con})$ and $\boldsymbol{y} = (y_1, (y)_{con})$ with $x_1 > y_1 > y > x$. There are z_1, z such that $y_1 > z_1 > y > z > x$. Now $\boldsymbol{W}(\boldsymbol{y}) > \boldsymbol{W}(z_1, (z)_{con})$ by WP, and $\boldsymbol{W}(z_1, (z)_{con}) \geqslant \boldsymbol{W}(x_1, (x)_{con})$ by HEF.

Finally, the Anonymity axiom is the usual equal treatment of all generations postulate.

Axiom 8 (*Anonymity* [AN]). For all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, if there exist $i, j \in \mathbb{N}$ such that $x_i = y_j$ and $x_j = y_i$, and for $k \in \mathbb{N} - \{i, j\}$, $x_k = y_k$, then $\mathbf{W}(\mathbf{x}) = \mathbf{W}(\mathbf{y})$.

2.2 Relationships and other auxiliary results

We proceed to state some relationships between HE and HEF under efficiency properties:

Lemma 1 *Any HE and Monotonic SWF satisfies HEF. Also, if either* $\boldsymbol{X} = l_\infty$ *or* $\boldsymbol{X} = Y^{\mathbb{N}}$ *with* $Y \subseteq \mathbb{R}$ *order-dense, then HE plus WP entail HEF*$^+$*, and HE plus WM entail HEF.*

Proof:

Asheim et al. (2012), Proposition 3, states a result similar to the first statement for social welfare relations. Its proof is direct and can be reproduced here.

Suppose that $\mathbf{W}$ is a SWF on either $\mathbf{X} = l_\infty$ or $\mathbf{X} = Y^{\mathbb{N}}$ with $Y \subseteq \mathbb{R}$ order-dense, and that $\mathbf{W}$ agrees with HE and WP (resp., WM). In order to check that $\mathbf{W}$ satisfies HEF$^+$ (resp., HEF), take $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, such that $\mathbf{x} = (x_1, (x)_{con})$ and $\mathbf{y} = (y_1, (y)_{con})$ with $x_1 > y_1 > y > x$. There are z_1, z_2 such that $y > z_1 > z_2 > x$.

Define $\mathbf{z} = (z_1, z_2, x, x, x, ...)$, such that $\mathbf{y} \gg \mathbf{z}$; by WP (resp., WM) we obtain $\mathbf{W}(\mathbf{y}) > \mathbf{W}(\mathbf{z})$ (resp., $\mathbf{W}(\mathbf{y}) \geqslant \mathbf{W}(\mathbf{z})$). Because $x_1 > z_1 > z_2 > x = x_2$ and $x_i = z_i$ for each $i \geqslant 2$, HE yields $\mathbf{W}(\mathbf{z}) \geqslant \mathbf{W}(\mathbf{x})$, and therefore $\mathbf{W}(\mathbf{y}) > \mathbf{W}(\mathbf{x})$ (resp., $\mathbf{W}(\mathbf{y}) \geqslant \mathbf{W}(\mathbf{x})$). □

Combining MON with reinforcements of HEF or HE is not complicated, as the next example shows.

Example 1 *The Rawlsian criterion* $\boldsymbol{W}_R(\boldsymbol{x}) = inf\,\{x_i : i = 1, 2, 3, ...\}$ *satisfies a reinforced version of MON (but not WD), Anonymity, HEF*$^+$*, and Hammond Equity.*

We end this subsection with a technical result that we use subsequently.

Lemma 2 *Suppose that* $\boldsymbol{W} : Y^{\mathbb{N}} \longrightarrow \mathbb{R}$ *(*$Y \subseteq \mathbb{R}$ *order-dense) satisfies HE and MON.*

(a) If $\mathbf{x}, \mathbf{y} \in \boldsymbol{X}$ *are such that* $x_j > y_j > y_k > x_k$ *for some* $j, k \in \mathbb{N}$ *and* $y_t \geqslant x_t$ *when* $j \neq t \neq k$*, then* $\boldsymbol{W}(\mathbf{y}) \geqslant \boldsymbol{W}(\mathbf{x})$ *.*

(b) If we further assume $y_s > x_s$ *for some* $j \neq s \neq k$ *and SP, then* $\boldsymbol{W}(\mathbf{y}) > \boldsymbol{W}(\mathbf{x})$ *.*

Proof:

Pick $\mathbf{z} \in \mathbf{X}$ such that $z_t = x_t$ when $j \neq t \neq k$, $z_j = y_j$, $z_k = y_k$.

Using MON, we obtain $\mathbf{W}(\mathbf{y}) \geqslant \mathbf{W}(\mathbf{z})$. If case (b) holds, then SP entails $\mathbf{W}(\mathbf{y}) > \mathbf{W}(\mathbf{z})$. In each instance, the conclusion follows because HE yields $\mathbf{W}(\mathbf{z}) \geqslant \mathbf{W}(\mathbf{x})$. □

3 Hammond Equity and efficient Social Welfare Functions

In this section, we show that the solution to the problem of combining the ethics of the Hammond Equity principle with efficiency under the Basu-Mitra approach depends on the domain of utility streams. In subsection 3.1, we show that the issue has been partially elucidated when the domain is $\mathbf{X} = [0,1]^{\mathbb{N}}$, and we complete the corresponding study. An analysis of the case where $\mathbf{X} = Y^{\mathbb{N}}$ with $Y = \mathbb{N}^*$ follows in subsection 3.2. We emphasize that for virtually unrestricted domains of feasible utilities, the Strong Pareto axiom is incompatible with HE under the Basu-Mitra approach as detailed in Theorem 1. Therefore, the negative conclusion in Basu and Mitra (2003), Theorem 1, remains valid when a consequentialist equity axiom such as HE replaces the procedural axiom AN.

3.1 The domain restriction $\mathbf{X} = [0,1]^{\mathbb{N}}$

For this subsection, we fix $\mathbf{X} = [0,1]^{\mathbb{N}}$. In this setting, Theorem 1 of Banerjee (2006) states that there is no SWF satisfying axioms HEF and WD. From this result and Lemma 1, we can easily deduce that there is no SWF on $[0,1]^{\mathbb{N}}$ that verifies HE, WM, and WD. Proposition 1 proves that a stronger incompatibility appears, because the Hammond Equity postulate is not compatible with a Restrictedly Weakly Dominant SWF either.

Proposition 1 *There are no SWFs on* $\boldsymbol{X} = [0,1]^{\mathbb{N}}$ *that verify both HE and RWD.*

Proof:

By way of contradiction, let $\mathbf{W} : [0,1]^{\mathbb{N}} \longrightarrow \mathbb{R}$ be HE and RWD. For each $0 < x < \frac{1}{4}$, we let $L(x) := \mathbf{W}(2x, x, 0, 0, ...)$ and $R(x) := \mathbf{W}(\frac{1+x}{2}, x, 0, 0, ...)$, such that $I(x) := (L(x), R(x))$ is nonempty because $\mathbf{W}$ is RWD.

In addition, $\frac{1}{4} > y > x > 0$ implies $I(x) \cap I(y) = \varnothing$, because

$$L(y) = \mathbf{W}(2y, y, 0, 0, ...) \geqslant \mathbf{W}(\frac{1+x}{2}, x, 0, 0, ...) = R(x)$$

by application of HE to $\frac{1+x}{2} > 2y > y > x$. This implication is impossible, because an uncountable number of different rational numbers are assigned. $\square$

As to the question of the possible compatibility between WP and HE, it has been solved in the negative in Alcantud (2012a).

3.2 The domain restriction $\mathbf{X} = Y^{\mathbb{N}}$, $Y = \{0, 1, 2, ...\}$

We investigate if it is possible to reconcile any version of Hammond Equity with WD (or stronger axioms) under the Basu-Mitra perspective and the assumption $\mathbf{X} = Y^{\mathbb{N}}$ with $Y = \{0, 1, 2, ...\}$.

In Theorem 1, we show that the incompatibility of SP and HE when $\mathbf{X} = [0, 1]^{\mathbb{N}}$ persists in the case in which $\mathbf{X} = \bar{Y}^{\mathbb{N}}$ and $\bar{Y}$ has enough elements as to make the Hammond Equity principle meaningful, even if these elements are not natural numbers.

Theorem 1 *If* $\bar{Y} \subseteq \mathbb{R}$ *and* $|\bar{Y}| \geqslant 4$*, there are no SWFs on* $\boldsymbol{X} = \bar{Y}^{\mathbb{N}}$ *that verify both HE and SP.*

Proof:

As a preliminary technical step, we need to produce an uncountable collection $\{E_i\}_{i \in I}$ of infinite proper subsets of $\mathbb{N}$ with two properties. First, for all $i, j \in I$ such that $i < j$, $E_i \subsetneq E_j$ and $E_j - E_i$ is infinite. Second, there exists q such that $q \in E_i$ for each index $i \in I$. To justify that such collection exists, we take an enumeration of the rational numbers in $(0, 1)$, namely, $\mathbb{Q} \cap (0, 1) = \{r_1, r_2, ...\}$, and then we set $I = (r_1, 1)$ and $E(i) = \{n \in \mathbb{N} : r_n < i\}$ for each $i \in I$ so that $q = r_1 \in E(i)$ for each $i \in I$.

Now we proceed by contradiction. Assume that $\mathbf{W}$ is a SWF on $\mathbf{X}$ that verifies HE and SP. To simplify the notation, we assume without loss of generality that $\{0, 1, 2, 3\} \subseteq \bar{Y}$. Define the following two utility streams associated with each $i \in I$:

$$r(i)_p = \begin{cases} 1 & \text{if } p \in E_i,\ p \neq q \\ 3 & \text{if } p = q \\ 0 & \text{otherwise} \end{cases}$$

$$l(i)_p = \begin{cases} 1 & \text{if } \; p \in E_i, \; p \neq q \\ 2 & \text{if } \; p = q \\ 0 & \text{otherwise} \end{cases}$$

For each $i \in I$, the open interval $(\mathbf{W}(l(i)), \mathbf{W}(r(i)))$ is not empty by SP, so it contains a rational number. We intend to check that the intervals associated with different indices $i, j \in I$ are disjoint, that is, that $j < i \Rightarrow \mathbf{W}(l(i)) > \mathbf{W}(r(j))$. This scenario produces the desired contradiction, because an uncountable number of distinct rational numbers is obtained.

Fix $k \in E_i - E_j$. We claim that Lemma 2 (b) applies to coordinates q and k of $l(i)$ and $r(j)$. Observe that $3 = r(j)_q > 2 = l(i)_q > 1 = l(i)_k > 0 = r(j)_k$. Also, when $q \neq p \neq k$, we have: $l(i)_p = r(j)_p$ if either $p \in E_i \cap E_j$ or $p \notin E_i \cup E_j$, and $l(i)_p = 1 > 0 = r(j)_p$ for every $p \in E_i$, $p \notin E_j$ (recall that there are an infinite number of elements in $E_i - E_j$). This ends the argument. $\square$

Despite this negative result, Theorem 2 assures that the conjunction of WP and WD can be combined with HE$^+$ even in the presence of Anonymity. We present this result for the sake of completeness, but it has a limited theoretical interest, because our construction must contradict MON: The combination of WD and MON entails SP, which is incompatible with AN. To prove Theorem 2, we provide the following auxiliary result:

Lemma 3 *The function $\nu(n) = \sum_{i=0,1,...,n} \frac{1}{2^i}$ $(n = 0, 1, 2, ...)$ is strictly increasing in n and satisfies: $x > y_2 \geqslant y_1 > z \Rightarrow \nu(y_1) - \nu(z) > \nu(x) - \nu(y_2)$.*

Proof:

Fix $x > y_2 \geqslant y_1 > z$. Some straightforward computations yield

$$\nu(y_1) - \nu(z) = \frac{1}{2^{z+1}} \left(1 + \frac{1}{2} + ... + \frac{1}{2^{y_1 - z - 1}}\right) \geqslant \frac{1}{2^{z+1}}$$

and

$$\nu(x) - \nu(y_2) = \frac{1}{2^{y_2+1}} \left(1 + \frac{1}{2} + ... + \frac{1}{2^{x - y_2 - 1}}\right) < \frac{1}{2^{y_2}}, \text{ because}$$

$$1 + \frac{1}{2} + ... + \frac{1}{2^{x - y_2 - 1}} < 2.$$

Because $y_2 \geqslant z + 1$, the conclusion follows. $\square$

Theorem 2 *If $Y = \{0, 1, 2, ...\}$, there are SWFs on $\boldsymbol{X} = Y^{\mathbb{N}}$ that satisfy HE$^+$, Anonymity, WP, and WD.*

Proof:

We closely follow Mitra and Basu's (2007) proof that there are Anonymous SWFs on $\mathbf{X} = Y^{\mathbb{N}}$ satisfying WP and WD. [1] The binary relation on $\mathbf{X}$ given by $\mathbf{x} \sim \mathbf{y}$, if and only if $x_i = y_i$ eventually is an equivalence relation. The equivalence class of $\mathbf{x}$ is denoted by $[\mathbf{x}]_\sim$. We select an element $g([\mathbf{x}]_\sim)$ from each equivalence class $[\mathbf{x}]_\sim$ in the quotient set $\frac{\mathbf{X}}{\sim}$. For simplicity, we write $g^{\mathbf{x}} = g([\mathbf{x}]_\sim)$, and as usual, $g^{\mathbf{x}} = (g_1^{\mathbf{x}}, g_2^{\mathbf{x}}, ...)$. Thus when $\mathbf{x}, \mathbf{y}$ satisfy $x_i = y_i$ eventually we determine that $g^{\mathbf{x}} = g^{\mathbf{y}}$.

Denote $A_N(\mathbf{x}) = \nu(x_1) + ... + \nu(x_N) - (\nu(g_1^{\mathbf{x}}) + ... + \nu(g_N^{\mathbf{x}}))$ for each $N \in \mathbb{N}$ and $\mathbf{x} \in \mathbf{X}$, and consider the function $h(\mathbf{x}) = lim_{N\to\infty}(A_N(\mathbf{x}))$, which is well defined because $A_N(\mathbf{x})$ is eventually constant (for any fixed $\mathbf{x}$). Then h is clearly Anonymous and Weakly Dominant. We next prove that h satisfies HE$^+$.

If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are such that $x_j > y_j \geqslant y_k > x_k$ for some $j, k \in \mathbb{N}$, and $x_t = y_t$ when $j \neq t \neq k$, our construction entails $g^{\mathbf{x}} = g^{\mathbf{y}}$. Therefore there is an index N_0 such that $A_N(\mathbf{y}) - A_N(\mathbf{x}) = \nu(y_j) - \nu(x_j) + \nu(y_k) - \nu(x_k)$ for each $N > N_0$. Now Lemma 3 yields $A_N(\mathbf{y}) - A_N(\mathbf{x}) > 0$ whenever $N > N_0$ and thus $h(\mathbf{y}) > h(\mathbf{x})$.

Finally, we define the SWF that satisfies our requirements by the expression:

$$\mathbf{L}_{AH}(\mathbf{x}) = \frac{1}{2} \cdot \frac{h(\mathbf{x})}{1 + |h(\mathbf{x})|} + min\{x_1, x_2, ...\}.$$

It is clear that $\mathbf{L}_{AH}$ is Anonymous because so is h. By mimicking Mitra and Basu's argument, we can check that it is WP and WD; the key point is that

$$H(t) := \frac{1}{2} \cdot \frac{t}{1 + |t|}$$

is strictly increasing, with values in $(-\frac{1}{2}, \frac{1}{2})$.

Thus whenever $\mathbf{y} \gg \mathbf{x}$, the fact that $min\{y_1, y_2, ...\} \geqslant min\{x_1, x_2, ...\} + 1$ implies $\mathbf{L}_{AH}(\mathbf{y}) > \mathbf{L}_{AH}(\mathbf{x})$. With the knowledge that $h(x)$ is WD, we can easily demonstrate that $\mathbf{L}_{AH}$ is WD. To prove that $\mathbf{L}_{AH}$ is HE$^+$, we select $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ such that $x_j > y_j > y_k > x_k$ for some $j, k \in \mathbb{N}$, and $x_t = y_t$ when $j \neq t \neq k$. Now $\mathbf{L}_{AH}(\mathbf{y}) > \mathbf{L}_{AH}(\mathbf{x})$ can be enforced through two inequalities:

[1] Their construction fulfils HEF too.

$h(\mathbf{y}) > h(\mathbf{x})$, as we proved previously, such that $H(h(\mathbf{y})) > H(h(\mathbf{x}))$, because H is strictly increasing; and $min\{y_1, y_2, ...\} \geqslant min\{x_1, x_2, ...\}$. $\square$

3.3 Application when $\mathbf{X} = [0,1]^{\mathbb{N}}$: Interfering properties of RWD evaluations

In analyzing the criteria for comparing allocations to a finite society, Hammond (1976) characterized the leximin ordering on the basis of AN, SP, and HE. Mariotti and Veneziani (2009) have proven that under AN and SP, HE is equivalent to a liberal non-interference property called the *Harm Principle* (HP). Thus an alternative characterization of the leximin social ranking can be stated in terms of AN, SP, and HP. Mariotti and Veneziani (2009) emphasize that this fact seems fairly surprising, considering that the HP does not embody any egalitarian consideration, whereas Hammond Equity is a strongly egalitarian property. Extensions of their analysis to the cases of the leximax criterion and infinitely lived societies appear in Mariotti and Veneziani (2011) and Lombardi and Veneziani (2009, 2012). They appeal to the appropriate versions of the Harm Principle and its counterpart, the *Individual Benefit Principle* (IBP): [2]

Axiom HP. Suppose $\mathbf{x}, \mathbf{y} \in \mathbf{X} = [0,1]^{\mathbb{N}}$ are eventually coincident and $\mathbf{x} \succ \mathbf{y}$. Consider two streams $\mathbf{x}', \mathbf{y}' \in \mathbf{X}$ such that for some $i \in \mathbb{N}$, $j \neq i$ implies $x'_j = x_j$ and $y'_j = y_j$. If $x'_i < x_i$ and $y'_i < y_i$, then $x'_i > y'_i$ implies $\mathbf{x}' \succ \mathbf{y}'$.

Axiom IBP. Suppose $\mathbf{x}, \mathbf{y} \in \mathbf{X} = [0,1]^{\mathbb{N}}$ are eventually coincident and $\mathbf{x} \succ \mathbf{y}$. Consider two streams $\mathbf{x}', \mathbf{y}' \in \mathbf{X}$ such that for some $i \in \mathbb{N}$, $j \neq i$ implies $x'_j = x_j$ and $y'_j = y_j$. If $x'_i > x_i$ and $y'_i > y_i$ then $x'_i > y'_i$ implies $\mathbf{x}' \succ \mathbf{y}'$.

In particular, preference continuities support characterizations of infinite extensions of the leximin criterion, on the basis of both HE (e.g. Asheim and Tungodden (2004b)) and adapted versions of HP. [3] Nevertheless, Lombardi

[2] For the respective versions when $\mathbf{X} = [0,1]^n$, $n \in \mathbb{N}$, we just need to suppress the unnecessary restriction to eventually coincident streams.

[3] Asheim and Zuber (2011) characterize strongly anonymous infinite extensions of the leximin criterion in terms of restricted continuity in the topological sense. Their use of the product topology is justified by means of prioritarianism for the worse-off, thus avoiding the appeal to HE too.

and Veneziani (2012), Theorem 3, show that in the evaluation of infinitely long streams by orderings, AN, SP, and preference continuity properties are incompatible with full non-interference (i.e., with the conjunction of HP and IBP). Restricting their analysis to a finite economy, Mariotti and Veneziani (2011) prove that a fully liberal non-interfering view of society leads to a dictatorship when WP is requested. However, Alcantud (2012b) proves that extending the horizon to infinity leads to a striking disappearance of this undesirable implication.

In this subsection, we take advantage of the techniques we have developed to prove a negative result that aligns with that Mariotti and Veneziani (2011) have demonstrated: RWD evaluations of $\mathbf{X} = [0,1]^{\mathbb{N}}$ must interfere with the interest of particular generations, in the sense that both HP and IBP are contradicted. Consider the following Lemma: [4]

Lemma 4 *(F. Maniquet) Let $\succcurlyeq$ be a SWO on $\boldsymbol{X} = [0,1]^{\mathbb{N}}$. If $\succcurlyeq$ verifies AN and HP, then $\succcurlyeq$ verifies HE.*

From Proposition 1 (no SWF on $\mathbf{X}$ verifies RWD and HE) and Lemma 4, we obtain a direct proof that RWD, AN, and HP are incompatible for SWFs; a twin argument proves the dual assertion for IBP. We now proceed to prove that even if we dispense with AN, a SWF that verifies RWD must contradict both HP and IBP.

Proposition 2 *There is no SWF on $\boldsymbol{X} = [0,1]^{\mathbb{N}}$ that verifies HP (resp., IBP) and RWD.*

Proof:

We prove the statement for HP and leave the dual proof for IBP to the reader.[5]

Step 1. If $\mathbf{W}$ on $\mathbf{X}$ verifies RWD, then there are $a, b, c \in (\frac{1}{8}, \frac{1}{2})$ such that $a < b < c$ and $\mathbf{W}(a, c, 0_{con}) < \mathbf{W}(b, b, 0_{con})$.

Suppose the contrary. For each $x \in (\frac{1}{4}, \frac{1}{2})$, we let $l(x) = \mathbf{W}(\frac{x}{2}, x, 0_{con})$ and $r(x) = \mathbf{W}(x, x, 0_{con})$. The open interval $i(x) = (l(x), r(x))$ is nontrivial due to RWD. Now for each $x < y$, $x, y \in (\frac{1}{4}, \frac{1}{2})$, we have $\frac{1}{8} < a = \frac{y}{2} < \frac{1}{4} <$

[4]This result by F. Maniquet was communicated to the authors by R. Veneziani.

[5]We note that the IBP case is simpler. First, recall that under the RWD axiom, HE is contradicted (cf., Proposition 1). Second, this information can similarly prove that IBP is contradicted too, as in Step 2 of our proof.

$b = x < c = y < \frac{1}{2}$, and by the assumption, $\mathbf{W}(x, x, 0_{con}) \leqslant \mathbf{W}(\frac{y}{2}, y, 0_{con}) \Leftrightarrow r(x) \leqslant l(y)$. But this outcome is impossible, because with each $x \in (\frac{1}{4}, \frac{1}{2})$, we would associate a different rational number.

Step 2. Take $\mathbf{W}$ on $\mathbf{X}$, verifying RWD and HP. Then, by Step 1, there exists $a, b, c \in (\frac{1}{8}, \frac{1}{2})$, such that $a < b < c$ and $\mathbf{W}(a, c, 0_{con}) < \mathbf{W}(b, b, 0_{con})$. We next obtain a contradiction.

For each $x \in [0, \frac{1}{8}]$, let $L(x) = \mathbf{W}(x, b, 0_{con})$, and $R(x) = \mathbf{W}(x, c, 0_{con})$. The open interval $I(x) = (L(x), R(x))$ is nontrivial, due to RWD. Thus we can claim $R(x) < L(y)$ if $0 \leqslant x < y \leqslant \frac{1}{8}$. Observe that $\mathbf{W}(b, b, 0_{con}) > \mathbf{W}(a, c, 0_{con})$, $b > \frac{1}{8} \geqslant y$, $a > \frac{1}{8} \geqslant y > x$, and thus HP applies to prove the claim. But this is impossible, because with each $x \in [0, \frac{1}{8}]$, we associate a different rational number. □

4 Summary of results, conclusions, and relation to prior literature

We have produced new arguments to contribute to a persistent debate: In combining equity and efficiency in the evaluation of infinite utility streams by social welfare functions, what properties can be guaranteed? Furthermore, what influence does the choice of the set of feasible utilities have? We conclude that if we are interested in imposing the HE spirit, then the existence of a non-degenerate interval as potential social states obliges at least RWD efficiency to incompatibility, whereas the appeal to $\mathbb{N}^*$ does not (and the problem of existence depends upon the precise version of the Pareto axiom). In particular, and for virtually unrestricted domains of feasible utilities, if we use evaluations of the streams that verify HE, then only restricted versions of Paretianity can be attained.

The following tables summarize results that served to motivate our discussion, and they also support comparisons of the differences in approaches when we vary the feasible utilities. Table 1 refers to the analysis of SWFs that verify AN, and Table 2 details the analysis under HE.

Table 1. Summary of results for domains of utility streams $Y^{\mathbb{N}}$ under AN

	$Y = \mathbb{N}^*$	$Y = [0,1]$
SP	Non-existence $\star$	Non-existence
WP + WD	Existence $\dagger$	Non-existence
WM + WD	Existence	Non-existence $\diamond$
WD	Existence	Existence $\ddagger$

Statement $\star$ is proven in Basu and Mitra (2003) when $|Y| > 1$. All $\dagger$, $\ddagger$ and $\diamond$ appear in Basu and Mitra (2007). We emphasize that the construction proving $\ddagger$ holds in $\mathbf{X} = l_\infty$ and fulfils HEF. The other statements in the table derive from $\diamond$ and $\dagger$.

When $Y = \mathbb{N}^*$, the Rawlsian criterion is MON, WP, HE, and AN, but Basu and Mitra (2007) prove that WP and AN are incompatible when $Y = [0,1]$. Dubey and Mitra (2011) prove that a social welfare function satisfying the Anonymity and Weak Pareto Axioms exists only in those domains that do not contain any set of the order type of the set of positive and negative integers.

Table 2. Summary of results for domains of utility streams $Y^{\mathbb{N}}$ under HE

	$Y = \mathbb{N}^*$	$Y = [0,1]$
SP	Non-existence $\star$	Non-existence
WP	Existence with MON and AN §	Non-existence $\ddagger$
WP + WD	Existence with AN $\diamond$	Non-existence
RWD	Existence with AN	Non-existence $\dagger$

Proposition 1 conveys statement $\dagger$. Case $\ddagger$ is proven in Alcantud (2011a). The Rawlsian criterion proves that § holds. Case $\star$ is proven by Theorem 1, which produces non-existence as long as Y has enough elements as to make the equity axiom meaningful. Case $\diamond$ holds by Theorem 2, which ensures HE^+. The other statements in the table derive from $\diamond$, $\dagger$, and $\ddagger$.

Proposition 1 generalizes Sakamoto (2011), Proposition 1, which states that for SWFs on $\mathbf{X} = [0,1]^{\mathbb{N}}$, a property called Altruistic Equity-2 (or AE-2) and WD are incompatible. The reason is that AE-2 plus WD implies the property called Strong Equity Principle, which is equivalent to HE under WD.

We can also report on the main results that arise when SWFs under the HEF axiom are investigated. For domains of utility streams of the form $[0,1]^{\mathbb{N}}$, Banerjee (2006) proves that a property weaker than WD must be contradicted. Turning to the case $Y^{\mathbb{N}}$ with $Y = \mathbb{N}^*$, Alcantud and García-Sanz (2010), in their Proposition 1, prove that SP can be guaranteed by an explicit construction, and their Theorem 1 proves that PP can be guaranteed by an anonymous construction, as well as that the dictatorship by the second (or third, fourth, ...) generations verifies MON, WP, and HEF.

We stress that the problem of the existence of criteria with certain properties differs from the problem of explicitly describing one such criterion. The need for such a distinction is emphasized by recent contributions, including Zame (2007), Theorem 4′, Lauwers (2010), and Dubey (2011), who respond to a conjecture by Fleurbaey and Michel (2003) about the constructibility of fair evaluations of infinite utility streams. In the words of Jacques Hadamard, these contributions demand that the debate distinguish "between what is *determined* and what can be *described*." [6]

[6] From a letter to E. Borel written in 1905 (Ewald (1996), pages 1084-1085).

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