Title	Surface waves in orthotropic incompressible materials
Author(s)	Destrade, Michel
Publication Date	2001-04-17
Publication Information	DESTRADE, M. (2001) 'Surface waves in orthotropic incompressible materials'. Journal of the Acoustical Society of America, 110 :837-840.
Publisher	Acoustical Society of America
Link to publisher's version	http://dx.doi.org/10.1121/1.1378346
Item record	http://hdl.handle.net/10379/4210
DOI	http://dx.doi.org/10.1121/1.1378346

Surface waves in orthotropic incompressible materials

Michel Destrade

2001

Abstract

The secular equation for surface acoustic waves propagating on an orthotropic incompressible half-space is derived in a direct manner, using the method of first integrals.

I INTRODUCTION

The problem of elastic waves propagating on the free surface of a semi-infinite elastic body is a well-covered research topic, initiated by Rayleigh [1] in his study of seismic waves within the context of classical linear elasticity. For anisotropic crystals, Barnett and Lothe [2] have drawn on the works of Stroh [3] to build a complete theory of surface waves based on an analogy between surface wave propagation and straight line dislocation motion. Extensive coverage and surveys of that topic can be found, for instance, in a textbook by Ting [4].

Recently, there has been some interest [5, 6, 7] in the study of wave propagation in anisotropic materials subjected to the constraint of *incompressibility*. The purpose of the present paper is to establish the secular equation for surface (Rayleigh) waves propagating on the free plane surface of an incompressible orthotropic half-space. A similar problem was solved by Chadwick [8] within the context of finite elasticity: he considered the propagation of small-amplitude surface waves in a finitely deformed incompressible material; the deformation was static and purely homogeneous, and the strain energy function for the incompressible nonlinearly elastic material was such that the deformed body presented orthotropic anisotropy. Following Nair and Sotiropoulos [7], the present article focuses on an orthotropic linearly elastic material for which the usual stress-strain relations are modified to take the incompressibility constraint into account, by adding an isotropic pressure term. These authors have argued that "the assumptions of incompressibility and orthotropy are applicable to several materials such as, for example, polymer Kratons, thermoplastic elastomers, rubber composites when low frequency waves are considered to justify the assumption of material homogeneity, etc." Other studies use these assumptions for the modeling of laminated composites made alternatively with reinforcing (filler) layers and matrix (binder) layers [9], or with stiff fibers and incompressible epoxy matrices [10].

The primary purpose of this paper is to show that the method of first integrals used by Mozhaev [11] to derive, in a rapid and elegant manner, the secular equation for surface waves in (compressible) orthotropic materials, can also be employed in the case of incompressible orthotropic materials. This can be achieved by applying the method of first integrals to a system of second order ordinary differential equations for the components of the *tractions* on surfaces parallel to the free surface, rather than for the components of the mechanical displacement (as in Ref. [11]). In the latter case, the pressure appears in the system of differential equations, whereas in the former case, it does not, and hence the number of unknowns is reduced from four (the pressure and the components of the mechanical displacement) to three (the components of the traction on surfaces parallel to the free surface). Also, the mechanical boundary conditions are easily written, because they correspond to the nullity of these traction components on the free surface of the half-space, and at infinite distance from this surface. A third advantage of this approach is that the assumption of plane strain [7] is not required a priori.

The paper is organized as follows. In Section II, the basic equations governing the propagation of elastic waves in an orthotropic incompressible material are recalled. In Section III, these equations are written for the case of surface acoustic waves. Then a system of six first order differential equations for the displacement and the traction components is derived. Eventually a system of three second order differential equations is found for the traction components. One of these three equations is trivially solved when the boundary conditions are applied. In Section IV, the method of first integrals [12, 11] is applied to the two remaining equations, and the secular equation for surface waves in orthotropic incompressible materials is quickly derived. As a check, the isotropic case is treated and Rayleigh's original equation [1] is recovered. Also, the correspondence between this paper's result and Chadwick's result [8] is shown. Finally in Section V, possible developments for this work are presented.

II PRELIMINARIES

First, the governing equations for an incompressible orthotropic elastic material are recalled. The material axes of the body are denoted by x_1, x_2, and x_3. The equations may be derived from the classical linearized equations of anisotropic elasticity [13] by adding an isotropic pressure term $p\mathbf{1}$ (say) to the nominal stress $\boldsymbol{\sigma}$ (say). Hence, for orthotropic incompressible elastic bodies [7],

$$\begin{aligned}
\sigma_{11} &= -p + C_{11}\epsilon_{11} + C_{12}\epsilon_{22} + C_{13}\epsilon_{33}, \\
\sigma_{22} &= -p + C_{12}\epsilon_{11} + C_{22}\epsilon_{22} + C_{23}\epsilon_{33}, \\
\sigma_{33} &= -p + C_{13}\epsilon_{11} + C_{23}\epsilon_{22} + C_{33}\epsilon_{33}, \\
\sigma_{32} &= 2C_{44}\epsilon_{32}, \quad \sigma_{13} = 2C_{55}\epsilon_{31}, \quad \sigma_{12} = 2C_{66}\epsilon_{12},
\end{aligned} \tag{1}$$

where ϵ's denote the strain components, and C's the elastic constants. The strain components are related to the displacement components u_1, u_2, u_3 through

$$\epsilon_{ij} = (u_{i,j} + u_{j,i})/2 \quad (i, j = 1, 2, 3). \tag{2}$$

Finally, the incompressibility constraint reads

$$u_{1,1} + u_{2,2} + u_{3,3} = 0, \tag{3}$$

and the equations of motion, in the absence of body forces, are written as

$$\sigma_{ij,j} = \rho u_{i,tt} \quad (i = 1, 2, 3), \tag{4}$$

where ρ is the mass density of the material, and the comma denotes differentiation. These are the equations established by Nair and Sotiropoulos [7]. These authors also note that for plane strain deformations, the strain-energy function density is positive definite when the following inequalities are satisfied,

$$C_{66} \geq 0, \quad C_{11} + C_{22} - 2C_{12} \geq 0. \tag{5}$$

III SURFACE WAVES

Here the equations of motion for a surface wave in a semi-infinite body made of an orthotropic incompressible elastic material are established. Attention is restricted to propagating inhomogeneous surface waves which are subsonic with respect to homogeneous body waves. The modelisation of the surface wave follows that of Mozhaev [11]: the plane wave propagates with speed v, wave number k, and corresponding displacement and pressure of the form

$$[u_j(x_1, x_2, x_3), p(x_1, x_2, x_3)] = [U_j(x_2), kP(x_2)]e^{ik(x_1 - vt)} \quad (j = 1, 2, 3), \tag{6}$$

where the U's and P are unknowns functions of x_2 alone. For these waves, the planes of constant phase are orthogonal to the x_1-axis, and the planes of constant amplitude are orthogonal to the x_2-axis. The stress-strain relations (1) reduce to

$$\begin{aligned} t_{11} &= -P + iC_{11}U_1 + C_{12}U_2', \\ t_{22} &= -P + iC_{12}U_1 + C_{22}U_2', \\ t_{33} &= -P + iC_{13}U_1 + C_{23}U_2', \\ t_{32} &= C_{44}U_3', \quad t_{13} = iC_{55}U_3, \quad t_{12} = C_{66}(U_1' + iU_2), \end{aligned} \tag{7}$$

where the prime denotes differentiation with respect to kx_2, and the t's are defined by

$$\sigma_{ij}(x_1, x_2, x_3) = kt_{ij}(x_2)e^{ik(x_1 - vt)} \quad (i, j = 1, 2, 3). \tag{8}$$

The surface $x_2 = 0$ is assumed to be free of tractions, and the mechanical displacement and pressure are assumed to be vanishing as x_2 tends to infinity. These conditions lead to the following boundary conditions,

$$t_{i2}(0) = 0, \quad U_i(\infty) = 0 \quad (i = 1, 2, 3), \quad P(\infty) = 0. \tag{9}$$

Finally, the equations of motion (4) and the incompressibility constraint (3) reduce to

$$\begin{aligned} & it_{11} + t'_{12} = -\rho v^2 U_1, \; it_{12} + t'_{22} = -\rho v^2 U_2, \; it_{13} + t'_{32} = -\rho v^2 U_3, \\ & iU_1 + U'_2 = 0. \end{aligned} \tag{10}$$

Note that a classical approach would be to substitute in this last equations, the expressions obtained earlier for the stress tensor components, which would lead to a system of four second order differential equations for the unknown functions U_1, U_2, U_3, P. Instead, the Stroh formalism is now used to derive a system of six first order differential equations for the components of the displacement and the tractions on the surface x_2 =const. Thus, introducing the notation

$$t_i = t_{i2} \quad (i = 1, 2, 3), \tag{11}$$

and using Eqs. (7)-(10), the system is found as

$$\begin{aligned} & U'_1 = -iU_2 + (1/C_{66})t_1, \quad U'_2 = -iU_1, \quad U'_3 = (1/C_{44})t_3, \\ & t'_1 = (C_{11} + C_{22} - 2C_{12} - \rho v^2)U_1 - it_2, \; t'_2 = -\rho v^2 U_2 - it_1, \; t'_3 = (C_{55} - \rho v^2)U_3. \end{aligned} \tag{12}$$

Now a system of three second order differential equations for t_1, t_2, t_3 is derived as follows. First, differentiation of $(12)_{4-6}$ yields relations between the t''_i and the u'_i, t'_i, or equivalently, using $(12)_{1-3}$ between the t''_i and the u_i, t'_i, t_i. Then, substitution for the u_i by their expression in terms of the t'_i, t_i obtained from $(12)_{4-6}$ is performed. Eventually it is found that the t''_i, t'_i, t_i $(i = 1, 2, 3)$ must satisfy the following equations,

$$\begin{aligned} & (\rho v^2)t''_1 - i(C_{11} + C_{22} - 2C_{12} - 2\rho v^2)t'_2 \\ & \quad +(C_{11} + C_{22} - 2C_{12} - \rho v^2)(1 - \rho v^2/C_{66})t_1 = 0, \\ & (C_{11} + C_{22} - 2C_{12} - \rho v^2)t''_2 + i(C_{11} + C_{22} - 2C_{12} - 2\rho v^2)t'_1 + \rho v^2 t_2 = 0, \\ & C_{44}t''_3 - (C_{55} - \rho v^2)t_3 = 0, \end{aligned} \tag{13}$$

and are subject to the following boundary conditions,

$$t_i(0) = t_i(\infty) = 0 \quad (i = 1, 2, 3). \tag{14}$$

The third differential equation in the system (13) is decoupled from the two others, and can be solved exactly. Taking the boundary conditions $(14)_3$ into account, it is seen that

$$t_3(x_2) = 0, \quad \text{for all } x_2, \tag{15}$$

and hence the motion is a pure mode [14] for the tractions on the surface x_2 = const. Now the coupled system of the two remaining equations may be solved.

IV SECULAR EQUATION

For surface waves in compressible orthotropic materials, Mozhaev [11] applied the method of first integrals to a system of two differential equations for the two nonzero components of the mechanical displacement. Here a similar procedure for the two nonzero components t_1, t_2 of the tractions on the surface x_2 =const. is followed, and the secular equation for surface waves in incompressible orthotropic materials is obtained in a direct manner.

The differential equations $(13)_{1,2}$ for t_1, t_2 are expressed as

$$\begin{aligned} &\xi t_1'' + i(\delta - 2\xi)t_2' - (\delta - \xi)(1 - \xi)t_1 = 0, \\ &(\delta - \xi)t_2'' - i(\delta - 2\xi)t_1' - \xi t_2 = 0, \end{aligned} \tag{16}$$

where ξ and δ are defined by

$$\xi = (\rho v^2)/C_{66}, \quad \delta = (C_{11} + C_{22} - 2C_{12})/C_{66}. \tag{17}$$

The speed given by $\xi = 1$ (that is, $\rho v^2 = C_{66}$) corresponds to the speed of a body (homogeneous) wave propagating in the x_1-direction, and gives therefore an upper bound for the speed of subsonic surface waves. Throughout the rest of paper, it is assumed that the surface wave travels with a speed distinct from that given by $\xi = \delta$ (that is, $\rho v^2 \neq (C_{11} + C_{22} - 2C_{12})/C_{66}$).

Now multiplication of $(16)_1$ by t_1' and $(16)_2$ by t_2', and integration between $x_2 = 0$ and $x_2 = \infty$, yields, using the boundary conditions (14),

$$\xi t_1'(0)^2 - 2i(\delta - 2\xi)\textstyle\int t_1' t_2' = 0, \quad \text{and} \quad (\delta - \xi)t_2'(0)^2 + 2i(\delta - 2\xi)\textstyle\int t_1' t_2' = 0, \tag{18}$$

so that

$$\xi t_1'(0)^2 + (\delta - \xi)t_2'(0)^2 = 0. \tag{19}$$

Similarly, multiplication of $(16)_1$ by $\xi t_1' + i(\delta - 2\xi)t_2$ and $(16)_2$ by $(\delta - \xi)t_2' - i(\delta - 2\xi)t_1$, and integration between $x_2 = 0$ and $x_2 = \infty$, yields

$$\begin{aligned} &\xi^2 t_1'(0)^2 + 2i(\delta - 2\xi)(\delta - \xi)(1 - \xi)\textstyle\int t_1 t_2 = 0, \\ &\qquad \text{and} \quad (\delta - \xi)^2 t_2'(0)^2 - 2i(\delta - 2\xi)\xi\textstyle\int t_1 t_2 = 0, \end{aligned} \tag{20}$$

so that

$$\xi^3 t_1'(0)^2 + (\delta - \xi)^3(1 - \xi)t_2'(0)^2 = 0. \tag{21}$$

Eqs.(19) and (21) form a trivial system of two equations for the unknowns $t_1'(0)^2$ and $t_2'(0)^2$, whose determinant must be zero:

$$\xi(\delta - \xi)[(\delta - \xi)^2(1 - \xi) - \xi^2] = 0. \tag{22}$$

It follows that the *secular equation* is given by

$$(\delta - \xi)^2(1-\xi) = \xi^2, \quad \text{i.e.} \quad (C_{11}+C_{22}-2C_{12}-\rho v^2)^2(C_{66}-\rho v^2) = C_{66}(\rho v^2)^2. \tag{23}$$

This equation constitutes the main result of the paper: the direct and explicit derivation of the secular equation for subsonic surface waves propagating in a semi-infinite body made of orthotropic incompressible linearly elastic material. It is worth mentioning that this result can be used for other types of anisotropy: Royer and Dieulesaint [15] have indeed proved that with respect to surface waves, results established for the orthotropic case may be applied to 16 different configurations, including cubic, tetragonal, and hexagonal anisotropy.

In order to justify the existence of a real wave speed, the secular equation (23) is expressed as

$$f(\xi) = 0, \quad \text{where} \quad f(\xi) = \xi^2 - (\delta - \xi)^2(1-\xi). \tag{24}$$

As noted earlier, for traveling subsonic surface waves, this secular equation is subject to

$$0 \leq \xi \leq 1. \tag{25}$$

Within this range, it is easy to prove that f is a monotonic increasing function of ξ, and that

$$f(0) = -\delta^2, \quad f(1) = 1. \tag{26}$$

It follows that the secular equation has a unique positive root in the interval (25).

For consistency purposes, the main result established in this paper is related to previous studies. First, attention is given to the isotropic limit, when $C_{11} = C_{22} = \lambda + 2\mu$, $C_{12} = \lambda$, $C_{66} = \mu$, where λ and μ are the classical Lamé moduli of elasticity. In this case, the secular equation, written for $\xi = \rho v^2/\mu$, reduces to

$$(4-\xi)^2(1-\xi) = \xi^2, \quad \text{or} \quad \xi^3 - 8\xi^2 + 24\xi - 16 = 0, \tag{27}$$

which is the well-known equation derived by Lord Rayleigh [1], by considering the incompressible limit ($\lambda = \infty$) for an isotropic linear elastic material.

Next, another previous result is put into perspective. Chadwick [8] has adapted the Stroh formalism to the theory of prestressed incompressible non-linearly elastic materials. Considering a material whose stored energy function is such that the body will present orthorhombic anisotropy once it has been subjected to a large pure homogeneous deformation, he obtained the secular equation for surface waves propagating in a principal direction as

$$[2(B + C - \overline{\sigma}) - \rho v^2][C(A - \rho v^2)]^{1/2} = (C - \overline{\sigma})^2 - C(A - \rho v^2), \tag{28}$$

where A, B, C are constants defined in terms of the strain energy, initial pressure, and initial stretch ratios, and $\overline{\sigma}$ is the normal stress applied on the surface $x_2 = 0$. When this surface is free of tractions, $\overline{\sigma} = 0$ and after squaring, Eq. (28) reduces to

$$(2B + C - A - \eta^2)^2(C - \eta^2) = C(\eta^2)^2, \tag{29}$$

where $\eta^2 = C - A + \rho v^2$. This equation may be formally compared to Eq. $(23)_2$, where η^2, C, and $2B - A$ play the role of ρv^2, C_{66}, and $C_{11} + C_{22} - C_{66} - 2C_{12}$, respectively.

Finally, Nair and Sotiropoulos [6] have obtained an *implicit* form of the secular equation for surface waves propagating in a monoclinic incompressible material. By taking the elastic coefficients C_{16} and C_{26} to be zero in their analysis, the reader may check that the explicit secular equation (23) is recovered.

V DISCUSSION

The secular equation for surface waves on an incompressible orthotropic half-space was derived directly. Hence it has been shown that a powerful method presented by Mozhaev [11], but which seems to have remained unnoticed, can be adapted to take the constraint of incompressibility into account.

For monoclinic or triclinic materials, the method of first integrals cannot be applied in the case of a three dimensional displacement. As demonstrated by Mozhaev [11], it leads to a trivial system of 18 equations for 18 unknowns, but the rank of the system turns out to be 17 at most, a fact which appears to have been overlooked by the author.

However, for *plane strain* deformations, some further results may be established. For instance, Sotiropoulos and Nair [5] have studied the reflection of plane elastic waves from a free surface in incompressible monoclinic materials with plane of symmetry at $x_3 = 0$, and Nair and Sotiropoulos [6] have considered interfacial waves with an interlayer in the same type of materials. In particular, they derived the secular equation for surface (Rayleigh) waves in an implicit form. The first integrals method makes it possible to write the secular equation in explicit form, as is proved in a forthcoming article. Possibly, interfacial (Stoneley) waves may also be investigated.

References

[1] Lord Rayleigh, " On waves propagated along the plane surface of an elastic solid," Proc. R. Soc. London, Sect. A. **17**, 4–11 (1885).

[2] D.M. Barnett and J. Lothe, “Free surface (Rayleigh) waves in anisotropic elastic half-spaces: The surface impedance method,” Proc. R. Soc. London, Sect. A. **402**, 135–152 (1985).

[3] A.N. Stroh, “Steady state problems in anisotropic elasticity,” J. Math. Phys. **41**, 77–103 (1962).

[4] T.C.T. Ting, “Anisotropic elasticity: theory and applications,” (Oxford University Press, New York, 1996).

[5] D.A. Sotiropoulos and S. Nair, “Elastic waves in monoclinic incompressible materials and reflection from an interface,” J. Acoust. Soc. Am. **105**, 2981–2983 (1999).

[6] S. Nair and D.A. Sotiropoulos, “Interfacial waves in incompressible monoclinic materials with an interlayer,” Mechs. Mat. **31**, 225–233 (1999).

[7] S. Nair and D.A. Sotiropoulos, “Elastic waves in orthotropic incompressible materials and reflection from an interface,” J. Acoust. Soc. Am. **102**, 102–109 (1997).

[8] P. Chadwick, “The application of the Stroh formalism to prestressed elastic media,” Maths. Mechs. Sol. **97**, 379–403 (1997).

[9] A.N.Guz’ and I.A. Guz’, “On the theory of stability of laminated composites,” Int. Appl. Mechs. **35**, 323–329 (1999).

[10] M. Sutcu, “Orthotropic and transversely isotropic stress-strain relations with built-in coordinate transformation,” Int. J. Solids Structures **29**, 503–518 (1992).

[11] V.G. Mozhaev, “Some new ideas in the theory of surface acoustic waves in anisotropic media,” IUTAM Symposium on anisotropy, inhomogeneity and nonlinearity in solids (D.F. Parker and A.H. England, eds.), 455–462 (Kluwer, Holland 1994).

[12] M.Y. Yu, “Surface polaritons in nonlinear media,” Phys. Rev. A **28**, 1855–1856 (1987).

[13] A. E. H. Love, “A treatise on the mathematical theory of elasticity,” (Cambridge University Press, England, 1927).

[14] P. Chadwick, “The existence of pure surface modes in elastic materials with orthorhombic symmetry,” J. Sound Vibr. **47**, 39–52 (1976).

[15] D. Royer and E. Dieulesaint, “Rayleigh wave velocity and displacement in orthorhombic, tetragonal, and cubic crystals,” J. Acoust. Soc. Am. **76**, 1438–1444 (1984).