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EVERYONE KNOWS THAT SOMEONE KNOWS: QUANTIFIERS OVER EPISTEMIC AGENTS

PAVEL NAUMOV AND JIA TAO

Abstract. Modal logic S5 is commonly viewed as an epistemic logic that captures the most basic properties of knowledge. Kripke proved a completeness theorem for the first-order modal logic S5 with respect to a possible worlds semantics. A multiagent version of the propositional S5 as well as a version of the propositional S5 that describes properties of distributed knowledge in multiagent systems has also been previously studied. This article proposes a version of S5-like epistemic logic of distributed knowledge with quantifiers ranging over the set of agents, and proves its soundness and completeness with respect to a Kripke semantics.

§1. Introduction. Several propositional modal logics including S5 were first proposed by Lewis and Langford [10]. The study of first-order modal logics was initiated in Barcan [2] by introducing a logical principle that connects modalities and quantifiers: $\forall x \Box\phi \to \Box\forall x\, \phi$. This principle is commonly referred to as the Barcan formula. Any propositional modal logic such as T, S4, etc. could be transformed into a first-order modal logic by extending the language and adding standard axioms and inference rules for quantifiers. Generally speaking, the Barcan formula is not provable in such systems. Thus, it is common to consider versions of predicate modal logics with and without the Barcan formula. Sometimes, the converse Barcan formula is considered as well [7]. An exception to this general rule is the modal logic S5. Prior showed that the Barcan formula is provable in the predicate version of this logical system [12].

Kripke proved the completeness of first-order S5 with respect to a possible world semantics [9]. Cresswell developed a technique for proving the completeness of first-order modal logics with respect to classes of Kripke models with constant domains [4]. Among other logical systems, his technique is also applicable to S5. Fine investigated a "second-order" version of S5 with quantifiers ranging over propositions [6].

Propositional modal logic S5, especially the multiagent version of this system, is often viewed as the default epistemic logic. Many epistemology-focused extensions of S5 have been proposed before. Of particular interest to us is an extension of S5 that captures properties of distributed knowledge [5, p.73]. This logical system investigates a modality $\Box_A\phi$ which stands for "a group of agents A has distributed knowledge of statement ϕ". Informally, a group of agents knows a statement distributively if the statement follows from the combination of the information available to the agents in this group. An example of a universal property of distributed knowledge captured in this system is $\Box_{a,b}\phi \wedge \Box_{b,c}\psi \to \Box_{a,b,c}(\phi \wedge \psi)$.

It states[1] that if agents a and b distributively know ϕ, and agents b and c distributively know ψ, then agent a, b, and c together distributively know $\phi \wedge \psi$.

In this article we add quantifiers over agents to the language of the logic of distributed knowledge. For example, in the language of our logical system one can write statement $\forall x\,(\Box_x \phi \to \Box_x \psi)$. This statement means that every agent who knows statement ϕ also knows statement ψ. It is different, for instance, from the statement $\forall x\,\Box_x \phi \to \forall x\,\Box_x \psi$, which says that if every agent knows ϕ, then every agent must know ψ. Another example of a statement in our language is $\forall x\,\forall y\,(\Box_{x,y}\phi \to \Box_x \phi \vee \Box_y \phi)$. It means that "if any two agents know statement ϕ distributively, then at least one of them must know ϕ alone". We assume that quantifiers range over a domain of agents specified in each model of our logical system.

An example of a universally true formula in our language is $\forall x\,(\Box_x \exists y\,\Box_y \phi \to \Box_x \phi)$, where variable y does not occur in formula ϕ. Informally, this statement means "if agent x knows that somebody knows ϕ, then agent x herself knows ϕ". We show that this statement is derivable in our logical system in Lemma 3.

The situation with the Barcan formula for quantifiers over epistemic agents is perhaps unexpected. As we show in Section 5, the Barcan formula is not true in its most general form: $\forall x\ \Box_A \phi \to \Box_A \forall x\ \phi$. However, this formula is true under the restriction $x \notin A$. We call this modified formula *the restricted Barcan formula*.

The main technical contribution of this article is a sound and complete logical system for reasoning about distributed knowledge with quantifiers over agents. This logical system consists of axioms and inference rules of propositional logic of distributed knowledge, and quantifier axioms and an inference rule similar to those in the predicate logic. The restricted Barcan formula is provable from these axioms.

Our proof of the completeness is built on several previous works. We adopt the derivation of the Barcan formula from Prior [12], we use a simplified version of C-forms from Cresswell's proofs [4] of completeness of constant-domain first-order modal logics, and we employ Sahlqvist's "unravelling" technique [13] to construct a multi-agent canonical Kripke model. Although not based on it, our work is also related to Fitting's article [8] on quantified logic of evidence where he adds quantifiers to the logic of justifications [1].

An alternative way to interpret variables in our logic is to consider them ranging over the set of *viewpoints* of agents rather than the set of agents themselves. For example, Charrier, Ouchet, and Schwarzentruber consider agents positioned on a plane at a certain point (x, y) with agents facing a certain direction θ [3]. Variables in this case could be interpreted as ranging over all possible triples (x, y, θ). Although it might be natural to assume that the number of agents is finite, the number of viewpoints that an agent can have is likely to be infinite. Having this more general setting in mind, in this article we make no restrictions on the cardinality of the domain of agents. Our proof of the completeness theorem yields a Kripke model with an infinite domain of agents just like most other proofs of completeness theorems for logics with quantifiers.

[1] We omit curly brackets when we list elements of a set in the subscript of a modality.

The article is organized as follows. Section 2 and Section 3 define the formal syntax and semantics of our logical system. Section 4 lists axioms and inference rules of our system, states the soundness theorem, and gives two examples of formal proofs in our logical system. Section 5 discusses the validity of the Barcan formula and the converse Barcan formula. Section 6 proves the completeness of our logical system with respect to the semantics introduced in Section 3. Section 7 concludes the article.

§2. Syntax. Throughout this article we assume a fixed countable set V of "variables" and a fixed at most countable set P of "propositions". Next we define[2] the language $\Phi(C)$ of our logical system for any given set of constants C.

Definition 1. *For any set C disjoint with set V, let set $\Phi(C)$ be the minimal set of formulae such that*

1. $P \subseteq \Phi(C)$,
2. *if* $\phi \in \Phi(C)$, *then* $\neg\phi \in \Phi(C)$,
3. *if* $\phi, \psi \in \Phi(C)$, *then* $\phi \to \psi \in \Phi(C)$,
4. *for any finite set* $C' \subseteq C$ *and any finite set* $V' \subseteq V$, *if* $\phi \in \Phi(C)$ *then* $\Box_{C'\cup V'}\phi \in \Phi(C)$,
5. *if* $x \in V$ *and* $\phi \in \Phi(C)$, *then* $\forall x\, \phi \in \Phi(C)$.

The next definition specifies an auxiliary operation of replacement (or substitution) of an element of a set by another element x.

Definition 2. *For any set A and any elements x and t, let*

$$A[t/x] = \begin{cases} (A \setminus \{x\}) \cup \{t\}, & \text{if } x \in A, \\ A, & \text{otherwise.} \end{cases}$$

For example, if element 2 is replaced in the set $\{1, 2, 3\}$ by element 4, then the result, denoted by $\{1, 2, 3\}[4/2]$ is the set $\{1, 3, 4\}$. At the same time, substitution $\{1, 2, 3\}[3/2]$ results in set $\{1, 3\}$. Finally, substitution $\{1, 2, 3\}[4/5]$ does not change the set because $5 \notin \{1, 2, 3\}$.

We now use the above definition to define substitution as an operation on formulae.

Definition 3. *For any formula $\phi \in \Phi(C)$, any variable $x \in V$, and any $t \in C \cup V$, let substitution $\phi[t/x]$ be defined recursively as follows. For each proposition $p \in P$, each variable $y \in V$, each set $A \subseteq C \cup V$, and all formulae $\psi, \chi \in \Phi(C)$,*

1. $p[t/x] = p$,
2. $(\neg\psi)[t/x] = \neg(\psi[t/x])$,
3. $(\psi \to \chi)[t/x] = \psi[t/x] \to \chi[t/x]$,
4. $(\Box_A\psi)[t/x] = \Box_{A[t/x]}(\psi[t/x])$,
5. $(\forall y\, \psi)[t/x]$ *is equal to* $\forall y\, \psi$ *if* $y = x$; *it is equal to* $\forall y\, (\psi[t/x])$ *otherwise.*

As an example, the result of substitution $(\Box_{x,y}\Box_{y,z}\forall y\ \Box_y p)[z/y]$ is formula $\Box_{x,z}\Box_z\forall y\ \Box_y p$.

[2]Modal languages often can be translated into first order languages with quantifiers over worlds. In our case, a natural translation leads to a two-sorted first order language with separate quantifiers for epistemic worlds and agents.

§3. Semantics. In this section we define Kripke-like models of our logical system. Compared to Kripke models for the propositional modal logic S5, our semantics also includes a function α that assigns values (agents) to all constants.

DEFINITION 4. *For any set C, a Kripke model is a tuple $\langle W, \mathcal{A}, \{\sim_a\}_{a\in\mathcal{A}}, \alpha, \pi\rangle$, where*

1. *W is an arbitrary set of "epistemic worlds",*
2. *$\mathcal{A}$ is a set of "agents",*
3. *$\sim_a$ is an ("indistinguishability") equivalence relation for each $a \in \mathcal{A}$,*
4. *$\alpha : C \to \mathcal{A}$ is a function that maps constants into agents,*
5. *$\pi : P \to \mathcal{P}(W)$ is a function that maps propositional variables into sets of epistemic worlds.*

In this article, we write $u \sim_X v$ if $u \sim_x v$ for each $x \in X$. The next definition introduces the update operation on an arbitrary function. This operation changes the value of the function at a single point.

DEFINITION 5.

$$f[x \mapsto w](t) = \begin{cases} w, & \textit{if } t = x, \\ f(t), & \textit{otherwise.} \end{cases}$$

Since formulae in our language might have free variables, the meaning of a formula could only be specified if values are assigned to all free variable in this formula. We represent this assignment by a function ρ that maps variables into agents. For any such function ρ, any epistemic world w and any formula ϕ, the satisfiable relation $(w, \rho) \Vdash \phi$ is specified by the definition below.

DEFINITION 6. *For any given set C, any Kripke model $\langle W, \mathcal{A}, \{\sim_a\}_{a\in\mathcal{A}}, \alpha, \pi\rangle$, and any function $\rho : V \to \mathcal{A}$,*

1. *$(w, \rho) \Vdash p$ if $w \in \pi(p)$,*
2. *$(w, \rho) \Vdash \neg\phi$ if $(w, \rho) \nVdash \phi$,*
3. *$(w, \rho) \Vdash \phi \to \psi$ if $(w, \rho) \nVdash \phi$ or $(w, \rho) \Vdash \psi$,*
4. *$(w, \rho) \Vdash \Box_A \phi$ if $(u, \rho) \Vdash \phi$ for each $u \in W$ such that $w \sim_{\alpha(A\cap C)\cup\rho(A\cap V)} u$,*
5. *$(w, \rho) \Vdash \forall x\, \phi$ if $(w, \rho[x \mapsto a]) \Vdash \phi$ for each $a \in \mathcal{A}$.*

We conclude this section with an auxiliary lemma that connects update and substitution operations. This lemma is used in the proof of the completeness.

LEMMA 1. *$(w, \rho[x \mapsto \alpha(c)]) \Vdash \phi$ iff $(w, \rho) \Vdash \phi[c/x]$.*

PROOF. We prove this statement by induction on the structural complexity of formula ϕ.

1. Suppose that formula ϕ is a proposition p. Then $\phi[c/x] = p$ by Definition 3. Thus, by Definition 6, $(w, \rho[x \mapsto \alpha(c)]) \Vdash \phi$ iff $(w, \rho[x \mapsto \alpha(c)]) \Vdash p$ iff $w \in \pi(p)$ iff $(w, \rho) \Vdash p$ iff $(w, \rho) \Vdash \phi[c/x]$.
2. Suppose that formula ϕ is a proposition p. Then $\phi[c/x] = p$ by Definition 3. Thus, $(w, \rho[x \mapsto \alpha(c)]) \Vdash \phi$ iff $(w, \rho[x \mapsto \alpha(c)]) \Vdash p$; and also $(w, \rho) \Vdash p$ iff $(w, \rho) \Vdash \phi[c/x]$. At the same time, by Definition 6, $(w, \rho[x \mapsto \alpha(c)]) \Vdash p$ iff $w \in \pi(p)$. Again by Definition 6, $w \in \pi(p)$ iff $(w, \rho) \Vdash p$. By combining the above statements, we get $(w, \rho[x \mapsto \alpha(c)]) \Vdash \phi$ iff $(w, \rho) \Vdash p$.

3. Suppose that formula ϕ has the form $\neg\psi$. Thus, by Definition 3, Definition 6, and the induction hypothesis, $(w, \rho[x \mapsto \alpha(c)]) \Vdash \phi$ iff $(w, \rho[x \mapsto \alpha(c)]) \Vdash \neg\psi$ iff $(w, \rho[x \mapsto \alpha(c)]) \nVdash \psi$ iff $(w, \rho) \nVdash \psi[c/x]$ iff $(w, \rho) \Vdash \neg(\psi[c/x])$ iff $(w, \rho) \Vdash (\neg\psi)[c/x]$ iff $(w, \rho) \Vdash \phi[c/x]$.
4. Suppose that formula ϕ has the form $\psi \to \chi$. By Definition 6, statement $(w, \rho[x \mapsto \alpha(c)]) \Vdash \psi \to \chi$ is equivalent to the disjunction of statement $(w, \rho[x \mapsto \alpha(c)]) \nVdash \psi$ and statement $(w, \rho[x \mapsto \alpha(c)]) \Vdash \chi$. By the induction hypothesis, this disjunction is in turn equivalent to the disjunction of statement $(w, \rho) \nVdash \psi[c/x]$ and statement $(w, \rho) \Vdash \chi[c/x]$. By Definition 6, the last disjunction is equivalent to $(w, \rho) \Vdash (\psi[c/x]) \to (\chi[c/x])$, which is equivalent to statement $(w, \rho) \Vdash (\psi \to \chi)[c/x]$ by Definition 3.
5. Suppose that formula ϕ has the form $\Box_A\psi$. By Definition 6, statement $(w, \rho[x \mapsto \alpha(c)]) \Vdash \Box_A\psi$ is equivalent to $(u, \rho[x \mapsto \alpha(c)]) \Vdash \psi$ being true for all $u \in W$ such that $w \sim_{\alpha(A\cap C)\cup\rho[x\mapsto\alpha(c)](A\cap V)} u$. By the induction hypothesis, the last statement is equivalent to $(u, \rho) \Vdash \psi[c/x]$ being true for all $u \in W$ such that $w \sim_{\alpha(A\cap C)\cup\rho[x\mapsto\alpha(c)](A\cap V)} u$. By Definition 2, it is also equivalent to $(u, \rho) \Vdash \psi[c/x]$ being true for all $u \in W$ such that $w \sim_{\alpha(A[c/x]\cap C)\cup\rho(A[c/x]\cap V)} u$, which is equivalent to $(w, \rho) \Vdash \Box_{A[c/x]}(\psi[c/x])$ by Definition 6. Finally, the last statement is equivalent to $(w, \rho) \Vdash (\Box_A\psi)[c/x]$ by Definition 3.
6. Suppose that formula ϕ has the form $\forall y\, \psi$, where $y \neq x$. By Definition 6, statement $(w, \rho[x \mapsto \alpha(c)]) \Vdash \forall y\, \psi$ is equivalent to the claim that $(w, (\rho[x \mapsto \alpha(c)])[y \mapsto a]) \Vdash \psi$ for each $a \in \mathcal{A}$. Since $x \neq y$, by Definition 5, the last statement is equivalent to $(w, (\rho[y \mapsto a])[x \mapsto \alpha(c)]) \Vdash \psi$ for each $a \in \mathcal{A}$, which, by the induction hypothesis, is equivalent to $(w, (\rho[y \mapsto a])) \Vdash \psi[c/x]$ for each $a \in \mathcal{A}$. The last statement, by Definition 6, is equivalent to $(w, \rho) \Vdash \forall y\, (\psi[c/x])$, which in turn is equivalent to $(w, \rho) \Vdash (\forall y\, \psi)[c/x]$ by Definition 3.
7. Suppose that formula ϕ has the form $\forall x\, \psi$. By Definition 6, statement $(w, \rho[x \mapsto \alpha(c)]) \Vdash \forall x\, \psi$ is equivalent to the claim that $(w, (\rho[x \mapsto \alpha(c)])[x \mapsto a]) \Vdash \psi$ for each $a \in \mathcal{A}$. By Definition 5, the last statement is equivalent to $(w, \rho[x \mapsto a]) \Vdash \psi$ for each $a \in \mathcal{A}$, which is equivalent to statement $(w, \rho) \Vdash \forall x\, \psi$ by Definition 6. The last statement is equivalent to $(w, \rho) \Vdash (\forall x\, \psi)[c/x]$ by Definition 3.

$\dashv$

§4. Axioms. In addition to the propositional tautologies in language $\Phi(C)$, our logical system contains the axioms below. As usual, term t is called free for a variable x in a formula ϕ if for any free occurrence of variable x in the formula ϕ replacing that occurrence by t does not place any variable in the term t into a scope of a quantifier.

1. Truth: $\Box_A\phi \to \phi$,
2. Negative Introspection: $\neg\Box_A\phi \to \Box_A\neg\Box_A\phi$,
3. Distributivity: $\Box_A(\phi \to \psi) \to (\Box_A\phi \to \Box_A\psi)$,
4. Monotonicity: $\Box_A\phi \to \Box_B\phi$, where $A \subseteq B$,
5. Specialization: $\forall x\, \phi \to \phi[t/x]$, where $t \in C \cup V$ is free for variable x in ϕ,

6. Uniformity: $\forall x\,(\phi \to \psi) \to (\phi \to \forall x\,\psi)$, where formula ϕ contains no free occurrences of variable x.

We write $\vdash_C \phi$ if formula ϕ can be derived from the above axioms using the Generalization, the Necessitation, and the Modus Ponens inference rules:

$$\frac{\phi}{\forall x\,\phi}, \qquad \frac{\phi}{\Box_A \phi}, \qquad \frac{\phi \quad \phi \to \psi}{\psi}.$$

For any $X \subseteq \Phi(C)$, we write $X \vdash_C \phi$ if ϕ is provable from the *theorems* of our system formulated in language $\Phi(C)$ combined with an additional set of axioms X using only the Generalization rule and the Modus Ponens rule. We write $\vdash \phi$ and $X \vdash \phi$ when the value of C is clear from the context.

THEOREM 1 (soundness). *For any $\phi \in \Phi(C)$, if $\vdash_C \phi$, then $(w, \rho) \Vdash \phi$ for each Kripke model $\langle W, \mathcal{A}, \{\sim_a\}_{a \in \mathcal{A}}, \alpha, \pi\rangle$, each world $w \in W$, and each function $\rho : V \to \mathcal{A}$.*

The proof of Theorem 1 consists of verifying the soundness of each axiom and each inference rule of our system. Although the language of our system is unique, the above list of axioms is a combination of the axioms of distributed knowledge logic [5, p.73] and the first-order axioms for quantifiers [11, p.62]. The proofs of the soundness of these axioms are no different from those of the soundness of their counterparts in modal and first-order logics.

The next lemma proves a well-known observation that so-called "positive introspection" principle is derivable from the rest of S5 axioms. We use this lemma in the proof of the completeness of our logical system.

LEMMA 2. $\vdash \Box_A \phi \to \Box_A \Box_A \phi$.

PROOF. Note that formula $\neg\Box_A\phi \to \Box_A\neg\Box_A\phi$ is an instance of the Negative Introspection axiom. Thus, $\vdash \neg\Box_A\neg\Box_A\phi \to \Box_A\phi$ by the law of contrapositive in propositional logic. Hence, $\vdash \Box_A(\neg\Box_A\neg\Box_A\phi \to \Box_A\phi)$ by the Necessitation inference rule. Thus, by the Distributivity axiom and the Modus Ponens inference rule,

$$\vdash \Box_A\neg\Box_A\neg\Box_A\phi \to \Box_A\Box_A\phi. \tag{1}$$

At the same time, $\Box_A\neg\Box_A\phi \to \neg\Box_A\phi$ is an instance of the Truth axiom. Thus, $\vdash \Box_A\phi \to \neg\Box_A\neg\Box_A\phi$ by contraposition. Hence, taking into account the following instance of the Negative Introspection axiom $\neg\Box_A\neg\Box_A\phi \to \Box_A\neg\Box_A\neg\Box_A\phi$, one can conclude that $\vdash \Box_A\phi \to \Box_A\neg\Box_A\neg\Box_A\phi$. The latter, together with statement (1), implies the statement of the lemma by the laws of propositional reasoning. $\dashv$

We conclude this section with a proof of the example from the introduction. We assume that $\exists y$ is an abbreviation for $\neg\forall y\,\neg$.

LEMMA 3. $\vdash \forall x\,(\Box_x \exists y\, \Box_y \phi \to \Box_x \phi)$, *where variable y does not occur in formula ϕ.*

PROOF. By the Truth axiom, $\vdash \Box_y\phi \to \phi$. Thus, by contraposition, $\vdash \neg\phi \to \neg\Box_y\phi$. Hence, by the Generalization inference rule, $\vdash \forall y\,(\neg\phi \to \neg\Box_y\phi)$. Then, due to the assumption that variable y does not occur in formula ϕ, by the

Uniformity axiom and the Modus Ponens inference rule, $\vdash \neg\phi \to \forall y\, \neg\Box_y\phi$. Thus, again by contraposition, $\vdash \neg\forall y\, \neg\Box_y\phi \to \phi$. Recall now that $\exists y$ is an abbreviation for $\neg\forall y\, \neg$. Hence, $\vdash \exists y\, \Box_y\phi \to \phi$. Then, by the Necessitation inference rule, $\vdash \Box_x(\exists y\, \Box_y\phi \to \phi)$. Thus, by the Distributivity axiom and the Modus Ponens inference rule, $\vdash \Box_x\exists y\, \Box_y\phi \to \Box_x\phi$. Therefore, $\vdash \forall x\, (\Box_x\exists y\, \Box_y\phi \to \Box_x\phi)$ by the Generalization inference rule. $\dashv$

§5. Restricted Barcan Formula. Barcan proposed a logical principle [2] that connects modalities and quantifiers: $\forall x\, \Box\phi \to \Box\forall x\, \phi$. Prior showed that this principle is derivable in the first-order version of S5 [12]. The natural translation of the Barcan formula into the language of our logical system is

$$\forall x\, \Box_A\phi \to \Box_A\forall x\, \phi. \tag{2}$$

In general, the Barcan formula (2) is not sound with respect to the semantics of our logical system specified in Definition 6. Indeed, consider the following instance of this formula: $\forall x\, \Box_x\Box_x p \to \Box_x\forall x\, \Box_x p$. Informally, this formula is not universally true because the knowledge of statement p by all agents does not imply that all agents know that everyone knows p. More formally, let us consider a Kripke model depicted in Figure 1. It has three epistemic worlds: w, u, and v and two agents a and b. Agent a cannot distinguish worlds w and u, and agent b cannot distinguish worlds u and v. Proposition p is satisfied in worlds w and u only. Let ρ be any function that maps variable x into agent a. That is, $\rho(x) = a$. In Lemma 4 and Lemma 5 we show that $(w, \rho) \Vdash \forall x\, \Box_x\Box_x p$, yet $(w, \rho) \nVdash \Box_x\forall x\, \Box_x p$.

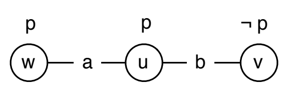

FIGURE 1. $(w, \rho) \nVdash \forall x\, \Box_x\Box_x p \to \Box_x\forall x\, \Box_x p$

LEMMA 4. $(w, \rho) \Vdash \forall x\, \Box_x\Box_x p$.

PROOF. By the definition of the model in Figure 1, we have $(w, \rho) \Vdash p$ and $(u, \rho) \Vdash p$. Thus, $(w, \rho) \Vdash \Box_a\Box_a p$ by Definition 6. Hence, $(w, \rho[x \mapsto a]) \Vdash \Box_x\Box_x p$ by Lemma 1. At the same time, $(w, \rho) \Vdash p$ also implies that $(w, \rho) \Vdash \Box_b\Box_b p$ by Definition 6. Hence, $(w, \rho[x \mapsto b]) \Vdash \Box_x\Box_x p$ by Lemma 1. Finally, $(w, \rho) \Vdash \forall x\, \Box_x\Box_x p$ follows from $(w, \rho[x \mapsto a]) \Vdash \Box_x\Box_x p$ and $(w, \rho[x \mapsto b]) \Vdash \Box_x\Box_x p$ by Definition 6. $\dashv$

LEMMA 5. $(w, \rho) \nVdash \Box_x\forall x\, \Box_x p$.

PROOF. By the definition of the model in Figure 1, we have $(v, \rho) \nVdash p$. Thus, $(u, \rho) \nVdash \Box_b p$ by Definition 6. Hence, $(u, \rho[x \mapsto b]) \nVdash \Box_x p$ by Lemma 1. Thus, $(u, \rho) \nVdash \forall x\, \Box_x p$ by Definition 6. Then, $(w, \rho) \nVdash \Box_a\forall x\, \Box_x p$ by Definition 6. Hence, $(w, \rho[x \mapsto a]) \nVdash \Box_x\forall x\, \Box_x p$ by Lemma 1. Recall that $\rho(x) = a$ by the choice of function ρ. Thus, $\rho[x \mapsto a] \equiv \rho$. Therefore, $(w, \rho) \nVdash \Box_x\forall x\, \Box_x p$. $\dashv$

As we have just seen, the Barcan formula (2) is not universally true for quantifiers over epistemic agents. However, this formula is true under the restriction $x \notin A$.

LEMMA 6. *If* $(w,\rho) \Vdash \forall x\, \Box_A \phi$*, then* $(w,\rho) \Vdash \Box_A \forall x\, \phi$*, for each Kripke model* $\langle W, \mathcal{A}, \{\sim_a\}_{a\in\mathcal{A}}, \alpha, \pi\rangle$*, each world* $w \in W$*, each function* $\rho : V \to \mathcal{A}$*, each formula* $\phi \in \Phi(C)$*, and each variable* $x \notin A$*.*

PROOF. Consider any $u \in W$ such that $w \sim_{\alpha(A\cap C)\cup\rho(A\cap V)} u$. By Definition 6, it suffices to show that $(u,\rho) \Vdash \forall x\, \phi$. Let $a \in \mathcal{A}$ be an arbitrary agent. By Definition 6, it suffices to show that $(u, \rho[x \mapsto a]) \Vdash \phi$.

By Definition 6, assumption $(w,\rho) \Vdash \forall x\, \Box_A \phi$ implies $(w, \rho[x \mapsto a]) \Vdash \Box_A \phi$. Thus, by Definition 6, $(v, \rho[x \mapsto a]) \Vdash \phi$ for each world $v \in W$ such that $w \sim_{\alpha(A\cap C)\cap\rho[x\mapsto a](A\cap V)} v$. Note that $\rho[x \mapsto a](A \cap V) = \rho(A\cap V)$ because $x \notin A$. Hence, $(v, \rho[x \mapsto a]) \Vdash \phi$ for each world $v \in W$ such that $w \sim_{\alpha(A\cap C)\cap\rho(A\cap V)} v$. In particular, $(u, \rho[x \mapsto a]) \Vdash \phi$. ⊣

We call the Barcan formula (2) under the restriction $x \notin A$ *the restricted Barcan formula*. Not only is the restricted Barcan formula true for quantifiers over epistemic agents, but it is also derivable from the axioms of our logical system introduced earlier. We prove this statement in the next lemma. Our proof of this lemma is a re-worked version of the original Prior's proof of derivability of the Barcan formula in the first-order version of S5 [12]. We have explicitly mentioned in the proof where the assumption $x \notin A$ is used.

LEMMA 7 (restricted Barcan formula). $\vdash \forall x\, \Box_A \phi \to \Box_A \forall x\, \phi$*, where* $x \notin A$*.*

PROOF. By the Specialization axiom, $\vdash \forall x\, \Box_A \phi \to \Box_A \phi[x/x]$. Then it follows from Definition 3 that $\vdash \forall x\, \Box_A \phi \to \Box_A \phi$. Thus, by contraposition in propositional logic: $\vdash \neg\Box_A \phi \to \neg\forall x\, \Box_A \phi$. Hence, by the Necessitation inference rule, $\vdash \Box_A(\neg\Box_A \phi \to \neg\forall x\, \Box_A \phi)$. Thus, $\vdash \Box_A\neg\Box_A \phi \to \Box_A\neg\forall x\, \Box_A \phi$ by the Distributivity axiom and the Modus Ponens inference rule. Then, by the Negative Introspection axiom, $\vdash \neg\Box_A \phi \to \Box_A\neg\forall x\, \Box_A \phi$. Hence, $\vdash \neg\Box_A\neg\forall x\, \Box_A \phi \to \Box_A \phi$ by contraposition. Thus, $\vdash \neg\Box_A\neg\forall x\, \Box_A \phi \to \phi$ by the Truth axiom. Then, by the Generalization inference rule, $\vdash \forall x\,(\neg\Box_A\neg\forall x\, \Box_A \phi \to \phi)$. Hence, $\vdash \neg\Box_A\neg\forall x\, \Box_A \phi \to \forall x\, \phi$, by the Uniformity axiom and the Modus Ponens rule; the Uniformity axiom can be applied in this setting because $x \notin A$ due to the assumption of the lemma. Thus, by the Necessitation inference rule, $\vdash \Box_A(\neg\Box_A\neg\forall x\, \Box_A \phi \to \forall x\, \phi)$. Then, by the Distributivity axiom and the Modus Ponens inference rule, $\vdash \Box_A\neg\Box_A\neg\forall x\, \Box_A \phi \to \Box_A\forall x\, \phi$. Hence, by the Negative Introspection axiom, $\vdash \neg\Box_A\neg\forall x\, \Box_A \phi \to \Box_A\forall x\, \phi$. Thus, by contraposition, $\vdash \neg\Box_A\forall x\, \phi \to \Box_A\neg\forall x\, \Box_A \phi$. Then, $\vdash \neg\Box_A\forall x\, \phi \to \neg\forall x\, \Box_A \phi$ by the Truth axiom. Therefore, $\vdash \forall x\, \Box_A \phi \to \Box_A\forall x\, \phi$ by contraposition. ⊣

Sometimes, the converse of the Barcan formula is considered as well [7]. The restricted form of the converse Barcan formula is also sound with respect to our semantics. This could be shown in the same fashion as the proof of Lemma 6. The provability of the converse Barcan formula in our logical system follows from the completeness theorem.

§6. Completeness. The rest of this article focuses on the proof of the completeness of our logical system. As stated in the introduction, this proof is built on prior works by Cresswell [4] and Sahlqvist [13].

6.1. Initial Observations. We start the proof of the completeness by making several technical observations about our formal system. The first of them is a version of the deduction theorem. Although the proof of this theorem closely follows the standard proof of the deduction theorem in the first-order logic, we include the complete proof in this article to show that changing from quantifiers over the elements of a domain to quantifiers over agents does not alter the proof.

Before stating the deduction theorem, recall that we write $X \vdash \phi$ if formula ϕ is provable from the *theorems* of our system combined with an additional set of axioms X using only the Generalization rule and the Modus Ponens rule.

LEMMA 8 (Deduction). *For any set of formulae $X \subseteq \Phi(C)$, any closed formula $\phi \in \Phi(C)$, and any formula $\psi \in \Phi(C)$, if $X, \phi \vdash \psi$, then $X \vdash \phi \to \psi$.*

PROOF. Let $\chi_1, \dots, \chi_n$ be a proof of formula ψ from additional axioms X, ϕ. We prove that $X \vdash \phi \to \chi_i$ by induction on i, where $0 \leq i \leq n$. Let us consider the following cases:

1. If $\vdash \chi_i$ or $\chi_i \in X$, then consider propositional tautology $\chi_i \to (\phi \to \chi_i)$. By the Modus Ponens inference rule, $X \vdash \phi \to \chi_i$.
2. If formula χ_i is derived from χ_j and $\chi_j \to \chi_i$ using the Modus Ponens inference rule, then formulae χ_j and $\chi_j \to \chi_i$ precede formula χ_i in the sequence $\chi_1, \dots, \chi_n$. Thus, by the induction hypothesis, $X \vdash \phi \to \chi_j$ and $X \vdash \phi \to (\chi_j \to \chi_i)$. Consider propositional tautology
$$(\phi \to (\chi_j \to \chi_i)) \to ((\phi \to \chi_j) \to (\phi \to \chi_i)).$$
By applying the Modus Ponens inference rule twice, we can conclude that $X \vdash \phi \to \chi_i$.
3. If formula χ_i has form $\forall x\, \chi_j$ for some $j < i$ and is derived from χ_j by the Generalization rule, then $X \vdash \phi \to \chi_j$ by the induction hypothesis. Hence, $X \vdash \forall x(\phi \to \chi_j)$ by the Generalization rule. Recall that ϕ is a closed formula. Thus, $\vdash \forall x\, (\phi \to \chi_j) \to (\phi \to \forall x\, \chi_j)$ by the Uniformity axiom. Therefore, $X \vdash \phi \to \forall x\, \chi_j$ by the Modus Ponens inference rule.

$\dashv$

The restricted Barcan formula in the form proven in Lemma 7 is easy to understand. Lemma 11 below states the same restricted Barcan formula in the form it is actually used in the proof of the completeness. Proving Lemma 11 from Lemma 7 is not a trivial task in itself. Our proof utilizes auxiliary Lemma 9 and Lemma 10.

LEMMA 9. $\vdash \forall x\, ((\phi \to \forall x\, \phi) \to \psi) \to (\neg\psi \to \forall x\, \phi)$*, where formula ψ contains no free occurrences of variable x.*

PROOF. Note that $\vdash \forall x\, ((\phi \to \forall x\, \phi) \to \psi) \to (((\phi \to \forall x\, \phi) \to \psi)[x/x])$ by the Specialization axiom. Thus, by Definition 3,

$$\vdash \forall x\, ((\phi \to \forall x\, \phi) \to \psi) \to ((\phi \to \forall x\, \phi) \to \psi). \tag{3}$$

At the same time, the formula $((p \to q) \to r) \to (\neg r \to p)$ is a propositional tautology. Thus,

$$\vdash ((\phi \to \forall x\, \phi) \to \psi) \to (\neg\psi \to \phi). \tag{4}$$

From (3) and (4) by the laws of propositional reasoning,

$$\vdash \forall x\, ((\phi \to \forall x\, \phi) \to \psi) \to (\neg\psi \to \phi).$$

By the Generalization inference rule,

$$\vdash \forall x\, (\forall x\, ((\phi \to \forall x\, \phi) \to \psi) \to (\neg\psi \to \phi)).$$

Note that variable x is not free in formula $\forall x\, ((\phi \to \forall x\, \phi) \to \psi)$. Thus, by the Uniformity axiom and the Modus Ponens inference rule,

$$\vdash \forall x\, ((\phi \to \forall x\, \phi) \to \psi) \to \forall x\, (\neg\psi \to \phi).$$

Recall that formula ψ contains no free occurrences of variable x due to the assumption of the lemma. Thus, again by the Uniformity axiom and the laws of propositional reasoning, $\vdash \forall x\, ((\phi \to \forall x\, \phi) \to \psi) \to (\neg\psi \to \forall x\, \phi)$. $\dashv$

LEMMA 10. *$\vdash \forall x\, ((\phi \to \forall x\, \phi) \to \psi) \to (\neg\psi \to \neg\forall x\, \phi)$.*

PROOF. Note that $\vdash \forall x\, ((\phi \to \forall x\, \phi) \to \psi) \to (((\phi \to \forall x\, \phi) \to \psi)[x/x])$ by the Specialization axiom. Thus, by Definition 3,

$$\vdash \forall x\, ((\phi \to \forall x\, \phi) \to \psi) \to ((\phi \to \forall x\, \phi) \to \psi). \tag{5}$$

At the same time, the formula $((p \to q) \to r) \to (\neg r \to \neg q)$ is a propositional tautology. Thus,

$$\vdash ((\phi \to \forall x\, \phi) \to \psi) \to (\neg\psi \to \neg\forall x\, \phi). \tag{6}$$

From (5) and (6) by the laws of propositional reasoning,

$$\vdash \forall x\, ((\phi \to \forall x\, \phi) \to \psi) \to (\neg\psi \to \neg\forall x\, \phi).$$

$\dashv$

LEMMA 11. *$\vdash \forall x\, \Box_A((\phi \to \forall x\, \phi) \to \psi) \to \Box_A\psi$, for each closed formula ψ and each variable $x \notin A$.*

PROOF. By the laws of propositional reasoning, Lemma 9 and Lemma 10 imply $\vdash \forall x\, ((\phi \to \forall x\, \phi) \to \psi) \to \psi$. Thus, by the Necessitation inference rule, $\vdash \Box_A((\forall x\, (\phi \to \forall x\, \phi) \to \psi) \to \psi)$. Hence, by the Distributivity axiom and the Modus Ponens inference rule,

$$\vdash \Box_A\forall x\, ((\phi \to \forall x\, \phi) \to \psi) \to \Box_A\psi. \tag{7}$$

At the same time, by the restricted Barcan formula (Lemma 7) and the assumption $x \notin A$,

$$\vdash \forall x\, \Box_A((\phi \to \forall x\, \phi) \to \psi) \to \Box_A\forall x\, ((\phi \to \forall x\, \phi) \to \psi).$$

Therefore, due to statement (7), by the laws of propositional reasoning,

$$\vdash \forall x\, \Box_A((\phi \to \forall x\, \phi) \to \psi) \to \Box_A\psi.$$

$\dashv$

6.2. Henkin Sets. This section defines Henkin sets for quantifiers over epistemic agents and proves their properties that are used in the proof of the completeness. Although this section does not closely follow Cresswell's work, most of the section is inspired by Cresswell [4].

DEFINITION 7. *A set of closed formulae $H \subseteq \Phi(C)$ is a Henkin set if for any formula $\gamma \in \Phi(C)$ with a single variable x, there is $c \in C$ such that c does not occur in γ and set H contains formula $\gamma[c/x] \to \forall x\, \gamma$.*

LEMMA 12. *For any at most countable set C and any consistent set of closed formulae $X \subseteq \Phi(C)$ there is an at most countable set $C' \supseteq C$ and a consistent Henkin set $H \subseteq \Phi(C')$ such that $X \subseteq H$.*

PROOF. Let $\gamma \in \Phi(C)$ be any propositional formula with a single variable x and $Y \subseteq \Phi(C)$ be any set of formulae. Suppose that $h \notin C$. It suffices to show that if set Y is consistent with respect to $\vdash_C$, then set $Y \cup \{\gamma[h/x] \to \forall x\, \gamma\}$ is consistent with respect to $\vdash_{C\cup\{h\}}$. Suppose the opposite, then $Y \vdash_{C\cup\{h\}} \neg(\gamma[h/x] \to \forall x\, \gamma)$. Consider a derivation of $\neg(\gamma[h/x] \to \forall x\, \gamma)$ from Y. Note that symbol h could be viewed as a variable rather than a constant. More formally, we can select a variable z not used anywhere in this derivation and replace constant h with this variable everywhere in the derivation. Thus, $Y \vdash_C \neg(\gamma[z/x] \to \forall x\, \gamma)$. Furthermore, the following two formulae are propositional tautologies:

$$\neg(\gamma[z/x] \to \forall x\, \gamma) \to \gamma[z/x],$$
$$\neg(\gamma[z/x] \to \forall x\, \gamma) \to \neg\forall x\, \gamma.$$

Thus, $Y \vdash_C \gamma[z/x]$ and $Y \vdash_C \neg\forall x\, \gamma$ by two applications of the Modus Ponens inference rule. The first statement, by the Generalization inference rule, implies that $Y \vdash_C \forall z\, (\gamma[z/x])$. Thus, by the Specialization axiom and the Modus Ponens inference rule, $Y \vdash_C (\gamma[z/x])[x/z]$. Hence, $Y \vdash_C \gamma$ by Definition 3 and because variable z does not occur anywhere in formula γ. Therefore, again by the Generalization inference rule, $Y \vdash_C \forall x\, \gamma$, which together with above observation $Y \vdash_C \neg\forall x\, \gamma$ implies the inconsistency of set Y with respect to $\vdash_C$. $\dashv$

In the rest of the article we assume that $\gamma_1, \gamma_2, \dots$ is an enumeration of all formulae with a single free variable in the language $\Phi(C)$. For any $i > 0$, let x_i denote the free variable in formula γ_i. The statement of the next lemma connects Henkin sets with the modalities.

LEMMA 13. *For each Henkin set $H \subseteq \Phi(C)$, each $A \subseteq C$, and each $\neg\Box_A\phi \in H$, there is a sequence $c_1, c_2, \dots \in C$ such that for each $n \geq 0$,*

$$H \vdash \neg\Box_A((\gamma_n[c_n/x_n] \to \forall x_n\, \gamma_n) \to (\dots((\gamma_1[c_1/x_1] \to \forall x_1\, \gamma_1) \to \phi)\dots)).$$

PROOF. First, note that the statement is true for $n = 0$ due to the assumption $\neg\Box_A\phi \in H$ of the lemma. We define sequence $c_1, c_2, \dots$ recursively. Suppose that we have already defined $c_1, \dots, c_n \in C$ such that

$$H \vdash \neg\Box_A((\gamma_n[c_n/x_n] \to \forall x_n\, \gamma_n) \to (\dots((\gamma_1[c_1/x_1] \to \forall x_1\, \gamma_1) \to \phi)\dots)).$$

Note that $x_{n+1} \notin A$ by the assumption $A \subseteq C$ and formula ϕ is closed by the assumption $\neg\Box_A\phi \in H$ of the lemma. Thus, by the contrapositive of Lemma 11, where ψ in Lemma 11 stands for formula

$$(\gamma_n[c_n/x_n] \to \forall x_n\ \gamma_n) \to (\dots((\gamma_1[c_1/x_1] \to \forall x_1\ \gamma_1) \to \phi)\dots),$$

we have

$$H \vdash \neg\forall x_{n+1}\Box_A((\gamma_{n+1} \to \forall\, x_{n+1}\gamma_{n+1}) \to ((\gamma_n[c_n/x_n] \to \forall x_n\ \gamma_n) \to (\dots ((\gamma_1[c_1/x_1] \to \forall x_1\ \gamma_1) \to \phi)\dots))).$$

Hence, it follows from Definition 7 that there is a constant $c_{n+1} \in C$ such that

$$H \vdash \neg\Box_A((\gamma_{n+1}[c_{n+1}/x_{n+1}] \to \forall\, x_{n+1}\gamma_{n+1}) \to ((\gamma_n[c_n/x_n] \to \forall x_n\ \gamma_n) \to (\dots((\gamma_1[c_1/x_1] \to \forall x_1\ \gamma_1) \to \phi)\dots))).$$

⊣

The last lemma in this section is our version of the standard lemma in modal logic used to create a “child node” in a Kripke model.

LEMMA 14. *Let H be any consistent Henkin subset of $\Phi(C)$. For any $\neg\Box_A\phi \in H$ there is a consistent Henkin set $H' \subseteq \Phi(C)$ such that*

$$\{\neg\phi\} \cup \{\Box_A\psi \mid \Box_A\psi \in H\} \cup \{\neg\Box_A\chi \mid \neg\Box_A\chi \in H\} \subseteq H'.$$

PROOF. Consider a sequence of constants $c_1, c_2, \dots \in C$ from the statement of Lemma 13. Let H' be set

$$\{\neg\phi\} \cup \{\Box_A\psi \mid \Box_A\psi \in H\} \cup \{\neg\Box_A\chi \mid \neg\Box_A\chi \in H\} \cup \{\gamma_i[c_i/x_i] \to \forall x_i\, \gamma_i \mid i \geq 1\}.$$

By Definition 7, set H' is a Henkin set. It suffices to prove that set H' is consistent. Assume the opposite. Hence, there are formulae $\Box_A\psi_1, \dots, \Box_A\psi_n \in H$ and formulae $\neg\Box_A\chi_1, \dots, \neg\Box_A\chi_m \in H$, and $k \geq 0$ such that

$$\Box_A\psi_1, \dots, \Box_A\psi_n, \neg\Box_A\chi_1, \dots, \neg\Box_A\chi_m, \gamma_k[c_k/x_k] \to \forall x_k\, \gamma_k, \dots, \gamma_1[c_1/x_1] \to \forall x_1\, \gamma_1 \vdash \phi.$$

Then, after $n + m + k$ applications of Lemma 8,

$$\vdash \Box_A\psi_1 \to (\dots(\Box_A\psi_n \to (\neg\Box_A\chi_1 \to \dots(\neg\Box_A\chi_m \to ((\gamma_k[c_k/x_k] \to \forall x_k\, \gamma_k) \to \dots((\gamma_1[c_1/x_1] \to \forall x_1\, \gamma_1) \to \phi)\dots))\dots))\dots).$$

Thus, by the Necessitation inference rule,

$$\vdash \Box_A(\Box_A\psi_1 \to (\dots(\Box_A\psi_n \to (\neg\Box_A\chi_1 \to \dots(\neg\Box_A\chi_m \to ((\gamma_k[c_k/x_k] \to \forall x_k\, \gamma_k) \to \dots((\gamma_1[c_1/x_1] \to \forall x_1\, \gamma_1) \to \phi)\dots))\dots))\dots)).$$

Hence, by applying the Distributivity axiom and the Modus Ponens inference rule $n + m$ times,

$$\Box_A\Box_A\psi_1, \dots, \Box_A\Box_A\psi_n, \Box_A\neg\Box_A\chi_1, \dots, \Box_A\neg\Box_A\chi_m \vdash \Box_A((\gamma_k[c_k/x_k] \to \forall x_k\, \gamma_k) \to \dots((\gamma_1[c_1/x_1] \to \forall x_1\, \gamma_1) \to \phi)\dots).$$

By Lemma 2 applied n times,

$$\Box_A\psi_1, \dots, \Box_A\psi_n, \Box_A\neg\Box_A\chi_1, \dots, \Box_A\neg\Box_A\chi_m \vdash \Box_A((\gamma_k[c_k/x_k] \to \forall x_k\, \gamma_k) \to \dots((\gamma_1[c_1/x_1] \to \forall x_1\, \gamma_1) \to \phi)\dots).$$

By applying the Negative Introspection axiom m times,

$$\begin{aligned}&\Box_A\psi_1,\dots,\Box_A\psi_n,\neg\Box_A\chi_1,\dots,\neg\Box_A\chi_m\\&\quad\vdash\Box_A((\gamma_k[c_k/x_k]\to\forall x_k\,\gamma_k)\to\dots((\gamma_1[c_1/x_1]\to\forall x_1\,\gamma_1)\to\phi)\dots).\end{aligned}$$

Hence, due to the choice of formulae $\Box_A\psi_1,\dots,\Box_A\psi_n$ and $\neg\Box_A\chi_1,\dots,\neg\Box_A\chi_m$,

$$H\vdash\Box_A((\gamma_k[c_k/x_k]\to\forall x_k\,\gamma_k)\to\dots((\gamma_1[c_1/x_1]\to\forall x_1\,\gamma_1)\to\phi)\dots).$$

The latter implies inconsistency of set H due to the choice of sequence $c_1,c_2,\dots$. This contradicts the assumption of the lemma. ⊣

6.3. Canonical Model. By a maximal consistent Henkin set $H_0\subseteq\Phi(C)$ we mean any maximal consistent subset of $\Phi(C)$ which is a Henkin set. In this section we construct a canonical Kripke model based on a maximal consistent Henkin set. Such a construction is not trivial even for *propositional* epistemic logic of distributed knowledge because one needs to use "unraveling" [13] or a similar technique. In this section we adapt the "unraveling" technique for our logic with quantifiers over epistemic agents to define the canonical Kripke model $K(C,H_0)=\langle W,C,\{\sim_c\}_{c\in C},\alpha,\pi\rangle$ for any set of constants C and any maximal consistent Henkin set $H_0\subseteq\Phi(C)$.

The key element of this technique is to define epistemic worlds not as maximal consistent sets of formulae, but as sequences of such sets satisfying certain conditions. The next definition shows how this is done.

Definition 8. *The set of epistemic worlds W is the set of all sequences $H_0,C_1,H_1,\dots,C_n,H_n$ such that*

1. $n\ge 0$,
2. $H_i\subseteq\Phi(C)$ *is a maximal consistent Henkin set for each* $i>0$,
3. C_i *is a finite subset of* C *for each* $i>0$,
4. $\{\phi\mid\Box_{C_i}\phi\in H_i\}\subseteq H_{i+1}$, *for each* $i\ge 0$.

The next lemma shows that the maximal consistent sets in a sequence representing an epistemic world share certain $\Box_A$-formulae.

Lemma 15. *For any $H_0,C_1,H_1,\dots,C_n,H_n\in W$, any $0\le k\le n$, and any closed formula $\Box_A\phi\in\Phi(C)$, if $A\subseteq C_i$ for each $k<i\le n$, then $\Box_A\phi\in H_k$ iff $\Box_A\phi\in H_n$.*

Proof. We prove the lemma by backward induction on k. If $k=n$, then the statement of the lemma is a logical tautology. Suppose that $k<n$. By the induction hypothesis $\Box_A\phi\in H_{k+1}$ iff $\Box_A\phi\in H_n$. Thus, it suffices to show that $\Box_A\phi\in H_k$ iff $\Box_A\phi\in H_{k+1}$.
($\Rightarrow$) Suppose that $\Box_A\phi\in H_k$. Then, $H_k\vdash\Box_A\Box_A\phi$ by Lemma 2. Thus, $H_k\vdash\Box_{C_{k+1}}\Box_A\phi$ by the Monotonicity axiom and the assumption $A\subseteq C_{k+1}$. Hence, $\Box_{C_{k+1}}\Box_A\phi\in H_k$ due to the maximality of set H_k. Therefore, $\Box_A\phi\in H_{k+1}$ by Definition 8.
($\Leftarrow$) Suppose that $\Box_A\phi\notin H_k$. Thus, $\neg\Box_A\phi\in H_k$ due to the maximality of set H_k. Hence, $H_k\vdash\Box_A\neg\Box_A\phi$ by the Negative Introspection axiom. Then, $H_k\vdash\Box_{C_i}\neg\Box_A\phi$ by the Monotonicity axiom and the assumption $A\subseteq C_i$. Hence, $\Box_{C_i}\neg\Box_A\phi\in H_k$ due to the maximality of set H_k. Thus, $\neg\Box_A\phi\in H_{k+1}$ by Definition 8. Therefore, $\Box_A\phi\notin H_{k+1}$ due to the consistency of set H_{k+1}. ⊣

Next, we define indistinguishability relations on epistemic worlds.

DEFINITION 9. *For any epistemic world* $w = H_0, C_1, H_1, \dots, C_n, H_n \in W$, *any epistemic world* $u = H_0, C'_1, H'_1, \dots, C'_m, H'_m \in W$, *and any* $c \in C$, *let* $w \sim_c u$ *if there is* $k \geq 0$ *such that*

1. $k \leq n$ *and* $k \leq m$,
2. $H_i = H'_i$ *and* $C_i = C'_i$ *for each* $0 < i \leq k$,
3. $c \in C_i$ *for each* $k < i \leq n$,
4. $c \in C'_i$ *for each* $k < i \leq m$.

COROLLARY 1. $\sim_c$ *is an equivalence relation on set* W *for each* $c \in C$.

For any epistemic world $w = H_0, C_1, H_1, \dots, C_n, H_n$, by $hd(w)$ we denote set H_n. The following lemma provides intuition behind the above definition of the indistinguishability relation.

LEMMA 16. *If* $w \sim_A u$, *then* $\Box_A\phi \in hd(w)$ *iff* $\Box_A\phi \in hd(u)$.

PROOF. Let $w = H_0, C_1, H_1, \dots, C_n, H_n$ and $u = H'_0, C'_1, H'_1, \dots, C'_m, H'_m$, where $H'_0 = H_0$. Assumption $w \sim_A u$ implies that $w \sim_a u$ for each $a \in A$. Thus, by Definition 9, for each $a \in A$ there is $k_a \geq 0$ such that

1. $k_a \leq n$ and $k_a \leq m$,
2. $H_i = H'_i$ and $C_i = C'_i$ for each $0 < i \leq k_a$,
3. $a \in C_i$ for each $k_a < i \leq n$,
4. $a \in C'_i$ for each $k_a < i \leq m$.

Consider $k_{max} = \max\{k_a \mid a \in A\}$; if $A = \varnothing$, then let $k_{max} = 0$. Thus,

1. $k_{max} \leq n$ and $k_{max} \leq m$,
2. $H_i = H'_i$ and $C_i = C'_i$ for each $0 < i \leq k_{max}$,
3. $A \subseteq C_i$ for each $k_{max} < i \leq n$,
4. $A \subseteq C'_i$ for each $k_{max} < i \leq m$.

By Lemma 15, we have $\Box_A\phi \in H_{k_{max}}$ iff $\Box_A\phi \in H_n$. By the same lemma, we also have $\Box_A\phi \in H'_{k_{max}}$ iff $\Box_A\phi \in H'_m$. Note that $H_{k_{max}} = H'_{k_{max}}$. Therefore, $\Box_A\phi \in H_n$ iff $\Box_A\phi \in H'_m$. In other words, $\Box_A\phi \in hd(w)$ iff $\Box_A\phi \in hd(u)$. ⊣

To finish the construction of the canonical model $K(C, H_0) = \langle W, C, \{\sim_c\}_{c\in C}, \alpha, \pi\rangle$, next we specify functions α and π.

DEFINITION 10. *For any* $c \in C$, *let* $\alpha(c) = c$.

DEFINITION 11. *For any* $p \in P$, *let* $\pi(p) = \{w \in W \mid p \in hd(w)\}$.

We conclude this section with the lemma that connects the satisfiability relation in the canonical model with the maximal consistent sets out of which this model is built.

LEMMA 17. $(w, \rho) \Vdash \phi$ *iff* $\phi \in hd(w)$ *for each* $\rho : V \to C$ *and each closed formula* $\phi \in \Phi(C)$.

PROOF. We prove this statement by induction on the structural complexity of formula ϕ. If formula ϕ is a proposition, then the required follows from Definition 6 and Definition 11. If formula ϕ is an implication or a negation, then the required follows from Definition 6 and the maximality and the consistency of the set $hd(w)$ in the standard way.

Suppose that ϕ is a closed universal formula. Recall that $\gamma_1, \gamma_2, \dots$ is an enumeration of all formulae in $\Phi(C)$ with a single free variable and $x_1, x_2, \dots$ are the corresponding free variables in formulae $\gamma_1, \gamma_2, \dots$. Thus, there must exist $n \geq 0$ such that ϕ is formula $\forall x_i\, \gamma_i$. Furthermore, by Definition 7, there must exist $c_0 \in C$ such that

$$(\gamma_i[c_0/x_i] \to \forall x_i\ \gamma_i) \in hd(w). \tag{8}$$

($\Rightarrow$) Assume that $(w, \rho) \Vdash \forall x_i\, \gamma_i$. Recall that C is the set of agents in the canonical model $K(C, H_0)$. Then, $(w, \rho[x_i \mapsto c]) \Vdash \gamma_i$ for each $c \in C$ by Definition 6. Hence, $(w, \rho[x_i \mapsto c_0]) \Vdash \gamma_i$. Then, $(w, \rho[x_i \mapsto \alpha(c_0)]) \Vdash \gamma_i$ by Definition 10. Thus, $(w, \rho) \Vdash \gamma_i[c_0/x_i]$ by Lemma 1. Then, by the induction hypothesis, $\gamma_i[c_0/x_i] \in hd(w)$. Hence, $hd(w) \vdash \forall x_i\ \gamma_i$ by the Modus Ponens inference rule using (8). Therefore, $\forall x_i\ \gamma_i \in hd(w)$ due to the maximality of the set $hd(w)$.
($\Leftarrow$) Suppose that $(w, \rho) \nVdash \forall x_i\, \gamma_i$. Then, by Definition 6, there must exist $c \in C$ such that $(w, \rho[x_i \mapsto c]) \nVdash \gamma_i$. Hence, $(w, \rho[x_i \mapsto \alpha(c)]) \nVdash \gamma_i$ by Definition 10. Then, $(w, \rho) \nVdash \gamma_i[c/x_i]$ by Lemma 1. Thus, $\gamma_i[c/x_i] \notin hd(w)$ by the induction hypothesis. Hence, $hd(w) \nvdash \gamma_i[c/x_i]$ due to the maximality of the set $hd(w)$. Therefore, $\forall x_i\ \gamma_i \notin hd(w)$ by the Specialization axiom and the Modus Ponens inference rule.

Finally, let us assume that ϕ is a closed formula of the form $\Box_A \eta$. Then, $A \subseteq C$ and η is also a closed formula.
($\Rightarrow$) Let $w = H_0, C_1, H_1, \dots, C_n, H_n$ where $H_n = hd(w)$ and suppose that $\Box_A \eta \notin hd(w)$. Thus, $\neg\Box_A \eta \in hd(w)$ due to the maximality of the set $hd(w)$. By Lemma 14, there is a consistent Henkin set H' such that

$$\{\neg\eta\} \cup \{\Box_A \psi \mid \Box_A \psi \in H_n\} \cup \{\neg\Box_A \chi \mid \neg\Box_A \chi \in H_n\} \subseteq H'.$$

Let H_{n+1} be a maximal consistent extension of set H'. Consider the sequence $u = H_0, C_1, H_1, \dots, C_n, H_n, A, H_{n+1}$. Then, $u \in W$ by Definition 8. Additionally, $w \sim_a u$ for each $a \in A$ by Definition 9. In other words, $w \sim_A u$. At the same time $\eta \notin hd(u)$ because $\neg\eta \in H' \subseteq H_{n+1} = hd(u)$ and set $hd(u)$ is consistent. Thus, $(u, \rho) \nVdash \eta$ by the induction hypothesis. Therefore, $(w, \rho) \nVdash \Box_A \eta$ by Definition 6.
($\Leftarrow$) Suppose that $\Box_A \eta \in hd(w)$. Consider an arbitrary $u \in W$ such that $w \sim_A u$. Then, $\Box_A \eta \in hd(u)$, by Lemma 16. Thus, $hd(u) \vdash \eta$, by the Reflexivity axiom. Then, $\eta \in hd(u)$, due to the maximality of the set $hd(u)$. Hence, $(u, \rho) \Vdash \eta$ by the induction hypothesis. Therefore, $(w, \rho) \Vdash \Box_A \eta$ by Definition 6. $\dashv$

6.4. Completeness Theorem. We are now ready to state and prove the completeness theorem for our logical system.

THEOREM 2. *For any at most countable set C and any closed formula $\phi \in \Phi(C)$, if $w \Vdash \phi$ for each epistemic world $w \in W$ of each Kripke model $\langle W, \mathcal{A}, \{\sim_a\}_{a \in \mathcal{A}}, \alpha, \pi\rangle$, then $\vdash_C \phi$.*

PROOF. Suppose that $\nvdash_C \phi$. Thus, $\{\neg\phi\}$ is a consistent subset of $\Phi(C)$. By Lemma 12, there is an at most countable set $C' \supseteq C$ and a consistent Henkin set $H \subseteq \Phi(C')$ such that $\neg\phi \in H$. Let H_0 be any maximal consistent subset of $\Phi(C')$ such that $H \subseteq H_0$. Consider the Kripke model $K(C', H_0)$ defined in Section 6.3 and let epistemic world w be the single-element sequence H_0. For any function ρ from variables to agents, $(w, \rho) \Vdash \neg\phi$ by Lemma 17. Therefore, $(w, \rho) \nVdash \phi$ by Definition 6. $\dashv$

§7. Conclusion. In this article we proposed a logical system for reasoning about quantifiers over epistemic agents. The main technical result of this article is the completeness theorem for this logical system.

In Kripke-like semantics of first order modal logics there are usually domains associated with each epistemic world. The Barcan formula and the converse Barcan formula are usually thought of as a syntactical way to capture the setting when the domains do not change from one epistemic world to another. The semantics of our logical system does not have separate domains for epistemic states. Instead, we have a set of agents for the entire model. One of the questions that we addressed in this article is how this setting affects the validity of the Barcan formula and of its converse. As we have shown, such a setting results in these formulae to be true only in a restricted form.

The completeness of this logical system with respect to the class of models with finitely many agents and the decidability of the system are open problems.

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