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# ON THE ASYMPTOTIC THEORY OF TESTS OF INDEPENDENCE BASED ON BIVARIATE LAYER RANKS

George G. Woodworth*

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University of Minnesota
Minneapolis, Minnesota

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0. Summary. Let $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_n$ be a sample drawn from a continuous bivariate population with distribution H. We define the $q^{\underline{th}}$ quadrant layer-rank of $\underset{\sim}{X}_j$, denoted by $\ell_{qj}$, $q = 1,\ldots,4$, $j = 1,\ldots,n$, to be the number of points $\underset{\sim}{X}_i$, $i = 1,\ldots,n$, such that $\underset{\sim}{X}_i - \underset{\sim}{X}_j$ is in the (closed) $q^{\underline{th}}$ quadrant (See figure 1.), and the $q^{\underline{th}}$ quadrant $r^{\underline{th}}$ layer statistic, denoted by $A_n^{(r)}(q), r = 1,\ldots,n$, $q = 1,\ldots,4$, to be the number of points with $q^{\underline{th}}$ quadrant layer ranks equal to r. (See figure 1.).

In this paper we investigate the properties of certain tests of independence of the marginals of H based on $3^{\underline{rd}}$ quadrant layer ranks, hereafter called layer rank tests, paying special attention to those based on linear combinations of $3^{\underline{rd}}$ quadrant* layer statistics. We prove asymptotic normality of the test statistics under the null and local alternative hypotheses, derive local asymptotic efficiencies (Pitman efficiencies) of these tests and show that in many cases an efficient test is found among the layer rank tests. We find the optimal (locally most powerful) number of the subclass of tests based on linear combinations of layer statistics and that of similar subclasses. Finally, we derive asymptotic efficiencies (Bahadur efficiencies) at distant alternatives.

*The results will apply with obvious modifications to tests based on $1^{\underline{st}}$, $2^{\underline{nd}}$ or $4^{\underline{th}}$-quadrant layer ranks, but not to tests which mix layer ranks from different quadrants.

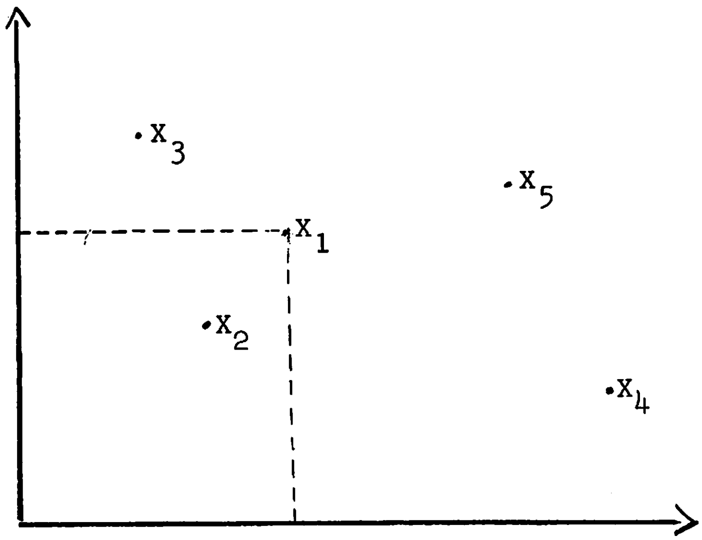


Fig. 1

A two dimensional sample of size 5.

Layer Ranks, $\ell_{qj}$.

| q \ j | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 1 | 1 | 1 |
| 2 | 2 | 2 | 1 | 5 | 2 |
| 3 | 2 | 1 | 1 | 1 | 3 |
| 4 | 2 | 2 | 5 | 1 | 2 |

For example, $\ell_{31} = 2$ since $\underset{\sim}{X}_2 - \underset{\sim}{X}_1$ and $\underset{\sim}{X}_1 - \underset{\sim}{X}_1$ are in the closed third quadrant.

Layer Statistics, $A_5^r(q)$.

| q \ r | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 3 | 1 | 1 | 0 | 0 |
| 2 | 1 | 3 | 0 | 0 | 1 |
| 3 | 3 | 1 | 1 | 0 | 0 |
| 4 | 1 | 3 | 0 | 0 | 1 |

1. <u>Introduction</u>. Tests based on layer ranks have been proposed at various times, some (but probably not all of them) are described here. The best known layer rank test of bivariate independence is the test based on Kendall's $\tau$-statistic, which, as we shall see later, is a linear function of the sum of the $3^{rd}$ quadrant layer ranks; tests of trend in a univariate time series based on layer statistics(1) were investigated by Foster and Stuart [ 7], who used the values $A_n^{(1)}(1), \ldots, A_n^{(1)}(4)$ associated with the first layer only. More recently Parent [18] investigated sequential tests based on layer ranks for equality of two populations(2) and for detecting the time at which the distribution of a sequence of independent observations changes. This paper bears little relation to the work of Foster and Stuart or of Parent and may be regarded as an extension of the theory of Kendall's $\tau$-test of independence.

Although the notions of layer ranks and layer statistics are probably not new, the first systematic investigation of the properties of layer statistics is recent, being that of Sobel and Barndorff-Nielsen [20], who derived the distribution of the layer statistics and similar quantities under the assumption that the components of the sampled random vector are independent. We now present the results from [20] needed for this paper; we share with [20] the assumption that the marginal distributions of H are continuous.

(1)Layer ranks in a sample from a time series are computed as in the bivariate case by treating time as the X-component and the value of the time series at time x as the Y-component of the two dimensional vector $X = (X,Y)$.

(2)A time series is generated by sampling alternatively from each population, layer ranks are defined as in footnote (1).

Let the random vector $\underset{\sim}{X}_j$ have components $(X_j, Y_j)$, $j = 1,2,\ldots,n$, let $Y_{[j]}$ be the Y-component of the vector with the $j^{th}$ smallest X-component $X_{(j)}$; if the marginals of $H$ are independent*, then $Y_{[1]},\ldots, Y_{[n]}$ are independent and identically distributed. Let $\ell_{(j)}$ be the $3^{rd}$ quadrant layer rank of $(X_{(j)}, Y_{[j]})$; clearly, $\ell_{(j)}$ is the rank of $Y_{[j]}$ among $Y_{[1]},\ldots, Y_{[j]}$, consequently, from the result of Dwass and Renyi, which also appears as Theorem 1.1 of Barndorff-Nielsen [2], we have:

Lemma 1.1: If the marginals of $H$ are independent, then the $\ell_{(j)}$ are independent and $P(\ell_{(j)} = i) = \frac{1}{j}$, $i = 1,\ldots,j$, $j = 1,\ldots,n$.

Statistics based on layer ranks have an invariance property which we now describe: Let $R_i$ and $S_i$ be the rank of $X_i$ among all the X's and $Y_i$ among all the Y's, $i = 1,\ldots,n$. It is evident that the layer ranks depend upon $(X_1,\ldots, X_n)$ through $(R_1,S_1),\ldots, (R_n,S_n)$ only. Suppose $H_0(u,v)$ is a continuous cdf. with uniform (0,1) marginals. Lehmann [14] defines non-parametric equivalence classes of bivariate cdf's. as follows:

$$\mathcal{H}(H_0) = \{H(x,y): H(x,y) = H_0(F(x),G(y)),$$
$$F \text{ and } G \text{ are continuous univariate cdf's}\}.$$

For example, if $H_\theta(x,y)$ is the bivariate normal cdf. with zero means, unit variances, and correlation $\theta$, then $H_\theta$ is contained in the class generated by $H_0(u,v) = H_\theta(\Phi^{-1}(u), \Phi^{-1}(v))$. As another example, $\mathcal{H}(uv)$ is the class of all cdf's of continuous bivariate random vectors with independent components.

From Lehmann [14], Theorem 7.1, we conclude that if $T$ is a statistic based only on layer ranks, then the distribution of $T$ is constant over the class $\mathcal{H}(H_0)$. For the sake of having a convenient term, we say

*I.e., $H(x,y) = F(x)G(y)$ for some distribution functions $F$ and $G$.

that $T$ is a marginal free statistic.

Now suppose that $\{H_\theta : \theta \in \Theta\}$ is a family of bivariate distributions. If $\mathcal{P}$ is a property of a sequence of marginal free statistics $\{T_n = T_n(X_1, \ldots, X_n)\}$ which follows from the assumption that $X_1, \ldots, X_n$ is a sample from $H_{\theta_n}$, $\theta_n \in \Theta$, $n = 1,2,\ldots,$ then $\mathcal{P}$ is also true if each $H_\theta$ is replaced by a member of its non-parametric class $\mathcal{H}(H_0)$, where $H_0(u,v) = H_\theta(F_\theta^{-1}(u), G_\theta^{-1}(v))$, and $F_\theta(x) = H_\theta(x,\infty)$ and $G_\theta(y) = H_\theta(\infty,y)$ are the marginals of $H_\theta$.

We conclude this section with a summary of the more interesting results of this paper; in an attempt to avoid being repetitious we use the symbol ucc to denote the qualifying phrase "under certain conditions".

In the next section we introduce a class of nonparametric statistics, called layer-rank statistics, of the form: $T_n(\underset{\sim}{C}) = n^{-\frac{1}{2}} \sum_{j=1}^{n} c_n\left(\frac{\ell(j)}{j+1}, \frac{j}{n+1}\right)$, where $c_n(u,v)$ is a function defined inside the unit square. In Section 3 the asymptotic distribution of a statistic of this type is investigated both under the null hypothesis (independence) and under "local" alternatives. An explicit expression for the Pitman efficiency of sequences of tests based on layer-rank statistics (layer-rank tests) is derived (ucc) and a table of Pitman efficiencies of various layer-rank tests against specific alternatives is presented (TableIII). From this expression for the Pitman efficiency, an explicit expression for a sequence of layer-rank tests which is asymptotically locally most powerful (ALMP) against a fixed but arbitrary family of alternatives is derived (ucc).

In Section 5 we consider a class of tests based on linear combinations of layer statistics (layer tests), which is a subclass of the class of layer-rank tests described above and contains the well-known Kendall's $\tau$ test. We show that (ucc) the problem of finding the layer test having maximum Pitman efficiency against a fixed but arbitrary family of alternatives is equivalent to solving a certain integral equation and the solution is explicitely obtained (ucc). As a special case it is shown that, against a certain family of alternatives, Kendall's $\tau$ has maximum Pitman efficiency not only among all layer tests but also among all tests.

Recalling the definition of $Y_{[1]}, \ldots, Y_{[n]}$ given earlier in this section and letting $R_{[j]}$ denote the rank of $Y_{[j]}$, $j = 1,\ldots,n$, among all the Y's, we note in Section 8 that (ucc) the locally most powerful test based on $R_{[1]},\ldots,R_{[n]}$ (we call such tests rank tests) is usually based on a statistic of the form $S_n(b) = n^{-\frac{1}{2}} \sum_{j=1}^{n} b_n(\frac{R_{[j]}}{n+1}, \frac{j}{n+1})$, where $b_n(u,v)$ is a function defined inside the unit square; a special case of this statistic was investigated by Bhuchongkul [3 ]. We show that (ucc) for every sequence of layer-rank tests based on statistics $T_n(\underset{\sim}{C})$ there is a corresponding sequence of rank tests based on $S_n(b_c)$ (and vice versa) and that the two sequences are indistinguishable in terms of Pitman efficiency; in other words, the Pitman efficiency of the tests based on $T_n(\underset{\sim}{C})$ with respect to the tests based on $S_n(b_c)$ is one against any family of alternatives (ucc).

Although one cannot assert the superiority of rank or layer-rank tests on the basis of Pitman efficiency, layer-rank tests have the advantage that a more comprehensive efficiency description (Bahadur efficiency) than that offered by Pitman efficiency can be computed for layer-rank tests but not (at least not easily) for rank tests. Bahadur efficiency gives asymptotic relative efficiencies for each fixed alternative in contrast to Pitman efficiency which measures relative efficiency only for alternatives "near" the null. In Section 6 we derive (ucc) explicit expressions for the Bahadur asymptotic relative efficiency, against a fixed alternative, of a sequence of layer-rank tests with respect to either another sequence of layer-rank tests or the likelihood ratio test. In addition, Bahadur efficiencies are computed for several layer-rank tests with respect to likelihood ratio tests against specific alternatives (for example, see Figures 4 and 5 ).

## 2. A Class of Test Statistics Based on Layer Ranks.

In order to motivate the class of non-parametric test statistics which we introduce below, we ask the reader to recall the univariate two sample problem. In that problem there are two populations X and Y with continuous CDF's F and G. We take a sample $X_1, \ldots, X_m$ of size m from the X-population and a sample $Y_1, \ldots, Y_n$ of size n from the Y-population and define $R_j$ to be the number of observations from either population less than or equal to $Y_j$. Two popular tests of $F = G$ versus $F < G$ are the Wilcoxon test and the Fisher-Yates test. Letting $N = m+n$, the test statistics are,

$$\text{Wilcoxon:} \quad T_N = \sum_{j=1}^{n} R_j$$

and

$$\text{Fisher-Yates:} \quad T_N = \sum_{j=1}^{n} \mu_{R_j|N} ,$$

where $\mu_{j|N}$ is the expected value of the $j^{\underline{th}}$ largest of N standard normal random variables. Note that both of these statistics are of the form:

$$(2.1) \qquad T_N = \sum_{j=1}^{n} h_N(R_j/N+1) ,$$

in the case of the Wilcoxon statistic the weight function $h_N(u)$ is $(N+1)u$ and for the Fisher-Yates statistic $h_N(u)$ is a step function given by:

$$(2.2) \qquad h_N(u) = \mu_{j|N} , \quad \frac{j-1}{N} \leq u < \frac{j}{N} , \quad j = 1,\ldots,n.$$

Now we return to the problem of testing independence in a bivariate distribution. Let $R_1, \ldots, R_n$ and $S_1, \ldots, S_n$ be the ranks of the X- and Y-components of a bivariate sample of size n (in the order observed). In [ 3], Bhuchongkul proposed test statistics of a form analogous to (2.1), namely:

(2.3) $$T_n = \sum_{j=1}^{n} J_n(R_j/n+1)\, L_n(S_j/n+1),$$ where $J_n$ and $L_n$ are

some weight functions defined on the interval (0,1); in particular, an analogue to the Wilcoxon statistic is obtained by setting $J_n(u) = L_n(u) = u$ and an analogue to the Fisher-Yates statistic by setting $J_n(u) = L_n(u) = h_n(u)$ defined in (2.2).

In this section we propose an entirely different class of test statistics, these statistics are related to the above in structural appearance but, as a class, seem to have an empty intersection with the class proposed by Buchongkul. Our statistics have a property which is distinctly advantageous from the theoretical point of view, namely: they can be expressed as sums of independent random variables under the null hypothesis (independence); moreover, whenever a statistic of Bhuchongkul's form is asymptotically locally most powerful* (ALMP) against some family of alternatives, then there exists an ALMP statistic of the form proposed by us.

Let $\{c_{nij},\ 1 \leqq i \leqq j \leqq n,\ n \geqq 1\}$ be a triple sequence of real numbers and for each $n \geqq 1$ let,

(2.4) $$c_n(u,v) = c_{nij}, \qquad \frac{i-1}{j} \leqq u < \frac{i}{j}, \quad \frac{j-1}{n} < v \leqq \frac{j}{n}, \quad 1 \leqq i \leqq j \leqq n.$$

Thus $\{c_n\}$ is a sequence of functions defined on the unit square. We use $\mathbf{L}^r$ and $\|\ \|_r$ to denote the space of $r^{th}$ power-Lebesgue integrable functions on the unit square and the corresponding norm; i.e., $\|g\|_r = (\iint |g(u,v)|^r du dv)^{\frac{1}{r}}$ (see footnote**) and $\mathbf{L}^r$ is the set of all

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*One advantage of Bhuchongkul's class of statistics is that the locally most powerful rank test is frequently in that class but never in the class we propose; we prove this remark in section 8.

**When the range of integration is not given, assume it to be (0,1).

functions $g$ such that $\|g\|_r < \infty$. We assume that there exists a function $c$ defined on the unit square such that

$$(2.5) \qquad 0 < \|c\|_2 < \infty \quad \text{and} \quad \|c_n - c\|_2 \to 0 \quad \text{as } n \to \infty.$$

We denote the sequence of weight functions $\{c, c_1, c_2, \ldots\}$ by $\underset{\sim}{C}$. We are interested in statistics of the following form*:

$$(2.6) \qquad T_n(\underset{\sim}{C}) = n^{-\frac{1}{2}} \sum_{j=1}^{n} c_{n,\ell_{(j)},j} = n^{-\frac{1}{2}} \sum_{j=1}^{n} c_n(\ell_{(j)}/j+1, j/n+1)$$

where $\ell_{(j)}$ is the layer rank of $(X_{(j)}, Y_{[j]})$ defined on page 4 . We include the argument $\underset{\sim}{C}$ to indicate the dependence of the statistics on the sequence of weight functions.

For convenience, we assume that

$$(2.7) \qquad \sum_{i=1}^{j} c_{n,i,j} = 0, \qquad 1 \leq j \leq n, \quad n \geq 1.$$

This assumption entails no loss of generality; for, if $\{c'_{n,i,j}\}$ satisfies (2.5) but not (2.7), let $c_{n,i,j} = c'_{n,i,j} - \bar{c}'_{n,j}$, where $\bar{c}'_{n,j} = \frac{1}{j} \sum_{i=1}^{j} c'_{n,i,j}$.

The new sequence satisfies (2.7), we now show that it also satisfies (2.5).

Let $\bar{c}'_n(v) = \frac{1}{j} \sum_{i=1}^{j} c'_{nij} = \int c'_n(u,v)du$, $\frac{j-1}{n} \leq v < \frac{j}{n}$, and $\bar{c}'(v) = \int c'(u,v)du$.

Note that $\int (\bar{c}'_n(v) - \bar{c}'(v))^2 dv = \int (\int (c'_n(u,v) - c'(u,v))du)^2 dv \leq \| c'_n - c' \|_2^2 \to 0$.

Thus, defining $c_n(u,v)$ by (2.4), we have, by the triangle inequality, $c_n(u,v) = c'_n(u,v) - \bar{c}'_n(v) \to c'(u,v) - \bar{c}'(v)$ in $\| \ \|_2$-norm.

Although in this paper we develop the asymptotic theory of statistics of the general form (2.6), we find certain special cases to be of particular interest, these are given by (2.10.1) and (2.10.2) below.

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*Compare this with (2.3).

Let $\{J_{ji},\ 1 \leq i \leq j,\ j \geq 1\}$ and $\{L_{nj},\ 1 \leq j \leq n,\ n \geq 1\}$ be double sequences of real numbers such that $\sum_{i=1}^{j} J_{ji} = 0$. We set

$$c_{nij}^{(1)} = J_{j,i}\, L_{n,j}, \qquad 1 \leq i \leq j \leq n, \tag{2.8.1}$$

and

$$c_{nij}^{(2)} = (J_{n,i} - \bar{J}_{n,j})L_{n,j}, \qquad 1 \leq i \leq j \leq n, \tag{2.8.2}$$

where $\bar{J}_{nj} = \Sigma_{i=1}^{j} J_{ni}/j$. If we define functions $J_n$ and $L_n$ on $(0,1)$ by

$$J_n(u) = J_{n,j}, \qquad \frac{j-1}{n} \leq u < \frac{j}{n}, \qquad j = 1,\ldots,n, \tag{2.9.1}$$

and

$$L_n(u) = L_{n,j}, \qquad \frac{j-1}{n} \leq u < \frac{j}{n} \qquad j = 1,\ldots,n, \tag{2.9.2}$$

then the test statistics of the form (2.6) which correspond to (2.8.1) and (2.8.2), call them $T_{n1}$ and $T_{n2}$, are:

$$T_{n1} = n^{-\frac{1}{2}} \sum_{j=1}^{n} J_{j,\ell_{(j)}} \cdot L_{n,j} = n^{-\frac{1}{2}} \sum_{j=1}^{n} J_j(\ell_{(j)}/j+1) \cdot L_n(j/n+1), \tag{2.10.1}$$

and

$$T_{n2} = n^{-\frac{1}{2}} \sum_{j=1}^{n} J_{n,\ell_{(j)}} L_{n,j} - n^{-\frac{1}{2}} \sum_{j=1}^{n} \bar{J}_{n,j} L_{n,j} \tag{2.10.2}$$

$$= n^{-\frac{1}{2}} \sum_{j=1}^{n} J_n(\ell_{(j)}/n+1)\, L_n(j/n+1) - K_n,$$

say. [illegible]

The form (2.9.1) arises quite naturally, since, for many families of distributions, there is an ALMP sequence of layer-rank tests based on statistics of this form. The form (2.9.2) with $L_{n,j} = 1$ is interesting since, as we show in Section 5, it is a linear combination of the layer statistics $A_n^1(3),\ldots, A_n^n(3)$. In particular, if we set $J_{n,j} = j/n$ and $L_{n,j} = 1$, then (2.10.2) becomes:

$$T_{n2} = n^{-3/2} \sum_{j=1}^{n} \ell_{(j)} = n^{-3/2} \sum_{1 \leq i \leq j \leq n}\sum z_{ij} , \qquad (2.11)$$

where $z_{ij} = 1$ or $0$ as $Y_{[i]} \leq Y_{[j]}$ or $Y_{[i]} > Y_{[j]}$, which, without the factor $n^{-3/2}$, is Kendall's $\tau$-statistic in the form given by Mann [16].

We would also like to point out that (2.8.1) and (2.8.2) have the following practical advantage: in order to be able to compute values of a statistic of the form (2.6) one would need a table containing the constants $c_{nij}$. If such a table were prepared for all $n \leq n_0$ it would in general contain about $n_0^3/6$ entries. But if one used statistics of the form (2.10.1) or (2.10.2) then the necessary table would contain only about $n_0^2$ entries.

We have not investigated the exact distribution of statistics of the general form $T_n(\underset{\sim}{c})$ introduced in (2.3), which is, of course, known for the special case of Kendall's $\tau$ mentioned above; however, in the next section we derive their limiting distributions both under the null and local alternative hypotheses.

3. Limiting Distributions and Pitman Efficiency of $T_n(\underset{\sim}{C})$. In Section 1 we pointed out that $T_n(\underset{\sim}{C})$ is a marginal free statistic. If the sample $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_n$, of which $T_n(\underset{\sim}{C})$ is a function, is drawn from $H_0(x,y) = F(x)G(y)$ then the distribution of $T_n(\underset{\sim}{C})$ is the same for any choice of F and G (provided they are continuous); to put it briefly: $T_n(\underset{\sim}{C})$ is distribution free under the (null) hypothesis of independence. We denote by $E_0$ and $\sigma_0^2$ the expectation and variance operators (operating on marginal free statistics) under the hypothesis of independence. Recalling (2.6) and applying Lemma 1.1 we have, by (2.7),

$$(3.1)\qquad E_0[T_n(\underset{\sim}{C})] = E_0[n^{-\frac{1}{2}} \sum_{j=1}^{n} c_{n,\ell_{(j)},j}] = n^{-\frac{1}{2}} \sum_{j=1}^{n} \frac{1}{j} \sum_{i=1}^{j} c_{n,i,j} = 0,$$

and, by (2.4) and (2.5),

$$(3.2)\qquad \sigma_0^2(T_n(\underset{\sim}{C})) = \frac{1}{n} \sum_{j=1}^{n} \sigma_0^2(c_{n,\ell_{(j)},j}) = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{j} \sum_{i=1}^{j} c_{n,i,j}^2$$

$$= \iint c_n^2(u,v)\,du\,dv \rightarrow \iint c^2(u,v)\,du\,dv = \|c\|_2^2 > 0.$$

Suppose $Z_n$ is a marginal free statistic based on a sample of size n. Adopting a standard notation*, we let $\mathcal{L}(Z_n|H_0)$ denote the probability law of $Z_n$ under the hypothesis of independence.

Theorem 3.1 Under the hypothesis of independence, if (2.2) and (2.4) hold, then $\mathcal{L}(T_n(\underset{\sim}{C})) \rightarrow N(0,\|c\|_2^2)$.

Proof: We verify the conditions of the Lindeberg-Feller (LF) Theorem*. For any $\epsilon > 0$, since $\|c\|_2 > 0$,

$$g_n(\epsilon) = \frac{1}{n} \sum_{\{c_{n,i,j}^2 \geq n\epsilon^2\}}\sum \frac{1}{j} c_{n,i,j}^2 = \iint_{\{c_n^2 \geq n\epsilon^2\}} c_n^2(u,v)\,du\,dv \rightarrow 0,$$

since (2.5) implies that $c_n{}^2$ is uniformly integrable. ▌

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*See Loéve [15] p. 201 for the $\mathcal{L}$-notation and p. 280 for the LF theorem.

We now consider alternatives to the hypothesis of independence and show that $T_n(\underset{\sim}{C})$ has, in the limit, a normal distribution even if the hypothesis of independence does not hold, provided the common distribution, $H$, of $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n$ approaches independence in a suitable way as $n \to \infty$.

For the rest of this section we shall be dealing with a fixed family $\{H_\theta; -\infty < \theta < \infty\}$ of continuous bivariate distribution functions indexed by a real parameter. We assume, without loss of generality, that $H_\theta(x,\infty)$, the marginal cdf of $X$, is independent of $\theta$. We let $H_\theta(x,\infty) = F(x)$, and we denote by $G_\theta(y|x)$ the conditional cdf of $Y$ given $X = x$ and assume that $G_\theta(y|x)$ is absolutely continuous with density $g_\theta(y|x)$ for all $\theta$ and almost all $(F)$ $x$. We assume, finally, that $\theta = 0$ corresponds to the hypothesis of independence and denote $G_0(y|x)$ and $g_0(y|x)$ by $G(y)$ and $g(y)$, respectively.

Under these assumptions, the likelihood ratio $r_\theta = dH_\theta/dH_0$ is given by: $r_\theta(x,y) = g_\theta(y|x)/g(y)$, almost surely $(H_0)$.

The behavior of the distribution of $T_n(\underset{\sim}{C})$, when $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n$ is a sample from $H_\theta$ and $\theta \to 0$ as $n \to \infty$, depends crucially upon the behavior of $r_\theta$ as $\theta \to 0$; and in order to obtain our results we must make certain assumptions about this behavior. In fact, we assume that

(3.3) $$\{\partial/\partial\theta\} r_\theta(x,y)\big|_{\theta=0} = s(x,y), \text{ say}$$

exists almost surely $(H_0)$, that

(3.4) $$\int_{-\infty}^{\infty} s(x,y)g(y)dy = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} s(x,y)dF(x) = 0,$$

almost surely $(H_0)$, that for some $\delta > 0$,

(3.5) $$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} |s(x,y)|^{2+2\delta}\, dF(x)\right]^{\frac{1}{1+\delta}} g(y)dy < \infty,$$

and finally, that

(3.6) $$\lim_{\theta\to 0} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{r_\theta^{\frac{1}{2}}(x,y) - 1}{\theta} - \frac{s(x,y)}{2}\right)^2 g(y)dydF(x) = 0.$$

Condition (3.6) is an adaptation of a similar condition of Matthes and Truax [17] (their (1.2)), and resembles (4.22) of Hájek [10]. Sufficient conditions for (3.3), (3.4), (3.5) and (3.6) in special cases are developed in Section 4.

We set $\theta_n = an^{-\frac{1}{2}}$ where $a \neq 0$ is fixed but arbitrary and define $H_n = H_{\theta_n}$. After some preliminary remarks about notation we present a lemma due to LeCam (Hájek [10] Lemma 4.2) which is our basic tool for proving asymptotic normality. We adopt the following notations in order to conform to those used by Hájek: $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n$ is, as usual, a sample drawn from a bivariate population; $P_n$ and $Q_n$ denote, respectively, the probability laws of the sample under the hypothesis of independence and under the alternative hypothesis that the bivariate population has cdf $H_n$, defined above.

For any statistic $Z_n = Z_n(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n)$, we denote by $\mathcal{L}(Z_n|P_n)$, $E(Z_n|P_n)$ and $\sigma^2(Z_n|Q_n)$ and $\mathcal{L}(Z_n|Q_n)$, $E(Z_n|Q_n)$ and $\sigma^2(Z_n|Q_n)$ the probability laws, means and variances of $Z_n$ under $P_n$ and $Q_n$, respectively.

Finally, setting $r_{nj} = r_{\theta_n}(X_j, Y_j)$, $j = 1, \ldots, n$, we define the following statistics:

(3.7) $$L_n = \sum_{j=1}^{n} \ln(r_{nj}), \qquad (\ln \text{ being the natural logarthm}),$$

(3.8) $$W_n = 2\sum_{j=1}^{n} (r_{nj}^{\frac{1}{2}} - 1),$$

and

(3.9) $$T_n = \theta_n \sum_{j=1}^{n} s(X_j, Y_j) = an^{-\frac{1}{2}} \sum_{j=1}^{n} s(X_j, Y_j) .$$

We state without proof (see Hájek [10] Lemma 4.2):

Lemma 3.1 (LeCam) If $\max_{1 \leq j \leq n} P_n(|r_{nj} - 1| > \epsilon) \to 0$ for every $\epsilon > 0$ and

$\mathcal{L}(W|P_n) \to N(-\frac{1}{2}\sigma^2, \sigma^2)$ for some $\sigma^2$, then

(1) if $Z_n \to 0$ in $P_n$-probability, then $Z_n \to 0$ in $Q_n$-probability,

(2) $W_n - L_n \to \frac{1}{4}\sigma^2$ in $P_n$-probability,

and

(3) if $\mathcal{L}(Z_n|P_n) \to N(\mu, b^2)$ and $\mathcal{L}(Z_n, L_n|P_n)$ tends to the bivariate normal with correlation coefficient $\rho$, then $\mathcal{L}(Z_n|Q_n) \to N(\mu+\rho b\sigma, b^2)$.

We now verify that the conditions of LeCam's lemma are satisfied in our case. The first condition follows from (3.6), since

$P_n(|r_{nj}-1| \geq \epsilon) = P_n(|r_{n1}-1| \geq \epsilon)$ and

$$(E[|r_{n1}-1| \; |P_n])^2 \leq E[|r_{n1}^{\frac{1}{2}}-1|^2|P_n] \cdot E[|r_{n1}^{\frac{1}{2}}+1|^2|P_n]$$

$$\leq E[|r_{n1}^{\frac{1}{2}}-1|^2|P_n] \cdot \{2E[|r_{n1}^{\frac{1}{2}}-1|^2|P_n] + 8\} \to 0.$$

Also by (3.6) we have,

$$(3.10) \qquad E(W_n|P_n) = 2n \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (r_{\theta_n}^{\frac{1}{2}}(x,y)-1)g(y)dF(x)dy$$

$$= 2n \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (g_{\theta_n}^{\frac{1}{2}}(y|x)-g^{\frac{1}{2}}(y))g^{\frac{1}{2}}(y)dF(x)dy$$

$$= -n \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (g_{\theta_n}^{\frac{1}{2}}(y|x)-g^{\frac{1}{2}}(y))^2 dF(x)dy$$

$$= -\frac{a^2}{\theta_n^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (r_{\theta_n}^{\frac{1}{2}}(x,y)-1)^2 g(y)dydF(x)$$

$$\to -\tfrac{1}{4} a^2 \iint s^2(x,y)g(y)dydF(x) = -\tfrac{1}{4}\sigma^2, \quad \text{say.}$$

Recalling (3.8), (3.9) and the fact that $\theta_n = an^{-\frac{1}{2}}$, we have

$$(3.11) \qquad \sigma^2[(T_n-W_n)|P_n] = 4a^2\sigma^2\left[\left(\frac{r_{\theta_n}^{\frac{1}{2}}(X_1,Y_1)-1}{\theta_n} - \frac{s(X_1,Y_1)}{2}\right)|P_n\right] \to 0.$$

Under $P_n$, by (3.4) and (3.5), $T_n$ is the sum of n independent and identically distributed random variables and has mean 0 and finite variance $\sigma^2 = a^2\iint s^2(x,y)g(y)dydF(x)$ so that $\mathcal{L}(W_n|P_n) \to N(-\frac{1}{4}\sigma^2, \sigma^2)$, which is the second condition of LeCam's lemma. We now use the conclusions of Lemma 3.1 to prove the asymptotic normality of $T_n(\underset{\sim}{C})$ under $Q_n$.

As a preliminary to the proof of asymptotic normality we introduce a special layer-rank statistic $T_n(\underset{\sim}{C}^*)$. We define

(3.12) $$c^*_{n,i,j} = E\{[s(X_{j|n},Y_{i|j}) - \frac{1}{j-1}\sum_{\alpha=1}^{j-1} s(X_{\alpha|n},Y_{i|j})]|P_n\},\ 1 \leq i \leq j \leq n,$$

where $X_{j|n} = X_{(j)}$ $1 \leq j \leq n$, $Y_{i|j}$ is the $i^{\underline{th}}$ largest of $Y_{[1]},\ldots,Y_{[j]}$, and $X_{[i]}, Y_{(i)}$, $1 \leq i \leq n$ are defined on p.4 ; and we let

(3.13) $$T_n(\underset{\sim}{C}^*) = an^{-\frac{1}{2}} \sum_{j=1}^{n} c^*_{n,\ell_{(j)},j} .$$

By (3.4), (2.7) is clearly satisfied by $\{c^*_{n,i,j}\}$; moreover, by Lemma I.3, (2.5) is also satisfied with q.m. limit $ac^*(u,v)$, where

(3.14) $$c^*(u,v) = s(F^{-1}(v),G^{-1}(u)) - \frac{1}{v}\int_{-\infty}^{F^{-1}(v)} s(x,G^{-1}(u))dF(x).$$

By Corollary I.5, Appendix I, and assumption (3.5), $E[(T_n-T_n(C^*))^2|P_n] \rightarrow 0$. Combining this with (3.11) and conclusion (2) of Lemma 3.1, we conclude that $L_n-W_n$ converges in $P_n$-probability to a constant, which, along with conclusion (3) of Lemma 3.1, implies the following:

Corollary 3.1 If (3.4), (3.5) and (3.6) hold and if $\{Z_n\}$ is a sequence of random variables such that

(1) $\mathcal{L}(Z_n|P_n) \rightarrow N(\mu,b^2)$,

and

(2) $\mathcal{L}(Z_n,T_n(\underset{\sim}{C}^*)|P_n)$ converges to a bivariate normal law with correlation $\rho$, then

$$\mathcal{L}(Z_n|Q_n) \rightarrow N(\mu+\rho b\sigma,b^2),$$

where $\sigma^2 = a^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s^2(x,y)g(y)dydF(x)$.

Now consider any sequence $\{T_n(\underset{\sim}{C})\}$ of layer-rank statistics of the form (2.6) satisfying (2.5) and (2.7) with limiting weight function $c(u,v)$. Recalling (3.14), we define

(3.15) $$\sigma(c,c^*) = \iint c(u,v)c^*(u,v)dudv,$$

and we have the following theorem:

<u>Theorem 3.2</u> If $\{T_n(C)\}$ is a sequence of layer-rank statistics of the form (2.6) satisfying (2.5) and (2.7) and if $s(x,y)$, defined by (3.3), satisfies (3.4), (3.5) and (3.6), then $\mathcal{L}(T_n(C)|Q_n) \to N(a\sigma(c,c^*),\|c\|_2^2)$.

<u>Proof</u>: By Theorem 3.1, $\mathcal{L}(T_n(C)|P_n) \to N(0,\|c\|_2^2)$ so, by Corollary 3.1 and the fact, stated in Corollary I.6, that $\|c^*\|_2^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s^2(x,y)g(y)\,dy\,dF(x)$, it suffices to show that $\mathcal{L}(T_n(C), T_n(C^*)|P_n)$ is asymptotically normal with correlation $\sigma(c,c^*)/\|c\|_2\|c^*\|_2$. We prove this by showing that for arbitrary numbers $t_1$ and $t_2$ $\mathcal{L}(t_1T_n(C)+t_2T_n(C^*)|P_n)$ is asymptotically normal with zero mean and variance $t_1^2\|c\|_2^2+2at_1t_2\sigma(c,c^*)+t_2^2a^2\|c^*\|_2^2$. And, since $t_1T_n(C)+t_2T_n(C^*)$ is of the form (2.6), clearly satisfies (2.7), and, by the triangle inequality, satisfies (2.5) with limiting weight function $t_1c(u,v)+t_2ac^*(u,v)$, the conclusion follows at once from Theorem 3.1. ▌

Let us assume that $\sigma(c,c^*) \geq 0$ (if not, replace $c_{nij}$ by $-c_{nij}$, $1 \leq i \leq j \leq n$). As we mentioned earlier, we are dealing with a fixed family of bivariate distributions $\{H_\theta;-\infty < \theta < \infty\}$ we propose to test $\theta = 0$ (independence) versus $\theta > 0$ on the basis of a sample of size $n$ by a test having rejection region of the form $T_n(C) \geq k$, $k$ a constant. (To test $\theta = 0$ vs. $\theta < 0$ we use $T_n(C) \leq k$.) From Theorem 3.1 it follows that an approximate $\alpha$-level test is obtained if we set $k = z_\alpha\|c\|_2$, where $z_\alpha$ is the upper $100\alpha$ percentile of the standard normal distribution.

From Theorem 3.2 it follows that the power of the approximate $\alpha$-level test based on $T_n(C)$ against $H_{\theta_n}$, $\theta_n = an^{-\frac{1}{2}}$, $\theta > 0$, is

$$Q_n(T_n(C) \geq z_\alpha\|c\|_2) \to 1-\Phi(z_\alpha-a\sigma(c,c^*)/\|c\|_2). \tag{3.16}$$

This may be stated in a more convenient form as follows: As $\theta \downarrow 0$, the sample size $n$ needed to achieve power $1-\beta$ $(\alpha < 1-\beta < 1)$ against the alternative $H_\theta$ is given by:

$$n \sim [(z_\alpha + z_\beta)\|c\|_2/\theta\sigma(c,c^*)]^2, \tag{3.17}$$

provided $\sigma(c,c^*) > 0$. To prove this simply set the right side of (3.16) equal to $1-\beta$ and note that $a = \theta n^{\frac{1}{2}}$.

Let $T_n(C')$ be another sequence of test statistics satisfying (2.4) and (2.7), and let $n(\alpha,\beta,\theta)$ and $n'(\alpha,\beta,\theta)$ be the smallest sample sizes required by $\alpha$-level tests of the form $T_n(C) \geqq k$ and $T_n(C') \geqq k$, respectively, to achieve power $1-\beta$ against $H_\theta$. From (3.17) we conclude that

$$\lim_{\theta \downarrow 0} [n(\alpha,\beta,\theta)/n'(\alpha,\beta,\theta)] = [\rho(c',c^*)/\rho(c,c^*)]^2, \tag{3.18}$$

where $\rho(c,c^*) = \sigma(c,c^*)/(\|c\|_2 \|c^*\|_2)$, provided both $\rho(c',c^*) > 0$ and $\rho(c,c^*) > 0$. The limit on the left of (3.18) is called the Pitman asymptotic relative efficiency (Pitman ARE) of the sequence $\{T_n(C')\}$ with respect to the sequence $\{T_n(C)\}$ against the family $\{H_\theta; \theta \geqq 0\}$. The modifications when one is testing $\theta = 0$ vs. $\theta < 0$ are obvious and will not be discussed.

The statistic $L_n$ given by (3.7) is just the log-likelihood ratio statistic. Applying Lemma 3.1 (3) with $Y_n = L_n$ we conclude, by an argument similar to the one by which we derived (3.17), that the sample size $n$ required by the $\alpha$-level likelihood ratio test to attain power $1-\beta$ at $H_\theta$ is given by

$$n \sim [(z_\alpha + z_\beta)/\theta\|c^*\|_2]^2 \quad \text{as } \theta \downarrow 0. \tag{3.19}$$

Here $n$ is clearly a lower bound on the corresponding sample size for any other test. The Pitman ARE of $\{T_n(C)\}$ with respect to the likelihood ratio test we shall call simply the <u>Pitman efficiency</u> of $\{T_n(C)\}$

and will denote by $e(\underset{\sim}{C})$. From (3.17) and (3.19) we have

$$(3.20) \qquad e(\underset{\sim}{C}) = [\rho(c,c^*)]^2.$$

We want to emphasize the fact that $e(\underset{\sim}{C})$ depends upon the family of distributions $\{H_\theta; -\infty < \theta < \infty\}$, through $c^*$.

An immediate consequence of (3.20) is the fact that the sequence $\{T_n(\underset{\sim}{C}^*)\}$ defined by (3.12) has Pitman efficiency one when used as a test of $\theta = 0$ vs. $\theta > 0$ in the family $\{H_\theta: -\infty < \theta < \infty\}$. We shall call any sequence of test statistics having this property <u>asymptotically locally most powerful</u> (ALMP) <u>against $\{H_\theta; \theta \geq 0\}$</u>. Specific examples of ALMP sequences of statistics are given in the next two sections; Table II page 39 summarizes various Pitman ARE values.

The main reason for using layer-rank tests is, presumably, that the family $\{H_\theta; -\infty < \theta < \infty\}$ of bivariate distributions, of which the distribution of the sample is a member, is in fact unknown. Thus, the above, despite its theoretical value, doesn't give a practical way of selecting the appropriate test statistic; nevertheless, one may be willing to assume that the distribution is at least approximated by some member of a specific family, the bivariate normal, say, and use the layer-rank test which is ALMP against that family.

So far we have considered testing one sided alternatives only; in testing $\theta = 0$ versus $\theta \neq 0$ one might use a rejection region of the form $T_n(\underset{\sim}{C}) \geq k_2$ or $\leq k_1$, where $k_1 \leq k_2$ are constants. The power of this test against the alternative $H_\theta$, $\theta = an^{-\frac{1}{2}}$, approaches

$$(3.21) \qquad \Phi((k_1 - a\sigma(c,c^*))/\|c\|_2) + 1 - \Phi((k_2 - a\sigma(c,c^*))/\|c\|_2).$$

Since (3.21) attains its min. at $a = (k_1+k_2)/2\sigma(c,c^*)$, the above test will be biased for sufficiently large $n$ if $k_1 \neq -k_2$ (possibly even when $k_1 = -k_2$); therefore, a necessary condition that it be unbiased is that $k_1 = -k_2$. If $k_2 = z_{\alpha/2}\|c\|_2$, the test will be approximately level $\alpha$. We do not know what optimal properties, if any, this test has.

Note that $\rho(c,c^*)$ is the limit of the correlation, under $H_0$, between $T_n(\underset{\sim}{C})$ and $T_n(\underset{\sim}{C}^*)$, the ALMP statistic. Thus (3.20) resembles (2.7) of Van Eeden [21] but the conditions that she requires are different from those we require.

4. <u>Asymptotically Locally Most Powerful Layer-Rank Tests for a Certain Class of Bivariate Distributions</u>. Suppose $F(x)$ and $G(y)$ are continuous univariate cdf's. We propose a family of bivariate distributions $\{H_\theta(x,y), -\infty < \theta < \infty\}$ specified as follows: the marginal cdf of $X$ is $F$ and the conditional cdf of $Y$ given $X = x$, for fixed $\theta$, is:

$$G(a(\theta)y - \theta b(x)), \tag{4.1}$$

where $a(\theta)$ and $b(x)$ are any real functions. To put it another way, if $Y(\theta)$ denotes the Y-component of the random vector $(X,Y)$ with cdf $H_\theta$, then

$$Y(\theta) = (\theta(b(X)+\xi)/a(\theta), \tag{4.2}$$

where $X$ and $\xi$ are independent random variables with cdf's $F$ and $G$, respectively. $X$ is, of course, observable but $\xi$ is not. As an example, let $F$ be normal, $N(0,1)$, $b(x) = x$, $G(y) = \Phi(y)$, and $a(\theta) = (1+\theta^2)^{\frac{1}{2}}$, then the bivariate distribution specified by (4.1) or (4.2) is normal with zero means, unit variances, and correlation coefficient $\theta(1+\theta^2)^{-\frac{1}{2}}$.

In this section we derive an ALMP sequence of layer-rank tests of $\theta = 0$ vs. $\theta > 0$ under certain conditions on the functions $F$, $G$, $a$, and $b$. First we shall state these conditions: we assume that $G$ has density $g$, that $g$ is positive on $(-\infty,\infty)$, that $g^{\frac{1}{2}}$ and $g$ are absolutely continuous*, and that $g$ satisfies Hájek's condition [10]:

$$\int_{-\infty}^{\infty} (g'(y)/g(y))^2\, g(y)dy < \infty. \tag{4.3}$$

If $a(\theta)$ is non-constant we assume that $g$ also satisfies:

*I.e., $g^{\frac{1}{2}}$ and $g$ are indefinite integrals of their derivatives.

(4.4) $$\int_{-\infty}^{\infty} y^2[g'(y)/g(y)]^2\, g(y)dy < \infty.$$

Our conditions on $b(x)$ are simply that there is a $\delta > 0$ such that

(4.5) $$\int_{-\infty}^{\infty} |b(x)|^{2+2\delta}\, dF(x) < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} b(x)dF(x) = 0 \text{ (see footnote)};$$

and, finally, we require that $a(0) > 0$ and $a'(0) = 0$. We assume, without loss of generality, that $a(0) = 1$.

Note that the likelihood ratio $r_\theta(x,y) = dH_\theta(x,y)/dH_0(x,y)$ is $a(\theta)g(a(\theta)y - \theta b(x))/g(y)$. Thus

(4.6) $$s(x,y) = \{\partial/\partial\theta\} r_\theta(x,y)\big|_{\theta=0} = -b(x)g'(y)/g(y),$$

so that, if the conditions of Theorem 3.2 are met, then an ALMP sequence of layer-rank tests is, according to the remarks in the paragraph following (3.20), obtained from (3.12) by setting:

$$c^*_{n,i,j} = E[J(U_{i|j})]\{E[b(X_{j|n})] - \frac{1}{j-1}\sum_{\alpha=1}^{j-1} E[b(X_{\alpha|n})]\}$$
$$= J_{j,i}\, L_{n,j}\text{, say,}$$

where $J(u) = -g'(G^{-1}(u))/g(G^{-1}(u))$, $U_{i|j}$ is the $i^{\text{th}}$ largest of $j$ independent uniform (0,1) random variables and $X_{j|n}$ is the $j^{\text{th}}$ largest of a sample of size $n$ from a univariate population with cdf $F$. The test statistic is, of course,

(4.7) $$T_n(\underset{\sim}{C}^*) = n^{-\frac{1}{2}} \sum_{j=1}^{n} J_{j,\ell(j)}\, L_{n,j}\,,$$

where $\ell(j)$ is the layer rank defined on p. 4 and the q.m. limit $c^*$, given by (3.14), is

(4.8) $$c^*(u,v) = J(u)[b(F^{-1}(v)) - \frac{1}{v}\int_{-\infty}^{F^{-1}(v)} b(x)dF(x)].$$

The Pitman ARE of any other sequence of layer-rank statistics with respect to the sequence (4.7) is obtained by inserting (4.8) into formula (3.20)

The following lemma, which parallels Hájek's [10] treatment of the univariate case, states that the conditions of Theorem 3.2 are satisfied in this bivariate case.

* This assumption causes no loss of generality; for, by the first paragraph of p.5 , we may make the transformation $Y' = Y - \theta\int b(x)dF(x)/a(\theta)$.

Lemma 4.1: If the conditions on $g$ stated in the sentence containing (4.3) hold, if either (4.4) holds or $a(\theta) = 1$, and if the conditions on a and b stated in the sentence containing (4.5) hold, then (3.4), (3.5) and (3.6) hold for $s(x,y)$ given by (4.6).

Proof: (3.5) is a trivial consequence of (4.3) and (4.5).

Since $0 = (d/dz)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} b(x)g(y+z)dydF(x)$, (3.4) will be shown to hold if we can show that $(d/dz)\int_{-\infty}^{\infty} g(y+z)dy|_{z=0} = \int_{-\infty}^{\infty} g'(y)dy$. Now,

$$0 = \int_{-\infty}^{\infty} (g(y+z)-g(y))/z \, dy = \int_{-\infty}^{\infty} (\int_{y}^{y+z} g'(w)dw/z)dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (g'(w)I_{(y,y+z]}(w)/z)dw \, dy,$$

where $I_A(y)$ is the indicator of the set A. Letting $\lambda^2$ denote Lebesgue measure on the plane and applying the Fubini theorem* we have

$$|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ((g'(w) \, I_{(y,y+z]}(w)/z)d\,\lambda^2(w,y)|$$

$$\leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (|g'(w)|I_{(y,y+z]}(w)/z)d\,\lambda^2(w,y) = \int_{-\infty}^{\infty} |g'(w)|dw$$

$$\leq [\int_{-\infty}^{\infty} (g'(w)/g(w))^2 \, g(w)dw]^{\frac{1}{2}} < \infty.$$

so that $g'(w)I_{(y,y+z]}(w)$ is $\lambda^2$-integrable. Therefore, applying the Fubini theorem once more, we have $0 = \int_{-\infty}^{\infty}(\int_{y}^{y+z} g'(w)dw/z)dy = \int_{-\infty}^{\infty}g'(w)dw$, which immediately implies the desired result.

Finally, to prove (3.6), note that since the difference quotient $(r_\theta^{\frac{1}{2}}(x,y)-1)/\theta \to -[b(x)g'(y)/g(y)]/2$ pointwise as $\theta \to 0$, it suffices, by the $L_r$-convergence theorem**, to show that

$$\lim_{\theta\to 0} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(r_\theta^{\frac{1}{2}}(x,y)-1)/\theta)^2 g(y)dydF(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s^2(x,y)g(y)dydF(x)/4.$$

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*Loéve [15] p. 136 Theorem B.

**Loéve [15] p. 163 Theorem C. Note: the form of this theorem found in the first and second editions is not suitable.

Now,

$$[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}((r_\theta^{\frac{1}{2}}(x,y)-1)/\theta)^2 g(y)dydF(x)]^{\frac{1}{2}}$$

$$= [\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{[a^{\frac{1}{2}}(\theta)g^{\frac{1}{2}}(a(\theta)y-\theta b(x))-g^{\frac{1}{2}}(y)]/\theta\}^2 \; dydF(x)]^{\frac{1}{2}}$$

$$\le \left|(a^{\frac{1}{2}}(\theta)-1)/\theta\right|[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(y)dydF(x)]^{\frac{1}{2}} \tag{4.9}$$

$$+[a(\theta)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(g^{\frac{1}{2}}(y)-g^{\frac{1}{2}}(a(\theta)y))^2 dydF(x)]^{\frac{1}{2}}$$

$$+[a(\theta)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(g^{\frac{1}{2}}(a(\theta)y-\theta b(x))-g^{\frac{1}{2}}(a(\theta)y))^2 dydF(x)]^{\frac{1}{2}}.$$

The first term of (4.9) approaches $a'(0)/2a(0) = 0$. Consider the second term, if $a(\theta) = 1$ it is zero, otherwise we have:

$$(a(\theta)/\theta^2)\int_{-\infty}^{\infty}(g^{\frac{1}{2}}(y)-g^{\frac{1}{2}}(a(\theta)y))^2 dy$$

$$= (a(\theta)/\theta^2)\int_{-\infty}^{\infty}\Big(\int_{\min(y,a(\theta)y)}^{\max(y,a(\theta)y)}(g'(z)/2g^{\frac{1}{2}}(z))dz\Big)^2 dy$$

$$\le (a(\theta)|a(\theta)-1|/\theta^2)\int_{-\infty}^{\infty}|y|\int_{\min(y,a(\theta)y)}^{\max(y,a(\theta)y)}(g'(z)/2g^{\frac{1}{2}}(z))^2 dz\,dy$$

$$= (a(\theta)|a(\theta)-1|/4\theta^2)\int_{-\infty}^{\infty}(g'(z)/g^{\frac{1}{2}}(z))^2\int_{\min(z,z/a(\theta))}^{\max(z,z/a(\theta))}|y|\,dydz$$

$$= (a(\theta)|a(\theta)-1|\,\Big|\frac{1}{a^2(\theta)}-1\Big|/8\theta^2)\int_{-\infty}^{\infty}z^2(g'(z)/g(z))^2 g(z)dz \;\to\; 0,$$

by the assumption that $\int_{-\infty}^{\infty}z^2(g'(z)/g(z))^2 g(z)dz < \infty$.

Now consider the last term of (4.9): for the sake of clarity we assume $\theta > 0$ and $b(x) > 0$ (if $\theta < 0$ or $b(x) \le 0$ the proof goes through with obvious modifications); for fixed $x$, we have:

$$\int_{-\infty}^{\infty}(g^{\frac{1}{2}}(y-\theta b(x))-g^{\frac{1}{2}}(y))^2 dy/\theta^2$$

$$= \int_{-\infty}^{\infty}\Big(\int_{y-\theta b(x)}^{y}(g'(z)/2g^{\frac{1}{2}}(z)dz\Big)^2 dy/\theta^2$$

$$\le \int_{-\infty}^{\infty} b(x)\int_{y-\theta b(x)}^{y}(g'(z)/2g^{\frac{1}{2}}(z))^2 dz\,dy/\theta$$

$$= \int_{-\infty}^{\infty} b^2(x)(g'(z)/g(z))^2\; g(z)dz/4 .$$

Integrating with $dF(x)$ and combining the result with (4.9), we have:

$$\limsup_{\theta\to 0} \iint((r_\theta^{\frac{1}{2}}(x,y)-1)/\theta)^2 g(y)\,dy\,dF(x)$$

$$\leq \iint s^2(x,y)g(y)\,dy\,dF(x)/4.$$

By Fatou's lemma, the reverse inequality holds for lim inf and the Lemma is proved.

We present below two examples illustrating the results of this section.

Example 4.1 Recall the specification (4.2) of the bivariate distribution $H_\theta$. We assume that $X$ and $\xi$ are both normally distributed with zero means and unit variances (if not* let $Y_1(\theta) = Y(\theta)/\sigma(\xi)$, $X_1 = X/\sigma(X)$, $b_1(x) = b(x\sigma(X))/\sigma(\xi)$, and $\xi_1 = \xi/\sigma(\xi)$.) If $b(x)$ is linear, then $H_\theta(x,y)$ is bivariate normal; as a slight generalization, we assume $b(x)$ is a $p^{\underline{th}}$ degree polynomial, in fact, we assume that

$$(4.11) \qquad b(x) = \sum_{k=1}^{p} b_k H_k(x),$$

where $H_k(x) = (-1)^k((d^k/dx^k)\varphi(x))/\varphi(x)$ is the $k^{\underline{th}}$ Hermite polynomial.

Since (4.3), (4.4), ( . ) and (4.5) are satisfied we can construct an ALMP sequence of layer-rank tests of the form (4.8), in fact, since $g$ is the standard normal density, the statistics are given by (4.7) with

$$(4.12) \qquad J_{j,i} = \mu_{i|j}, \qquad 1 \leq i \leq j \leq n$$

and

$$(4.13) \qquad L_{n,j} = \sum_{k=1}^{p} b_k\{EH_k(X_{j|n}) - \frac{1}{j-1}\sum_{\alpha=1}^{j-1} EH_k(X_{\alpha|n})\},$$

where $\mu_{i|j}$ is the mean of the $i^{\underline{th}}$ largest of $j$ standard normal

*I.e., if the variances are not one.

random variables and $X_{j|n}$ is the $j^{\text{th}}$ largest of $n$ standard normal random variables. The limiting weight function $c^*(u,v)$ (see (4.8)) is

$$\Phi^{-1}(u) \sum_{k=1}^{p} b_k\{H_k(\Phi^{-1}(v)) - \frac{1}{v}\int_{-\infty}^{\Phi^{-1}(v)} H_k(x)\varphi(x)dx\} \tag{4.14}$$

$$= \Phi^{-1}(u) \sum_{k=1}^{p} b_k H_k^*(\Phi^{-1}(v)) = c(u,v;\underset{\sim}{b}), \text{ say,}$$

where $H_k^*(x) = H_k(x) + H_{k-1}(x)(\varphi(x)/\Phi(x))$ $k = 1,2,\ldots, p$ and $\underset{\sim}{b} = (b_1,b_2,\ldots,b_p)$. It is pleasant to note that the functions $H_k^*$, $k = 1,2,\ldots,$ are orthogonal with respect to $\varphi(x)$, as we now demonstrate:

$$\int_{-\infty}^{\infty} H_k^*(x)H_{k'}^*(x)\varphi(x)dx = \int_{-\infty}^{\infty} H_k(x)H_{k'}(x)\varphi(x)dx$$

$$+ \int_{-\infty}^{\infty}(H_{k-1}(x)H_{k'}(x) + H_k(x)H_{k'-1}(x))\,\frac{\varphi^2(x)}{\Phi(x)}\,dx$$

$$+ \int_{-\infty}^{\infty} H_{k-1}(x)H_{k'-1}(x)\,\frac{\varphi^3(x)}{\Phi^2(x)}\,dx$$

$$= k!\delta_{kk'} - \int_{-\infty}^{\infty}\left(\frac{d}{dx}\,(H_{k-1}(x)H_{k'-1}(x)\varphi^2(x))\right)\frac{1}{\Phi(x)}\,dx$$

$$+ \int_{-\infty}^{\infty} H_{k-1}(x)H_{k'-1}(x)\,\frac{\varphi^3(x)}{\Phi^2(x)}\,dx = k!\delta_{k,k'},$$

where $\delta_{k,k'}$ is the Kroneker $\delta$. Thus, if $\underset{\sim}{b}' = (b_1',b_2',\ldots,b_p')$, then, recalling (4.14),

$$\iint c(u,v;\underset{\sim}{b})c(u,v;\underset{\sim}{b}')dudv = \Sigma_{k=1}^{p} k!b_k b_k'. \tag{4.15}$$

We use this result as follows: Suppose we assume the model of this example with $b(x)$ given by (4.11) to be the true specification of $H_\theta$ and employ the ALMP layer-rank test for $\theta = 0$ vs $\theta > 0$, namely (4.7) with $J_{j,i}$ and $L_{n,j}$ given by (4.12) and (4.13). If this model is not correct and in fact $b(x)$ is given by (4.11) but with coefficients $(b_1',b_2',\ldots,b_p')$, then (3.20) implies that the ARE of the test we are using compared to the ALMP test corresponding to the true specification is:

$$\left(\sum_{k=1}^{p} k!b_k b_k'\right)^2 / \left(\sum_{k=1}^{p} k!b_k^2\right)\left(\sum_{k=1}^{p} k!b_k'^2\right).$$

We have tabulated in TableIIIthe constants $L_{n,j}$, $1 \leq j \leq n \leq 20$, given by (4.13) for the special case $b(x) = x$, which gives the ALMP layer-rank test of $\theta = 0$ vs $\theta > 0$ (positive correlation) in the bivariate normal distribution. We call this the Normal Scores Layer-Rank Test; if $T_n(\underset{\sim}{C})$ is the test statistic, then, by (4.15) and the remarks following Theorem 3.2, an approximate $\alpha$-level test has rejection region $T_n(\underset{\sim}{C}) \geq z_\alpha$. In Figure 2 we illustrate the use of Table I by computing the Normal Scores Layer-Rank Test statistic for a sample of size 10.

Example 4.2 Let X be a positive random variable with distribution F and let G be an absolutley continuous cdf with density g where $g(y) = 0$ $(y \leq 0)$, $g(y) > 0$ $(y > 0)$. Suppose the conditional cdf of Y given $X = x$ is $G(Y/X^\theta)$. From the remarks at the end of Section 1 we conclude that the properties of statistics based on layer ranks are unchanged if we make the transformation $Y' = \ell n(Y)$. But the conditional cdf of $Y'$ given $X = x$ is $G(\exp(y'-\theta\ell n(x)))$, which is in the form (4.1) given above but with $G(y)$ replaced by $G(\exp(y))$.

We can now specialize the results of this section to obtain an ALMP layer-rank test of $\theta = 0$ vs $\theta > 0$, in fact if

$$J(u) = -1-g'(G^{-1}(u))\cdot G^{-1}(u)/g(G^{-1}(u))$$

and

$$c^*_{nij} = EJ(U_{i|j})\{E\ell n(X_{j|n}) - \frac{1}{j-1}\sum_{i=1}^{j-1}\ell n(X_{\alpha|j})\}$$

then the sequence of tests based on $T_n(\underset{\sim}{C}^*)$ is ALMP against $\theta > 0$ and has limiting weight function

$$c^*(u,v) = J(u)\{\ell n(F^{-1}(v)) - \frac{1}{v}\int_0^{F^{-1}(v)} \ell n(x)dF(x)\}$$

provided the following conditions are met:

Figure 2

Computation of the Normal Scores Layer-Rank Test Statistic for a Sample of Size 10.

| $j$ | $X_j$ | $Y_j$ | $Y_{[j]}$ | $\ell_{(j)}$ | $\mu_{\ell_{[j]}\mid j}$ | $L^*_{10,j}$ |
|---|---|---|---|---|---|---|
| 1 | -.221 | -3.238 | .196 | 1 | 0 | — |
| 2 | -2.454 | 2.044 | 2.044 | 2 | .56419 | .53739 |
| 3 | .089 | 1.183 | .589 | 2 | 0 | .61399 |
| 4 | .931 | -1.741 | -3.202 | 1 | -1.02938 | .68963 |
| 5 | .361 | 2.649 | -3.238 | 1 | -1.16296 | .77031 |
| 6 | -.559 | -3.202 | 1.183 | 5 | .64176 | .86159 |
| 7 | -4.816 | .196 | 2.649 | 7 | 1.35218 | .97108 |
| 8 | .784 | .401 | .401 | 4 | -.15251 | 1.11266 |
| 9 | 2.576 | -.764 | -1.741 | 3 | -.57197 | 1.31887 |
| 10 | -1.232 | .589 | -.764 | 4 | -.37576 | 1.70972 |
| | | | | | -1.15535 | |

$$T_{10}(\underset{\sim}{c}^*) = \sum_{j=1}^{10} \mu_{\ell_{(j)}\mid j} \, L^*_{10,j}/\sqrt{10}$$

$$= -1.155/3.162$$

$$= -.365$$

The $\mu_{i|j}$ values are from Sarhan and Greenberg [19] Table 10B.1 and the $L^*_{10,j}$ values are from Table I of this paper.

$$\int_0^1 J^2(u)du = \int_0^\infty (1+yg'(y)/g(y))^2 g(y)dy < \infty$$

and

$$\int_0^1 |\ell n(x)|^{2+\delta} dF(x) < \infty, \quad \text{for some } \delta > 0 \text{ and*} \int_{-\infty}^{\infty} \ell n(x)dF(x) = 0.$$

We remark that the transformation $Y' = \ell n(Y)$ was made in order to apply the results of this section and need not be made to compute the test statistic since it is invariant under this transformation.

As a more specific example suppose $G(y) = 1-e^{-y}$ so that, given $X$ and $\theta$, $Y$ is exponential with scale parameter $X^\theta$; $G(y/x^\theta)$ is a possible model for the conditional distribution of the lifetime $Y$ of an object when the lifetime depends stochastically on an additional observable random variable $X$. If (4.7) is satisfied, then an ALMP sequence of layer-rank tests for testing $\theta = 0$ vs $\theta > 0$ is $\{T_n(\underset{\sim}{C})\}$, defined in (2.3) with

$$c^*_{nij} = \left(\sum_{\beta=1}^{i} \frac{1}{j-\beta+1} - 1\right)\{E\ell n(X_{j|n}) - \frac{1}{j-1}\sum_{\alpha=1}^{j-1} E\ell n(X_{\alpha|n})\}$$

and the limiting weight function is

$$c^*(u,v) = -(1+\ell n(1-u))\{\ell n(F^{-1}(v) - \frac{1}{v}\int_0^{F^{-1}(v)} \ell n(x)dF(x)\}.$$

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*But see the footnote on p.22.

5. Asymptotically Locally Most Powerful Layer Tests. In Section 0 we defined the ($3^{rd}$ quadrant) layer statistics $(A_n^{(1)}, \ldots, A_n^{(n)})$, $A_n^{(r)}$ being the number of sample points with layer rank $r$. A layer test of $\theta = 0$ vs $\theta > 0$ in the bivariate family $\{H_\theta; \theta \geq 0\}$ is a test which rejects for large values of a statistic of the form

$$T_{n2}(\underset{\sim}{J}) = n^{-\frac{1}{2}} \sum_{r=1}^{n} A_n^{(r)} J_{n,r} - K_n$$

where $\underset{\sim}{J} = \{J_{n,r}; 1 \leq r \leq n\}$ is a double sequence of real numbers and $K_n$ is selected so that $E_0[T_{n2}(\underset{\sim}{J})] = 0$.

Since

$$T_{n2}(\underset{\sim}{J}) = n^{-\frac{1}{2}} \sum_{r=1}^{n} \sum_{\{j:\ell_{(j)}=r\}} J_{n,r} - K_n = n^{-\frac{1}{2}} \sum_{j=1}^{n} J_{n,\ell_{(j)}} - K_n;$$

$T_{n2}(\underset{\sim}{J})$ is a layer-rank statistic of the form (2.10.2), with $L_{n,j} = 1$, and $K_n = n^{-\frac{1}{2}} \sum_{j=1}^{n} \frac{1}{j} \sum_{i=1}^{j} J_{n,i}$.

As usual, we define a step function on (0,1)

$$J_n(u) = J_{n,j}, \qquad \frac{j-1}{n} \leq u < \frac{j}{n}, \qquad 1 \leq j \leq n. \tag{5.1}$$

We require that there exist a function $J(u)$ on (0,1) such that for some $\delta > 0$, letting $\|g\|_r = \int_0^1 |g(u)|^r du$, we have

$$\|J\|_{2+\delta} < \infty \quad \text{and} \quad \|J_n - J\|_{2+\delta} \to 0 \tag{5.2}$$

The asymptotic theory of layer-rank tests (in particular $T_{n2}(\underset{\sim}{J})$) developed in Section 3 was based on bivariate "limiting weight functions" $c(u,v)$. In the present case, Lemma I.7.2 (Appendix I) and (5.2) imply that

$$c(u,v) = J(uv) - \frac{1}{v} \int_0^v J(w) dw.$$

Therefore if the family* $\{H_\theta; \theta \geq 0\}$ satisfies (3.3), (3.4), (3.5) and (3.6), then (3.20) implies that the Pitman ARE, $e(\underset{\sim}{J})$, of $T_{n2}(\underset{\sim}{J})$ with

*We assume this family to be fixed but arbitrary and that $\theta = 0$ corresponds to the hypothesis of independence.

respect to an ALMP sequence of tests is:

$$(5.3)\qquad e(J) = \frac{[\iint J(uv)c^*(u,v)dudv]^2}{\|c^*\|_2^2 \iint [J(uv) - \frac{1}{v}\int_0^v J(w)dw]^2 dudv},$$

provided† $\iint J(uv)c^*(u,v)dudv > 0$, where $c^*(u,v)$ is given by (3.14).

If $J(t)$ is absolutely continuous $0 < t < 1$ and $\lim_{t\to 0} tJ(t) = 0 = \lim_{t\to 1}(1-t)J(t)$, then

$$(5.4)\qquad \iint [J(uv) - \frac{1}{v}\int_0^v J(w)dw]^2 dudv$$

$$= \iint_{t\le v} \frac{1}{v}[J(t)-J(v) + \frac{1}{v}\int_0^v wJ'(w)dw]^2 dtdv = -\int [\frac{1}{v}\int_0^v wJ'(w)dw]^2$$

$$- 2\iint_{t\le v} \frac{t}{v}[J(t)-J(v)]J'(t)dtdv$$

$$= -2\int\int\int_{w_1\le w_2\le v} \frac{1}{v^2} w_1 w_2 J'(w_1)J'(w_2)dvdw_1dw_2$$

$$- 2\iint_{t\le v} t\ell n(v)J'(v)J'(t)dvdt$$

$$= 2\iint_{u\le v} J'(u)J'(v)u(v-1-\ell n(v))dudv = \iint J'(u)J'(v)K(u,v)dudv,$$

where

$$K(u,v) = \begin{cases} u(v-1-\ell n(v)) & u \le v \\ K(v,u) & u \ge v \end{cases}$$

is a symmetric positive definite kernel*.

Moreover, if $\lim_{t\to 0} tJ(t)c^*(\frac{t}{v},v) \to 0$ for almost all $v$, then

*In fact it is the limit of the covariance kernel of the stochastic process $Z_n(t) = n^{-\frac{1}{2}} \Sigma_{r=1}^{[nt]} A_n^r$, $0 < t < 1$.

†It is also necessary to assume that neither c nor J is zero almost surely.

$$(5.5)\qquad \iint J(uv)c^*(u,v)\,du\,dv = \iint_{t\le v} \frac{1}{v} J(t)c^*(\tfrac{t}{v},v)\,dt\,dv$$

$$= \iint_{t\le v} \frac{1}{v} J'(t) \int_t^v c^*(\tfrac{w}{v},v)\,dw\,dt\,dv$$

$$= \int J'(t)\Big[\int_t^1 \int_w^1 \frac{1}{v} c^*(\tfrac{w}{v},v)\,dv\,dw\Big]dt$$

$$= \int J'(u)\gamma(u)\,du,$$

where

$$(5.6)\qquad \gamma(u) = \int_u^1 \int_w^1 \frac{1}{v} c^*(\tfrac{w}{v},v)\,dv\,dw, \qquad 0 < u < 1.$$

By combining (5.3), (5.4), and (5.5), we obtain another expression for the Pitman ARE of $T_{n2}(J)$,

$$(5.7)\qquad e(J)\|c^*\|_2^2 = \frac{[\int J'(u)\gamma(u)\,du]^2}{\iint J'(u)J'(v)K(u,v)\,du\,dv}$$

The sequence of layer tests based on the statistics $\{T_{n2}(\underset{\sim}{J})\}$ will be ALMP among all layer tests if the derivative of its J-function (see 5.2) maximizes the right side of (5.7) and $\int J'(u)\gamma(u)du > 0$. This sequence is in general not ALMP among all tests.

We now derive the J-function whose derivative maximizes the right side of (5.7). On the space of real-valued functions defined on (0,1) we introduce an inner product

$$(\gamma_1,\gamma_2)_K = \iint \gamma_1(u)\gamma_1(v)K(u,v)\,du\,dv,$$

and a norm

$$(\|\gamma_1\|_K)^2 = (\gamma_1,\gamma_1)_K.$$

Define $\gamma$ by (5.6); if there is a $\gamma^*$ such that

$$(5.8)\qquad \gamma(u) = \int \gamma^*(v)K(v,u)\,dv,$$

then (5.7) becomes

$$(5.9) \qquad e(\underset{\sim}{J})\|c^*\|_2^2 = \left(\frac{(J',\gamma^*)_K}{\|J'\|_K}\right)^2 ,$$

from which it is clear that $J' = \pm\gamma^*$ maximizes $e(\underset{\sim}{J})$. Since we also require that $0 < \int J'(u)\gamma(u)du = \iint J'(u)\gamma^*(v)K(u,v)dudv$, the correct solution is $J' = \gamma^*$. Thus, the problem reduces to solving the integral equation (5.8) or, in view of the remarks just above, to solving

$$(5.10) \qquad \gamma(u) = \int J'(v)K(v,u)dv = (u-1-\ln(u))\int_0^u vJ'(v)dv + u\int_u^1 (v-1-\ln(v))J'(v)dv.$$

By taking the first two derivatives of (5.10) and solving the resulting system of equations for $J'$, one can easily verify that the solution of (5.10) is:

$$(5.11) \qquad J'(u) = \frac{\gamma''(u)}{\ln(u)} - \frac{\gamma'(u)}{u\ln^2(u)} + \frac{\gamma(u)}{u^2\ln^2(u)} ,$$

where, $\gamma$ is given by (5.6). Hence,

$$(5.12) \qquad J(u) = \frac{\gamma'(u)}{\ln(u)} - \int_u^1 \frac{\gamma(w)}{w^2\ln^2(w)}dw .$$

If $\|J\|_{2+\delta} < \infty$ for some $\delta > 0$, then we can construct a sequence of layer test statistics $\{T_{n2}(\underset{\sim}{J})\}$ which is ALMP among all layer tests. We do this by finding a double sequence $\{J_{n,r}; 1 \le r \le n\}$ such that $J_n$, defined by (5.1), converges in $2+\delta^{\text{th}}$ moment to J. By an obvious generalization of Hájek [9 ] Lemma 6.1 one such choice is

$$(5.13) \qquad J_{n,r} = EJ(U_{r|n}),$$

where $U_{r|n}$ is the $r^{\text{th}}$ largest of n uniform (0,1) random variables; another possibility is simply

$$(5.14) \qquad J_{n,r} = J(\tfrac{r}{n+1}).$$

Lemma 5.1 Let $J_{n,r} = J(\frac{r}{n+1})$ and $\|J\|_{2+\delta} < \infty$. If $J$ is continuous on $(0,1)$ and there is a number $u_0$, $0 < u_0 < \frac{1}{2}$, such that $|J|$ is non-increasing on $(0,u_0]$ and non-decreasing on $[1-u_0,1)$, then $\|J_n - J\|_{2+\delta} \to 0$.

Proof: For any $\epsilon < u_0$ it is clear that $\int_\epsilon^{1-\epsilon} |J_n(u)-J(u)|^{2+\delta}du \to 0$.

Consider

$$\int_0^\epsilon |J_n(u)|^{2+\delta}du \leq \frac{1}{n}\sum_{r=1}^{[n\epsilon]+1} |J(\frac{r}{n+1})|^{2+\delta}$$

$$\leq \int_0^{\epsilon+\frac{1}{n}} |J(u)|^{2+\delta}du + \int_0^{\frac{1}{n}} |J(u)|^{2+\delta}du$$

$$\to \int_0^\epsilon |J(u)|^{2+\delta}du.$$

Since the latter can be made arbitrarily small and a similar result holds for the upper tail, the Lemma is proved.

Let $\{T_{n2}(\underset{\sim}{J})\}$ be a sequence of layer test statistics using the weights given by (5.13) or (5.14) with $J$ given by (5.12). Since $J' = \gamma^*$, (5.8) and (5.9) imply that the Pitman efficiency of $\{T_{n2}(\underset{\sim}{J})\}$ is

$$(5.15) \qquad e(\underset{\sim}{J}) = \frac{(J',\gamma^*)_K}{\|c^*\|_\lambda^2} = \frac{\int J'(u)\gamma(u)du}{\int\int(c^*(u,v))^2 dudv} .$$

Example 5.1 Kendall's $\tau$ and related statistics.

Consider the following family of bivariate cdf's:

$$(5.16) \qquad \{H_\theta : H_\theta(x,y) = F(x)G(y)(1+\theta(1-F^m(x))(1-G^m(y))), \ -\frac{1}{m} \leq \theta \leq \frac{1}{m^2}, \quad m \geq 1\}.$$

The marginals of $H_\theta$ are $F$ and $G$ and since the properties of any layer-rank statistic are marginal free we can work with the following family*: $\{H_\theta : H_\theta(x,y) = xy(1+\theta(1-x^m)(1-y^m)), 0 < x, y < 1, \ -\frac{1}{m} \leq \theta \leq \frac{1}{m^2}, \quad m \geq 1\}$.

For this family $s(x,y) = (1-(m+1)x^m)(1-(m+1)y^m)$ (see (3.3)); consequently,

*Since we are concerned with testing $\theta=0$ versus $\theta>0$ the negative values of $\theta$ are immaterial.

the optimal limiting weight function is $c^*(u,v) = (-v^m+(m+1)(uv)^m)m$ (see (3.14)). Also, (5.6) becomes

$$\gamma(u) = u(1-u^m)+mu^{m+1}\ell n(u),$$

so that $J'$, given by (5.10), becomes

$$J'(u) = m^2(m+1)u^{m-1},$$

and, except for an arbitrary constant,

$$J(u) = m(m+1)u^m.$$

In view of the remarks in the paragraph containing (5.3) and (5.14) and Lemma 5.1 either of the following will give sequence of layer tests which is ALMP among all layer tests:

(5.17) $$J_{n,r} = EJ(U_{r|n}) = m(m+1)\frac{(r+m-1)(r+m-2)\ldots r}{(n+m)(n+m-1)\ldots(n+1)}$$

or

(5.18) $$J_{n,r} = J\left(\frac{r}{n+1}\right) = m(m+1)\left(\frac{r}{n+1}\right)^m .$$

The Pitman efficiency of a layer test using either of the above weights is given by (5.15) and is:

$$e(\underset{\sim}{J}) = \frac{(m+1)\int u^{m-1}[u(1-u^m)+mu^{m+1}\ell n(u)]du}{\iint(v^m-(m+1)(uv)^m)^2 dudv} = 1.$$

Thus the ALMP layer test for testing $\theta = 0$ vs $\theta > 0$ in the family (5.16) is in fact ALMP among all tests.

In particular if $m = 1$ in (5.16) then (5.17) and (5.18) reduce to $J_{n,r} = \frac{2r}{n+1}$ and $\frac{(n+1)n^{\frac{1}{2}}}{2}(T_n(\underset{\sim}{J})-K_n) = \sum_{j=1}^{n} A_n^{(r)} = \sum_{j=1}^{n} \ell_{(j)}$, which is essentially Kendall's $\tau$-statistic (see (2.11)). Thus Kendall's $\tau$-statistic

is ALMP for testing $\theta = 0$ vs $\theta > 0$ in the family*
$\{H_\theta; H_\theta(x,y) = F(x)G(y)(1+\theta(1-F(x))(1-G(y))\}$. We remark that there are other families against which $\tau$ is ALMP, for example, a family of the type considered in Section 4 with $G_\theta(y|x) = (1-e^{-(y-\theta x)})^{-1}$ $-\infty < y < \infty$, $0 < x < 1$, and $F(x) = x$, $0 < x < 1$.

Example 5.2 The ALMP layer test against the bivariate normal alternative.

Let $H_\theta(x,y)$ be a bi-normal cdf with correlation $\theta(1+\theta^2)^{-\frac{1}{2}}$ (see Section 4). From (3.14), $c^*(u,v) = \Phi^{-1}(u)[\Phi^{-1}(v) + \frac{\varphi(\Phi^{-1}(v))}{v}]$, where $\Phi$ and $\varphi$ are the standard normal cdf and density, respectively. Leaving out the details of its derivation from (5.6) and (5.11), we claim that the optimal J-function is given by:

$$(5.19)\qquad J(u) = \int_u^1 \int_v^1 \frac{[\varphi(\Phi^{-1}(w))-w\Phi^{-1}(w)]\varphi(\Phi^{-1}(\frac{u}{w}))}{w(v\ln(v))^2} dwdv$$

$$- \int_u^1 \frac{[\varphi(\Phi^{-1}(v))+v\Phi^{-1}(v)]\Phi^{-1}(\frac{u}{v})}{v^2} dv .$$

J(u) is tabulated in Table VIII and is also presented in graphical form in Figure 3. If one defines $J_{n,r}$, $r = 1,\ldots,n$, by (5.14) then the layer test statistic $T_{n2}(J)$ defined by (5.2) can be computed either from the graph of J or from Table VIII.

We have computed the Pitman efficiency of this test and found it to be approximately .955. It has been conjectured that this number is $\frac{3}{\pi} \doteq .95493$ but we have made no progress in proving or disproving the conjecture.

Tables I and II on pages 38 and 39 summarize the examples discussed in this and the preceding section.

*The reader should compare this with the fact that the Wilcoxon statistic is ALMP in the univariate two sample problem for testing $F = G$ against $G = F(1-\theta(1-F))$, $\theta > 0$.

Fig. 3 Graph of J(u) ([illegible]) for the ALMP layer test against the normal alternative.

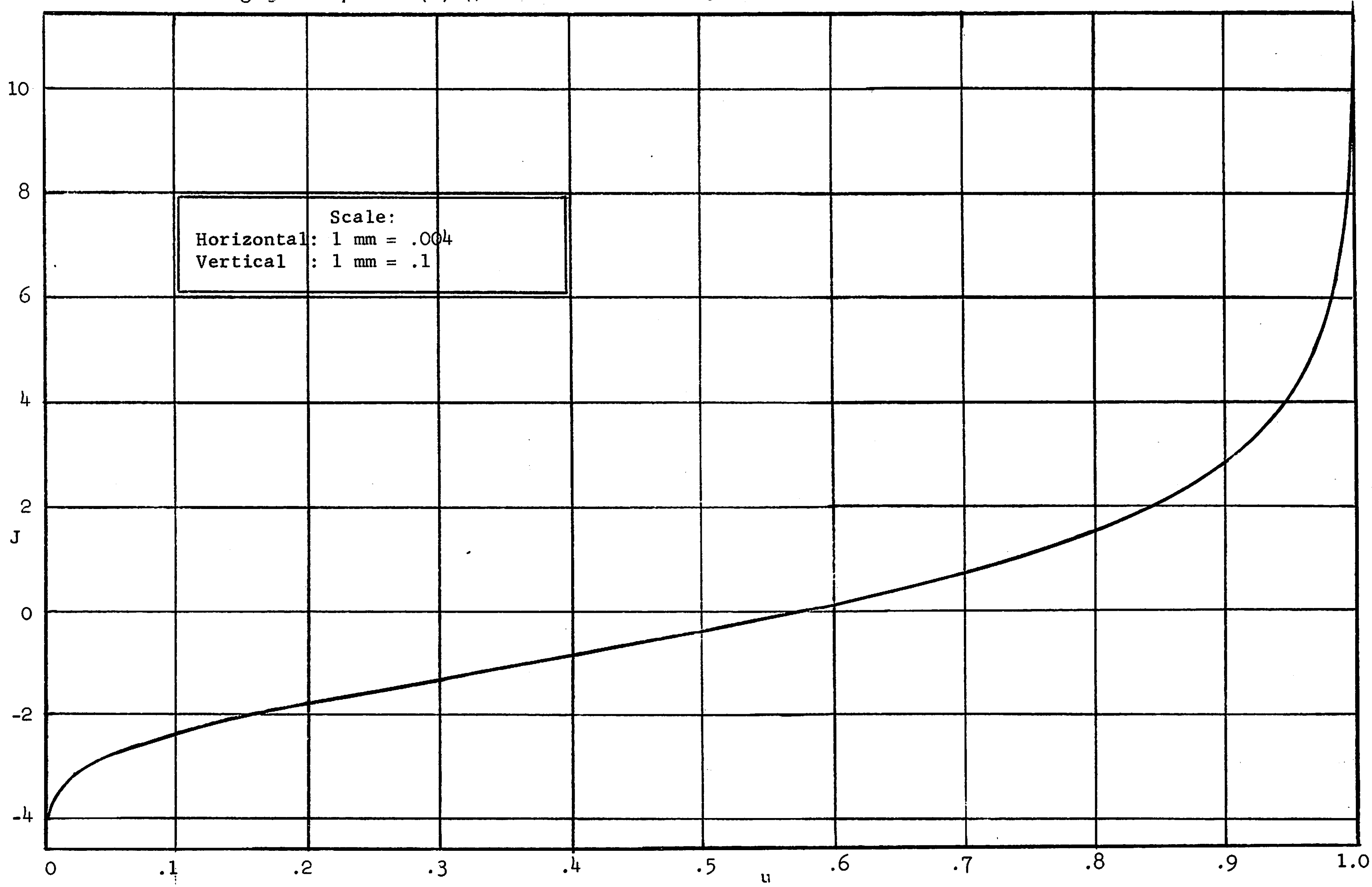

Table II : Several bivariate families and their ALMP layer-rank tests.

| Name of family | Member of family corresponding to parameter value $\theta$. | ALMP Layer-rank test statistic $T_n(\underset{\sim}{C}^*) = n^{-\frac{1}{2}} \sum_{j=1}^{n} c^*_{n,\ell(j),j}$, where $c^*_{n,i,j}$ is: | Q.m. limit $c^*(u,v)$ (see(2.5)). |
|---|---|---|---|
| $\mathcal{N}(\underset{\sim}{b})$ | $G_\theta(y\|x)^{(1)} = \Phi(a(\theta)y - \theta \sum_{k=1}^{p} b_k H_k(x))^{(2)}$, $F_\theta(x)^{(1)} = \Phi(x)$, $a(\theta) = (1+\theta^2)^{\frac{1}{2}}$ | $\mu_{i\|j}\{\sum_{k=1}^{p} b_k E[H^*_k(Z_{j\|n})]\}$ (3) | $\Phi^{-1}(u)\{\sum_{k=1}^{p} b_k H^*_k(\Phi^{-1}(v))\}$ |
| $\mathcal{N}(1)$ | Bivariate normal with correlation $\rho = \theta(1+\theta^2)^{-\frac{1}{2}}$ | $\mu_{i\|j}[\mu_{i\|j} - \frac{1}{j-1}\sum_{\alpha=1}^{j-1} \mu_{\alpha\|n}]$ | $\Phi^{-1}(u)[\Phi^{-1}(v) - \frac{\varphi(\Phi^{-1}(v))}{v}]$ |
| $\mathcal{U}(m)$ | $H_\theta(x,y)^{(1)} = F(x)G(y)[1+\theta(1-F^m(x))(1-G^m(y))]$ | $c^*_{n,i,j} = J_{n,i} - \sum_{\alpha=1}^{j} J_{n,\alpha}$, with $J_{n,i}$, $1 \le i \le n$, given by (5.17) or (5.18) | $m[(m+1)(uv)^m - v^m]$ |
| $\mathcal{U}(1)$ | The above with m = 1. | $\frac{2i}{n+1} - \frac{j(j+1)}{n+1}$ (4) | $2uv - v$ |
| $\mathcal{E}(F)$ | $G(y\|x) = \exp[y/x^\theta]$, $F_\theta(x) = F(x)$ | $(\sum_{\beta=1}^{i} \frac{1}{j-\beta+1} - 1)E[\ell n(X_{j\|n}) - \frac{1}{j-1}\sum_{\alpha=1}^{j-1} \ell n(X_{\alpha\|n})]^{(5)}$ | $-[1+\ell n(1-u)]\cdot[\ell n(F^{-1}(v)) - \frac{1}{v}\int_0^{F^{-1}(v)} \ell n(x)dF(x)]$ |
| $\mathcal{E}(\frac{x}{r})$ | The above with $F(x) = \frac{x}{r}$, $0 \le x \le r$, $r>0$, fixed. | $\sum_{\beta=1}^{i} \frac{1}{j-\beta+1} - 1$ (6) | $-[1+\ell n(1-u)]$ |

(1) $G_\theta(y|x) = P_\theta(Y \le y | X \le x)$, $F_\theta(x) = P_\theta(X \le x)$, and $H_\theta(x,y) = P_\theta(X \le x, Y \le y)$. (2) $H_k(x)$ is the $k^{\underline{th}}$ Hermite polynomial. (3) $Z_{i|j}$ is the $i^{\underline{th}}$ largest of j standard normal random variables and $\mu_{i|j}$ is its mean. This expression was obtained from (4.13) by means of Lemma I.1. $H^*_k(x) = H_k(x) + H_{k-1}(x)\frac{\varphi(x)}{\Phi(x)}$. (4) This statistic is essentially Kendall's $\tau$ (see example 5.1). (5) $X_{j|n}$ is the $j^{\underline{th}}$ largest of a sample of size n from F. (6) By Lemma I.1.

Table III: values of $\sigma(c,c^*)$ for computing Pitman efficiencies*

| | $\mathcal{N}(\underline{b})$ | $\mathcal{N}(1)$ | $\mathcal{U}(m)$ | $\mathcal{U}(1)$ | $\mathcal{E}(F)$ | $\mathcal{E}(\frac{x}{\rho})$ |
|---|---|---|---|---|---|---|
| $\mathcal{N}(\underline{b}')$ | $\sum_{k=1}^{p} b_k b'_k k!$ | | | | | |
| $\mathcal{N}(1)$ | $b_1^2$ | $1$ | | | | |
| $\mathcal{U}(m')$ | $(\mu_{m+1\mid m+1})(m+1)\cdot[\sum_{k=1}^{p} b_k \int_{-\infty}^{\infty}\Phi^{m-1}(x)H_{k-1}(x)\varphi^2(x)dx]$ | $(\mu_{m'+1\mid m'+1})^2$ | $\frac{(mm')^2}{(m+m'+1)^2}$ | | | |
| $\mathcal{U}(1)$ | $\frac{1}{\pi}\sum_{k=0}^{[p]} b_{2k+1}(-\tfrac{1}{4})^k \frac{(2k)!}{k!}$ | $\frac{1}{\pi} \approx .31831$ | $\frac{m^2}{(m+2)^2}$ | $\frac{1}{9}$ | | |
| $\mathcal{E}(F)$ | $[\int_{-\infty}^{\infty}\frac{\varphi^2(y)}{\Phi(y)}dy]\cdot[\sum_{k=1}^{p} b_k \int_0^1 H_k(\Phi^{-1}(v))\ln(F^{-1}(v))dv]$ | $[\int_{-\infty}^{\infty}\frac{\varphi^2(y)}{\Phi(y)}dy]\cdot\int_0^1 \Phi^{-1}(v)\ln(F^{-1}(v))dv$ | $(\sum_{j=1}^{m}\frac{1}{j+1})(m+1)\cdot\int_0^{\infty}F^m(x)\ln(x)dF(x)$ | $\int_0^{\infty}F(x)\ln(x)dF(x)$ | $\int_0^{\infty}\ln^2(x)dF(x)$ | |
| $\mathcal{E}(\frac{x}{\rho})$ | $[\int_{-\infty}^{\infty}\frac{\varphi^2(y)}{\Phi(y)}dy]\cdot[\sum_{k=1}^{p} b_k \int_{-\infty}^{\infty}H_{k-1}(x)\frac{\varphi^2(x)}{\Phi(x)}dx]$ | $\int_{-\infty}^{\infty}\frac{\varphi^2(y)}{\Phi(y)} \approx .90320$ | $\frac{m}{m+1}\sum_{j=1}^{m}\frac{1}{j+1}$ | $\frac{1}{4}$ | $\int_0^{\infty}\ln(F(x))\ln(x)dF(x)$ | $1$ |

*This table contains values of $\sigma(c,c^*)$ (see (3.15)); $c$ is the q.m. limit (see (3.14)) corresponding to the ALMP layer-rank test against the family of alternatives named in the row heading (see Table ) and $c^*$ is the q.m. limit corresponding to the ALMP layer-rank test against the family of alternatives named in the column heading.

The Pitman efficiency $(\rho(c,c^*))^2 = \frac{(\sigma(c,c^*))^2}{\sigma(c,c)\sigma(c^*,c^*)}$ of $T_n(\underline{C})$ with respect to $T_n(\underline{C}^*)$ against the alternative for which $T_n(\underline{C}^*)$ is ALMP is easily computed from this table. For example, the efficiency of Kendall's $\tau$ compared to the normal scores layer-rank test is $(\frac{3}{\pi})^2 \doteq .912$ and the efficiency of ALMP statistic against $\mathcal{E}(\frac{x}{\rho})$ compared to Kendall's $\tau$ is $(\frac{3}{4})^2$.

6. Asymptotic Relative Efficiencies at Fixed Alternatives. Suppose there are two sequences of test statistics $T_i = \{T_{ni}, n \geq 1\}$, $i = 1,2$, for testing $\theta = 0$ vs $\theta > 0$ in the family $\{H_\theta; \theta \geq 0\}$ of bivariate cdf's (we assume that the tests reject for large values of $T_{ni}$, $i = 1,2$). Assuming the tests are consistent, we define $n_i(\theta,\alpha,\beta)$, $i = 1,2$, to be the smallest sample size required by a level $\alpha$ test in the sequence $T_i$ to achieve power $1-\beta$ against the alternative $H_\theta$. We may call the ratio $n_2(\theta,\alpha,\beta)/n_1(\theta,\alpha,\beta)$ the exact relative efficiency* of $T_1$ with respect to $T_2$. Since this exact efficiency is in general difficult to evaluate various asymptotic relative efficiencies, each giving some idea of the behavior of this ratio, have been proposed.** Pitman's ARE ($\lim_{\theta\to 0} \frac{n_2(\theta,\alpha,\beta)}{n_1(\theta,\alpha,\beta)}$) is usually a number independent of $\alpha$ and $\beta$ and gives information about the exact efficiency only for $\theta$ near 0. Another ARE, which we call Bahadur*** (exact) ARE, is defined as:

$$(6.1) \qquad \lim_{\alpha\to 0} \frac{n_2(\theta,\alpha,\beta)}{n_1(\theta,\alpha,\beta)}, \quad \theta > 0, \quad 0 < \beta < 1, \ \theta, \beta \text{ fixed},$$

provided the limit exists. Bahadur ARE seems particularly appropriate for significance testing in which one is interested in as large a significance $(1-\alpha)$ as possible while maintaining reasonable power.

In this section we derive the Bahadur ARE of a layer-rank test with respect to another layer-rank test or the likelihood ratio test.

For fixed $\beta$ let us denote the Bahadur efficiency of $\{T_{n1}\}$ with respect to $\{T_{n2}\}$ by $e(\theta,T_1,T_2)$. We shall show that in the case of layer-rank or likelihood ratio tests $e(\theta,T_1,T_2)$ doesn't depend on $\beta$ (provided $0 < \beta < 1$). Our derivation of $e(\theta,T_1,T_2)$ for these tests

*See Hodges and Lehmann [11] for a discussion of this notion.

**By Bahadur [1], Hodges and Lehmann[11] and Chernoff [4], to mention a few.

***But see Gleser [8], who uses the term slightly differently.

is similar to that of Klotz [13] and is based on Theorem 1 of Feller [5]; we prove below the version of Feller's theorem needed for this paper, since the original version is proved in great generality and is hard to apply here.

Consider the statistic $T_n(\underset{\sim}{C})$ defined by (2.6), where (2.5) and (2.7) hold. Letting $c_0 = c$ and recalling (2.4), we define the following functions for $n = 0,1,2,\ldots,$ and real h,

$$\psi_{c_n}(h) = \iint \exp[hc_n(u,v)]dudv, \tag{6.2}$$

$$\mu_{c_n}(h) = \int \frac{\int c_n(u,v)\exp[hc_n(u,v)]du}{\int \exp[hc_n(u,v)]du} dv, \tag{6.3}$$

$$\mu_{c_n}^{(i)}(h) = \int \frac{\int |c_n(u,v)|^i \exp[hc_n(u,v)]du}{\int \exp[hc_n(u,v)]du} dv, \quad i = 2,3, \tag{6.4}$$

and

$$m_{c_n}(h) = \iint |c_n(u,v)|^3 \exp[hc_n(u,v)]dudv. \tag{6.5}$$

We assume that in addition to (2.5) the following holds:

$$\| c_n - c \|_3 \to 0, \tag{6.6}$$

or, equivalently*, that $m_{c_n}(0) \to m_c(0)$. We denote by $I(\underset{\sim}{C})$ the h-interval** on which $m_c(h)$ is finite and $m_{c_n}(h) \to m_c(h)$.

<u>Theorem 6.1</u> (Feller) For any $x > 0$ if there is an $h_x$ in the interior of $I(\underset{\sim}{C})$ such that $\mu_c(h_x) = x$, then for any sequence $x_n \to x$

$$-\lim_{n\to\infty} \frac{1}{n} \ln[P(n^{\frac{1}{2}}T_n(\underset{\sim}{C}) \geq nx_n | H_0)] = xh_x - \int \ln[\int \exp(h_x c(u,v))du]dv,$$

where $P(\cdot|H)$ denotes the prob. measure corresponding to an infinite sequence of observations from a population with cdf H.

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*By the $L_r$ convergence theorem and the fact that $c_n \to c$ a.s.

**See Appendix II.

Proof: $n^{\frac{1}{2}}T_n(\underset{\sim}{C}) = \sum_{j=1}^{n} c_{n,\ell_{(j)},j}$ is a sum of bounded, independent random variables under $H_0$; this fact is crucial to the argument. We let

$$f_{nj}(h) = E[\exp(hc_{n,\ell_{(j)},j})] = \frac{1}{j}\sum_{i=1}^{j}\exp(hc_{nij})$$

and define, for arbitrary $h \in I(\underset{\sim}{C})$, $Z_{n1}(h),\ldots, Z_{nn}(h)$ to be independent random variables such that, for each pair j,n with $j \leq n$,

$$P(Z_{nj}(h) = c_{nij}) = \frac{\exp(hc_{nij})}{j\, f_{nj}(h)}, \qquad 1 \leq i \leq j \leq n.$$

By Lemma II. 1, if

$$S_n(h) = \sum_{j=1}^{n} Z_{nj}(h),$$

then

$$P_n[n^{\frac{1}{2}}T_n(\underset{\sim}{C}) \geq nx_n] = [\prod_{j=1}^{n} f_{nj}(h)][\int_{nx_n}^{\infty}\exp(-hz)dP(S_n(h) \leq z)]. \tag{6.7}$$

Let us first find an asymptotic expression for the second factor on the right of (6.7). It is easy to see that

$$E\, S_n(h)/n = \mu_{c_n}(h), \tag{6.8}$$

and

$$\sigma^2(S_n(h))/n = \mu_{c_n}^{(2)}(h) - \int\left\{\frac{\int c_n(u,v)\exp[hc_n(u,v)]du}{\int\exp[hc_n(u,v)]du}\right\}^2 dv \tag{6.9}$$

By Lemma II.2 (ii), the first term on the right side of (6.9) converges to $\mu_c^{(2)}(h)$ uniformly on compact subsets of $I(\underset{\sim}{C})$. Using arguments similar to those in the proof of Lemma II.2 one can easily prove that a similar result holds for the second term on the right side of (6.9). Thus,

$$\sigma^2(S_n(h))/n \rightarrow \mu_c^{(2)}(h) - \int\left\{\frac{\int c(u,v)\exp[hc(u,v)]du}{\int\exp[hc(u,v)]du}\right\}^2 dv$$

$= \sigma^2(h)$, say,

and the convergence is uniform on compact subsets of $I(\underset{\sim}{C})$.

is clear that $\sigma_c^2(h) > 0$ unless $c$ is degenerate*. Also, since

$$(6.10) \qquad \frac{1}{n}\sum_{j=1}^{n} E|S_n(h)-n\mu_{c_n}(h)|^3 \le 8\mu_{c_n}^{(3)}(h) \to 8\mu_c^{(3)}(h),$$

uniformly on compact subsets of $I(\underset{\sim}{C})$, by Lemma II.2.; it follows from the continuity of $\mu_c^{(3)}(h)$ that the quantity on the left of (6.10) is uniformly bounded on any compact subset of $I(\underset{\sim}{C})$. Combining this with (6.9) and (6.8), we conclude from the normal approximation theorem** that

$$(6.11) \qquad P\{S_n(h) \le z\} \doteq \Phi\left(\frac{z-n\mu_{c_n}(h)}{n^{\frac{1}{2}}\sigma(h)}\right) + R_n(z),$$

where $R_n(z) = O(n^{-\frac{1}{2}})$, uniformly in $h$ on any compact subset of $I(\underset{\sim}{C})$.

It is clear that $\mu_{c_n}(h)$ is continuous on $I(\underset{\sim}{C})$, $n = 0,1,\ldots.$ Since, by Lemma II.2, $\mu_{c_n}(h) \to \mu_c(h)$ uniformly on any compact subset of $I(\underset{\sim}{C})$, and since $x_n \to x$, it is easy to see that for large enough $n$ there is an $h_n \in I(\underset{\sim}{C})$ such that $\mu_{c_n}(h_n) = x_n$ and $h_n \to h_x$.

Since (6.7) holds for any $h \in I(\underset{\sim}{C})$ we may set $h = h_n$, and the second factor of (6.7), in view of (6.11) becomes,

$$(6.12) \qquad \int_{nx_n}^{\infty} \exp(-hz)d\Phi\left(\frac{3-nx_n}{n^{\frac{1}{2}}\sigma(h_n)}\right) + \int_{nx_n}^{\infty} \exp(-h_n z)dR_n(z)$$

$$= \left(\frac{1}{n^{\frac{1}{2}}\sigma(h_n)}\right)\int_0^{\infty} \exp[-h_n(z+nx_n)]\varphi\left(\frac{3}{n^{\frac{1}{2}}\sigma(h_n)}\right) dz + R_n^*(h), \quad \text{say.}$$

Integrating by parts, we obtain

$$R_n^*(h_n) = -R_n(z)\exp(-h_n z)\Big|_{nx_n}^{\infty} + \int_{nx_n}^{\infty} R_n(z)\exp(-h_n z)dz.$$

Since $x_n \to x > 0$, we can assume $nx_n > 0$.

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*c is degenerate if $c(u,v)$ is a function of $v$ only (hence = 0).

**Loéve [15] p. 288.

Thus,

$$|R_n^*(h_n)| = \mathcal{O}(n^{-\frac{1}{2}})\exp(-nh_nx_n)$$

and (6.12), the second factor of (6.7), becomes

$$(6.13) \qquad \exp(-nh_nx_n)[\int_0^\infty \exp(-n^{\frac{1}{2}}h_n\sigma(h_n)z)\varphi(z)dz + \mathcal{O}(n^{-\frac{1}{2}})]$$

$$= \exp(-nh_nx_n)\left[\frac{1-\Phi(n^{\frac{1}{2}}h_n\sigma(h_n))}{\varphi(n^{\frac{1}{2}}h_n\sigma(h_n))} \quad 2\pi + \mathcal{O}(n^{-\frac{1}{2}})\right] .$$

$\sigma(h)$ given by (6.9) is clearly continuous on $I(\underset{\sim}{C})$ thus $\sigma(h_n) \to \sigma(h_x) > 0$. By Lemma II.4 $\mu_c(h)$ is strictly increasing. Since $\mu_c(h_x) = x > 0 = \mu_c(0)$, it follows that $h_0 > 0$. Thus, by the Feller-Laplace expansion of Mill's ratio,

$$\frac{1-\Phi(n^{\frac{1}{2}}h_n\sigma(h_n))}{\varphi(n^{\frac{1}{2}}h_n\sigma(h_n))} = \frac{1}{n^{\frac{1}{2}}h_n\sigma(h_n)} + \mathcal{O}(n^{-\frac{3}{2}}).$$

Combining this with (6.13) we have, finally,

$$(6.14) \qquad \int_{nx_n}^\infty \exp(-h_nz)dP(S_n(h_n) \leq z) = \exp(-nh_nx_n)\,\mathcal{O}(n^{-\frac{1}{2}}).$$

Thus,

$$-\frac{1}{n}\ln[P_n(n^{\frac{1}{2}}T_n(\underset{\sim}{C}) \geq nx_n)] = -\frac{1}{n}\sum_{j=1}^n f_{nj}(h_n) + h_nx_n + \frac{1}{n}\ln(\mathcal{O}(n^{-\frac{1}{2}}))$$

$$= h_nx_n - \int \ln[\exp(h_nc_n(u,v))du]dv + \mathcal{O}(1)$$

$$\to xh_x - \int \ln[\int \exp(h_xc(u,v))du]dv,$$

by Lemma II.5, and the Lemma is proved.

Suppose there is a finite constant $\eta_c(\theta)$ such that

$n^{-\frac{1}{2}}T_n(\underset{\sim}{C}) = \frac{1}{n}\sum_{j=1}^n c_{n,\ell_{(j)},j} \to \eta_c(\theta)$ in $H_\theta$-probability (see Appendix III for a discussion of this point). Let us select $k_n$ so that the test which rejects $H_0$ in favor of $H_\theta$ when $T_n(\underset{\sim}{C}) \geq k_n$ has power $1-\beta$;

i.e., $P\{T_n(\underset{\sim}{C}) \geq k_n | H_\theta\} = 1-\beta$. Since $n^{-\frac{1}{2}} T_n(\underset{\sim}{C}) \to \eta_c(\theta)$ in $H_\theta$-probability it is easy to see that $k_n = n^{-\frac{1}{2}}[\eta_c(\theta) + (1)]$.

Letting $\alpha_n = P\{T_n(\underset{\sim}{C}) \geq k_n | H_0\}$ denote the type I error of this test, we obtain from Theorem 6.1,

<u>Corollary 6.1</u> If there is an $h \in I(\underset{\sim}{C})$ such that

$$\mu_c(h_\theta) = \eta_c(\theta), \tag{6.15}$$

then

$$-\frac{1}{n} \ln(\alpha_n) \to h_\theta \eta_c(\theta) - \int \ln[\int \exp(hc(u,v))du]dv, \tag{6.16}$$

where $\mu_c(h)$ is given by (6.3).

Letting $e_c(\theta)$ denote the right side of (6.16), we obtain the following asymptotic $(\alpha \to 0,\ \theta,\ \beta \text{ fixed})$ expression for the sample size $n(\theta,\alpha,\beta)$ required by the $\alpha$-level test of the form $T_n(\underset{\sim}{C}) \geq k$ to attain power $1-\beta$ at the alternative $H_\theta$:

$$n(\theta,\alpha,\beta) \sim \frac{\ln(\alpha)}{e_c(\theta)} \quad . \tag{6.17}$$

Thus, the Bahadur ARE of $\{T_n(\underset{\sim}{C}_1)\}$ with respect to $\{T_n(\underset{\sim}{C}_2)\}$ given by (6.1) is simply

$$e(\theta,\{T_n(C_1)\},\{T_n(C_2)\}) = \frac{e_{c_1}(\theta)}{e_{c_2}(\theta)} \quad . \tag{6.18}$$

Now let us consider the likelihood ratio test or, equivalently, the test which rejects for large values of

$$L_n = n^{-\frac{1}{2}} \sum_{j=1}^{n} Z_j \ ,$$

where $Z_j = \ln(r_\theta(X_j,Y_j))$, $j = 1,\ldots,n$, $(X_1,Y_1),\ldots,(X_n,Y_n)$ is a sample either from $H_0$ or $H_\theta$, and $r_\theta = dH_\theta/dH_0$ is the likelihood ratio. Let

$$\eta_Z(\theta) = \int\int Z dH_\theta . \tag{6.19}$$

If $\eta_Z(\theta)$ is finite (hence exists), then $n^{-\frac{1}{2}}L_n = \frac{1}{n}\sum_{j=1}^{n} Z_j \rightarrow \eta_c(\theta)$ in $H_\theta$-probability*. Thus, if the test $L_n \geq k_n$ has power $1-\beta$ at $H_\theta$, then $k_n = n^{\frac{1}{2}}(\eta_Z(\theta)+O(1))$ and the type I error $\alpha_n$ is given by:

$$\alpha_n = P[T_n \geq k_n | H_0].$$

Let $I(Z)$ denote the interval of real numbers $h$ on which $\iint |z|^3 \exp(hZ)dH_0 < \infty$. Using the methods of this section, it is easy to show that if there is an $h_0$ in the interior of $I(Z)$ such that

$$\eta_Z(\theta) = \frac{\iint Z\exp(h_0 Z)dH_0}{\iint \exp(h_0 Z)dH_0}, \tag{6.20}$$

then

$$-\frac{1}{n}\ln(\alpha_n) \rightarrow h_0\eta_Z(\theta) - \ln[\iint \exp(h_0 Z)dH_0]. \tag{6.21}$$

But since $Z = \ln(r_\theta(x,y))$ and $r_\theta = dH_\theta/dH_0$, (6.20) can be put in the form:

$$\iint \ln(r_\theta)r_\theta dH_0 = \frac{\iint \ln(r_\theta)(r_\theta)^{h_0}dH_0}{\iint (r_\theta)^{h_0}dH_0}.$$

Thus, $h_0 = 1$ is a solution of (6.20) and (6.21) becomes:

$$-\frac{1}{n}\ln(\alpha_n) \rightarrow \eta_Z(\theta),$$

where $\eta_Z(\theta)$ is given by (6.19). We conclude that the sample size required by the $\alpha$-level likelihood ratio test to attain power $1-\beta$ against $H_\theta$ has the following asymptotic expression as $\alpha \rightarrow 0$ with $\theta$, $\beta$ fixed:

$$n \sim \frac{\ln(\alpha)}{\eta_Z(\theta)}.$$

Consequently, the Bahadur efficiency of $T_n(\underset{\sim}{C})$ with respect to the

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*I.e., the probability-measure corresponding to an infinite sequence of observations from a population with cdf $H_\theta$.

likelihood ratio statistic is given by:

$$e(\theta,\{T_n(\underset{\sim}{C})\},L) = \frac{e_c(\theta)}{\eta_Z(\theta)}, \tag{6.22}$$

where

$$\eta_Z(\theta) = \iint Z dH_\theta = \iint r_\theta \ell n(r_\theta) dH_\theta \tag{6.23}$$

and $e_c(\theta)$ is the right side of (6.16).

Example 6.1. Kendall's $\tau$.

If we set $c_{n,i,j} = 2(i-\frac{j(j+1)}{2})/(n+1)$, $1 \leq i \leq j \leq n$, then $T_n(\underset{\sim}{C})$ given by (2.6) is essentially Kendall's $\tau$-statistic. Moreover, $c_n(u,v)$ defined by (2.4) converges to $c(u,v) = 2(uv - \frac{v}{2})$, uniformly on any set of the form $0 \leq u \leq 1$, $v_0 \leq v \leq 1$ $(v_0 > 0)$. Also, since $|c_n(u,v)| \leq 3$, $m_{c_n}(h) \to m_c(h) < \infty$ for all real h (see (6.5)) so that $I(\underset{\sim}{C}) = (-\infty,\infty)$ (see the sentence just before Theorem 6.1). Thus, by Corollary III.2, for any continuous bivariate cdf $H_\theta$,

$$n^{-\frac{1}{2}}T_n(\underset{\sim}{C}) \to \iint (H_\theta(x,y) - \frac{F_\theta(x)}{2})dH_\theta(x,y) = \eta_\tau(\theta), \text{ say,} \tag{6.24}$$

in $H_\theta$-probability, where $F_\theta(x) = H_\theta(x,\infty)$. Thus, the condition stated in the last paragraph of p.44 is satisfied. If we let $(X_1,Y_1)$ and $(X_2,Y_2)$ be independent bivariate random variables with cdf $H_\theta$, then $\eta_\tau(\theta)$ can be put in the form:

$$\eta_\tau(\theta) = P[X_1 \leq X_2 \text{ and } Y_1 \leq Y_2 | H_\theta] - \frac{1}{4}. \tag{6.25}$$

The right side of (6.16), call it $e_\tau(\theta)$, becomes

$$e_\tau(\theta) = h\,\eta_\tau(\theta) - (1 - \frac{1}{h}\int_0^h \frac{t\,dt}{e^t - 1} - \frac{1}{4}h) + \frac{h}{2} - \ell n(\frac{e^h - 1}{h}), \tag{6.26}$$

where h is the solution of:

$$(6.27)\qquad \eta_\tau(\theta) = \frac{1}{4} - \frac{1}{h}\left(1 - \frac{1}{h}\int_0^h \frac{t\,dt}{e^t - 1}\right).$$

By combining (6.26) and (6.27), we can put (6.26) in the form:

$$(6.28)\qquad e_\tau(\theta) = 2h\eta_\tau(\theta) + \frac{h}{2} + \ln\left(\frac{e^h - 1}{h}\right).$$

We have tabulated (Table IV) the right sides of (6.27) and (6.28) as functions of h; the use of this table is illustrated below.

Now consider two specific families of alternatives.

a) A family against which Kendall's $\tau$ is ALMP: the family given by (5.16) with $m = 1$. It is interesting to compute the Bahadur efficiency (6.22) of Kendall's $\tau$ with respect to the likelihood ratio statistic at fixed values of $\theta$ in this family. The quantity $\eta_Z(\theta)$ is, in this case, given by

$$(6.29)\qquad \eta_Z(\theta) = -\frac{1}{2}\int_{-1}^{1} \ln(1+\theta w)(1+\theta w)\ln(|w|)\,dw.$$

For the cdf $H_\theta(x,y) = F(x)G(y)[1+\theta(1-F(x))(1-G(y))]$ (6.25) becomes:

$$\eta_\tau(\theta) = \frac{\theta}{18}.$$

By means of Tables IV,VI we compute values of the ratio $\frac{e_\tau(\theta)}{\eta_Z(\theta)}$. For example, in Table IV we find at $\eta_\tau(\theta) \doteq .01385$ that $e_\tau(\theta) \doteq .003459$ so that $\theta = 18.\eta_\tau(\theta) \doteq .2494$. We find in Table VI that at $\theta \doteq .2494$, $\eta_Z(\theta) \doteq .003468$, thus the Bahadur efficiency of $\tau$ with respect ot the likelihood ratio statistic at $\theta \doteq .249$ in the specified family is approximately $(.003459)/(.003468) \doteq .997$. (see Figure 5 ).

b) The bivariate normal family with correlation $\rho = \theta(1+\theta^2)^{-\frac{1}{2}}$. $\eta_Z(\theta)$ (6.23) is, in this case, $\frac{1}{2}\ln(1+\theta^2)$.

Since, in this case, (6.25) becomes:

$$(6.30)\qquad \eta_\tau(\theta) = \frac{1}{2\pi}\arctan(\theta),$$

*Which we have not indicated in the table.

we can, using Tables IV and V, compute values of the Bahadur efficiency of $\tau$ compared to the likelihood ratio statistic against normal alternatives for specific values of the parameter $\theta$ (see figure 4 ).

Example 6.2. The Normal Scores Layer-Rank Test.

In Example 4.1 we derived a layer-rank which is ALMP against the bivariate normal alternative. The test statistic is

$$T_n(\underset{\sim}{C}) = n^{-\frac{1}{2}} \sum_{j=1}^{n} \mu_{\ell_{(j)}|j} L^*_{n,j} ,$$

where

$$L^*_{n,j} = \begin{cases} \mu_{j|n} - \frac{1}{j-1} \sum_{i=1}^{j-1} \mu_{i|n} & j > 1 \\ 0 & j = 1 \end{cases}$$

and $\mu_{i|j}$ $1 \leq i \leq j \leq n$ is the expected value of the $i^{\underline{th}}$ largest of $j$ standard normal random variables. The q.m. limit of $c_n(u,v)$, given by (4.8), is

$$c(u,v) = \Phi^{-1}(u)[\Phi^{-1}(v) + \frac{\varphi[\Phi^{-1}(v))}{v}].$$

Since $\frac{\varphi(x)}{\Phi(x)} \sim |x|$ as $x \to -\infty$ and is bounded on any set of the form $x \geq x_0 > -\infty$ , we have, for any $r \geq 0$,

$$\iint |c(u,v)|^r dudv = [\int_{-\infty}^{\infty} y^r \varphi(y)dy][\int_{-\infty}^{\infty}(x+ \frac{\varphi(x)}{\Phi(x)})^r \varphi(x)dx] < \infty,$$

and for any $h$ $(-\infty < h < \infty)$

$$\iint \exp(h|c(u,v)|)dudv = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(h|y(x+ \frac{\varphi(x)}{\Phi(x)})|)\varphi(y)\varphi(x)dydx < \infty.$$

Thus, by the Schwarz inequality, the set $A$ of Lemma II.6 is $(-\infty,\infty)$; consequently $I(\underset{\sim}{C}) = (-\infty,\infty)$.

We shall be dealing with two families of alternatives (see parts a) and b) of the previous example) and conjecture* that for a bivariate cdf $H_\theta$ in either of these families

$$n^{-\frac{1}{2}} T_n(\underset{\sim}{C}) \rightarrow \iint \Phi^{-1}\left(\frac{H_\theta(x,y)}{F(x)}\right)\left[\Phi^{-1}(F(x)) + \frac{\varphi(\Phi^{-1}(F(x)))}{F(x)}\right] dH_\eta(x,y)$$

$$= \eta_c(\theta), \quad \text{say,}$$

in $H_\theta$-probability.

Letting $L(x) = x + \frac{\varphi(x)}{\Phi(x)}$, we see that the right side of (6.16) is

$$e_c(\theta) = h_\theta \eta_c(\theta) - \int_{-\infty}^{\infty} \ln\left[\int_{-\infty}^{\infty} \exp(h_\theta y L(x))\varphi(y)dy\right]\varphi(x)dx$$

$$= h_\theta \eta_c(\theta) - \tfrac{1}{2} h_\theta^2 \int_{-\infty}^{\infty} L^2(x)\ (x)dx = h_\theta \eta_c(\theta) - \tfrac{1}{2} h_\theta^2.$$

Moreover, since $h_\theta$ satisfies (6.15), we have

$$\eta_c(\theta) = \int_{-\infty}^{\infty} \frac{\int_{-\infty}^{\infty} y L(x) \exp(h_\theta y L(x))\varphi(y)dy}{\int_{-\infty}^{\infty} \exp(h_\theta(yL(x))\varphi(y)dy} \varphi(x)dx$$

$$= h_\theta \int_{-\infty}^{\infty} L^2(x)\varphi(x)dx = h_\theta .$$

Thus,

$$e_c(\theta) = \tfrac{1}{2}(\eta_c(\theta))^2.$$

We have computed Bahadur efficiencies of this statistic with respect to the likelihood ratio statistic for the two families of alternatives considered in Example 6.1 (see Figures 4 and 5 ); the reader will find in Tables IV, V, VI, and VII values of $e_c(\theta)$ and $\eta_z(\theta)$, for both of the above statistics and both of the above families of distributions, from which the Bahadur efficiencies in Figures 4 and 5 were computed.

*See Appendix III.

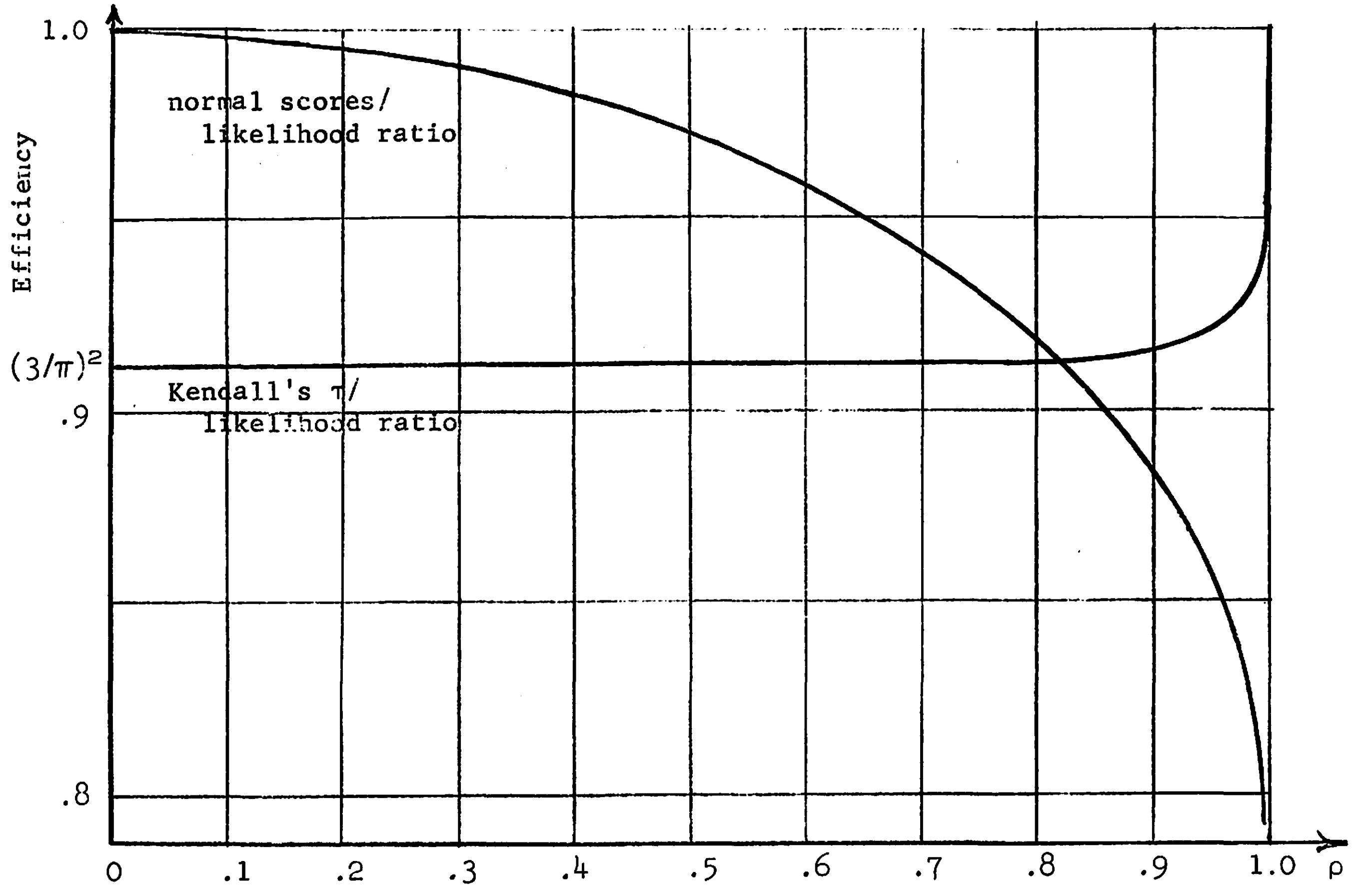


Fig. 4 Bahadur efficiencies for normal alternatives.

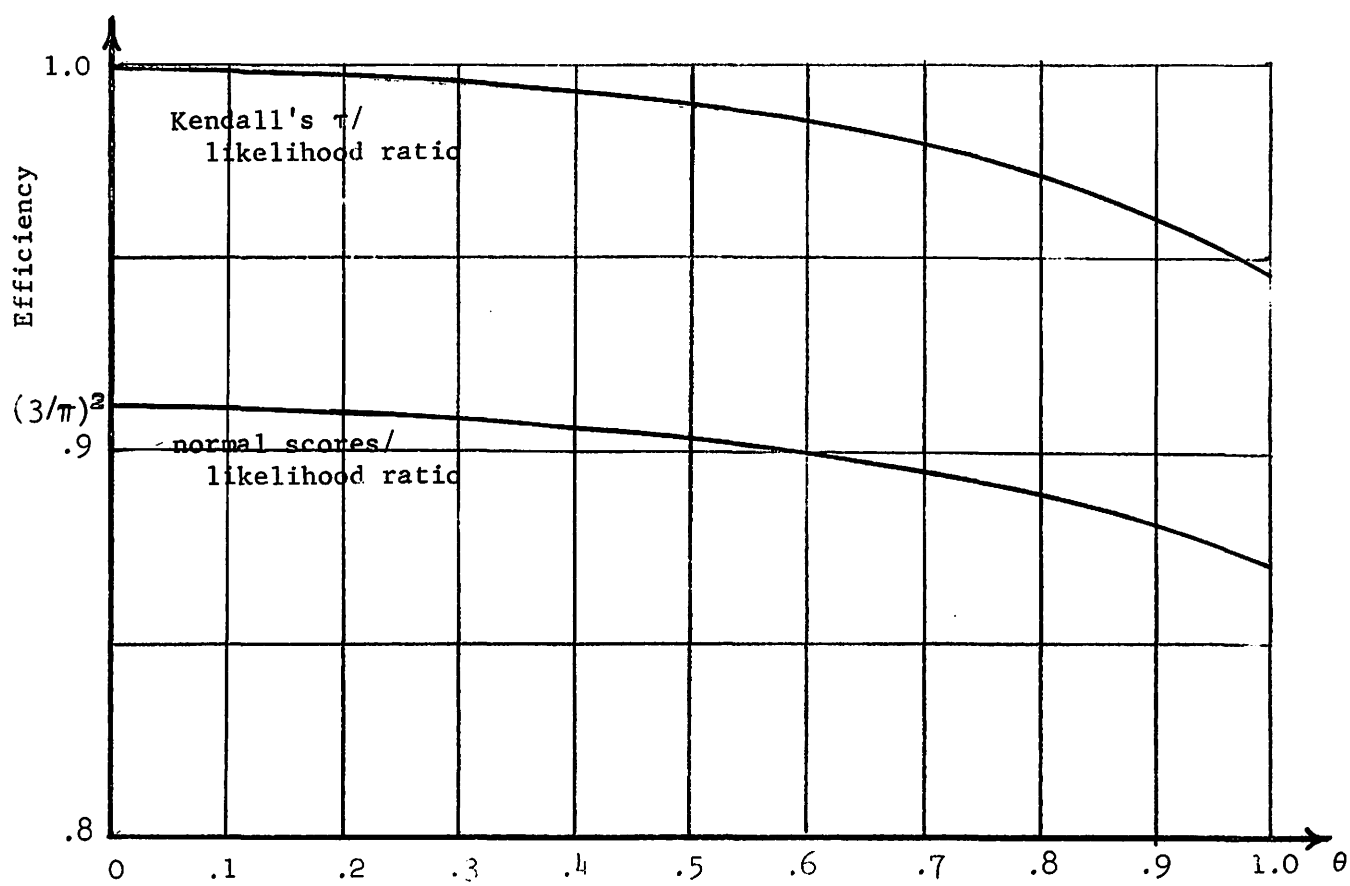


Fig. 5 Bahadur efficiencies for the alternative
$H_\theta = FG(1 + \theta(1 - F)(1 - G))$.

## 7. Some Remarks on the Small Sample Properties of Layer-Rank Tests.

Consider a family $\mathcal{H} = \{H_\theta : -\infty < \theta < \infty\}$ of continuous bivariate distributions, where $H_0(x,y) = F(x)G(y)$. Let $G_\theta(y|x)$ be the conditional cdf of $Y$ given $X = x$. We say that $Y$ is stochastically increasing (decreasing) in $X$ for fixed $\theta$ if $G_\theta(y|x)$ is non-increasing (non-decreasing) in $x$.

Lemma 7.1 If $T_n(\underset{\sim}{C})$ is a test statistic of the form (2.3), if $c_{nij} \leq c_{ni'j}$ for all $i \leq i'$, $1 \leq j \leq n$, and if $Y$ is stochastically increasing or decreasing in $X$ as $\theta > 0$ or $\theta < 0$, then the test $T_n(\underset{\sim}{C}) \geq k$ is unbiased.

Proof: In view of the marginal free nature of $T_n(\underset{\sim}{C})$ we can assume without loss of generality that the marginal of $X$ is independent of $\theta$.

Recall the definitions of $Y_{[i]}$, $X_{(i)}$, $\ell_{(i)}$, $i = 1,\ldots,n$, and let $y_{(i)}$, $x_{(i)}$, $\ell^*_{(i)}$, $i = 1,\ldots,n$, denote realizations of these random variables. Let $V_1,\ldots, V_n$ be independent uniform (0,1) random variables independent of $X_1,\ldots, X_n$. We define on (0,1), for fixed $\theta$ and $x$,

(7.1) $$y_\theta(v;x) = \inf\{y : G_\theta(y|x) = v\}.$$

$y_\theta(v;x)$ is strictly increasing in $v$ for fixed $x$ and $\theta$ and is non-decreasing in $x$ for fixed $\theta > 0$ and $v$, by the assumption that $Y$ is stochastically increasing in $X$ for $\theta > 0$. Note that $y_0(v;x) = y_0(v)$ is independent of $x$.

Clearly, the random vectors $(X_{(i)}, y_\theta(V_i;X_{(i)}))$, $i = 1,\ldots,n$, have the same distribution as $(X_{(i)},Y_{[i]})$ when the sample is taken from $H_\theta$. Thus, knowing $\theta$, for each realization $(v_1,\ldots, v_n; x_{(1)},\ldots, x_{(n)})$ one can construct the corresponding realization

$$(y_{[1]},\ldots, y_{[n]};x_{(1)},\ldots, x_{(n)}) = (y_\theta(v_1;x_{(1)}),\ldots,y_\theta(v_n;x_{(n)});x_{(1)},\ldots,x_{(n)}).$$

Now consider $\ell^*_{(j)}(\theta)$, the corresponding realization of the layer-rank $\ell_{(j)}$; letting $z(x) = 1(0)$ as $x \geq (<)\ 0$, we have

$$(7.2)\qquad \ell^*_{(j)}(\theta) = \sum_{i=1}^{j} z(y_{[j]}-y_{[i]})$$

$$= \sum_{i=1}^{j} z(y_\theta(v_j;x_{(j)}) - y_\theta(v_i;x_{(i)}))$$

$$\geq \sum_{i=1}^{j} z(y_\theta(v_j;x_{(i)}) - y_\theta(v_i;x_{(i)}))$$

$$= \sum_{i=1}^{j} z(y_0(v_j) - y_0(v_i))$$

but the latter is $\ell^*_j(0)$, the corresponding realization of $\ell_{(j)}$ when $\theta = 0$. The conditional probabilities of the two realizations $\ell^*_{(j)}(\theta)$ and $\ell^*_{(j)}(0)$ given $x_1,\ldots, x_n$ are the same since they depend only on the V's and not on $\theta$. To summarize, if we condition on the X-values, then for each realization $\ell^*_j(0)$ of $\ell_{(j)}$, $j = 1,\ldots,n$ when $\theta = 0$ there corresponds an equiprobable realization $\ell^*_j(\theta)$ $j = 1,\ldots,n$ when $\theta > 0$ and moreover $\ell^*_j(0) \leq \ell^*_j(\theta)$, $j = 1,\ldots,n$. Since $T_n(\underset{\sim}{C})$ is non-decreasing in $\ell_{(j)}$ for each $j$ this immediately implies that, for any k,

$$P_0\{T_n(\underset{\sim}{C}) \geq k | x_1,\ldots, x_n\} \leq P_\theta\{T_n(\underset{\sim}{C}) \geq k | x_1,\ldots, x_n\},$$

and, since the distribution of $x_1,\ldots, x_n$ is independent of $\theta$, this implies that

$$P_0\{T_n(\underset{\sim}{C}) \geq k\} \leq P_\theta\{T_n(\underset{\sim}{C}) \geq K\}.$$

The reverse inequality is proved similarly when $\theta < 0$ and the Lemma is proved.

Suppose that for each $\theta$ there is a strictly indreasing function $m(y;\theta)$ on the range of Y. Since layer-rank statistics are marginal free* we can define a new family of bivariate distributions call it $\mathcal{H}(m)$ by setting $Y' = m(Y;\theta)$ for each $\theta$ and the distributional properties of any layer-rank statistic will be unchanged.

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*See Section 1.

We prove below that under certain conditions a layer-rank test is not only unbiased but also has a monotone power function; first, however it is necessary to introduce a certain property of families of distributions.

Consider a family $\mathcal{H} = \{H_\theta : -\infty < \theta < \infty\}$ of bivariate distributions. Defining $y_\theta(v;x)$ by (7.1), we say that $\mathcal{H}$ satisfies condition (7.3) if for $x_1 \leq x_2$ and any $v_1$ and $v_2$

(7.3) (i) $y_\theta(v_2;x_2) - y_\theta(v_1;x_1)$ is either negative or non-decreasing in $\theta$, and

(ii) the marginal distribution of $X$ doesn't depend on $\theta$.

Corollary 7.1 Let $\mathcal{H} = \{H_\theta : -\infty < \theta < \infty\}$ be a family of bivariate distributions. If there is a family $\{m(\cdot;\theta)\}$ of transformations of $Y$ as described above such that the transformed family $\mathcal{H}(m)$ satisfies condition (7.3) and if $T_n(\underset{\sim}{C})$ satisfies the conditions of Lemma 7.1 then the test $T_n(\underset{\sim}{C})$ has a monotone non-decreasing power function.

Proof: Let $\theta_1 < \theta_2$ and in the proof of Lemma 7.1 change (7.2) to:

$$\begin{aligned}
\ell^*_{(j)}(\theta_2) &= \sum_{i=1}^{j} z(y_{[j]} - y_{[i]}) \\
&= \sum_{i=1}^{j} z(y_{\theta_2}(v_j;x_{(j)}) - y_{\theta_2}(v_i;x_{(i)})) \\
&\geq \sum_{i=1}^{j} z(y_{\theta_1}(v_j;x_{(j)}) - y_{\theta_1}(v_i;x_{(i)})) \\
&= \ell^*_{(j)}(\theta_1).
\end{aligned}$$

The difficulty of verifying condition (7.3) makes this Corollary rather impractical in its present form; nevertheless, we are able to apply it to families of the form (4.2). In fact, if we set $Y' = a(\theta)Y$, then $y_\theta(v;x)$ becomes $G^{-1}(v) + \theta b(x)/a(\theta)$ and condition (7.3) reduces to the requirement that $b(x)$ and $\theta/a(\theta)$ be non-decreasing in $x$ and $\theta$, respectively. Thus in particular the power function of any test

$T_n(\underset{\sim}{C}) \geq k$ such that $c_{nij} \leq c_{ni'j}$, $i \leq i'$, $1 \leq j \leq n$, has a monotone power function against the normal alternative, since in that case we can select $b(x) = \Phi^{-1}(x)$ and $\theta/a(\theta) = \theta/\sqrt{1+\theta^2}$.

Note that since $G_\theta(y_\theta(v;x)|x) = v$, we have $(\partial/\partial\theta)y_\theta(v;x) = -((\partial/\partial\theta)G_\theta(y|x))/g_\theta(y|x)$, where $y = y_\theta(v;x)$. Thus a sufficient condition for (7.3) (i) is

$$(7.4) \qquad \frac{(\partial/\partial_\theta)G_\theta(y_1|x_1)}{g_\theta(y_1|x_1)} \geq \frac{(\partial/\partial_\theta)G_\theta(y_2|x_2)}{g_\theta(y_2|x_2)}, \quad \text{when } x_1 \leq x_2, \; y_1 \leq y_2.$$

We remark, finally, that if a layer-rank statistic satisfies the conditions of Corollary 7.1 for some bivariate family and if it has non-zero Pitman efficiency (3.20), then it follows from (3.16) and Corollary 7.1 that $T_n(\underset{\sim}{C})$ is consistent against any $\theta > 0$ in that family.

## 8. Comparison of Layer-Rank Tests with Rank Tests.

In this section, as in Section 3, we will be dealing with a family of bivariate cdf's $\{H_\theta : -\infty < \theta < \infty\}$ and we shall assume that (3.3), (3.4), (3.5) and (3.6) hold for this family. We show first that the locally most powerful (LMP) rank test statistic is in a class of statistics proposed by Hoeffding [12] of which those studied by Bhuchongkul [3] form a subclass.

Let $\underset{\sim}{R} = (R_{[1]}, \ldots, R_{[n]})$ be the ordinary (not layer) ranks of $Y_{[1]}, \ldots, Y_{[n]}$ (see p. 4 for a definition of the $Y_{[j]}$). If $r = (r_1, \ldots, r_n)$ is a permutation of $(1,2,\ldots,n)$ and if $r_{\alpha_i} = i$, $i = 1,\ldots,n$, then, using notations introduced on p.13 , we have

$$P_\theta(R = r) = E_\theta[P_\theta(Y_{[\alpha_1]} \le \ldots \le Y_{[\alpha_n]} | X_{(1)}, \ldots, X_{(n)})] \tag{8.1}$$

$$= n! \int_{x_1 \le \ldots \le x_n} \int \int_{y_1 \le \ldots \le y_n} \int \prod_{i=1}^{n} r_\theta(x_i, y_{r_i}) g(y_i) dy_i dF(x_i)$$

$$= \frac{1}{n!} E_0[\prod_{i=1}^{n} r_\theta(X^0_{j|n}, Y^0_{r_j|n})],$$

where $X^0_{j|n}$ is the $j^{\underline{th}}$ largest of a sample $X^0_1, \ldots, X^0_n$ from $F(x)$ and $Y^0_{j|n}$ is the $j^{\underline{th}}$ largest of a sample $Y^0_1, \ldots, Y^0_n$ from $G(y)$ and the $X^0$'s and $Y^0$'s are independent (the $X^0$'s and $Y^0$'s should not be confused with the sample $(X_1,Y_1), \ldots, (X_n,Y_n)$, drawn from a population with cdf $H_\theta$, from which the ranks $R_{[1]}, \ldots, R_{[n]}$ are computed.)

The LMP rank test rejects for values $\underset{\sim}{r}$ of $\underset{\sim}{R}$ giving large values of $(\partial/\partial\theta) P_\theta(\underset{\sim}{R} = \underset{\sim}{r})|_{\theta=0}$. Thus, the following lemma implies that any test which rejects for large values of

$$S_n(b^*) = n^{-\frac{1}{2}} \sum_{j=1}^{n} E[s(X^0_{j|n}, Y^0_{r_j|n})] \tag{8.2}$$

is LMP (the notation $S_n(b^*)$ is explained below ), where s is given by (3.3).

Lemma 8.1 If (3.4), (3.5) and (3.6) hold, then $\{\partial/\partial\theta\}P_\theta(\underset{\sim}{R} = \underset{\sim}{r})|_{\theta=0} = \frac{1}{n!}\sum_{j=1}^{n} E[s(X^0_{j|n}, Y^0_{r_j|n})]$.

Proof: For compactness of notation we let $r_{\theta_j} = r_\theta(X^0_{j|n}, Y^0_{r_j|n})$. From (8.1) we obtain

$$n!\,\frac{P_\theta(\underset{\sim}{R}=\underset{\sim}{r})-1/n!}{\theta} \tag{8.3}$$

$$= E[\prod_{j=1}^{n} r_{\theta j}-1]/\theta = E[(\prod_{j=1}^{n} r^{\frac{1}{2}}_{\theta j}-1)^2 + 2(\prod_{j=1}^{n} r^{\frac{1}{2}}_{\theta j}-1)]/\theta$$

$$= \theta\, E[\sum_{j=1}^{n}(\prod_{i=j+1}^{n} r^{\frac{1}{2}}_{\theta j})(\frac{r^{\frac{1}{2}}_{\theta j}-1}{\theta})]^2 + 2\,E[\sum_{j=1}^{n}(\prod_{i=j+1}^{n} r^{\frac{1}{2}}_{\theta j})(\frac{r^{\frac{1}{2}}_{\theta j}-1}{\theta})].$$

Consider the first term in the last member of (8.3)

$$\theta\, E[\sum_{j=1}^{n}(\prod_{i=j+1}^{n} r^{\frac{1}{2}}_{\theta i})(\frac{r^{\frac{1}{2}}_{\theta j}-1}{\theta})]^2$$

$$\leq n\,\theta \sum_{j=1}^{n} E(\prod_{i=j+1}^{n} r_{\theta i})\, E(\frac{r^{\frac{1}{2}}_{\theta j}-1}{\theta})^2$$

$$\leq n(n!)^4\theta \sum_{j=1}^{n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\frac{r^{\frac{1}{2}}_{\theta}(x,y)-1}{\theta})^2\, g(y)dydF(x)$$

$\to 0$ as $\theta \to 0$, by (3.6). (Bear in mind that $n$ is fixed.)

Now consider the second term in the last member of (8.3)

$$2\sum_{j=1}^{n} E[(\prod_{i=j+1}^{n} r^{\frac{1}{2}}_{\theta i})(\frac{r^{\frac{1}{2}}_{\theta j}-1}{\theta})] = \sum_{j=1}^{n} E[(\prod_{i=j+1}^{n} r^{\frac{1}{2}}_{\theta i})\,(X_{j|n}, Y_{r_j|n})] + d(\theta),$$

(do not confuse $r_j$ with $r_{\theta j}$),

where

$$|d(\theta)|^2 = 4\left|\sum_{j=1}^{n} E\left[\left(\prod_{i=j+1}^{n} r_{\theta i}^{\frac{1}{2}}\right)\left\{\left(\frac{r_{\theta j}^{\frac{1}{2}}-1}{\theta}\right) - \frac{S(X_{j|n},Y_{r_j|n})}{2}\right\}\right]\right|^2$$

$$\leq 4n \sum_{j=1}^{n} E\left[\prod_{i=j+1}^{n} r_{\theta i}\right] E\left[\left(\frac{r_\theta^{\frac{1}{2}}(X_{j|n},Y_{r_j|n})-1}{\theta}\right) - \frac{S(X_{j|n},Y_{r_j|n})}{2}\right]^2$$

$$\leq 4n(n!)^2 \sum_{j=1}^{n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\left(\frac{r^{\frac{1}{2}}(x,y)-1}{\theta}\right) - \frac{S(x,y)}{2}\right]^2 g(y)\,dy\,dF(x)$$

$$\to 0 \text{ as } \theta \to 0.$$

Since the limit as $\theta \to 0$ of the first member of (8.3) is $n!\{\partial/\partial\theta\}P_\theta(R=r)|_{\theta=0}$, the Lemma is proved.

We define the function $b^*(u,v)$ in the unit square as follows:

(8.4) $$b^*(u,v) = s(F^{-1}(u),G^{-1}(v)).$$

Conditions (3.4) and (3.5) imply that $b^*(u,v)$ is square integrable and that $\int b^*(u,v)du = \int b^*(u,v)dv = 0$.

Let us consider a more general situation in which we are given an arbitrary square integrable function $b(u,v)$ for which $\int b(u,v)du = \int b(u,v)dv = 0$. We define

(8.5) $$b_{n,i,j} = E[b(U_{i|n},V_{j|n})],$$

where $U_{i|n}$ is the $i^{\underline{th}}$ largest of $n$ uniform (0,1) random variables $U_1,\ldots,U_n$, $V_{j|n}$ is the $j^{\underline{th}}$ largest of $n$ uniform (0,1) random variables and the U's and V's are independent.

We define the rank statistic $S_n(b)$ as follows:

(8.6) $$S_n(b) = n^{-\frac{1}{2}} \sum_{j=1}^{n} b_{n,R_{[j]},j}.$$

Note that if we define a bivariate step function:

$$b_n(u,v) = b_{n,i,j}, \quad \frac{i-1}{n} \leq u < \frac{i}{n}, \quad \frac{j-1}{n} \leq v < \frac{j}{n}, \quad 1 \leq i, \quad j \leq n,$$

then

(8.7) $$S_n(b) = n^{-\frac{1}{2}} \sum_{j=1}^{n} b_n\left(\frac{R_{[j]}}{n+1}, \frac{j}{n+1}\right).$$

Lemma 8.6 states that $S_n(b^*)$, given by (8.5) and (8.6) with $b^*$ given by (8.4), is the LMP rank statistic for testing $\theta = 0$ vs $\theta > 0$ in the family $\{H_\theta: \theta \geq 0\}$. We shall show that $S_n(b^*)$ is also ALMP and find an expression for the Pitman ARE of one such statistic with respect to another by which we will compare them with layer-rank statistics.

For any square integrable functions $b_1$ and $b_2$, defined on the unit square, we let (as in Section 3) $\rho(b_1,b_2) = \iint b_1(u,v)b_2(u,v)dudv$. Recalling the definition of $P_n$ and $Q_n$ given on p.14 we have

Lemma 8.2 If s, defined by (3.3), satisfies (3.4), (3.6) and (3.5) with $\delta = 0$ and if $b$ is a square integrable function on the unit square such that $\int b(u,v)du = \int b(u,v)dv = 0$, then

(1) $\mathcal{L}(S_n(b)|P_n) \rightarrow N(0,\|b\|_2^2)$ and

(2) $\mathcal{L}(S_n(b)|Q_n) \rightarrow N(a\rho(b,b^*)\|b\|_2^2 \|b^*\|_2^2, \|b\|_2^2)$.

Proof: To prove (1) we introduce two statistics:

$$S_{n1}(b) = n^{-\frac{1}{2}} \sum_{j=1}^{n} b_n(G(Y_{[j]}), \frac{j}{n+1}),$$

and

$$S_{n2}(b) = n^{-\frac{1}{2}} \sum_{j=1}^{n} b_n(G(Y_{[j]}), F(X_{(j)})).$$

We remark that under $P_n$ $G(Y_{[j]}) = U_j$, $j = 1,\ldots,n$, are independent uniform (0,1) random variables as are $F(X_j) = V_j$, $j = 1,\ldots,n$ and the U's and V's are independent. Since $E_0[b_n(U_j, \frac{j}{n+1})] = E_0[b_n(\frac{R_{[j]}}{n+1}, \frac{j}{n+1})] = 0$, we have by Hájek [ 9] Lemma 2.1, (8.8) $$E_0[S_n(b) - S_{n1}(b)]^2$$

$$= \frac{1}{n} \sum_{j=1}^{n} E_0[b_n(U_j, \frac{j}{n+1}) - b_n(\frac{R_{[j]}}{n+1}, \frac{j}{n+1})]^2$$

$$\le \frac{1}{n} \sum_{j=1}^{n} 2 \max_{1\le i\le n} \frac{b_{nij}}{n^{\frac{1}{2}}} [2 \frac{1}{n} \sum_{i=1}^{n} b^2_{n,i,j}]^{\frac{1}{2}}$$

$$\le 2^{3/2} [\frac{1}{n^2} \sum_{j=1}^{n} \max_{1\le i\le n} b^2_{nij}]^{\frac{1}{2}} [\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} b^2_{n,i,j}]^{\frac{1}{2}}$$

An examination of the proof of Hájek [ 9] Lemma 6.1 will convince the reader that $b_n(u,v) \to b(u,v)$ in q.m. and this implies the uniform integrability of $b_n$. Thus

(8.9) $$\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} b^2_{n,i,j} = \iint b^2_n(u,v)dudv \to \|b\|^2_2 < \infty$$

and

(8.10) $$\frac{1}{n^2} \sum_{j=1}^{n} \max_{1\le i\le n} b^2_{n,i,j} = \iint_{A_n} b^2_n(u,v)dudv \to 0,$$

since $A_n = \bigcup_{j=1}^{n} \{(u,v): \frac{j-1}{n} \le v < \frac{i}{n}, \frac{i(j)-1}{n} \le u < \frac{i(j)-1}{n}, |b_{n,i(j),j}| = \max_{1\le i\le n} |b_{nij}|\}$

is a set whose Lebesgue measure approaches zero as $n \to \infty$. From (8.8), (8.9) and (8.10) we conclude that $E_0[S_n(b)-S_{n1}(b)]^2 \to 0$.

Also

(8.11) $$E_0[S_{n1}(b)-S_{n2}(b)]^2$$

$$= E_0\{E_0([S_{n1}(b)-S_{n2}(b)]^2 | X_{(1)}, \ldots, X_{(n)})\}$$

$$= \frac{1}{n} \sum_{j=1}^{n} E_0\{\int [b_n(u, \frac{j}{n+1}) - b_n(u, F(X_{(j)}))]^2 du\}$$

$$= E_0\{\int [b_n(u, \frac{R_1'}{n+1}) - b_n(u, V_1)]^2 du\}$$

where $R_1'$ is the rank of $V_1 = F(X_1)$ among $V_1, \ldots, V_n$. By Hájek [ 9]

Lemma 2.1 the last term of (8.11) is smaller than

$$2^{3/2}\int\{[\max_{1\le j\le n}|b_n(u, \tfrac{j}{n+1})|/n^{\frac{1}{2}}][\tfrac{1}{n}\sum_{j=1}^{n} b_n^2(u, \tfrac{j}{n+1})]^{\frac{1}{2}}\}du$$

$$\le 2^{3/2}[\tfrac{1}{n}\int \max_{1\le j\le n} b_n^2(u, \tfrac{j}{n+1})du]^{\frac{1}{2}}\ [\iint b_n^2(u,v)dudv]^{\frac{1}{2}} \rightarrow 0.$$

Combining this with the previous result, we conclude that $E_0[S_n(b)-S_{n2}(b)]^2 \rightarrow 0$. Since

$$S_{n2}(b) = n^{-\frac{1}{2}}\sum_{j=1}^{n} b_n(G(Y_{[j]}), F(X_{(j)}))$$

$$= n^{-\frac{1}{2}}\sum_{j=1}^{n} b_n(G(Y_j), F(X_j))$$

is a sum of independent and identically distributed random variables, (1) is proved.

To prove part (2) we introduce

$$S_{n3}(b) = n^{-\frac{1}{2}}\sum_{j=1}^{n} b(G(Y_{[j]}), F(X_{(j)})) = n^{-\frac{1}{2}}\sum_{j=1}^{n} b(G(Y_j), F(X_j)).$$

It is clear that $E_0[S_{n3}(b)-S_{n2}(b)]^2 \rightarrow 0$. Note that $T_n$ defined in (3.9) is $aS_{n3}(b^*)$; therefore, by Lemma 3.1 part (2) and (3.11), we may substitute $aS_{n3}(b^*)$ for $L_n$ in Lemma 3.1 part (3).

It is clear that $\mathcal{L}(S_{n3}(b), aS_{n3}(b^*)|P_n)$ is asymptotically normal with correlation $\iint b(u,v)b^*(u,v)dudv/\|b\|_2\|b^*\|_2 = \rho(b,b^*)$. Therefore, by Lemma 3.1 part (3) with $L_n$ replaced by $S_{n3}(b^*)$, $\mathcal{L}(S_{n3}(b)|Q_n) \rightarrow N(a\rho(b,b^*)\|b\|_2\|b^*\|_2, \|b\|_2^2)$, and part (1) of this lemma follows from this combined with Lemma 3.1 part (1) and the fact that $E_0[S_{n3}(b)-S_n(b)]^2 \rightarrow 0$.

We conclude from Lemma (8.2) by arguments similar to those used in Section 4 that $B_n(b^*)$ is ALMP and that the Pitman ARE of any two Bhuchongkul statistics $S_n(b_1)$ and $S_n(b_2)$, say, is

$$e(S_n(b_1), S_n(b_2)) = \frac{\rho(b_1,b^*)}{\rho(b_2,b^*)}, \tag{8.12}$$

where $b^*$ is defined by (8.4). Thus the ARE of $S_n(b)$ compared to the layer-rank statistic $T_n(\underset{\sim}{C})$ defined in Section 2, is

$$(8.13)\qquad e(S_n(b),\, T_n(\underset{\sim}{C})) = \frac{\rho(b,b^*)}{\rho(c,c^*)},$$

where $c^*$ is given by (3.14).

It is easy to see that $c^*(u,v) = b^*(u,v) - \frac{1}{v}\int_0^v b^*(u,w)dw$; thus

$$(8.14)\qquad \iint c(u,v)c^*(u,v)dudv = \iint [c(u,v) - \int_v^1 \frac{c(u,w)}{w}dw]b^*(u,v)dudv.$$

Also

$$\|c^*\|^2 = \iint [b^*(u,v)]^2 dudv - 2\iint\limits_{w\leq v}\int \frac{1}{v}\, b(u,v)dwdvdu + \iint [\frac{1}{v}\int_0^v b(u,w)dw]^2 dudv,$$

and since,

$$\iint [\frac{1}{v}\int_0^v b(u,w)dw]^2 dudv = 2\iint\limits_{w_1<w_2<v}\int \frac{1}{v^2}\, b(u,w_1)b(u,w_2)dvdw_1dw_2du$$

$$= 2\iint\limits_{w_1<w_2}\frac{1}{w_2}\, b(u,w_1)b(u,w_2)dw_1dw_2du - \iiint b(u,w_1)b(u,w_2)dw_1dw_2du$$

$$= 2\iint\limits_{w\leq v}\int \frac{1}{v}\, b(u,v)b(u,w)dwdvdu,$$

we have

$$(8.15)\qquad \|c^*\|_2 = \|b^*\|_2.$$

Combining (8.14) and (8.15) and letting $b_c(u,v) = c(u,v) - \int_v^1 \frac{c(u,w)}{w}dw$ we have

$$(8.16)\qquad \rho(c,c^*) = \rho(b_c,b^*).$$

Since $\|c\|_2 < \infty$ implies $\|b_c\|_2 < \infty$ and $\iint c(u,v)du = \iint c(u,v)dv = 0$ implies $\iint b_c(u,v)du = \iint b_c(u,v)dv = 0$, the rank statistic $S_n(b_c)$ satisfies the conditions of Lemma 8.2. Thus, by (8.16) the ARE of

$S_n(b_c)$ with respect to $T_n(\underset{\sim}{C})$ is $\frac{\rho(b_c, b^*)}{\rho(c, c^*)} = 1$ for any $b^*$. We emphasize that this is true for any family of bivariate distributions satisfying (3.3), (3.4), (3.5), and (3.6). In other words, the statistics $T_n(C)$ and $S_n(b_c)$ are indistinguishable in terms of Pitman ARE.

Some ALMP layer-rank tests and their equivalent ALMP rank tests are listed below; $\mu_{i|j}$ is the mean of the $i^{th}$ largest of j normal r.v.

| Layer-Rank Test | Rank Test |
|---|---|
| Kendall's $\tau$: $T_n(\underset{\sim}{C}) = n^{-3/2} \sum_{j=1}^{n} (\ell_{(j)} - \frac{j}{2})$ | Spearman's $\rho$ (Rank Correlation): $S_n(b_c) = n^{-5/2} \sum_{j=1}^{n} j(R_{[j]} - \frac{n}{2})$ |
| Normal Scores: $T_n(\underset{\sim}{C}) = n^{-1/2} \sum_{j=1}^{n} \mu_{\ell_{(j)}\|j} L^*_{n,j}$ $(L_{n,j} = \mu_{j\|n} - \frac{1}{j-1} \sum_{j=1}^{n} \mu_{i\|j})$ | Normal Scores: $S_n(b_c) = n^{-1/2} \sum_{j=1}^{n} \mu_{R_{[j]}\|n} \mu_{j\|n}$ |

Appendix I $\mathbf{L}_r$-Convergence of Certain Functions.

In this section we denote by $b(u,v)$ a square integrable function whose domain is the unit square and which satisfies

$$\int b(u,v)du = 0. \tag{I.1}$$

We make the following definitions:

$$\bar{b}(u,v) = \left(\int_0^v b(u,w)dw\right)/v, \tag{I.2}$$

$$b_n(u,v) = E[b(U_{i|j},V_{j|n})], \quad \frac{i-1}{j} \le u < \frac{i}{j}, \quad \frac{j-1}{n} \le v < \frac{j}{n}, \tag{I.3}$$

$$\bar{b}_n(u,v) = \begin{cases} \frac{1}{j-1}\sum_{\alpha=1}^{j-1} E[b(U_{i|j},V_{\alpha|n}], & \frac{i-1}{j} \le u < \frac{i}{j},\ \frac{j-1}{n} \le v < \frac{j}{n},\ v \ge \frac{1}{n} \\ 0 & v < \frac{1}{n} \end{cases} \tag{I.4}$$

where $U_{i|j}$ is the $i^{\underline{th}}$ largest of $j$ independent uniform (0,1) random variables, $V_{j|n}$ is the $j^{\underline{th}}$ largest of $n$ independent uniform (0,1) random variables $(V_1,\ldots, V_n)$ and the U's and V's are independent.

Lemma I.1 If $b$ is square integrable and satisfies (I.1) and if $\bar{b}_{n,i,j} = E[\bar{b}(U_{i|j},V_{j|n})]$, $1 \le i \le j \le n$, then $\bar{b}_n(u,v) = \bar{b}_{n,i,j}$, $\frac{i-1}{j} \le u < \frac{i}{j}$, $\frac{j-1}{n} \le v < \frac{j}{n}$.

Proof: By Feller [6] p. 163 (10.9), we have for $j < 1$

$$E\,\frac{1}{j-1}\sum_{\alpha=1}^{j-1} b(U_{i|j},V_{\alpha|n})$$

$$= \frac{n}{j-1}\sum_{\alpha=1}^{j-1}\binom{n-1}{\alpha-1}\int_0^1 E[b(U_{i|j},v)]v^{\alpha-1}(1-v)^{n-\alpha}dv$$

$$= \frac{n!}{(j-1)!(n-j)!}\int_0^1 E[b(U_{i|j},v)]\int_v^1 w^{j-2}(1-w)^{n-j}dwdv$$

$$= \frac{n!}{(j-1)!(n-j)!}\int_0^1 \frac{1}{w}\int_0^w E[b(U_{i|j},v)]dv\, w^{j-1}(1-w)^{n-j}dw$$

$$= E[\bar{b}(U_{i|j}, V_{j|n})].$$

For $j = 1$, since $U_{1|1} = U_1$, $E[\bar{b}(U_{1|1}, V_{1|n}] = 0$; therefore, the lemma is proved.

Lemma I.2 If $b$ is continuous a.e. then $b_n \to b$ a.e. and $\bar{b}_n \to \bar{b}$ a.e.

Proof: $b_n(u,v) = E[b(U_{i|j}, V_{j|n})]$ where $\frac{i-1}{j} \leq u < \frac{i}{j}$ and $\frac{j-1}{n} \leq v < \frac{j}{n}$ or, equivalently, $i = [ju]+1$ and $j = [nv]+1$. For fixed $(u,v)$ let $\beta_n(x,y;u,v)$ denote the joint density of $(U_{i|j}, V_{j|n})$. It is easy to see that, for any $\epsilon > 0$, $\beta_n(x,y;u,v)$ aproaches zero uniformly in $(x,y)$ for $|x-u| > \epsilon$, $|y-v| > \epsilon$. From this and the integrability of $b$ it follows that $b_n(u,v) = \iint b(x,y)\beta_n(x,y;u,v)dxdy \to b(u,v)$.

Note that $\bar{b}$ is clearly a.e. continuous and that

$$\begin{aligned} \iint|\bar{b}(u,v)|dudv &= \iint[\frac{1}{v}\int_0^v b(u,w)dw]^2 dudv \\ &\leq \iint \frac{1}{v}\int_0^v |b(u,w)|dw\ dudv \\ &= \iint \ell n(\frac{1}{w})|b(u,w)|dwdu \\ &\leq [\int \ell n^2(w)dw]^{\frac{1}{2}}\ [\iint b^2(u,w)dwdu] < \infty. \end{aligned}$$

Thus, by Lemma II.1 an argument identical to that used for $b_n$ implies that $\bar{b}_n \to \bar{b}$ a.s. ▌

The reader will recall that we use $\mathbf{L}_r$ and $\|\cdot\|_r$ to denote the space of $r$-th integrable functions defined in the unit square and the norm of the space.

Lemma I.3 (1) If $b(u,v) \in \mathbf{L}_2$, then $\|b_n - b\|_2 \to 0$, and (2) if there is a $\delta > 0$ such that $\int(\int|b(u,v)|^{2(1+\delta)}dv)^{\frac{1}{1+\delta}}du < \infty$, then $\|\bar{b}_n - \bar{b}\|_2 \to 0$.

Proof: (1) By Lemma I.2 and the $L_r$-convergence theorem*, it suffices to

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*Loéve [15] p. 163.

show that $\|b_n\|_2 \to \|b\|_2$. By Fatou's lemma*, $\liminf_{n\to\infty} \|b_n\|_2 \geq \|b\|_2$.

On the other hand, using Jensen's inequality, we have,

$$\|b_n\|_2^2 = \frac{1}{n} \sum_{j=1}^{j} \frac{1}{j} \sum_{i=1}^{j} (E[b(U_{i|j},V_{j|n})])^2$$

$$\leq \frac{1}{n} \sum_{j=1}^{n} \frac{1}{j} \sum_{i=1}^{j} E[b^2(U_{i|j},V_{j|n})] = \|b\|_2^2 .$$

Since

$$\iint(\bar{b}(u,v))^2 dudv \leq \iint \frac{1}{v} \int_0^v b^2(u,w)dwdudv$$

$$\leq \iint(\frac{1}{v})^{\frac{1}{1+\delta}}[\int_0^1 |b(u,w)|^{2+2\delta}\, dw]^{\frac{1}{1+\delta}}\, dudv < \infty,$$

part (2) follows from Lemma I.1 by an identical argument.

Remark: It is clear that the condition $\int[\int|b(u,v)|^{2+2\delta}dv]^{\frac{1}{1+\delta}}du < \infty$ can be replaced by a weaker condition:

(I.5) $\|\bar{b}\|_2 < \infty.$

*Ibid. p. 125, B.

Let $U_1,\dots,U_n$ and $V_1,\dots,V_n$ be independent uniform (0,1) random variables. Suppose $V_{\alpha_1} \le V_{\alpha_2} \le \dots \le V_{\alpha_n}$, it is clear that $U_{\alpha_1},\dots,U_{\alpha_n}$ are independent uniform (0,1) random variables and are independent of the V's. Let $V_{\alpha_j} = V_{j|n}$, $j = 1,\dots,n$, let $U_{i|j}$ be the $i^{\underline{th}}$ largest of $U_{\alpha_1},\dots,U_{\alpha_j}$ and let $\ell_{(j)}$ be the rank of $U_{\alpha_j}$ among $U_{\alpha_1},\dots,U_{\alpha_j}$. We define the statistics

(I.6) $$Z_n = n^{-\frac{1}{2}} \sum_{j=1}^{n} b(U_j,V_j) = n^{-\frac{1}{2}} \sum_{j=1}^{n} b(U_{\alpha_j},V_{j|n}),$$

and, recalling (I.4) and (I.3),

$$Z_n^* = n^{-\frac{1}{2}} \sum_{j=1}^{n} [b_n(\frac{\ell_{(j)}}{j+1}, \frac{j}{n+1}) - \bar{b}_n(\frac{\ell_{(j)}}{j+1}, \frac{j}{n+1})].$$

Lemma I.4 Under the conditions of Lemma I.3(2), if $\int b(u,v)du = \int b(u,v)dv = 0$, then $E(Z_n-Z_n^*)^2 \to 0$.

Proof: It is clear that $EZ_n = 0$, thus $EZ_n^2 = \iint b^2(u,v)dudv$. Letting $c_n(u,v) \equiv b_n(u,v)-\bar{b}_n(u,v)$ and noting that $\ell_{(j)}$ depends only on $U_{\alpha_1},\dots,U_{\alpha_j}$, we have

(I.7) $$E[Z_nZ_n^*] = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{j} \sigma[b(U_{\alpha_i},V_{i|n}), c_n(\frac{\ell_{(j)}}{j+1}, \frac{j}{n+1})].$$

From Lemma 1.1 and the fact that $U_{\alpha_j} = U_{\ell_{(j)}|j}$, we obtain

(I.8) $$\sigma[b(U_{\alpha_j},V_{j|n}), c_n(\frac{\ell_{(j)}}{j+1}, \frac{j}{n+1})]$$

$$= \frac{1}{j} \sum_{i=1}^{j} E[b(U_{i|j},V_{j|n})] c_n(\frac{i}{j+1}, \frac{j}{n+1})$$

$$= \frac{1}{j} \sum_{i=1}^{j} b_n(\frac{i}{j+1}, \frac{j}{n+1}) c_n(\frac{i}{j+1}, \frac{j}{n+1}).$$

Letting $\ell'$ be the rank of $U_{\alpha_i}$ among $U_{\alpha_1},\dots,U_{\alpha_i}$, we have, for $i < j$,

(I.9)

$$\sigma[b(U_{\alpha_i}, V_{i|n}), (\cdots c)_n(\frac{\ell(j)}{j+1}, \frac{j}{n+1})]$$

$$= \frac{1}{j(j-1)} \sum_{\ell \neq} \sum_{\ell'} Eb(U_{\ell'|j}, V_{i|n})(\cdots c_n)(\frac{\ell}{j+1}, \frac{j}{n+1})$$

$$= -\frac{1}{j(j-1)} \sum_{\ell=1}^{j} Eb(U_{\ell|j}, V_{i|n})(\cdots c_n)(\frac{\ell}{j+1}, \frac{j}{n+1})$$

$$+ [\frac{1}{j} \sum_{\ell'=1}^{j} Eb(U_{\ell'|j}, V_{i|n})][\frac{1}{j-1} \sum_{\ell=1}^{j} (\cdots c_n)(\frac{\ell_j}{j+1}, \frac{j}{n+1})$$

$$= -\frac{1}{j(j-1)} \sum_{i=1}^{j} b_n(\frac{i}{j+1}, \frac{i}{n+1})(\cdots c_n)(\frac{i}{j+1}, \frac{j}{n+1}),$$

since $\frac{1}{j} \sum_{\ell'=1}^{j} Eb(U_{\ell'|j}, V_{j|n}) = E[\int b(u, V_{j|n})du] = 0.$

Inserting (I.9) and (I.8) into (I.7), we obtain,

$$E[Z_n Z_n^*] = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{j} \sum_{i=1}^{j} [(\cdots c_n)(\frac{i}{j+1}, \frac{j}{n+1})]^2$$

$$= \iint [(b_n(u,v) - \bar{b}_n(u,v)]^2 dudv \to \iint [b(u,v) - \bar{b}(u,v)]^2 dudv,$$

by Lemma I.2.

Finally,

$$E[Z_n^*]^2 = \iint [b_n(u,v) - \bar{b}_n(u,v)]^2 dudv \to \iint [b(u,v) - \bar{b}(u,v)]^2 dudv.$$

Thus,

$$E(Z_n - Z_n^*)^2$$

$$\to \iint [b(u,v)]^2 dudv - \iint [b(u,v) - \bar{b}(u,v)]^2 dudv$$

$$= 2\iint b(u,v)\bar{b}(u,v) dudv - \iint [\bar{b}(u,v)]^2 dudv = 0,$$

since

$$\iint[\bar{b}(u,v)]^2dudv = 2\iint \int_{0<w_1<w_2<v}\int \frac{1}{v^2} b(u,w_1)b(u,w_2)dw_1dw_2dudv$$

$$= 2\iint_{0<w_1<w_2<1}\int (\frac{1}{w_2}-1)b(u,w_1)b(u,w_2)dw_1dw_2du$$

$$= 2\iint[\frac{1}{v}\int_0^v b(u,w)dw]b(u,v)dudv$$

$$= 2\iint \bar{b}(u,v)b(u,v)dudv. \blacksquare$$

Suppose that $(X_1,Y_1),\ldots,(X_n,Y_n)$ is a sample from a bivariate population with continuous CDF $F(x)G(y)$, where $G$ has density $g$. It is well known that if we set $U_j = G(Y_j)$, $V_j = F(X_j)$, $j = 1,\ldots,n$, then the U's and V's are independent uniform (0,1) random variables. If $s(x,y)$ is a function satisfying (3.4) and (3.5), then the function $b(u,v) = s(F^{-1}(v),G^{-1}(u))$ satisfies the conditions of Lemma I.4, from which we obtain the following:

Corollary I.5 Let $T_n$ and $T_n(\underset{\sim}{C}^*)$ be given by (3.9) and (3.13), respectively. If $s(x,y)$ satisfies (3.4) and (3.5), then $E[T_n-T_n(\underset{\sim}{C}^*)]^2 \to 0$.

Corollary I.6 If $\bar{b}$ is square integrable and if $\int b(u,v)du = \int b(u,v)dv = 0$, then $\|b\|_2 = \|b-\bar{b}\|_2$.

Let us define $J_n(u)$ by (2.9.1) and $c_n^{(2)}(u,v)$ by (2.4) with $c_{n,i,j}^{(2)}$ given by (2.8.2) and $L_{n,j} = 1$.

Lemma I.7 If there is an a.s. continuous function J such that, for some $\delta > 0$, $\|J\|_{2+2\delta} < \infty$ and $\|J_n - J\|_{2+2\delta} \to 0$, then $\|c_n^{(2)} - c^{(2)}\|_2 \to 0$, where $c^{(2)}(u,v) = J(uv) - \frac{1}{v}\int_0^v J(w)dw$.

Proof. In view of the remarks following (2.7), it suffices to prove that $c_n'(u,v)$, given by (2.4) with $c_{n,i,j} = J_{n,i}$, approaches $J(uv)$ in q.m.

Letting $v_n = \frac{j}{n} = \frac{[nv]}{n}$, it is clear from (2.9.1) that $c_n'(u,v) = J_n(uv_n)$.

Thus,

$$\|c_n' - J(uv)\|_2$$

$$= [\int\int (J_n(uv_n) - J(uv))^2 dudv]^{\frac{1}{2}}$$

$$\leq [\int\int (J_n(uv_n) - J(uv_n))^2 dudv]^{\frac{1}{2}} + [\int\int (J(uv_n) - J(uv))^2]^{\frac{1}{2}}$$

$$= R_{n1} + R_{n2}, \text{ say.}$$

Now,

$$R_{n1} = [\int \frac{1}{v_n} \int_0^{v_n} (J_n(u) - J(u))^2 dudv]^{\frac{1}{2}}$$

$$\leq [\int v_n^{-\frac{1}{1+\delta}} dv]^{\frac{1}{2}} [\int_0^1 |J_n(u) - J(u)|^{2+2\delta} du]^{\frac{1}{2+\delta}}$$

$$= [\frac{1}{n} \sum_{j=1}^{n} (\frac{n}{j})^{\frac{1}{1+\delta}}]^{\frac{1}{2}} \|J_n - J\|_{2+2\delta} \to 0,$$

and, for any $\epsilon > 0$,

$$R_{n2} = [\int\int (J(uv_n) - J(uv))^2 dudv]^{\frac{1}{2}}$$

$$\leq [\int_0^\epsilon \int (J(uv_n) - J(uv))^2 dudv]^{\frac{1}{2}}$$

$$+ [\int_\epsilon^1 \frac{1}{v_n} \int_0^{v_n} (J(u) - J(u \frac{v}{v_n}))^2 dudv]^{\frac{1}{2}}$$

$$= R_{n21} + R_{n22}, \text{ say.}$$

Since

$$R_{n21} \leq [\int_0^\epsilon \int J^2(uv_n) dudv]^{\frac{1}{2}} + [\int_0^\epsilon \int J^2(uv) dudv]^{\frac{1}{2}}$$

$$\leq \{[\int_0^\epsilon v_n^{-\frac{1}{1+\delta}} dv]^{\frac{1}{2}} + [\int_0^\epsilon v^{-\frac{1}{1+\delta}} dv]^{\frac{1}{2}}\} \|J\|_{2+2\delta},$$

can be made arbitrarily small by selecting $\epsilon$ small enough and since

$$R_{n22} \leq (\frac{2}{\epsilon})^{\frac{1}{2}} [\int_\epsilon^1 \int (J(u) - J(u \frac{v_n}{v}))^2 dudv]^{\frac{1}{2}}, \; n \geq \frac{2}{\epsilon},$$

and $\frac{v_n}{v} \to 1$ uniformly in $v \geq \epsilon > 0$ it follows from the a.s. continuity and square integrability of $J$ that $R_{n22} \to 0$ for any $\epsilon > 0$, and the lemma is proved.

## Appendix II Properties of Moment Generating and Related Functions.

Let $F_1, \ldots, F_n$ be the cdf's of independent random variables $X_1, \ldots, X_n$. We denote by $f_i(h) = E\exp(hX_i) = \int \exp(hx)dF_i(x)$, $i = 1, \ldots, n$, the moment generating function of $X_i$. For each $h$ such that $f_1(h), \ldots, f_n(h)$ are finite we define $Z_i(h)$, $i = 1, \ldots, n$, to be independent random variables such that

$$dP(Z_i(h) \leq z) = \frac{\exp(hz)}{f_i(h)} dP(X_i \leq z), \quad i = 1, \ldots, n.$$

Lemma II.1 If $S_n(h) = \sum_{i=1}^{n} Z_i(h)$, then

$$P[\sum_{i=1}^{n} X_i \geq x] = \prod_{i=1}^{n} f_i(h) \int_x^{\infty} \exp(-hz)dP(S_n(h) \leq z).$$

Proof: $$\int_x^{\infty} \exp(-hz)dP(S_n(h) \leq z) = \int_{z_1+\ldots+z_n \geq x} \cdots \int \sum_{i=1}^{n} \exp(-hz_i)dP(Z_i(h) \leq z_i)$$

$$= [\prod_{i=1}^{n} \frac{1}{f_i(h)}] \int_{z_1+\ldots+z_n \geq x} \cdots \int \prod_{i=1}^{n} dP(X_i \leq z_i). \blacksquare$$

We next prove various properties of the functions $\psi_{c_n}, \mu_{c_n}, \mu_{c_n}^{(i)}$, $i=2,3$, and $m_{c_n}$, $n = 0,1,\ldots$, defined by (6.2)-(6.5).

Lemma II.2 If $I(\underset{\sim}{C}) = \{h: m_{c_n}(h) \to m_c(h)\}$, then (i) if $I(\underset{\sim}{C})$ is non-empty it is an interval containing the origin,

(ii) $\psi_{c_n} \to \psi_c$, $\mu_{c_n} \to \mu_c$, $\mu_{c_n}^{(i)} \to \mu_c^{(i)}$, $i = 2,3$, uniformly in $h$ on any compact subset, A, of $I(\underset{\sim}{C})$.

(iii) $\psi_c$, $\mu_c$, and $\mu_c^{(i)}$, $i = 1,2$, are uniformly bounded in h on A.

Proof: From Jensen's inequality and (2.4) we obtain

$$\int \exp(hc_n(u,v))du \geq 1. \tag{II.1}$$

Suppose $h_1 \in I(\underset{\sim}{C})$, $h_1 \geq 0$, and let $0 \leq h \leq h_1$, then, again from Jensen's inequality, we have for any $m \geq 0$, $i = 0, \ldots, 3$,

$$\int\int_{\{|c_n|\geq m\}} [|c_n(u,v)|^i \exp(hc_n(u,v))/\int\exp(hc_n(w,v))dw]dudv$$

$$\leq \left[\int\int_{\{|c_n|\geq m\}} [|c_n(u,v)|^3 \exp(hc_n(u,v))/\int\exp(hc_n(w,v))dw]dudv\right]^{\frac{i}{3}}$$

$$\leq \left[\int\int_{\{|c_n|\geq m\}} |c_n(u,v)|^3 \exp(hc_n(u,v))dudv\right]^{\frac{i}{3}}$$

$$\leq \left[\int\int_{\{|c_n|\geq m\}} |c_n(u,v)|^3 dudv + \int\int_{\{|c_n|\geq m\}} |c_n(u,v)|^3 \exp(h_1(u,v)dudv\right]^{\frac{i}{3}}$$

which, by (2.2), (6.6) and the assumption that $h_1 \in I(\underset{\sim}{C})$, implies* that $\psi_{c_n}$, $\mu_{c_n}$, $\mu_{c_n}^{(i)}$, $i = 2,3$, and $m^*_{c_n}$ are uniformly integrable uniformly in $h$, $0 \leq h \leq h_1$, a similar result being true if $h_1 \leq 0$. Clearly, this implies* (i), (ii) and (iii). ∎

In order to prove the monotonicity of $\mu_c(h)$ we require the following result:

Lemma II.3 Let $X$ be a real random variable with distribution $F$ and finite mean, $\mu$. If $g(x)$ is a non-decreasing, a.e. Finite function on the line, then $\sigma(X,g(X)) \geq 0$.

This result is so obvious that one must classify it as statistical folklore; nevertheless, the only proof of which we are aware is the following, which is due to Sobel [22]:

Proof: Let $EX = \mu$; $-\infty < \mu < \infty$ and $g$ non-decreasing imply that $|g(\mu)| < \infty$.

$$\sigma(X,g(X)) = \int_{-\infty}^{\infty}(x-\mu)g(x)dF(x)$$

$$= \int_{-\infty}^{\mu}(x-\mu)g(x)dF(x) + \int_{\mu}^{\infty}(x-\mu)g(x)dF(x)$$

$$\geq g(\mu)\int_{-\infty}^{\mu}(x-\mu)dF(x) + g(\mu)\int_{-\infty}^{\mu}(x-\mu)dF(x) = 0.$$ ∎

We are now ready to prove the strict monotonicity of $\mu(h)$.

*See Loéve [15], p. 163; $L_r$-convergence Theorem.

Lemma II.4 $\mu_c(h)$ is strictly increasing inside $I(\underset{\sim}{C})$ provided $c$ is non-degenerate*.

Proof: We show that $\mu_c(h)$ has a positive derivation inside $I(\underset{\sim}{C})$. Suppose $h_1 > 0$, $h_1 \in I(\underset{\sim}{C})$, for fixed $h$ and any $\delta$ such that $0 \leq h < h+\delta \leq h_1$, if we define, for each $v$, the density function $f(u;v) = \exp(hc(u,v)/\int\exp(hc(w,v))dw$, then, suppressing the arguments of $c$, we have

$$(\text{II.2}) \qquad (\mu_c(h+\delta) - \mu_c(h))/\delta$$

$$= \int\left(\frac{\int e^{hc}du}{\int e^{(h+\delta)c}du} \quad \int c\,\frac{(e^{c\delta}-1)}{\delta}\,F(u,v)du \right.$$

$$\left. - \int cf(u,v)du \int \frac{e^{c\delta}-1}{\delta}\,f(u,v)du\right) dv.$$

From the inequality $|e^x-1| \leq |x|(e^x+1)$ we have, for $r = 0,1$, $c^r(\exp(c\delta)-1)/\delta f(u,v) \leq |c|^2(\exp(c(h+\delta)) + \exp(hc)) \leq 2|c|^2(1+\exp(h_1c))$. Thus, by the dominated convergence theorem, if $r = 1$ or $2$,

$$\int c^r(u,v)(\exp(\delta c(u,v))-1)/\delta f(u;v)du \to \int c^{r+1}(u,v)f(u;v)du, \quad \text{as } \delta \to 0,$$

and, similarly, $\int\exp((h+\delta)c(u,v))du \to \int\exp(hc(u,v))du$, as $\delta \to 0$. Therefore, the integrand in (II.2) converges to $\int c^2(u,v)f(u;v)du - (\int c(u,v)f(u;v)du)^2$, as $\delta \to 0$.

Applying (II.1), the inequality $|e^x-1| \leq |x|(e^x+1)$, and Lemma II.3 with $X = c(U,v)$ where, for each fixed $v$, $U$ is a random variable with density $f(u;v)$, we conclude that the integrand in (II.1) is bounded by $2\int|c(u,v)|^2(\exp((h+\delta)c(u,v)) + \exp(hc(u,v)))du$ $\leq 4\int|c(u,v)|^2(1+\exp(h_1c(u,v)))du < \infty$. Therefore, by the dominated convergence theorem,

---

*c(u,v) is degenerate if it is a function of $v$ only.

$$(d/dh)\mu(h) = \int[\int c^2(u,v)f(u;v)du - (\int c(u,v)f(u;v)du)^2]dv,$$

which is positive for non-degenerate $c$.

Lemma II.5

$$\int \ell n(\int \exp(hc_n(u,v))du)dv \to \int \ell n(\int \exp(hc(u,v))du)dv,$$

uniformly on compact subsets of $I(\underset{\sim}{C})$.

Proof: Suppose $0 > h_1$, $h_1 \in I^*$. We show that $\ell n(\int \exp(hc_n(u,v))du)$ is unifôrmly integrable uniformly in $h$, for $0 \leq h \leq h_1$. Recall (II.1) and let

(II.3) $$A_n(M) = \{v: \int \exp(hc_n(u,v))du \geq m\}$$

$$0 \leq \int_{A_n(m)} \ell n(\int \exp(hc_n(u,v))du)dv$$

$$\leq \int_{A_n(m)} dv\{\ell n(\int \int_{A_n(m)} \exp(hc_n(u,v))dudv) - \ell n(\int_{A_n(m)} dv)\}$$

Since $\exp(hc_n) \leq 1+\exp(h_1c_n)$ and $A_n(m) \subset \{v: \int \exp(h_1c_n(u,v))du \geq m-1\}$, if follows from the uniform integrability of $\int \exp(h_1c_n(u,v))du$ that

$$\int \int_{A_n(m)} \exp(hc_n(u,v))dudv$$

can be made arbitrarily small uniformly in $h$ and $n$ by selecting $m$ large enough. Therefore, the last term in (II.3) is bounded by

$$- \int_{A_n(m)} dv\ \ell n(\int_{A_n(m)} dv),$$

from which we obtain the desired result. ▌

Suppose $b(u,v)$ is a function defined and square integrable on the unit square such that $\int b(u,v)du = 0$ and that $c_n(u,v) = b_n(u,v)-\overline{b}_n(u,v)$, where $b_n$ and $\overline{b}_n$ are defined in (I.3) and (I.4). Letting $c(u,v) = b(u,v)-\overline{b}(u,v)$, $\overline{b}$ being defined by (I.2), we have:

Lemma II.6 If $A = \{h: \iint |c(u,v)|^3 \exp(h|c(u,v)|)dudv < \infty\}$, then $A \subset I(C)$.

Proof: $I(C)$ is the set on which

(II.4) $$\iint |c_n(u,v)|^3 \exp(hc_n(u,v))dudv \to \iint |c(u,v)|^3 \exp(hc(u,v))dudv.$$

Since by Lemma I.1 $c_n(u,v) \to c(u,v)$ a.s., if

(II.5) $$\iint |c_n(u,v)|^3 \exp(h|c_n(u,v)|)dudv \to \iint |c(u,v)|^3 \exp(h|c(u,v)|)dudv,$$

then $|c_n|^3 \exp(h|c_n|)$ is uniformly integrable*. But this clearly implies the uniform integrability of $|c_n|^3 \exp(hc_n)$ which, in turn, inplies (II.4). Thus it is sufficient to show that (II.5) is true for every $h$ in $A$.

By Fatou's lemma we have for any $h$,

$$\liminf \iint |c_n(u,v)|^3 \exp(h|c_n(u,v)|)dudv$$

$$\geq \iint |c(u,v)|^3 \exp(h|c(u,v)|dudv.$$

On the other hand, applying Lemma I.1,

(II.6) $$\iint |c_n(u,v)|^3 \exp(h|c_n(u,v)|)dudv$$

$$= \frac{1}{n} \sum_{j=1}^{n} \frac{1}{j} \sum_{i=1}^{j} |c_{nij}|^3 \exp(h|c_{nij}|)$$

$$= \frac{1}{n} \sum_{j=1}^{n} \frac{1}{j} \sum_{i=1}^{j} |E[c(U_{i|j}, V_{j|n})]|^3 \exp(h|E[c(U_{i|j}, V_{j|n}]|)$$

$$\leq \frac{1}{n} \sum_{j=1}^{n} \frac{1}{j} \sum_{i=1}^{i} E|c(U_{i|j}, V_{j|n})|^3 \exp(hE|c(U_{i|j}, V_{j|n})|).$$

$$\leq \frac{1}{n} \sum_{j=1}^{n} \frac{1}{j} \sum_{i=1}^{j} E[|c(U_{i|j}, V_{j|n})|^3 \exp(h|c(U_{i|j}, V_{j|n})|)],$$

*Loéve [15] p. 163 C.

by Lemma II.3 (with $X = |c(U_{i|j}, V_{j|n})|^3$ and $g(x) = \exp(hx)$). Since the last term of (II.6) equals $\iint |c(u,v)|^3 \exp(h|c(u,v)|)dudv$, the lemma is proved.

## Appendix III  Probability Limit of $n^{-\frac{1}{2}}T_n(\underset{\sim}{C})$.

Let $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_n$, $\underset{\sim}{X}_j = (X_j, Y_j)$, $j = 1, \ldots, n$, be a sample from $H_\theta$ and let $H_n(x,y)$ be the empirical cdf corresponding to the sample; i.e., $nH_n(x,y)$ is the number of sample points to the left of and below the point $(x,y)$. $F_n(x) = H_n(x,\infty)$ is the empirical cdf of the X-coordinate of the sample points. Clearly $H_n(\underset{\sim}{X}_j)$ is the ($3^{\underline{rd}}$ quadrant) layer-rank of $X_j$ and $F_n(X_j)$ is the rank of $X_j$ among $X_1, \ldots, X_n$; thus, recalling the definition of $\ell_{(j)}$ (p. 5), we have:

$$n^{-\frac{1}{2}}T_n(\underset{\sim}{C}) = n^{-1} \sum_{j=1}^{n} c_n\left(\frac{\ell_{(j)}}{j+1}, \frac{j}{n+1}\right) \tag{III.1}$$

$$= n^{-1} \sum_{j=1}^{n} c_n\left(\frac{nH_n(X_j,Y_j)}{nF_n(X_j)+1}, \frac{nF_n(X_j)}{n+1}\right)$$

$$= \iint c_n\left(\frac{nH_n}{nF_n+1}, \frac{nF_n}{n+1}\right) dH_n .$$

Let $P_\theta$ denote the probability measure induced by an infinite sequence of observations from $H_\theta$. It seems evident, in view of the Glivenko-Cantelli Lemma, that the only reasonable $P_\theta$-probability limit of $n^{-\frac{1}{2}}T_n(\underset{\sim}{C})$ is:

$$\eta_c(\theta) = \iint c\left(\frac{H_\theta}{F_\theta}, F_\theta\right) dH_\theta, \tag{III.2}$$

where $F_\theta(x) = H_\theta(x,\infty)$; nevertheless, we were not able to find very satisfactory sufficient conditions that this be the case and are forced to offer the following somewhat impractical result:

<u>Lemma III.1</u>  If

(i) $\iint \left[c_n\left(\frac{nH_n}{nF_n+1}, \frac{nF_n}{n+1}\right) - c\left(\frac{nH_n}{nF_n+1}, \frac{nF_n}{n+1}\right)\right] dH_n \to 0$

in probability as $n \to \infty$,

(ii) $E_\theta \left| c\left(\frac{nH_n(X_1)}{nF_n(X_1)+1}, \frac{nF_n(X_1)}{n+1}\right) - c\left(\frac{H_\theta(X_1)}{F_\theta(X_1)}, F_\theta(X_1)\right)\right| \to 0$, as $n \to \infty$,

and

(iii) if the right side of (III.2) is finite, then $n^{-\frac{1}{2}}T_n(\underset{\sim}{C}) \to \eta_c(\theta)$ in $P_\theta$-probability.

Proof: (i) and (ii) imply that $n^{-\frac{1}{2}}T_n(\underset{\sim}{C}) - \frac{1}{n}\sum_{j=1}^{n} c\left(\frac{H_\theta(X_j,Y_j)}{F_\theta(X_j)}, F_\theta(X_j)\right) \to 0$, in $P_\theta$-probability. Since $n^{-1}\Sigma_{j=1}^{n} c\left(\frac{H_\theta(X_j,Y_j)}{F_\theta(X_j)}, F_\theta(X_j)\right)$ is the average of $n$ independent and identically distributed random variables with finite mean, $\eta_c(\theta)$, the result follows from the weak law of large numbers. ▌

The simplest way to satisfy (i) is to set $c_{nij} = c\left(\frac{i}{j+1}, \frac{j}{n+1}\right)$; however, in several important applications, in particular Kendall's $\tau$-statistic, we have a sequence $\underset{\sim}{C}$ such that $\sup_{n,v}|c_n(u,v)| \leq m < \infty$ for all $n$ and $c_n(u,v) \to c(u,v)$ uniformly in $u$ and $v$ on any set of the form $v \geqq v_0 > 0$. In this case, for any $v_0 > 0$,

$$\iint \left| c_n\left(\frac{nH_n}{nF_n+1}, \frac{nF_n}{n+1}\right) - c\left(\frac{nH_n}{nF_n+1}, \frac{nF_n}{n+1}\right)\right| dH_n$$

$$\leq \sup_{v \geqq v_0} |c_n(u,v)-c(u,v)| + m\int_{\{F_n \geqq v_0\}} dF_n \to mv_0, \text{ almost surely } (P_\theta).$$

Consequently, condition (i) of Lemma (III.1) holds.

Because of the boundedness of $c$, (iii) holds, and we now show that if $c$ is continuous on the unit square, then (ii) holds. Let $H_{n-1}$ denote the empirical cdf of the sample

$\underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_n$, then $nH_n(X_1,Y_1) = (n-1)H_{n-1}(X_1,Y_1) + 1.$

Since $\sup_{x,y}|H_{n-1}(x,y)-H_\theta(x,y)| \to 0$ almost surely $(P_\theta)$ and $c$ is uniformly continuous, it follows that $c(\frac{nH_n(X_1,Y_1)}{nF_n(X_1)+1}, \frac{nF_n(X_1)}{n+1}) - c(\frac{H_\theta(X_1,Y_1)}{F_\theta(X_1)}, F_\theta(X_1))$

converges to zero on any set of the form $\{F(X_1) \geq v_0\}$, $v_0 > 0$.

Therefore (ii) holds. To summarize, we have:

Corollary (III.2) If $\sup_{u,v}|c_n(u,v)| \leq m < \infty$, $n = 0,1,\ldots,$ and $c_n \to c$ uniformly on any set of the form $\{v \geq v_0\}$, $v_0 > 0$, then $n^{-\frac{1}{2}}T_n(\underset{\sim}{C}) \to \eta_c(\theta)$, given by (III.2), in $P_\theta$-probability.

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Table III Weight Factors For Normal Scores Layer-Rank Test

| n | j | $L^*_{n,j}$ |
|---|---|---|
| 2 | 2 | 1.12833 |
| 3 | 2 | .84628 |
| | 3 | 1.25942 |
| 4 | 2 | .73237 |
| | 3 | .96020 |
| | 4 | 1.37250 |
| 5 | 2 | .66794 |
| | 3 | .82898 |
| | 4 | 1.04767 |
| | 5 | 1.45370 |
| 6 | 2 | .62545 |
| | 3 | .75290 |
| | 4 | .90504 |
| | 5 | 1.11899 |
| | 6 | 1.52065 |
| 7 | 2 | .59481 |
| | 3 | .70206 |
| | 4 | .82075 |
| | 5 | .96826 |
| | 6 | 1.17928 |
| | 7 | 1.57754 |
| 8 | 2 | .57138 |
| | 3 | .66509 |
| | 4 | .76370 |
| | 5 | .87780 |
| | 6 | 1.02255 |
| | 7 | 1.23152 |
| | 8 | 1.62697 |
| 9 | 2 | .55271 |
| | 3 | .63668 |
| | 4 | .72190 |
| | 5 | .81595 |
| | 6 | .92729 |
| | 7 | 1.07018 |
| | 8 | 1.27763 |
| | 9 | 1.67064 |
| 10 | 2 | .53739 |
| | 3 | .61399 |
| | 4 | .68963 |
| | 5 | .77031 |
| | 6 | .86159 |
| | 7 | .97108 |
| | 8 | 1.11266 |
| | 9 | 1.31887 |
| | 10 | 1.70972 |
| 11 | 2 | .52452 |
| | 3 | .59534 |
| | 4 | .66375 |
| | 5 | .73490 |
| | 6 | .81281 |
| | 7 | .90223 |
| | 8 | 1.01043 |
| | 9 | 1.15099 |
| | 10 | 1.35618 |
| | 11 | 1.74508 |
| 12 | 2 | .51350 |
| | 3 | .57964 |
| | 4 | .64243 |
| | 5 | .70641 |
| | 6 | .77479 |
| | 7 | .85084 |
| | 8 | .93895 |
| | 9 | 1.04617 |
| | 10 | 1.18592 |
| | 11 | 1.39023 |
| | 12 | 1.77734 |
| 13 | 2 | .50391 |
| | 3 | .56620 |
| | 4 | .62445 |
| | 5 | .68348 |
| | 6 | .74410 |
| | 7 | .81060 |
| | 8 | .88533 |
| | 9 | .97246 |
| | 10 | 1.07893 |
| | 11 | 1.21802 |
| | 12 | 1.42154 |
| | 13 | 1.80699 |
| 14 | 2 | .49548 |
| | 3 | .55451 |
| | 4 | .60904 |
| | 5 | .66297 |
| | 6 | .71865 |
| | 7 | .77801 |
| | 8 | .84319 |
| | 9 | .91693 |
| | 10 | 1.00332 |
| | 11 | 1.10918 |
| | 12 | 1.24771 |
| | 13 | 1.45051 |
| | 14 | 1.83441 |
| 15 | 2 | .47797 |
| | 3 | .54423 |
| | 4 | .59263 |
| | 5 | .64591 |
| | 6 | .69712 |
| | 7 | .75094 |
| | 8 | .80896 |
| | 9 | .87314 |
| | 10 | .94612 |
| | 11 | 1.03191 |
| | 12 | 1.13728 |
| | 13 | 1.27532 |
| | 14 | 1.47747 |
| | 15 | 1.85164 |
| 16 | 2 | .48125 |
| | 3 | .53510 |
| | 4 | .58383 |
| | 5 | .63103 |
| | 6 | .67862 |
| | 7 | .72798 |
| | 8 | .77714 |
| | 9 | .83747 |
| | 10 | .90088 |
| | 11 | .97326 |
| | 12 | 1.05857 |
| | 13 | 1.16352 |
| | 14 | 1.30112 |
| | 15 | 1.50265 |
| | 16 | 1.88372 |
| 17 | 2 | .47516 |
| | 3 | .52690 |
| | 4 | .57335 |
| | 5 | .61793 |
| | 6 | .66247 |
| | 7 | .70820 |
| | 8 | .75623 |
| | 9 | .80769 |
| | 10 | .86394 |
| | 11 | .92674 |
| | 12 | .99863 |
| | 13 | 1.08355 |
| | 14 | 1.18812 |
| | 15 | 1.32533 |
| | 16 | 1.52629 |
| | 17 | 1.90606 |
| 18 | 2 | .46962 |
| | 3 | .51950 |
| | 4 | .56379 |
| 18 | 5 | .60616 |
| | 6 | .64814 |
| | 7 | .69086 |
| | 8 | .73526 |
| | 9 | .78429 |
| | 10 | .83297 |
| | 11 | .88863 |
| | 12 | .95093 |
| | 13 | 1.02243 |
| | 14 | 1.10699 |
| | 15 | 1.21126 |
| | 16 | 1.34811 |
| | 17 | 1.54856 |
| | 18 | 1.92709 |
| 19 | 2 | .46454 |
| | 3 | .51276 |
| | 4 | .55543 |
| | 5 | .59582 |
| | 6 | .63556 |
| | 7 | .67570 |
| | 8 | .71707 |
| | 9 | .76045 |
| | 10 | .80668 |
| | 11 | .85674 |
| | 12 | .91187 |
| | 13 | .97378 |
| | 14 | 1.04494 |
| | 15 | 1.12921 |
| | 16 | 1.23318 |
| | 17 | 1.36969 |
| | 18 | 1.56961 |
| | 19 | 1.94695 |
| 20 | 2 | .45988 |
| | 3 | .50659 |
| | 4 | .54769 |
| | 5 | .58637 |
| | 6 | .62418 |
| | 7 | .66211 |
| | 8 | .70093 |
| | 9 | .74129 |
| | 10 | .82288 |
| | 11 | .82949 |
| | 12 | .87904 |
| | 13 | .93376 |
| | 14 | .99533 |
| | 15 | 1.06620 |
| | 16 | 1.15021 |
| | 17 | 1.25392 |
| | 18 | 1.39013 |
| | 19 | 1.58955 |
| | 20 | 1.96576 |

Table IV

$\eta_\tau$ and $e_\tau$ values for Kendall's $\tau$ for selected† h-values

| $\eta_\tau(\theta)$ | $e_\tau(\theta)$ | $\eta_\tau(\theta)$ | $e_\tau(\theta)$ | $\eta_\tau(\theta)$ | $e_\tau(\theta)$ |
|---|---|---|---|---|---|
| $.0_{2}1389$* | $.0_{4}3472$ | .05223* | .05000 | .1400 | .4109 |
| $.0_{2}2778$* | $.0_{3}1389$ | .05471 | .05496 | .1448 | .4453 |
| $.0_{2}5553$ | $.0_{3}5552$ | .05716* | .06011 | .1473* | .4642 |
| $.0_{2}8326$* | $.0_{2}1248$ | .05958 | .06544 | .1502 | .4866 |
| .01109 | $.0_{2}2217$ | .06552 | .07952 | .1551 | .5269 |
| .01385* | $.0_{2}3459$ | .07013 | .09151 | .1600 | .5697 |
| .01523 | $.0_{2}4182$ | .07350* | .1009 | .1615* | .5837 |
| .01661* | $.0_{2}4973$ | .07461 | .1041 | .1651 | .6180 |
| .01935 | $.0_{2}6756$ | .08004 | .1205 | .1700 | .6679 |
| .02208* | $.0_{2}8805$ | .08321* | .1307 | .1749 | .7222 |
| .02344* | $.0_{2}9927$ | .08528 | .1376 | .1783* | .7621 |
| .02480 | .01112 | .09032 | .1554 | .1801 | .7830 |
| .02750* | .01368 | .09325* | .1663 | .1852 | .8492 |
| .03019 | .01651 | .09516 | .1736 | .1901 | .9195 |
| .03153* | .01801 | .1007 | .1961 | .1950 | .9954 |
| .03286* | .01958 | .1025* | .2037 | .2000 | 1.082 |
| .03552 | .02290 | .1051 | .2152 | .2041* | 1.160 |
| .03684* | .02464 | .1102 | .2384 | .2050 | 1.179 |
| .03946 | .02832 | .1126* | .2501 | .2185* | 1.511 |
| .04076* | .03024 | .1150 | .2619 | .2307* | 1.978 |
| .04206* | .03222 | .1202 | .2894 | .2338* | 2.149 |
| .04464 | .03634 | .1231* | .3052 | .2361* | 2.295 |
| .04720* | .04069 | .1252 | .3171 | .2402* | 2.638 |
| .04846* | .04294 | .1305 | .3486 | .2429* | 2.965 |
| .04973 | .04524 | .1348 | .3760 | | |

†See (6.27) and (6.28) for definitions of $\eta_\tau$ and $e_\tau$. We have selected the h-value to give $\eta_\tau$ values in the range .005-.045 in steps of approximately .005.

*These values are included because they occur either in Table V or Table VI.

## Table V

$\eta_z$-values for the normal likelihood-ratio test* for selected** $\rho$-values.

| $\rho = \theta(1+\theta^2)^{-\frac{1}{2}}$ | $\frac{\tan^{-1}(\theta)}{2\pi}$ | $\eta_z(\theta)$ | $\rho = \theta(1+\theta^2)^{-\frac{1}{2}}$ | $\frac{\tan^{-1}(\theta)}{2\pi}$ | $\eta_z(\theta)$ |
|---|---|---|---|---|---|
| .05229 | $.0_28326$ | $.0_21369$ | .7990 | .1473 | .5086 |
| .09556 | .01523 | $.0_24587$ | .8000 | —— | .5108 |
| .1000 | —— | $.0_25025$ | .8493 | .1615 | .6388 |
| .1468 | .02344 | .01089 | .9000 | —— | .8304 |
| .1968 | .03153 | .01976 | .9003 | .1783 | .8320 |
| .2000 | —— | .02041 | .9100 | —— | .8804 |
| .2533 | .04076 | .03317 | .9200 | —— | .9367 |
| .2998 | .04846 | .04708 | .9300 | —— | 1.001 |
| .3000 | —— | .04716 | .9400 | —— | 1.075 |
| .3515 | .05716 | .06593 | .9500 | —— | 1.164 |
| .4000 | —— | .08718 | .9587 | .2041 | 1.258 |
| .4001 | .06552 | .08723 | .9600 | —— | 1.273 |
| .4518 | .07350 | .1142 | .9700 | —— | 1.414 |
| .4993 | .08321 | .1434 | .9800 | —— | 1.614 |
| .5000 | —— | .1438 | .9805 | .2185 | 1.626 |
| .5529 | .09325 | .1826 | .99 | —— | 1.959 |
| .6000 | —— | .2231 | .9926 | .2307 | 2.110 |
| .6004 | .1025 | .2235 | .9948 | .2338 | 2.286 |
| .6499 | .1126 | .2701 | .9950 | —— | 2.304 |
| .6986 | .1231 | .3348 | .9961 | .2361 | 2.436 |
| .7000 | —— | .3367 | .9981 | .2402 | 2.785 |
| .7494 | .1349 | .4123 | .9990 | .2429 | 3.116 |

*See example 6.1 (b).

**The $\rho$-values with entries in the second column were in fact computed by means of (6.30) from $\eta_T(\theta)$ values in a larger version of Table IV and correspond to $\eta_T(\theta)$ values in Table IV; the $\rho$-values without entries in the second column are .1(.1) .9(.01) .99 and .995.

## Table VI

$\eta_z(\theta)$-values* at selected** $\theta$-values for the likelihood ratio test of $\theta=0$ vs. $\theta > 0$ in the family $\{H_\theta = FG(1+\theta(1-F)(1-G)),\ 0 \leq \theta \leq 1\}$.

| $\theta$ | $\theta/18$ | $\eta_z(\theta)$ | $\theta$ | $\theta/18$ | $\eta_z(\theta)$ |
|---|---|---|---|---|---|
| .05000 | $.0_22778$ | $.0_31389$ | .5435 | .03019 | .01672 |
| .09996 | $.0_25553$ | $.0_35554$ | .5916 | .03286 | .01988 |
| .1000 | —— | $.0_35559$ | .6000 | —— | .02047 |
| .1499 | $.0_28326$ | $.0_21249$ | .6631 | .03684 | .02514 |
| .1997 | .01109 | $.0_22221$ | .7000 | —— | .02812 |
| .2000 | —— | $.0_22228$ | .7103 | .03946 | .02898 |
| .2494 | .01385 | $.0_23468$ | .7571 | .04206 | .03310 |
| .2989 | .01661 | $.0_24991$ | .8000 | —— | .03715 |
| .3000 | —— | $.0_25028$ | .8035 | .04464 | .03750 |
| .3483 | .01935 | $.0_26790$ | .8495 | .04720 | .04217 |
| .3975 | .02208 | $.0_28863$ | .8951 | .04973 | .04714 |
| .4000 | —— | $.0_28977$ | .9000 | —— | .04770 |
| .4464 | .02480 | .01121 | .9402 | .05223 | .05241 |
| .4951 | .02750 | .01383 | .9848 | .05471 | .05800 |
| .5000 | —— | .01411 | 1.000 | —— | .06000 |

*See (6.29) ( ).

**The $\theta$-values with entries in the second column correspond to $\eta_T(\theta)$-values in Table IV; the $\theta$-values without entries in the second column are .1(.1) 1.0.

Table VII

$e_c(\theta)$-values for the normal-scores layer-rank test.

A. Against the normal alternative with correlation $\rho = \theta(1+\theta^2)^{-\frac{1}{2}}$

| $\rho$ | $e_c(\theta) = \frac{1}{2}(\eta_c(\theta))^2$ |
|---|---|
| .1 | $.0_25020$ |
| .2 | .02033 |
| .3 | .04671 |
| .4 | .08569 |
| .5 | .1399 |
| .6 | .2141 |
| .7 | .3172 |
| .8 | .4696 |
| .9 | .7361 |
| .91 | .7767 |
| .92 | .8221 |
| .93 | .8735 |
| .94 | .9328 |
| .95 | 1.003 |
| .96 | 1.009 |
| .97 | 1.198 |
| .98 | 1.352 |
| .99 | 1.615 |
| .995 | 1.877 |

B. Against the alternative $H_\theta = FG(1+\theta(1-F)(1-G))$

| $\theta$ | $e_c(\theta) = \frac{1}{2}(\eta_c(\theta))^2$ |
|---|---|
| .1 | $.0_35067$ |
| .2 | $.0_22028$ |
| .3 | $.0_24570$ |
| .4 | $.0_28140$ |
| .5 | .01275 |
| .6 | .01842 |
| .7 | .02517 |
| .8 | .03303 |
| .9 | .04203 |
| 1.0 | .06000 |

Table VIII

Weight function J of the ALMP layer test against the normal alternative*.

| | .001 | .002 | .003 | .004 | .005 | .006 | .007 | .008 | .009 | .010 |
|---|---|---|---|---|---|---|---|---|---|---|
| .00 | -4.131 | -3.955 | -3.846 | -3.764 | -3.698 | -3.642 | -3.593 | -3.550 | -3.510 | -3.474 |

| | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| .0 | - ∞ | -3.474 | -3.202 | -3.025 | -2.887 | -2.771 | -2.670 | -2.579 | -2.496 | -2.419 |
| .1 | -2.346 | -2.278 | -2.213 | -2.151 | -2.091 | -2.033 | -1.977 | -1.922 | -1.869 | -1.817 |
| .2 | -1.766 | -1.716 | -1.666 | -1.618 | -1.570 | -1.523 | -1.476 | -1.429 | -1.383 | -1.338 |
| .3 | -1.292 | -1.247 | -1.202 | -1.157 | -1.112 | -1.067 | -1.023 | -.9780 | -.9333 | -.8885 |
| .4 | -.8436 | -.7985 | -.7533 | -.7079 | -.6623 | -.6164 | -.5702 | -.5237 | -.4769 | -.4296 |
| .5 | -.3820 | -.3338 | -.2851 | -.2360 | -.1862 | -.1359 | -.0849 | -.0332 | .0193 | .0726 |
| .6 | .1268 | .1818 | .2379 | .2950 | .3532 | .4126 | .4733 | .5353 | .5988 | .6639 |
| .7 | .7307 | .7993 | .8698 | .9425 | 1.017 | 1.094 | 1.175 | 1.258 | 1.344 | 1.434 |
| .8 | 1.528 | 1.626 | 1.729 | 1.837 | 1.952 | 2.073 | 2.202 | 2.340 | 2.488 | 2.649 |
| .9 | 2.824 | 3.017 | 3.234 | 3.478 | 3.759 | 4.090 | 4.496 | 5.019 | 5.757 | 7.023 |

| | .000 | .002 | .004 | .006 | .008 | .010 | .012 | .014 | .016 | .018 |
|---|---|---|---|---|---|---|---|---|---|---|
| .97 | 5.019 | 5.145 | 5.280 | 5.425 | 5.583 | 5.757 | 5.949 | 6.165 | 6.408 | 6.690 |

| | .000 | .001 | .002 | .003 | .004 | .005 | .006 | .007 | .008 | .009 |
|---|---|---|---|---|---|---|---|---|---|---|
| .99 | 7.023 | 7.217 | 7.432 | 7.676 | 7.959 | 8.293 | 8.702 | 9.229 | 9.972 | 11.231 |

*See Example 5.2.

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