A General Multivariate Exponentially Weighted Moving Average Control Chart

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This paper proposes a general multivariate exponentially weighted moving average chart, in which the smoothing matrix is full, instead of one having only diagonal elements. The average run length properties of this scheme are examined for a diverse set of quality control environments and information needed to design the chart is provided. The performance of the scheme is measured by estimating the ARL and comparing it to the traditional diagonal multivariate EWMA chart. The comparison results show that allowing non-zero off-diagonal elements in the weight matrix of the new chart can improve its performance compared to the current standard of a diagonal weight matrix. In particular, there is a substantial improvement when the process shifts out-of-control in the start-up stage. The performance of the chart is illustrated with some medical data. Univariate methods are far inferior to multivariate on this data set, and the full EWMA proposed outperforms the diagonal EWMA.

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INTRODUCTION

The standard Shewhart charts provide good performance in detecting large changes in a process, but are much less effective for smaller but persistent changes. Other complementary charting methods that fill this gap include the exponentially weighted moving average (EWMA) chart. Since Roberts (1959) introduced the EWMA control chart for a univariate normal process with independent and identical distribution, its properties has been evaluated numerically and analytically for a variety of situations (Robinson and Ho, 1978; Hunter, 1986; Crowder, 1989; Lucas and Saccucci, 1990).

Lowry *et al.* (1992) extended the original univariate EWMA procedure to a multivariate control chart scheme for controlling the mean of a multivariate normal process. The multivariate EWMA (MEWMA) chart is a straightforward vector generalization of the corresponding univariate procedure, using a smoothing matrix instead of the scalar smoothing constant of the EWMA. Current MEWMA practice seems to be confined to using a smoothing matrix with zero off-diagonal elements, and generally equal diagonal elements (Lowry *et al.*, 1992). We will use the abbreviation DEWMA for this MEWMA with nonzero elements only on the diagonal of the smoothing matrix. A general MEWMA chart with an unrestricted smoothing matrix is proposed in this paper. We will use the abbreviation of FEWMA for the multivariate EWMA chart with a full smoothing matrix.

There are a number of differences between the DEWMA and the FEWMA charts. In particular, if μ, μ_0 and Σ represent the out-of-control mean vector, the in-control mean vector and the covariance matrix of measurements, the DEWMA chart is directionally invariant — that is its ARL depends on these three quantities only through the value of the noncentrality parameter

$$\eta_c = \sqrt{\delta \Sigma^{-1} \delta}\,. \quad \text{where the mean shift } \delta = \mu - \mu_0$$

The FEWMA chart does not have this directional invariance property — its performance is affected by the direction of shift and the covariance structure as well as the noncentrality. Thus we will study the ARL performance of the FEWMA scheme for various correlation structures and mean shift directions.

The FEWMA scheme is described in the next section, and the following section contains numerical results, including the ARL performance of FEWMA. In a subsequent section, some conclusions are presented.

GENERAL MULTIVARIATE EWMA CHARTS

Suppose that the successive p-component vectors of measurements $\{\mathbf{x}_n, n=1,2, \ldots\}$ are independent and identically distributed multivariate normal random vectors $\mathbf{x}_n \sim N(\mu,\Sigma)$. The in-control mean vector is μ_0. A FEWMA vector is defined by

$$\mathbf{y}_n = \mathbf{R}(\mathbf{x}_n - \mu_0) + (\mathbf{I} - \mathbf{R})\mathbf{y}_{n-1} \qquad (1)$$

for $n = 1,2,\ldots$ where $\mathbf{y}_0 = \mathbf{0}$ and $\mathbf{R}$ is the smoothing matrix.

The DEWMA chart of Lowry *et al.* used a diagonal matrix with elements $\{0 < r_i \le 1,\ i = 1,2,\ldots,p\}$ for $\mathbf{R}$. Unless there is any reason to weight the quality characteristic measurements differently, all diagonal elements of $\mathbf{R}$ are set to the same value -- that is, $r_1 = r_2 = \ldots = r_p = r$. This DEWMA control scheme gives an out-of-control signal as soon as

$$T_n^2 = \mathbf{y}_n' \Sigma_{y_n}^{-1} \mathbf{y}_n > h \qquad (2)$$

where $h > 0$ is a control limit and

$$\Sigma_{y_n} = \frac{r[1-(1-r)^{2n}]}{2-r}\Sigma \qquad (3)$$

The covariance matrix Σ_{y_n} is not constant, but varies with n. As n increases, it tends to the asymptotic covariance matrix

$$\Sigma_{y_n} = \frac{r}{2-r}\Sigma . \qquad (4)$$

Often the process stays in control for a sufficiently long period to make Σ_{y_n} effectively indistinguishable from its asymptotic form, and for this reason the DEWMA is often designed to use the asymptotic ('steady state') form.

A natural extension to the DEWMA chart is to allow non-zero off-diagonal elements in **R.** With non-zero off-diagonal elements in **R**, the covariance matrix of the FEWMA vector $\mathbf{y}_n$ is more complicated than that of the MEWMA procedure. It may be computed most easily recursively by the recursion $\Sigma_{y_0} = 0$;

$$\Sigma_{\mathbf{y}_n} = \mathbf{R}\Sigma\mathbf{R}' + (\mathbf{I}-\mathbf{R})\Sigma_{\mathbf{y}_{n-1}}(\mathbf{I}-\mathbf{R})' \qquad (5)$$

As n increases, this converges to the asymptotic value Σ_∞ given by the solution to the linear system

$$\Sigma_\infty - (\mathbf{I}-\mathbf{R})\Sigma_\infty(\mathbf{I}-\mathbf{R})' = \mathbf{R}\Sigma\mathbf{R}'$$

provided the matrix **R** has all eigenvalues less than 1 (Cullen, 1972).

From here on we will assume, mainly for notational convenience, that the elements of x have been standardized to mean zero and standard deviation 1. As in the diagonal case, in the absence of some good reason to the contrary, we will treat the different measurements symmetrically, so if r_{ij} is the (i,j) th element of **R,** it is natural (and will be done here) to restrict the values to equal diagonal elements $r_{ii} = r_{on}$ for $i = 1,2,\ldots,p$ and to equal off-diagonal elements $r_{ij} = r_{off}$ for $i,j = 1,2,\cdots,p$ and $i \neq j$. It is convenient when studying the impact of off-diagonal weights to fix the total weight of each variable. We will define this total weight of a variable by the row sums of **R**

$$r = \sum_{j=1}^{p} r_{ij} = r_{on} + (p-1)r_{off}, \quad \forall i. \qquad (6)$$

Since it seems inappropriate to use off-diagonal weights greater than the on-diagonal weight, this study uses the matrix **R** such that $r_{off} = cr_{on}$ for $0 \le c < 1$. We will characterize such a FEWMA scheme by its r and c. Given these,

$$r_{on} = \frac{r}{1+(p-1)c} \text{ and } r_{off} = \frac{cr}{1+(p-1)c}.$$

The diagonal DEWMA chart using (3) corresponds to $c = 0$. Table 1 contains an example of the full smoothing matrix **R** with $r = 0.1$ and $c = 0.75$.

ARL PERFORMANCE OF GENERAL MULTIVARIATE EWMA CHARTS

The performance of a control chart is commonly measured by its average run length (ARL), defined as the average number of observations from the time of occurrence of a shift to the time that the chart signals. We can distinguish two interesting ARL's – the 'initial state' ARL assumes that the shift occurs the instant that the chart is started, so the data are off-center from the very first observation. The 'steady state' ARL assumes that the chart has run long enough for the covariance matrix $\Sigma_{\mathbf{y}_n}$ to reach its asymptotic level. These two possibilities represent in some sense the extremes between which all others lie. Since in practice processes often start out off-center due to start-up problems or ineffective control action for the previous signal, the reaction of the chart to immediate shifts, as measured by the 'initial state' ARL, is perhaps the more generally relevant of these two measures.

It is a feature of the general FEWMA is that, unlike the DEWMA, it is not affine invariant. It responds differently to shift of the same magnitude but in different directions, and so by suitable choice of the parameters r and c, it can be made more sensitive to mean shifts. ARL performance of FEWMA control schemes depends on the correlation structure of **x**, the direction and size of the shift in mean, and the c and r parameters of the smoothing matrix **R**.

To illustrate the potential for improvement in going beyond the diagonal form of the EWMA, consider the detection of a shift in four-component data vectors by the diagonal scheme r=0.1, c=0 and the scheme r=0.1 c=0.75, which has the same total weight on the most recent observation but which has large off-diagonal elements. We will vary correlation structures – 'Ind' refers to independent components in x; the structure 'P_8' has all correlation equal to 0.8 and in the structure 'M_8' variables i and j have correlation –0.8 if i-j is odd, and +0.8 if it is even. We also vary the direction of shift – 'equal' corresponds to all elements of δ being equal; in 'single' just one is shifted and the remainder are zero, and for

'Symmetric' the first two are a positive constant while the last two are the negative of this constant.

Some ARLs to detection of a shift with η=0.4 and in-control ARL of 300 follow. All have standard errors of approximately 0.4. For this noncentrality, the DEWMA has a steady-state ARL of 60.2, and an initial-state ARL of 58.5.

	Steady-state			Initial state		
	P-8	Ind	M-8	P-8	Ind	M-8
Equal	45.2	45.3	45.5	41.6	41.8	41.4
Single	57.7	53.4	54.5	35.4	36.6	36.3
Symmetric	58.1	58.7	57.6	35.7	35.9	34.4

Noteworthy features are:

- The FEWMA beats the DEWMA in all cases.
- The benefit is particularly large for 'initial state' shifts, where the ARL's are 30 to 40% lower.
- The correlation structure appears to have little effect.

Theoretical analysis – small shifts

These details of performance improvements raise the question of whether these features hold in general. This question is not easily answered in general terms as the performance of the FEWMA involves a complex mix of steady-state and transient behavior. We can however make considerable progress for the case δ small. This case is particularly interesting since it is for moderate shifts that control is most needed, since simple Shewhart charts easily detect very large shifts.

If the shift is not large, then its detection will generally require a number of out-of-control data vectors. From the initial state, n points after the shift, the vector $\mathbf{y_n}$ follows a multivariate normal distribution (recall that we are assuming μ_0=0)

$$\mathrm{N}[\sum_{i=1}^{n}(\mathbf{I}-\mathbf{R})^{i}\mathbf{R}\delta,\Sigma_{\mathbf{y}_n}].$$

As n increases, this approaches $\mathrm{N}[\delta,\Sigma_\infty]$, so the noncentrality is $\delta\Sigma_\infty^{-1}\delta$. With the supposition that detecting the shift will require more than a handful of observations, we may use this asymptotic distribution to gain some understanding of the impact of different choices of the parameters r and c. First, we fix r and ask whether using a particular non-zero c leads to better performance than the DEWMA with c=0. Writing $\Sigma_\infty(c)$ for the asymptotic covariance matrix considered as a function of c, we conclude that the FEWMA will outperform the DEWMA if δ lies in some direction such that $\delta\Sigma_\infty^{-1}(c)\delta > \delta\Sigma_\infty^{-1}(0)\delta$ and will underperform it otherwise.

We can explore the limiting relative performance of the two charts for small δ by solving the simultaneous diagonalization problem

$$[\Sigma_\infty(c) - \lambda_i \Sigma_\infty(0)]\, \mathbf{a}_i = \mathbf{0}$$

If δ lies in the principal direction $\mathbf{a}_i$ and the corresponding λ_i is greater than 1, then the DEWMA can be expected to outperform the FEWMA for small shifts. If it is less than 1, then the FEWMA is the winner. This prediction of relative performance is an approximate one since it depends on the assumption that the signal does not occur very shortly after the shift, but it is nevertheless very useful.

We illustrate this with the 5-component M-8 configuration, evaluating at r =0.1 and c=0.75. The eigenvalues and corresponding eigenvectors are:-

	Principal direction i				
	1	2	3	4	5
λ	1.612	0.060	0.060	0.060	0.058
a	3.62	3.98	1.65	6.69	-0.69
	5.25	0.00	6.69	-1.65	1.34
	3.62	3.98	-1.65	-6.69	-0.69
	5.25	0.00	-6.69	1.65	1.34
	3.62	-7.96	0.00	0.00	-0.69

This suggests that for a shift in the first of these directions (proportional to the vector (0.7,1,0.7,1,0.7), the DEWMA should out-perform the FEWMA, the eigenvalue being larger than 1. For shifts in any of the other principal directions though, the FEWMA should outperform the DEWMA. Furthermore, while λ_1 is not substantially larger than 1, the other λ_i are much smaller than 1, leading to the conclusion that while the FEWMA will never be much worse than the FEWMA, it may be much better, and furthermore is much better in "most" directions.

To check these predictions, we evaluated the ARL for shifts in the direction $\mathbf{a}_1$ and $\mathbf{a}_5$, setting the noncentrality to 0.2. With an in-control ARL of 300, this gave the out-of-control ARL's

$c = 0.75$		$c = 0$	
Steady state	Initial state	Steady state	Initial state

Direction	a_1	168	157	169	167
	a_5	152	124	166	167

As the scheme with c=0 is affine invariant, the four ARL's on the right are identical except for random variation in the simulation. The first direction is that in which the FEWMA is predicted to perform worst relative to the DEWMA. However it matches the DEMWA in the steady state, and beats it modestly in the initial state. The best direction for the FEWMA shows a substantial improvement in the initial state and a smaller improvement in the steady state. Despite the improvement in performance where the theory predicted none, these results indicate that the theory is helpful in predicting better and worse behavior of the FEWMA relative to the DEWMA.

In the light of this, we list the eigenvalues of some scenarios. To the P_8, M_8 and Ind scenarios, we have added a fourth – the covariance matrix labeled 'Random' is a matrix of independent uniforms multiplied by its transpose.

p	Structure	Eigenvalues				
4	P_8	1.0000	0.0734	0.0734	0.0734	
	Ind	1.0000	0.0734	0.0734	0.0734	
	M-8	1.0000	0.0734	0.0734	0.0734	
	Random	1.4588	0.0734	0.0734	0.0712	
5	Ind	1.0000	0.0596	0.0596	0.0596	
	P_8	1.0000	0.0596	0.0596	0.0956	0.0596
	M_8	1.6118	0.0596	0.0596	0.0596	0.0582
	Random	2.9092	0.0596	0.0596	0.0596	0.0573

In all cases, at most one eigenvalue reached or exceeded the value 1, the remainder being substantially less than 1. The main message we believe can be drawn from these figures is that "most" directions correspond to the situation $\delta\Sigma_\infty^{-1}(c)\delta > \delta\Sigma_\infty^{-1}(0)\delta$ in which the FEWMA outperforms the DEWMA, but it is certainly possible for the reverse to happen.

Some empiric performance evaluation

The performance of the FEWMA is clearly quite a complicated function of the various parameters involved. Potential users are therefore best advised to check the performance of particular choices of *r* and *c* in their particular setting – that is to say, using their actual process correlation matrix and some anticipated shifts that are to be guarded against. This experimentation need only be done once, at the design phase. It is greatly facilitated by a computer program FEWMA, discussed in a companion report (Hawkins *et al.* 2002), which allows trail scenarios to be assessed quickly.

It is helpful though to provide some general guidelines. To investigate the joint impact of these factors, we chose seven correlation structures and three shift directions. First is the multivariate normal process with no correlation, denoted IND. The other six correlation structures are categorized into two classes: the positive type, in which all pairs of variables have an equal positive correlation, and the mixed type, in which variables i and j for $i \neq j$ have a negative correlation if $i + j$ is odd and a positive correlation if $i + j$ is even. The common correlation is set to one of three values – 0.2, 0.5, 0.8, giving rise to three positive-correlation settings P-2, P-5, P-8, and three mixed settings M-2, M-5, M-8. An example of the correlation type of M-8 is illustrated in Table 1. The out-of-control mean process is modeled with three shift directions:

Equal Shift, in which all components of μ are equal;

Symmetric Shift, which differs from Equal Shift in that the first half of the components of μ has different signs to the second half;

Single Shift, in which only a single component of μ is non-zero.

In lieu of an attempt at theoretical optimization of the parameters, we conducted empirical study instead, evaluating the ARL for six sizes of shift corresponding to the noncentrality parameter values η_c = 0.1, 0.2, 0.4, 0.8, 1.6, 3.2 respectively. These were estimated from 10,000 simulation runs using the same independent random seed number for each case considered.

First, we investigate the properties of the ARL performance of FEWMA charts for a 4-variate normal processes using the control limits which result in ARL = 300 for in-control processes in both states for independent 10,000 simulation runs. Table 2 contains the estimated values of the control limit h for the seven correlation types, which were computed using a binary search method with linear interpolation. For the directionally invariant charts (c = 0) the control limit h is independent of the correlation structures. For non-zero c, the estimated values of h vary only slightly with the correlation types. The differences between the h values for the seven types are less than 0.05 in most of the cases considered. The EWMA control limit increases with r, and decreases with c for a fixed r. Table 3 - 5 display the averages and standard deviations of the estimated ARL's for the seven correlation types for the FEWMA charts of $r = 0.06, 0.1, 0.5$ and $c = 0, 0.75$. Using the h values of in-control ARL = 300 in Table 2, these results were obtained from total of 70,000 simulation runs by applying the seven types to 10,000 runs each. Each entry in the table represents the average of seven ARL's, one for each of the seven correlation types. The values in parentheses are the standard deviations of these seven ARL's.

For the steady state out-of-control, these tables show that the best choice of r and c depends somewhat on the distances and directions of the shift in the mean vector. In Equal Shift, the chart appears to perform best with $r = 0.06$ and $c = 0.75$ for $\eta_c \geq 0.8$, $r = 0.2$ and $c = 0.75$ for $\eta_c = 1.6$, and $r = 0.5$ with the virtually identical out-of-control ARL's for different c's at the largest shift distance. In the other shift directions, the shortest ARL is given by the non-diagonal smoothing schemes of $r = 0.06$ and $c = 0.5$ for $\eta_c \leq 0.2$, $r = 0.06$ and $c = 0.25$ for $\eta_c = 0.4$, but the charts of diagonal smoothing matrices with $r = 0.1, 0.2, 0.5$ show the best performance for $\eta_c = 0.8, 1.6, 3.2$ respectively. The FEWMA scheme generally yields a shorter ARL with larger c for a given r in Equal Shift, but the ARL increases with increasing c for relatively large shifts in the other shift directions. When shifting a small distance, the ARL is reduced more by using a nonzero c in Symmetric and Single Shifts than Equal Shift. For the initial state out-of control, the FEWMA scheme appears to always have shorter ARL for smaller r and larger c in all the shift directions. The initial state benefits more than the steady state from using the non-diagonal components. As shown in the results of the standard deviations (the values in parenthesis), the variation in ARL for the different classes of correlation type is not much different between the diagonal and non-diagonal smoothing schemes, except for small shifts with larger r's in Single Shift. For these cases, the non-diagonal scheme varies considerably more than the diagonal one. The performance of the FEWMA charts when $c \neq 0$ appears to be sensitive to the shift directions of the process mean. For example, when $\eta_c = 0.2$, the chart of $r = 0.1$ and $c = 0.5$ has the ARL's of 148.7, 118.7, 122.7 for the steady state and 138.7, 103.6, 107.5 for the initial state in Equal, Symmetric and Single Shifts respectively.

The values of the control limit h for the FEWMA charts of $r = 0.1$ were estimated for the 10 in-control ARL's with run-length increment of 100 from in-control ARL = 100. The cases of $p = 2, 3, 4, 5, 10$ were considered for two correlation types, M-5 and P-5. All the thresholds were computed such that the chart results in having the specified in-control ARL for the independent 10,000 simulation runs. Table 6 illustrates the h values obtained for in-control ARL = 100, 300, 500, 1000 of the 10 lengths considered. The h values differ a little between the correlation types with the maximum difference of less than 0.1 (the differences in most of the cases considered are less than 0.05). The EWMA control limit has a smaller value for larger c, and for the steady state than for the initial state. The simulation results indicate that a control limit of the FEWMA chart can be approximated with a linear function of the in-control ARL. Table 7 contains the estimated values of h using the function

$$h = a\ln(\text{ARL}) + b. \tag{7}$$

for $c = 0.75$ and the P-5 correlation type. The parameters of (7) for the results in Table 7 were estimated using only three points of in-control ARL = 100, 500, 1000, while the estimated values of h ($\hat{h}$) are compared with the values computed using simulation (h_{sim}) for the 10 in-control ARL's considered.

Based on independent 10,000 simulation runs, the comparison of the FEWMA charts of r = 0.1 were made for p = 2, 3, 5, 10 using the M-5 and P-5 correlation types by shifting the mean vector in two extreme directions of Equal and Single Shifts. The control schemes were designed to give an out-of-control signal when the test statistic is greater than the threshold h of in-control ARL = 300 in Table 6. Table 8 and 9 displays the comparison results for the out-of-control processes in the Equal and Single Shift directions respectively. From these tables, the performance of the chart for p = 2, 3, 5, 10 is very similar to that for p = 4.

Table 10 presents the optimal values of r ($r_{\min}$) for the FEWMA charts using three large values of c for p = 4 and in-control ARL = 300, 500, 1000 in Equal Shift for the steady state. This search was made with increments of 0.001 when $r \le 0.03$ and increments of 0.01 when $r > 0.03$ using 10,000 independent simulation runs. In Table 10, $\text{ARL}_{\min}$ represents the minimum out-of-control ARL at the shift of interest and the ranges of r values are corresponding to those values that differ with the minimum ARL in $\pm 1\%$ of the $\text{ARL}_{\min}$ value. The optimal values of r are very similar for all the three values of c, and larger for the diagonal scheme than for the non-diagonal one except for very small shift. From the results in the previous investigation, FEWMA charts may use smaller r and larger c for better ARL performance in the initial state. However, it may result in computational instability to use extreme values close to the boundary for r and c. We recommend to choose the values in $r > 0.001$ and $c < 0.99$. In the steady state, larger values of c for the FEWMA chart yields shorter ARL for Equal Shift, but it is not true for the other shift directions. For some shift directions such as Single and Symmetric Shifts, the FEWMA chart does not give better ARL performance for larger values of c, and the diagonal scheme is even more effective in detecting large shifts in the process mean. Given a correlation structure and shift direction of process mean for quality control environment, we can design the optimal FEWMA chart for a specified in-control run length by using a simulation approach.

EXAMPLE

We illustrate the method with some ambulatory monitoring data. Here, the subject indefinitely wears instrumentation that measures and records heart function every 15 minutes. The data set is part of a record some 7 years long. At the first stage, the 15 minute traces are reduced to week-long summary statistics; these are the numbers we study. For reasons sketched in Hawkins and Olwell (1998) Chapter 8 (which also provides detail on the source of the data and more description), we rejected the first year's data, and used the following two years' data to estimate the parameters. For brevity, we restrict the analysis to four measures – MESORs (location statistics) of systolic blood pressure (SBP), diastolic blood pressure (DBP), mean arterial pressure (MAP) and heart rate (HR). These had the correlation matrix:

	SBPM	DBPM	MAPM
DBPM	0.9329		
MAPM	0.9532	0.9571	
HRM	0.4995	0.4788	0.5242

An initial check by simultaneous diagonalization of $\Sigma_\infty(0.75)$, $\Sigma_\infty(0)$ with $r = 0.1$ gave the eigenvalues 1.0664, 0.0734, 0.0734 and 0.0730, showing that there is much potential but little downside risk in using the FEWMA rather than the DEWMA for control of the vectors.

There is particular interest in increases in the blood pressure measures (SBP, DBP and MAP) as these lead to increased risk of stroke. So we used the interactive program FEWMA to calibrate a FEWMA with r =0.1, c =0.75 for an in-control ARL of 300, corresponding to a false signal roughly once every six years, and evaluated its performance for a shift of (0.2, 0.2, 0.2, 0) – that is, a 0.2 standard deviation increase in each of these three blood pressure measures but not the heart rate. FEWMA's output included: -

```
This gives root noncentrality       0.237
DEWMA steady state noncentrality    1.034
FEWMA steady state noncentrality    3.323
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suggesting the benefit from using FEWMA rather than DEWMA – a more than trebling of the non-centrality – is quite dramatic.

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Estimated h is    11.182 with CI    11.060    11.283
OOC ARL is    77.727  with CI    75.625    79.830
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showing that the ARL to detection of this quite small shift would be about a year and a half. Repeating the performance calculations using the DEWMA gives an out-of-control ARL of 130 weeks, nearly twice as long, confirming the FEWMA's performance improvement.

We then ran the FEWMA on the data starting from week 161 of the sequence. The first 20 cases' data are listed in Table 11, and the resulting values of y_n and T_n^2 are in Table 12. The FEWMA broke through the control limit at week 165, returned, and then went through convincingly and stayed above for the remainder of the 4 year history. What is striking is that none of the individual components of y_n was even close to the control limit, which would be approximately 0.72. Thus this shift was one that could only be seen using a multivariate approach to the measures, and not a univariate one.

CONCLUSIONS

This study suggests extending the multivariate EWMA technique of Lowry, *et al.* (1992) by using a general matrix for the smoothing weight coefficient rather than restricting it to be diagonal. Some theoretical calculations for small shifts give an indication of the circumstances under which the full smoothing matrix will improve performance, but these diagnostics are best confirmed with actual ARL calculations as part of any plan to implement the FEWMA.

Whereas the diagonal scheme of MEWMA is directionally invariant, the ARL performance of the non-diagonal smoothing scheme is affected by the direction of the shift, and by the correlation structure, thereby complicating the chart design. Using non-diagonal components for the smoothing matrix creates modest additional computational requirements, but offers a practical advantage of improving the performance in detecting a shift in the process mean for many cases of quality control environment.

This paper demonstrates the potential utility of the general FEWMA. To turn this into actual performance in a particular setting with a particular covariance matrix, target mean shift and in-control ARL requires additional design tools. At this stage, theoretical understanding of the interplay between these factors in determining the chart's performance is incomplete. We do not however see this as a fatal flaw. In a companion manuscript, we set out a computer program FEWMA that can be used as such a design tool, allowing the user to experiment with different choices of the weighting parameters to determine which will give the best performance for shifts of particular interest. In conjunction with the general insights of this paper, we believe this program turns the FEWMA into a useful tool for multivariate control.

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KEY WORDS: Multivariate EWMA, Average run length, Non-diagonal smoothing matrix, Steady state, Initial state, Out-of-control, Correlation structure, Shift direction.

Table 1. Covariance matrix Σ_{y_n} of MEWMA vector y_n with $r = 0.1$ and $c = 0.75$ for negative correlation type M-8 in time n.

R of $r = 0.1$ and $c = 0.75$				Σ of M-8			
0.031				1			
0.023	0.031			−0.8	1		
0.023	0.023	0.031		0.8	−0.8	1	
0.023	0.023	0.023	0.031	−0.8	0.8	−0.8	1
Σ_{y_n}							
($n = 101$)				($n = 201$)			
0.0055				0.0061			
0.0000	0.0055			−0.0005	0.0061		
0.0049	0.0000	0.0055		0.0054	−0.0005	0.0061	
0.0000	0.0049	0.0000	0.0055	−0.0005	0.0054	−0.0005	0.0061
($n = 301$)				Steady state ($n=\infty$)			
0.0063				0.0063			
−0.0006	0.0063			−0.0007	0.0063		
0.0055	−0.0006	0.0063		0.0055	−0.0007	0.0063	
−0.0006	0.0055	−0.0006	0.0063	−0.0007	0.0055	−0.0007	0.0063

Table 2. h values of in-control ARL = 300 for FEWMA's of $p = 4$.

(Steady State)

r	c	M-8	M-5	M-2	IND	P-2	P-5	P-8
	0	12.80	12.80	12.80	12.80	12.80	12.80	12.80
0.06	0.25	11.38	11.38	11.39	11.39	11.40	11.40	11.39
	0.5	10.17	10.17	10.15	10.13	10.14	10.15	10.13
	0.75	8.81	8.79	8.79	8.78	8.78	8.77	8.77
	0	13.83	13.83	13.83	13.83	13.83	13.83	13.83
0.10	0.25	12.62	12.62	12.60	12.60	12.60	12.60	12.60
	0.5	11.48	11.45	11.46	11.46	11.49	11.49	11.46
	0.75	10.18	10.17	10.13	10.12	10.15	10.15	10.10
	0	14.84	14.84	14.84	14.84	14.84	14.84	14.84
0.20	0.25	13.92	13.94	13.94	13.94	13.95	13.96	13.95
	0.5	13.02	13.00	13.01	13.03	13.02	13.00	12.98
	0.75	11.81	11.82	11.79	11.79	11.83	11.83	11.77
	0	15.65	15.65	15.65	15.65	15.65	15.65	15.65
0.50	0.25	15.17	15.18	15.18	15.18	15.19	15.17	15.20
	0.5	14.52	14.51	14.50	14.50	14.51	14.51	14.51
	0.75	13.53	13.52	13.53	13.52	13.50	13.50	13.50

(Initial State)

r	c	M-8	M-5	M-2	IND	P-2	P-5	P-8
	0	13.05	13.05	13.05	13.05	13.05	13.05	13.05
0.06	0.25	11.89	11.90	11.91	11.92	11.93	11.92	11.90
	0.5	11.10	11.08	11.06	11.06	11.05	11.06	11.06
	0.75	10.39	10.38	10.36	10.35	10.36	10.36	10.38
	0	13.95	13.95	13.95	13.95	13.95	13.95	13.95
0.10	0.25	12.89	12.89	12.90	12.90	12.89	12.90	12.88
	0.5	12.05	12.02	12.05	12.04	12.04	12.03	12.04
	0.75	11.25	11.24	11.25	11.24	11.22	11.23	11.23
	0	14.89	14.89	14.89	14.89	14.89	14.89	14.89
0.20	0.25	14.06	14.07	14.06	14.07	14.07	14.08	14.07
	0.5	13.27	13.27	13.29	13.28	13.28	13.28	13.26
	0.75	12.40	12.43	12.43	12.42	12.43	12.43	12.39
	0	15.66	15.66	15.66	15.66	15.66	15.66	15.66
0.50	0.25	15.21	15.22	15.23	15.22	15.22	15.21	15.24
	0.5	14.59	14.59	14.59	14.58	14.60	14.60	14.59
	0.75	13.76	13.77	13.78	13.77	13.74	13.75	13.76

Table 3. Averages (standard deviations) of ARL's of FEWMA's for seven correlation types using h of in-control ARL = 300 for Equal Shift of p = 4.

Equal Shift (Steady State)

r	c	η_c=0.1	η_c=0.2	η_c=0.4	η_c=0.8	η_c=1.6	η_c=3.2
0.06	0	230.7 (0.7)	136.7 (0.7)	52.9 (0.1)	18.8 (0.0)	8.0(0.0)	3.9(0.0)
	0.75	218.0 (1.0)	120.8 (0.5)	46.1 (0.1)	16.5 (0.0)	7.0(0.0)	3.4(0.0)
0.20	0	265.0 (0.8)	197.0 (0.7)	87.8 (0.4)	22.4 (0.1)	6.3(0.0)	2.7(0.0)
	0.75	257.1 (2.0)	181.2 (1.8)	78.2 (0.5)	21.2 (0.1)	6.0(0.0)	2.5(0.0)
0.50	0	285.4 (0.2)	249.3 (1.0)	157.5 (0.6)	49.6 (0.2)	8.1(0.0)	2.2(0.0)
	0.75	281.8 (1.7)	238.4 (1.1)	142.7 (1.1)	46.1 (0.2)	8.5(0.0)	2.2(0.0)

Equal Shift (Initial State)

r	c	η_c=0.1	η_c=0.2	η_c=0.4	η_c=0.8	η_c=1.6	η_c=3.2
0.06	0	227.7 (0.3)	131.3 (0.7)	47.0 (0.1)	14.1 (0.0)	4.4(0.0)	1.6(0.0)
	0.75	198.8 (1.5)	98.3 (0.6)	33.1 (0.2)	10.3 (0.0)	3.4(0.0)	1.4(0.0)
0.20	0	264.4 (0.6)	196.3 (1.0)	86.3 (0.6)	21.1 (0.0)	5.2(0.0)	1.8(0.0)
	0.75	255.1 (1.8)	170.2 (1.3)	66.3 (0.4)	15.9 (0.1)	4.3(0.0)	1.5(0.0)
0.50	0	285.8 (0.3)	249.5 (0.8)	157.5 (0.7)	49.1 (0.2)	7.8(0.0)	1.9(0.0)
	0.75	280.6 (1.7)	235.9 (1.2)	135.1 (0.8)	39.2 (0.2)	6.5(0.0)	1.7(0.0)

Table 4. Averages (standard deviations) of ARL's of FEWMA's for seven correlation types using h of in-control ARL = 300 for Symmetric Shift of $p = 4$.

Symmetric Shift (Steady State)

r	c	η_c=0.1	η_c=0.2	η_c=0.4	η_c=0.8	η_c=1.6	η_c=3.2
0.06	0	235.1 (1.8)	137.7 (0.3)	52.9 (0.1)	18.8 (0.0)	8.1(0.0)	3. (0.0)
	0.75	210.8 (0.6)	132.1 (0.1)	71.1 (0.1)	36.7 (0.1)	18.7(0.0)	9. (0.0)
0.20	0	266.2 (0.9)	197.5 (0.8)	85.8 (0.3)	22.2 (0.0)	6.4(0.0)	2. (0.0)
	0.75	210.5 (0.8)	120.4 (0.3)	56.5 (0.1)	26.6 (0.0)	13.0(0.0)	6. (0.0)
0.50	0	285.8 (0.7)	249.0 (0.7)	156.9 (0.8)	48.5 (0.1)	8.3(0.0)	2. (0.0)
	0.75	226.9 (0.8)	129.0 (0.6)	52.3 (0.1)	21.0 (0.0)	9.5(0.0)	4. (0.0)

Symmetric Shift (Initial State)

r	c	η_c=0.1	η_c=0.2	η_c=0.4	η_c=0.8	η_c=1.6	η_c=3.2
0.06	0	232.1(2.0)	131.9(0.4)	47.1(0.1)	14.2(0.0)	4.5(0.0)	1.6(0.0)
	0.75	171.4(0.8)	85.3(0.3)	32.6(0.1)	10.7(0.0)	3.5(0.0)	1.4(0.0)
0.20	0	265.5(0.6)	196.8(0.7)	84.4(0.3)	21.0(0.1)	5.3(0.0)	1.8(0.0)
	0.75	196.6(0.9)	101.7(0.3)	39.7(0.2)	13.3(0.1)	4.2(0.0)	1.5(0.0)
0.50	0	286.2(0.8)	249.4(0.8)	156.6(0.9)	48.3(0.1)	7.9(0.0)	1.9(0.0)
	0.75	221.7(1.2)	121.3(0.8)	44.9(0.1)	14.6(0.0)	4.7(0.0)	1.7(0.0)

Table 5. Averages (standard deviations) of ARL's of FEWMA's for seven correlation types using h of in-control ARL = 300 for Single Shift of p = 4.

Single Shift (Steady State)

r	c	η_c=0.1	η_c=0.2	η_c=0.4	η_c=0.8	η_c=1.6	η_c=3.2
0.06	0	232.0(1.0)	137.5(0.5)	52.9(0.2)	18.8(0.0)	8.0(0.0)	3.9(0.0)
	0.75	211.9(0.9)	129.0(1.6)	65.1(3.3)	30.1(3.4)	13.8(2.4)	6.7(1.4)
0.20	0	267.1(0.6)	196.9(0.9)	87.5(0.7)	22.3(0.1)	6.3(0.0)	2.7(0.0)
	0.75	217.2(3.6)	127.3(4.0)	58.6(1.3)	25.5(0.6)	11.0(1.1)	4.9(0.8)
0.50	0	285.5(0.9)	248.3(0.9)	156.8(0.3)	48.9(0.5)	8.2(0.0)	2.2(0.0)
	0.75	234.4(4.5)	142.6(8.0)	58.6(4.1)	22.5(0.9)	9.4(0.1)	4.0(0.4)

Single Shift (Initial State)

r	c	η_c=0.1	η_c=0.2	η_c=0.4	η_c=0.8	η_c=1.6	η_c=3.2
0.06	0	228.6(1.4)	131.3(0.4)	47.0(0.2)	14.1(0.1)	4.4(0.0)	1.6(0.0)
	0.75	175.2(2.1)	86.4(1.0)	32.3(0.2)	10.5(0.0)	3.5(0.0)	1.4(0.0)
0.20	0	267.0(0.8)	196.5(0.7)	86.0(0.7)	21.0(0.1)	5.2(0.0)	1.8(0.0)
	0.75	205.4(4.0)	109.6(4.6)	42.1(1.7)	13.5(0.2)	4.2(0.0)	1.5(0.0)
0.50	0	285.8(1.1)	248.5(1.0)	156.6(0.3)	48.4(0.4)	7.9(0.0)	1.9(0.0)
	0.75	230.5(4.7)	135.3(8.2)	51.0(3.9)	16.0(0.9)	4.9(0.1)	1.7(0.0)

Table 6. h values of FEWMA's of $r = 0.1$ for various in-control ARL's.

(Steady State)

p	ARL	$c = 0$	$c = 0.25$ M-5	$c = 0.25$ P-5	$c = 0.5$ M-5	$c = 0.5$ P-5	$c = 0.75$ M-5	$c = 0.75$ P-5
	300	9.57	9.19	9.20	8.76	8.79	8.24	8.25
2	500	10.76	10.39	10.40	9.99	9.99	9.50	9.52
	1000	12.32	11.95	11.95	11.61	11.62	11.16	11.16
	300	11.77	10.97	10.99	10.16	10.21	9.23	9.24
3	500	13.04	12.28	12.28	11.55	11.58	10.62	10.70
	1000	14.73	14.03	14.05	13.32	13.37	12.53	12.57
	300	13.83	12.62	12.60	11.45	11.49	10.17	10.15
4	500	12.99	14.05	14.03	13.01	13.01	11.77	11.75
	1000	16.97	15.95	15.95	14.97	14.99	13.86	13.86
	300	15.74	14.07	14.08	12.61	12.61	10.87	10.86
5	500	17.10	15.58	15.61	14.28	14.29	12.68	12.65
	1000	18.93	17.60	17.59	16.40	16.46	14.98	15.02
	300	24.08	20.38	20.36	17.34	17.35	13.59	13.56
10	500	25.75	22.46	22.46	19.74	19.80	16.32	16.35
	1000	27.90	25.01	25.00	22.73	22.75	19.84	19.86

Initial State

p	ARL	$c = 0$	$c = 0.25$ M-5	$c = 0.25$ P-5	$c = 0.5$ M-5	$c = 0.5$ P-5	$c = 0.75$ M-5	$c = 0.75$ P-5
	300	9.69	9.33	9.34	8.97	9.00	8.59	8.58
2	500	10.82	10.47	10.48	10.11	10.11	9.71	9.73
	1000	12.35	12.00	12.00	11.67	11.68	11.28	11.28
	300	11.89	11.19	11.21	10.57	10.59	9.93	9.96
3	500	13.11	12.40	12.41	11.77	11.80	11.09	11.18
	1000	14.77	14.09	14.11	13.46	13.50	12.79	12.82
	300	13.95	12.89	12.90	12.02	12.03	11.24	11.23
4	500	15.22	14.23	14.22	13.33	13.36	12.48	12.50
	1000	17.01	16.00	16.03	15.15	15.17	14.27	14.27
	300	15.87	14.45	14.47	13.41	13.41	12.47	12.45
5	500	17.19	15.82	15.86	14.78	14.79	13.75	13.75
	1000	18.97	17.72	17.71	16.65	16.72	15.60	15.61
	300	24.25	21.31	21.30	19.60	19.66	18.32	18.36
10	500	25.85	23.03	23.04	21.19	21.24	19.66	19.69
	1000	27.94	25.28	25.27	23.47	23.49	21.79	21.86

Table 7. Estimated control limits of FEWMA chart of $r = 0.1$ and $c = 0.75$ using linear functions of logarithm of in-control ARL for P-5 correlation type.

(Steady State)

	$p=2$		$p=3$		$p=4$		$p=5$		$p=10$	
ARL	h_{sim}	$\hat{h}$	h_{sim}	$\hat{h}$	h_{sim}	$\hat{h}$	h_{sim}	$\hat{h}$	h_{sim}	$\hat{h}$
100	5.54	5.54	6.19	6.19	6.61	6.61	7.01	7.01	8.08	8.08
200	7.25	7.25	8.14	8.13	8.83	8.82	9.42	9.44	11.45	11.64
300	8.25	8.26	9.24	9.27	10.15	10.12	10.86	10.86	13.56	13.73
400	8.98	8.97	10.09	10.07	11.05	11.04	11.87	11.87	15.14	15.20
500	9.52	9.52	10.70	10.70	11.75	11.75	12.65	12.65	16.35	16.35
600	9.96	9.95	11.21	11.19	12.34	12.31	13.29	13.27	17.34	17.27
700	10.33	10.32	11.62	11.61	12.80	12.77	13.80	13.80	18.11	18.05
800	10.65	10.63	11.97	11.97	13.19	13.18	14.25	14.26	18.78	18.73
900	10.92	10.91	12.29	12.29	13.54	13.54	14.66	14.66	19.35	19.33
1000	11.16	11.16	12.57	12.57	13.86	13.86	15.02	15.02	19.86	19.86

(Initial State)

	$p=2$		$p=3$		$p=4$		$p=5$		$p=10$	
ARL	h_{sim}	$\hat{h}$	h_{sim}	$\hat{h}$	h_{sim}	$\hat{h}$	h_{sim}	$\hat{h}$	h_{sim}	$\hat{h}$
100	6.30	6.30	7.62	7.62	8.75	8.75	9.95	9.95	15.66	15.66
200	7.70	7.78	9.06	9.15	10.27	10.37	11.47	11.59	17.28	17.40
300	8.58	8.64	9.96	10.05	11.23	11.31	12.45	12.54	18.36	18.41
400	9.25	9.25	10.64	10.69	11.94	11.98	13.17	13.22	19.12	19.13
500	9.73	9.73	11.18	11.18	12.50	12.50	13.75	13.75	19.69	19.69
600	10.13	10.14	11.59	11.61	12.96	12.97	14.23	14.24	20.22	20.26
700	10.49	10.48	11.97	11.98	13.35	13.36	14.64	14.65	20.70	20.74
800	10.80	10.78	12.29	12.29	13.69	13.70	15.00	15.01	21.11	21.16
900	11.05	11.04	12.57	12.57	14.00	14.00	15.32	15.33	21.51	21.53
1000	11.28	11.28	12.82	12.82	14.27	14.27	15.61	15.61	21.86	21.86

Table 8. ARL Performance of FEWMA's of $r = 0.1$ using h of in-control ARL = 300 for Equal Shift for two correlation types.

(Steady State)

p	c	M-5						P-5					
		η_c=0.1	η_c=0.2	η_c=0.4	η_c=0.8	η_c=1.6	η_c=3.2	η_c=0.1	η_c=0.2	η_c=0.4	η_c=0.8	η_c=1.6	η_c=3.2
	0	229.6	133.1	47.8	15.1	6.0	2.9	231.7	133.4	48.1	15.0	6.0	2.9
2	0.25	227.8	129.8	46.9	14.9	5.9	2.8	230.2	131.0	47.0	14.9	5.9	2.8
	0.5	225.4	127.2	45.5	14.6	5.8	2.8	226.0	127.8	45.9	14.6	5.8	2.8
	0.75	219.3	123.9	44.1	14.3	5.7	2.7	221.1	122.8	44.3	14.3	5.7	2.7
	0	237.4	146.0	54.7	16.9	6.6	3.1	239.9	145.7	54.8	16.9	6.6	3.1
3	0.25	235.1	143.1	54.0	16.9	6.6	3.1	237.5	141.3	53.1	16.6	6.5	3.1
	0.5	234.7	141.1	54.2	17.1	6.6	3.1	234.3	136.4	51.6	16.3	6.3	3.0
	0.75	233.5	140.4	54.1	17.3	6.6	3.1	225.7	130.3	49.2	15.6	6.1	2.9
	0	256.4	172.4	67.2	20.0	7.5	3.5	257.0	174.1	67.6	20.1	7.5	3.5
5	0.25	249.8	165.8	66.6	20.4	7.6	3.5	250.0	163.5	65.0	19.9	7.4	3.4
	0.5	247.2	163.4	66.0	20.6	7.5	3.4	247.6	156.7	62.8	19.4	7.1	3.3
	0.75	244.2	159.4	65.1	19.8	7.2	3.3	240.2	150.8	59.2	18.1	6.7	3.1
	0	269.3	199.7	87.8	25.4	9.0	4.2	267.7	197.9	87.4	25.3	9.0	4.2
10	0.25	261.5	193.4	88.6	27.2	9.4	4.1	263.3	192.6	88.9	27.3	9.3	4.1
	0.5	260.1	190.1	88.7	26.7	8.9	3.9	261.5	191.9	88.6	26.8	8.8	3.9
	0.75	245.4	173.0	76.5	22.3	7.7	3.5	243.8	173.1	76.0	22.4	7.7	3.5

(Initial State)

p	c	M-5						P-5					
		η_c=0.1	η_c=0.2	η_c=0.4	η_c=0.8	η_c=1.6	η_c=3.2	η_c=0.1	η_c=0.2	η_c=0.4	η_c=0.8	η_c=1.6	η_c=3.2
	0	229.6	130.6	44.8	12.7	4.0	1.5	231.0	131.1	45.2	12.6	4.0	1.5
2	0.25	227.2	126.0	42.8	12.1	3.8	1.4	228.0	126.6	43.3	12.1	3.8	1.4
	0.5	221.8	122.4	40.9	11.6	3.7	1.4	223.3	123.2	41.3	11.7	3.7	1.4
	0.75	215.5	116.7	38.5	11.1	3.6	1.4	216.4	115.9	38.4	11.0	3.6	1.4

	0	235.1	142.6	51.4	14.2	4.4	1.6	239.1	142.9	51.7	14.3	4.4	1.6
3	0.25	231.9	137.8	48.9	13.3	4.1	1.5	234.9	137.3	48.2	13.4	4.1	1.5
	0.5	232.5	133.1	46.2	12.4	3.9	1.5	230.9	128.6	44.3	12.5	3.9	1.5
	0.75	227.4	124.6	42.2	11.5	3.7	1.4	219.8	118.1	40.3	11.5	3.7	1.4
	0	253.2	168.2	63.8	17.0	5.0	1.8	255.2	171.4	64.2	17.1	5.1	1.8
5	0.25	245.6	157.1	58.2	15.2	4.6	1.6	247.9	156.0	57.0	15.2	4.6	1.6
	0.5	240.6	147.8	52.6	13.6	4.2	1.5	238.4	144.1	49.6	13.5	4.2	1.5
	0.75	230.9	134.0	44.8	12.1	3.8	1.5	225.4	127.6	42.5	11.9	3.8	1.5
	0	267.1	197.4	83.2	22.0	6.2	2.1	264.4	196.1	83.4	22.0	6.3	2.1
10	0.25	255.9	179.3	70.9	17.9	5.1	1.8	255.7	178.4	71.2	17.9	5.2	1.8
	0.5	246.9	158.6	57.5	14.8	4.5	1.6	253.2	164.0	58.3	15.0	4.5	1.6
	0.75	236.2	136.5	46.0	12.5	4.0	1.5	238.7	140.4	46.6	12.6	4.0	1.5

Table 9. ARL Performance of FEWMA's of $r = 0.1$ using h of in-control ARL = 300 for Single Shift for two correlation types.

(Steady State)

p	c	M-5						P-5					
		η_c=0.1	η_c=0.2	η_c=0.4	η_c=0.8	η_c=1.6	η_c=3.2	η_c=0.1	η_c=0.2	η_c=0.4	η_c=0.8	η_c=1.6	η_c=3.2
2	0	231.7	133.4	48.1	15.0	6.0	2.9	225.9	130.1	48.2	15.3	6.0	2.9
2	0.25	225.0	126.6	46.0	15.1	6.2	3.0	219.9	119.8	44.8	15.8	6.7	3.3
	0.5	218.7	119.3	44.8	15.4	6.3	3.0	210.4	111.8	44.2	17.2	7.6	3.7
	0.75	211.1	115.2	44.9	15.9	6.4	3.1	201.1	109.5	48.2	20.4	9.0	4.3
	0	242.4	146.6	54.6	17.1	6.6	3.1	240.3	145.5	55.4	17.1	6.6	3.1
3	0.25	227.9	128.2	49.0	17.6	7.4	3.6	221.6	124.8	48.2	18.0	7.8	3.8
	0.5	215.8	120.2	49.0	19.4	8.3	4.0	209.0	115.1	48.8	20.7	9.4	4.6
	0.75	208.7	119.6	54.3	22.7	9.5	4.4	200.1	116.0	56.3	26.2	12.1	5.8
	0	254.8	172.0	67.4	20.1	7.5	3.5	259.2	175.4	67.0	20.2	7.5	3.5
5	0.25	230.3	138.6	55.8	21.4	9.3	4.5	230.3	135.3	54.3	21.6	9.7	4.8
	0.5	219.7	128.9	58.5	25.2	11.2	5.3	216.1	125.2	57.6	26.4	12.6	6.3
	0.75	218.1	135.2	68.5	31.3	13.4	6.0	215.6	133.1	70.3	35.0	17.1	8.4
	0	267.7	198.9	88.5	25.4	9.0	4.2	268.7	201.4	88.1	25.4	9.0	4.2
10	0.25	238.7	150.1	67.6	29.2	13.4	6.5	234.1	145.8	66.4	29.4	14.0	7.0
	0.5	230.6	148.6	77.0	37.4	17.5	8.2	227.1	146.2	76.9	38.5	19.2	9.8
	0.75	227.8	158.7	90.6	45.6	20.9	9.2	225.2	157.5	92.2	49.3	25.2	12.8

(Initial State)

p	c	M-5						P-5					
		η_c=0.1	η_c=0.2	η_c=0.4	η_c=0.8	η_c=1.6	η_c=3.2	η_c=0.1	η_c=0.2	η_c=0.4	η_c=0.8	η_c=1.6	η_c=3.2
	0	231.0	131.1	45.2	12.6	4.0	1.5	226.8	128.3	45.4	12.8	4.0	1.5
2	0.25	222.1	122.2	42.0	12.1	3.8	1.4	216.3	115.0	40.3	12.2	3.9	1.5
	0.5	214.6	113.6	39.1	11.6	3.7	1.4	204.8	104.6	37.3	11.8	3.8	1.4
	0.75	206.2	105.7	36.9	11.1	3.6	1.4	191.1	95.4	35.3	11.3	3.6	1.4
	0	240.1	143.6	51.5	14.4	4.4	1.6	239.0	142.8	52.0	14.4	4.4	1.6
3	0.25	223.7	122.3	43.2	13.2	4.2	1.5	219.0	119.1	42.6	13.1	4.2	1.5
	0.5	210.6	110.1	39.3	12.4	4.0	1.5	200.6	103.7	38.4	12.3	3.9	1.5

	0.75	191.8	97.7	36.4	11.6	3.7	1.4	184.0	93.4	35.8	11.6	3.7	1.4
	0	253.1	169.3	63.8	17.2	5.1	1.8	258.0	172.6	63.5	17.3	5.1	1.8
5	0.25	226.1	128.8	47.0	14.6	4.6	1.6	225.7	126.2	45.4	14.5	4.6	1.6
	0.5	207.2	109.5	41.2	13.2	4.2	1.5	201.1	104.7	39.8	13.2	4.2	1.5
	0.75	188.7	96.9	36.7	11.9	3.8	1.5	182.7	92.2	35.7	11.9	3.9	1.5
	0	266.6	197.1	84.4	21.9	6.3	2.1	267.4	197.5	84.0	21.9	6.2	2.1
10	0.25	227.0	132.2	49.9	16.4	5.2	1.8	224.8	127.8	49.0	16.1	5.2	1.8
	0.5	198.0	106.7	41.8	14.1	4.5	1.6	201.3	105.4	41.5	13.9	4.6	1.7
	0.75	175.5	90.3	35.8	12.2	4.1	1.5	177.1	90.0	35.5	12.1	4.1	1.6

Table 10. Optimal FEWMA schemes of in-control ARL's 300, 500, 1000 with a fixed c for p = 4 and steady state in Equal Shift.

(ARL = 300)		$\eta_c = 0.1$	$\eta_c = 0.2$	$\eta_c = 0.4$	$\eta_c = 0.8$	$\eta_c = 1.6$	$\eta_c = 3.2$
$c = 0.0$	ARL_{min}	199.1	111.5	49.4	18.31	6.34	2.15
	r_{min}	0.01	0.015	0.03	0.09	0.23	0.63
	r range	0.005–0.014	0.01–0.019	0.022–0.04	0.07–0.1	0.18–0.25	0.55–0.72
$c = 0.85$	ARL_{min}	167.7	86.3	38.2	15.50	5.82	2.13
	r_{min}	0.006	0.011	0.023	0.07	0.19	0.6
	r range	0.004–0.009	0.008–0.014	0.016–0.03	0.05–0.08	0.15–0.23	0.45–0.68
$c = 0.90$	ARL_{min}	166.2	85.2	37.3	14.91	5.63	2.07
	r_{min}	0.006	0.011	0.023	0.07	0.2	0.6
	r range	0.005–0.008	0.008–0.016	0.017–0.03	0.06–0.08	0.18–0.22	0.48–0.68
$c = 0.95$	ARL_{min}	165.9	84.2	36.3	14.18	5.34	1.97
	r_{min}	0.006	0.01	0.025	0.08	0.2	0.6
	r range	0.005–0.01	0.009–0.016	0.019–0.04	0.06–0.1	0.16–0.25	0.49–0.7

(ARL = 500)		$\eta_c = 0.1$	$\eta_c = 0.2$	$\eta_c = 0.4$	$\eta_c = 0.8$	$\eta_c = 1.6$	$\eta_c = 3.2$
$c = 0.0$	ARL_{min}	276.7	142.6	57.8	20.65	6.91	2.30
	r_{min}	0.01	0.015	0.03	0.09	0.2	0.61
	r range	0.004–0.007	0.009–0.016	0.02–0.03	0.06–0.09	0.16–0.24	0.49–0.65
$c = 0.85$	ARL_{min}	232.9	86.3	46.2	17.85	6.45	2.30
	r_{min}	0.005	0.011	0.023	0.07	0.19	0.52
	r range	0.004–0.008	0.006–0.012	0.015–0.03	0.04–0.09	0.14–0.22	0.44–0.60
$c = 0.90$	ARL_{min}	230.6	85.2	44.9	17.24	6.25	2.25
	r_{min}	0.005	0.011	0.023	0.07	0.19	0.54
	r range	0.005–0.008	0.007–0.013	0.017–0.03	0.04–0.09	0.14–0.22	0.45–0.60
$c = 0.95$	ARL_{min}	228.9	84.2	43.6	16.31	5.95	2.17
	r_{min}	0.006	0.01	0.025	0.07	0.19	0.56
	r range	0.005–0.01	0.007–0.013	0.017–0.04	0.05–0.1	0.14–0.23	0.41–0.65

(ARL = 1000)		$\eta_c = 0.1$	$\eta_c = 0.2$	$\eta_c = 0.4$	$\eta_c = 0.8$	$\eta_c = 1.6$	$\eta_c = 3.2$
$c = 0.0$	ARL_{min}	412.5	187.9	69.9	23.58	7.68	2.49
	r_{min}	0.004	0.015	0.024	0.07	0.18	0.51

	r range	0.003–0.005	0.006–0.011	0.017–0.03	0.05–0.08	0.14–0.22	0.43–0.61
$c = 0.85$	ARL_{min}	354.3	151.7	58.3	21.57	7.32	2.52
	r_{min}	0.006	0.009	0.018	0.05	0.17	0.47
	r range	0.003–0.005	0.006–0.011	0.013–0.024	0.04–0.07	0.13–0.2	0.38–0.56
$c = 0.90$	ARL_{min}	351.7	149.4	57.0	20.60	7.16	2.49
	r_{min}	0.006	0.009	0.019	0.05	0.16	0.48
	r range	0.004–0.006	0.006–0.011	0.014–0.025	0.04–0.07	0.12–0.2	0.35–0.57
$c = 0.95$	ARL_{min}	352.1	147.07	55.3	19.69	6.85	2.40
	r_{min}	0.006	0.009	0.021	0.06	0.17	0.47
	r range	0.003–0.006	0.007–0.011	0.017–0.025	0.05–0.07	0.13–0.2	0.31–0.64

Table 11. Ambulatory monitoring data

Week	SBP	DBP	MAP	HR	Week	SBP	DBP	MAP	HR
161	122.96	74.26	93.63	81.35	162	127.77	76.71	96.68	86.08
163	124.54	75.01	96.51	82.10	164	123.06	73.81	94.44	81.33
165	125.73	76.03	95.82	81.82	166	129.08	77.58	98.16	85.62
167	130.24	78.81	100.59	87.31	168	129.03	78.10	99.24	86.94
169	129.31	78.90	100.02	85.19	170	128.07	77.78	97.64	84.41
171	130.24	78.58	99.91	85.67	172	128.50	76.72	98.62	85.33
175	128.62	78.75	98.98	83.08	176	130.70	78.94	100.16	85.02
177	126.65	77.28	96.96	84.61	178	128.82	77.13	98.24	87.00
179	124.40	73.42	94.81	84.19	180	125.71	75.97	96.32	84.34
181	125.50	76.02	96.57	86.72	182	123.44	75.58	96.44	83.98
183	122.31	73.38	93.04	80.14	184	122.08	73.00	92.58	81.60
185	124.25	75.68	94.96	80.71	186	124.50	75.12	95.33	80.83
187	125.02	76.72	97.74	81.65	188	125.28	76.26	96.10	82.14
189	126.75	77.08	97.81	80.47	190	124.55	76.73	96.25	81.35
191	132.39	80.23	101.23	87.75	194	128.52	78.75	99.45	83.38

Table 12. FEWMA of ambulatory monitoring data

Week	Components of y vector				T^2
161	-0.167	-0.164	-0.168	-0.157	6.613
162	-0.167	-0.164	-0.171	-0.146	10.436
163	-0.257	-0.253	-0.257	-0.230	7.188
164	-0.396	-0.391	-0.394	-0.359	10.412
165	-0.449	-0.441	-0.447	-0.408	11.933
166	-0.386	-0.380	-0.388	-0.341	11.066
167	-0.265	-0.259	-0.265	-0.213	9.037
168	-0.194	-0.187	-0.193	-0.133	9.945
169	-0.126	-0.115	-0.122	-0.062	9.877
170	-0.120	-0.108	-0.119	-0.052	11.818
171	-0.050	-0.038	-0.049	0.020	11.488
172	-0.042	-0.034	-0.041	0.033	10.787
175	-0.027	-0.014	-0.025	0.047	11.005
176	0.040	0.052	0.041	0.112	10.722
177	0.002	0.019	0.003	0.083	13.546
178	0.023	0.037	0.023	0.113	15.163
179	-0.106	-0.096	-0.107	-0.003	15.303
180	-0.164	-0.152	-0.164	-0.050	17.267
181	-0.194	-0.180	-0.193	-0.063	21.173
182	-0.276	-0.254	-0.267	-0.129	24.111