BREAKDOWN IN NONLINEAR REGRESSION

by

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ABSTRACT

The breakdown point is considered an important measure of the robustness of a linear regression estimator. This paper addresses the concept of breakdown in nonlinear regression. Because it is not invariant to nonlinear reparameterization, the usual definition of the breakdown point is inadequate for use in nonlinear regression. We introduce the breakdown function, and based upon it, a new definition of the breakdown point. For the linear regression model, our definition of the breakdown point coincides with the usual definition. For most nonlinear regression functions, we show that the breakdown point of the least squares estimator is 1/n. We prove that for a large class of unbounded regression functions the breakdown point of the least median of squares or the least trimmed sum of squares estimator is close to 1/2. For monotonic regression functions of the type $g(\alpha + \beta x)$ where g is bounded above and/or below, we establish upper and lower bounds for the breakdown points that depend on the data.

KEY WORDS: Breakdown function, Invariant to reparametrization, Least median of squares estimator, Least squares estimator, Least trimmed sum of squares estimator.

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1. INTRODUCTION

This article is concerned with extending breakdown analysis from linear to nonlinear regression. There has been considerable recent interest in high-breakdown point (HBP) estimators for linear regression. Because of their relative immunity to masking, HBP estimators are effective in outlier detection (Atkinson 1986; Rousseeuw and van Zomeren 1989). For inference when the sample contains outliers among the predictor variables, HBP estimators, especially those that also possess a bounded influence function, are much more "stable" than other estimators (Simpson, Ruppert, and Carroll 1989). "Stability" means that the conclusions of the analysis are not radically altered by deletion of one, or a few, data points.

In linear regression, the basic HBP estimators are S-estimators, that is, estimators defined by minimizing a robust scale measure of the residuals (Rousseeuw and Yohai 1984). Other HBP estimators use iterative algorithms with S-estimators as starting values. The most studied S-estimator is the least median of squared residuals (LMS) estimator (Rousseeuw 1984), which, as its name implies, minimizes the median of the squared residuals. Another well-known S-estimator is the least trimmed sum of squared (LTS) estimator which minimizes the sum of the $[\![n/2]\!] + 1$ smallest squared residuals, where n is the sample size and $[\![\cdot]\!]$ is the greatest integer function.

Although the *definitions* of the LMS, LTS, and other S-estimators are easily extended to nonlinear regression, analysis of breakdown properties and computation of these estimators in the nonlinear case is far from trivial. Here we consider only breakdown; for a discussion of computation, see Stromberg (1989).

The usual method for studying properties of a nonlinear model is linear approximation in a neighborhood of the true parameter. Approximation by a linear model does *not* work for the analysis of breakdown properties since these are determined by large (i.e., nonlocal) changes in the estimated parameters.

Donoho and Huber (1983) define the *finite sample breakdown point*, ϵ_n^* of an estimator to be the smallest proportion of data that must be changed to cause an infinite perturbation of the estimate. More precisely, let

$\chi_n \equiv$ the observed data set consisting of the sample points $\{z_1, \ldots, z_n\}$,

$\hat{\theta}_n(\chi_n) \equiv$ an estimator $\hat{\theta}_n$ evaluated at χ_n,

and

$D_n^m \equiv$ the set of all data sets, χ_n^m, obtained by replacing any m points in χ_n with arbitrary values.

Then the finite-sample breakdown point of $\hat{\theta}_n$ at χ_n is

$$\epsilon_n^*(\hat{\theta}_n(\chi_n)) \equiv \min_{1 \le m \le n} \left\{ \frac{m}{n};\ \sup_{\chi_n^m \in D_n^m} \left\| \hat{\theta}_n(\chi_n^m) - \hat{\theta}_n(\chi_n) \right\| = \infty . \right\}. \tag{1.1}$$

We will use the the nonlinear regression model

$$y_i = h(x_i, \theta_o) + \epsilon_i \quad i=1,2, \ldots, n, \tag{1.2}$$

where the x_i are k-dimensional vectors of explanatory variables, ϵ_i are independent and identically distributed random variables with mean 0 and unknown variance σ^2, θ_o is an unknown p-dimensional element of parameter space Θ, and h is a known model function that is assumed continuous in θ for each x. In the definition of ϵ_n^*, let $z_i = (y_i, x_i)$.

For several reasons, the usual breakdown point ϵ_n^* is of limited use in nonlinear regression settings. First of all, regardless of the regression function, if the parameter space is bounded, then $\epsilon_n^* = 1$. Thus ϵ_n^* is not an appropriate definition of the breakdown point for bounded parameter spaces. Note also that ϵ_n^* is not invariant to reparameterization. Consider the equivalent nonlinear regression models (1.2) and

$$y_i = h(x_i, g(\omega_0)) + \epsilon_i, \quad i=1,2,\ldots,n \tag{1.3}$$

where g is a continuous function such that $\theta_0 = g(\omega_0)$. An estimator is invariant to reparameterization if $\hat{\theta}_n = g(\hat{\omega}_n)$. A breakdown point is invariant to reparameterization if it is the same for models (1.2) and (1.3).

To see that ϵ_n^* is not invariant to reparameterization, consider the regression model:

$$y_i = x_i \omega + \epsilon_i$$

where ω is an angle in $(-\frac{\pi}{2}, \frac{\pi}{2})$ and x_i has appropriate units. Investigator A calculates the finite sample breakdown point directly for the least-squares estimator and finds that it is one, since ω can't take on arbitrarily large values. Investigator B reparameterizes using $\theta = \tan^{-1}(\omega)$. Since θ can take all real values, he finds that the finite sample breakdown point for the least-squares estimator is $\frac{1}{n}$. ϵ_n^* is, however, invariant to linear reparametrization, which explains its suitability for linear regression.

Section 2 introduces a new concept, the breakdown function, and uses it to give a new definition of the breakdown point. In linear regression, this new breakdown point is the same as the usual definition; see Theorem 1. Section 3 discuss breakdown of the ordinary least-squares estimator. We find that, in general, the least-squares estimator can be broken down by a single outlier, just as in linear regression; see Theorem 2. Section 4 discusses breakdown of the LMS and LTS estimators. The breakdown points of these estimators depend upon the model and are not necessarily near 50% as in the linear case, but these estimators do seem acceptable in terms of breakdown. The Appendix contains proofs of all theorems presented in Sections 2 through 4.

2. A NEW FINITE SAMPLE BREAKDOWN POINT

To rectify the deficiencies of ϵ_n^*, we define breakdown in terms of the estimated regression function, not the estimated parameter. Specifically, we define the finite sample breakdown function, ϵ_n' at $\hat{\theta}_n(\chi_n)$ under regression function h as

$$\epsilon_n'(x,h,\hat{\theta}_n(\chi_n)) = \min_{0\le m\le n}\left\{\tfrac{m}{n};\ \sup_{\chi_n^m \in D_n^m} \left|h(x,\hat{\theta}_n(\chi_n^m)) - h(x,\hat{\theta}_n(\chi_n))\right| = \sup_{\theta} \left|h(x,\theta) - h(x,\hat{\theta}_n(\chi_n))\right|\right\}.$$

We then define the finite sample breakdown point, ϵ_n' by

$$\epsilon_n'(h,\hat{\theta}_n(\chi_n)) = \inf_{\text{nontrivial } x}\left\{\epsilon_n'(x,h,\hat{\theta}_n(\chi_n))\right\},$$

where x is nontrivial if there exists θ,θ' in Θ with $h(x,\theta) \neq h(x,\theta')$. In linear regression we suppress h, writing $\epsilon_n'(x,\hat{\theta}_n(\chi_n))$ for the finite sample breakdown function and $\epsilon_n'(\hat{\theta}_n(\chi_n))$ for the breakdown point.

It is rather easy to establish that for any estimator that is invariant to reparameterization, e. g., the least-squares, LMS, and LTS estimators, ϵ_n' is also invariant to reparameterization.

Our first theorem shows that in linear regression, where ϵ_n^* is an accepted measure of robustness, ϵ_n' and ϵ_n^* coincide.

<u>Theorem 1</u>

In a linear regression where $\Theta = \mathbb{R}^p$ and $h(x,\theta) = x^T\theta$, we have $\epsilon_n'(\hat{\theta}_n(\chi_n)) = \epsilon_n^*(\hat{\theta}_n(\chi_n))$.

3. BREAKDOWN PROPERTIES OF THE LEAST-SQUARES ESTIMATOR

In nonlinear regression, we will see that the finite sample breakdown point for the LMS estimator depends on the regression function h as well as the sample χ_n. This is not the case for the least-squares estimator. The following theorem establishes that for most regression functions, the finite sample breakdown point of the least-squares estimator, denoted $\hat{\theta}_n^{LS}$, is $\frac{1}{n}$.

Theorem 2

In the nonlinear regression setting defined in (1.2) assume there exists a nontrivial x and that

$$\hat{\theta}_n^{LS} \text{ exists for all } \chi_n^1 \in D_n^1. \tag{3.1}$$

Then

$$\epsilon_n'(h, \hat{\theta}_n^{LS}(\chi_n)) = \frac{1}{n}.$$

4. BREAKDOWN PROPERTIES OF THE LEAST MEDIAN OF SQUARES AND LEAST TRIMMED SQUARES ESTIMATORS

In this section, we investigate the finite sample breakdown properties of the least median of squares estimator, $\hat{\theta}^{LMS}$, and the least trimmed squares, $\hat{\theta}^{LTS}$, for various nonlinear models. We begin by presenting a theorem that can be used to establish the finite sample breakdown point of both estimators for many nonlinear regression functions. We introduce the following new notation:

$r_i^2(\theta)$, i=1,2, ..., n $\equiv$ the squared residuals based on χ_n.

$r_{(i)}^2(\theta)$, i=1,2, ..., n $\equiv$ the ordered $r_i^2(\theta)$.

$\tilde{r}_i^2(\theta)$, i=1,2, ..., n $\equiv$ the squared residuals based on χ_n^m, an arbitrary element of D_n^m.

$\tilde{r}_{(i)}^2(\theta)$, i=1,2, ..., n $\equiv$ the ordered $\tilde{r}_i^2(\theta)$.

$k \equiv [\![\frac{n}{2}]\!] + 1.$

We define the LMS estimator by

$$\hat{\theta}_n^{LMS} = \arg\min_{\theta} \; r_{(k)}^2(\theta)$$

and the LTS estimator by

$$\hat{\theta}_n^{LTS} = \arg\min_{\theta} \sum_{i=1}^{k} r_{(i)}^2(\theta)$$

Theorem 3

Let $m \in \{1, 2, \ldots, [\![\frac{n}{2}]\!]\}$. Suppose that for some fixed x,

(1) $$\sup_{\theta} |h(x,\theta)| = \infty,$$

and

(2) there exist $\tau_m \subset \{1,\ldots,n\}$ with $(n - [\![\frac{n}{2}]\!] + m)$ elements such that

$$\lim_{M \to \infty} \inf_{\theta: |h(x,\theta)| > M} \left\{ \inf_{i \in \tau_m} | h(x_i,\theta) | \right\} = \infty.$$

Then

$$\epsilon_n'(x,h,\hat{\theta}_n^{LTS}) = \epsilon_n'(x,h,\hat{\theta}_n^{LMS}) \geq \frac{(m+1)}{n}.$$

Corollary 1

If the conditions (1) and (2) Theorem 3 hold for all nontrivial x, then

$$\epsilon_n'(h,\hat{\theta}_n^{LTS}) = \epsilon_n'(h,\hat{\theta}_n^{LMS}) \geq \frac{m+1}{n}.$$

Application: The Michaelis-Menten nonlinear regression model is given by:

$$y_i = h_{mm}(x_i,\theta) = \frac{Vx_i}{K + x_i}, \quad i = 1, 2, \ldots n,$$

where $\theta = (V,K)^T$, V and K are nonnegative parameters, and $x_i > 0$ for all i. We can apply the previous theorem to find the breakdown point of $\hat{\theta}^{LTS}$ and $\hat{\theta}^{LMS}$ for the Michaelis-Menten model.

It is clear that for any nontrivial, i. e., nonzero, x, condition (1) of the theorem is satisfied. Because $K > 0$, for any $x > 0$, $h_{mm}(x,\theta) \to \infty$ is equivalent to $V \to \infty$, which, in turn, is equivalent to $h_{mm}(x_i,\theta) \to \infty$ for all i. Therefore (2) of the theorem holds for all $m \in \{1,\ldots,[\![\frac{n}{2}]\!]\}$. Thus

$$\epsilon_n'(x, h_{mm}, \hat{\theta}_n^{LTS}) = \epsilon_n'(x, h_{mm}, \hat{\theta}_n^{LMS}) \geq \frac{[\![\frac{n}{2}]\!]+1}{n}. \quad (4.1)$$

Clearly $\frac{[\![\frac{n}{2}]\!]+1}{n}$ is the maximum possible breakdown point so equality holds in (4.1). Applying the corollary

$$\epsilon_n'(h_{mm}, \hat{\theta}_n^{LTS}) = \epsilon_n'(h_{mm}, \hat{\theta}_n^{LMS}) = \frac{[\![\frac{n}{2}]\!]+1}{n}.$$

It is interesting that the breakdown point for the Michaelis-Menten model is 1/n larger than for the straight-line regression model. The Michaelis-Menten model is constrained to pass through (0,0), and this in effect gives an extra "good point".

Next we consider the class of models with the form

$$h(x,\theta) = g(\alpha + \beta x),\ \alpha \in \mathbb{R},\ \beta \in \mathbb{R},\ \text{and}\ \theta = (\alpha,\beta)^T$$

where g is monotonically increasing in x. (Since both g and y can be multiplied by -1, the results presented here hold for montonically decreasing g as well.) Models of this type fit into one of the following subclasses:

G_a; where $\lim_{x \to -\infty} g_a(x) = -\infty$ and $\lim_{x \to \infty} g_a(x) = \infty$ if $g_a \in G_a$.

G_b; where $\lim_{x \to -\infty} g_b(x) = \underline{g}_b > -\infty$ and $\lim_{x \to \infty} g_b(x) = \infty$ if $g_b \in G_b$.

G_c; where $\lim_{x \to -\infty} g_c(x) = \underline{g}_c > -\infty$ and $\lim_{x \to \infty} g_c(x) = \bar{g}_c < \infty$ if $g_c \in G_c$.

Note that for g_b, we can subtract $\underline{g}_b$ from y and g, thus we can take $\underline{g}_b = 0$. Also, for g_c, we can subtract $\underline{g}_c$ from y and divide by ($\bar{g}_c$ - g), thus we can take $\underline{g}_c = 0$ and $\bar{g}_c = 1$. These standardizations will be used in the following theorems.

For models in the class G_a, we can establish the breakdown point exactly by applying the corollary to Theorem 3.

<u>Theorem 4</u>

For a regression function g_a in G_a,

$$\epsilon_n'(g_a,\hat{\theta}_n^{LTS}) = \epsilon_n'(g_a,\hat{\theta}_n^{LMS}) = [\![\tfrac{n}{2}]\!]/n.$$

$G_b(x)$ and $G_c(x)$ are classes of models where the breakdown point depends on how good a fit exists for the original data - at least there exists upper and lower bounds depending upon the goodness of fit. In Theorems 5 and 6, bounds are established for $\epsilon_n'(g_b,\hat{\theta}_n^{LMS}(\chi_n))$ and $\epsilon_n'(g_b,\hat{\theta}_n^{LTS}(\chi_n))$ when $g_b \in G_b$.

Theorem 5

For a regression function g_b in G_b, let M_1 be the maximum integer m such that

$$\inf_{\theta \text{ in } \Theta} r^2_{(k+m)}(\theta) < y^2_{([\![\frac{n}{2}]\!]-m)}.$$

Then, if $x_1,\ldots,x_n$ are all distinct,

$$\epsilon'_n(g_b,\hat{\theta}_n^{LMS}(\chi_n)) > \frac{M_1}{n}. \tag{4.2}$$

Let M_2 be the minimum integer m such that

$$r^2_{(k)}(\hat{\theta}_n^{LMS}) > y^2_{(k - m)}. \tag{4.3}$$

Then

$$\epsilon'_n(g_b,\hat{\theta}_n^{LMS}(\chi_n)) \le \frac{M_2}{n}. \tag{4.4}$$

Theorem 6

For g_b in G_b, Let M_1 be the maximum integer m such that

$$\inf_{\theta} \sum_{i=1}^{(k+m)} r^2_{(i)}(\theta) < \sum_{i=1}^{([\![\frac{n}{2}]\!]-m)} y^2_{(i)}.$$

Then, if $x_1,\ldots,x_n$ are all distinct,

$$\epsilon'_n(g_b,\hat{\theta}_n^{LTS}(\chi_n)) > \frac{M_1}{n}.$$

Let M_2 be the minimum integer m such that

$$\sum_{i=1}^{k} r^2_{(k)}(\hat{\theta}_n^{LTS}) > \sum_{i=1}^{k-m} y^2_{(i)}.$$

Then

$$\epsilon'_n(g_b,\hat{\theta}_n^{LTS}(\chi_n)) \le \frac{M_2}{n}.$$

Remark: Consider fixed x_i and θ. As y_i converges to $g_b(x_i,\theta)$ for $i = 1,\ldots,n$, eventually M_1 equals $[\![\frac{n}{2}]\!] - 1$ and M_2 equals $[\![\frac{n}{2}]\!]$ in Theorems 5 and 6, and therefore

$$\epsilon_n'(g_b,\hat{\theta}_n^{LMS}(\chi_n)) = \epsilon_n'(g_b,\hat{\theta}_n^{LTS}(\chi_n)) = [\![\tfrac{n}{2}]\!].$$

Thus, if the uncontaminated data fit well, one expects a high breakdown point .

In Theorems 7 and 8, we establish bounds for $\epsilon_n'(g_c,\hat{\theta}_n^{LMS}(\chi_n))$ and $\epsilon_n'(g_c,\hat{\theta}_n^{LTS}(\chi_n))$ when g_c is in G_c.

<u>Theorem 7</u>

Suppose that $0 < y_i < 1$ for all i and that $x_1,\ldots,x_n$ are all distinct. Let g_c be a continuous and strictly increasing regression model in G_c. Define $z_i = \min(y_i, (1 - y_i))$ and let M_1 be the maximum integer $m < k-1$ such that

$$\inf_{\theta} r^2_{(k+m)}(\theta) < z^2_{([\![\frac{n}{2}]\!]-m)}.$$

Then

$$\epsilon_n'(g_c,\hat{\theta}_n^{LMS}(\chi_n)) > \frac{M_1}{n}.$$

Define $v_i^2(x) = (y_i - I_{\{x_i \geq x\}})^2$. Let $\{v^2_{(i)}(x)\}$ be the ordered values of $\{v_i^2(x)\}$. Fix x^* and let $M_2(x^*)$ be the smallest integer $m < k$ such that

$$v^2_{(k-m)}(x^*) < \inf_{\theta} r^2_{(k-[\![\frac{m+1}{2}]\!])}(\theta). \tag{4.5}$$

Then

$$\epsilon_n'(x^*, g_c,\hat{\theta}_n^{LMS}) \leq \frac{M_2(x^*)}{2}.$$

Remark: If $\epsilon \leq y_i \leq 1 - \epsilon$ and $\left|r_i(\hat{\theta}_n^{LMS})\right| < \epsilon$ for some $\epsilon > 0$ and all i, then $M_1 = [\![\frac{n}{2}]\!] - 1$ and the breakdown point is $[\![\frac{n}{2}]\!] / n$.

Theorem 8

Suppose that $0 < y_i < 1$ for all i and that $x_1,\ldots,x_n$ are all distinct. Let g_c be a continuous and strictly increasing regression model in G_c. Define $z_i = \min(y_i, (1 - y_i))$ and let M_1 be the maximum integer $m < k-1$ such that

$$\inf_{\theta} \sum_{i=1}^{k+m} r^2_{(i)}(\theta) < \sum_{i=1}^{[\![\frac{n}{2}]\!]-m} z^2_{(i)}.$$

Then

$$\epsilon_n'(g_c,\hat{\theta}_n^{LTS}(\chi_n)) > \frac{M_1}{n}.$$

Define $v_i^2(x) = (y_i - I_{\{x_i \geq x\}})^2$. Let $\{v^2_{(i)}(x)\}$ be the ordered values of $\{v_i^2(x)\}$. Fix x^* and let $M_2(x^*)$ be the smallest integer $m < k$ such that

$$\sum_{i=1}^{k-m} v^2_{(i)}(x^*) < \inf_{\theta} \sum_{i=1}^{k-[\![\frac{m+1}{2}]\!]} r^2_{(i)}(\theta). \tag{4.9}$$

Then

$$\epsilon_n'(x^*, g_c,\hat{\theta}_n^{LTS}) \leq \frac{M_2(x^*)}{2}.$$

Remark: As in the remark following Theorem 6, it is possible to establish that the exact breakdown point is $[\![n/2]\!]/n$ whenever $M_1 = [\![n/2]\!] - 1$. When $M_1 < [\![n/2]\!] - 1$, none of Theorems 5-8 can be used to find the exact breakdown point because taking $M_1 + 1 = M_2$ in any of them would lead to a contradiction. To see this, suppose that the conditions of Theorem 5 hold and that $M_1 < [\![n/2]\!] - 1$. Let $M_1 + 1 = M_2 = M$. By the first equation in Theorem 5,

$$\inf_{\theta \text{ in } \Theta} r^2_{(k+M-1)}(\theta) < y^2_{([\![\frac{n}{2}]\!]-M+1)} = y^2_{(k-M)} \tag{4.10}$$

By (4.3)

$$r^2_{(k)}(\hat{\theta}_n^{LMS}) > y^2_{(k-M)}. \tag{4.11}$$

Combining (4.10) and (4.11)

$$\inf_{\theta \text{ in } \Theta} r^2_{(k+M-1)}(\theta) < r^2_{(k)}(\hat{\theta}_n^{LMS})$$

This cannot happen so $M_1 + 1 < M_2$ implying that the breakdown bounds cannot establish the exact breakdown point. Considering Theorems 6-8 similarly justifies the remark.

5. AN EXAMPLE

Theorem 4 indicates that when the regression function is unbounded above and below, breakdown for both $\hat{\theta}_n^{LMS}$ and $\hat{\theta}_n^{LTS}$ occurs in much the same way that it does in linear regression. Theorems 5 through 8 make it clear that the breakdown point for high breakdown estimators for bounded regression functions depends on the behavior of the data points that are near the boundary of the regression function. We will use an artificial data set and the least median of squares estimator to illustrate the situation. Suppose we wish to fit the continuous logistic regression model given by

$$g_c(\alpha + \beta x) = \frac{1}{1+\exp(-(\alpha+\beta x))}$$

to the following data set:

x:	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
y:	.0219	.0001	.1064	.1738	.3315	.4893	.6899	.8075	.9136	.9999	.9999	.9803	.9998

Because of the number of points that are near zero or one, transforming to linear regression will induce outliers, thus we chose not to transform the data. Using computational methods developed by the authors (Stromberg 1989) and Theorems 7, we found the breakdown function bounds for $\hat{\theta}_n^{LMS}$ given in Figure 5.1. The upper bound on the breakdown function indicates that modifying as few as four points will cause breakdown of $\hat{\theta}_n^{LMS}$. As an example, we will show one way to cause breakdown at x=-0.5. (Depicted by the dashed line in Figure 5.1.) Move the first and fourth data points to $(-0.5 - \frac{1}{s}, 0)$. Then move the 9^{th} and 12^{th} data points to $(-0.5 + \frac{1}{s}, 1)$ and let $s \to \infty$. Note that breakdown was achieved in this example by allowing the modified points to remain in [0,1]. The least squares estimator will break down if one y value approaches infinity, but it requires modifying n−1 points with range in [0,1] to break down the least squares estimator.

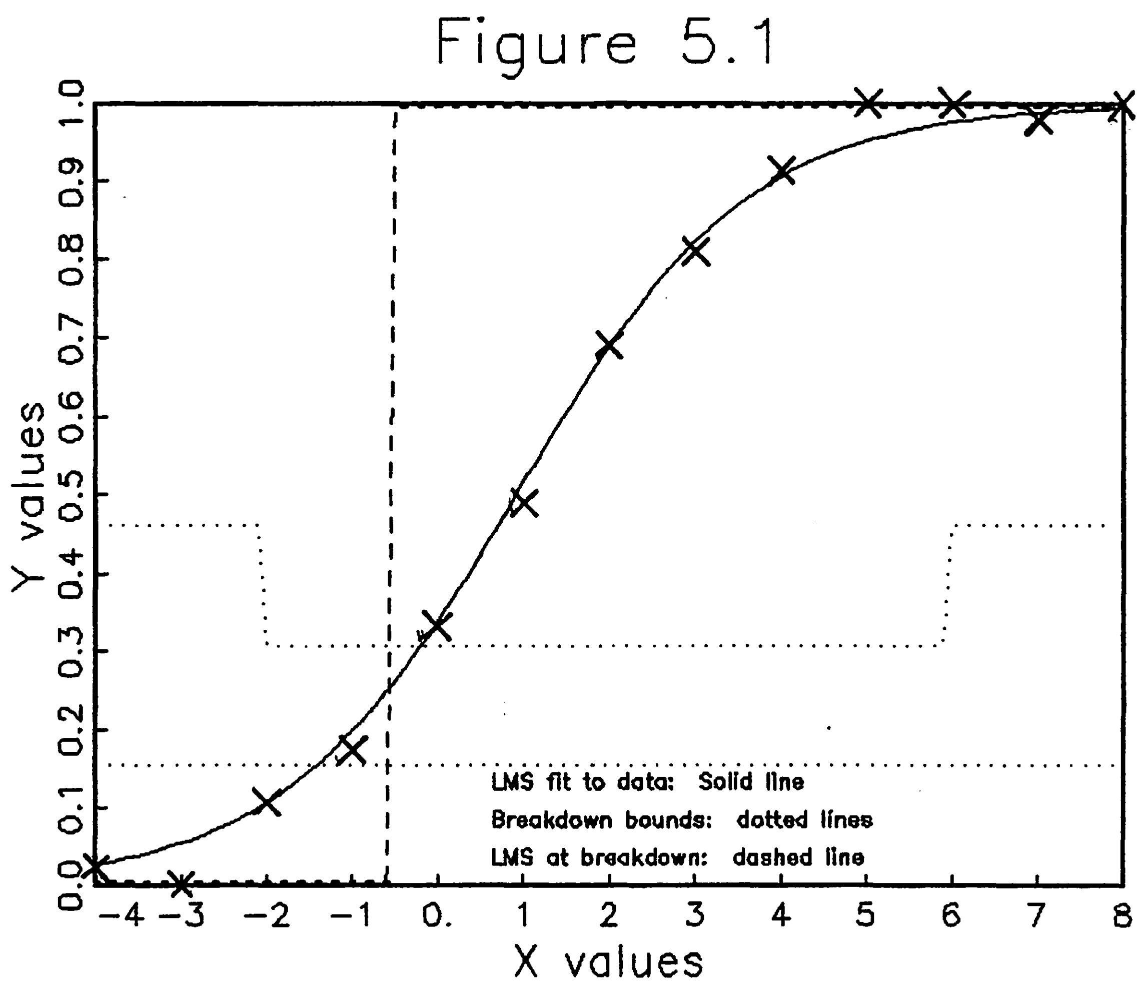

Figure 5.1
Y values
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
−4
−3
−2
−1
0.
1
2
3
4
5
6
7
8
X values
LMS fit to data: Solid line
Breakdown bounds: dotted lines
LMS at breakdown: dashed line

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APPENDIX: PROOFS

Proof of Theorem 1

Claim 1: $\epsilon'_n(\hat{\theta}_n) \geq \epsilon^*_n(\hat{\theta}_n)$.

Suppose that $\epsilon'_n(\hat{\theta}_n) = \frac{m}{n}$. Then for some $x \neq 0$,

$$\sup_{\chi_n^m} \left| x^T\hat{\theta}_n(\chi_n^m) - x^T\hat{\theta}_n(\chi_n) \right| = \infty,$$

where the sup is over D_n^m. This implies

$$\sup_{\chi_n^m} \|x^T\| \left\| \hat{\theta}_n(\chi_n^m) - \hat{\theta}_n(\chi_n) \right\| = \infty,$$

but $\|x^T\| < \infty$ so

$$\sup_{\chi_n^m} \left\| \hat{\theta}_n(\chi_n^m) - \hat{\theta}_n(\chi_n) \right\| = \infty.$$

This implies claim 1.

Claim 2: $\epsilon'_n(\hat{\theta}_n) \leq \epsilon^*_n(\hat{\theta}_n)$.

In order to verify the claim, we need only show $\epsilon'_n(x,\hat{\theta}_n) \leq \epsilon^*_n(\hat{\theta}_n)$ for some x. Suppose that $\epsilon^*_n(\hat{\theta}_n) = \frac{m}{n}$, then

$$\sup_{\chi_n^m} \left\| \hat{\theta}_n(\chi_n^m) - \hat{\theta}_n(\chi_n) \right\| = \infty.$$

Thus, for at least one coordinate of $\hat{\theta}_n(\chi_n^m)$, denoted $\theta_{nj}(\chi_n^m)$,

$$\limsup_{\chi_n^m} \left| \theta_{nj}(\chi_n^m) \right| = \infty.$$

Let x^1 be a p-vector of $p-1$ zeros and one 1 at coordinate j. Note that

$$\sup_{\chi_n^m} \left| x^{1T}\hat{\theta}_n(\chi_n^m) - x^{1T}\hat{\theta}_n(\chi_n) \right| = \infty,$$

thus $\epsilon'_n(x^1,\hat{\theta}_n) \leq \frac{m}{n}$, so claim 2 is verified.

By claims 1 and 2, $\epsilon'_n(\hat{\theta}_n) = \epsilon^*_n(\hat{\theta}_n)$. □

Proof of Theorem 2

Fix x^* that is nontrivial. To prove the theorem, it suffices to show that

$$\epsilon_n'(x^*, h, \hat{\theta}_n^{LS}(\chi_n)) = \frac{1}{n}. \tag{A.1}$$

Define

$$J = \sup_{\theta \in \Theta} \mid h(x^*,\theta) - h(x^*,\hat{\theta}_n^{LS}(\chi_n)) \mid .$$

$J > 0$ since x^* is nontrivial. Fix ϵ in the interval $(0,J)$ and define

$$\begin{aligned} J(\epsilon) &= J - \epsilon \quad \text{if } J < \infty \\ &= 1/\epsilon \quad J = \infty. \end{aligned}$$

Define

$$C(\epsilon) = \{ \theta : \mid h(x^*,\theta) - h(x^*,\hat{\theta}_n^{LS}(\chi_n) \mid \leq J(\epsilon) \}.$$

Note that $C(\epsilon)$ is nonempty. Since ϵ is arbitrary, to prove (A.1) it suffices to prove the existence of y^* with the following property: If we replace one observation, say (x_1,y_1), by (x^*,y^*), then the resulting least-squares estimator, which exists by (3.1), is *not* in $C(\epsilon)$.

Fix $\theta_\epsilon \notin C(\epsilon)$. Either

$$h(x^*,\theta_\epsilon) > h(x^*, \hat{\theta}_n^{LS}(\chi_n)) + J(\epsilon) \tag{A.2}$$

or $h(x^*,\theta_\epsilon) < h(x^*, \hat{\theta}_n^{LS}(\chi_n)) - J(\epsilon)$. Without loss of generality, assume (A.2). Then as $y \to \infty$,

$$\inf_{\theta \in C(\epsilon)} [y - h(x^*,\theta)]^2 - [y - h(x^*,\theta_\epsilon)]^2 \to \infty. \tag{A.3}$$

By (A.3) there exists y^* such that

$$\begin{aligned} &\inf_{\theta \in C(\epsilon)} [y^* - h(x^*,\theta)]^2 \\ &> [y^* - h(x^*,\theta_\epsilon)]^2 + \sum_{i=2}^{n} [y_i - h(x_i,\theta_\epsilon)]^2, \end{aligned}$$

which proves that $\hat{\theta}_n^{LS}(\{(y^*,x^*), (y_2,x_2),\ldots,(y_n,x_n)\})$ is not in $C(\epsilon)$. □

Proof of Theorem 3

For all θ

$$\operatorname*{med}_{1\le i\le n} \tilde{r}_i^2(\theta) \ge \frac{1}{k} \sum_{i=1}^{k} \tilde{r}_{(i)}^2(\theta). \tag{A.4}$$

Notice that $\{ \tilde{r}_{(i)}^2(\theta)$: $i = 1,2, \ldots ,k\}$ contains at least (k−m) elements of $\{r_{(i)}^2(\theta);\ i = 1,\ldots,n\}$ and therefore, since $(k - m) + \text{card}(\tau_m) = n + 1$, at least one element of $\{r_i^2(\theta)\text{: } i\in \tau_m\}$. Therefore, by (A.4) and assumption (2)

$$\lim_{M\to\infty}\left\{ \inf_{\theta:\, |\, h(x,\theta)\, |\, \ge M} \left\{ \operatorname*{med}_{1\le i\le n} \tilde{r}_i^2(\theta) \right\}\right\} = \infty. \tag{A.5}$$

This implies that

$$h(x,\hat{\theta}_n^{LMS}(\chi_n^m))$$

remains in a compact set as χ_n^m varies over D_n^m. But since by assumption (1),

$$\sup_{\theta} \left| h(x,\theta) - h(x,\hat{\theta}_n^{LMS}(\chi_n))\right| = \infty, .$$

so that m points cannot cause breakdown. Therefore,

$$\epsilon_n'(x,h,\hat{\theta}_n^{LMS}) \ge \frac{m+1}{n}. \tag{A.6}$$

Replacing $\operatorname*{med}_{1\le i\le n}$ with $\sum_{i=1}^{k}$ in (A.4) and (A.5),

$$\epsilon_n'(x,h,\hat{\theta}_n^{LTS}) \ge \frac{m+1}{n}.$$

Using (A.6),

$$\epsilon_n'(x,h,\hat{\theta}_n^{LTS}) = \epsilon_n'(x,h,\hat{\theta}_n^{LMS}) \qquad \square$$

Proof of Corollary 1

Definition of $\epsilon_n'(h,\hat{\theta}_n^{LTS})$ and $\epsilon_n'(h,\hat{\theta}_n^{LTS})$. □

Proof of Theorem 4

We can use the corollary to Theorem 3 to establish the breakdown point for models of the form $g_a(x)$. For any nontrivial x, (1) of Theorem 3 is satisfied by the definition of $g_a(x)$. Since $x_1,\ldots,x_n$ are distinct, if $|g_a(x,\theta_j)| \underset{j\to\infty}{\to} \infty$, then for all $i \in \{1,2,\ldots,n\}$ except possibly one, $|g_a(x_i,\theta_j)| \underset{j\to\infty}{\to} \infty$. This implies that (2) of Theorem 3 is satisfied with $m = [\![\frac{n}{2}]\!] - 1$. Thus the corollary implies

$$\epsilon_n'(g_a,\hat\theta_n^{LTS}) = \epsilon_n'(g_a,\hat\theta_n^{LMS}) \geq [\![\tfrac{n}{2}]\!]/n.$$

By using one original point, breakdown can be caused by modifying $[\![\frac{n}{2}]\!]$ points thus the conclusion of the theorem holds. □

Proof of Theorem 5

For any nontrivial fixed x, assume that modifying M_1 points will cause breakdown. Thus there exists a sequence of modified data sets χ_n^{m1}, χ_n^{m2},... determining LMS estimators $\hat\theta_1,\hat\theta_2,\ldots$ such that

$$g_b(\hat\alpha_s+\hat\beta_s x) \to \infty \text{ as } s \to \infty.$$

Consider the residuals in the modified data sets. Since the x_i's are distinct, as $s \to \infty$ at most one of the original data points can have zero residual though which point could depend on s. For each of the other data points, $|\hat\alpha_s+\hat\beta_s x_i|$ is arbitrarily large, so that the squared residual approaches y_i^2 or ∞. Thus, as $s \to \infty$, the smallest possible set of ordered squared residuals are M_1+1 zeros, then $y_{(1)}^2$, $y_{(2)}^2,\ldots$. Thus, as $s \to \infty$ $\tilde r_{(k)}^2(\theta) \geq y^2_{([\![\frac{n}{2}]\!]-M_1)}$. But, by assumption,

$$\inf_{\theta \text{ in } \Theta} r^2_{(k+M_1)}(\theta) < y^2_{([\![\frac{n}{2}]\!]-M_1)},$$

and thus altering M_1 points can not cause breakdown. Therefore, for all nontrivial fixed x,

$$\epsilon_n'(x,g_b,\hat\theta_n^{LMS}(\chi_n)) > \frac{M_1}{n}, \text{ and thus } \epsilon_n'(g_b,\hat\theta_n^{LMS}(\chi_n)),$$

proving (4.2).

To prove (4.4), first note that by (4.3) there exists $\epsilon > 0$ such that

$$r^2_{(k)}(\hat\theta_n^{LMS}) > y^2_{(k-M_2)} + 2\epsilon. \tag{A.7}$$

Without loss of generality we can and will assume that $x_i > 0$ for all i, because adding a constant to each x_i is merely a reparametrization. Now suppose we replace M_2 points by $\{(x_i^*, y_i^*)\}$ where for some s,

$$x_i^* = -i \text{ and } y_i^* = g_b(-s\, x_i^*), \text{ for } i = 1,\ldots, M_2.$$

Now let $\theta_s^* = (0, -s)^T$. Then for all large s, we have

$$\tilde{r}^2_{(k)}(\theta_s^*) \leq y^2_{(k-M_2)} + \tfrac{\epsilon}{2}, \tag{A.8}$$

since M_2 residuals from θ_s^* are 0, and for $x > 0$, $g_b(x,\theta_s^*)$ tends to 0 as $s \to \infty$.

Let $\hat{\theta}_s$ be the LMS estimator for the contaminated data, which depends upon s. Now suppose that breakdown does not occur as $s \to \infty$. This implies that as $s \to \infty$, $g_b(x_i^*,\hat{\theta}_s)$ stays bounded for each $i=1,\ldots, M_2$. Then for any $\epsilon > 0$,

$$\tilde{r}^2_{(k)}(\hat{\theta}_s) \geq r^2_{(k)}(\hat{\theta}_s) - \epsilon \text{ for all large s}, \tag{A.9}$$

because the residuals of (y_i^*, x_i^*) from $\hat{\theta}_s$ tend to ∞. Since $\hat{\theta}^{LMS}$ minimizes $r^2_{(k)}(\theta)$, using (A.7), (A.8), and (A.9) we obtain,

$$\tilde{r}^2_{(k)}(\hat{\theta}_s) \geq r^2_{(k)}(\hat{\theta}^{LMS}) - \epsilon \geq y^2_{(k-M_2)} + \epsilon \geq \tilde{r}^2_{(k)}(\theta_s^*) + \tfrac{\epsilon}{2},$$

which is a contradiction to the fact that $\hat{\theta}_s$ is the LMS estimator for the contaminated data. Therefore, breakdown does occur. □

<u>Proof of Theorem 6</u>

The proof follows by making the following modifications to the proof of Theorem 5:

Replace $r^2_{(k)}(\theta)$ with $\sum_{i=1}^{k} r^2_{(i)}(\theta)$.

Replace $\tilde{r}^2_{(k)}(\theta)$ with $\sum_{i=1}^{k} \tilde{r}^2_{(i)}(\theta)$.

Replace $r^2_{(k+M_1)}(\theta)$ with $\sum_{i=1}^{k+M_1} r^2_{(i)}(\theta)$.

Replace $r^2_{(k-M_2)}(\theta)$ with $\sum_{i=1}^{k-M_2} r^2_{(i)}(\theta)$.

Proof of Theorem 7

The lower bound is established by making minor modifications to the proof of the lower bound in Theorem 5.

To establish the upper bound, let $x_{.5}$ be the solution to

$$.5 = g_c(\hat{\alpha}_n^{LMS}(\chi_n) + \hat{\beta}_n^{LMS}(\chi_n)x_{.5}).$$

Assume that $\hat{\beta}_n^{LMS}(\chi_n) \geq 0$. (The case $\hat{\beta}_n^{LMS}(\chi_n) < 0$ is analogous and will not be covered.)

Assume that $x^* \leq x_{.5}$, since the other case is analogous. To simplify notation, let $M_2 = M_2(x^*)$. Consider a sequence $\{\chi_n^{M_2,i}: i = 1, 2,\dots\}$ of perturbed data sets and corresponding LMS estimators $\hat{\theta}_1, \hat{\theta}_2,\dots$. Since $x^* \leq x_{.5}$, breakdown occurs at x^* if

$$g_c(\hat{\alpha}_s + \hat{\beta}_s x^*) \to 1.$$

Fix $x^{**} < x^*$ so that there are no x's in $[x^{**}, x^*)$.

The s^{th} perturbed data set is constructed as follows: Take M_2 of the original observations not corresponding to $v^2_{(1)}(x^{**}), \dots, v^m_{(k-M_2)}(x^{**})$. Let $[\![M_2/2]\!]$ of their replacements have x's in $(x^{**}, x^{**}-\frac{1}{s})$ and y values equal to zero. Let the remaining replacements have x's in $(x^{**}, x^{**}+\frac{1}{s})$ and y values equal to 1.

Now suppose that as $s \to \infty$,

$$g_c(\hat{\alpha}_s + \hat{\beta}_s x) \to \begin{cases} 1 & \text{if } x > x^{**} \\ 0 & \text{if } x < x^{**} \end{cases}, \tag{A.10}$$

which implies breakdown at x in $(x^{**}, x_{.5}]$ and in particular at x^*. Because g_c is continuous and stictly increasing from 0 to 1, one can always find $(\hat{\alpha}_s, \hat{\beta}_s)$ so that (A.10) holds.

Then the residuals from $\hat{\theta}_s$ tend to 0 at the perturbed data points, and therefore

$$\tilde{r}_{(k)}(\hat{\theta}_s) \to v^2_{(k-M_2)}(x^{**}) = v^2_{(k-M_2)}(x^*), \tag{A.11}$$

where the equality holds because none of the original x values are in $[x^{**}, x^{*})$.

On the other hand, if (A.10) does not hold, then at least $[\![\frac{M_2}{2}]\!]$ of the absolute residuals will have a lim inf of at least 1/2 as $s \to \infty$. Therefore, since the original y's are in (0, 1), for all large s at most $[\![\frac{M_2+1}{2}]\!]$ of the perturbed points correspond to

$$\tilde{r}^2_{(1)}(\hat{\theta}_s), \ldots, \tilde{r}^2_{(k)}(\hat{\theta}_s).$$

Consequently,

$$\tilde{r}^2_{(k)}(\hat{\theta}_s) \geq \inf_{\theta} r^2_{(k-[\![\frac{M_2+1}{2}]\!])}(\theta) \tag{A.12}$$

for large s.

By (A.11) and (A.12), (4.5) implies (A.10) and therefore breakdown at x^*. □

<u>Proof of Theorem 8</u>

By making modifications similar to those used to prove Theorem 6, the proof follows from the the proof of Theorem 7.