Measurable, Nonleavable Gambling Problems

by

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Abstract

The optimal return function for a Borel measurable gambling problem with a bounded utility function was shown by Strauch (1967) to be universally measurable when the problem is leavable in the sense that the gambler may terminate play at any time. The same is shown here for the more general class of nonleavable problems.

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1. Introduction. The gambling theory of Dubins and Savage [8] takes place in a very general finitely additive framework in which a player is not restricted to measurable strategies. Thus the optimal return function V assigns to each fortune x the supremum of the utilities $u(\sigma)$ taken over all strategies σ available including nonmeasurable ones. Our main concern in this paper will be Borel and, more generally, analytic, gambling problems which are measurable and countably additive in a sense to be made precise in the next section. For such problems, it is natural to inquire, as Dubins and Savage did, whether the function V is measurable and whether it equals the function V_M which assigns to each x the supremum of $u(\sigma)$ taken over only the measurable σ available at x.

These questions were first considered for the class of leavable problems in which a player can effectively stop at any fortune x because the Dirac delta measure $\delta(x)$ is available there. Dubins and Savage themselves gave positive answers in the leavable case under assumptions of compactness and continuity [8, Theorem 2.16.1]. Strauch [18] formulated the notion of a Borel measurable gambling problem and again found positive answers. Later these results were generalized to the class of analytic gambling problems by Dellacherie, Meyer and Traki [7,13] and by Dubins and Sudderth [9].

The major result here is that V is measurable and $V=V_M$ for analytic problems in the general, nonleavable case. Another result is a new characterization of V and a transfinite inductive scheme for calculating it.

2. Preliminaries. Let (F,Γ,u) be a gambling problem in the sense of Dubins and Savage [8]. That is, the fortune space F is a nonempty set; the gambling house Γ is a mapping which assigns to each $x \in F$ a nonempty set, $\Gamma(x)$, of finitely additive probability measures defined on all subsets of F; and the utility

function u is a bounded, real-valued function with domain F. A strategy σ available at x is a sequence σ_0, $\sigma_1, \ldots$ such that $\sigma_0 \in \Gamma(x)$ and, for $n \geq 1$, σ_n is a mapping with domain F^n such that $\sigma_n(x_1,\ldots,x_n) \in \Gamma(x_n)$ for every $(x_1,\ldots,x_n) \in F^n$ [8, pp.11-12]. Dubins and Savage show [8, section 2.8] that every strategy σ determines a finitely additive probability measure, also denoted by σ, on the algebra of clopen subsets of the history space $H = F \times F \times \ldots$. (Here F is given the discrete topology and H the product topology.) A gambler with initial fortune x may choose any σ available at x and the coordinate process $h = (h_1,h_2,\ldots)$ on H with distribution σ is then thought of as the gambler's sequence of fortunes.

Let us recall briefly the two general approaches taken by Dubins and Savage. In the first approach, a player starting at $x \in F$ selects a strategy σ available at x and a stop rule t [8, p.20]. The pair $\pi = (\sigma,t)$ is a policy available at x and the utility of π is

$$u(\pi) = u(\sigma,t) = \int u(h_t)d\sigma,$$

the expected utility under σ at the time of stopping. The optimal return function is defined to be

(2.1) $$U(x) = \text{maximum of } \sup u(\pi) \text{ and } u(x),$$

where the supremum is over all π available at x [8, section 2.10].

In the second approach, a player starting at x selects a strategy σ available there as before but is not allowed to stop. Rather the utility of σ is defined to be

$$u(\sigma) = \inf_s \sup_{t \geq s} u(\sigma,t) = \limsup_t u(\sigma,t),$$

where the "lim sup" is taken over the directed set of stop rules t. The optimal return function is now defined as

(2.2) $V(x) = \sup u(\sigma)$

where the supremum is over all σ available at x [7, pp. 39-41].

The second approach essentially includes the first. This is because the return function U for a problem (F,Γ,u) is equal to the return function corresponding to V for the problem (F, Γ', u) where $\Gamma'(x) = \Gamma(x) \cup \{\delta(x)\}$ for every x [8, corollary 3.3.3.].

Assume from now on that F is a Borel set by which we mean a Borel subset of a complete, separable metric space. Let $\mathcal{P}(F)$ be the collection of countably additive probability measures defined on the sigma-field $\underset{\sim}{B}(F)$ of Borel subsets of F. Then $\mathcal{P}(F)$ is also a Borel set when equipped with the usual weak star topology (See, for example, Parthasarathy [15, chapter 2] or Dellacherie and Meyer [7, chapter III, 60 to 62].) Next assume that, for every $x \in F$ and every $\gamma \in \Gamma(x)$, the measure γ is countably additive when restricted to $\underset{\sim}{B}(F)$ and, for simplicity, identify γ with its restriction to $\underset{\sim}{B}(F)$. Assume further that Γ is analytic in the sense that the set $\{(x,\gamma): \gamma \in \Gamma(x)\}$ is an analytic subset of $F \times \mathcal{P}(F)$. (Recall that an analytic set is the continuous image of a Borel set.) Finally assume that the utility function u is bounded and upper analytic in the sense that $\{x: u(x) > a\}$ is an analytic set for every real number a. A problem (F,Γ,u) satisfying the assumptions of this paragraph is called analytic. The class of such problems includes the Borel measurable problems of Strauch [18] and is essentially the same as the class studied by Dellacherie and Meyer [7] except that they assume leavability and allow u to be unbounded above. A related class of dynamic programming problems was investigated by Blackwell, Freedman, and Orkin [1].

A strategy $\sigma = (\sigma_0,\sigma_1,...)$ is called measurable if , for $n = 1,2,...$, the mapping σ_n: $F^n \to \mathcal{P}(F)$ is universally measurable; i.e. measurable with respect to the completion of every probability measure on $\underset{\sim}{B}(F^n)$. Every measurable strategy σ determines a countably additive probability measure $\mu(\sigma)$ on the sigma-field $\underset{\sim}{B}(H) = \underset{\sim}{B}(F) \times \underset{\sim}{B}(F) \times \ldots$ of Borel subsets of H. That is, the $\mu(\sigma)$ - marginal distribution of h_1 is σ_0 and, for every $n \geq 1$ and $(x_1,\ldots,x_n) \in F^n$, the $\mu(\sigma)$ - conditional distribution of h_{n+1} given $h_1 = x_1,\ldots,h_n = x_n$ is $\sigma_n(x_1,\ldots,x_n)$. For simplicity, σ is written for $\mu(\sigma)$ below. (For a measurable σ, it is natural to consider $\bar{u}(\sigma) = \lim_t \sup u(\sigma,t)$ where the "lim sup" is taken over Borel measurable stop rules t. However, $\bar{u}(\sigma) = u(\sigma)$ [20, Theorem 3.2].) For $x \in F$, let $\Sigma(x)$ be the collection of all measurable strategies σ available at x and define the optimal return from measurable strategies $V_M(x)$ as

$$V_M(x) = \sup\{u(\sigma): \sigma \in \Sigma(x)\}.$$

Here is our main result.

<u>Theorem</u> 2.1. If (F,Γ,u) is analytic, then $V = V_M$ and V is upper analytic.

Most of the paper is devoted to the proof which relies, in part, on a similar result about U.

A policy $\pi = (\sigma,t)$ is measurable if σ is measurable and the stop rule $t: H \to \{1,2,\ldots\}$ is Borel measurable. For each $x \in F$, let $\Pi(x)$ be the collection of measurable policies available at x and define

$$U_M(x) = \max\{u(x), \sup\{u(\pi): \pi \in \Pi(x)\}\}.$$

The next result was proved by Strauch [18] for Borel problems and by Dubins and Sudderth [9, section 6] for analytic problems. The essential elements of the proof are also in Dellacherie and Meyer [7] and a generalization is in Maitra, Purves and Sudderth [11, Theorem 4.8].

Theorem 2.2. If (F,Γ,u) is analytic, then $U = U_M$ and U is upper analytic.

The proof of Theorem 2.1 will also rely on the definition by induction over the ordinals of a collection of functions which decrease to V. Define first an operator T which assigns to every bounded function $w: F \to R$ the bounded function $Tw: F \to R$ where, for $x \in F$,

(2.3) $$(Tw)(x) = \sup w(\pi)$$

and the supremum is over all policies π available at x. Next define

(2.4) $$Q_0 = Tu$$

and, for every positive ordinal ξ, let

(2.5) $$Q_\xi = T(u \wedge (\inf\{Q_\eta : \eta < \xi\})).$$

Finally, set

(2.6) $$Q = \inf_\xi Q_\xi.$$

Similarly defined systems of functions were considered by Dellacherie [6].

The next section presents a theorem of Moschovakis from effective, descriptive set theory. In section 4 the theorem is applied to show that $Q = Q_{\omega_1}$ and Q is upper analytic. We show $V \le Q$ in section 5 and $V_M \ge Q$ in section 6. Obviously $V_M \le V$ so it will follow that $V=V_M=Q$. A characterization of V is given in section 7 and section 8 has some remarks and open questions.

3. A theorem of Moschovakis. The proof of Theorem 2.1 depends on a result from the theory of inductive definability. To formulate the result, let Z be a set and Φ be a mapping from subsets of Z to subsets of Z. Say that Φ is a monotone operator if, whenever $E_1 \subseteq E_2 \subseteq Z$, then $\Phi(E_1) \subseteq \Phi(E_2)$. Define the iterates of Φ

by transfinite induction as follows:

(3.1) $$\Phi^{\xi} = \Phi(\bigcup_{\eta<\xi} \Phi^{\eta})$$

where ξ is any ordinal. It is easy to verify that Φ^{∞}, the <u>least fixed point</u> of Φ, is given by $\cup\{\Phi^{\eta}:\eta < \kappa\}$, where κ is the least cardinal greater than the cardinality of Z.

Suppose Z is a Borel set and Φ is a monotone operator on Z. Say that Φ <u>respects</u> <u>coanalytic</u> <u>sets</u> if, whenever Y is a Polish space and C is a coanalytic subset of $Y \times Z$, then the set

(3.2) $$C^{*} = \{(y,z) \in Y \times Z: z \in \Phi(C_y)\}$$

is also coanalytic. (Here $C_y = \{z: (y,z) \in C\}$.)

<u>Theorem</u> 3.1. Let Φ be a monotone operator on a Borel set Z and suppose Φ respects coanalytic sets. Then

(a) Φ^{∞} is a coanalytic subset of Z,

(b) $\Phi^{\infty} = \Phi^{\omega_1} = \bigcup_{\xi<\omega_1} \Phi^{\xi}$.

Part (a) is a special case of a very general result of Moschovakis [14, 7C.8, p.414]. Part (b) is not stated explicitly in [14], but it can be deduced from results there as was done by Louveau [10]. A related result is in Dellacherie [4].

4. <u>The function Q</u>. The following theorem states the properties of Q we will need to prove Theorem 2.1.

Theorem 4.1. The function Q equals Q_{ω_1}, is upper analytic, and satisfies the functional equation

$$(4.1) \qquad Q = T(u \wedge Q).$$

The proof will use several lemmas. The first concerns the operator T defined in (2.3). Notice that Tw differs from the return function U for the problem (F, Γ,w) only in that the maximum is not taken with the utility function w. Nevertheless it is not difficult to use Theorem 2.2 to establish an analogous result for Tw.

Lemma 4.2. If w is bounded and upper analytic, then so is Tw and

$$(Tw)(x) = \sup \{w(\pi): \pi \in \Pi(x)\},$$

for every $x \in F$.

Proof: Consider the problem (F_0,Γ_0,u_0) where $F_0 = F \times \{0,1\}$; $u_0(x,0) = \inf w$, $u_0(x,1) = w(x)$ for $x \in F$; $\Gamma_0(x,0) = \Gamma_0(x,1) = \{\gamma \times \delta(1): \gamma \in \Gamma(x)\}$ for $x \in F$ (i.e. the first coordinate moves according to a gamble available in Γ and the second moves to 1). It is easy to check that $(Tw)(x) = U_0(x,0)$. The lemma then follows from Theorem 2.2. □

Here is an immediate corollary of Lemma 4.2 and (2.5).

Corollary 4.3. For $0 \le \xi < \omega_1$, Q_ξ is upper analytic.

The previous lemma permits us to calculate Tw by taking the supremum just over the class of measurable policies $\pi = (\sigma,t)$. The next lemma records a nice result of Strauch [18, Theorem 2] which is then used to reduce the class even further.

Lemma 4.3. There is a countable set C of Borel measurable stop rules such that, for every Borel stop rule t and every probability measure $\sigma \in \mathcal{P}(H)$, there exist $t_1, t_2, \ldots$ in C satisfying $\sigma[\lim_n t_n = t] = 1$.

For each $x \in F$, let $\tilde{\Pi}(x)$ be the collection of policies (σ,t) in $\Pi(x)$ such that $t \in C$.

Lemma 4.4. For w bounded and upper analytic and $x \in F$,

$$(Tw)(x) = \sup\{w(\pi): \pi \in \tilde{\Pi}(x)\}.$$

Proof: Let $\pi = (\sigma,t) \in \Pi(x)$. By Lemma 4.3 there exist $t_n \in C$ such that $t_n \to t$ σ-almost surely. So t_n eventually equals t and, hence, $w(h_{t_n}) \to w(h_t)$ σ-almost surely. Thus, by the dominated convergence theorem,

$$w(\sigma,t) = \int w(h_t)d\sigma = \lim_n \int w(h_{t_n})d\sigma = \lim_n w(\sigma,t_n). \qquad \square$$

It is convenient to use Lemma 4.4 to define a new gambling house $\tilde{\Gamma}$ for which the operator T corresponds to the one-day optimal return function

$$(4.2) \qquad (\tilde{\Gamma}^1 w)(x) = \sup\{\gamma w: \gamma \in \tilde{\Gamma}(x)\}.$$

To define $\tilde{\Gamma}$, use $\gamma = \sigma h_t^{-1}$ to denote the distribution of h_t under σ and, for $x \in F$, set

$$(4.3) \qquad \tilde{\Gamma}(x) = \{\gamma: (\exists(\sigma,t) \in \tilde{\Pi}(x))(\gamma = \sigma h_t^{-1})\}.$$

The next lemma records an obvious fact for future reference.

Lemma 4.5. The operators T and $\tilde{\Gamma}^1$ agree on bounded, upper analytic w.

The last lemma of this section establishes that $\tilde{\Gamma}$ is an analytic house. Enumerate the elements of C as $r_1, r_2, \ldots$.

Lemma 4.6. For k = 1,2,..., the set

$$A_k = \{(x,\gamma,\sigma) \in F \times \mathcal{P}(F) \times \mathcal{P}(H): \sigma \in \Sigma(x),\ \gamma = \sigma h_{r_k}^{-1}\}$$

is analytic and so is the set $\{(x,\gamma): \gamma \in \tilde{\Gamma}(x)\}$.

Proof: That A_k is analytic follows from the facts that $\{(x,\sigma): \sigma \in \Sigma(x)\}$ is an analytic subset of $F \times \mathcal{P}(H)$ by a theorem of Dellacherie [5, Théorème 3] (cf. also Sudderth [19, Theorem 2.1]) and that the mapping $\sigma \to \sigma h_{r_k}^{-1}$ is Borel. Hence, the projection of A_k onto $F \times \mathcal{P}(F)$, namely the set

$$\tilde{\Gamma}_k = \{(x,\gamma) : (\exists\sigma \in \Sigma(x))(\gamma = \sigma h_{r_k}^{-1})\}, \tag{4.4}$$

is also analytic. Consequently,

$$\{(x,\gamma) : \gamma \in \tilde{\Gamma}(x)\} = \bigcup_k \tilde{\Gamma}_k$$

is analytic too. □

Assume for the rest of this section that $0 \le u \le 1$. Since u is bounded, there is no real loss of generality. Notice that the function Q_ξ now takes values in the unit interval also.

The completion of the proof of Theorem 4.1 will rely on Theorem 3.1. To apply the latter theorem, take I to be the unit interval, set Z equal to $F \times I$ and define Φ on the power set of Z by

$$\Phi(E) = \{(x,a) \in Z: \sup\{(\gamma\times\lambda)^*(E^c \cap G) : \gamma \in \tilde{\Gamma}(x)\} \le a\}, \tag{4.5}$$

where λ is Lebesgue measure on I, $(\gamma \times \lambda)^*$ is the outer measure associated with the product measure $\gamma \times \lambda$, and $G = \{(x,a) : u(x) > a\}$.

Lemma 4.7. Φ is monotone and respects coanalytic sets.

Proof: It is trivial to check that Φ is monotone. So let Y be a Polish space and let C be a coanalytic subset of $Y \times Z$. To see that the set C^* of (3.2) is coanalytic, define a Borel Markov kernel K on $\mathcal{P}(F) \times \underset{\sim}{B}(Z)$ by $K(\gamma,B)=(\gamma\times\lambda)(B)$. The mapping

$$(y,\gamma) \to K(\gamma, G \cap C_y^c)$$

is upper analytic since it is the composition of the Borel mapping

$$(y,\gamma) \to \delta(y) \times K(\gamma,.)$$

from $Y \times \mathcal{P}(F)$ into $\mathcal{P}(Y \times Z)$ with the upper analytic mapping

$$\mu \to \mu((Y \times G) \cap C^c)$$

from $\mathcal{P}(Y \times Z)$ into [0,1]. (cf. [1],[7] or [11]). Consequently, the set

$$C^* = \{(y,x,a): K(\gamma, G \cap C_y^c) \le a \text{ for all } \gamma \in \tilde{\Gamma}(x)\}$$

is coanalytic. □

To see how the operator Φ is related to the operator T, let $w: F \to [0,1]$ and define

$$E(w) = \{(x,a): w(x) \le a\}.$$

Lemma 4.8. If w is upper analytic, then $\Phi(E(w)) = \{(x,a): T(u \wedge w) \le a\}$.

Proof: For $\gamma \in \mathcal{P}(F)$ and $E = E(w)$,

$$(\gamma \times \lambda)(E^c \cap G) = \int \lambda((E^c \cap G)_x)\gamma(dx)$$

$$= \int (w \wedge u)(x)\gamma(dx).$$

The result follows from (4.5) and Lemma 4.5. □

Now notice that

$$\Phi^0 = \Phi(\phi) = \{(x,a) : \gamma u \le a \text{ for all } \gamma \in \tilde{\Gamma}(x)\}$$
$$= \{(x,a) : Q_0(x) \le a\},$$

so that, by Corollary 4.3, Lemma 4.8, and induction

$$\Phi^\xi = \{(x,a) : Q_\xi(x) \le a\} \tag{4.6}$$

for $0 \le \xi < \omega_1$.

Proof of Theorem 4.1: Let $w = \inf_{\xi<\omega_1} Q_\xi$ and $E = E(w) = \{(x,a): w(x) \le a\}$.

By (4.6) and Theorem 3.1(b),

$$E = \bigcup_{\xi<\omega_1} \Phi^\xi = \Phi^\infty . \tag{4.7}$$

So, by Theorem 3.1(a), Φ^∞ is coanalytic and, hence, w is upper analytic. Apply Φ to (4.7) and use Lemma 4.8 to obtain

$$\Phi^\infty = \{(x,a) : T(u \wedge w)(x) \le a\}$$
$$= \{(x,a) : Q_{\omega_1}(x) \le a\}.$$

Thus Q_{ω_1} is upper analytic. Apply Lemma 4.8 again, this time with $w = Q_{\omega_1}$, to see that $T(u \wedge Q_{\omega_1}) = Q_{\omega_1}$ and, hence, $Q = Q_{\omega_1}$. □

5. The proof that $V_M \ge Q$. To show that V_M dominates the function Q, it suffices, by Theorem 4.1, to establish the following result.

Theorem 5.1. If $L:F \to R$ is bounded, upper analytic, and $T(u \wedge L) \ge L$, then $V_M \ge L$.

The idea of the proof is to construct, for a given x, a strategy $\sigma \in \Sigma(x)$ whose utility $u(\sigma)$ is almost as large as $L(x)$. The construction will be based on two lemmas.

To state the first lemma, define a <u>measurable family of policies</u> to be a mapping π which assigns to each x a policy $\pi(x) = (\bar{\sigma}(x), t(x)) \in \Pi(x)$ in such a way that $t(x)(h)$ is jointly universally measurable in x and h and, for $n \geq 0$, $\bar{\sigma}(x)_n (h_1,\ldots,h_n)$ is jointly universally measurable in x and $(h_1,\ldots,h_n)$. Say that the family π ϵ - <u>conserves</u> L if

$$(u \wedge L)(\pi(x)) \geq L(x) - \epsilon$$

for all x.

<u>Lemma</u> 5.2. For every $\epsilon > 0$, there is a measurable family of policies π which ϵ - conserves L.

<u>Proof</u>: By Lemma 4.5 and the hypothesis $T(u \wedge L) \geq L$,

$$\tilde{\Gamma}^1(u \wedge L) \geq L.$$

So, by Lemma 6.4 of Dubins and Sudderth [9], there is a universally measurable mapping γ from F to $\mathcal{P}(F)$ such that $\gamma(x) \in \tilde{\Gamma}(x)$ and $\gamma(x)(u \wedge L) \geq L(x) - \epsilon$ for all $x \in F$. By the definition of $\tilde{\Gamma}$, $\gamma(x)$ corresponds to the distribution of $h_{t(x)}$ under some policy $\pi(x)$ in $\tilde{\Pi}(x)$. It remains to select such a policy measurably.

Let A_k be the analytic set of Lemma 4.6 and use the Yankov-Von Neumann selection theorem (cf.[1] or [13]) to get a universally measurable mapping

$$g_k : F \times \mathcal{P}(F) \to \mathcal{P}(H)$$

such that

$$(x,\gamma,g_k(x,\gamma)) \in A_k$$

for every (x,γ) in $\tilde{\Gamma}_k$, the projection of A_k onto $F \times \mathcal{P}(F)$. Then, for $x \in F$, let $k(x)$ be the least k such that $(x,\gamma(x)) \in \tilde{\Gamma}_k$ and, for $k = k(x)$, define

$$t(x) = r_k, \ \bar{\sigma}(x) = g_k(x,\gamma(x)).$$

Now let $\pi(x) = (\bar{\sigma}(x), t(x)$ for each x. By construction, $t(x)(h)$ is jointly universally measurable in x and h, and $\bar{\sigma}$ is a universally measurable mapping from F to $\mathcal{P}(H)$. It follows from Lemma 2.2 of [11] that the mappings $\bar{\sigma}(x)_n(h_1,\ldots,h_n)$ can be chosen to be universally measurable. □

Now fix $x_0 \in F$ and $\epsilon > 0$. To prove Theorem 5.1, it suffices to find $\sigma \in \Sigma(x_0)$ such that

(5.1) $$u(\sigma) \geq L(x_0) - \epsilon.$$

To obtain σ, first choose $\epsilon_0, \epsilon_1, \ldots$ to be positive numbers such that $\Sigma\epsilon_n < \epsilon$. Then, for each n, use Lemma 5.2 to get a measurable family of policies $\pi_n = (\sigma^n, t_n)$ which ϵ_n - conserves L. We will take σ to be the <u>sequential composition of the</u> π_n <u>starting from</u> x_0. Intuitively, σ follows $\sigma^0(x_0)$ up to time $t_0(x_0)$, then switches to $\sigma^1(h_{t_0}(x_0))$ and so on. To be precise, first define stop rules $s_0 < s_1 < \ldots$ by setting

$$s_0(h) = t_0(x_0)(h)$$

$$s_{n+1}(h) = s_n(h) + t_{n+1}(h_{s_n})(h_{s_n+1}, h_{s_n+2}, \ldots) .$$

Plainly the s_n are universally measurable. Now let

$$\sigma_0 = \sigma^0(x_0)_0$$

$$\sigma_n(h_1,\ldots,h_n) = \sigma^0(x_0)_n(h_1,\ldots,h_n) \text{ if } n < s_0(h),$$

$$= \sigma^{k+1}(h_{s_k})_{n-s_k}(h_{s_k+1},\ldots,h_n) \text{ if } s_k(h) \le n < s_{k+1}(h),$$

where $h = (h_1,\ldots,h_n,\ldots)$.

(The related notion of a "composite policy" is discussed in [8, p.22].)

The next lemma will establish (5.1) and complete the proof of Theorem 5.1.

Lemma 5.3. Let $x \in F$, let $\pi_0,\pi_1,\ldots$ be measurable families of policies and let σ be the sequential composition of the π_n at x. Assume $\epsilon_0,\epsilon_1,\ldots$ are positive numbers such that, for every n, π_n ϵ_n-conserves L. Then, for every stop rule s, there is a stop rule $t \ge s$ such

(5.2) $$u(\sigma,t) \ge L(x) - \Sigma\epsilon_n .$$

Proof: The proof is by induction on the structure of h_s and the inductive hypothesis is taken to include all strategies σ constructed by sequential composition.

If h_s has structure one, then $s = 1 \le t_0(x)$ and $t=t_0(x)$ will satisfy (5.2).

Suppose h_s has structure α and assume the inductive hypothesis for stop rules s' for which $h_{s'}$ has structure smaller than α. Define the stop rule t as follows: If $s(h) \le t_0(h)$, let $t(h) = t_0(h)$. If $s(h) > t_0(h)$, let $\bar{s}(h)$ be the conditional stop rule $s[p_{t_0}(h)] = s[h_1,\ldots,h_{t_0}]$ which is defined by

$$\bar{s}(h)(h') = s(h_1,\ldots,h_{t_0}, h_1',h_2',\ldots) - t_0(h).$$

Now the structure of $h_{\bar{s}(h)}$ is smaller than α (cf.[8, Theorem 2.9.3]). Apply the inductive hypothesis to the conditional strategy $\bar{\sigma}(h) = \sigma[p_{t_0}(h)]$, which is the sequential composition of $\pi_1,\pi_2,\ldots$ at h_{t_0}, to obtain a stop rule

$\bar{t}(h)$, depending only on $(\bar{\sigma}(h), \bar{s}(h))$, such that $\bar{t}(h) \geq \bar{s}(h)$ and

$$u(\bar{\sigma}(h), \bar{t}(h)) \geq L(h_{t_0}) - (\epsilon_1 + \epsilon_2 + \ldots).$$

Then set

$$t(h) = t_0(h) + \bar{t}(h)(h_{t_0+1}, h_{t_0+2}, \ldots).$$

Finally, condition on p_{t_0} to get

$$\begin{aligned} u(\sigma,t) &= \int_{s\leq t_0} u(h_{t_0})d\sigma + \int_{s>t_0} u(\bar{\sigma},\bar{t})\, d\sigma \\ &\geq \int (u\wedge L)(h_{t_0})d\sigma - (\epsilon_1 + \epsilon_2 + \ldots) \\ &= (u\wedge L)(\pi_0(x)) - (\epsilon_1 + \epsilon_2 + \ldots) \\ &\geq L(x) - (\epsilon_0 + \epsilon_1 + \ldots). \end{aligned}$$

□

6. <u>The proof that</u> $Q \geq V$.

It suffices to show that

$$V \leq Q_\eta \tag{6.1}$$

for every ordinal number η. The proof is by induction over the ordinals.

For every strategy σ available at x,

$$u(\sigma) \leq \sup_t u(\sigma,t) \leq Q_0(x).$$

Take the supremum over σ to see that $V \leq Q_0$.

Now assume (6.1) holds for every $\eta < \xi$. Let $\epsilon > 0$ and let σ be a strategy available at x. By a result of Dubins and Savage [8, Theorem 3.7.1], there is a stop rule t_0 such that for every stop rule $t \geq t_0$,

$$\sigma\{h: u(h_t) \geq V(h_t) + \epsilon\} < \epsilon.$$

Thus, for $t \geq t_0$,

$$\begin{aligned} u(\sigma,t) &\leq (u\wedge V)(\sigma,t) + \epsilon(1 + 2\sup|u|) \\ &\leq T(u\wedge \inf_{\eta<\xi} Q_\eta) + \epsilon(1 + 2\sup|u|), \end{aligned}$$

$$= Q_\xi(x) + \epsilon(1 + 2\sup|u|),$$

where the second line is by the inductive assumption and the third by (2.5). Since ϵ is arbitrary, $u(\sigma)$ is no larger than $Q_\xi(x)$. Consequently, $V(x)$ is also bounded above by $Q_\xi(x)$.

This completes the proof that $Q \geq V$. As mentioned in section 2, it now follows that $V = V_M = Q$ and Theorem 2.1 is immediate from Theorem 4.1.

7. A characterization of V. No assumptions of measurability or countable additivity are needed for the results of this section. So let (F,Γ,u) be a classical gambling problem in the sense of [8]. Dubins and Savage [8, pp. 41-42] characterized V as the least excessive function w such that $w(\sigma) \geq u(\sigma)$ for every strategy σ available. Here is a new characterization.

Theorem 7.1. The function V is the largest, bounded function $w: F \to R$ such that

$$T(u \wedge w) = w. \tag{7.1}$$

The proof uses two lemmas which may have some independent interest.

Lemma 7.2. The function Q is the largest, bounded function $w: F \to R$ such that (7.1) holds.

Proof: It is clear from (2.5) that $Q_\xi \leq Q_\eta$ when $\xi \leq \eta$. Thus, if κ is the least cardinal greater than the cardinality of R^F, then there is an ordinal α less than κ such that $Q_\alpha = Q_{\alpha+1}$ and $Q_{\alpha+1} = T(u \wedge Q_\alpha)$. Therefore $Q = Q_\alpha$ is a solution of (7.1).

Now let w be any solution of (7.1). Then $Q_0 = Tu \geq T(u \wedge w) = w$. And, if

$Q_\eta \geq w$ for all $\eta < \xi$, then

$$Q_\xi = T(u \wedge \inf_{\eta<\xi} Q_\eta) \geq T(u \wedge w) = w.$$

So $Q = \inf Q_\xi \geq w$. □

Lemma 7.3. The functions Q and V are the same.

Outline of proof: This was proved above for measurable problems. However, the proof in section 6 that $Q \geq V$ is completely general. Also, it is easy to adapt the proof in section 5 that $V_M \geq Q$ to show $V \geq Q$ in general. (Given $x \in F$ and $\epsilon > 0$, one constructs σ available at x such that $u(\sigma) > Q(x) - \epsilon$. The construction is similar to that in section 5, but somewhat simpler because there are no measurability concerns.) □

The theorem is immediate from the two lemmas. A result related to Theorem 7.1 is discussed by Dellacherie [4, Théorème 27].

8. Remarks. It seems likely that the results established here for a bounded utility function u are also true for $u \geq 0$. However, Theorem 2.1 cannot be proved for $u \leq 0$. As was shown in [11], the statement that $V = V_M$ is undecidable for $u \leq 0$ even in the special case when Γ is leavable.

If u is bounded, then, for every strategy σ,

$$u(\sigma) = \int u^* d\sigma$$

where $u^*(h) = \limsup u(h_n)$ as was shown by Chen [3] and Sudderth [20]. (For nonmeasurable σ, the integral above was defined in Purves and Sudderth [17].) Thus Theorem 2.1 says that, for a measurable problem, the supremum over σ of the integral of u^* with respect to σ is the same whether taken over all σ or only measurable σ available at x. Does this remain true when u^* is replaced by an arbitrary bounded, Borel measurable function g? Some further information about this question is in [12].

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