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MULTIPLICITY AND REPRESENTATION THEORY OF PURELY NON-DETERMINISTIC STOCHASTIC PROCESSES

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1. Introduction. In this paper we study the multiplicity theory of a wide class of purely non-deterministic weakly stationary processes and show how this theory provides a natural means of obtaining representations of continuous parameter processes that are extensions of the well known result due to K. Karhunen [10]. Our work can be described as a unified time domain analysis that applies equally to finite dimensional and infinite dimensional stationary processes. The earliest time domain analysis of a (univariate) continuous parameter weakly stationary process was made by O. Hanner in a remarkably original paper [6]. More recently, in the light of the extensive development of multidimensional stationary processes, it has appeared desirable to separate time domain studies from the spectral, and consequently, interest in the former has revived. As an example, we mention the paper of P. Masani and J. Robertson [11] whose approach makes extensive use of the Cayley transform associated with the unitary group of the process. The extension of this method to finite dimensional stationary processes has been carried out by J. Robertson in his thesis [14]. The earlier work of E. G. Gladyshev also belongs to the same order of ideas [5]. Hanner's paper, nevertheless, has remained an isolated piece of work and his method has apparently given the impression of being *ad hoc*. As a matter of fact, as we have shown elsewhere [9], Hanner's ideas reveal an intimate connection with multiplicity theory. Thus the generalization of Hanner's approach to multidimensional (even infinite dimensional) stationary processes is to be sought in the development of the multiplicity theory of the process, i.e., in the study of the self adjoint operator A of the process and its spectral types. This is one of the central problems discussed in this paper, in Sections 4, 5, and 6.

In recent years a theory of representation of purely non-deterministic (possibly non-stationary) processes has been introduced by H. Cramér and also by T. Hida ([1], [2], [3], [7]). Following the technique of the latter author it is easy to extend the main representation theorem of [7] to the processes considered by us. This is done in Sections 2 and 3. Our purpose in doing so is to compare the representation theorem of the Hida-Cramér theory (Theorem 2.2 of this paper) with the result of Section 5 which is essentially independent of Sections 2 and 3. The generalized Hanner approach leads naturally to a definition of multiplicity which is seen to be identical with the concept of the multiplicity of the process introduced by Hida. Indeed it is shown that every spectral type belonging to A has this multiplicity. Further discussion of this question is deferred to Section 5. Section 6 brings to light the natural role of multiplicity as a generalization of the rank of a stationary finite dimensional process.

We consider stochastic processes of the following kind.

Let Φ be a Hausdorff space satisfying the second countability axiom but otherwise arbitrary. We shall say that $\underline{x}_t$ $(-\infty < t < \infty)$ is a stochastic process on Φ if for each φ in Φ, $\underline{x}_t(\varphi)$ is a complex-valued random variable with mean zero and $\mathcal{E}|\underline{x}_t(\varphi)|^2$ finite. The process $\{\underline{x}_t\}$ $(-\infty < t < \infty)$ on Φ is called weakly stationary (or briefly, stationary) if for all φ, ψ in Φ and arbitrary real numbers s, t and τ we have

$$\mathcal{E}[\underline{x}_{t+\tau}(\varphi)\ \overline{\underline{x}_{s+\tau}(\psi)}] = \mathcal{E}[\underline{x}_t(\varphi)\ \overline{\underline{x}_s(\psi)}] .$$

The covariance function $\mathcal{E}[\underline{x}_t(\varphi)\ \overline{\underline{x}_s(\psi)}]$ of the process depends on t-s, φ and ψ . The definition of a discrete parameter process $\{\underline{x}_n\}$ is similarly given. It should be noted that the stationarity considered here is a temporal one and does not involve Φ . Nevertheless, it is sufficiently general and useful for our purpose since it includes as special cases many stationary random processes of practical interest. For instance, if Φ is a q-dimensional euclidean (or unitary) space and $\underline{x}_t(\varphi)$ is linear with respect to φ for each t , then the $\underline{x}_t$-process can be regarded as a q-vector stationary process (see [15]); if Φ is an infinite dimensional locally convex linear space and $\underline{x}_t(\varphi)$ is again supposed linear in φ (with probability one), then $\underline{x}_t$ is a weak stochastic process on Φ . On the other hand, stationary processes $\underline{x}_t$ as defined above include those that are not linear in φ (indeed Φ itself need not be a linear space). Such processes can serve as useful models for certain problems in meteorology (e.g. see [8]).

Associated with the $\underline{x}_t$ process (not assumed to be stationary) are the following spaces:

(a) the (Hilbert) space of the process $H(\underline{x})$, defined to be $\mathfrak{S}[\underline{x}_t(\varphi),$ $t \in T, \varphi \in \Phi]$, the subspace of $L_2(\Omega, P)$ generated by the family of random variables $\underline{x}_t(\varphi)$ as t and φ vary respectively over T and Φ ;

(b) the subspace $H(\underline{x}; t)$ of $H(\underline{x})$ given by $H(\underline{x}; t) = \mathfrak{S}[\underline{x}_\tau(\varphi)$, $\tau \leq t$, and $\varphi \in \Phi]$ for every real t .

We say that $\underline{x}_t$ on Φ is purely non-deterministic if $H(\underline{x};-\infty)$, the intersection of the subspaces $H(\underline{x}; t)$ for all $t \in T$ is trivial.

The process $\underline{x}_t$ is said to be deterministic if for each t $H(\underline{x}; t) = H(\underline{x}; -\infty)$.

In the concluding part of the paper we consider in greater detail Hilbert space valued processes since they are, perhaps, mathematically the simplest examples of infinite dimensional processes. If the process $\{\underline{x}_t\}$ is a weak process on the Hilbert space Φ its representation is already given by Theorem 5.1. In Sections 7, 8 and 9 we make the stronger assumption that for each t, $\underline{x}_t$ is a random element in the dual of a separable Hilbert space Φ, satisfying the further requirement that $\mathcal{E}\|\underline{x}_t\|^2$ is finite. Strengthened versions (involving random, Hilbert spaces valued integrals) of the representation theorems of Sections 2 and 5 are obtained in Section 9.

The problem of relating the multiplicity of a stationary process with spectral theory and of actually determining the multiplicity in concrete instances will be studied in a later paper.

THE HIDA-CRAMER THEORY.

2. Representations of stochastic processes on Φ.

Although our main interest will be in the study of continuous parameter weakly stationary processes we begin by considering representations of arbitrary second order purely non-deterministic processes $\underline{x}_t(\varphi)$ on Φ. It can be easily seen that the results stated in this section contain as special cases those of H. Cramér [2] and of T. Hida [7] (if Gaussian assumptions are made). They will, however, be stated without proof since they are proved by following essentially the method of the latter author. Our only reason for including them here is for the purpose of relating the representation and the definition of multiplicity given in this section with similar concepts for stationary processes obtained in Sections 5 and 6. For the sake of completeness we begin with the following "Wold decomposition" of $\underline{x}_t$.

Proposition 2.1. If $\{\underline{x}_t, t \in T\}$ is a stochastic process on Φ, then

$$\underline{x}_t(\varphi) = \underline{x}_t^{(1)}(\varphi) + \underline{x}_t^{(2)}(\varphi) \quad \text{for each } \varphi \in \Phi \text{ where}$$

(i) $\{\underline{x}_t^{(1)}\}$ is a deterministic and $\{\underline{x}_t^{(2)}\}$, a purely non-deterministic process on Φ; and

(ii) $H(\underline{x}^{(1)})$ is orthogonal to $H(\underline{x}^{(2)})$.

Observe that the topological assumptions concerning Φ in no way enter into the proof of this result.

Writing $J = T \times \Phi$, $\alpha = (t, \varphi)$, $\beta = (s, \psi)$ $(\alpha, \beta \in J)$ define $K(\alpha, \beta) = \mathcal{E}[\underline{x}_t(\varphi)\,\overline{\underline{x}_s(\psi)}]$. Then, clearly, K is a covariance function on $J \times J$. Let us denote by $H(K)$, the reproducing kernel Hilbert-space of functions defined on J whose reproducing kernel is K. Let

$H(K; t) = \mathfrak{S}[K(\cdot, \alpha), \alpha \in J_t]$, i.e. the subspace of $H(K)$ generated by $\{K(\cdot, \alpha), \alpha \in J_t\}$ where $J_t = \{(u, \varphi), u \leq t \text{ and } \varphi \in \Phi\}$. It is well known that there exists an isometry, which we denote by V, from $H(K)$ to $H(\underline{x})$ taking functions $K(\cdot, \alpha)$ into the random variables $\underline{x}_t(\varphi)$ and such that $VH(K; t) = H(\underline{x}; t)$.

The following assumptions (A) will be basic for our purpose:

(A.1) The space $H(\underline{x})$ is separable;

(A.2) $H(\underline{x}; -\infty) = \{0\}$.

Condition (A.2) is equivalent to the process $\{\underline{x}_t\}$ being purely non-deterministic, while the following lemma gives sufficient conditions on the r.v.s $\underline{x}_t(\varphi)$ for (A.1) to hold.

Lemma 2.1. Suppose that for each $t, (-\infty < t < \infty)$

(i) $\underline{x}_t(\varphi)$ is continuous in quadratic mean relative to the topology of Φ, and

(ii) the random variables $\underline{x}_{t-0}(\varphi)$ and $\underline{x}_{t+0}(\varphi)$ exist (in quadratic mean) for each $\varphi \in \Phi$.

Then $H(\underline{x})$ is separable.

This result is a generalization of a lemma due to Cramér [2] and takes as its starting point the fact, proved there, that for each φ, the set of all discontinuity points of the one-dimensional process $\{\underline{x}_t(\varphi), t \in T\}$ is at most denumerable.

Proof. It suffices to prove that there exists a countable dense set H_o in $\{\underline{x}_t(\varphi), t \in T, \varphi \in \Phi\}$. Let $\Phi_o = \{\varphi_k\}$ be a countable, everywhere dense set in Φ. The set D_k of discontinuities of the one-dimensional process $\underline{x}_t(\varphi_k)$ is at most denumerable. We shall show that $H_o = \{\underline{x}_u(\varphi_k)\ \varphi_k \in \Phi_o, u \in \bigcup_k D_k, \text{ or } u \text{ rational}\}$ is a dense subset of

$\{\underline{x}_t(\varphi) \quad t \in T, \quad \varphi \in \Phi\}$. Since H_o has at most denumerable elements, the proof of the lemma will be complete once we establish the preceding assertion. For τ and φ fixed, consider an element $\underline{x}_\tau(\varphi)$ and let $\mathcal{E}$ be an arbitrary positive number. By (i), there exists a $\varphi_k \in \Phi_o$ such that $\mathcal{E}|\underline{x}_\tau(\varphi) - \underline{x}_\tau(\varphi_k)|^2 < \mathcal{E}/2$. If τ is a discontinuity point of the one-dimensional process $\{\underline{x}_t(\varphi_k)\}$ $(t \in T)$, then since $\underline{x}_\tau(\varphi_k) \in H_o$, the proof will be complete. On the other hand, if τ is not a discontinuity point of $\{\underline{x}_t(\varphi_k)\}$ then there exists a rational number r such that $\mathcal{E}|\underline{x}_\tau(\varphi_k) - \underline{x}_r(\varphi_k)|^2 < \mathcal{E}/2$. This implies that $\mathcal{E}|\underline{x}_\tau(\varphi) - \underline{x}_r(\varphi_k)|^2 < 2\mathcal{E}$ and since $\underline{x}_r(\varphi_k) \in H_o$ the proof is complete.

It might be remarked in passing that if $\underline{x}(t) = [x_1(t), \ldots, x_q(t)]$ is a q-dimensional process such that the random variables $x_i(t-0)$, $x_i(t+0)$ exist for $i = 1, \ldots, q$, the conditions of Lemma 2.1 are fulfilled if we take Φ to be q-dimensional Euclidean space and define $\underline{x}_t(\varphi) = \sum_{i=1}^{q} x_i(t)\,\varphi_i$, φ being the vector $(\varphi_1, \ldots, \varphi_q)$. In other words, Lemma 1 of [2] is a special case of Lemma 2.1. In view of the isometry V between $H(K)$ and $H(\underline{x})$, the assumptions (A) are equivalent to corresponding assumptions concerning the spaces $H(K)$ and $H(K; -\infty)$. Let us introduce the spaces $H^*(K;\, t) = \bigcap_{n=1}^{\infty} H(K;\, t + \frac{1}{n})$. We then have $H^*(K; -\infty) = \{0\}$ and $H(K) = H^*(K; \infty)$, the smallest subspace containing all the $H^*(K;\, t)$.

The spaces $H^*(\underline{x};\, t)$ are similarly introduced. Let $\hat{E}(t)$ denote the projection operator from $H(K)$ onto $H^*(K;\, t)$ and $E(t)$ the projection from $H(\underline{x})$ onto $H^*(\underline{x};\, t)$. It then follows easily that the families $\{\hat{E}(t)\ ,\ -\infty < t < \infty\}$ and $\{E(t), -\infty < t < \infty\}$ are right continuous resolutions of the identity in the respective Hilbert spaces $H(K)$ and $H(\underline{x})$.

The two results which follow are proved as in [7]. We omit the proof, which is essentially based on the Hellinger-Hahn decomposition of the self-adjoint operators $\hat{A}$ and A defined respectively on $H(K)$ and $H(\underline{x})$ by the resolutions of the identity introduced above. Observe that while the parameter set T of the process is always either the real line or the set of all integers, the resolution of the identity $\{E(t)\}$ determined by the process is defined for all real t.

Theorem 2.1. Let assumptions (A) be satisfied. Then each element $K(\cdot, \alpha)$ (α in J) of $H(K)$ has the following representation

$$K(\cdot, \alpha) = \sum_{n=1}^{M_o} \int_{-\infty}^{t} G_n(\alpha, u)\, dE(u)\, f^{(n)} + \sum_{t_j \le t} \sum_{\ell=1}^{M_j} a_{j\ell}(\alpha)\, g_{j\ell}$$

where the symbols introduced have the following meaning:

(a) $\{f^{(n)}\}$ is a sequence of elements in $H(K)$ with the following properties:

(i) The inner product $(E(\Delta_1)f^{(n)}, E(\Delta_2)f^{(m)}) = 0$ whenever Δ_1 and Δ_2 are disjoint intervals or $m \neq n$;

(ii) For each n, $G_n(\alpha, \cdot) \in L_2(\rho_n)$ where $\rho_n(\Delta) = \| E(\Delta)f^{(n)}\|^2$,
$\sum_{n=1}^{M_o} \int |G_n(\alpha, u)|^2\, d\rho_n(u) < \infty$ and $\rho_1 \gg \rho_2 \gg \ldots$ etc.

(b) For each $j = 1, 2, \ldots$ the sequence $\{g_{j\ell}\}$ ($\ell = 1, \ldots, M_j$) are the eigenvectors of the self-adjoint operator $\hat{A}$ corresponding to the eigenvalue t_j and such that

$$\sum_{j=1}^{\infty} \sum_{\ell=1}^{M_j} |a_{j\ell}(\alpha)|^2 \| g_{j\ell}\|^2 < \infty .$$

The elements $\{g_{j\ell}\}$ further, form a complete orthonormal system in the subspace $[E(t_j) - E(t_j - 0)]\, H(K)$ with

$(g_{j\ell}, g_{im}) = 0$ if $i \neq j$.

For $\alpha = (t, \varphi)$ writing $\Gamma_n(\varphi; t, u) = G_n(\alpha, u)$ and $b_{j\ell}(\varphi; t) = a_{j\ell}(\alpha)$ we obtain the following representation for the process $\underline{x}_t$ on Φ.

Theorem 2.2. If conditions (A) hold we have the following representation for $\underline{x}_t$. For each t and φ, with probability one

$$(2.1)\quad \underline{x}_t(\varphi) = \sum_{n=1}^{M_o} \int_{-\infty}^{t} \Gamma_n(\varphi; t, u)\, d\, z_n(u) + \sum_{t_j \leq t} \sum_{\ell=1}^{M_j} b_{j\ell}(\varphi; t)\, \xi_{j\ell} ,$$

where

(a) $z_n(u)$ $(-\infty < u < \infty)$ for each n, is an orthogonal random function with the further property that $\mathcal{E}[z_m(u)\, \overline{z_n(v)}] = 0$ for $m \neq n$ and $\mathcal{E}|z_n(\Delta)|^2 = \rho_n(\Delta)$. Further, the functions Γ_n and ρ_n satisfy the conditions stated in the preceding theorem;

(b) The random variables $\xi_{j\ell}$ $(\ell = 1, \ldots, M_j$ and $j = 1, 2, \ldots)$ are mutually orthogonal with

$$\sum_{j=1}^{\infty} \sum_{\ell=1}^{M_j} \sigma_{j\ell}^2 \,|\, b_{j\ell}(\varphi; t)|^2 \text{ finite, where } \sigma_{j\ell}^2 = \mathcal{E}|\xi_{j\ell}|^2 .$$

Definition. The cardinal number $M = \max\, [M_0, \sup_j M_j]$ is called the multiplicity of the stochastic process $\underline{x}_t$ on Φ.

It is to be noted that M can be infinite, in which case of course M is aleph null. The corresponding series that occur in our work are then to be treated as infinite series.

If T is the set of integers it is easy to see that M_o is necessarily zero and $t_j = j$.

3. <u>Canonical and proper canonical representations</u>.

The representation obtained in Theorem 2.2 has the following property. For $s < t$,

$$(3.1) \quad E(s)\, \underline{x}_t(\varphi) = \sum_{1}^{M_o} \int_{-\infty}^{s} \Gamma_n(\varphi;\, t,\, u)\, dz_n(u) + \sum_{t_j \le s} \sum_{\ell=1}^{M_j} b_{j\ell}(\varphi;\, t)\, \xi_{j\ell} \ .$$

A representation satisfying (3.1) will be called canonical. From the form of (2.1), it follows that $H(\underline{x};\, t) \subset \mathfrak{S}[H(\underline{z};\, t) \cup H(\underline{\xi};\, t)]$ where $H(\underline{\xi};\, t) = \mathfrak{S}[\xi_{j\ell},\ \ell = 1, 2, \ldots, M_j,\ t_j \le t]$. For applications of the theory, however, it is more useful to consider canonical representations for which,

$$(3.2) \quad \mathfrak{S}[H(\underline{z};\, t) \cup H(\underline{\xi};\, t)] = H(\underline{x};\, t) \quad \text{for all } t \ .$$

Following Hida, we refer to a representation with property (3.2) as proper canonical. In [7], Hida was concerned with proper canonical representations of multiplicity one. In order to be able to discuss the multiplicity theory of the more general processes considered by us it is necessary to establish the existence of a proper canonical representation of arbitrary multiplicity equivalent to the one given by Theorem 2.2. This we do in Theorem 3.1.

For the representation of Theorem 2.2 define the processes $B_n(u)$ as follows:

(i) If both M_o and $\operatorname{Sup}_j M_j$ are infinite, then

$$B_n(u) = z_n(u) + \sum_{t_j \le u} \xi_{jn} \quad \text{for } n = 1, 2, \ldots \text{ ad inf} \ .$$

(ii) If M_o is finite and $M_o \le \operatorname{Sup}_j M_j$, let

$$B_n(u) = z_n(u) + \sum_{t_j \le u} \xi_{jn} \quad \text{for } n = 1, 2, \ldots M_o$$

$$= \sum_{t_j \le u} \xi_{jn} \qquad \text{for } M_o < n \le \operatorname{Sup}_j M_j \ .$$

(iii) In the remaining cases define

$$B_n(u) = z_n(u) + \sum_{t_j \le u} \xi_{jn} \qquad (n = 1, 2, \ldots, \operatorname{Sup}_j M_j)$$

$$= z_n(u) \qquad \operatorname{Sup}_j M_j \le n \le M_o .$$

With the above notation we rewrite (2.1) as

$$(3.3) \qquad \underline{x}_t(\varphi) = \sum_{n=1}^{M} \int_{-\infty}^{t} G_n(\varphi;\ t,\ u)\ dB_n(u), \quad \text{where}$$

$M = \max(\operatorname{Sup}_j M_j,\ M_o)$. What the functions G_n stand for is clear from the context.

Also, $H(\underline{B};\ t) = \mathfrak{S}[H(\underline{z};\ t) \cup H(\underline{\xi};\ t)]$. A representation of the form (3.3) will be denoted by $\{G_n,\ B_n\}_1^M$.

Theorem 3.1. Let $\{G_n,\ B_n\}_1^M$ be a canonical representation. Then there exists a proper canonical representation $\{\tilde{G}_n,\ \tilde{B}_n\}_1^M$ such that for every $(\varphi,\ t)$, $\underline{x}_t(\varphi) = \sum_{n=1}^{M} \int_{-\infty}^{t} \tilde{G}_n(\varphi;\ t,\ u)\ d\tilde{B}_n(u)$ with probability one.

Proof. Let $\rho_n(\Delta) = \mathcal{E}|B_n(\Delta)|^2$. For each φ and t , and every measurable subset S of $(-\infty,\ t]$, define the measure $\mu^{(n)}_{(t,\varphi)}(S) = \int_S |G_n(\varphi;\ t,\ u)|^2\ d\rho_n(u)$. Then for each n , the measure $\mu^{(n)}$ given by $\mu^{(n)}(S) = \bigvee_{(t,\varphi)} \mu^{(n)}_{(t,\varphi)}(S)$ (see [7]) is absolutely continuous with respect to ρ_n . Let $N_n = \{u \mid \frac{d\mu^{(n)}}{d\rho_n}(u) > 0\}$ and $\tilde{B}_n(S)$ be the random set function with variance function $\tilde{\rho}_n$ and defined by the stochastic integral $\tilde{B}_n(S) = \int_S I_{N_n}(u)\ dB_n(u)$. Further, set $\tilde{G}_n(\varphi;\ t,\ u) = G_n(\varphi;\ t,\ u)$ for all φ, t and u and consider the sum, $\underline{y}_t(\varphi) = \sum_1^M \int_{-\infty}^{t} \tilde{G}_n(\varphi; t, u) d\tilde{B}_n(u)$. If M is infinite, the right hand side series is easily seen to be convergent in quadratic mean. From the fact that

$$\frac{d\mu^{(n)}_{(t,\varphi)}}{d\mu^{(n)}}(u)\ \frac{d\mu^{(n)}}{d\rho_n}(u) = |G_n(\varphi;\ t,\ u)|^2$$ for each t, φ and n , it is easy to deduce that

$$\int_{-\infty}^{t} [1-I_{N_n}(u)]^2 |G_n(\varphi;\ t,\ u)|^2\ d\rho_n(u) = 0\ .$$

Thus for all t, φ,

$$(3.4)\qquad \mathcal{E}|\underline{x}_t(\varphi) - \underline{y}_t(\varphi)|^2 = \sum_{n=1}^{M} \int_{-\infty}^{t} |\ 1 - I_{N_n}(u)|^2 |G_n(\varphi;\ t,\ u)|^2\ d\rho_n(u) = 0\ .$$

From (3.4) we find that for every t and φ

$$(3.5)\qquad \underline{x}_t(\varphi) = \underline{y}_t(\varphi) \quad \text{with probability one}$$

and that

$$(3.6)\qquad H(\underline{x};\ s) = H(\underline{y};\ s) \quad \text{for all} \quad s \in T\ .$$

A similar argument also yeilds that for every measurable subset S of $(-\infty,\ t]$,

$$(3.7)\qquad \sum_{n=1}^{M} \int_S |G_n(\varphi;t,u)|^2 d\rho_n(u) = \sum_{n=1}^{M} \int_S |\tilde{G}_n(\varphi;t,u)|^2 d\tilde{\rho}_n(u)\ .$$

Since $\mathcal{E}[\tilde{B}_n(\Delta)\ \tilde{B}_m(\Delta')] = 0$ for $\Delta \neq \Delta'$ or $n \neq m$, we have

$$H(\underline{\tilde{B}};\ t) = \sum_{1}^{M} \oplus\ H(\tilde{B}_n;\ t)\ .$$

Therefore, to establish that $\{\tilde{G}_n,\ \tilde{B}_n\}$ is proper canonical, it suffices to show that $H(\tilde{B}_n;\ t) \subset H(\underline{x};\ t)$ for all n and t . Now suppose that there is a t and an n , such that

$$H(\tilde{B}_n;\ t) \not\subset H(\underline{x};\ t)\ .$$

Then we can find a non-zero element $z \in H(\tilde{B}_n;\ t)$ which is orthogonal to $H(\underline{x};\ t)$. Let $s'' \in T$ be arbitrary and $s \leq s' \leq t$. By the canonical property of $\{G_n,\ B_n\}$, (3.5), (3.6) and (3.7), the projection of $\underline{x}_{s''}(\varphi)$ onto $H(\underline{x};\ s')$ is given by $\sum_{1}^{M} \int_{-\infty}^{s'} \tilde{G}_n(\varphi;\ s'',\ u)\ d\tilde{B}_n(u)$. But $z \perp H(\underline{x};t)$

and $z = \int_{-\infty}^{t} h(u)\, dB_n(u)$ with $h \in L_2(\tilde{\rho}_n)$ (see [4], pp. 426-28). Hence

$$(3.8) \quad \int_{-\infty}^{s'} \tilde{G}_n(\varphi;\, s'',\, u)\, \overline{h(u)}\, d\tilde{\rho}_n(u) = 0 \quad \text{for all} \quad s'',\ \varphi \ .$$

Using a similar argument with s we obtain

$$(3.9) \quad \int_{s}^{s'} \tilde{G}_n(\varphi;\, s'',\, u)\, \overline{h(u)}\, d\tilde{\rho}_n(u) = 0 \quad \text{for all} \quad s'' \quad \text{and} \quad \varphi \ .$$

Proceeding as in Theorem I.2 of [7], it can be shown that (3.9) implies

$$\rho_n\{N(h) \cap N_n\} = 0 \quad \text{where} \quad N(h) = \{u \mid h(u) \neq 0\} \quad .$$

Hence,

$$\mathcal{E}|z|^2 = \int_{-\infty}^{t} |h(u)|^2 d\tilde{\rho}_n(u) = \int_{-\infty}^{t} I_{N_n}(u)\, |h(u)|^2 d\rho_n(u) = \int_{N_n \cap N(h)} |h(u)|^2 d\rho_n(u) = 0,$$

contradicting the assumption that $z \neq 0$.

<u>Remarks</u>. (i) The relation obtained in (3.5) is an equivalence relation. Hence we shall refer to $\{\tilde{G}_n,\, \tilde{B}_n\}_1^M$ as a proper canonical representation equivalent to $\{G_n,\, B_n\}_1^M$.

(ii) By definition of $\tilde{B}_n$ and the fact that $\frac{d\tilde{\rho}_n}{d\rho_n}(u) = I_{N_n}^2(u)$ if $\tilde{\rho}_n \equiv 0$, we obtain $I_{N_n}(u) = 0$ a.e. ρ_n . But this will imply $\rho_n\left\{\frac{d\mu^{(n)}}{d\rho_n}(u) > 0\right\} = 0$. Hence $|G_n(\varphi;\, t,\, u)|^2$ which equals $\frac{d\mu^{(n)}_{(t,\varphi)}}{d\mu^{(n)}}(u)\, \frac{d\mu^{(n)}}{d\rho_n}(u)$ vanishes almost everywhere $[\rho_n]$, i.e., for every φ and t $G_n(\varphi;\, t,\, u) = 0$ a.e. with respect to ρ_n , contradicting the fact that M is the multiplicity of $\{G_n,\, B_n\}_1^M$. Thus the representation $\{\tilde{G}_n,\, \tilde{B}_n\}$ also has multiplicity M .

(iii) Finally, from the definition of $\tilde{B}_n$ we have

$$\tilde{B}_n(S) = \int_S I_{N_n}(u)\, dz_n(u) + \sum_{t_j \in N_n \cap S} \xi_{jn}$$

$$= \tilde{z}_n(S) + \sum_{t_j \in S} \tilde{\xi}_{jn}$$

say, where $\tilde{\xi}_{jn} = \xi_{jn}$ if $t_j \in N_n$, and 0 otherwise. Hence the proper canonical representation obtained can again be put in the form of (2.1).

$\underline{\text{WEAKLY}}$ $\underline{\text{STATIONARY}}$ $\underline{\text{STOCHASTIC}}$ $\underline{\text{PROCESSES}}$ $\underline{\text{ON}}$ $\underline{\Phi}$

We now turn to the central task of this paper, the study of the multiplicity theory of weakly stationary processes on Φ. As we shall see, this theory applies also to a class of infinite dimensional stationary processes and shows that in the study of the latter, the idea of multiplicity naturally supplants that of rank.

Before proceeding to the discrete parameter case whose results we shall need in Section 6 we make the following observations concerning the Wold decomposition of continuous parameter stationary processes on Φ. If for every real h, we define

$$T_h \underline{x}_t(\varphi) = \underline{x}_{t+h}(\varphi) ,$$

where t is an arbitrary real number and $\varphi \in \Phi$, it is easy to see that this definition can be extended so that T_h becomes a unitary operator. Indeed, $\{T_h\}$ $(-\infty \quad h, +\infty)$ is a group of unitary operators and for all real a and h

$$T_h E(a) = E(a+h) T_h .$$

Using this fact and proposition 2.1 we are able to state the following proposition:

If $\{\underline{x}_t\}$ is a weakly stationary process on Φ then there exist weakly stationary processes on Φ, $\{\underline{x}_t^{(1)}\}$ and $\{\underline{x}_t^{(2)}\}$ such that

(1) $\underline{x}_t(\varphi) = \underline{x}_t^{(1)}(\varphi) + \underline{x}_t^{(2)}(\varphi)$ for every t,

(2) $\{\underline{x}_t^{(1)}\}$ is deterministic $\{\underline{x}_t^{(2)}\}$ is purely non-deterministic, and

(3) $H(\underline{x}^{(1)})$ and $H(\underline{x}^{(2)})$ are orthogonal.

4. Discrete parameter processes. Let $\underline{x}_n$ $(n = 0, \pm 1, \ldots)$ be a purely non-deterministic stationary process on Φ. Since we want $H(\underline{x})$ to be separable, we shall assume that for each n, $\underline{x}_n(\cdot)$ is continuous in quadratic mean in the Φ-topology. If in Theorem 2.2, T is the set of integers then the resolution of identity of the process is given by

$$E_t = \sum_{n \leq t} (p_n - p_{n-1}) \quad \text{where } p_n \text{ is the projection onto } H(\underline{x};\, n)\ .$$

The self-adjoint operator A then has a purely discrete spectrum, having each integer as an eigenvalue and $H(\underline{x};\, n) \ominus H(\underline{x};\, n-1)$ as the invariant subspaces. The multiplicity M of the process is therefore given by

$$M = \sup_n \,[\dim \{H(\underline{x};\, n) \ominus H(\underline{x};\, n-1)\}\,]\ .$$

The following two lemmas show that $\dim \{H(\underline{x};\, n) \ominus H(\underline{x};\, n-1)\}$ is independent of n. Let $g_n(\varphi) = \underline{x}_n(\varphi) - p_{n-1}\underline{x}_n(\varphi)$.

Lemma 4.1. $H(\underline{x};\, n) \ominus H(\underline{x};n-1) = \mathfrak{S}[g_n(\varphi),\ \varphi \in \Phi]$ $(n = 0, \pm 1, \ldots)$.

Lemma 4.2. For arbitrary integers m and n, there exists a unitary operator T_m such that,

$$T_m \mathfrak{S}\,[g_n(\varphi),\ \varphi \in \Phi\,] = \mathfrak{S}[g_{m+n}(\varphi),\ \varphi \in \Phi]\ .$$

To prove Lemma 4.1 it is enough to show that $H(\underline{x};n) = H(\underline{x};n-1) \oplus \mathfrak{S}[g_n(\varphi),\ \varphi \in \Phi\,]$. But this is true from the definition of $g_n(\varphi)$.

For the proof of Lemma 4.2, we consider the group $\{T_m\}$ of unitary operators given by

$$T_m\, \underline{x}_n(\varphi) = \underline{x}_{m+n}(\varphi) \quad \text{for all } n \text{ and } \varphi\ .$$

It can be easily verified that

$$T_m \, p_{n-1}\underline{x}_n(\varphi) = p_{m+n-1}\underline{x}_{m+n}(\varphi) \quad . \quad \text{Hence,} \quad T_m \, g_n(\varphi) = g_{m+n}(\varphi)$$

and the proof is complete.

For the process $\underline{x}_n$ of this section we now have the following result.

Theorem 4.1 $\underline{x}_n(\varphi) = \sum_{\ell=1}^{M} \sum_{m\leq n} b_\ell(\varphi; m\text{-}n)\, \xi_\ell(m)$, **where**

(i) $M = \dim[H(\underline{x}; n) \ominus H(\underline{x}; n-1)]$ is the multiplicity of the process,

(ii) For each ℓ, $\{\xi_\ell(m)\}$ $(m = 0, \pm 1, \dots)$ has stationary orthogonal increments and $\mathcal{E}[\xi_\ell(m)\, \overline{\xi_k(m')}] = 0$ if $k \neq \ell$. Furthermore,

$$\sum_{\ell=1}^{M} \sum_{m\leq 0} |b_\ell(\varphi; m)|^2 \, \mathcal{E}\, |\xi_\ell(m)|^2 \text{ is finite and}$$

(iii) $\sum_{i=1}^{M} \oplus H(\xi_i; n) = H(\underline{x}; n)$ for all n.

Proof: From Theorems 2.2, 3.1 and the remarks preceding Lemma 4.1 about the resolution of the identity in $H(\underline{x})$, we have

$$(4.1) \quad \underline{x}_n(\varphi) = \sum_{m\leq n} \sum_{\ell=1}^{M} b'_\ell(\varphi; n,m)\, \xi'_\ell(m) \quad \text{with} \quad H(\underline{x}; n) = \sum_1^M \oplus H(\xi'_i; n) \; .$$

By Lemma 4.1 and (4.1)

$$\mathfrak{S}[g_n(\varphi), \varphi\in\Phi, m\leq n] = H(\underline{x}; n) = \sum_1^M \oplus H(\xi_i; n) \; .$$

In particular

$$\mathfrak{S}[g_o(\varphi); \varphi \in \Phi] = \mathfrak{S}[\xi'_\ell(0), \quad \ell = 1, 2, \dots M] \quad .$$

Hence, if we define

$\xi_\ell(m) = T_m \; \xi'_\ell(0)$, we have

$$\mathfrak{S}[\xi'_\ell(m), \quad \ell = 1, 2, \dots M] = \mathfrak{S}[\xi_\ell(m), \ell = 1, 2, \dots M] \; ,$$

since

$$T_m \mathfrak{S}[\xi_\ell'(0), \quad \ell = 1, 2, \ldots M] = T_m \mathfrak{S}[g_o(\varphi), \varphi \quad \Phi] = \mathfrak{S}[g_m(\varphi), \varphi \in \Phi].$$

Therefore, $H(\underline{x}; n) = \sum_1^M \bigoplus H(\xi_i; n)$ and hence

$$x_o(\varphi) = \sum_{\ell=1}^M \sum_{m \leq 0} b_\ell(\varphi; m)\, \xi_\ell(m), \quad \text{with}$$

$$\sum_{\ell=1}^M \sum_{m \leq 0} |b_\ell(\varphi; m)|^2 \mathcal{E}|\xi_\ell(m)|^2 < \infty$$

$$\underline{x}_n(\varphi) = T_n\, \underline{x}_o(\varphi) = \sum_{\ell=1}^M \sum_{m \leq 0} b_\ell(\varphi; m)\, \xi_\ell(m+n)$$

$$= \sum_{\ell=1}^M \sum_{m \leq n} b_\ell(\varphi; m-n)\, \xi_\ell(m).$$

5. Continuous parameter, weakly stationary processes. We shall give in this section the generalization of what we believe to be the essence of Hanner's ideas underlying his time domain analysis of one-dimensional stationary processes. The desired generalization will turn out to be based on a study of the properties of the maximal spectral type of the operator A of the process and its multiplicity, thus effecting a unity with the work presented in Sections 2 and 3.

It is convenient to recall at this point some of the terminology of multiplicity theory in a separable Hilbert space H. Let A be any self adjoint operator with spectral measure function $E(.)$. For any element f in H let ρ_f be the finite measure on the Borel sets of line (sometimes also called the spectral function) given by $\rho_f(\Delta) = ||E(\Delta)f||^2$. The family of all finite measures on the line is divided into equivalence classes by the relation of equivalence between measures (equivalence here means mutual absolute continuity). If ρ is used to denote the equivalence class to which the measure ρ_f belongs, ρ will be called the spectral type of f with respect to A. ρ is also referred to as the spectral type belonging to A. If elements f and g are such that $\rho_f \equiv \rho_g$, they obviously have the same spectral type ρ. We shall say that the spectral type ρ dominates the spectral type σ $(\rho > \sigma$ or $\sigma < \rho)$ if any (and thus every) measure belonging to σ is absolutely continuous with respect to any measure belonging to ρ. ρ and σ are said to be independent spectral types if for any spectral type ν such that $\nu < \rho$ and $\nu < \sigma$ we have $\nu = 0$. An element f is said to be of maximal spectral type ρ (with respect to A) if for every g in H $\rho_g \ll \rho_f$. The subspace $\mathfrak{S}\{E(\Delta)f, \Delta$ ranging over all finite intervals$\}$ is called the cyclic subspace (with respect to A) generated by f. If this subspace coincides with H, f is called a cyclic or generating element of A and A is called cyclic. Also if f is a generating element of A, f is of maximal spectral type and the latter is referred to as the spectral type of the (cyclic) operator A. It is to be noted that if A is any self adjoint operator (since H is separable) there always exists a maximal spectral type belonging to A.

Any system of mutually cyclic parts of A of type ρ is called an orthogonal system of type ρ relative to A. An orthogonal system of type ρ which cannot be enlarged by adding to it more cyclic parts of A is called maximal. It is a known result of this theory that all maximal systems of type ρ have the same cardinal number. This uniquely determined cardinal number is defined to be the multiplicity of the spectral type ρ with respect to A.

Finally we need the notion of a uniform spectral type. The spectral type ρ $(\neq 0)$ is said to be uniform if every non-zero type σ dominated by ρ has the same multiplicity as ρ itself. Most of the above definitions have been taken from the article by A. I. Plessner and V. A. Rohlin [12] to which the reader is also referred for further details.

When dealing with continuous parameter processes, we assume not only that $\underline{x}_t(\varphi)$ is continuous in q.m. in the topology of Φ but that for each $\varphi \in \Phi$, the complex valued univariate process $[\underline{x}_t(\varphi)]$ $(-\infty < t < +\infty)$ is continuous in q.m. in t. We shall refer to this as condition (C). It is easy to see that if (C) holds, the assumptions of Lemma 2.1 are valid so that the separability condition (A.1) is satisfied. In addition, it follows from condition (C) that the group $[T_h]$ introduced in Section 4 is strongly continuous. We recall from Section 4

$$T_h E(t) = E(t+h) T_h \tag{5.1}$$

for all real t, h. As in [9], (5.1) is the basic relation between the operator A and the unitary group of the process which we propose to exploit in our time domain analysis. We shall prove the central theorem on representation by means of a number of lemmas. The first group of lemmas concerns the properties of spectral types.

<u>Lemma 5.1</u> If f is any element of $H(\underline{x})$ then, $\rho_f \ll \mu$, the Lebesque measure.

<u>Proof:</u> Let us define for every real t, and every measurable set S of the real line $\rho_f^{(t)}(S) = \rho_f(S-t) = ||E(S-t)f||^2$. From (5.1), however, $\rho_f^{(t)}(S)$ equals $||E(S)\, T_t f||^2$. Hence by the strong continuity of the group $\{T_t\}$, $\rho_f^{(t)}(S)$ converges to $\rho_f(S)$ as $t \rightarrow 0$. The assertion of the lemma now follows from a

result due to N. Wiener and R. C. Young [See Saks [16], p.91].

Let $f^{(1)}$ be a maximal element of A, i.e., an element of maximal spectral type with respect to A. and u any positive number. If we define

(5.2) $\quad g_a^b = \{E(b) - E(a)\} \int_A^B T_h E(\Delta_0)\, f^{(1)} dh,$ where $\Delta_0 \subset (0,u)$, $A < a-u$

$B > b$ and the integral is taken as in [6], we observe that g_a^b can be identified with Hanner's $Z(I_a{}^b)$ with $z = E(\Delta_0) f^{(1)}$ in the formula (3.2) of [6] (p.166). We remark that g_a^b does not depend on A and B as long as these limits of integration satisfy the stated inequalities. We give here the properties of g_a^b which follow from those of $Z(I_a{}^b)$ [See [6], p.167]. For $a < b < c$, we have

(5.3) $\quad g_a^b + g_b^c = g_a^c,$

(5.4) $\quad g_a^b$ is orthogonal to g_b^c,

and for arbitrary t,

(5.5) $\quad T_t\, g_a^b = g_{a+t}^{b+t}$.

It follows from (5.3), (5.4) and (5.5) that

(5.6) $\quad ||g_a^b||^2 = \tau(b-a)$ where τ is a non-negative number that does not depend on the interval $(a,b]$.

Lemma 5.2. There exists a finite interval $\Delta_0 \subset (0,u]$ such that g_0^u as defined in (5.2) is different from zero.

Proof: We follow Hanner closely in proving this lemma ([6], Proposition C). Suppose $g_0^u = 0$, then for every $z' \in H(\underline{x})$ and every $\Delta_0 \subset (0,u]$, $\mathcal{E}[g_0^u \overline{z}'] = 0$. Hence, if $z = \omega(s_1,t_1)$ and $z' = \omega(s_2,t_2)$, where $\omega(s,t) = \{E(t) - E(s)\} f^{(1)}$ for $s < t$, then from the fact that $\mathcal{E}[g_0^u\, \overline{z}'] = 0$, we have

(5.7) $\quad \int_{-u}^{u} \mathcal{E}[T_h \omega(s_1,t_1) . \overline{\omega(s_2,t_2)}] dh = 0 \qquad (0 < s_1,t_1,s_2,t_2 \leq u).$

But for δ such that $0 < \delta < \frac{1}{2}u$, $\mathcal{E}[T_h\omega(0,u)\ \overline{\omega(\delta,u-\delta)}] = \mathcal{E}|T_h\ \omega(\delta,u-\delta)|^2$ is a continuous function of h which converges, as $h \to 0$ to $\mathcal{E}|\omega(\delta,\ u-\delta)|^2$. Now, $\omega(\delta,\ u-\delta) = 0$ implies that $[E(\delta) - E(u-\delta)]\ f^{(1)} = 0$ and hence $[E(\delta) - E(u-\delta)]\ f = 0$ for all $f \in H(\underline{x})$, giving $H(\underline{x};\delta) \ominus H(\underline{x};\ u-\delta) = \{0\}$. This contradicts the fact that the $\underline{x}_t$ -process is purely non-deterministic. Therefore we can find a γ $(0 < \gamma < u)$ such that

$$L = \int_{-\gamma}^{\gamma} \mathcal{E}[T_h\ \omega(0,u)\ \overline{\omega(\delta,\ u-\delta)}]\ dh \neq 0.$$ Let $t_0 = \delta < t_1 < \dots \leq t_n = u-\delta$

be a finite subdivision of the interval $(\delta,\ u-\delta]$. Then

$$L = \sum_1^n \int_{-\gamma}^{\gamma} \mathcal{E}[T_h\ \omega(0,u) . \overline{\omega(t_{i-1},\ t_i)}]\ dh.$$

Let $$M = \sum_1^n \int_{-\gamma-(t_i-t_{i-1})}^{\gamma+(t_i-t_{i-1})} \mathcal{E}[T_h\ \omega(t_{i-1}-\gamma,\ t_i+\gamma)\ \overline{\omega(t_{i-1},\ t_i)}]\ dh$$

$$= \sum_1^n \int_{-u}^{+u} \mathcal{E}[T_h\ \omega(t_{i-1}-\gamma,\ t_i+\gamma)\ \overline{\omega(t_{i-1},\ t_i)}]\ dh$$ which is zero from (5.7).

Now $|M-L| \leq 2u||\ \omega(0,u)||\max_i||\omega(t_{i-1},\ t_i)||$. But

$$\omega(t_{i-1},\ t_i) + P_{H(\underline{x};\ t_i,u)}\ f^{(1)} = P_{H(\underline{x};\ t_{i-1},u)}\ f^{(1)}$$ and $\omega(t_{i-1},\ t_i)$ is

orthogonal to $H(\underline{x};\ t_i,u)\ f^{(1)}$. Hence, $||\omega(t_{i-1},\ t_i)||^2 = ||P_{H(x;\ t_{i-1},u)}f^{(1)}||^2 - ||P_{H(\underline{x};\ t_i,u)}\ f^{(1)}\ ||^2$. Since $||P_{H(\underline{x};\ t,u)}f^{(1)}||^2$ is a continuous function of t, we make $||\omega(t_{i-1},\ t_i)\ ||$ as small as we please by taking a fine enough subdivision. Hence $M = L$. But $M = 0$ and $L \neq 0$. We arrive at a contradiction, thus proving the lemma.

Henceforth, Δ_0 will denote a fixed subinterval of $(0,u]$, such that $||g_0^u||^2 \neq 0$ where in (5.2) we take $(a,b] = (0,u]$.

Suppose that $0 < b < u$ and consider $g_0^b = [E(b) - E(0)] \int_{A'}^{B'} T_h E(\Delta_0) f^{(1)} dh$,

where $A' < -u$, $B' > b$. Since the definition of g_0^b is independent of this particular choice of A', B', we have

$g_0^b = [E(b) - E(0)]\, g_0^u = [E(b) - E(0)] \int_A^B T_h E(\Delta_0) f^{(1)} dh$, where $A < -u$ and $B > u$. Also from (5.3) and (5.4), $g_0^u = g_0^b + g_b^u$ with g_b^u orthogonal to g_0^b. Hence $||g_0^u||^2 = ||g_0^b||^2 + ||g_b^u||^2$. If $g_0^b = 0$, we have from (5.6) that $\tau u = \tau(u - b)$ where $\tau \neq 0$ by Lemma 5.2. Since u and b are distinct positive numbers, the above relation is absurd and thus $g_0^b \neq 0$. On the other hand, if $b > u$ then again (5.3) and (5.4) imply that $g_0^b = g_0^u + g_u^b$ with g_0^u being orthogonal to g_u^b. Therefore $||g_0^b||^2 = ||g_0^u||^2 + ||g_u^b||^2$ thus giving $g_0^b \neq 0$ for all positive b. Finally if $b' < 0$, then from (5.5), $T_\beta\, g_{b'}^0 = g_0^\beta$ where $\beta = -b'$. From previous arguments $g_0^\beta \neq 0$. Hence $g_{b'}^0 \neq 0$. Thus $g_0^b \neq 0$ if $b > 0$ and $g_{b'}^0 \neq 0$ if $b' < 0$. We therefore obtain $\tau \neq 0$ in (5.6), since for any $(c,d]$, $T_{-c} g_c^d = g_0^{d-c} \neq 0$.

Lemma 5.3. The spectral measure $\rho_{g_a^b} = \tau \mu^I$, $(I = (a,b])$, where $\mu^I(S) = \mu(I \cap S)$ for every measurable subset S of the real line.

Proof: Let Δ be any finite interval. Then $\rho_{g_a^b}(\Delta) = ||E(\Delta) g_a^b||^2$. Therefore, from (5.2), $\rho_{g_a^b}(\Delta) = ||E(\Delta \cap I) g_a^b||^2$, which equals zero if $\Delta \cap I = \emptyset$ and, from (5.6), is equal to $\tau\, \mu(\Delta \cap I)$ if $\Delta \cap I \neq \emptyset$. The result follows immediately from the definition of μ^I.

The definition of g_a^b can obviously be adjusted to make $\tau = 1$. From now on we shall assume that this has been done.

Lemma 5.4. If ρ is the maximal spectral type of A, then $\rho \equiv \mu$.

Proof: It suffices to prove that if $f^{(1)}$ is a maximal element then $\rho_{f^{(1)}} \equiv \mu$. From the maximality of $f^{(1)}$ and the fact, shown in Lemma 5.3, that $\rho_{g_a^b} = \mu^I$ for an arbitrary interval $I = (a,b]$, it follows that $\mu \ll \rho_{f^{(1)}}$. An appeal to Lemma 5.1 completes the proof.

We next define a complex-valued process $\xi_1(a)$ for all real a, as follows:

$$\xi_1(a) = -g_a^0 \quad \text{if} \quad a < 0$$

$$\xi_1(0) = 0$$

$$\xi_1(a) = g_0^a \quad \text{if} \quad a > 0.$$

If we set $\xi_1(I) = \xi_1(b) - \xi_1(a)$ for every interval $I = (a,b]$, it follows from (5.3) and (5.4) that

$$\xi_1(I) = g_a^b \quad . \tag{5.8}$$

It is easy to see that $\{\xi_1(t)\}$ $(-\infty < t < +\infty)$ is a stochastic preocess with stationary orthogonal increments and $\mathcal{E}|\xi_1(\Delta)|^2 = \mu(\Delta)$. Let us write $H(\xi_1) = \mathfrak{S}\{\xi_1(\Delta), \Delta$ ranging over all finite subintervals of real line$\}$ and $H(\xi_1;t) = \mathfrak{S}\{\xi_1(\Delta), \Delta$ ranging over all finite intervals contained in $(-\infty, t]\}$. Then by (5.5) it follows that for every real t, $T_t P_{H(\xi_1)} = P_{H(\xi_1)} T_t$. If we now define

$$\underline{x}_t^{(1)}(\varphi) = P_{H(\xi_1)} \underline{x}_t(\varphi), \tag{5.9}$$

then the $\underline{x}_t^{(1)}$-process is stationary and $T_t \underline{x}_s^{(1)}(\varphi) = \underline{x}_{s+t}^{(1)}(\varphi)$ for all s and φ. Furthermore, since ξ_1 is a process with orthogonal increments, we have

$H(\xi_1) = H(\xi_1;t) \oplus \mathfrak{S}\{\xi_1(\Delta), \Delta \subset (t, +\infty)\} = H(\xi_1;t) \oplus \mathfrak{S}\{g_a^b, t < a \leq b < +\infty\}$

from (5.8). But, by definition of g_a^b, $\underline{x}_t(\varphi) \perp \mathfrak{S}\{g_a^b, t < a \leq b < \infty\}$ so that $\underline{x}_t(\varphi) = P_{H(\xi_1;t)}\, \underline{x}_t(\varphi)$ for all t, φ. Since from (5.8) and (5.2), $\xi_1(\Delta) \in H(\underline{x};t)$ for every finite interval Δ lying in $(-\infty, t]$, we have $H(\underline{x}^{(1)}; t) \subset H(\underline{x}; t)$. Hence the $\underline{x}_t^{(1)}$ -process is purely non-deterministic.

<u>Lemma 5.5</u> For every real t and φ in Φ, $\underline{x}_t^{(1)}(\varphi) = \int_{-\infty}^{t} F_1(\varphi; u - t) d\xi_1(u)$ where $\int_{-\infty}^{0} |F_1(\varphi; u)|^2 \, d\mu(u) < \infty$.

<u>Proof:</u> Since $\underline{x}_0^{(1)}(\varphi) \in H(\xi_1; 0)$, it has the stochastic integral representation

$$\underline{x}_0^{(1)}(\varphi) = \int_{-\infty}^{0} F_1(\varphi;u)\, d\xi_1(u) \quad \text{with} \quad \int_{-\infty}^{0} |F_1(\varphi;u)|^2 d\mu(u) \text{ finite (See [4], pp. 425-28).}$$

The $\underline{x}_t^{(1)}$-process is stationary and $T_t\, \xi_1(\Delta) = \xi_1(\Delta + t)$ from (5.5) and (5.8); hence

$$\underline{x}_t^{(1)}(\varphi) = T_t \underline{x}_0^{(1)}(\varphi) = \int_{-\infty}^{0} F_1(\varphi;u)\, d\xi_1(u + t) = \int_{-\infty}^{t} F_1(\varphi;\, u-t) d\xi_1(u) \; .$$

For every $\varphi \in \Phi$ and t real, set $\underline{y}_t^{(1)}(\varphi) = \underline{x}_t(\varphi) - \underline{x}_t^{(1)}(\varphi)$. Then $T_t\, \underline{y}_s^{(1)}(\varphi) = \underline{y}_{s+t}^{(1)}(\varphi)$ and $H(\underline{y}^{(1)};t) \subset H(\underline{x};t)$. Hence the $\underline{y}_t^{(1)}$ -process is also weakly stationary and purely non-deterministic. From (5.9) we have $\underline{y}_t^{(1)} = \underline{x}_t(\varphi) - P_{H(\xi_1)}\, \underline{x}_t(\varphi)$ which implies that for all t, φ, $\underline{y}_t^{(1)}(\varphi) \perp H(\xi_1)$. Since $H(\underline{x}^{(1)}) \subset H(\xi_1)$ it follows that for every t and s

(5.10) $H(\underline{y}^{(1)};s) \perp H(\underline{x}^{(1)};t)$

<u>Lemma 5.6</u> $H(\underline{x};t) = H(\underline{x}^{(1)};t) \oplus H(\underline{y}^{(1)};t)$ for each t.

<u>Proof:</u> Since $H(\underline{x}^{(1)};t) \oplus H(\underline{y}^{(1)};t) \subset H(\underline{x};t)$, we need to show only that $H(\underline{x}^{(1)};t) \oplus H(\underline{y}^{(1)};t) \supset H(\underline{x};t)$. But this follows from the fact that for $\varphi \in \Phi$, $\underline{x}_\tau(\varphi) = \underline{x}_\tau^{(1)}(\varphi) + \underline{y}_\tau^{(1)}(\varphi)$ which belongs to $H(\underline{x}^{(1)};t) \oplus H(\underline{y}^{(1)};t)$ for for $t \geq \tau$.

Lemma 5.7 Let a and b be arbitrary real numbers. If we write

$H(\underline{x}^{(1)};a,b) = H(\underline{x}^{(1)};b) \ominus H(\underline{x}^{(1)};a)$ then

$$(5.11)\qquad H(\underline{x}^{(1)};a,b) = \mathfrak{S}\{g_\alpha^\beta,\ a < \alpha \le \beta \le b\} = \mathfrak{S}\{(E(\beta) - E(\alpha))g_a^b\ a < \alpha \le \beta \le b\}$$

Proof: The second half of relation (5.11) is obvious since $[E(\beta) - E(\alpha)]g_a^b = g_\alpha^\beta$

for $a < \alpha \le \beta \le b$. To prove the first part we proceed as follows: For

$a < t \le b$ and $\varphi \in \Phi$, $\underline{x}_t^{(1)}(\varphi) - P_{H(\underline{x}^{(1)};a)}\, x_t^{(1)}(\varphi) = P_{H(\xi_1;t)}\underline{x}_t(\varphi) - P_{H(\underline{x}^{(1)};a)}\underline{x}_t^{(1)}(\varphi)$

From Lemma 5.6 and (5.10),

$\underline{x}_t^{(1)}(\varphi) - P_{H(\underline{x}^{(1)};a)}\underline{x}_t^{(1)}(\varphi) = P_{H(\xi_1;t)}\, \underline{x}_t(\varphi) - P_{H(\underline{x};a)}\, P_{H(\xi_1;t)}\, \underline{x}_t(\varphi)$. Furthermore, for $a \le t$, writing $H(\xi_1;a,t) = H(\xi_1;t) \ominus H(\xi_1;a)$

$$(5.12)\qquad H(\xi_1;t) = H(\xi_1;a) \oplus H(\xi_1;a,t) \text{ and } \underline{x}_\tau(\varphi) \perp H(\xi_1;a,t).$$

The latter assertion follows from (5.8) and the definition of g_a^b. Thus, we have

$P_{H(\underline{x};a)}\, P_{H(\xi_1;t)} = P_{H(\underline{x};a)}\, \{P_{H(\xi_1;a)} + P_{H(\xi_1;a,t)}\} = P_{H(\underline{x};a)}\, P_{H(\xi_1;a)}$. Further,

since $H(\xi_1;a) \subset H(\underline{x};a)$, we have

$\underline{x}_t^{(1)}(\varphi) - P_{H(\underline{x}^{(1)};a)}\, \underline{x}_t^{(1)}(\varphi) = P_{H(\xi_1;t)}\, \underline{x}_t(\varphi) - P_{H(\xi_1;a)}\, \underline{x}_t(\varphi)$. Hence

$H(\underline{x}^{(1)};a,b) \subset H(\xi_1;a,b)$ which from (5.8) is the same as

$\mathfrak{S}\{g_\alpha^\beta,\ a < \alpha \le \beta \le b\}$. To complete the proof we have only to observe, because of Lemma 5.5, that for $a < \alpha \le \beta \le b$, g_α^β is in $H(\underline{x};a,b)$ and is orthogonal to $H(\underline{y}^{(1)};a,b)$.

Let $\hat{\underline{x}}_t^{(1)}(\varphi) = \underline{x}_t^{(1)}(\varphi) - P_{H(\underline{x}^{(1)};a)}\, \underline{x}_t^{(1)}(\varphi)$. From Lemma 5.7, it follows that

$a < t \le b$ and $\varphi \in \Phi$.

$$(5.13)\qquad \hat{\underline{x}}_t^{(1)} = \int_a^t F(\varphi;\ t,u)\, d\, E(u) g_a^b \quad \text{where } \int_a^b |F(\varphi;t,u)|^2 d\mu(u) \text{ is finite.}$$

We are now in a position to prove the following result

Lemma 5.8 The operator A is reduced by $H(\underline{x}^{(i)};a,b)$.

Proof: It suffices to prove that for $a < t \leq b$ and $\varphi \in \Phi$, $A\hat{\underline{x}}_t^{(i)}(\varphi) \in H(\underline{x}^{(i)};a,b)$ since $H(\underline{x}^{(1)};a,b) = \mathfrak{S}\{\underline{x}_t^{(1)}(\varphi),\ \varphi \in \Phi\ a < t \leq b\}$. From (5.13)

$$A\hat{\underline{x}}^{(1)}(\varphi) = \int_a^t u\, F(\varphi;\, t,u)d\, E(u)g_a^b \quad \text{where } F(\varphi;\, t,u) \in L_2(\mu^I). \quad \text{Hence}$$

$A\hat{\underline{x}}_t^{(1)}(\varphi) \in \mathfrak{S}\{(E(\beta) - E(\alpha))g_a^b\ a < \alpha < \beta \leq b\}$ since $u\, F(\varphi;t,u) \in L_2(\mu^I)$.

From the preceding lemma it now follows that $A\hat{\underline{x}}_t^{(i)}(\varphi) \in H(\underline{x}^{(i)};a,b)$.

Lemma 5.9 $H(\underline{x}^{(i)})$ reduces the operator A.

Proof: From the properties of the resolution of the identity corresponding to A, we have

$$E(\Delta)A = A\, E(\Delta) \tag{5.14}$$

for every finite subinterval $\Delta = (a,b]$. If w is any element belonging to $\mathscr{D}_A \cap H(\underline{x}^{(1)})$ (which is non-empty) where $\mathscr{D}_A$ is the domain of A, then from Lemma 5.8 we have

$$E(\Delta)Aw = A\, E(\Delta)w = A\, P_{H(\underline{x}^{(i)};a,b)}\, w \in H(\underline{x}^{(i)};a,b).$$

Now letting $a = n - 1$, $b = n$ and $\Delta_n = (n - 1, n]$ we obtain

$$Aw = \sum_{n=-\infty}^{\infty} E(\Delta_n)Aw \in \sum_{n=-\infty}^{\infty} \oplus\, H(\underline{x}^{(i)};\, n\ 1,\, n) = H(\underline{x}^{(1)}).$$

Let $A^{(i)}$ be the reduction of A to $H(\underline{x}^{(i)})$. Then (Lemma 5.8) clearly $A^{(i)}$ is reduced by $H(\underline{x}^{(i)};\, a,b)$. We denote this operator on $H(\underline{x}^{(1)};a,b)$ by $A_I^{(i)}$ $(I = (a,b])$. An immediate implication of Lemma 5.6 is that $A_I^{(1)}$ is a cyclic operator with generating element g_a^b. We recall from Lemma 5.3 that the spectral function of g_a^b is given by $\rho_{g_a^b} = \mu^I$.

Now let $I_j = (a_j, b_j]$ $(j = 1,2,\ldots)$ be disjoint intervals whose union is the real line. If ρ_j denotes the spectral type of the operator $A_j^{(1)}$. (which we write here for $A_{I_j}^{(1)}$) then it is easy to verify that the ρ_j's are independent spectral types. For let j and m be arbitrary $(j \neq m)$ and suppose that σ is a measure whose spectral type is dominated by both ρ_j and ρ_m. For all $k \neq j$ since $\mu^{I_j}(I_k) = 0$ we have $\sigma(I_k) = 0$. But $\sigma(I_j)$ is also equal to zero since $\mu^{I_m}(I_j) = 0$. Hence $\sigma = 0$. Summarizing all the above facts we find that we have a representation of $A^{(1)}$ as the orthogonal sum of cyclic operators $A_{I_j}^{(1)}$ whose corresponding spectral types ρ_j are independent.

It then follows that ([12] p. 152), $A^{(1)}$ itself is cyclic and since the spectral function μ^{I_j} belongs to the type ρ_j for each j we can conclude moreover that the spectral type of $A^{(1)}$ is equivalent to μ. From Lemma 5.4 it follows that the spectral type of $A^{(1)}$ is equal to ρ, the maximal spectral type of A.

Let us recall that $H(\underline{x}) = H(\underline{x}^{(1)}) \oplus H(\underline{y}^{(1)})$ and the self-adjoint operator A is reduced by $H(\underline{x}^{(1)})$. Hence A can be written as the orthogonal sum of the reduced operators, $A = A_{H(\underline{x}^{(1)})} + A_{H(\underline{y}^{(1)})}$.

Now, $A_{H(\underline{y}^{(1)})}$, a self-adjoint operator on $H(\underline{y}^{(1)})$ is the operator of the weakly stationary non-deterministic process $\{\underline{y}_t^{(1)}, -\infty < t < +\infty\}$. We may, therefore, apply the above analysis to this process replacing $H(\underline{x})$ by $H(\underline{y}^{(1)})$ and A by $A'_{H(\underline{y}^{(1)})}$. We then have, $H(\underline{y}^{(1)}) = H(\underline{x}^{(2)}) \oplus H(\underline{y}^{(2)})$, where the $\underline{x}_t^{(2)}$ process is constructed from the $\underline{y}_t^{(1)}$ -process in the same way as the $\underline{x}_t^{(1)}$ -process is obtained from the given $\underline{x}_t$ -process. The $\underline{y}_t^{(2)}$ -process is stationary and purely non-deterministic. We also have the orthogonal

decomposition
$$A = A^{(1)} + A^{(2)} + A_{H(\underline{y}^{(2)})},$$

where $A^{(i)} = A_{H(\underline{x}^{(i)})}$. Continuing the above procedure we arrive at the following relations,

(5.15) $\quad H(\underline{x}) = H(\underline{x}^{(1)}) \oplus H(\underline{x}^{(2)}) \oplus \ldots \oplus H(\underline{x}^{(M)}).$

(5.16) $\quad A = A^{(1)} + A^{(2)} + \ldots + A^{(M)}.$

where $\underline{x}_t^{(i)}(\varphi) = P_{H(\xi_i)}\underline{x}_t(\varphi)$ and $\{\xi_i(u), -\infty < u < +\infty\}$ are mutually orthogonal processes with stationary orthogonal increments. The operators $A^{(i)}$ are cyclic, all having the same spectral type ρ (the maximal spectral type of A). Further M is a cardinal number at most equal to $\aleph_0$.

Also from Lemmas 5.5, 5.6 and 5.7, we have

(5.17) $$\underline{x}_t^{(i)}(\varphi) = \int_{-\infty}^{t} F_i(\varphi;u-t)\, d\xi_i(u)$$

with

(5.18) $$H(\underline{x};t) = \sum_{i=1}^{H} \oplus\, H(\underline{x}^{(i)};t) = \sum_{n=1}^{M} \oplus\, H(\xi_i;t).$$

Let $f^{(i)}$ be the generating element of $A^{(i)}$. Since $\{E(b) - E(a)\}\, f^{(i)} = P_{H(\underline{x}^{(i)};a,b)}\, f^{(i)}$, clearly $H(\underline{x}^{(i)})$ is the cyclic subspace generated by $f^{(i)}$, i.e.,

(5.19) $\quad H(\underline{x}^{(i)}) = \mathfrak{S}\{E(\Delta)\, f^{(i)},\ \Delta$ ranging over all finite subintervals of the real line$\}$. We also have $\rho_{f^{(i)}} \equiv \mu$. From (5.15) and (5.19), we have

$$H(\underline{x}) = \sum_{i=1}^{M} \oplus\, \mathfrak{S}\{E(\Delta) f^{(i)},\ \Delta \text{ ranging over all finite subintervals}\}$$ and

(5.20) $\quad \rho_f(1) \equiv \rho_f(2) \equiv \cdots \equiv \rho_f(M).$

Hence, it follows that M is the multiplicity of the $\underline{x}_t$-process. (See Section 2 where this notion is defined). Assembling all the results of this section together we observe that we have established the following basic representation theorem.

Theorem 5.1 Let $\underline{x}_t$ $(-\infty < t < +\infty)$ be a weakly stationary, purely non-deterministic process on Φ satisfying (C). Then

$$(5.21) \qquad \underline{x}_t(\varphi) = \sum_{i=1}^{M} \int_{-\infty}^{t} F_i(\varphi; u-t)\, d\xi_i(u),$$

where,

(i) M is the multiplicity of the process,

(ii) each $\xi_i(u)$ is a process with stationary orthogonal increments (homogeneous process) and the ξ_i's are mutually orthogonal. Furthermore,

$$H(\underline{x};t) = \sum_{i=1}^{M} \oplus H(\xi_i;t) \text{ for every real } t, \text{ and } \sum_{1}^{M} \int_{-\infty}^{0} |F_i(\varphi;u)|^2 d\mu(u)$$

is finite.

It can be easily seen that the homogeneous processes ξ_i, $(i=1,2,\ldots M)$ of the representation (5.21) are uniquely determined upto a unitary equivalence.

The above theorem is a generalization of the Karhunen representation to stationary stochastic processes $\underline{x}_t$ on Φ. This result also generalizes the Rozanov-Gladyshev representation for q-dimensional stationary processes as will be seen in the next section. The reader will observe that (5.21) has been derived essentially independently of the Hida representation (2.1) and the latter is referred to at the end of the proof only for the purpose of identifying M as the multiplicity of the process. Indeed, the whole point of the problem is to study the maximal spectral type and to construct the homogeneous processes $\xi_i(u)$. Once (5.21) has been obtained, however, it is easy to discover the special properties that the representation possesses in this case, e.g to see that all the elements $f^{(i)}$ occuring in it are equivalent, with a common spectral type equivalent to μ. Moreover, starting with the ξ_i's one can construct without difficulty a sequence $\{f^{(i)}\}$ for the representation

(2.1) of Section 2. This can be done as follows: It is clear that the elements f_i $(i = 1,\dots,M)$ occurring in the proof of Theorem 5.1 and with the property that they have all the same spectral type equivalent to μ (see(5.19 and (5.20)) can be chosen as the elements in the Hida representation of $\underline{x}_t$. If we now set

$$\xi_i(\Delta) = \int_\Delta \left[\frac{d\rho_{f_i}}{d\mu}(u)\right]^{-\frac{1}{2}} dE(u)f_i \quad ,$$

it is easy to verify that the ξ_i are mutually orthoganal random set functions each having μ as its measure function, and that (Δ being a finite interval)

$$E(\Delta)f_i = \int_\Delta \left[\frac{d\rho_{f_i}}{d\mu}(u)\right]^{\frac{1}{2}} d\xi_i(u).$$

If we now make the appropriate substitution in (2.1) and compare it with the representation (5.21) it follows that for each t and φ

$$F_i(\varphi;t,u) = F_i(\varphi;\, u-t)\left[\frac{d\rho_{f_i}}{d\mu}(u)\right]^{-\frac{1}{2}} \quad (i=1,..,M)$$

a.e. with respect to μ.

Thus, for stationary processes, the generalization of the approach of Hanner given in Theorem 5.1 leads to a deeper analysis which includes the proof of (5.19) and (5.20) and yields directly the representation we seek. It is interesting to explore further the connection between ρ and M. The following discussion presents another aspect of the problem and provides additional information.

<u>Theorem 5.2</u> ρ is a uniform spectral type with (uniform) multiplicity M.

<u>Proof</u>: We use the ideas of Plessner and Rohlin [12]. It will first be shown that ρ has multiplicity M. Let $\{A'_\beta\}$ be an orthogonal system of type ρ and cardinality M', i.e., a system of orthogonal cyclic parts A_β' of the operator A, the spectral type of each cyclic operator A'_β being ρ. According to to the terminology of [12] M is the multiplicity of ρ if we can prove that

$M' \leq M$. Observe that neither M nor M' can exceed $\aleph_0$, for otherwise we would arrive at a contradiction of the fact that $H(\underline{x})$ is separable. Furthermore, there is obviously nothing to prove if $M = \aleph_0$. Thus the only case to be considered is when M is a finite cardinal. If possible let $M' > M$. We shall show that this leads to a contradiction. Let h_i $(i=1,\dots,M)$ be a generating element of the subspace $H(\underline{x}^{(i)})$ and h'_β $(\beta = 1,\dots,M')$ be similarly a generating element of the cyclic subspace corresponding to A'_β. Clearly, there is no loss of generality in supposing that all these elements have the same spectral function, say, ρ'. From (5.15) and (5.19) it follows that for each β we have

$$h'_\beta = \sum_{i=1}^{M} \int F_{i\beta}(u)\, dE(u) h_i \quad \text{where} \quad \sum_i \int |F_{i\beta}(u)|^2\, d\rho'(u) \quad \text{is}$$

finite. For every measurable set Δ we obtain

$$\mathcal{E}\{E(\Delta)h'_\beta,\ h'_\gamma\} = \int_\Delta \sum_{i=1}^{M} F_{i\beta}(u)\, \overline{F_{i\gamma}(u)}\, d\rho'(u).$$

The left hand side of the above relation is zero if $\beta \neq \gamma$ and equals $\rho'(\Delta)$ if $\beta = \gamma$. Hence for u not belonging to a set $N_{\beta\gamma}$ of zero ρ'-measure we have

$$\sum_{i=1}^{M} F_{i\beta}(u)\, \overline{F_{i\gamma}(u)} = \delta_{\beta\gamma}.$$

Since M' is at most $\aleph_0$ the set $N = \bigcup_{\beta,\gamma} N_{\beta\gamma}$ is measurable and $\rho'(N) = 0$. Choosing a fixed point u_0 in the complement of N we see that

$$(5.22) \qquad \sum_{i=1}^{M} F_{i\beta}(u_0)\, \overline{F_{i\gamma}(u_0)} = \delta_{\beta\gamma} \quad \text{for all } \beta,\ \gamma.$$

If we now set $a_\beta = \{F_{1\beta}(u_0),\dots, F_{M\beta}(u_0)\}$, the relations (5.22) imply that the a_β are M' orthonormal vectors in M dimensional unitary space. Hence M' cannot exceed M. In other words ρ has multiplicity M.

The proof that the spectral type ρ is uniform is achieved by a modification of the above argument. The reader will no doubt, observe that the conclusion about uniformity rests on the fact that the orthogonal system $[A^{(i)}, i=1,\dots M]$ is not only maximal but that the orthogonal sums of the $A^{(i)}$ is equal to A (see (5.16)).

Let σ by any spectral type dominated by ρ. The only change we make in the proof given above is to let $\{A'_\beta\}$ be an orthogonal system of type σ and cardinality M'. Let h'_β be a generating element of the cyclic subspace of A'_β. Assuming, as we may that the h_i have all the same spectral function ρ' and that the h'_β have the same spectral function σ' we obtain the relations

$$(5.23) \qquad \sum_{i=1}^{M} F_{i\beta}(u)\,\overline{F_{i\gamma}(u)} = \frac{d\sigma'}{d\rho'}(u)\ \delta_{\beta\gamma}, \quad \text{where } u \notin N \text{ and } \frac{d\sigma'}{d\rho'}$$

is the Radon-Nikodyn derivative of σ' with respect to ρ'. Since the set $S = \{u: \frac{d\sigma'}{d\rho'}(u) > 0\}$ has positive ρ' -measure we can choose u_0 in $S \cap N^c$ when as before N is the set of zero ρ'-measure. Substituting u_0 for u in the relations (5.23), we are again led to the conclusion that $M' \leq M$. Thus it has been shown that the multiplicity of any spectral type dominated by ρ is equal to the multiplicity of ρ . Hence ρ is a uniform spectral type.

Remark: It follows at once from the theorem just proved that every spectral type belonging to the operator A of the stationary process $\underline{x}_t$ has multiplicity M.

To find the funtions F_i and the value of M in the representation (5.21) in specific instances one would have to consider, individually, concrete examples of spaces Φ and purhaps have to assume additional properties of the process $\underline{x}_t$ such as linearity in φ. The study of some of these questions we postpone to a later paper. However, since it is important to relate our work to recent developments in the theory of multidimensional stationary processes we consider in the next section the case when Φ is a q-dimensional unitary space.

6. Multiplicity as a Generalization of Rank. In the theory of finite dimensional weakly stationary processes the notion of rank plays a conceptually essential role. Zasuhin, in 1941, was the first to define the rank of a q-dimensional, discrete parameter stationary process as the rank of the $(q \times q)$ "error matrix" (See [18]). More recently, the definition of rank for a continuous parameter process has been given by Gladyshev [5] to be the rank of the discrete parameter process associated with the process. This point of view has been further explored in the recent thesis of Robertson [14]. It is also well known in the literature that the rank of the process is equal to the rank of the spectral density matrix. (See [15] where the rank is defined this way and [14].)

We shall show in this section that the multiplicity M occurring in the representation given in Theorem 5.1 constitutes a generalization of rank in the following sense: If $\underline{x}_t$ is a weakly stationary process on Φ where Φ may be infinite dimensional (and $\underline{x}_t(\varphi)$ itself may or may not be linear in φ) then M is equal to the multiplicity of the associated discrete process (Theorem 6.1). In the case where Φ is a q-dimensional unitary space and $\underline{x}_t(\varphi)$ is linear in φ, so that we are dealing with a q-dimensional stationary process, it is shown in Theorem 6.2 that the multiplicity equals the rank of the process and the representation of Theorem 5.1 coincides with that obtained in [5] and [14].

The connection between multiplicity and spectral theory for infinite dimensional stationary processes $\underline{x}_t$ will be considered in a later paper.

If $\{\underline{x}_t\}$ $(-\infty < t < +\infty)$ is a given stationary stochastic process on Φ satisfying condition (C), then for each φ, the one dimensional weakly stationary process $\{\underline{x}_t(\varphi)\}$ is continuous in q.m. and hence for fixed φ, $\underline{x}_t(\varphi) \doteq \int_{-\infty}^{+\infty} e^{it\lambda} d_\lambda G(\lambda) \underline{x}_0(\varphi)$ where $\{G(\lambda), -\infty < \lambda < +\infty\}$ is a resolution of the identity of the unitary group $\{T_h\}$ of the $\underline{x}_t$ process.

With the process $\{\underline{x}_t(\varphi)\}$ (for fixed φ) is associated a discrete parameter process, (6.1) $\tilde{\underline{x}}_n(\varphi) = \int_{-\pi}^{\pi} e^{in\lambda}\, d_\lambda G(\tfrac{1}{2\pi} \tan^{-1}\lambda)\, \underline{x}_0(\varphi)$, $(n = 0, \pm 1, \ldots)$ [[4], [11]].

Let us now write for each φ and t, $H_\varphi(x;t) = \mathfrak{S}\{x_\tau(\varphi), \tau \leq t\}$ and $H_\varphi(\tilde{x};m) = \mathfrak{S}\{\tilde{\underline{x}}_n(\varphi), n \leq m\}$ (m any integer). We have for all φ, $H_\varphi(x;+\infty) = H_\varphi(\tilde{x};+\infty)$ and $H_\varphi(x;0) = H_\varphi(\tilde{x};0)$ (See [4], [11]). Therefore,

(6.2) $H(\underline{x}; +\infty) = H(\tilde{\underline{x}}; +\infty)$ and $H(\underline{x};0) = H(\tilde{\underline{x}};0)$.

From stationarity and (6.2), the following lemma is immediate.

<u>Lemma 6.1</u>: $\{\underline{x}_t, -\infty < t < +\infty\}$ is deterministic if and only if $\{\tilde{\underline{x}}_n, n = 0, \pm 1, \ldots\}$ is deterministic.

We recall here two lemmas from [5] which will be frequently used in what follows. It should be observed that in Lemma (G_2) stated below the process can be infinite-dimensional. Its proof, however, involves no change and is an easy consequence of (6.2).

<u>Lemma</u> (G_1). If $\{\eta_t\}$ is a one-dimensional weakly stationary, continuous in q.m., purely non-deterministic process, then the $\tilde{\eta}_n$- process is purely non-deterministic.

<u>Lemma</u> (G_2). If $\{\underline{\eta}_t\}$ and $\{\underline{\zeta}_t\}$ are stationary processes on Φ satisfying condition (C) and such that $H(\underline{\eta};t) \subset H(\underline{\zeta};t)$ for all t, then $H(\tilde{\underline{\eta}};m) \subset H(\tilde{\underline{\zeta}};m)$ for every m and conversely.

We shall now obtain from Theorem 5.1, a representation for the $\tilde{\underline{x}}_n$- process. The notation will be that of Section 5. Let us define for each $i = 1,2\ldots M$,

(6.3) $\underline{x}_t^{(i)}(\varphi) = \int_{-\infty}^{t} F_i(\varphi;u-t)\,d\xi_i(u)$, where the right hand side expression is the term appearing in the representation (5.21) of $\underline{x}_t(\varphi)$. Consider now the process $h^{(i)}(t) = \int_{-\infty}^{t} e^{s-t}\,d\xi_i(s)$ $(-\infty < t < +\infty)$. Then $\{h^{(i)}(t)\}$ is a one dimensional stationary stochastic process with $T_t h^{(i)}(0) = h^{(i)}(t)$. Furthermore, since

$\xi_i(t) - \xi_i(s) = \{h^{(i)}(t) - h^{(i)}(s)\} + \int_s^t h^{(i)}(u)du$ $(s < t)$, it follows that

for all t

$$(6.4) \qquad H(\xi_i;t) = H(h^{(i)};t) \quad (i = 1,2,\ldots.M).$$

The $h_t^{(i)}$- process which is obviously continuous in q.m., is also purely non-deterministic, since from (6.4), $\bigcap_{-\infty}^{+\infty} H(h^{(i)};t) = \bigcap_{-\infty}^{+\infty} H(\xi_i;t) \subset \bigcap_{-\infty}^{+\infty} H(\underline{x};t)$. The discrete parameter process $\{\tilde{h}^{(i)}(m)\}$ is thus purely non-deterministic and therefore has a moving average representation given by

$$(6.5) \qquad \tilde{h}^{(i)}(m) = \sum_{\ell=0}^{\infty} b_i(\ell)u_i(m-\ell) ,$$

where

$$(6.6) \qquad H(\tilde{h}^{(i)};m) = \mathfrak{S}\{u_i(m-\ell),\ 0 \leq \ell < +\infty\}$$ and $\{u_i(m)\}$ (for fixed i) is a process with stationary orthogonal increments. From (6.2), (6.4), (6.6) and the mutual orthogonality of $\{\xi_i(n)\}$, it follows that the processes $\{u_i(n)\}$ $(i=1,2\ldots.M)$ are mutually orthogonal. Also from (6.3) and (6.4), $H(\underline{x}^{(i)};t) \subset H(h^{(i)};t)$ for each t. But from Lemma (G_2) and (6.6), $H(\tilde{\underline{x}}^{(i)};m)$ is a subspace of $\mathfrak{S}\{u_i(m-\ell),\ \ell=0,1,2\ldots.\}$. Hence

$$(6.7) \qquad \tilde{x}_m^{(i)}(\varphi) = \sum_{\ell=0}^{\infty} c_i(\varphi;\ell)u_i(m-\ell) .$$

From (6.3) the $\{x_t^{(i)}(\varphi)\}$ process is stationary and continuous in q.m. with $T_t\underline{x}_s^{(i)}(\varphi) = x_{s+t}^{(i)}(\varphi)$. Hence $\underline{x}_t^{(i)}(\varphi) = \int_{-\infty}^{+\infty} e^{it\lambda} d_\lambda G(\lambda)\underline{x}_0^{(i)}(\varphi)$. Furthermore,

$$(6.8) \qquad \underline{x}_t(\varphi) = \sum_1^M \underline{x}_t^{(i)}(\varphi) \quad \text{for every } t;$$

where the (possibly) infinite series converges in q.m., since $\sum_1^M \mathcal{E}|\underline{x}_t^{(i)}(\varphi)|^2$ is finite. Also,

$$(6.9)\qquad \tilde{\underline{x}}_n^{(i)}(\varphi) = \int_{-\pi}^{\pi} e^{in\lambda} d_\lambda G(\tfrac{1}{2\pi} \tan^{-1}\lambda)\underline{x}_0(\varphi) .$$

Since $S \doteq \int_{-\pi}^{\pi} e^{i\lambda} d_\lambda G(\frac{1}{2\pi}\tan^{-1}\lambda)$ is a bounded linear (in fact, unitary) operator on $H(\underline{x})$ from (6.8) [with t=0], (6.9) and (6.1), we have

$$(6.10)\qquad \tilde{\underline{x}}_n(\varphi) = \sum_1^M \tilde{\underline{x}}_n^{(i)}(\varphi) .$$

From (6.7) and (6.10) $\tilde{\underline{x}}_n(\varphi) = \sum_{i=1}^{M} \sum_{\ell=-\infty}^{n} C_i(\varphi;n-\ell)u_i(\ell)$. From Theorem 5.1 and (6.4) $H(\underline{x};t) = \sum_{i=1}^{M} \oplus H(\xi_i;t) = \sum_{i=1}^{M} \oplus H(h^{(i)};t)$. In other words

$$(6.11)\qquad H(\underline{x};t) = \mathfrak{S}\{h^{(i)}(\tau),\ \tau \le t,\ i = 1,2,\dots M\} .$$

From Lemma (G_2), (6.11) and (6.6) we have

$$(6.12)\qquad H(\tilde{\underline{x}};m) \doteq \sum_{i=1}^{M} \oplus H(\tilde{h}^{(i)};m) = \sum_{i=1}^{M} \oplus \mathfrak{S}\{u_i(m-\ell),\ \ell = 0,1,2\dots\} .$$

(6.11) and (6.12) imply (see Theorem 4.1) that

$$(6.13)\qquad M = \dim\{H(\tilde{\underline{x}};n) \ominus H(\tilde{\underline{x}};n-1)\} .$$

We summarize the above results.

Theorem 6.1. Let $\underline{x}_t(-\infty < t < +\infty)$ be a stationary, purely non-deterministic process satisfying condition (C). Then its multiplicity is equal to the common dimension of the subspaces $H(\tilde{\underline{x}};n) \ominus H(\tilde{\underline{x}};n-1)$.

The above discussion pertaining to multiplicity is very general since we have been dealing with weakly stationary processes on an arbitrary Hausdorff space, satisfying the second countability axiom. It is instructive to consider the case when Φ is a finite dimensional unitary space and the process $\underline{x}_t$ is linear on Φ. We have referred to the fact that some recent work of H. Cramér [2] can be regarded as a special case of the results of Section 2. In [2], Cramér also includes a brief discussion of the stationary case and shows that the multiplicity of the q-dimensional process does not exceed q. We shall now deduce from Theorem 6.1 that the multiplicity is actually equal to the rank of the process. This corollary (Theorem 6.2), incidentally, provides an alternative proof of a theorem due to Gladyshev (Theorem 1, [5]).

Suppose $\{e_i\}$ $(i=1,2,\ldots q)$ is an orthonormal basis in Φ. If $\{\underline{x}_t\}$ is a weakly stationary process linear in φ then, if $\varphi = \sum_{i=1}^{q} a_i e_i$, $\underline{x}_t(\varphi) = \sum_{i=1}^{q} a_i x_i(t)$ where $x_i(t) = \underline{x}_t(e_i)$. Now, $(x_1(t), x_2(t),\ldots x_q(t))$ is a q-dimensional process which is weakly stationary. Since $\{\underline{x}_t\}$ satisfies condition (C), $\{x_i(t)\}$ $(i=1,2\ldots q)$ are continuous in q.m. Also, if $(x_1(t), x_2(t),\ldots,x_q(t))$ is a q-dimensional weakly stationary process continuous in q.m. then there corresponds a stationary process $\{\underline{x}_t\}$ on the q-dimensional unitary space Φ which is linear in φ and satisfies condition (C); [viz., $\underline{x}_t(\varphi) = \sum_{i=1}^{q} a_i x_i(t)$ if φ is the vector $(a_1,a_2,\ldots,a_q)$]. Furthermore, $H(\underline{x};t) = \mathfrak{S}[x_i(u),\ u \leq t,\ i = 1,2,\ldots,q]$.

<u>Theorem 6.2</u>. Let $(x_1(t),x_2(t),\ldots,x_q(t))$ be a continuous in q.m., purely non-deterministic, weakly stationary process. Then

$$x_i(t) = \sum_{i=1}^{M} \int_{-\infty}^{t} F_{in}(u-t)d\xi_i(u)$$

where the ξ_i-processes and the number M are as introduced in Theorem 5.1,

$$\mathfrak{S}[x_i(u),\ u \leq t,\ i=1,2,\ldots,q] = \sum_{i=1}^{M} \oplus H(\xi_i;t)$$ and M is the rank of the process.

<u>Proof</u>: All the assertions of the theorem follow immediately upon setting $\varphi = e_i$ in the representation obtained in Theorem 5.1. It remains only to show that M is the rank of the process. From Theorem 6.1 and Lemma 4.1 it follows that $M = \dim[H(\tilde{\underline{x}};n) \ominus H(\tilde{\underline{x}};n-1)] = \dim\{\mathfrak{S}[\tilde{g}_n(\varphi),\ \varphi\epsilon\Phi]\}$. Writing $\tilde{\mathfrak{S}}_n = \mathfrak{S}[\tilde{x}_i(m),\ m \leq n,$ integer, $\ell=1,2,\ldots,q]$ and $\tilde{g}_i(n) = \tilde{x}_i(n) - P_{\tilde{\mathfrak{S}}_{n-1}}\tilde{x}_i(n)$ we find that

$\tilde{g}_n(\varphi) = \sum_{i=1}^{q} a_i \tilde{g}_i(n)$. Therefore, $M = \dim \mathfrak{S}[g_i(n),\ i=1,2,\ldots,q]$. But the latter quantity is the rank of the $q \times q$ "error matrix" with elements $\mathcal{E}\tilde{g}_i(0)\tilde{g}_j(0)$, $(i,j = 1,2,\ldots,q)$, i.e., the rank of the process $(\tilde{x}_1(n),\tilde{x}_2(n),\ldots,\tilde{x}_q(n))$ [[18]]. Hence the multiplicity M of $\underline{x}_t$-process (Theorem 5.1) equals its rank.

Theorems 4.1, 5.1 and 6.1, apply to weakly stationary processes $\underline{x}_t$ on a Hausdorff space Φ. The only assumptions on the process is that it satisfies condition (C) and is purely non-deterministic, while no condition is imposed on Φ other than that its topology satisfy the second countability axiom. If, in particular, Φ is a locally convex linear space (e.g. if Φ is an infinite-dimensional separable Hilbert space) with a countable basis $\{e_i\}$ and if $\underline{x}_t(\varphi)$ is linear in φ (e.g. $\underline{x}_t$ is a <u>weak</u> process on Φ) then we may consider the $\underline{x}_t$-process as having an infinite number of components $x_t^{(i)} = \underline{x}_t(e_i)$ $(i = 1,2,\ldots,)$. Thus we may conclude from these results and Theorem 6.2 that for infinite-dimensional processes the representation given in Theorem 5.1 is a generalization of the Karhunen-Gladyshev representation and that the multiplicity is the appropriate generalization of rank.

HILBERT-SPACE VALUED PROCESSES

7. **Preliminaries**. In Theorem 2.2, and for the stationary case in Theorem 5.1 we obtained a representation of the purely non deterministic process on an arbitrary Hausdorff space Φ. Suppose now that Φ is a locally convex linear space and that for each t, $\underline{x}_t$ is a random variable taking values in Φ', the dual space of Φ; i.e., for each t, there exists a mapping $\underline{x}_t$ from Ω to Φ' such that (i) $< x_t, \varphi >$ [$< \varphi, \varphi' >$ denotes the value of the functional φ' at φ] is a random variable on Ω, and (2) for all $\varphi \epsilon \Phi$, $\underline{x}_t(\varphi)\ [\omega] = < \underline{x}_t(\omega), \varphi >$ with probability one. As is well-known these assumptions are stronger than the ones made in the concluding paragraph of Section 6 dealing with weak processes. We shall call $\{\underline{x}_t\}$ defined as above a process in Φ'. The definitions of deterministic and purely non-deterministic processes in Φ' are the same as the ones given in the Introduction.

By a representation of a purely non-deterministic process $\{\underline{x}_t\}$ in Φ', we mean a process $\{\underline{y}_t\}$ in Φ' such that, $\underline{x}_t = \underline{y}_t$ with probability one for each t and $\underline{y}_t$ represents a "moving average" over the present and past of $\underline{x}_t$-process analogous to what was obtained in Theorem 2.2. In this section we confine our attention to the case in which Φ is a real separable Hilbert space and refer to $\{\underline{x}_t\}$ as a process in Φ. Although this is the only case studied in detail here, we feel that a similar theory can be developed to cover more general situations, e.g., where Φ is a separable, reflexive Banach space or a nuclear space. The last mentioned problem could well have points of contact with recent work of K. Urbanik and others on the representation of purely non-deterministic homogeneous generalized random fields ([17]).

We shall also make the stronger assumption that $\mathcal{E}||\underline{x}_t||^2$ is finite for each t, with the help of which we are able to prove a strengthened form of the

Wold decomposition stated in Section 2.

Proposition 7.1. Let $\{\underline{x}_t\}$ be a process in Φ with $\mathcal{E}||\underline{x}_t||^2 < \infty$, for each t. Then, with probability one we have $\underline{x}_t = \underline{x}_t^{(1)} + \underline{x}_t^{(2)}$ and $\underline{x}_t^{(i)}$ $i=1,2$ which are defined except possibly for an ω-set of probability zero, have the following properties:

(1) $\{\underline{x}_t^{(1)}\}$ and $\{\underline{x}_t^{(2)}\}$ are processes in Φ with $\mathcal{E}||\underline{x}_t^{(i)}||^2 < \infty$ $(i = 1,2)$;

(2) $H(\underline{x}^{(1)})$ is orthogonal to $H(\underline{x}^{(2)})$, and

(3) $\{\underline{x}_t^{(1)}\}$ is deterministic and $\{\underline{x}_t^{(2)}\}$ is purely non-deterministic.

Proof: The process $\tilde{\underline{x}}_t(\varphi) = < \underline{x}_t, \varphi >$ is a stochastic process on Φ. Hence Proposition 2.1 gives us $\tilde{\underline{x}}_t(\varphi) = \tilde{\underline{x}}_t^{(1)}(\varphi) + \tilde{\underline{x}}_t^{(2)}(\varphi)$. It suffices to show that $\tilde{\underline{x}}_t^{(i)}(\varphi) = < \underline{x}_t^{(i)}, \varphi >$ $(i = 1,2)$ where $\{x_t^{(i)}\}$ are processes in Φ with the above mentioned properties. This is achieved by means of the following lemma.

Lemma 7.1. Let $\{\underline{x}_t\}$ be a process in Φ and let P be a projection operator onto an arbitrary subspace M of $H(\underline{x};t)$. Then there exists an almost everywhere weakly measurable mapping $\underline{x}_{t,p}$ from Ω to Φ such that with probability one $< \underline{x}_{t,p}, \varphi > = P < \underline{x}_t, \varphi >$ for every $\varphi \in \Phi$.

Proof: Let t be fixed. It is well-known that our assumptions on $\underline{x}_t$ imply that for all φ_1, φ_2 in Φ $\mathcal{E}[< \underline{x}_t, \varphi_1 > \cdot < \underline{x}_t, \varphi_2 >] = < B_t \varphi_1, \varphi_2 >$, where B_t is an S-operator (see [13]). Choosing a complete orthonormal (C.O.N.) system of eigenelements corresponding to the eigenvalues $\{\lambda_n\}$ of B_t and observing that B_t has finite trace, we obtain $\sum_1^\infty [P < \underline{x}_t(\omega), \varphi_n >]^2 < \infty$. This implies that there is an ω--set N of zero probability such that

$$(7.1) \qquad \sum_1^\infty [P < \underline{x}_t(\omega), \varphi_n >]^2 \text{ is finite, if } \omega \notin N.$$

For every $\varphi \in \Phi$ and $\omega \notin N$, define

$$(7.2) \qquad \eta_{t,p}(\varphi)[\omega] = \sum_{n=1}^{\infty} < \varphi, \varphi_n > [P < \underline{x}_t(\omega), \varphi_n >] \quad .$$

Then $\eta_{t,p}$ is an a.e. weakly measurable, bounded linear functional on Φ. Hence, $\eta_{t,p}(\varphi)[\omega] = < \eta_{t,p}(\omega), \varphi >$ for $\omega \notin N$. Clearly, for each φ, $\mathcal{E}[P < \underline{x}_t(\omega),\varphi> - < \eta_{t,p}(\omega),\varphi >]^2 = 0$ and from (7.1), $||\eta_{t,p}(\omega)||^2$ is finite. If $\{\chi_m\}$ is any other C.O.N. system then following the above argument we obtain an a.e. weakly measurable function $\zeta_{t,p}$ from Ω to Φ such that $< \zeta_{t,p}(\omega),\varphi > = P < \underline{x}_t(\omega),\varphi >$, $||\zeta_{t,p}(\omega)||^2 < \infty$ and $\mathcal{E}[P < \underline{x}_t(\omega),\varphi > - < \zeta_{t,p}(\omega), \varphi >]^2 = 0$ for every φ. Thus we have

$$(7.3) \qquad ||\eta_{t,p}(\omega) - \zeta_{t,p}(\omega)||^2 = \sum_1^{\infty} \mathcal{E}[< \eta_{t,p}(\omega) - \zeta_{t,p}(\omega), \chi_m >]^2 = 0$$

since for every φ, $\mathcal{E}[< \eta_{t,p}(\omega),\varphi > - < \zeta_{t,p}(\omega),\varphi >]^2 = 0$. Let $\mathcal{L}_2(\Omega,P)$ be the space of weakly measurable functions g from Ω to Φ, satisfying $\mathcal{E}||g(\omega)||^2 < \infty$ (strictly speaking, equivalence classes of functions, see Section 8). From (7.3) we see that $\eta_{t,p}$ and $\zeta_{t,p}$ are elements of the same equivalence class, say, $\underline{x}_{t,p}$ belonging to $\mathcal{L}_2(\Omega,P)$. Identifying $\underline{x}_{t,p}$ with any of its elements we have $< \underline{x}_{t,p},\varphi > = P < \underline{x}_t,\varphi >$.

Since $\tilde{\underline{x}}_t^{(1)}(\varphi) = P_{H(\underline{x};-\infty)} <x_t,\varphi>$ and $\tilde{x}_t^{(2)}(\varphi) = P_{H(\underline{x};t)\cap H^{\perp}(\underline{x}:-\infty)} < x_t,\varphi >$, it follows from the lemma that there exist processes $\{\underline{x}_t^{(1)}\}$, $\{\underline{x}_t^{(2)}\}$ in Φ, defined for each t, except possibly on a null ω-set such that $\tilde{x}_t^{(i)}(\varphi) = < x_t^{(i)},\varphi >$ for i = 1,2. Obviously, $\{\underline{x}_t^{(i)}\}$ satisfy all the other desired properties.

Before proving the representation theorem for purely non-deterministic processes $\underline{x}_t$ in Φ, we need to introduce stochastic integrals taking values in Φ, which we shall call Stochastic Pettis integrals.

8. Stochastic Pettis integrals. Let $(A, \mathfrak{A}, \mu)$ be an arbitrary σ-finite measure space and $\mathcal{L}_2(A,\mu)$ be the set of all weakly measurable functions g from A to Φ such that $\int ||g(a)||^2 d\mu(a)$ is finite. It is well known that upon identifying functions which are equal almost everywhere $[\mu]$ (i.e., setting f = g if $\int ||f(a) - g(a)||^2 d\mu(a) = 0$), $\mathcal{L}_2(A,\mu)$ becomes a Hilbert space with inner product given by

$$(g_1, g_2)_{\mathcal{L}_2(A,\mu)} = \int \langle g_1(a), g_2(a) \rangle d\mu(a).$$

The norm of g will be denoted by $||g||_{\mathcal{L}_2(A,\mu)}$. It is easy to show that $\mathcal{L}_2(A,\mu)$ is separable if the Hilbert space $L_2(A,\mu)$ of real functions square integrable with respect to μ is separable. In particular, if A = T, the real line and μ is a σ-finite measure on Borel sets then the Hilbert space $\mathcal{L}_2(T,\mu)$ is separable. In what follows we write $\mathcal{L}_2(\mu)$ for $\mathcal{L}_2(T,\mu)$.

Lemma 8.1 Let z be a real orthogonal random set function with $\mathcal{E}[z(\Delta)]^2 = \rho(\Delta)$. If $g \in \mathcal{L}_2(\rho)$, then there exists an a.e. $[\rho]$ weakly measurable mapping J(g) from Ω to Φ with the following properties:

(8.1) $\quad J(g) \in \mathcal{L}_2(\Omega,P)$;

if g_1, g_2 are any elements of $\mathcal{L}_2(\rho)$ and c_1, c_2 are real numbers then

(8.2) $\quad J(c_1 g_1 + c_2 g_2) = c_1 J(g_1) + c_2 J(g_2)$,

the equality holding in the sense of $\mathcal{L}_2(\Omega,P)$;

for every $\varphi \in \Phi$,

(8.3) $\quad \langle J(g),\varphi \rangle = \int \langle g(t),\varphi \rangle dz(t)$ with probability one, where the right hand side integral is an ordinary stochastic integral.

The element $J(g)$ is called the Stochastic Pettis Integral of $g(t)$ with respect to z and is written $\int g(t)dz(t)$. We also have

(8.4) $$\mathcal{E}[< \int g_1(t)dz(t), \int g_2(t)dz(t) >] = \int < g_1(t), g_2(t) > d\rho(t).$$

Proof: Let $\{\varphi_k\}$ be a C.O.N. system in Φ and let g be any element of $\mathcal{L}_2(\rho)$. Strictly speaking, each g represents an equivalence class belonging to $\mathcal{L}_2(\rho)$ and it is clear that elements of this equivalence class give rise to the same stochastic integral $\int <g(t),\varphi_k > dz(t)$ since the latter is itself defined up to an equivalence. Denoting it (more precisely, a random variable belonging to the equivalence class) by $L(g, \varphi_k)$ we have

$$\sum_{k=1}^{\infty} \mathcal{E}\,[L(g,\varphi_k)]^2 = \sum_{k=1}^{\infty} \int < g\ (t),\varphi_k >^2 d\rho(t) < \infty, \text{ so that}$$

$$\sum_{k=1}^{\infty} [L(g\ ,\varphi_k)[\omega]\ ]^2 < \infty$$ except possibly when ω in a set N of zero ρ-measure. If, for any φ, we now set

$$L(g\ ,\varphi)[\omega] = \sum_{k=1}^{\infty} <\varphi,\varphi_k > L(g, \varphi_k)\ [\omega], \quad (\omega \notin N),$$ it follows that $L(g, \varphi)[\omega]$ is a bounded linear functional on Φ. Hence we obtain

$$L(g,\varphi)[\omega] = < J_1(g)[\omega]\ , \varphi >,$$

where $J_1(g)[\omega] \in \Phi$. It is further easy to see that $J_1(g)\ [.]$ is a.e. weakly measurable and that $\mathcal{E}||J_1(g)[\omega]||^2$ is finite. It is evident that we have relied on the choice of a particular C.O.N. system in our definition of $J_1(g)$. However, if $\{\psi_m\}$ is any other C.O.N system in Φ and $J_2(g)\ [.]$ is the corresponding a.e. weakly measurable mapping, then we have

$$\mathcal{E}||J_1(g)[\omega] - J_2(g)[\omega]||^2 = 0, \text{ i.e., } \quad ||J_1(g) - J_2(g)||_{\mathcal{L}_2(\Omega,\mathfrak{P})} = 0.$$

In other words, $J_1(g)$ and $J_2(g)$ belong to the same equivalence class, say $J(g)$, of $\mathcal{L}_2(\Omega,\mathfrak{P})$. Thus, the equivalence class $J(g)$ in $\mathcal{L}_2(\Omega,\mathfrak{P})$ is unambiguously defined for each g in $\mathcal{L}_2(\rho)$ and further $||g||_{\mathcal{L}_2(\rho)} = ||J(g)||_{\mathcal{L}_2(\Omega,\mathfrak{P})}$. For every $g \in \mathcal{L}_2(\rho)$, the corresponding element $J(g)$ of $\mathcal{L}_2(\Omega,\mathfrak{P})$ will be called the stochastic Pettis integral of g with respect to the orthogonal process z and will be denoted by $\int g(t)dz(t)$. The assertions (8.2)-(8.4) of the lemma are easy to verify.

If z_1, z_2 are orthogonal random set functions with measure functions ρ_1 and respectively and are further mutually orthogonal then it can be shown that

$$\mathcal{E}\Big[\int g_1(t)dz_1(t) \; , \; \int g_2(t)dz_2(t) \Big] = 0 \text{ for } g_1 \in \mathcal{L}_2(\rho_1) \text{ and } g_2 \in \mathcal{L}_2(\rho_2).$$

The proof follows by the definition of the Pettis integral.

The following result will be useful in the next section.

__Lemma 8.2.__ Let $z_k(k = 1,2\ldots)$ be mutually orthogonal processes with orthogonal increments and with respective measure functions ρ_k. If $g_k \in \mathcal{L}_2(\rho_k)$ are such that

$$(8.4) \qquad \sum_{k=1}^{\infty} \int ||g_k(t)||^2 \, d\rho_k(t) \text{ is finite, } \qquad \text{then}$$

$$\sum_{k=1}^{\infty} \int g_k(t)dz_k(t) \qquad \text{is an element of } \mathcal{L}_2(\Omega,\mathfrak{P}) \text{ (the series of}$$

Stochastic Pettis integrals converging in the $\mathcal{L}_2(\Omega,\mathfrak{P})$ sense), and for every $\varphi \in \Phi$,

$$(8.5) \qquad < \sum_{k=1}^{\infty} \int g_k(t)dz_k(t),\varphi > = \sum_{k=1}^{\infty} \int < g_k(t),\varphi > dz_k(t) \text{ with}$$

Probability one.

<u>Proof:</u> It is clear from the definition of $\int g_k(t)dz_k(t)$ that $\{\zeta_m\}$ where

$\zeta_m = \sum_1^m \int g_k(t)dz_k(t)$ is a Cauchy sequence of elements in $\mathcal{L}_2(\Omega, P)$; since

$(m' > m)$,

$$||\zeta_{m'} - \zeta_m||^2_{\mathcal{L}_2(\Omega, P)} = \sum_m^{m'} \int ||g_k(t)||^2 d\rho_k(t) \to 0$$

by (8.4). Hence the limit (in $\mathcal{L}_2(\Omega, P)$ sense) of ζ_m exists which we denote by

$\sum_1^\infty \int g_k(t)dz_k(t)$. The other conclusions of the lemma are similarly proved.

9. Representation Theorems For Purely Non-deterministic Hilbert Space-valued Processes. In this section we consider a purely non-deterministic process $\{\underline{x}_t\}$ in Φ, with $\mathcal{E}||\underline{x}_t||^2$ finite. As in Section 2 we confine ourselves to the continuous parameter case. The representation we seek for $\underline{x}_t$ is obtained in terms of Stochastic Pettis integrals. Since

$\mathcal{E}[<\underline{x}_t,\varphi> - <\underline{x}_t,\psi>]^2 \leq \mathcal{E}||\underline{x}_t||^2\,||\varphi-\psi||^2$, it follows that $\underline{x}_t$ -process is continuous in the topology of Φ. Hence, from Lemma 2.1, the space $H(\underline{x})$ is separable provided the limits $\underline{x}_{t-0}(\varphi)$ and $\underline{x}_{t+0}(\varphi)$ exist for each $\varphi \in \Phi$. We shall refer to this condition as assumption (B).

Theorem 9.1. Let $\{\underline{x}_t\}$ be a purely non-deterministic process in Φ with $\mathcal{E}||\underline{x}_t||^2$ finite and satisfying assumption (B). Then for each t, with probability one

(9.1) $$\underline{x}_t = \sum_{1}^{M_0} \int_{-\infty}^{t} F_n(t,u)d\,z_n(u) + \sum_{t_j \leq t} \sum_{\ell=1}^{M_j} b_{j\ell}(t)\,\xi_{j\ell}$$

where M_0, $M_{j\ell}$ the processes z_n and the random variables $\xi_{j\ell}$ have the same meaning as in Theorem 2.2.

Furthermore, for each t,

(9.2) $F_n(t,.) \in \mathcal{L}_2(\rho_n)$, ρ_n being the measure function of z_n, and $b_{j\ell}(t) \in \Phi$ for every j, ℓ;

(9.3) $$\sum_{n=1}^{M_0} \int_{-\infty}^{t} ||F_n(t,u)||^2\,d\rho_n(u) < \infty ;$$

(9.4) $$\sum_{j=1}^{\infty} \sum_{\ell=1}^{M_j} ||b_{j\ell}(t)||^2\,\mathcal{E}(\xi_{j\ell}^2) < \infty; \quad \text{and}$$

(9.5) $H(\underline{x};t) = \mathfrak{S}\{H(\underline{z};t) \cup H(\underline{\xi};t)\}$ for every t, where

$H(\underline{z};t) = \mathfrak{S}[z_n(u) \mid u \leq t, n=1,..,M_0]$ and $H(\underline{\xi};t) = \mathfrak{S}[\xi_{j\ell} \mid \ell=1,..,M_j, t_j \leq t]$.

Proof: Since $\langle \underline{x}_t, \varphi \rangle$ is a S.P. on Φ Theorem 2.2 applies without any change to it. Furthermore, it has been shown in Section 3 that the representation for $\langle \underline{x}_t, \varphi \rangle$ can be chosen to be proper canonical without changing the numbers M_0 and M_j and hence without affecting the multiplicity M of the process. This accounts for the conclusion (9.5) of the theorem. In order to prove the remaining assertions we need to use the additional hypothesis in the present case, viz., that $\mathcal{E}||\underline{x}_t||^2 < \infty$.

From Theorem 2.2, we obtain

$$(9.6) \qquad \sum_{n=1}^{M_0} \sum_{k=1}^{\infty} \int_{-\infty}^{t} F_n^2(\varphi_k;t,u)d\rho_n(u) \leq \mathcal{E}\,||\,\underline{x}_t||^2 < \infty, \qquad \text{where}$$

$\{\varphi_k\}$ is a C.O.N. system in Φ. A fortiori, there exists a set A_n of ρ_n - measure zero such that for $u \notin A_n$,

$$(9.7) \qquad \sum_{k=1}^{\infty} F_n^2\,(\varphi_k;t,u) \quad < \infty.$$

For $\varphi \in \Phi$ setting $c_k = \langle \varphi, \varphi_k \rangle$, we obtain from (9.7)that for $u \notin A_n, \sum_k c_k F_n(\varphi_k;t,u)$ converges and is in fact, equal to $F_n(\varphi;t,u)$ a.e. $[\rho_n]$. Hence $F_n(\varphi;t,u)$ is a bounded linear functional on Φ for $u \notin A_n$. We may therefore write $F_n(\varphi;t,u) = \langle F_n(t,u),\varphi \rangle$, where $F_n(t,u)$ is an element of Φ and moreover, $F_n(t,\cdot)$ is an element of $\mathcal{L}_2(\rho_n)$. From (9.6) we have

$$(9.8) \qquad \sum_{n=1}^{M_0} \int_{-\infty}^{+\infty} ||\,F_n(t,u)||^2 d\rho_n(u) < \infty.$$

Since $\mathcal{E}||\underline{x}_t||^2$ is finite it follows that for all j and ℓ there exists a bounded linear functional $b_{j\ell}(t)$ such that for each t, (9.9) $$b_{j\ell}(\varphi;t) = \langle b_{j\ell}(t),\varphi\rangle \text{ with } \sum_{j,\ell} ||b_{j\ell}(t)||^2\sigma_{j\ell}^2 < \infty.$$

By (9.8), Lemma 8.2 and (9.9), we have

$$\underline{x}_t = \sum_{n=1}^{M_0} \int_{-\infty}^{t} F_n(t,u)d\, z_n(u) + \sum_{tj \le t} \sum_{\ell=1}^{M_j} b_{j\ell}(t)\xi_{j\ell}$$

The corresponding results for weakly stationary (see Introduction for definition of stationarity) Φ -valued processes are stated below without proof.

Theorem 9.2. A discrete parameter weakly stationary, purely non deterministic, process in Φ, with $\mathcal{E}||\underline{x}_t||^2 < \infty$, has the following representation.

$$\underline{x}_n = \sum_{m=-\infty}^{n} \sum_{\ell=1}^{M} b_\ell(n-m)\ \xi_\ell(m).$$

Here M is the multiplicity of $\{\underline{x}_n\}$

(i) the discrete parameter processes $\{\xi_\ell(m)\}$ $(\ell=1,..,M)$ have orthogonal increments and are mutually orthogonal;

(ii) $H(\underline{x};n) = \sum_{i=1}^{M} \oplus H(\xi_i;n)$ for each n,

(iii) $b_\ell(n-m) \in \Phi$ with $\sum_{m=-\infty}^{0} \sum_{\ell=1}^{M} ||b_\ell^2(m)||^2\ \mathcal{E}[\xi_\ell^2(m)] < \infty$.

The number M is the multiplicity associated with the Stochastic process.

Theorem 9.3 Let $\{\underline{x}_t\}$ be a continuous parameter weakly stationary process with values in Φ satisfying the assumptions of Theorem 9.1 and condition (C). Then for each t with probability one,

(9.12) $$\underline{x}_t = \sum_{n=1}^{M} \int_{-\infty}^{t} F_n(u-t)d\xi_n(u).$$

In this representation

(i) the ξ_n's are mutually orthogonal and each ξ_n is a homogeneous orthogonal random set function (with Lebesque measure μ for its measure function),

(ii) $H(\underline{x};t) = \sum_{i=1}^{M} \oplus H(\xi_i;t)$ for every t,

(iii) M is the multiplicity of the process, and

(iv) $F_n(u-t) \in \mathcal{L}_2(\mu) \quad (n=1,\ldots,M)$ such that

$$\sum_{n=1}^{M} \int_{-\infty}^{0} ||F_n(u)||^2 \, d\mu(u) < \infty.$$

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