Locally Coherent Rates of Exchange

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Abstract

A theory of coherence is formulated for rates of exchange between events. The theory can be viewed as a generalization of de Finetti's theory of coherence as well as Holzer's theory of conditional coherence. Coherent rates of exchange on a fixed Boolean algebra are in one-to-one correspondence with finitely additive conditional probability measures on the algebra. Results of Renyi and Krauss on conditional probability spaces are used to show that coherent rates of exchange are generated by ordered families of finitely additive measures, possibly infinite measures. This provides an interpretation of improper prior distributions in terms of coherence. An extension theorem is proved and gives a generalization of extension theorems for finitely additive probability measures.

Key words and phrases: coherence, conditional probability, finite additivity, improper priors.

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1. Heuristics.

Suppose the sample space Ω for some chance experiment is the set of points on the real line. A statistician believes that sets having the same finite, positive Lebesgue measure are equally likely; so Lebesgue measure, μ_1, might be used as an improper prior. However, the statistician also feels that finite sets of the same cardinality are equally likely. Now Lebesgue measure gives all such sets measure zero and so counting measure, μ_0, seems more appropriate for finite sets. Finally the statistician feels that sets having the same positive density are equally likely, where the density of a set A is the limit

$$\mu_2(A) = \lim_{x \to \infty} \frac{1}{2x} \int_{-x}^{x} 1_A(t)\mu_1(dt)$$

when the limit exists. Now if $\mu_2(A) > 0$, both $\mu_0(A)$ and $\mu_1(A)$ are infinite. In the past statisticians wishing to express vague prior information have often chosen an improper distribution such as μ_1, which assesses all "large" sets as having infinite mass. Some have used finitely additive proper priors like μ_2 which give all "small" sets mass zero.

Is there a way of expressing these opinions simultaneously and of assessing their coherence? To answer these questions, we propose a theory of exchange rates. The idea is that, if two sets are believed to be equally likely, the statistician should be willing to trade a prospective payoff on the one for an equal payoff on the other. The usual theory of coherence involves comparing a payoff on each event to a payoff on the whole sample space (a sure thing). This theory is inadequate for comparing two events both of which are infinitesimally small in relation to the whole space. The theory of exchange rates makes such comparisons quite natural.

The appropriate notion of coherence for exchanges involving a finite number of "small" sets cannot be the usual one of avoiding a sure loss. The union of all the sets involved in any such exchange can again be "small." We will call a rate locally coherent if no exchange involving a finite number of sets results in a loss on all of their union. (Formal definitions are in the next section.)

Every measure μ determines a natural exchange rate between sets of finite positive measure; $\mu(A)$ one-dollar payoffs on A are worth $\mu(B)$ one-dollar payoffs on B. Thus the theory of coherence for rates of exchange will also apply to measures including improper ones like Lebesgue measure. (This idea that $\mu(A)/\mu(B)$ is the relative value of a ticket on A to one on B is mentioned by Hartigan [6, p. 15].)

2. Definitions and summary of results.

Let Ω be the sample space for a chance experiment and let $\underset{\sim}{D}$ be a collection of pairs (A,B) of subsets of Ω such that the second element B is not empty. A rate of exchange r on $\underset{\sim}{D}$ is a mapping from $\underset{\sim}{D}$ to $[0,\infty]$. Associated to each pair $(A,B) \in \underset{\sim}{D}$ is the simple exchange

$$S_{A,B} = r(A,B)B - A.$$

(In this expression and in the sequel, events and their indicator functions are identified.) We imagine that a bookie offers such simple exchanges to a gambler. If B occurs, the bookie pays \$r(A,B) to the gambler, and if A occurs, the gambler pays the bookie \$1. If neither A nor B occurs, no money changes hands. (Some readers may wish to interpret $r(A,B)$ as the bookie's odds on A against B.)

An exchange e is any well-defined linear combination of simple exchanges. (The usual conventions are made about arithmetic operations with ∞ and $-\infty$. In particular, $\infty - \infty$ and $0 \cdot \infty$ are not defined. By the way, we could avoid the use of infinite numbers by interpreting a rate $r(A,B) = \infty$ as meaning that the bookie will accept any exchange $rB - A$ where $r > 0$.) Let λ be a real-valued function defined on $\underset{\sim}{D}$ which is zero except for a finite number of pairs (A,B) and let

$$e(\lambda) = \sum_{(A,B)} \lambda(A,B) S_{A,B}, \tag{2.1}$$

assuming the sum is well-defined. Every exchange e is of this form for some λ. For each λ, let the support of λ be the set $\text{supp}(\lambda) = \cup\{A \cup B: \lambda(A,B) \neq 0\}$. Notice that, for an exchange $e(\lambda)$, no money changes hands if $\text{supp}(\lambda)$ does not occur. This suggests the following notion of coherence for the bookie.

Definition. The rate of exchange r is locally coherent if there is no exchange $e(\lambda)$ which is strictly positive on $\text{supp}(\lambda)$.

Notice that if $e(\lambda) > 0$ on supp (λ), then $e(\lambda)$ has a positive infimum on $\text{supp}(\lambda)$. This is because exchanges have only finitely many possible values.

We use the term "local coherence" rather than "coherence" because the bookie is required to avoid losses on certain proper subsets of the outcome space. The usual theory of de Finetti [2,3] only requires the bookie to avoid sure losses on the whole space. In an interesting paper [15], Smith develops a notion of "consistency" which is related to local coherence.

There is a simple relationship between the de Finetti theory and that presented here. Let $\underset{\sim}{C}$ be the collection of sets A such that $(A,\Omega) \in \underset{\sim}{D}$ and set $p(A) = r(A,\Omega)$. Then

$$S_{A,\Omega} = p(A) - A$$

and we can regard \$p(A) as the bookie's price for a ticket worth \$1 if A occurs. The support of any exchange involving Ω will, of course, be Ω. Thus if r is a coherent rate of exchange, then p will be coherent in the sense of de Finetti (i.e. no linear combination of exchanges $S_{A,\Omega}$ is everywhere positive.) However, the converse is easy to disprove. For example, r could be incoherent when $\underset{\sim}{C}$ is empty or r could be incoherent because of bad behavior on p-null sets.

A number of authors, including de Finetti, have studied notions of conditional coherence. In a recent paper [8] which is closely related to this one, Holzer introduces a notion of conditional coherence for a real-valued mapping $P(\cdot|\cdot)$ with domain $\underset{\sim}{E}$, a collection of pairs (A,B) of subsets of Ω such that the second element B is not empty. If one sets $r(A \cap B,B) = P(A|B)$, then r is unambiguously defined on the collection $\underset{\sim}{D} = \{(A \cap B,B): (A,B) \in \underset{\sim}{E}\}$. Also, r is locally coherent if and only if P is conditionally coherent in the sense of Holzer. Thus the conditionally coherent previsions of Holzer are in one-to-one correspondence with locally coherent rates of exchange whose domains satisfy the requirement that $(A,B) \in \underset{\sim}{D}$ implies $A \subset B$. This is a quite natural restriction if the exchange rate is viewed in terms of conditional probability. However, it rules out many exchanges which we wish to consider.

A stronger requirement than local coherence is that a bookie avoid exchanges which are positive somewhere and non-negative everywhere.

<u>Definition</u>. The rate of exchange r is <u>strictly coherent</u> if there is no exchange e with $e \geq 0$ on all of Ω with strict inequality holding somewhere.

The notion of strict coherence was studied by Kemeny [9] in the context of betting odds rather than rates of exchange.

The next section establishes some of the basic properties of locally coherent exchange rates. In section 4 it is shown that locally coherent rates of exchange on an algebra of sets are in one-to-one correspondence with conditional probability measures. This correspondence together with Renyi's characterization of conditional probabilities in terms of linearly ordered families of measures leads to an analogous characterization of locally coherent rates in section 5. This characterization is useful for the interpretation of improper priors and also results in a simple characterization of strictly coherent rates of exchange. It is shown in the final section that a locally coherent rate defined on an arbitrary domain can always be extended to the algebra of all subsets.

No attempt is made here to develop a theory of local coherence for statistical models comparable to the coherence theories of Heath, Lane, and Sudderth [7,11,12]. Such a theory is no doubt possible and would be of interest to us.

3. Elementary properties of locally coherent rates.

Assume in this section that the domain $\underset{\sim}{D}$ of the rate of exchange r consists of all pairs (A,B), where A and B are elements of a ring $\underset{\sim}{F}$ of subsets of Ω and B is not empty. This assumption about the domain of r is not necessary for the following proposition, as will follow from the extension theorem of section 6.

Theorem 3.1. Let r be a locally coherent rate of exchange. Then the following are true whenever the quantities are well-defined:

(i) $r(A,A) = 1$

(ii) $r(A_1 \cup A_2, B) = r(A_1,B) + r(A_2,B)$ if $A_1 \cap A_2 = \emptyset$,

(iii) $r(A_1,B) \le r(A_2,B)$ if $A_1 \subset A_2$,

(iv) $r(A,B)r(B,C) = r(A,C)$

(v) $r(A,B) = r(B,A)^{-1}$

(vi) $r(A,B) = r(A,C)r(B,C)^{-1}$

(vii) $r(A,B_1) \ge r(A,B_2)$ if $B_1 \subset B_2$.

Proof: (i) if $r(A,A) > 1$, then $r(A,A)A - A > 0$ on A. If $r(A,A) < 1$, then $-[r(A,A)A - A] > 0$ on A.

(ii) Suppose the left-hand-side is larger than the right and is a finite number. Then there is an $\varepsilon > 0$ such that

$$\delta \equiv r(A_1 \cup A_2, B) - (1+\varepsilon)[r(A_1,B) + r(A_2,B)] > 0.$$

(If $r(A_1 \cup A_2, B) = \infty$, it can be replaced by a finite number for which the inequality still holds.) Consider the exchange

$$e = \{r(A_1 \cup A_2, B)B - (A_1 \cup A_2)\} - (1+\varepsilon)[r(A_1, B)B - A_1] - (1+\varepsilon)[r(A_2, B)B - A_2]$$
$$= \delta B + \varepsilon(A_1 \cup A_2).$$

Then $e > 0$ on $B \cup A_1 \cup A_2$, a contradiction.

A contradiction is reached by a similar argument if the right-hand-side is assumed larger than the left.

(iii) By (ii),

$$r(A_2, B) = r(A_1, B) + r(A_2 - A_1, B).$$

(iv) Suppose the left-hand-side is larger than the right. Choose ε in (0,1) so that

$$\delta \equiv (1-\varepsilon)r(A,B)r(B,C) - (1+\varepsilon)r(A,C) > 0.$$

(If $r(A,B)$ or $r(B,C)$ equals ∞, replace them by real numbers which preserve the inequality.) Consider the exchange

$$e = [r(A,B)B - A] + (1-\varepsilon)r(A,B)[r(B,C)C - B] - (1+\varepsilon)[r,(A,C)C - A]$$
$$= \delta C + \varepsilon r(A,B)B + \varepsilon A.$$

Then $e > 0$ on $A \cup B \cup C$, a contradiction. (Notice $r(A,B) > 0$ if the left side of

(iv) is larger than the right.)

Next suppose the right-hand-side of (iv) is larger than the left. Choose ε in (0,1) so that

$$\delta \equiv (1-\varepsilon)r(A,C) - (1+\varepsilon)r(A,B)r(B,C) > 0.$$

(If $r(A,C) = \infty$, replace it by a finite number, and if $r(A,B) = 0$, replace it by a positive number so that the inequality still holds.) Consider the exchange

$$\begin{aligned} e &= -[r(A,B)B-A] - (1+\varepsilon)r(A,B)[r(B,C)C-B] + (1-\varepsilon)[r(A,C)C-A] \\ &= \delta C + \varepsilon r(A,B)B + \varepsilon A. \end{aligned}$$

Then $e > 0$ on $A \cup B \cup C$, a contradiction. (In the case where $r(A,B) = 0$, replace its second occurence in the definition of e by the positive number used to replace it in the definition of δ.)

(v) If $0 < r(A,B) < \infty$, the desired equality follows from (i) and (iv). If $r(A,B) = 0$ and $r(B,A) < \infty$, then

$$1 = r(A,A) = r(A,B)r(B,A) = 0,$$

a contradiction.

Similarly, if $r(A,B) = \infty$, we must have $r(B,A) = 0$ to avoid a contradiction.

(vi) By (v), $r(B,C)^{-1} = r(C,B)$. Now use (iv).

(vii) Use (iii) and (v). □

4. Conditional probability and rates of exchange.

Let $\underset{\sim}{B}$ be an algebra of subsets of Ω and let $\underset{\sim}{B}^0$ be the collection of non-empty sets in $\underset{\sim}{B}$.

Definition 4.1. A conditional probability P on $\underset{\sim}{B}$ is a mapping $P = P(\cdot|\cdot)$ from $\underset{\sim}{B}\times\underset{\sim}{B}^0$ to the real numbers satisfying

(a) $P(\cdot|B)$ is a finitely additive probability measure on $\underset{\sim}{B}$ for every $B \in \underset{\sim}{B}^0$,

(b) $P(A\cap B|C) = P(A|C)P(B|A\cap C)$ for A, B in $\underset{\sim}{B}$, C, $A\cap C$ in $\underset{\sim}{B}^0$.

This definition is from Krauss [10] and is essentially that of Renyi [14] except that countable additivity of the conditional measures is not required here.

A rate of exchange r with domain $\underset{\sim}{B}\times\underset{\sim}{B}^0$ is said to be a rate of exchange on $\underset{\sim}{B}$.

The result of this section is that locally coherent rates of exchange and conditional probabilities on an algebra can be viewed as different aspects of the same objects. Together with Holzer's equivalence property ([8], Theorem 5.3), it also shows the equivalence of these notions with his coherent conditional previsions on an algebra.

Theorem 4.1. (i) If r is a locally coherent rate of exchange on $\underset{\sim}{B}$ and P is defined by

$$P(A|B) = r(A\cap B, B) \quad \text{if } A\cap B \neq \emptyset$$
$$= 0 \quad \text{if } A\cap B = \emptyset$$

for $A \in \underset{\sim}{B}$, $B \in \underset{\sim}{B}^0$, then P is a conditional probability on $\underset{\sim}{B}$.

(ii) If P is a conditional probability on $\underset{\sim}{B}$ and r is defined by

$$r(A,B) = \frac{P(A|A\cup B)}{P(B|A\cup B)} \quad \text{if } P(B|A\cup B) > 0$$

$$= \infty \quad \text{if } P(B|A\cup B) = 0$$

for $A \in \underset{\sim}{B}$, $B \in \underset{\sim}{B}^0$, then r is a locally coherent rate of exchange on $\underset{\sim}{B}$.

(iii) The mappings $r \longrightarrow P$ and $P \longrightarrow r$ defined in (i) and (ii) are inverses of each other and therefore define a one-to-one corrrespondence.

<u>Proof</u>: (i) Use (i),(ii), and (iv) of Theorem 3.1.

(ii) Let $e = e(\lambda)$ be an exchange with $C = \text{supp}(\lambda)$. Write

$$e = \sum_{i=1}^{n} \lambda_i S_i,$$

where $\lambda_i = \lambda(A_i,B_i) \neq 0$, $S_i = r(A_i,B_i)B_i - A_i$, $A_i \in \underset{\sim}{B}$, $B_i \in \underset{\sim}{B}^0$ for $i = 1,\ldots, n$, and $C = \bigcup_{i=1}^{n} (A_i \cup B_i)$.

In order to reach a contradiction, assume

$$\inf_C e > 0.$$

An immediate consequence is that, if $r(A_i,B_i) = \infty$, then $\lambda_i > 0$.

Let $E(\cdot|C)$ be the expectation operator corresponding to the measure $P(\cdot|C)$.

To reach a contradiction, it suffices to show

$$(4.1) \qquad E(\lambda_i S_i | C) \leq 0$$

for $i = 1, \ldots, n$, for then $E(e|C) \leq 0$. To prove (4.1), we will consider three cases and, to simplify notation, we will omit the subscript 'i'.

<u>Case 1</u>. $0 < r(A,B) < \infty$.

In this case,

$$E(\lambda S|C) = \lambda[r(A,B)P(B|C) - P(A|C)] = 0$$

because

$$r(A,B) = \frac{P(A|A\cup B)}{P(B|A\cup B)} = \frac{P(A|C)}{P(B|C)} \text{ if } P(A\cup B|C) > 0.$$

To verify the last equality, use (b) in the definition of conditional probability to calculate

$$\begin{aligned} (4.2) \qquad P(A|C) &= P(A\cup B)\cap A|C) \\ &= P(A\cup B|C)P(A|(A\cup B)\cap A) \\ &= P(A\cup B|C)P(A|A\cup B) \end{aligned}$$

and similarly

$$P(B|C) = P(A \cup B|C)P(B|A \cup B).$$

<u>Case 2</u>. $r(A,B) = 0$.

By the definition of r in (ii), $P(A|A \cup B) = 0$, and then by the calculation in (4.2), $P(A|C) = 0$. Hence, $E(\lambda S|C) = 0$.

<u>Case 3</u>. $r(A,B) = \infty$.

As was remarked above, $\lambda > 0$ in this case. Also, $P(B|A \cup B) = 0$ and hence $P(B|C) = 0$. We make the usual convention that integrals over sets of measure zero are also zero and conclude that

$$E(\lambda S|C) = -\lambda P(A|C) \leq 0. \quad \square$$

It will follow from Theorem 4.1 and the extension theorem of section 6 that a locally coherent rate of exchange on an arbitrary domain $\underset{\sim}{D}$ is consistent with some conditional probability on the algebra of all subsets. However, the correspondence will not in general be one-to-one.

5. The Renyi ordering, improper priors, and strict coherence.

For a conditional probability P on an algebra $\underset{\sim}{B}$, there is a natural ordering of nonempty events: $A \leq B$ if and only if $P(B|A \cup B) > 0$ and $A < B$ if and only if $P(A|A \cup B) = 0$. This is a linear ordering with associated equivalence relation $A \sim B$ if and only if both $P(A|A \cup B)$ and $P(B|A \cup B)$ are positive. This ordering was introduced by Renyi [14] for his countably additive conditional probabilities and studied by Krauss [10] in the general finitely additive setting.

Suppose r is a locally coherent rate of exchange on $\underset{\sim}{B}$ and P is the associated conditional probability as in Theorem 4.1.

Lemma 5.1. If A and B are nonempty members of $\underset{\sim}{B}$, then

(i) $A \sim B$ if and only if $0 < r(A,B) < \infty$
if and only if $0 < r(B,A) < \infty$

(ii) $A < B$ if and only if $0 = r(A,B)$
if and only if $r(B,A) = \infty$

(iii) $A \leq B$ if and only if $r(A,B) < \infty$
if and only if $0 < r(B,A)$.

Proof: Use Theorem 4.1 and Theorem 3.1(v). □

Let $[B]$ be the equivalence class of B under $\sim$ and set Γ equal to the collection of all equivalence classes. For $\alpha,\beta \in \Gamma$, write $\alpha \leq \beta$ when $A \leq B$ for some $A \in \alpha$, $B \in \beta$.

Theorem 5.1 (Renyi, Krauss). The set Γ of equivalence classes is linearly ordered under $\leq$. For each $\alpha \in \Gamma$, there is a finitely additive measure m_α on $\underset{\sim}{B}$ which is unique up to proportionality and such that

(i) $0 < m_\alpha(B) < \infty$ for $B \in \alpha$

(ii) $m_\alpha(B) = 0$ for $[B] < \alpha$

(iii) $m_\alpha(B) = \infty$ for $\alpha < [B]$

(iv) $r(A,B) = m_\alpha(A)/m_\alpha(B)$ if $B \in \alpha$, $A \in \underset{\sim}{B}$

(v) If $\alpha < \beta$, then $m_\alpha(B) < \infty \Rightarrow m_\beta(B) = 0$.

Conversely, suppose Γ is a linearly ordered set and $\{m_\alpha, \alpha \in \Gamma\}$ is a family of measures on $\underset{\sim}{B}$ satisfying (v). Suppose also that, for every nonempty $B \in \underset{\sim}{B}$, there is an $\alpha \in \Gamma$ such that $0 < m_\alpha(B) < \infty$. For that α, which is unique by (v), define

$$r(A,B) = m_\alpha(A)/m_\alpha(B) \tag{5.1}$$

for all $A \in \underset{\sim}{B}$. Then r is a locally coherent rate of exchange on $\underset{\sim}{B}$.

The proof of this result can be found in Renyi [14] and Krauss [10] although these authors work with conditional probabilities rather than the equivalent rates. The proof is not difficult, and the measure m_α on the equivalence class [B] is just $r(\cdot,B)$ up to a proportionality constant.

Example 5.1. Let m be a finitely additive measure on an algebra $\underset{\sim}{B}$ and define

$$r(A,B) = m(A)/m(B)$$

whenever the right-hand-side is well-defined. ($\underset{\sim}{B}$ could be the algebra of Borel sets in R^n and m could be Lebesgue measure on $\underset{\sim}{B}$.)

Example 5.2. Let $\underset{\sim}{B}$ be the Borel subsets of the real line; let μ_0 be counting measure; let μ_1 be Lebesgue measure; and let μ_2 be any finitely additive extension of the density to $\underset{\sim}{B}$. (See section 1.) It is easily verified that $\mu_i(B) < \infty \Rightarrow \mu_j(B) = 0$ for $i < j$ and $B \in \underset{\sim}{B}$. Define

$$r_i(A,B) = \mu_i(A)/\mu_i(B), \quad i = 1,2,3$$

whenever the right-hand-side is well-defined. The r_i agree on any points which lie in the domains of more than one and so we can let $r(A,B) = r_i(A,B)$ on the domain of r_i.

Example 5.3. Let $\underset{\sim}{B}$ be the Borel subsets of R^n and, for $0 \leq \alpha \leq n$, let m_α be α-dimensional Hausdorff measure on $\underset{\sim}{B}$. Define a rate r by equation (5.1) whenever the denominator is finite and positive.

The rates defined in all three examples are locally coherent. This follows from the second half of Theorem 5.1 together with the following lemma.

Let $\{m_\alpha, \alpha \in I\}$ be a family of finitely additive measures on $\underset{\sim}{B}$. Say the family is linearly ordered if it satifies condition (v) of Theorem 5.1, and call the family complete if, for each $B \in \underset{\sim}{B}^0$, there is an $\alpha \in I$ such that $0 < m_\alpha(B) < \infty$.

Lemma 5.2. Every linearly ordered family is contained in a complete, linearly ordered family.

Proof: By Zorn's Lemma, there is a maximal linearly ordered family $\{m_\alpha, \alpha \in J\}$ containing the given family. Suppose it is not complete. Then there is a set B

ε $\underset{\sim}{B}$ such that $m_\alpha(B)$ is 0 or ∞ for every $\alpha \in J$. Let $J_\infty = \{\alpha \in J: m_\alpha(B) = \infty\}$ and $J_0 = \{\alpha \in J: m_\alpha(B) = 0\}$. Then $c = (J_\infty, J_0)$ is a Dedekind cut of J and we can adjoin c to Γ setting $\Gamma' = \Gamma \cup \{c\}$ with the ordering on Γ' to satisfy $\alpha < c < \beta$ for $\alpha \in J_\infty$, $\beta \in J_0$. Let $\underset{\sim}{F}$ be the collection of all sets $A \in \underset{\sim}{B}$ such that $m_\beta(A) = 0$ for some $\alpha \in J_\infty$ and let $\underset{\sim}{N}$ be the collection of $A \in \underset{\sim}{B}$ such that $m_\beta(A) = 0$ for all $\beta \in J_0$. Then

$$\underset{\sim}{F} \cup \{B\} \subset \underset{\sim}{N}.$$

Define $\underset{\sim}{A} = \underset{\sim}{B} \cap B$ to be the algebra of sets in $\underset{\sim}{B}$ which are subsets of B. Then $\underset{\sim}{A} \cap \underset{\sim}{F}$ is a proper ideal in $\underset{\sim}{A}$ and, consequently, there is a finitely additive probability measure m on $\underset{\sim}{A}$ which annihilates $\underset{\sim}{A} \cap \underset{\sim}{F}$. Define m_c on $\underset{\sim}{B}$ by setting

$$\begin{aligned} m_c(A) &= m(A \cap B) \quad \text{if } A \in \underset{\sim}{N}, \\ &= \infty \quad \text{if } A \notin \underset{\sim}{N}, \end{aligned}$$

for $A \in \underset{\sim}{B}$. Then $\{m_\alpha, \alpha \in \Gamma'\}$ is a linearly ordered family contradicting the maximality of $\{m_\alpha, \alpha \in \Gamma\}$. □

The converse half of Theorem 5.1 shows how to construct a locally coherent rate r from a complete, linearly ordered family of measures. There is a more recent technique of Carlson [1] which makes it possible to obtain a locally coherent rate from any complete family after the index set is well-ordered.

Theorem 5.2. Let I be a well-ordered set and let $\{m_\alpha, \alpha \in I\}$ be a complete family of finitely additive measures on $\underset{\sim}{B}$. For $A \in \underset{\sim}{B}$, $B \in \underset{\sim}{B}^0$, let $\alpha(B)$ be the least $\alpha \in I$ such that $0 < m_\alpha(B) < \infty$, and define

$$
\begin{aligned}
P(A|B) &= m_{\alpha(B)}(A\cap B)/m_{\alpha(B)}(B),\\
r(A,B) &= m_{\alpha(A\cup B)}(A)/m_{\alpha(A\cup B)}(B) \quad \text{if } m_{\alpha(A\cup B)}(B) > 0,\\
&= \infty \quad \text{if not.}
\end{aligned}
$$

Then (i) P is a conditional probability on $\underset{\sim}{B}$ and (ii) r is the locally coherent rate of exchange associated with P.

Proof: (i) Part (a) of definition 4.1 is obvious. To check (b), notice that, if $m_{\alpha(C)}(A\cap C) > 0$, then $\alpha(A\cap C) = \alpha(C)$, and

$$
\begin{aligned}
P(A|C)P(B|A\cap C) &= \frac{m_{\alpha(C)}(A\cap C)}{m_{\alpha(C)}(C)} \cdot \frac{m_{\alpha(C)}(A\cap B\cap C)}{m_{\alpha(C)}(A\cap C)}\\
&= P(A\cap B|C).
\end{aligned}
$$

If $m_{\alpha(C)}(A\cap C) = 0$, then $P(A|C) = 0 = P(A\cap B|C)$ and (b) holds.

(ii) This is easily verified using the formula in Theorem 4.1(ii). □

The construction of Theorem 5.2 makes it easy to define countably additive conditional probabilities, a problem found difficult by Krauss [10, p. 236].

Apply Lemma 5.2 and Theorem 5.1 to a singleton {m} as in Example 5.1 to see that every improper (or proper) prior m is consistent with a locally coherent rate of exchange.

Not every m determines a strictly coherent rate, but it is now easy to characterize those which do.

Theorem 5.3. A rate of exchange r on $\underset{\sim}{B}$ is strictly coherent if and only if there is a finitely additive measure m on $\underset{\sim}{B}$ such that, for every $A \in \underset{\sim}{B}$ and every $B \in \underset{\sim}{B}^0$, $0 < m(B) < \infty$ and $r(A,B) = m(A)/m(B)$.

Proof: Suppose r is strictly coherent. Then r is certainly locally coherent. Let $\{m_\alpha, \alpha \in \Gamma\}$ be the family given by Theorem 5.1. We need to show that Γ contains only a single element. Suppose to the contrary that $\alpha, \beta \in \Gamma$ with $\alpha < \beta$. Choose sets $A \in \alpha$, $B \in \beta$. Then $r(A,B) = m_\beta(A)/m_\beta(B) = 0/m_\beta(B) = 0$. Thus the exchange

$$\begin{aligned} e &= -(r(A,B)B - A) \\ &= A \end{aligned}$$

is everywhere nonnegative and positive on A, contradicting strict coherence.

For the converse, suppose m is a measure on $\underset{\sim}{B}$ which is everywhere finite and positive on $\underset{\sim}{B}^0$, and that $r(A,B) = m(A)/m(B)$ for $A \in B$, $B \in \underset{\sim}{B}^0$. Then every simple exchange and, hence, every exchange has integral zero with respect to m. Thus no exchange e can be everywhere nonnegative and somewhere positive. (The set where $e > 0$ would belong to $\underset{\sim}{B}^0$ and have positive measure under m.) □

Kemeny [9] argues that strict coherence is a reasonable requirement in his framework. It seems a bit stringent to us, because, in view of Theorem 5.2, it would rule out even proper, countably additive priors on an algebra such as the

Borel subsets of the unit interval.

6. An extension theorem.

Let r be a rate of exchange defined on an arbitrary domain $\underset{\sim}{D}$ consisting of pairs (A,B) in $\underset{\sim}{B}\times\underset{\sim}{B}^0$. (Recall that $\underset{\sim}{B}^0 = \underset{\sim}{B}\setminus\{\emptyset\}$.)

Theorem 6.1. If r is locally coherent then r has a locally coherent extension to all of $\underset{\sim}{B}\times\underset{\sim}{B}^0$.

This theorem extends several results in the literature including de Finetti's theorem on the extension of coherent previsions [3, p. 78] and Holzer's theorem on the extension of coherent conditional previsions [8]. Our theorem is closely related to that of Holzer, but the proof will be quite different.

The proof will be given in several lemmas and is based on the study of a partial order $\dot{\leq}$ of the elements of $\underset{\sim}{B}$ which would correspond to the Renyi ordering if $\underset{\sim}{D}$ were already all of $\underset{\sim}{B}\times\underset{\sim}{B}^0$. Until the very last step in the proof we will assume that r takes only finite values in $[0,\infty)$.

To define the ordering, first let $\underset{\sim}{E}$ be the linear space of all exchanges $e(\lambda)$ as defined in (2.1). Associate to each such exchange $e(\lambda)$ the sets

$$E^+(\lambda) = [e(\lambda) > 0],$$

$$E^-(\lambda) = [e(\lambda) \leq 0] \cap \operatorname{supp}(\lambda).$$

Then, for A,B ε $\underset{\sim}{B}$, define A $\dot{\leq}$ B if there is an exchange $e(\lambda)$ ε $\underset{\sim}{E}$ with

(6.1) $$E^+(\lambda) \supset A\backslash B \text{ and } E^-(\lambda) \subset B.$$

(Notice $E^+(\lambda) \overset{\bullet}{\leq} E^-(\lambda)$ in this ordering.)

For $B \in \underset{\sim}{B}$, define $\underset{\sim}{F}_B = \{A \in \underset{\sim}{B}: \ A \overset{\bullet}{\leq} B\}$.

Lemma 6.1. $\underset{\sim}{F}_B$ is an ideal in the Boolean algebra $\underset{\sim}{B}$, and $B \in \underset{\sim}{F}_B$.

Proof: For the second assertion, consider the exchange $e(\lambda)$ which is indentically zero and has $E^+(\lambda) = E^-(\lambda) = \emptyset$. Clearly, $E^+(\lambda) \supset B\backslash B$ and $E^-(\lambda) \subset B$. So $B \overset{\bullet}{\leq} B$.

To prove the first assertion, we must verify these two properties:

(a) $A_1 \subseteq A \in \underset{\sim}{F}_B \Rightarrow A_1 \in \underset{\sim}{F}_B$,

(b) $A_1, A_2 \in \underset{\sim}{F}_B \Rightarrow A_1 \cup A_2 \in \underset{\sim}{F}_B$.

Property (a) is obvious because the exchange $e(\lambda)$ satisfying (6.1) will still work if A is replaced by A_1.

To prove (b), notice that we may assume that $(A_1 \cup A_2) \cap B = \emptyset$. Assume this and find $e(\lambda_1)$, $e(\lambda_2)$ in $\underset{\sim}{E}$ so that $A_i \subset E^+(\lambda_i)$, $i = 1,2$, and $B \supset E^-(\lambda_1) \cup E^-(\lambda_2)$. Let $e(\lambda_3) = e(\lambda_1) + e(\lambda_2)$.

Suppose $w \in A_1$. Then $e(\lambda_1)(w) > 0$. Also, $w \notin B$ whence $e(\lambda_2)(w) \geq 0$. Thus $A_1 \subset E^+(\lambda_3)$. Similarly, $A_2 \subset E^+(\lambda_3)$ and, consequently,

(6.2) $$A_1 \cup A_2 \subset E^+(\lambda_3).$$

Now suppose $w \in E^-(\lambda_3)$. Then $w \in \text{supp}(\lambda_1) \cup \text{supp}(\lambda_2)$ and $e(\lambda_1) + e(\lambda_2) \leq 0$.

To get a contradiction, suppose $w \notin B$ and say, $w \in \text{supp}(\lambda_1)$. Then $e(\lambda_1)(w) > 0$ (because $B \supset E^-(\lambda_1)$) and, hence, $e(\lambda_2)(w) < 0$. But then $w \in E^-(\lambda_2) \subset B$, a contradiction. We must conclude that

$$E^-(\lambda_3) \subseteq B. \tag{6.3}$$

Property (b) follows from (6.2) and (6.3). □

<u>Lemma 6.2</u>. If $A_1 \dot{\lesssim} A_2$ and $A_2 \in \underset{\sim}{F}_B$, then $A_1 \in \underset{\sim}{F}_B$. (In other words, $\dot{\lesssim}$ is transitive.)

<u>Proof</u>: Write

$$A_1 = (A_1 \setminus (A_2 \cup B)) \dot{\cup} (A_1 \cap (A_2 \cup B)).$$

By lemma 6.1, $A_2 \cup B \in \underset{\sim}{F}_B$ and $A_1 \cap (A_2 \cup B) \in \underset{\sim}{F}_B$. Thus, by Lemma 6.1 again, we need only show that

$$A_1 \setminus (A_2 \cup B) \in \underset{\sim}{F}_B \tag{6.4}$$

and we can and do assume that

$$A_1 \cap (A_2 \cup B) = \emptyset. \tag{6.5}$$

To establish (6.4), find $e(\lambda_1)$ and $e(\lambda_2)$ in $\underset{\sim}{E}$ such that: $E^-(\lambda_2) \subset B$,

$E^+(\lambda_2) \supset A_2 \backslash B$; $E^-(\lambda_1) \subset A_2$, $E^+(\lambda_1) \supset A_1 \backslash A_2 = A_1$ (by (6.5)).

Then $E^+(\lambda_2) \supset E^-(\lambda_1) \backslash B$ or $e(\lambda_2) > 0$ on $E^-(\lambda_1) \backslash B$. Hence, there is a positive number α such that $\alpha e(\lambda_2) + e(\lambda_1) > 0$ on $E^-(\lambda_1) \backslash B$. Define $e(\lambda_3) = \alpha e(\lambda_2) + e(\lambda_1)$. It now suffices to verify (a) $E^-(\lambda_3) \subset B$ and (b) $E^+(\lambda_3) \supset A_1$.

For (a), suppose $w \in E^-(\lambda_3)$. Then either $w \in E^-(\lambda_1)$ or $w \in E^-(\lambda_2)$. By the choice of α, $w \notin E^-(\lambda_1) \not\subset B$, and, by the choice of $e(\lambda_2)$, $E^-(\lambda_2) \subset B$.

For (b), suppose $w \in A_1$. Then $e(\lambda_1)(w) > 0$. Also, by (6.5), $w \notin B$ so that $e(\lambda_2)(w) \geq 0$. Hence, $e(\lambda_3) > 0$. □

Assume from now on that r is locally coherent. For the proof of Theorem 6.1 we can also assume that the domain $\underset{\sim}{D}$ of r includes every pair (A,A) for $A \in \underset{\sim}{B}$ and that $r(A,A) = 1$. Obviously the addition of these pairs to $\underset{\sim}{D}$ will not introduce any incoherency.

The next lemma is the key to the proof. It produces a measure μ which will play the role of one of the measures occuring in Theorem 5.1.

Lemma 6.3. Let $B \in \underset{\sim}{B}^0$. Then the following are true.

(a) There is a non-zero finitely additive measure μ defined on $\underset{\sim}{F}_B$ with values in $[0,\infty)$ such that

$$\mu(e) = \int e\,d\mu = 0$$

for every exchange $e = e(\lambda) \in \underset{\sim}{E}$ satisfying $\text{supp}(\lambda) \in \underset{\sim}{F}_B$.

(b) For any μ as in (a), $\mu(B) > 0$.

(c) If $C \in \underset{\sim}{F}_B$ and $\underset{\sim}{F}_C = \underset{\sim}{F}_B$, then, for any μ as in (a), $\mu(C) > 0$.

Proof: (a) Let $C \in \underset{\sim}{F}_B$ and $C \supset B$. (Such sets exist. For example, take $C = B$.) By definition of $\underset{\sim}{F}_B$, there is an exchange $e(\lambda_C) \in \underset{\sim}{E}$ with $B \supset E^-(\lambda_C)$, $E^+(\lambda_C) \supset C \backslash B$. Set $D = \text{supp}(\lambda_C) \cup B$.

There is a probability measure ν_D defined on $\underset{\sim}{B}$ such that (i) $\nu_D(D) = 1$ and (ii) $\nu_D(e(\lambda)) = 0$ for $e(\lambda) \in \underset{\sim}{E}$ with $\text{supp}(\lambda) \subset D$. This follows from applying a separating hyperplane theorem ([5], p. 417) to separate the collection of exchanges $\underset{\sim}{E}_D = \{e(\lambda) \in \underset{\sim}{E}: \text{supp}(\lambda) \subset D\}$ from the cone $\underset{\sim}{D}^+$ of bounded functions defined on D which have a positive infimum on D. $\underset{\sim}{D}^+$ is open in the sup norm topology and the assumption that r is locally coherent implies that $\underset{\sim}{D}^+$ and $\underset{\sim}{E}_D$ are disjoint. In addition to (i) and (ii), ν_D also satisfies (iii) $\nu_D(B) > 0$.

To see this, suppose to the contrary that $\nu_D(B) = 0$. Then $\nu_D(E^-(\lambda_C)) = 0$ and, by (ii),

$$0 = \nu_D(e(\lambda_C)) = \int_{E^+(\lambda_C)} e(\lambda_C) d\nu_D.$$

Now the infimum of $e(\lambda_C)$ on $E^+(\lambda_C)$ is positive and so it follows that $\nu_D(E^+(\lambda_C)) = 0$. But then $\nu_D(D) = 0$, contradicting (i).

Next define the measure μ_C on $\underset{\sim}{B}$ by

$$\mu_C(A) = \nu_D(A \cap D)/\nu_D(B)$$

for $A \in \underset{\sim}{B}$ and regard μ_C as a linear functional on the linear space λ whose elements are finite linear combinations of indicator functions of sets in $\underset{\sim}{B}$.

Notice that, by (ii), $\mu_C(e(\lambda)) = 0$ when $e(\lambda) \in \underset{\sim}{E}$ and $\text{supp}(\lambda) \subset C$ (because $C \subset D$).

Order $\underset{\sim}{F}_B$ by inclusion and let μ be a limit point of the net $\{\mu_C: C \in \underset{\sim}{F}_B\}$ of $[0,\infty]$-valued functions on $\underset{\sim}{L}$ where such functions are given the topology of pointwise convergence.

We must verify that μ restricted to $\underset{\sim}{F}_B$ has the properties listed in (a) of the theorem. Clearly, μ is additive because each μ_C is; $\mu(B) = 1$ because $\mu_C(B) = 1$ for all C and consequently μ is not the zero measure; $\mu(e(\lambda)) = 0$ if $\text{supp}(\lambda) \in \underset{\sim}{F}_B$ because $\mu_C(e(\lambda))$ is eventually zero. It remains to be shown that $\mu(A) < \infty$ for $A \in \underset{\sim}{F}_B$. To see this, it suffices to show $\mu(A \backslash B) < \infty$ because $\mu(A \cap B) \leq \mu(B) = 1$. Choose an exchange $e = e(\lambda) \in \underset{\sim}{E}$ satisfying (6.1). Then $\mu(E^-(\lambda)) \leq \mu(B) = 1$ and, hence,

$$\int_{E^-(\lambda)} e d\mu > -\infty. \tag{6.6}$$

Now $E^+(\lambda) \dot{\leq} E^-(\lambda) \subseteq B$. So, by Lemmas 6.1 and 6.2, $E^+(\lambda)$, $E^-(\lambda)$ and, consequently, $\text{supp}(\lambda)$ are elements of $\underset{\sim}{F}_B$. Thus

$$0 = \mu(e(\lambda)) = \int_{E^-(\lambda)} e d\mu + \int_{E^+(\lambda)} e d\mu \tag{6.7}$$

and, by (6.6) and (6.7),

$$\int_{E^+(\lambda)} e d\mu < +\infty.$$

Use (6.1) and the fact that $e = e(\lambda)$ has a positive infimum on $E^+(\lambda)$ to get

$$\mu(A\backslash B) \le \mu(E^{+}(\lambda)) < \infty,$$

which is the inequality we needed.

(b) Suppose μ has the properties listed in (a). If $\mu(B) = 0$, then an argument like that in the preceeding paragraph shows $\mu(A) = 0$ for all $A \in \underset{\sim}{F}_B$, a contradiction.

(c) Suppose $C \in \underset{\sim}{F}_B = \underset{\sim}{F}_C$. Then μ has the properties listed in (a) when B is replaced there by C. So $\mu(C) > 0$ by an application of (b). □

Notice that, if A_1 and A_2 are elements of $\underset{\sim}{F}_B$ and if the pair (A_1,A_2) is in the domain $\underset{\sim}{D}$ of r, then part (a) of Lemma 6.3 applied to the exchange $e = r(A_1,A_2)A_2 - A_1$ shows that

$$r(A_1,A_2)\mu(A_2) = \mu(A_1). \tag{6.8}$$

Recall that r has been assumed to be a locally coherent exchange rate defined on $\underset{\sim}{D} \subset \underset{\sim}{B} \times \underset{\sim}{B}^0$ with values $[0,\infty)$. Order the collection of such exchange rates by saying that r_1 dominates r_2 if $\underset{\sim}{D}_1 \supset \underset{\sim}{D}_2$ and $r_1|\underset{\sim}{D}_2 = r_2$. An obvious application of Zorn's Lemma shows there is a maximal locally coherent exchange rate with values in $[0,\infty)$. Assume now that r is such an exchange rate.

Lemma 6.4. Let $C \in \underset{\sim}{F}_B$. Then $(C,B) \in \underset{\sim}{D}$ and $r(C,B) = \mu(C)/\mu(B)$ where μ is the measure in Lemma 6.3. Hence μ is uniquely determined by r up to a proportionality constant.

Proof: The formula for $r(C,B)$ is immediate from (6.8) once we show $(C,B) \in \underset{\sim}{D}$.

Suppose to the contrary that $(C,B) \notin \underset{\sim}{D}$. Let $\underset{\sim}{D}' = \underset{\sim}{D} \cup \{(C,B)\}$ and define r' be an exchange rate which agrees with r on $\underset{\sim}{D}$ and has $r'(C,B) = \mu(C)/\mu(B)$.

Because r is maximal, r' cannot be locally coherent. So there exists an exchange

$$e(\lambda') = \pm \left(\frac{\mu(C)}{\mu(B)} B - C\right) + e(\lambda)$$

where $e(\lambda)$ is an exchange for r such that $e(\lambda') > 0$ on $\text{supp}(\lambda') = \text{supp}(\lambda) \cup B \cup C$. It follows that if

$$e(\lambda') = \frac{\mu(C)}{\mu(B)} B - C + e(\lambda),$$

then $B \supset E^-(\lambda)$, $C \setminus B \subset E^+(\lambda)$, and $\mu(C) > 0$. So in this case $B \gtrsim E^-(\lambda) \gtrsim E^+(\lambda)$ and, by Lemma 6.1, $\text{supp}(\lambda) \in \underset{\sim}{F}_B$. Then, by Lemma 6.3(a),

$$0 < \int e(\lambda')d\mu = \frac{\mu(C)}{\mu(B)} \mu(B) - \mu(C) + \int e(\lambda)d\mu = 0,$$

a contradiction. Similarly, if

$$e(\lambda') = C - \frac{\mu(C)}{\mu(B)} B + e(\lambda),$$

then $C \supset E^-(\lambda)$ and $B \setminus C \subset E^+(\lambda)$. So in this case, $C \gtrsim E^-(\lambda) \gtrsim E^+(\lambda)$ and $\text{supp}(\lambda) \in \underset{\sim}{F}_C \subset \underset{\sim}{F}_B$. Once again

$$0 < \int e(\lambda')d\mu = 0,$$

a contradiction. We must conclude that $(C,B) \in \underset{\sim}{D}$. □

For sets $A,B \in \underset{\sim}{B}$, define $A \doteq B$ if $A \dot{\leq} B$ and $B \dot{\leq} A$.

<u>Lemma 6.5</u>. Let $A \in \underset{\sim}{B}$, $B \in \underset{\sim}{B}^0$, and let μ be the measure of Lemma 6.3.

(a) $A \in \underset{\sim}{F}_B$ if and only if $r(A,B)$ is defined (as a finite number).

(b) If $A \in \underset{\sim}{F}_B$, then $A \doteq B$ if and only if $\mu(A) > 0$ if and only if $0 < r(A,B) < \infty$.

<u>Proof</u>: (a) If $A \in \underset{\sim}{F}_B$, then $r(A,B)$ is well-defined by Lemma 6.4. Conversely, if $r(A,B)$ is defined, consider the exchange

$$e(\lambda) = -r(A,B)B + A.$$

Clearly, $B \supset E^-(\lambda)$ and $A\backslash B \subset E^+(\lambda)$. Hence, $A \dot{\leq} B$.

(b) $A \doteq B \Rightarrow \mu(A) > 0$ (by Lemma 6.3(c))

$$\Rightarrow r(A,B) = \mu(A)/\mu(B) \in (0,\infty).$$

Finally, if $A \in \underset{\sim}{F}_B$ and $0 < r(A,B) < \infty$, consider the exchange $e(\lambda) = r(A,B)B - A$. Then $A \supset E^-(\lambda)$ and $B\backslash A \subset E^+(\lambda)$. So $A \dot{\geq} B$. □

<u>Lemma 6.6</u>. The order $\dot{\leq}$ is complete on $\underset{\sim}{B}$; that is, given A_1, A_2 in $\underset{\sim}{B}$, either $A_1 \dot{\leq} A_2$ or $A_2 \dot{\leq} A_1$.

Proof: Since $\emptyset \dot{\lesssim} A$ for all A, assume that A_1 and A_2 are nonempty. Let $B = A_1 \cup A_2$. Then A_1, A_2 are in $\underset{\sim}{F}_B$ and, by Lemma 6.4

$$r(A_1 \cup A_2, B) = 1 = r(A_1, B) + r(A_2, B).$$

Hence, either $r(A_1,B) > 0$ or $r(A_2,B) > 0$ and, by Lemma 6.5, either $A_1 \doteq B \gtrsim A_2$ or $A_2 \doteq B \gtrsim A_1$. □

The relation $\doteq$ is an equivalence relation on $\underset{\sim}{B}$. (Symmetry and reflexivity are clear; transitivity follows from the transitivity of $\gtrsim$ (Lemma 6.2).) The quotient space $\underset{\sim}{B}/\doteq$ of equivalence classes is linearly ordered in a complete fashion by the order induced on $\underset{\sim}{B}/\doteq$ by $\dot{\lesssim}$. For $\alpha \in \underset{\sim}{B}/\doteq$, pick a representative $B_\alpha \in \alpha$ and let μ_α be the measure on $\underset{\sim}{F}_{B_\alpha}$ given by Lemma 6.3. For $C \in \underset{\sim}{B} \setminus \underset{\sim}{F}_{B_\alpha}$, define $\mu_\alpha(C)$ to be ∞. The family $\{\mu_\alpha\}$ has the properties required for the Renyi-Krauss representation of Theorem 5.1.

Lemma 6.7. Each μ_α is a finitely additive measure on $\underset{\sim}{B}$ and the family $\{\mu_\alpha\}$ is complete and linearly ordered.

Proof: Easy using the preceeding lemmas. □

Apply the converse half of Theorem 5.1 to complete the proof of Theorem 6.1 for the case under consideration in which r takes on only finite values.

Finally, suppose $\underset{\sim}{D} \subset \underset{\sim}{B} \times \underset{\sim}{B}^0$ and $r: \underset{\sim}{D} \longrightarrow [0,\infty]$ is locally coherent. Let $\underset{\sim}{C}$

be the collection of pairs (B,A) such that (A,B) $\in \underset{\sim}{D}$ and $r(A,B) = \infty$. Define

$$\underset{\sim}{D}' = \underset{\sim}{D} \cup \underset{\sim}{C}$$

and define r' on $\underset{\sim}{D}'$ by $r'|\underset{\sim}{D} = r$ and $r'(B,A) = 0$ for (B,A) $\in \underset{\sim}{C}$.

Lemma 6.8. r' is locally coherent on $\underset{\sim}{D}'$.

Proof: Suppose $r(A_i,B_i) = \infty$ for $i = 1,\ldots, n$ and there is an exchange

$$e(\lambda') = \sum_{i=1}^{n} c_i[B_i - 0A_i] + e(\lambda)$$

where $e(\lambda)$ is based on r and $e(\lambda') \geq \varepsilon > 0$ on supp(λ'). We will reach a contradiction to the local coherence of r by finding an exchange $e(\lambda'')$ based on r such that $e(\lambda'') \geq \varepsilon/2$ on supp(λ'').

We can assume $c_i > 0$ for $i = 1,\ldots, n$. (If some $c_i < 0$, then the term $c_i[B_i - 0A_i]$ is nowhere positive and can be deleted from $e(\lambda')$.) Set

$$e(\lambda'') = \frac{\varepsilon}{2n} \sum_{i=1}^{n} [\infty B_i - A_i] + e(\lambda).$$

Then

$$e(\lambda'') \geq e(\lambda') - \varepsilon/2 \geq \varepsilon/2$$

and supp(λ'') = supp(λ'). □

Now let

$$\underline{D}' = \{(A,B) \in \underline{D}' : r'(A,B) < \infty\}$$

and let

$$r'' = r' | \underline{D}''.$$

Obviously, r'' is locally coherent and has only finite values. By the case already treated, r'' has a locally coherent extension $\bar{r}$ to $\underline{B} \times \underline{B}^0$. But $\bar{r}$ also extends r because $\bar{r}(A,B) = 0$ implies $\bar{r}(B,A) = \infty$ (Theorem 3.1(v)).

The proof of Theorem 6.1 is now complete.

References

1. Carlson, Tim (1985). A solution of Ulam's problem on relative measure. Proc. Amer. Math. Soc. 94 129-134.

2. de Finetti, Bruno (1937). La Prevision, ses lois logique, ses sources subjectives. Ann. Inst. H. Poincare 7 1-68.

3. de Finetti, Bruno (1972). Probability, Induction and Statistics. Wiley, New York.

4. Dubins, Lester E. (1975). Finitely additive conditional probabilities, conglomerability, and disintegrations. Ann. Prob. 3 89-99.

5. Dunford, Nelson and Schwartz, Jacob J. (1958). Linear Operators, Part I: General Theory. Interscience, New York.

6. Hartigan, John (1983). Bayes Theory. Springer-Verlag, New York.

7. Heath, David and Sudderth, William (1978). "On finitely additive priors, coherence, and extended admissibility," Ann. Statist. 6 333-345.

8. Holzer, S. (1985). "On coherence and conditional prevision," Bollettino U.M.I., Analisi Funzionale e Applicazioni, Serie VI, Vol. IV-C, N. 1.

9. Kemeny, John G. (1955). "Fair bets and inductive probabilities," J. Symbolic Logic 20 263-273.

10. Krauss, P.H. (1968). "Representation of conditional probability measures on Boolean algebras," Acta Mathematica Academial Scientiarum Hungaricae 19 (3-4) 229-241.

11. Lane, David and Sudderth, William (1983). "Coherent and continuous inference," Ann. Statist. 11 114-120.

12. Lane, David and Sudderth, William (1984). "Coherent predictive inference," Sankhya 46 Series A 166-185.

13. Regazzini, Eugenio (1983). "Coherent conditional probabilities, finite additivity, extensions," Technical Report Dipart Sc. Statist., University of Bologna.

14. Renyi, A. (1956). "On conditional probability spaces generated by a dimensionally ordered set of measures," Theory of Probability and its Applications, I 61-71.

15. Smith, Cedric A.B. (1961). "Consistency in statistical inference and decision," J. Royal Stat. Soc. (B), 23, 1-37.